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SUMMARY

This study is an extension of the work that was initiated by Huang (2001) and further developed by Huang (2004) for simulation of stochastic processes using Karhunen-Loeve (K-L) expansion. The scope of this research is limited to non-Gaussian simulation.

When the random process is indexed over a domain that is much larger than the correlation distance, K-L expansion will approach the popular spectral representation. It follows that the non-Gaussian K-L expansion proposed by Phoon et al. (2002a and 2005) could be applied to the spectral representation as a special case. This special case is of pragmatic interest because the tedious eigenvalue problem in K-L expansion is replaced by FFT. Numerical examples with different target spectral density functions and different target marginal distribution functions are illustrated to demonstrate the capability of this spectral representation within the more general K-L framework. Results show that non-Gaussian K-L expansion can generate different processes satisfying the same target spectral density function and the same target marginal distribution function, which is potentially capable of providing a better fit to observed data.

The difference in simulating the spectral representation using the non-Gaussian K-L approach and the standard translation approach is investigated. Theoretically, the commonly used non-Gaussian translation process exists only when the prescribed target spectral density function and marginal distribution function have a compatible relationship. The situation with incompatible functions was previously handled by correcting the original target spectral density function using spectral preconditioning. Examples with both compatible and incompatible spectral density function and marginal distribution function show that non-Gaussian K-L expansion can generate
different processes satisfying the same target spectral density function and the same
target marginal distribution function regardless of their compatibility. Results also
show that the previously used spectral preconditioning technique will change the
original target processes significantly for those with strongly incompatible target
functions. Therefore, the K-L scheme has an advantage over the translation approach
because it has no requirements for the compatibility of the target spectral density
function and the target marginal distribution function.

Finally, the non-Gaussian K-L expansion technique is further extended to
simulate multi-dimensional non-Gaussian stochastic fields. Since non-Gaussian K-L
technique has been successfully applied to the spectral representation for simulation
of highly skewed non-Gaussian processes, it is straightforward to simulate multi-
dimensional non-Gaussian stochastic fields using multi-dimensional spectral
representation within the more general K-L framework. Numerical examples
pertaining to non-Gaussian stochastic fields with different marginal distribution
functions are presented. It is demonstrated that different non-Gaussian stochastic
fields satisfying the same target spectral density function and the same target marginal
distribution function can be generated using the non-Gaussian K-L expansion method,
and that the K-L simulated non-Gaussian stochastic fields are in general non-
translation fields. Numerical results also show that the initial spectral random
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<tr>
<td>$a$</td>
<td>range of stochastic process</td>
</tr>
<tr>
<td>$A$</td>
<td>skewness statistics</td>
</tr>
<tr>
<td>$b_1, b_2$</td>
<td>parameters in the spectral density function of target stochastic field</td>
</tr>
<tr>
<td>$B$</td>
<td>kurtosis statistics</td>
</tr>
<tr>
<td>$C(\tau)$</td>
<td>covariance function of stationary process</td>
</tr>
<tr>
<td>$C(x_1, x_2)$</td>
<td>covariance function of non-stationary process</td>
</tr>
<tr>
<td>$D$</td>
<td>bounded domain of stochastic process</td>
</tr>
<tr>
<td>$E$</td>
<td>K-L generated non-Gaussian processes or fields from exponential initial spectral random variables</td>
</tr>
<tr>
<td>$E[\cdot]$</td>
<td>mathematical expectation</td>
</tr>
<tr>
<td>$f_i(x)$</td>
<td>(1) eigenfunction of covariance function</td>
</tr>
<tr>
<td></td>
<td>(2) harmonic function for stochastic process</td>
</tr>
<tr>
<td>$f_{i_1i_2\cdots q}(x_1, x_2, \cdots, x_q)$</td>
<td>harmonic function for $q$-dimensional stochastic field</td>
</tr>
<tr>
<td>$F$</td>
<td>marginal distribution function of non-Gaussian process</td>
</tr>
<tr>
<td>$F_x$</td>
<td>marginal distribution function $F$ for different value of $x$</td>
</tr>
<tr>
<td>$\hat{F}_M^{(k)}$</td>
<td>cumulative distribution function of simulated non-Gaussian process</td>
</tr>
<tr>
<td>$\tilde{F}_{M_1M_2\cdots M_q}^{(k)}$</td>
<td>cumulative distribution function of simulated non-Gaussian field</td>
</tr>
<tr>
<td>$g(x, \theta)$</td>
<td>Gaussian stochastic process</td>
</tr>
</tbody>
</table>
G  K-L generated non-Gaussian process or field from independent Gaussian initial spectral random variables

G1  K-L generated non-Gaussian field from correlated Gaussian initial spectral random variables with exponential covariance

G2  K-L generated non-Gaussian field from correlated Gaussian initial spectral random variables with square exponential covariance

$H_0$  null hypothesis

$H_1$  alternative hypothesis

$I_{(\text{event})}$  indicator function, 1 if event is true and 0 otherwise

$k$  (1) iteration number

(2) number of the dimension of stochastic vector

$m$  sample number

$M$  number of terms in K-L expansion or spectral representation

$M_1, M_2, \cdots, M_q$  number of terms in K-L expansion for $q$-dimensional stochastic fields

$n$  sample size

$N$  number of sampled points in each realization

$p$  parameter in beta process

$q$  (1) parameter in beta process

(2) number of the dimension of non-Gaussian field

$Q$  triangular matrix after the Cholesky decomposition of $T$
\( R_{\text{gs}}(\tau) \) correlation function of underlying Gaussian process

\( R'_{\text{gs}}(\tau) \) corrected correlation function of underlying Gaussian process

\( R_{\text{mn}}(\tau) \) correlation function of stochastic process \( \sigma(x, \theta) \)

\( R'_{\text{mn}}(\tau) \) corrected correlation function of \( \sigma(x, \theta) \)

\( R_{\text{mn}}(\tau_1, \tau_2) \) correlation function of two-dimensional stochastic field

\( R_{\text{mn}}(\tau_1, \tau_2, \cdots, \tau_q) \) correlation function of \( q \)-dimensional stochastic field

\( S_{\text{gs}}(\omega) \) power spectral density of underlying Gaussian process

\( S'_{\text{gs}}(\omega) \) corrected power spectral density of underlying Gaussian process

\( S_{\text{mn}}(\omega) \) power spectral density of stochastic process \( \sigma(x, \theta) \)

\( S'_{\text{mn}}(\omega) \) corrected power spectral density of \( \sigma(x, \theta) \)

\( S_{\text{mn}}(\omega_1, \omega_2) \) power spectral density of target two-dimensional field

\( \hat{S}_{\text{mn}}(\omega_1, \omega_2) \) power spectral density of simulated two-dimensional stochastic field

\( S_{\text{mn}}(\omega_1, \omega_2, \cdots, \omega_q) \) power spectral density of \( q \)-dimensional stochastic field

\( T \) correlation matrix

\( T \) non-Gaussian translation processes or fields

\( T_0 \) length of stochastic process

\( T_{x_0}, T_{x_10}, \cdots, T_{x_q0} \) length of stochastic field along different directions

\( u \) parameter in beta process

\( U \) vector containing ones

\( V_i(\theta) \) random variables in spectral representation
\( V_{\theta_{1,2}}(\theta) \) random variables in two-dimensional stochastic field

\( V_{\theta_{1,2,\ldots,q}}(\theta) \) random variables in \( q \)-dimensional stochastic field

\( W_i(\theta) \) random variables in spectral representation

\( W_{\theta_{1,2}}(\theta) \) random variables in two-dimensional stochastic field

\( W_{\theta_{1,2,\ldots,q}}(\theta) \) random variables in \( q \)-dimensional stochastic field

\( x \) position of stochastic process

\( x_1, x_2 \) (1) two arbitrary positions of stochastic process

(2) different directions of two-dimensional field

\( x_1, x_2, \ldots, x_q \) different directions of \( q \)-dimensional field

\( X \) matrix containing realizations of \( \xi_i(\theta) \)

\( X' \) uncorrelated matrix transformed from matrix \( X \)

\( y_{\min}, y_{\max} \) parameters in beta process

\( \alpha \) level of significance.

\( \beta_{1k}^2 \) skewness statistics

\( \beta_{2k} \) kurtosis statistics

\( \gamma_i \) eigenvector of covariance matrix

\( \Gamma(\cdot) \) gamma function

\( \delta_{ij} \) Kronecker-delta function

\( \Delta \omega \) frequency interval

\( \Delta \omega_1, \Delta \omega_2 \) frequency interval along \( \omega_1, \omega_2 \)

\( \Delta \omega_1, \Delta \omega_2, \ldots, \Delta \omega_q \) frequency interval along \( \omega_1, \omega_2, \ldots, \omega_q \)

\( \varsigma(\tau) \) covariance function of stationary non-Gaussian process
\( \zeta_{\text{max}}(\tau) \) upper bound of \( \zeta(\tau) \)

\( \zeta_{\text{min}}(\tau) \) lower bound of \( \zeta(\tau) \)

\( \zeta(x_1, x_2) \) covariance function of non-stationary non-Gaussian process

\( \zeta_{\text{max}}(x_1, x_2) \) upper bound of \( \zeta(x_1, x_2) \)

\( \zeta_{\text{min}}(x_1, x_2) \) lower bound of \( \zeta(x_1, x_2) \)

\( \overline{\eta}_M^{(k)}(x) \) average value of \( \eta_M^{(k)}(x, \theta_m) \)

\( \eta_M^{(k)}(x, \theta_m) \) sample function obtained by mapping in the \( k \)th iteration of K-L expansion for non-Gaussian process

\( \overline{\eta}_{M_1 M_2 \cdots M_q}^{(k)}(x_1, x_2, \cdots x_q) \) average value of \( \eta_{M_1 M_2 \cdots M_q}^{(k)}(x_1, x_2, \cdots x_q, \theta_m) \)

\( \eta_{M_1 M_2 \cdots M_q}^{(k)}(x_1, x_2, \cdots x_q, \theta_m) \) sample function obtained by mapping in the \( k \)th iteration of K-L expansion for non-Gaussian field

\( \theta \) element in the sample space

\( \Theta(\cdot) \) function of covariance function \( \rho(\tau) \)

\( \lambda_e \) parameter in the shifted exponential process

\( \lambda_i \) eigenvalue of covariance function

\( \mu_e \) parameter in the shifted exponential process

\( \mu_g \) mean of underlying Gaussian process \( g(x, \theta) \)

\( \mu_l \) parameter in the shifted lognormal process

\( \mu_v \) mean of stochastic vector \( \nu \)

\( \mu_{\nu} \) mean of stochastic process \( \nu(x, \theta) \)

\( \nu \) stochastic vector
\[ \xi_i(\theta) \quad \text{K-L random variables} \]
\[ \xi_i^{(k)}(\theta) \quad \text{K-L random variables in the } k^{\text{th}} \text{ iteration} \]
\[ \xi_{ij\cdots i_q}(\theta) \quad \text{K-L random variables for } q\text{-dimensional stochastic field} \]
\[ \xi_{ij\cdots i_q}^{(k)}(\theta) \quad \text{K-L random variables in the } k^{\text{th}} \text{ iteration for } q\text{-dimensional stochastic field} \]
\[ \rho(\tau) \quad \text{covariance function of underlying Gaussian process} \]
\[ \sigma_g \quad \text{standard deviation of underlying Gaussian process} \]
\[ \sigma_i \quad \text{standard deviation of stochastic process} \]
\[ \sigma_{ij} \quad \text{standard deviation of two-dimensional field} \]
\[ \sigma_{ij\cdots i_q} \quad \text{standard deviation of } q\text{-dimensional field} \]
\[ \sigma^2 \quad \text{variance of stochastic process } \sigma(x,\theta) \]
\[ \Sigma_{\nu} \quad \text{covariance matrix of stochastic vector } \nu \]
\[ \tau \quad \text{lag distance} \]
\[ \tau^* \quad \text{special value of } \tau \]
\[ \tau_1, \tau_2 \quad \text{lag distance for two-dimensional stochastic field} \]
\[ \tau_1, \tau_2, \cdots, \tau_q \quad \text{lag distance for } q\text{-dimensional stochastic field} \]
\[ \Phi(\cdot) \quad \text{cumulative distribution function of standard Gaussian variate} \]
\[ \omega \quad \text{frequency} \]
\[ \omega_{\text{max}} \quad \text{maximum cut-off frequency} \]
\[ \omega_1, \omega_2 \quad \text{frequency axis for two-dimensional stochastic field} \]
\[ \omega_1, \omega_2, \cdots, \omega_q \quad \text{frequency axis for } q\text{-dimensional stochastic field} \]
\( \omega_{j_{\text{max}}} \) (\( j = 1, 2, \ldots q \))  
maximum cut-off frequency for \( q \)-dimensional field

\( \Omega, A, P \)  
probability space

\( \sigma(x, \theta) \)  
stochastic process

\( \overline{\sigma}(x) \)  
mean of stochastic process

\( \sigma_{M}(x, \theta) \)  
approximated process from truncated K-L expansion

\( \sigma_{M}^{(k)}(x, \theta) \)  
approximated process in the \( k^{\text{th}} \) iteration

\( \sigma(x_{1}, x_{2}, \theta) \)  
two-dimensional stochastic field

\( \sigma(x_{1}, x_{2}, \ldots x_{q}, \theta) \)  
\( q \)-dimensional stochastic field

\( \sigma_{M_{1}M_{2} \ldots M_{q}}(x_{1}, x_{2}, \ldots x_{q}, \theta) \)  
approximated field from truncated K-L expansion

\( \sigma_{M_{1}M_{2} \ldots M_{q}}^{(k)}(x_{1}, x_{2}, \ldots x_{q}, \theta) \)  
approximated field in the \( k^{\text{th}} \) iteration

\( \phi[\cdot] \)  
joint probability density function of correlated standard Gaussian variables

\( \varepsilon \)  
acceptable error in the variance of stochastic field
CHAPTER 1

INTRODUCTION

1.1 Background

Numerous reliability-based design (RBD) codes which take into consideration the stochastic nature of material properties have been put into practice for routine structural design (e.g., ACI 1983; BSI 1972; CSA 1974; NKB 1978) since the mid-1970s. However, the geotechnical design community has been slow in assimilating this new design methodology. Part of the reason lies in the difficulty of assessing the variability of soil properties that are needed for these new RBD procedures. In RBD, the uncertainties in the soil properties are modeled as stochastic processes or fields.

A significant amount of work has been devoted to developing methodologies for the characterization and simulation of stochastic processes. Most of these methodologies focus on the Gaussian process which is completely determined by its mean value and covariance function and can be simulated very efficiently and uniquely using different methods, e.g., auto-regressive moving average (ARMA) (Samaras et al., 1985); spectral representation (Shinozuka and Jan, 1972); Karhunen-Loeve expansion (Ghanem and Spanos, 1991a); and wavelet (Zeldin and Spanos, 1996).

However, the Gaussian assumption may not be appropriate in a number of geotechnical engineering problems because most of the soil properties in geotechnical
engineering applications are physical quantities that can only assume positive values, e.g., elastic modulus, shear modulus, and cone tip resistance. A Gaussian assumption to model them leads to a non-zero probability of obtaining negative values for such quantities. This non-zero probability is usually very small for small coefficient of variation (in short COV, and is defined by the standard deviation divided by the mean value), but it can become significant for larger values of COV. Unlike the uncertainties in structural material properties which fall within a narrow range, geotechnical properties exhibit a large COV. For example, the COV for soil modulus and the undrained shear strength of clays was found to be in the range of 20-70% and 10-60%, respectively (Phoon and Kulhawy, 1999). If these quantities are modeled as Gaussian processes, it implies that a substantial number of realizations will take on negative values. Serious numerical problems can arise consequently during the process of analyzing the system and erroneous results may be obtained. Currently, these non-negative properties for soil are modeled as lognormal processes, which have a simple relationship with the Gaussian processes. However, even within the third-order statistics of the observed data, this lognormal model has its limitation because it cannot deal with those data having negative skewness. It is therefore obvious that the development of algorithm for generating non-Gaussian stochastic processes is of theoretical and practical significance.

A major difficulty in dealing with non-Gaussian processes has been the complexity of their characterization. Generally, a stochastic process is fully characterized only when the entire family of joint multi-dimensional density functions
is specified. This information is encapsulated in the second-order statistics for Gaussian processes. However, for non-Gaussian processes, no such simplification exists in general. Hence, the target of many simulation algorithms has been to match the specified lower-order moments or marginal probability distribution function and the covariance function or equivalently, the power spectral density function.

At present, simulation of non-Gaussian processes is mostly based on memoryless nonlinear transforms of some underlying Gaussian process. Such processes are known as translation processes (Grigoriu, 1984). Theoretically, a translation process exists only when the prescribed target covariance function and marginal distribution function have a compatible relationship. However, there is no physical reason to ensure that the target covariance function and marginal distribution function from real data must satisfy this translation restriction. Furthermore, it is well known that a non-Gaussian process cannot be defined uniquely by the first two moments, but a non-Gaussian translation process is unique. Therefore, the commonly used translation process may not produce a process that could match observed non-Gaussian data.

Phoon et al. (2002a) suggested using the Karhunen-Loeve (K-L) expansion with non-Gaussian random variables to produce non-Gaussian processes based on prescribed covariance functions and marginal distribution functions using an iterative scheme. This K-L method has been refined to produce good results for highly skewed non-Gaussian marginal distributions (Phoon, et al., 2005). The main feature of this technique is that the target covariance function is maintained, while the
probability distributions of the K-L random variables are updated iteratively. In principle, this approach is very attractive because it can be extended to non-stationary and multi-dimensional fields in a unified way. However, the application of this K-L expansion technique and its relationship with the existing non-Gaussian translation processes have not been fully investigated.

1.2 Objective and Scope

Since in general the observed data for soil properties in real geotechnical problems may be non-Gaussian non-translational, the commonly used translation method may not produce a process that could match observed non-Gaussian data. The K-L scheme is an alternative which has the potential to circumvent the theoretical restrictions imposed by the popular translation approach. Therefore, the objective of this study is to model the observed data more realistically using K-L expansion.

K-L expansion with uncorrelated non-Gaussian random variables has been successfully applied to the simulation of highly skewed non-Gaussian processes (Phoon et al., 2002a and 2005). When the random process is indexed over a domain that is much larger than the correlation distance, K-L expansion will approach the spectral representation. This special case is of pragmatic interest because the power spectral density function is less costly to evaluate (via FFT of the covariance function) than the eigenvalues and eigenfunctions in the K-L expansion. Therefore, in the first part of this study, the non-Gaussian K-L expansion proposed by Phoon et al. (2002a and 2005) will be applied to the spectral representation as a special case.
Subsequently, the difference in simulating the spectral representation using the non-Gaussian K-L approach and the standard translation approach will be investigated. Theoretically, a non-Gaussian translation process exists only when the prescribed spectral density function and the prescribed marginal distribution function have a compatible relationship. However, compatibility is not guaranteed if the target spectral density function and the target marginal distribution function are specified independently. The situation with incompatible functions was previously handled by simulating the translation processes that only match the target spectral density function approximately. Hence, a crucial point of this study is to demonstrate that K-L expansion can be used to address the situation with incompatible target functions where the convenient translation method may not be applicable. The comparison between K-L processes and translation processes will also be made for non-Gaussian processes with compatible target functions.

Finally, the non-Gaussian K-L expansion technique will be further extended to simulate multi-dimensional non-Gaussian stochastic fields. Since non-Gaussian K-L expansion technique has been successfully applied to the spectral representation for simulation of highly skewed non-Gaussian processes, it is relatively straightforward to simulate multi-dimensional non-Gaussian fields using multi-dimensional spectral representation within a more general K-L framework. Numerical examples will be used to demonstrate the capability of this new method. Comparisons between the K-L simulated fields and the translation fields will also be made.
1.3 Organization

Chapter 2 summarizes a review on the literature which covers simulation methods related to the research in this thesis for both Gaussian and non-Gaussian processes.

Chapter 3 introduces the simulation algorithm for non-Gaussian K-L expansion. An extension of the non-Gaussian K-L expansion to spectral representation is presented. Numerical examples are provided to demonstrate the capability of this spectral representation within the more general K-L framework.

Chapter 4 demonstrates the simulation of non-Gaussian processes with incompatible spectral density functions and marginal distribution functions using non-Gaussian K-L expansion. The difference between this K-L approach and the traditional approach will be studied.

Chapter 5 presents the simulation algorithm for multi-dimensional non-Gaussian stochastic fields using spectral representation within a more general K-L framework. Numerical examples will be used to demonstrate the capability of this method. Comparisons between the K-L simulated fields and the translation fields will be made.

Chapter 6 summaries the key research findings and recommends the scope for future study.
CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

Monte Carlo simulation (MCS) is the only currently available universal methodology for solving problems involving stochastic properties. Though once considered impractical because of its major computational expense, MCS is now steadily gaining favor after the widespread availability of inexpensive computational systems. Currently, MCS is the most widely used probabilistic approach in geotechnical problems (Ohtomo and Shinozuka, 1990; Popescu, 1995; Popescu et al., 1997 and 2005; Fenton et al., 2005; Fenton and Griffiths, 2002 and 2005; Low, 2005). The usefulness of MCS is based on the fact that the best situation to represent the probability distribution of a certain random quantity is to have a corresponding large population of data. The implementation of this method consists of numerically simulating a population corresponding to the basic random quantities in the problem, solving the deterministic problem associated with each member of that population, and obtaining a population corresponding to the random response quantities. This population can then be used to obtain the statistics of the response.

In this method, the most important part is the generation of sample functions to account for the uncertainties involved in the problem which are usually modeled as stochastic processes. The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes. This chapter
reviews the current simulation methods for stochastic processes.

### 2.2 Overview of Simulation of Gaussian Processes

Simulation of Gaussian processes is well established. In general, Gaussian processes are mostly simulated by means of: (1) auto-regressive moving average (ARMA) (Samaras et al., 1985; Mignolet, 1987; Mignolet and Spanos, 1987 and 1992; Naganuma et al., 1987; Spanos and Mignolet, 1987, 1990 and 1992); (2) spectral representation (Shinozuka and Jan, 1972; Yang, 1972; Shinozuka, 1974 and 1987; Shinozuka and Deodatis, 1991; Grigoriu, 1993a and 1993b); (3) Karhunen–Loeve expansion (Ghanem and Spanos, 1991a and 1991b; Gutierrez et al., 1992; Li and Der Kiureghian, 1993; Zhang and Ellingwood, 1994; Huang et al., 1999; Huang, 2001; Huang, 2004; Phoon et al., 2002b and 2004); and (4) wavelet expansion (Gurly and Kareem, 1994; Zeldin, 1996; Zeldin and Spanos, 1996; Spanos and Rao, 2001). Huang (2001) and Huang (2004) made quite extensive review of simulation methods for Gaussian processes. Thus, in this thesis, two methods related to the research in this thesis are reviewed here.

#### 2.2.1 Karhunen-Loeve expansion

The K-L expansion approximates a Gaussian process by a linear combination of orthogonal deterministic functions with independent standard Gaussian random variables where the orthogonal deterministic functions and their magnitude are the eigenfunctions and eigenvalues of the covariance function, respectively. The
efficiency of K-L expansion for simulating random processes hinges crucially on the availability of accurate eigenvalues and eigenfunctions of the covariance function. The eigensolution of the covariance function involves the solution of a Fredholm integral equation. The solution for such an equation can be obtained analytically for some special cases. In general, a numerical solution has to be implemented. An efficient wavelet-Galerkin scheme has been proposed by Phoon et al. (2002b) which has been demonstrated to be computationally equivalent to using wavelet directly for stochastic expansion and simulating the correlated random coefficients using eigenvalue and eigenfunction decomposition (Phoon et al., 2004).

Other expansions involving orthogonal functions such as the Legendre polynomials also may be used to represent the random process. Such expansions can avoid the problem of solving the integral equation for eigenfunctions, but more terms are required to achieve the same accuracy as in K-L expansion. Zhang and Ellingwood (1994) showed that expanding a random process on any orthogonal base is equivalent to expanding the random process using the K-L expansion.

### 2.2.2 Spectral representation

The spectral representation method represents the Gaussian stochastic processes using a set of trigonometric functions and corresponding Gaussian random coefficients. The Gaussian random coefficients are uncorrelated only if the stochastic processes are assumed stationary and the length of the process is infinite or periodic (Stark and Woods, 1994). This method generates the sample functions of
stochastic processes with highly accurate spatial statistics because of the periodicity and orthogonality of the trigonometric functions used for expansion. Yang (1972) showed that the Fast Fourier Transform (FFT) technique can be used to improve the computational efficiency dramatically. This method can be readily extended to simulate multi-dimensional, multivariate processes from one-dimensional, univariate processes. The extension of this method to non-stationary processes has been implemented by introducing the evolutionary spectral density concept developed by Priestley (1965).

2.3 Overview of Simulation of Non-Gaussian Processes

Simulation methods for non-Gaussian stochastic processes can be grouped into two general classes (Deodatis and Micaletti, 2001). The first class seeks to generate sample functions of non-Gaussian processes according to their prescribed lower order moments and power spectral density function (Gurley, 1997; Gurley et al., 1997; Poirion, 2001) while the second class is based on the prescribed marginal probability distribution and the power spectral density function. Methods of the second class are relatively more challenging than those of the former class since matching a prescribed marginal probability distribution is equivalent to matching an infinite number of prescribed moments.

Therefore, only the simulation methods in the second class are examined in this review. These methods include: (1) translation processes (Yamazaki and Shinozuka, 1988; Grigoriu, 1995 and 1998; Deodatis and Micaletti, 2001; Arwade,
(2005; Ferrante et al., 2005); (2) polynomial chaos expansion (Sakamoto and Ghanem, 2002a and 2002b; Puig et al., 2002; Puig and Akian, 2004; Xiu and Karniadakis, 2002a, 2002b and 2003; Xiu et al., 2002); (3) non-Gaussian K-L expansion (Phoon et al., 2002a and 2005). Only two methods related to the research in this thesis are reviewed here.

### 2.3.1 Translation processes

The definition of a general non-Gaussian process is based on its finite dimensional distributions. This characterization of a non-Gaussian process is impractical because it is difficult to construct approximations of random processes from their finite dimensional distributions. Currently, a number of available techniques for simulating non-Gaussian processes represent them as nonlinear transformations of some underlying Gaussian processes:

\[
\varphi(x, \theta) = F^{-1} \circ \Phi[g(x, \theta)]
\]  

(2.1)

where \( \varphi(x, \theta) \) is the target non-Gaussian process with marginal distribution \( F \); \( g(x, \theta) \) is the underlying Gaussian process; and \( \Phi(\cdot) \) is the cumulative distribution function of standard Gaussian variate.

The process \( \varphi(x, \theta) \) defined by Equation (2.1) is known as translation process (Grigoriu, 1984). Grigoriu (1995 and 1998) demonstrates that the finite dimensional distributions of translation processes are determined by those of the underlying Gaussian processes. A stochastic process may not exist if its finite dimensional distributions are not compatible as required by Kolmogorov’s conditions.
Therefore, transformation from an underlying Gaussian process is a practical necessity since high-order finite dimensional distributions are frequently unavailable.

The challenge in simulation of translation processes is to determine a Gaussian covariance function that would yield the target non-Gaussian covariance function. Direct solution of the relationship between the covariance of the target translation process and that of the underlying Gaussian process involves solving an integral equation (Grigoriu, 1995 and 1998) or an equivalent differential equation given by Price’s theorem (Deutsch, 1962). This direct approach is not easy to apply in practice.

In many related simulation algorithms, the determination of the appropriate underlying Gaussian processes which will produce the specified non-Gaussian content typically involved an iterative procedure. Yamazaki and Shinozuka (1988) were among the first to propose a simulation algorithm for stationary non-Gaussian translation processes. His approach consisted of correcting the power spectral density (PSD) function (Fourier transform of covariance function) of the underlying Gaussian process until the PSD function of the target non-Gaussian process is achieved. This simulation algorithm was modified by Deodatis and Micaletti (2001) to simulate highly skewed non-Gaussian processes.

The standard stationary translation process as defined by Equation (2.1) has been recently extended to the non-stationary case (Ferrante, et al., 2005):

\[ \sigma(x, \theta) = F_x^{-1} \circ \Phi[g(x, \theta)] \]  

(2.2)
where \( F_x \) is the prescribed marginal cumulative distribution function of \( \sigma(x, \theta) \) which may not be identical for different \( x \).

The limitation of the translation process is that it requires the target covariance function (or equivalently, spectral density function) and the target marginal distribution function to be compatible. However, if the target covariance function and the target marginal distribution function are specified independently, compatibility is not guaranteed and the translation process satisfying these two target functions simultaneously may not exist.

2.3.2 Non-Gaussian Karhunen-Loeve expansion

Phoon et al. (2002a) suggested using the K-L expansion with non-Gaussian random variables to produce the desired non-Gaussian process. The principal difficulty is that the distributions of the non-Gaussian K-L random variables are not the same as the prescribed non-Gaussian distribution for the process and seems to require an iterative solution approach. These iterative steps have been proposed by Phoon et al. (2002a) to compute for the unknown K-L distributions. This K-L method has been refined to produce very good results for highly skewed non-Gaussian marginal distributions (Phoon et al., 2005). The key elements of the method are: (1) back-calculate the marginal distributions of the K-L random variables empirically by using orthogonality of the eigenfunctions and (2) reducing product-moment correlations between the K-L random variables by using a modified Latin hypercube scheme. The feature of this technique is that the target covariance
function is maintained, while the probability distributions of the K-L random variables are updated iteratively. This approach can be extended readily to non-stationary and multi-dimensional fields in a unified way. Furthermore, it has the potential to simulate different non-Gaussian processes satisfying the same target covariance function and marginal distribution.

Unlike the commonly used translation method which is fundamentally related to the multivariate Gaussian probability structure, the non-Gaussian K-L expansion is based on a different concept and has the potential to avoid the restriction imposed by the translation approach.

2.4 Summary

The literature review for simulation methods for stochastic processes is provided in this chapter. Simulation of Gaussian processes is a prerequisite for the non-Gaussian translation method. Techniques for Gaussian processes simulation have been well established, and two of them which are strongly related to the research in this thesis, K-L expansion and spectral representation, are reviewed. For non-Gaussian simulation methods, the currently most-used translation method is first introduced. However, this method has limitations to model realistically observed data because it requires the target covariance function and the target marginal distribution function to be compatible. The non-Gaussian K-L expansion is then reviewed. This scheme is conceptually different from current methods and has the potential to circumvent the theoretical restrictions of the popular translation approach.
CHAPTER 3
EXTENSION OF NON-GAUSSIAN KARHUNEN-LOEVE EXPANSION

3.1 Introduction

Karhunen-Loeve (K-L) expansion represents a stochastic process as a set of uncorrelated random variables multiplied by a corresponding set of orthogonal deterministic functions where these orthogonal deterministic functions and their magnitudes are the eigenfunctions and eigenvalues of the covariance function, respectively. For Gaussian processes, the random variables are independent standard Gaussian random variables. However, for non-Gaussian processes, the distributions of the random variables do not follow those of the target processes.

Phoon et al. (2002a and 2005) suggested using the K-L expansion with non-Gaussian random variables to produce non-Gaussian processes based on the prescribed covariance function and marginal distribution function using an iterative scheme. The main feature of this technique is that the target covariance function is maintained, while the probability distributions of the K-L random variables are updated iteratively.

When the random process is indexed over a domain that is much larger than the correlation distance, K-L expansion will approach the spectral representation (Van Trees, 1968). It follows that the K-L simulation method proposed by Phoon et al. (2002a and 2005) could be applied to the spectral representation as a special case.
This special case is of pragmatic interest because the power spectral density function is less costly to evaluate (via FFT of the covariance function) than the eigenvalues and eigenfunctions in the K-L expansion. Hence, the spectral representation is preferred whenever it is applicable.

In this chapter, non-Gaussian K-L expansion has been extended to the case of spectral representation. Numerical examples are presented to demonstrate the capability of this spectral representation within the more general K-L framework.

3.2 Simulation Algorithm for Non-Gaussian Karhunen-Loeve Expansion

A random process \( \varphi(x, \theta) \) defined on a probability space \( (\Omega, A, P) \) and indexed on a bounded domain \( D \), having a mean \( \overline{\varphi}(x) \) and a finite variance \( \sigma^2 \), can be approximated using the following finite K-L expansion:

\[
\varphi_M(x, \theta) = \overline{\varphi}(x) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) f_i(x)
\]  

(3.1)

where \( M \) is the number of K-L terms; \( \lambda_i \) and \( f_i(x) \) are the eigenvalues and eigenfunctions of the covariance function \( C(x_1, x_2) \), satisfying the second kind Fredholm integral equation

\[
\int_D C(x_1, x_2) f_i(x_2) dx_2 = \lambda_i f_i(x_1)
\]

(3.2)

and \( \xi_i(\theta) \) is a set of uncorrelated K-L random variables with zero-mean and unit-variance, that is

\[
E[\xi_i(\theta)] = 0
\]

(3.3)
\[ E[\xi_i(\theta)\xi_j(\theta)] = \delta_{ij} \]  

(3.4)

where \( \delta_{ij} \) is the Kronecker-delta function.

If \( \sigma(x, \theta) \) is a Gaussian process, \( \xi_i(\theta) \) is a vector of uncorrelated standard Gaussian random variables. For \( \sigma(x, \theta) \) with an arbitrarily prescribed marginal distribution, the distributions of \( \xi_i(\theta) \) are not identical, and do not follow the distribution of the target non-Gaussian process. Hence, an iterative procedure is required to estimate these unknown non-Gaussian random variables.

The following iterative steps were proposed to compute these unknown K-L random variables (Phoon et al., 2002a):

1. Generate \( n \) sample functions of the non-Gaussian process:

\[ \sigma^{(k)}_{M}(x, \theta_m) = \bar{\sigma}(x) + \sum_{i=1}^{M} \sqrt{\lambda_i \varphi^{(k)}_i(\theta_m)} f_i(x) \quad m = 1, 2, \ldots n \]  

(3.5)

where \( k \) = iteration number and \( m \) = sample number. The initial random variables \( \varphi_i(\theta) \) in this step are chosen as some known distributed random variables, such as Gaussian, lognormal, uniform, etc. They are assumed to be independent for convenience because they can be readily obtained through MATLAB command.

2. Estimate the empirical cumulative marginal distribution function as:

\[ \hat{F}^{(k)}_{M}(y | x) = \frac{1}{n} \sum_{m=1}^{n} I(\sigma^{(k)}_{M}(x, \theta_m) \leq y) \]  

(3.6)

where \( I(event) \) = indicator function = 1 if event is true and 0 otherwise.

3. Transform each sample function to match the target marginal distribution \( F \):

\[ \eta^{(k)}_{M}(x, \theta_m) = F^{-1}\left[ \hat{F}^{(k)}_{M}(x, \theta_m) \right] \]  

(3.7)
4. Estimate the next generation of $\tilde{\xi}_i(\theta)$ as:

$$\tilde{\xi}_i^{(k+1)}(\theta_m) = \frac{1}{\sqrt{\lambda_i}} \int_D \left[ \eta_M^{(k)}(x, \theta_m) - \overline{\eta}_M^{(k)}(x) \right] f_i(x) \, dx$$  

(3.8)

where $\overline{\eta}_M^{(k)}(x)$ is the average value of $\eta_M^{(k)}(x, \theta_m)$. This step back-calculates the K-L random variables using orthogonality of the eigenfunctions.

5. Standardize $\tilde{\xi}_i^{(k+1)}(\theta)$ to unit-variance. Note that $\tilde{\xi}_i^{(k+1)}(\theta)$ is a zero-mean vector by virtue of Equation (3.8). A modified Latin hypercube orthogonalization technique (Phoon et al., 2005) is applied to reduce the product-moment correlations between all columns of $\tilde{\xi}_i^{(k+1)}(\theta)$ for the same $k+1$. This technique is described below, assuming that the realizations of $\tilde{\xi}_i(\theta)$ are stored in an $n \times M$ matrix $X$:

(i) Compute the $M \times M$ product-moment covariance matrix of $X$:

$$T = \frac{X^T X}{n-1} - \frac{X^T U U^T X}{n(n-1)}$$  

(3.9)

where $U$ is a $n \times 1$ vector containing ones.

(ii) Obtain an uncorrelated realization matrix $X'$ by:

$$X' = X Q^{-1}$$  

(3.10)

where

$$Q^T Q = T$$  

(3.11)

(iii) Re-order the realizations in each column of $X$ to follow the ranking of realizations in each column of $X'$.

6. Repeat steps (1) through (5) until the sample functions achieved the target marginal distribution.
3.3 Relationship between Karhunen-Loeve Expansion and Spectral Representation

In many cases, stationary processes characterized over an infinite interval are of interest. When the process to be represented is stationary and the observation interval is infinite, Equation (3.2) becomes:

$$\lambda f(x_2) = \int_{-\infty}^{\infty} C(x_2 - x_1) f(x_1) dx_1 \quad -\infty < x_2 < \infty$$  \hspace{1cm} (3.12)

Equation (3.12) is analogous to the linear filtering problem with input \( f(x) \) and impulse response \( C(x_2 - x_1) \) such that the output is \( f(x) \) with a change in gain given by \( \lambda \). It is well known from elementary linear circuit theory that this requirement can be met by making \( f(x) = e^{i\omega x} \).

Substituting into Equation (3.12) yields:

$$\lambda = \int_{-\infty}^{\infty} C(x_2 - x_1) e^{-i\omega(x_2 - x_1)} dx_1 = S_{\omega}(\omega)$$  \hspace{1cm} (3.13)

Thus the eigenvalue for a particular \( \omega \) is the value of the two-sided power spectral density function \( S_{\omega}(\omega) \) of the process at that \( \omega \).

Van Trees (1968) has shown that for the case of a finite length process defined in \([-a, a]\) with large \( a \):

$$\lambda_k \approx S_{\omega}(\omega_k) = S_{\omega}\left(\frac{\pi k}{a}\right)$$  \hspace{1cm} (3.14)

$$f_k(x) = \frac{1}{\sqrt{2a}} e^{i\omega_k a x}$$  \hspace{1cm} (3.15)

Substituting Equations (3.14) and (3.15) into Equation (3.1), results in:

$$\varphi(x, \theta) = \bar{\varphi}(x) + \sum_{k=1}^{M} \sqrt{\frac{1}{2a}} S_{\omega}\left(\frac{\pi k}{a}\right) e^{i\omega_k a x} \xi_k(\theta)$$  \hspace{1cm} (3.16)
which is the commonly used spectral representation for the simulation of stationary
Gaussian random process (Shinozuka and Deodatis, 1991; Grigoriu, 1993a). Thus
for large $a$, the K-L expansion reduces to the spectral representation method. Huang
et al. (2001) shows that spectral representation produces almost the same result as
K-L expansion for first-order Markov process when the domain is 200 times longer
than the correlation length.

3.4 Spectral Representation Using Non-Gaussian Random
Variables

Non-Gaussian K-L expansion with uncorrelated non-Gaussian random
variables has been successfully applied to the simulation of highly skewed
non-Gaussian processes. The key elements of the method are: (1) back-calculate the
marginal distributions of the K-L random variables empirically by using the
orthogonality of eigenfunctions and (2) minimize the product-moment correlations
between the K-L random variables by using a modified Latin hypercube
orthogonalization scheme.

Since spectral representation is a limiting case of K-L expansion, it follows
that the K-L simulation method proposed by Phoon et al. (2002a and 2005) could be
applied to the popular spectral representation. This special case is of pragmatic
interest because eigenvalues are replaced by the power spectral density function and
eigenfunctions are known trigonometric functions. Hence, a tedious eigenvalue
problem is replaced by FFT. In addition, the integral involved in the evaluation of
the next generation of K–L random variables as shown in Equation (3.8) is also replaced by FFT.

A one-dimensional, univariate, stationary random process \( \sigma(x, \theta) \) with zero-mean and two-sided power spectral density function \( S_{\sigma\sigma}(\omega) \) can be approximated as follows:

\[
\sigma(x, \theta) = \sum_{i=0}^{\infty} \sigma_i [V_i(\theta) \cos(\omega_i x) + W_i(\theta) \sin(\omega_i x)]
\]

(3.17)

in which \( \theta \) is an element in the sample space, \( V_i(\theta) \) and \( W_i(\theta) \) are uncorrelated random variables with zero-mean and unit-variance, and

\[
\sigma_i = \left(2S_{\sigma\sigma}(\omega_i) \Delta \omega\right)^{1/2}
\]

(3.18)

where \( \Delta \omega \) is the discretization interval along the frequency axis:

\[
\omega_i = i \Delta \omega
\]

(3.19)

Equation (3.17) can also be viewed as a discrete approximation of the spectral representation (Cramér, 1939). If \( \sigma(x, \theta) \) is a Gaussian process, \( V_i(\theta) \) and \( W_i(\theta) \) are independent standard Gaussian random variables. In the general (non-Gaussian) case, the distributions of \( V_i(\theta) \) and \( W_i(\theta) \) are not the same as the prescribed non-Gaussian marginal distribution of the process \( \sigma(x, \theta) \). More importantly, \( V_i(\theta) \) and \( W_i(\theta) \) are not independent, even though they are uncorrelated. The dependence between \( V_i(\theta) \) and \( W_i(\theta) \) must be preserved because, otherwise, the representation becomes Gaussian if refined by increasing the number of harmonics indefinitely. Thus far, there is no feasible method of constructing the multivariate distribution of \( V_i(\theta) \) and \( W_i(\theta) \) for the general case. Hence, it is not surprising
that the translation strategy is the most popular.

Given that the spectral representation is a limiting case of the K-L expansion, it should be possible to simulate non-Gaussian processes directly from Equation (3.17) without using translation. Following the notations in section 3.2, Equation (3.17) is rewritten as follows:

\[
\mathbf{\sigma}_M(x, \theta) = \sum_{i=0}^{M-1} \mathbf{\sigma}_i \xi_i^T(\theta) f_i(x)
\]  

(3.20)

in which

\[
\xi_i(\theta) = \begin{bmatrix} V_i(\theta) \\ W_i(\theta) \end{bmatrix}
\]  

(3.21)

and

\[
f_i(x) = \begin{bmatrix} \cos(\omega_i x) \\ \sin(\omega_i x) \end{bmatrix}
\]  

(3.22)

\(M\) is the number of terms satisfying:

\[
\Delta \omega = \frac{\omega_{\text{max}}}{M}
\]  

(3.23)

in which \(\omega_{\text{max}}\) (maximum cut-off frequency) is selected based on the acceptable error in the variance of the process and \(\Delta \omega\) is selected to produce a process of adequate length \(T_0\):

\[
T_0 = \frac{2\pi}{\Delta \omega}
\]  

(3.24)

The similar iterative steps as in section 3.2 are used to compute the unknown spectral random variables \(V_i(\theta)\) and \(W_i(\theta)\):

1. Generate \(n\) sample functions of the non-Gaussian process:
\( \sigma_M^{(i)}(x, \theta_m) = \sum_{i=0}^{M-1} \sigma_i \xi_i^{(i)}(\theta_m)f_i(x) \quad m = 1, 2, \ldots n \) \hspace{1cm} (3.25)

2. Estimate the empirical cumulative marginal distribution function as:

\( \hat{F}_M^{(i)}(y | x) = \frac{1}{n} \sum_{m=1}^{n} I(\sigma_M^{(i)}(x, \theta_m) \leq y) \) \hspace{1cm} (3.26)

3. Transform each sample function to match the target marginal distribution \( F \):

\( \eta_M^{(i)}(x, \theta_m) = F^{-1}\hat{F}_M^{(i)}[\sigma_M^{(i)}(x, \theta_m)] \) \hspace{1cm} (3.27)

4. Estimate the next generation of \( \xi_i(\theta) \) as:

\[ \xi_i^{(k+1)}(\theta_m) = \frac{1}{\sigma_i} \int_{\xi_i^{-1}(\eta_M^{(i)}(x, \theta_m) - \eta_M^{(i)}(x))} f_i(x)dx \] \hspace{1cm} (3.28)

As noted previously, this integral can be evaluated efficiently using FFT.

5. Standardize \( \xi_i^{(k+1)}(\theta) \) to unit-variance and reduce the correlations between all columns of the realizations of \( \xi_i^{(k+1)}(\theta) \) using modified Latin hypercube orthogonalization technique. Note that the realizations here are \( n \times 2M \), not \( n \times M \) as in section 3.2.

6. Repeat steps (1) through (5) until the sample functions achieved the target marginal distribution.

3.5 Numerical Examples

In this section, examples are illustrated to demonstrate the effectiveness of the spectral representation within the more general K-L framework.

Two marginal distribution functions are selected:

1. Beta distribution
where $F(x; p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x z^{p-1}(1-z)^{q-1} dz$ (3.29)

and $u$ is given by

\[ u = \frac{y - y_{\text{min}}}{y_{\text{max}} - y_{\text{min}}} \] (3.30)

in which

\[ y_{\text{min}} = \mu - \sigma \sqrt{\frac{p(p+q+1)}{p}} \] (3.31)

and

\[ y_{\text{max}} = \mu + \sigma \sqrt{\frac{q(p+q+1)}{p}} \] (3.32)

$\mu$ and $\sigma$ are the mean and standard deviation of the target Beta distributed process. For simulation, the distribution parameters are chosen to be $p = 4$ and $q = 2$ so that the mean is zero and the variance is one. Note that the realizations of this distribution are bounded between $y_{\text{min}} = -3.74$ and $y_{\text{max}} = 1.87$.

2. Shifted exponential distribution

\[ F(y; \mu, \lambda) = 1 - e^{-\lambda(y - \mu)} \] (3.33)

where $\lambda = 1$ and $\mu = -1$. The values of the distribution parameters $\lambda$ and $\mu$ are selected to produce zero-mean and unit-variance.

Figure 3.1 shows the target marginal probability density functions (PDF) for both cases.

Two spectral density functions are selected:

1. Exponential function

\[ S_{\text{exp}}(\omega) = \frac{1}{\pi(1 + \omega^2)}, \quad -\infty < \omega < \infty \] (3.34)
2. Square exponential function

\[ S_{\text{sq}}(\omega) = \frac{1}{2\sqrt{\pi}} e^{-\frac{\omega^2}{4}} , \quad -\infty < \omega < \infty \]  \hspace{1cm} (3.35)

Figure 3.2 shows the target spectral density functions for both cases.

For exponential spectral density function, the maximum cut-off frequency and the number of frequency intervals are chosen as \( \omega_{\text{max}} = 5.12\pi \) and \( M = 128 \), respectively. For square exponential spectral density function, the corresponding parameter values are chosen as \( \omega_{\text{max}} = 2\pi \) and \( M = 128 \), respectively. In both cases, the number of sampled points in each realization is chosen as \( N = 256 \). Therefore, the Nyquist frequency is exactly \( \omega_{\text{max}} / 2\pi \). Two thousand realizations (\( n = 2000 \)) are simulated.

Non-Gaussian K-L expansion outlined in Section 3.4 can be used to simulate the target process with exponential spectral density function and Beta marginal distribution function. Figure 3.3a shows that the simulated process matches both the spectral density function and marginal probability density function with their corresponding targets. Note that Gaussian initial spectral random variables are used here.

The non-Gaussian process defined by exponential spectral density function and Beta marginal distribution function can also be simulated from the non-Gaussian K-L expansion starting with initial spectral random variables with distribution other than Gaussian. Exponentially distributed initial spectral random variables are used here. Figure 3.3b shows that the simulated process also satisfies both the prescribed
target spectral density function and prescribed target probability density function.

Since K-L simulated processes from Gaussian and exponential initial random variables both satisfy the same target spectral density function and the same target marginal distribution function, it is natural to check whether they are the same. Figure 3.4 compares the cumulative distribution of the extreme values in each realization of the K-L simulated non-Gaussian processes from Gaussian and exponential initial random variables, showing that they are different processes. Figure 3.5 shows that they have different cumulative first time-to-failure distribution with failure defined as process exceeding threshold $= 1.7$. The result supports the conclusions in Figure 3.4.

Therefore, different processes can be generated by non-Gaussian K-L expansion satisfying the same target spectral density function and the same target marginal distribution function. The non-uniqueness of the target non-Gaussian process is reasonable because it is well known that a non-Gaussian process cannot be defined uniquely by the first two moments.

Figures 3.6-3.8 show the corresponding results of Figures 3.3-3.5 for the target process with square exponential spectral density function and Beta marginal distribution function. Results also show that different processes can be generated by non-Gaussian K-L expansion satisfying the same target spectral density function and the same target marginal distribution function.

Figures 3.9-11 and Figures 3.12-3.14 show the results for example with exponential spectral density function and shifted exponential marginal distribution
function and example with square exponential spectral density function and shifted exponential marginal distribution function, respectively. The results further support the above conclusion.

3.6 Multivariate Gaussianity Test

Figure 3.15 shows the cumulative distribution function of a typical spectral random variable \( V_{i0} \) of K-L simulated processes from Gaussian and exponential initial random variables using a Gaussian probability plot. The linear plots obtained for those from Gaussian initial random variables imply that they are indeed Gaussian random variables. It is found that spectral random variables at other frequencies are also Gaussian for the K-L simulated processes from Gaussian initial random variables. For those generated using exponential initial random variables, the spectral random variables are not Gaussian for any frequency.

The principal component method (Srivastava, 1984) based on the measure of skewness and kurtosis is used to test the multivariate normality for spectral random variables of the K-L simulated processes from Gaussian initial random variables. For a random \( k \)-vector \( \nu \) with mean vector \( \mu_\nu \) and covariance matrix \( \Sigma_\nu \), the measure of skewness \( (\beta_{1k}^2) \) and kurtosis \( (\beta_{2k}^2) \), respectively, can be represented by:

\[
\beta_{1k}^2 = k^{-1} \sum_{i=1}^{p} \left\{ E \left[ y'_i (\nu - \mu_\nu) \right] / \lambda_i^{3/2} \right\}^2 
\]

\[
\beta_{2k}^2 = k^{-1} \sum_{i=1}^{p} \left\{ E \left[ y'_i (\nu - \mu_\nu) \right]^4 / \lambda_i^2 \right\} 
\]

(3.36)

(3.37)
where $E[\cdot]$ denotes mathematical expect value; $\lambda_i$ is the eigenvalue and $\gamma_i$ is eigenvector of $\Sigma$, $i = 1, 2, \cdots, k$.

If $\nu$ is a multivariate Gaussian random vector,

$$
\beta_{1k}^2 = 0 \quad (3.38)
$$

$$
\beta_{2k} = 3 \quad (3.39)
$$

Let

$$
A = (nk/6)\beta_{1k}^2 \quad (3.40)
$$

$$
B = (nk/24)^{1/2}(\beta_{2k} - 3) \quad (3.41)
$$

For large sample size $n$, $A$ is chi-square distributed with $k$ degrees of freedom and $B$ follows a standard Gaussian distribution.

Two-sided hypothesis tests are performed for the spectral random variables from Gaussian initial random variables:

$H_0$: The spectral random variables from Gaussian initial are multivariate normals.

$H_1$: The spectral random variables from Gaussian initial are not multivariate normals.

where $H_0$, $H_1$ denotes the null hypothesis and alternative hypothesis, respectively.

For a specific level of significance $\alpha$, e.g., $\alpha = 0.01$, the acceptance regions for skewness statistics $A$ and kurtosis statistics $B$ are $(200.5875, 316.9194)$ and $(-2.5758, 2.5758)$, respectively.

Results are shown in Table 3.1 for the spectral random variables from Gaussian initial seed distribution for different examples. It can be seen that for the level of significance $\alpha = 0.01$, the kurtosis statistics for all examples fall outside the acceptance region, which leads to the rejection of the null hypothesis. Therefore, the
spectral random variables from Gaussian initial seed distribution are not multivariate normals at the level of significance $\alpha = 0.01$. Although the spectral random variables are uncorrelated Gaussian (maximum correlation between the spectral random variables from Gaussian initial random variables for these examples are $6.22 \times 10^{-4}$, $6.39 \times 10^{-4}$, $6.12 \times 10^{-4}$ and $3.09 \times 10^{-3}$, respectively), they do not pass the multivariate Gaussianity test. Hence, the spectral random variables are not independent. This explains why the summation shown in Equation (3.17) does not converge to Gaussian as required by Central Limit Theorem.

### 3.7 Summary

K-L expansion with uncorrelated non-Gaussian random variables has been successfully applied to the simulation of processes with highly skewed non-Gaussian marginal distributions. When the random process is indexed over a domain that is much larger than the correlation distance, it is well known that the K-L expansion will approach the spectral representation. It follows that the K-L simulation method proposed by Phoon et al. (2002a and 2005) could be applied to the spectral representation as a special case. This special case is of pragmatic interest because the power spectral density function is less costly to evaluate than the eigenvalues and eigenfunctions in the K-L expansion.

In this chapter, non-Gaussian K-L expansion has been extended to the case of spectral representation. Numerical results with different target spectral density functions and marginal distribution functions are presented. It is demonstrated that
non-Gaussian K-L expansion can generate different processes satisfying the same target spectral density function and the same target marginal distribution function, which is potentially capable of providing a better fit to observed data.
Table 3.1  Results of multivariate Gaussianity test for spectral random variables from Gaussian initial random variables

<table>
<thead>
<tr>
<th>Spectral density function</th>
<th>Examples</th>
<th>Skewness statistics</th>
<th>Kurtosis statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \beta_{1p}^2 )</td>
<td>( A )</td>
</tr>
<tr>
<td>Exponential</td>
<td>Beta</td>
<td>0.0029</td>
<td>250.40</td>
</tr>
<tr>
<td>Square exponential</td>
<td>Beta</td>
<td>0.0034</td>
<td>289.72</td>
</tr>
<tr>
<td>Exponential</td>
<td>Shifted exponential</td>
<td>0.0075</td>
<td>637.62</td>
</tr>
<tr>
<td>Square exponential</td>
<td>Shifted exponential</td>
<td>0.0089</td>
<td>756.50</td>
</tr>
</tbody>
</table>
Figure 3.1  Target marginal probability density function (PDF)
Figure 3.2  Target spectral density function

(a) Exponential

(b) Square exponential
Figure 3.3  Spectral density function and PDF of K-L simulated processes versus their corresponding targets for example with exponential spectral density function and Beta PDF: (a) Gaussian initial; (b) exponential initial
Figure 3.4 Comparison of cumulative distribution of extreme value in each realization for K-L simulated processes with exponential spectral density function and Beta PDF from Gaussian and exponential initial random variables.
Figure 3.5  Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 1.7 for K-L simulated processes with exponential spectral density function and Beta PDF from Gaussian and exponential initial random variables.
Figure 3.6 Spectral density function and PDF of K-L simulated processes versus their corresponding targets for example with square exponential spectral density function and Beta PDF: (a) Gaussian initial; (b) exponential initial
Figure 3.7 Comparison of cumulative distribution of extreme value in each realization for K-L simulated processes with square exponential spectral density function and Beta PDF from Gaussian and exponential initial random variables.
Figure 3.8 Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 1.7 for K-L simulated processes with square exponential spectral density function and Beta PDF from Gaussian and exponential initial random variables
Figure 3.9 Spectral density function and PDF of K-L simulated processes versus their corresponding targets for example with exponential spectral density function and shifted exponential PDF: (a) Gaussian initial; (b) exponential initial
Figure 3.10  Comparison of cumulative distribution of extreme value in each realization for K-L simulated processes with exponential spectral density function and shifted exponential PDF from Gaussian and exponential initial random variables
Figure 3.11  Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 3.5 for K-L simulated processes with exponential spectral density function and shifted exponential PDF from Gaussian and exponential initial random variables.
Figure 3.12  Spectral density function and PDF of K-L simulated processes versus their corresponding targets for example with square exponential spectral density function and shifted exponential PDF: (a) Gaussian initial; (b) exponential initial.
Figure 3.13  Comparison of cumulative distribution of extreme value in each realization for K-L simulated processes with square exponential spectral density function and shifted exponential PDF from Gaussian and exponential initial random variables.
Figure 3.14 Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 3.5 for K-L simulated processes with square exponential spectral density function and shifted exponential PDF from Gaussian and exponential initial random variables.
Figure 3.15 Gaussian probability plot of spectral random variables for K-L simulated processes from Gaussian and exponential initial random variables:
(a) exponential spectral density function and Beta PDF
(b) square exponential spectral density function and Beta PDF
(c) exponential spectral density function and shifted exponential PDF
(d) square exponential spectral density function and shifted exponential PDF
CHAPTER 4
SIMULATION OF NON-GAUSSIAN PROCESSES WITH INCOMPATIBLE SPECTRAL DENSITY FUNCTION AND MARGINAL DISTRIBUTION FUNCTION

4.1 Introduction

Currently, most of the available methods for simulation of non-Gaussian processes are based on translation process (Grigoriu, 1984), which invokes the memoryless nonlinear transforms of some underlying Gaussian process. Since a translation process is characterized by the covariance of the underlying Gaussian process, efforts have been devoted to determine the unknown underlying Gaussian covariance function from the known non-Gaussian target. These attempts include the iterative correction of the underlying Gaussian spectral density function (Yamazaki and Shinozuka, 1988; Deodatis and Micaletti, 2001), explicit solution based on the Price theorem (Grigoriu, 1995 and 1998; Arwade, 2005; Ferrante et al., 2005), and Hermite polynomial chaos expansion (Sakamoto and Ghanem, 2002a and 2002b; Puig et al., 2002; Puig and Akian, 2004).

Most of the works assume that correlation in the non-Gaussian process can be specified using a suitably correlated Gaussian process. However, this is not always possible. For an arbitrarily prescribed marginal distribution function and an arbitrarily prescribed covariance function (or equivalently, spectral density function), the underlying Gaussian process which would yield the target covariance function
after transform may not exist. The reason is that translation processes do not form the complete $L_2$ functions in the random space.

In general, a non-Gaussian process can be translational or non-translational. Theoretically, a translation process exists when the prescribed covariance function and the prescribed marginal distribution function have a compatible relationship. However, if the target covariance function and the target marginal distribution function are specified independently, compatibility is not guaranteed and the convenient translation method may not be applicable. The challenge of developing practical simulation methods for cases where translation does not hold has not been explicitly addressed and the assumption in current published works (Deodatis and Micaletti, 2001; Puig et al., 2002) is that it is always possible to construct a translation process which is close enough to the original target process.

Phoon et al. (2002a and 2005) suggested using the K-L expansion with non-Gaussian K-L random variables to simulate non-Gaussian stochastic processes. The non-Gaussian K–L algorithm has also been applied to the popular spectral representation. However, the differences in simulating the spectral representation using the non-Gaussian K–L approach and the standard translation approach has not been investigated.

In this chapter, it will be shown that the non-Gaussian K-L algorithm can be used to simulate non-Gaussian processes with both compatible and incompatible spectral density functions and marginal distribution functions. Therefore, comparing with the commonly used translation approach, the K-L scheme is a more robust
method because it has no requirements for the compatibility of the target spectral
density and marginal distribution functions.

4.2 Compatibility between Covariance Function and Marginal
Distribution Function

Non-Gaussian processes are mostly simulated by matching a prescribed
marginal distribution function and covariance function (or equivalently, the spectral
density function). For translation approach, these two quantities cannot be
prescribed arbitrarily, but must satisfy certain conditions.

Let $g(x, \theta)$ be a stationary zero-mean Gaussian process with unit-variance
and covariance function $\rho(\tau)$. A non-Gaussian process $\varphi(x, \theta)$ satisfying a
prescribed marginal cumulative distribution function $F$ can be constructed using the
following transformation:

$$\varphi(x, \theta) = F^{-1} \circ \Phi[g(x, \theta)]$$

where $\Phi(\cdot)$ denotes the cumulative distribution of the standard Gaussian variate.

The correlation function of the non-Gaussian translation process $R_{\tau \varphi}(\tau)$ is related to
the covariance of the underlying Gaussian process

$$R_{\tau \varphi}(\tau) = E[\varphi(x, \theta)\varphi(x + \tau, \theta)]$$

$\mathbb{E}[.]$ denotes the expected value, and
\[
\phi[z_1, z_2; \rho(\tau)] = \frac{1}{2\pi(1 - \rho(\tau)^2)^{1/2}} \exp \left\{ -\frac{z_1^2 + z_2^2 - 2z_1z_2\rho(\tau)}{2[1 - \rho(\tau)^2]} \right\}
\]  \quad (4.4)

Let
\[
\zeta(\tau) = \frac{R_{\omega\omega}(\tau) - E[\omega(x, \theta)]}{R_{\omega\omega}(0) - E[\omega(x, \theta)]}
\]  \quad (4.5)

be the covariance function of \( \omega(x, \theta) \).

For a non-Gaussian process with prescribed marginal distribution function and covariance function, there is always a memoryless transformation \( F^{-1} \circ \Phi(\cdot) \) such that the target non-Gaussian process has the specified distribution. Unfortunately, it is not always possible to find a covariance function of the underlying Gaussian process yielding the covariance function of the target non-Gaussian process. The compatibility of spectral density function and marginal distribution function has been discussed by Deodatis and Micaletti (2001). A prescribed marginal distribution function and covariance function are deemed incompatible if no translation process satisfying these two quantities exists. There are two types of incompatibilities.

### 4.2.1 Type I: bivariate level incompatibility

Grigoriu (1995 and 1998) has demonstrated that the covariance of the underlying Gaussian process and the stationary translation process matches at 0 and 1, but covariance of the translation process has a lower bound greater than -1 unless the transform is an odd function. However, the condition for odd transform is seldom satisfied and most of the cases are neither odd nor even transforms in practice. Therefore, if the target covariance takes values outside this admissible lower bound, it
means that there are no values of $\rho(\tau)$ in Equation (4.2) that will produce the corresponding target $R_{\text{me}}(\tau)$. Hence, there is no translation process for this specific combination of target covariance function and marginal distribution function. This type of incompatibility is called bivariate level incompatibility (or Type I for brevity).

It has been demonstrated by Grigoriu (1995 and 1998) that the covariance function $\zeta(\tau)$ is non-decreasing in $\rho(\tau)$. Consequently, $\zeta(\tau)$ is bounded by

$$
\zeta_{\text{min}}(\tau) \leq \zeta(\tau) \leq \zeta_{\text{max}}(\tau)
$$

(4.6)

where $\zeta_{\text{min}}(\tau)$, $\zeta_{\text{max}}(\tau)$ can be obtained by setting $\rho(\tau)$ equals to -1 and 1, respectively. This yields:

$$
\zeta_{\text{max}}(\tau) = 1
$$

(4.7)

$$
\zeta_{\text{min}}(\tau) = \frac{E\left[F^{-1} \circ \Phi(g)\right] \cdot E\left[F^{-1} \circ \Phi(-g)\right] - E\left[F^{-1} \circ \Phi(g)\right]^2}{E\left[F^{-1} \circ \Phi(g)^2\right] - E\left[F^{-1} \circ \Phi(g)\right]^2}
$$

(4.8)

$\zeta_{\text{min}}(\tau)$ is greater than -1 unless $F^{-1} \circ \Phi(\cdot)$ is an odd function.

More significantly, the upper bound of $\zeta(\tau)$ could be less than 1 in non-stationary translation cases. Ferrante, et al. (2005) has extended the standard stationary translation process defined by Equation (4.1) to the non-stationary case:

$$
\sigma(x, \theta) = F_{x}^{-1} \circ \Phi[\sigma(x, \theta)]
$$

(4.9)

where $\sigma(x, \theta)$ is the prescribed marginal cumulative distribution function of $\sigma(x, \theta)$ which may not be identical for different $x$.

For this non-stationary case, Equation (4.2) can be generalized to:

$$
\mu_{\sigma}(x_{i}) \mu_{\sigma}(x_{2}) + \sigma_{\sigma}(x_{i}) \sigma_{\sigma}(x_{2}) \zeta(x_{1}, x_{2})
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_{x_{1}}^{-1} \circ \Phi(z_{1}) \cdot F_{x_{2}}^{-1} \circ \Phi(z_{2}) \cdot \phi(z_{1}, z_{2}; \rho(\tau)) \, dz_{1} \, dz_{2}
$$

(4.10)

where $\mu_{\sigma}(x_{i})$, $\sigma_{\sigma}(x_{i})$ are mean and standard deviation of $\sigma(x_{i}, \theta)$ ($i = 1, 2$),
respectively, and \( \zeta(x_1, x_2) \) is the covariance function of \( \sigma(x, \theta) \). The bounds for \( \zeta(x_1, x_2) \) can be obtained (Arwade, 2005; Ferrante et al., 2005):

\[
\zeta_{\text{min}}(x_1, x_2) = \frac{\mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1) \cdot F_{x_2}^{-1} \circ \Phi(g_2)] - \mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)] \cdot \mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]}{\sqrt{\mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)]^2 - \mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)]^2} \cdot \sqrt{\mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]^2 - \mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]^2}}
\]

(4.11)

\[
\zeta_{\text{max}}(x_1, x_2) = \frac{\mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1) \cdot F_{x_2}^{-1} \circ \Phi(g_2)] - \mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)] \cdot \mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]}{\sqrt{\mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)]^2 - \mathbb{E}[F_{x_1}^{-1} \circ \Phi(g_1)]^2} \cdot \sqrt{\mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]^2 - \mathbb{E}[F_{x_2}^{-1} \circ \Phi(g_2)]^2}}
\]

(4.12)

where \( F_i, g_i \) denotes \( F_{x_i} \) and \( g(x_i, \theta) \), respectively, \((i = 1, 2)\).

To illustrate the above, consider a two-dimensional vector with the following lognormal marginal distribution (Phoon and Nadim, 2004)

\[
F(y; \mu_g, \sigma_g, \mu) = \Phi\left( \frac{\log_y(y - \mu) - \mu_g}{\sigma_g} \right)
\]

(4.13)

The results in Figure 4.1 show that the upper bound will be less than 1 when \( \sigma_{g_1} \) and \( \sigma_{g_2} \) are different. When \( \sigma_{g_1} \) and \( \sigma_{g_2} \) are large and also significantly different from each other, the intervals defined by \( \zeta_{\text{min}}(x_1, x_2) \) and \( \zeta_{\text{max}}(x_1, x_2) \) become narrower.

### 4.2.2 Type II: multivariate level incompatibility

If a target covariance function has values that lie outside the bounds of \( \zeta(\tau) \), there will be no translation process satisfying this target covariance function and the corresponding marginal distribution function simultaneously. However, even when \( \zeta(\tau) \) is in the range of \([\zeta_{\text{min}}(\tau), \zeta_{\text{max}}(\tau)]\), the translation process may still not exist. The reason is that although there is a corresponding \( \rho(\tau) \) for any value of \( \tau \) in this
case, the covariance matrix $\{\rho(\mathbf{r})\}$ thus obtained may not be non-negative definite and hence cannot be admitted as a covariance function for the underlying Gaussian process. This type of incompatibility is known as multivariate level incompatibility (or Type II for brevity).

The condition of non-negative definiteness for the covariance matrix $\{\rho(\mathbf{r})\}$ needs to be checked. Many methods can be used for this purpose, four of which are:

1. $\{\rho(\mathbf{r})\}$ must satisfy the condition that for all vector $\mathbf{Z} \in \mathbb{R}^n$, $\mathbf{Z}^T \{\rho(\mathbf{r})\} \mathbf{Z} \geq 0$;
2. The determinant of $\{\rho(\mathbf{r})\}$ is non-negative for any rank;
3. The smallest eigenvalue of $\{\rho(\mathbf{r})\}$ is non-negative;
4. The spectral density function of $\{\rho(\mathbf{r})\}$ is non-negative.

A three-dimensional vector example is illustrated for this kind of incompatibility (Grigoriu, 1995 and 1998). Consider the target marginal distribution function

$$F(y) = \Phi[\text{sign}(y)|y|^{\beta}], \quad y \in \mathbb{R}$$

(4.14)

The relationship between $\zeta(\mathbf{r})$ and $\rho(\mathbf{r})$ is:

$$\zeta(\mathbf{r}) = \frac{1}{\beta} \rho(\mathbf{r}) \left[ \beta + 2 \rho(\mathbf{r})^2 \right]$$

(4.15)

where $\zeta(\mathbf{r})$ ranges from -1 to 1. Therefore, there is no Type I incompatibility.

Consider the specific target covariance matrix

$$\{\zeta(\mathbf{r})\} = \begin{bmatrix}
1 & 0.1908 & -0.8316 \\
0.1908 & 1 & 0.1908 \\
-0.8316 & 0.1908 & 1
\end{bmatrix}$$

(4.16)

To match this covariance matrix, the covariance of the underlying Gaussian process can be computed term by term using Equation (4.15) as:
This matrix is not non-negative definite because the smallest eigenvalue is -0.0685, and consequently is invalid as a covariance matrix leading to Type II incompatibility. Therefore, no translation vector exists for the prescribed covariance matrix and marginal distribution function.

For a prescribed marginal distribution $F$ and covariance function $\zeta(\tau)$, a translation process exists only when the target covariance function $\zeta(\tau)$ is within the admissible bounds, and the corresponding $\{\rho(\tau)\}$ is non-negative definite. However, the prescribed marginal distribution function and covariance function are often given independently or there is no physical reason to ensure that they must satisfy translation restrictions in real data. For a given marginal distribution function, it may be compatible with a particular covariance function and incompatible with another covariance function. On the other hand, for a given covariance function, it may be compatible with a particular marginal distribution function and incompatible with another marginal distribution function.

### 4.3 Spectral Preconditioning for Weakly Incompatible Spectral Density Function and Marginal Distribution Function

For a target non-Gaussian process with incompatible spectral density function and marginal distribution function, a spectral preconditioning procedure is proposed by Deodatis and Micaletti (2001), as summarized below:
1. Compute the Wiener-Khintchine transform of $S_{gg}(\omega)$ to determine $R_{gg}(\tau)$, the autocorrelation function of $\sigma(x, \theta)$;

2. Fix $\tau = \tau^*$;

3. Set $\Theta[\rho] = R_{gg}(\tau^*)$;

4. Find the value of $\rho$ that produces $\Theta[\rho] = R_{gg}(\tau^*)$ using the relationship established through Equation (4.2);

5. Set $R_{gg}(\tau^*) = \rho$;

6. Repeat Steps 2–5 for different values of $\tau = \tau^*$, until $R_{gg}(\tau)$ is constructed;

7. Compute the inverse Wiener-Khintchine transform of $R_{gg}(\tau)$ to obtain $S_{gg}(\omega)$;

8. Set the negative values of $S_{gg}(\omega)$ equal to zero, resulting in a corrected spectral density function $S'_{gg}(\omega)$;

9. Compute the Wiener-Khintchine transform of $S'_{gg}(\omega)$ to get $R'_{gg}(\tau)$, the autocorrelation function of the underlying Gaussian process;

10. Use $R'_{gg}(\tau)$ and Equation (4.2) to get $R'_{gg}(\tau)$, the corrected autocorrelation function of the target non-Gaussian process $\sigma(x, \theta)$;

11. Compute the inverse Wiener-Khintchine transform of $R'_{gg}(\tau)$ to obtain $S'_{gg}(\omega)$.

By performing these spectral preconditioning steps, the original target $S_{gg}(\omega)$ has been changed to $S'_{gg}(\omega)$, which is compatible with the prescribed non-Gaussian marginal distribution function, since it corresponds to a non-negative underlying Gaussian spectral density function $S'_{gg}(\omega)$.

This method only works for Type II incompatibility, since it will break down
in step 3 for Type I incompatibility. However, even within the scope of Type II incompatibility, it only works well for weakly incompatible spectral density function and marginal distribution function, where the corrected target spectral density function is almost the same as the original one. For those with strongly incompatible spectral density function and marginal distribution function, the target spectral density function will be greatly changed after preconditioning and consequently, the properties of the target non-Gaussian process will also change significantly.

4.4 Numerical Examples

4.4.1 Example with strongly incompatible spectral density function and marginal distribution function

A highly skewed non-Gaussian marginal distribution (shifted lognormal) is selected:

\[
F(y; \mu_g, \sigma_g, \mu_l) = \Phi\left(\frac{\log_e(y - \mu_l) - \mu_g}{\sigma_g}\right)
\]  

(4.18)

where \( \mu_g = -0.7707, \sigma_g = 1 \) and \( \mu_l = 0.7629 \). The values of the distribution parameters \( \mu_g, \sigma_g, \) and \( \mu_l \) are selected to produce zero-mean and unit-variance. The marginal probability density function is plotted in Figure 4.2, together with the standard Gaussian probability density function for comparison.

The target spectral density function is

\[
S_{mn}(\omega) = \frac{1}{4} \sigma_\sigma^2 \omega^2 e^{-2|\omega|}, \quad -\infty < \omega < \infty
\]  

(4.19)

where \( \sigma_\sigma \) denotes standard deviation of the stochastic process and \( b \) is directly
related to the correlation distance of the stochastic process. The values of $\sigma_\pi$ and $b$ are 1 and 5, respectively.

The values used in this example for $\omega_{\text{max}}$ (maximum cut-off frequency), $M$ (number of frequency intervals), $T_\text{0}$ (length of each realization), $N$ (number of sampled points in each realization), and $n$ (number of realizations) are:

$$\omega_{\text{max}} = \pi$$  \hspace{1cm} (4.20)

$$M = 128$$  \hspace{1cm} (4.21)

$$T_\text{0} = 256$$  \hspace{1cm} (4.22)

$$N = 256$$  \hspace{1cm} (4.23)

$$n = 10000$$  \hspace{1cm} (4.24)

The sampling interval along $x$ axis = $T_\text{0}/N = 256/256$. Therefore, the Nyquist frequency is 0.5 Hz, which is exactly $\omega_{\text{max}}/2\pi$.

To verify the compatibility between $F$ and $S_{\sigma\sigma}(\omega)$, the autocorrelation function $R_{\sigma\sigma}(\tau)$ (Figure 4.3a left panel, solid line) is first computed from $S_{\sigma\sigma}(\omega)$ via Wiener-Khintchine transform. For translation processes with marginal distribution function defined by Equation (4.18), the relationship between $R_{\sigma\sigma}(\tau)$ and the autocorrelation function of the underlying Gaussian process $R_{gg}(\tau)$ is available in closed-form:

$$R_{gg}(\tau) = \frac{\log_e \left[ 1 + \left( e^{\frac{\sigma^2}{\sigma_g^2}} - 1 \right) R_{\sigma\sigma}(\tau) \right]}{\sigma^2_g}$$ \hspace{1cm} (4.25)

$R_{gg}(\tau)$ is plotted in left panel of Figure 4.3a (dashed line).

The inverse Wiener-Khintchine transform is next performed on $R_{gg}(\tau)$ to
obtain the spectral density function \( S_{gg}(\omega) \) of this calculated underlying Gaussian autocorrelation function (see Figure 4.3a right panel, dashed line). It can be readily observed that \( S_{gg}(\omega) \) takes negative value. Therefore, \( F \) and \( S_{\varpi\varpi}(\omega) \) as given by Equations (4.18) and (4.19) are incompatible (Type II).

A corrected spectral density function \( S'_{gg}(\omega) \) is obtained by setting the negative values in \( S_{gg}(\omega) \) equal to zero, shown in right panel of Figure 4.3b (solid line). After performing the Wiener-Khintchine transform, the corresponding autocorrelation function \( R'_{gg}(\tau) \) of the corrected Gaussian spectral density function \( S'_{gg}(\omega) \) can be obtained, as shown in left panel of Figure 4.3b (solid line). The autocorrelation function of the corrected target non-Gaussian process \( R'_{\varpi\varpi}(\tau) \) is available in closed-form

\[
R'_{\varpi\varpi}(\tau) = \frac{e^{\sigma^2_\varpi R'(\tau)} - 1}{e^{\sigma^2_\varpi} - 1} \tag{4.26}
\]

Finally, the corrected spectral density function \( S'_{\varpi\varpi}(\omega) \) can be obtained by inverse Wiener-Khintchine transform of \( R'_{\varpi\varpi}(\tau) \). \( R'_{\varpi\varpi}(\tau) \) and \( S'_{\varpi\varpi}(\omega) \) are plotted as dashed lines in left and right panels of Figures 4.3c, respectively. It can be seen that the variance of the corrected target process has changed to a value greater than 1.

To avoid this variance change, a standardization procedure is applied on \( S'_{gg}(\omega) \) before performing Wiener-Khintchine transform:

\[
S'_{gg}(\omega) = \frac{S'_{gg}(\omega)}{\int_{-T/2}^{T/2} S'_{gg}(\omega) d\omega} \tag{4.27}
\]

The other steps are exactly the same as those discussed above. Figure 4.4 shows the corresponding results for Figure 4.3 when the spectral preconditioning procedure is
combined with the standardization step.

The new non-Gaussian spectral density function $S'_{\text{gs}}(\omega)$ is compatible with the prescribed non-Gaussian marginal probability distribution because it corresponds to an admissible underlying Gaussian spectral density function $S_{\text{gs}}'(\omega)$.

Non-Gaussian K-L expansion can be used to simulate the original target process with incompatible spectral density function and marginal distribution function without spectral preconditioning. Figure 4.5a shows that the simulated process matches both the original spectral density function and marginal probability density function very well. Note that Gaussian initial spectral random variables are used here.

At the same time, non-Gaussian K-L expansion also can be used to simulate the corrected target process with compatible spectral density function and marginal distribution function. Figure 4.5b shows that the simulated process also matches both the corrected spectral density function and marginal probability density function very well. Note that the same Gaussian initial spectral random variables are used here.

It was assumed by Deodatis and Micaletti (2001) that the correction of $S_{\text{gs}}(\omega)$ to $S'_{\text{gs}}(\omega)$ would not alter the functional form of $S_{\text{gs}}(\omega)$ to any significant degree. Fortunately, it is possible to check the influence of the correction using the non-Gaussian K-L expansion. Figure 4.6 compares the up-crossing rates at different thresholds for K-L simulated processes from original and corrected targets showing that these two processes have different up-crossing rates. It is inferred that
the properties of the original target process has been changed by the spectral
preconditioning procedure.

The non-Gaussian process defined by Equations (4.18) and (4.19) can also be
obtained from the non-Gaussian K-L expansion starting with initial spectral random
variables with distribution other than Gaussian. Note that Equations (4.18) and (4.19)
are incompatible and thus no translation process exists satisfying these two Equations
simultaneously. Exponentially distributed initial spectral random variables are used
here. Figure 4.7 shows that the simulated process also satisfies both the prescribed
target spectral density function and prescribed target probability density function.

Figure 4.8 shows that the K-L simulated processes from Gaussian and
exponential initial random variables have different cumulative first time-to-failure
distribution with failure defined as process exceeding threshold = 4.5. Therefore,
different processes can be generated by non-Gaussian K-L expansion satisfying the
same target spectral density function and the same target marginal distribution
function which are strongly incompatible from the point of view of a translation
approach. Since the target spectral density function is incompatible with the target
marginal distribution function, it is inferred that both the K-L simulated processes are
non-translation processes.

4.4.2 Example with weakly incompatible spectral density function and
marginal distribution function

Another shifted lognormal marginal distribution function is selected:
\[
F(y; \mu_g, \sigma_g, \mu_t) = \Phi \left( \frac{\log_e (y - \mu_t) - \mu_g}{\sigma_g} \right)
\]  
(4.28)

where \( \mu_g = 0.2381 \), \( \sigma_g = 0.6 \) and \( \mu_t = 1.5191 \). The values of the distribution parameters \( \mu_g \), \( \sigma_g \), and \( \mu_t \) are also selected to produce zero-mean and unit-variance. The marginal probability density function is plotted in Figure 4.9, together with the standard Gaussian probability density function for comparison.

The same target spectral density function as defined in Equation (4.19) is selected. All the parameters take the same values as in section 4.4.1, as defined by Equations (4.20)-(4.24).

The same procedures are performed to check the compatibility between \( F \) and \( S_m(\omega) \), and results are plotted in Figure 4.10. It can be seen that the calculated underlying Gaussian spectral density function takes negative values (see Figure 4.10a right panel, dashed line). Therefore, \( F \) and \( S_m(\omega) \) are incompatible. Figure 4.11 shows the corresponding results for Figure 4.10 when the spectral preconditioning procedure is combined with the standardization step. It can be seen that the corrected target spectral density function is almost the same as the original target, which is different from the case in section 4.4.1 where the corrected target spectral density function is quite different from the original target.

Non-Gaussian K-L expansion is used to simulate processes with both the original and corrected target spectral density functions. Figure 4.12 shows that the simulated processes match the spectral density function and marginal probability density function with their corresponding targets. Note that the same Gaussian initial spectral random variables are used in both cases here.
Comparison between the K-L simulated processes from both original and corrected targets has been made. Figure 4.13 shows that both processes have almost the same up-crossing rates at different thresholds, indicating that the correction of the target spectral density function does not change the properties of the original target process significantly. Figure 4.14 shows that they have almost the same cumulative distribution of first time-to-failure with failure defined as process exceeding threshold $= 4.5$. The results confirm that the process with the corrected target is almost the same as that with the original target. Figure 4.15 shows the cumulative distribution of maximum and minimum values in each realization of the K-L simulated non-Gaussian processes. The results further support the conclusion in Figures 4.13 and 4.14.

The non-Gaussian process defined by Equations (4.28) and (4.19) can also be obtained by K-L expansion starting with exponentially distributed initial spectral random variables. Figure 4.16 shows that the simulated process also satisfies both the prescribed target spectral density function and prescribed target probability density function.

Since K-L simulated processes from Gaussian and exponential initial random variables both satisfy the same target spectral density function and the same target marginal distribution function, it is natural to check whether they are the same. Figure 4.17 shows that they have different cumulative distribution of first time-to-failure with failure defined as process exceeding threshold $= 4.5$. Hence, non-Gaussian K-L expansion can also generate different processes satisfying the same
target spectral density function and the same target marginal distribution function which are weakly incompatible.

4.5 Comparison between Karhunen-Loeve Processes and Translation Processes

In this section, a numerical example is selected to compare the K-L simulated processes with the translation process. The same marginal distribution function as defined by Equation (4.18) is selected. To ensure the existence of the translation process, the spectral density function of the underlying Gaussian process is prescribed, rather than the spectral density function of the non-Gaussian translation process:

\[ S_{gg} (\omega) = \frac{125}{4} \omega^2 e^{-|\omega|}, \quad -\infty < \omega < \infty \]  

(4.29)

All the parameters take the same values as in section 4.4, as defined by Equations (4.20)-(4.24).

The corresponding translation process (denoted by T) can be generated by first simulating the underlying Gaussian process and then transforming as defined by Equation (4.1). The spectral density function of the target non-Gaussian translation process can be obtained, as shown in Figure 4.18 (solid line).

Non-Gaussian processes following Equation (4.18) as the target marginal distribution function and Figure 4.18 (solid line) as the target spectral density function can be simulated using the non-Gaussian K-L simulation algorithm. The spectral random variables can be initialized using different distributions before the iteration. Gaussian and exponential initial spectral random variables are considered here, and
the generated non-Gaussian processes are denoted by G and E, respectively. Results
in Figure 4.19 show that both spectral density function and marginal distribution
function of the simulated processes G and E almost coincide with the respective
targets regardless of the initial spectral random variable distributions.

Since G and E both satisfy the same spectral density function and the same
marginal distribution function as those of the translation process T, it is natural to
investigate whether G and E are translation processes. No theoretical results are
available at present, but it is possible to numerically evaluate some features of these
processes that are of engineering significance.

Figure 4.20 compares the cumulative distribution of first time-to-failure with
failure defined as process exceeding threshold = 4.5 for T, G and E. It seems that E
is different from both T and G, whereas T and G are very similar, indicating that E is a
non-translation process. Figure 4.21 shows the cumulative distribution of maximum
value in each realization of the non-Gaussian processes. The results support the
conclusions in Figure 4.20.

Figure 4.22 compares the up-crossing rates at different thresholds for T, G and
E. It can be seen that the up-crossing rates for both G and E are slightly different from
T (Figure 4.22a), however, the difference becomes more distinct for a longer process
with larger period $T_0 = 512$ (Figure 4.22b). In Figure 4.22b, the sampling interval
along $x$ axis remains the same as that in Figure 4.22a, leading to large discretization
points because of longer period. Comparing with Figures 4.22a and 4.22b, it can be
seen that the up-crossing rates at different thresholds for K-L simulated processes (G
and E) are not constant. Their difference from those of the corresponding translation process is more distinct for larger discretization points, as seen in Figure 4.22b. Results in Figure 4.22 show that both G and E are non-translation processes.

Therefore, K-L expansion can simulate different processes satisfying the same target spectral density function and the same target marginal distribution function using initial spectral random variables with different distributions. When the spectral random variables are initialized as exponential random variables at the start of the K-L iterations, the resulting process is clearly non-translational. When the spectral random variables are initialized as Gaussian random variables, the conclusion is less clear but there is still evidence to show that the resulting process is also non-translational.

4.6 Summary

Theoretically, a translation process exists only when the prescribed target spectral density function and marginal distribution function have a compatible relationship. The situation with incompatible functions was previously handled by correcting the original target spectral density function using a spectral preconditioning procedure. Consequently, the corrected target spectral density function is not identical to the original target. The difference may be significant in some cases.

The non-Gaussian K-L expansion has been used to simulate the target processes with both compatible and incompatible spectral density and marginal distribution functions. Numerical examples with incompatible spectral density
functions and marginal distribution functions demonstrated that different processes can be generated satisfying the same target spectral density function and the same target marginal distribution function which is incompatible from the point of view of translation approach. Results also show that the previously used spectral preconditioning technique will change the original target process significantly for those with strongly incompatible target functions. The example with compatible target functions shows that non-Gaussian K-L expansion can also be used to generate different processes satisfying both the target spectral density function and the target marginal distribution function, and that the K-L generated non-Gaussian processes are generally non-translation processes.

Comparing with the commonly used translation approach, the K-L expansion is therefore a more robust method for simulation of non-Gaussian processes because it has no requirements for the compatibility of the target spectral density function and the target marginal distribution function. It can generate different processes satisfying the same spectral density function and the same marginal distribution function regardless of their compatibility.
Figure 4.1 Relationship between the covariance functions of translation process and the underlying Gaussian process.

(a) $\sigma_{g_1} = 0.3$; $\sigma_{g_2} = 0.3$

(b) $\sigma_{g_1} = 1$; $\sigma_{g_2} = 1$

(c) $\sigma_{g_1} = 1$; $\sigma_{g_2} = 0.3$

(d) $\sigma_{g_1} = 2$; $\sigma_{g_2} = 0.3$
Figure 4.2  PDF of zero-mean unit-variance shifted lognormal distribution versus standard Gaussian density function for example with strongly incompatible spectral density function and marginal distribution function.
Figure 4.3 Spectral preconditioning proposed by Deodatis and Micaletti for strongly incompatible spectral density function and marginal distribution function:
(a) original non-Gaussian target and calculated underlying Gaussian processes
(b) calculated and corrected underlying Gaussian processes
(c) original and corrected non-Gaussian target processes
Figure 4.4  Spectral preconditioning proposed by Deodatis and Micaletti for strongly incompatible spectral density function and marginal distribution function (with standardization of corrected underlying spectral density function):
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<table>
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<tr>
<td>(a)</td>
<td>original non-Gaussian target and calculated underlying Gaussian processes</td>
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<tr>
<td>(b)</td>
<td>calculated and corrected underlying Gaussian processes</td>
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<tr>
<td>(c)</td>
<td>original and corrected non-Gaussian target processes</td>
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Figure 4.5 Spectral density function and PDF of K-L simulated processes from Gaussian initial random variables versus their corresponding targets for example with strongly incompatible spectral density function and marginal distribution function: (a) original non-Gaussian targets; (b) corrected non-Gaussian targets
Figure 4.6  Comparison of up-crossing rates for K-L simulated processes with original and corrected non-Gaussian targets for example with strongly incompatible spectral density function and marginal distribution function.
Figure 4.7  K-L simulated process from exponential initial random variables versus original non-Gaussian targets for example with strongly incompatible spectral density function and marginal distribution function.
Figure 4.8  Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 4.5 for K-L simulated processes from Gaussian and exponential initial random variables for example with strongly incompatible spectral density function and marginal distribution function
Figure 4.9 PDF of zero-mean unit-variance shifted lognormal distribution versus standard Gaussian density function for example with weakly incompatible spectral density function and marginal distribution function.
Figure 4.10 Spectral preconditioning proposed by Deodatis and Micaletti for weakly incompatible spectral density function and marginal distribution function: (a) original non-Gaussian target and calculated underlying Gaussian processes

(a)

(b)

(c)
(b) calculated and corrected underlying Gaussian processes
(c) original and corrected non-Gaussian target processes
Figure 4.11 Spectral preconditioning proposed by Deodatis and Micaletti for weakly incompatible spectral density function and marginal distribution function (with standardization of corrected underlying spectral density function):
| (a) original non-Gaussian target and calculated underlying Gaussian processes |
| (b) calculated and corrected underlying Gaussian processes |
| (c) original and corrected non-Gaussian target processes |
Figure 4.12  Spectral density function and PDF of K-L simulated processes from Gaussian initial random variables versus their corresponding targets for example with weakly incompatible spectral density function and marginal distribution function: (a) original non-Gaussian targets; (b) corrected non-Gaussian targets
Figure 4.13  Comparison of up-crossing rates for K-L simulated processes with original and corrected non-Gaussian targets for example with weakly incompatible spectral density function and marginal distribution function.
Figure 4.14 Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 4.5 for K-L simulated processes with original and corrected non-Gaussian targets for example with weakly incompatible spectral density function and marginal distribution function.
Figure 4.15 Comparison of cumulative distribution of extreme value in each realization for K-L simulated processes with original and corrected non-Gaussian targets for example with weakly incompatible spectral density function and marginal distribution function.
Figure 4.16 K-L simulated process from exponential initial random variables versus original non-Gaussian targets for example with weakly incompatible spectral density function and marginal distribution function
Figure 4.17 Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 4.5 for K-L simulated processes from Gaussian and exponential initial random variables for example with weakly incompatible spectral density function and marginal distribution function.
Figure 4.18  Spectral density function of target non-Gaussian translation process computed from the prescribed version belonging to the underlying Gaussian process
Figure 4.19  Spectral density function and PDF of K-L simulated processes G and E versus their corresponding targets
Figure 4.20  Comparison of cumulative distribution of first time-to-failure with failure defined as process exceeding threshold = 4.5 for stochastic processes T, G and E
Figure 4.21  Comparison of cumulative distribution of maximum value in each realization for stochastic processes T, G and E
Figure 4.22  Comparison of up-crossing rates for stochastic processes T, G and E

(a) $T_0 = 256$

(b) $T_0 = 512$
CHAPTER 5
SIMULATION OF MULTI-DIMENSIONAL NON-GAUSSIAN
STOCHASTIC FIELDS USING KARHUNEN-LOEVE
EXPANSION

5.1 Introduction

Techniques for simulating non-Gaussian stochastic fields have received considerable attention recently. This is due to the fact that the quantities arising in practical engineering problems (e.g., material and geometric properties of structural systems, soil properties in geotechnical engineering applications, wind loads, waves) exhibit non-Gaussian probabilistic characteristics, and that these quantities are best modeled as stochastic fields.

As in the case of simulating non-Gaussian processes, most of the existing techniques for simulating non-Gaussian fields are based on the translation concept, which invokes memoryless nonlinear transforms of some underlying Gaussian fields. These methods are developed by Yamazaki and Shinozuka (1988), Grigoriu (1998), Deodatis and Micaletti (2001), Sakamoto and Ghanem (2002), Masters and Gurley (2003), and Lagaros et al. (2005). Among these methods, the inherent limitation of the translation based approach still exists that the specified target covariance function and the specified target marginal distribution function are supposed to have a compatible relationship.

As discussed in Chapter 4, the K-L expansion is a more robust method for
simulation of non-Gaussian processes because it has no requirements for the compatibility of the target spectral density function and the target marginal distribution function. The non-Gaussian K-L expansion technique has been successfully applied to the spectral representation for simulation of highly skewed non-Gaussian processes within a more general K-L framework. However, this K-L scheme has been limited in the simulation of one-dimensional stochastic processes. In this chapter, an extension to simulation of multi-dimensional non-Gaussian stochastic fields using K-L expansion will be discussed.

5.2 Spectral Representation of Multi-dimensional Stochastic Fields

5.2.1 Spectral representation of two-dimensional stochastic fields

Let $\sigma(x_1, x_2, \theta)$ be a two-dimensional, univariate (2D-1V), homogeneous stochastic field with zero-mean, autocorrelation function $R_{\sigma\sigma}(\tau_1, \tau_2)$, and power spectral density function $S_{\sigma\sigma}(\omega_1, \omega_2)$. Therefore, the following relations hold:

$$E[\sigma(x_1, x_2, \theta)] = 0$$  \hspace{1cm} (5.1)

$$E[\sigma(x_1, x_2, \theta) \cdot \sigma(x_1 + \tau_1, x_2 + \tau_2, \theta)] = R_{\sigma\sigma}(\tau_1, \tau_2)$$  \hspace{1cm} (5.2)

$$S_{\sigma\sigma}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\sigma\sigma}(\tau_1, \tau_2) \cdot e^{-i(\omega_1 \tau_1 + \omega_2 \tau_2)} d\tau_1 d\tau_2$$  \hspace{1cm} (5.3)

$$R_{\sigma\sigma}(\tau_1, \tau_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{\sigma\sigma}(\omega_1, \omega_2) \cdot e^{i(\omega_1 \tau_1 + \omega_2 \tau_2)} d\omega_1 d\omega_2$$  \hspace{1cm} (5.4)

where $\tau_1$ and $\tau_2$ are the separation distances along the $x_1$ and $x_2$ directions respectively, and $\omega_1$ and $\omega_2$ are the corresponding wave numbers. Equations (5.3) and (5.4) constitute the two-dimensional version of the Wiener-Khintchine transform.
Therefore, a real-valued 2D-1V homogeneous stochastic field \( \sigma(x_1, x_2, \theta) \) with zero-mean and power spectral density function \( S_{\sigma\sigma}(\omega_1, \omega_2) \) can be approximated as follows (Shinozuka and Deodatis, 1996):

\[
\sigma(x_1, x_2, \theta) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \sigma_{i_1 i_2} \left[ V_{i_1 i_2}(\theta) \cos(\omega_{i_1} x_1 + \omega_{i_2} x_2) + W_{i_1 i_2}(\theta) \sin(\omega_{i_1} x_1 + \omega_{i_2} x_2) \right] \quad (5.5)
\]

in Equation (5.5), \( V_{i_1 i_2}(\theta) \) and \( W_{i_1 i_2}(\theta) \) are uncorrelated random variables with zero-mean and unit-variance, and

\[
\sigma_{i_1 i_2} = \sqrt{2 S_{\sigma\sigma}(\omega_{i_1}, \omega_{i_2}) \cdot \Delta \omega_1 \cdot \Delta \omega_2} \quad (5.6)
\]

where \( \Delta \omega_1 \) and \( \Delta \omega_2 \) are the discretization interval along the frequency axis \( \omega_1 \) and \( \omega_2 \), respectively:

\[
\omega_{i_1} = i_1 \cdot \Delta \omega_1 \quad (5.7)
\]

\[
\omega_{i_2} = i_2 \cdot \Delta \omega_2 \quad (5.8)
\]

Equation (5.5) is the discrete approximation of the spectral representation for 2D-1V homogeneous stochastic field (Shinozuka and Deodatis, 1996).

### 5.2.2 Spectral representation of multi-dimensional stochastic fields

Let \( \sigma(x_1, x_2, \ldots, x_q, \theta) \) be a \( q \)-dimensional, univariate \((qD-1V)\), homogeneous stochastic field with zero-mean, autocorrelation function \( R_{\sigma\sigma}(\tau_1, \tau_2, \ldots, \tau_q) \), and power spectral density function \( S_{\sigma\sigma}(\omega_1, \omega_2, \ldots, \omega_q) \). Therefore, Equations (5.1)-(5.4) can be generalized to (Shinozuka and Deodatis, 1996):

\[
E[\sigma(x_1, x_2, \ldots, x_q, \theta)] = 0 \quad (5.9)
\]
\[ E[\sigma(x_1, x_2, \ldots, x_q, \theta) \cdot \sigma(x_1 + \tau_1, x_2 + \tau_2, \ldots, x_q + \tau_q, \theta)] = R_{\sigma\sigma}(\tau_1, \tau_2, \ldots, \tau_q) \]  

(5.10)

\[ S_{\sigma\sigma}(\omega_1, \omega_2, \ldots, \omega_q) \]

\[ = \frac{1}{(2\pi)^q} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{\sigma\sigma}(\tau_1, \tau_2, \ldots, \tau_q) \cdot e^{-i(\omega_1 \tau_1 + \omega_2 \tau_2 + \cdots + \omega_q \tau_q)} d\tau_1 d\tau_2 \cdots d\tau_q \]  

(5.11)

\[ R_{\sigma\sigma}(\tau_1, \tau_2, \ldots, \tau_q) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{\sigma\sigma}(\omega_1, \omega_2, \ldots, \omega_q) \cdot e^{i(\omega_1 \tau_1 + \omega_2 \tau_2 + \cdots + \omega_q \tau_q)} d\omega_1 d\omega_2 \cdots d\omega_q \]  

(5.12)

where \( \tau_1, \tau_2, \ldots, \tau_q \) are the separation distances along the \( x_1, x_2, \ldots, x_q \) directions respectively, and \( \omega_1, \omega_2, \ldots, \omega_q \) are the corresponding wave numbers. Equations (5.11) and (5.12) constitute the \( q \)-dimensional version of the Wiener-Khintchine transform pair.

The spectral representation of a \( q \)-dimensional stochastic field is a straightforward extension of the two-dimensional case. Therefore, a real-valued \( q \)-D-1V homogeneous stochastic field \( \sigma(x_1, x_2, \ldots, x_q, \theta) \) with zero-mean and power spectral density function \( S_{\sigma\sigma}(\omega_1, \omega_2, \ldots, \omega_q) \) can be approximated as follows:

\[ \sigma(x_1, x_2, \ldots, x_q, \theta) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} \sigma_{i_1 \cdots i_q} \left[ V_{i_1 \cdots i_q}(\theta) \cos(\omega_{i_1} x_1 + \omega_{i_2} x_2 + \cdots + \omega_{i_q} x_q) \ight. \]

\[ + W_{i_1 \cdots i_q}(\theta) \sin(\omega_{i_1} x_1 + \omega_{i_2} x_2 + \cdots + \omega_{i_q} x_q) \]  

(5.13)

in Equation (5.13), \( V_{i_1 \cdots i_q}(\theta) \) and \( W_{i_1 \cdots i_q}(\theta) \) are uncorrelated random variables with zero-mean and unit-variance, and

\[ \sigma_{i_1 \cdots i_q} = \sqrt{2S_{\sigma\sigma}(\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_q})} \cdot \Delta\omega_{i_1} \cdot \Delta\omega_{i_2} \cdots \Delta\omega_{i_q} \]  

(5.14)

where \( \Delta\omega_{i_1}, \Delta\omega_{i_2}, \ldots, \Delta\omega_{i_q} \) are the discretization intervals along the frequency axis \( \omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_q} \), respectively:

\[ \omega_{ij} = i_j \cdot \Delta\omega_j, \quad j = 1, 2, \ldots, q \]  

(5.15)
5.3 Simulation of Multi-dimensional Non-Gaussian Stochastic Fields Using Karhunen-Loeve Expansion

As in the case of one-dimensional stochastic processes simulation, if the target stochastic field \( \sigma(x_1, x_2, \ldots, x_q, \theta) \) in Equation (5.13) is a Gaussian field, the spectral random variables \( V_{i_1i_2\cdots i_q}(\theta) \) and \( W_{i_1i_2\cdots i_q}(\theta) \) are independent standard Gaussian random variables. In the general case, the distributions of \( V_{i_1i_2\cdots i_q}(\theta) \) and \( W_{i_1i_2\cdots i_q}(\theta) \) are not the same as the prescribed non-Gaussian marginal distribution of the field \( \sigma(x_1, x_2, \ldots, x_q, \theta) \). More importantly, \( V_{i_1i_2\cdots i_q}(\theta) \) and \( W_{i_1i_2\cdots i_q}(\theta) \) are not independent, even though they are uncorrelated. The dependence between \( V_{i_1i_2\cdots i_q}(\theta) \) and \( W_{i_1i_2\cdots i_q}(\theta) \) must be preserved because, otherwise, the representation becomes Gaussian if refined by increasing the number of harmonics indefinitely. Thus far, there is no feasible method of constructing the multivariate distribution of \( V_{i_1i_2\cdots i_q}(\theta) \) and \( W_{i_1i_2\cdots i_q}(\theta) \) for the general case. Hence, it is not surprising that the translation strategy is the most popular.

Phoon et al. (2002a and 2005) suggested using the Karhunen-Loeve (K-L) expansion with non-Gaussian random variables to produce non-Gaussian processes. The non-Gaussian K-L technique has been successfully applied to the spectral representation for simulation of highly skewed non-Gaussian processes within a more general K-L framework. The main advantage of this K-L scheme is that it does not require the target spectral density function and the target marginal distribution functions to be compatible. It has been shown in Chapter 4 that it can generate different processes satisfying the same target spectral density function and the same
marginal distribution function regardless of their compatibility. However, this K-L scheme has not been demonstrated with examples beyond one-dimensional processes.

In principle, the non-Gaussian K-L expansion technique can be extended straightforwardly to multi-dimensional spectral representation. Following the same notations as in Chapter 3, Equation (5.13) is rewritten as follows:

\[
\sigma_{M_1M_2\cdots M_q}(x_1, x_2, \cdots, x_q, \omega) = \sum_{i_1=0}^{M_1-1} \sum_{i_2=0}^{M_2-1} \cdots \sum_{i_q=0}^{(M_q-1)} \sigma_{i_1i_2\cdots i_q} \cdot \xi_{i_1i_2\cdots i_q}^T(\omega) \cdot f_{i_1i_2\cdots i_q}(x_1, x_2, \cdots, x_q)
\]

in which

\[
\xi_{i_1i_2\cdots i_q}(\omega) = \begin{bmatrix} V_{i_1i_2\cdots i_q}(\omega) \\ W_{i_1i_2\cdots i_q}(\omega) \end{bmatrix}
\]

and

\[
f_{i_1i_2\cdots i_q}(x_1, x_2, \cdots, x_q) = \begin{bmatrix} \cos(\omega_{i_1} x_1 + \omega_{i_2} x_2 + \cdots + \omega_{i_q} x_q) \\ \sin(\omega_{i_1} x_1 + \omega_{i_2} x_2 + \cdots + \omega_{i_q} x_q) \end{bmatrix}
\]

\(M_1, M_2, \cdots, M_q\) are the number of terms satisfying:

\[
\Delta \omega_j = \frac{\omega_{j_{\text{max}}}}{M_j}, \quad j = 1, 2, \ldots q
\]

in which \(\omega_{j_{\text{max}}}(j = 1, 2, \ldots q)\) are determined based on the acceptable error in the variance of the fields, and the following criterion is usually used to estimate their values:

\[
\int_0^{\omega_{1_{\text{max}}}} \int_0^{\omega_{2_{\text{max}}}} \cdots \int_0^{\omega_{q_{\text{max}}}} S_{\text{mean}}(\omega_1, \omega_2, \cdots, \omega_q) \cdot \omega_1 d\omega_2 \cdots d\omega_q = (1 - \epsilon) \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} S_{\text{mean}}(\omega_1, \omega_2, \cdots, \omega_q) \cdot \omega_1 d\omega_2 \cdots d\omega_q
\]

where \(\epsilon << 1\) (eg, \(\epsilon = 0.01\)) such that the power spectral density function \(S_{\text{mean}}(\omega_1, \omega_2, \cdots, \omega_q)\) is assumed to be zero outside the region defined by:
\[-\omega_{j_{\text{max}}} \leq \omega_j \leq \omega_{j_{\text{max}}}, \quad j = 1, 2, \ldots q \]  
(5.21)

and \(\Delta \omega_1, \Delta \omega_2, \ldots, \Delta \omega_q\) are selected to produce a field with adequate length along \(x_1, x_2, \ldots, x_q\) axes, respectively:

\[T_{x_0} = \frac{2\pi}{\Delta \omega_j}, \quad j = 1, 2, \ldots q \]  
(5.22)

The following iterative steps are used to compute the unknown spectral random variables \(V_{\omega_{x_1} \cdots \omega_{x_q}}(\theta)\) and \(W_{\omega_{x_1} \cdots \omega_{x_q}}(\theta)\):

1. Generate \(n\) sample functions of the non-Gaussian process:

\[a_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta) = \sum_{i_1=0}^{M_1-1} \sum_{i_2=0}^{M_2-1} \cdots \sum_{i_q=0}^{M_q-1} \sigma_{i_1i_2\cdots i_q} \cdot \xi_{i_1i_2i_q}^{(k)}(\theta) \cdot f_{\omega_{x_1} \cdots \omega_{x_q}}(x_1, x_2, \ldots, x_q), \quad m = 1, 2, \ldots n \]  
(5.23)

2. Estimate the empirical cumulative marginal distribution function as:

\[
\hat{F}_{M_1M_2\cdots M_q}(y | x_1, x_2, \ldots, x_q) = \frac{1}{n} \sum_{m=1}^{n} I\left(a_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta_m) \leq y \right)
\]  
(5.24)

3. Transform each sample function to match the target marginal distribution \(F\):

\[
\eta_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta_m) = F^{-1}\left[\hat{F}_{M_1M_2\cdots M_q}(a_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta_m))\right]
\]  
(5.25)

4. Estimate the next generation of \(\xi_{ij}(\theta)\) as:

\[
\xi_{i_1i_2\cdots i_q}^{(k+1)}(\theta) = \frac{1}{\sigma_{i_1i_2\cdots i_q}} \int_{-\frac{T_{x_1}}{2}}^{\frac{T_{x_1}}{2}} \int_{-\frac{T_{x_2}}{2}}^{\frac{T_{x_2}}{2}} \cdots \int_{-\frac{T_{x_q}}{2}}^{\frac{T_{x_q}}{2}} \eta_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta_m) \cdot f_i(x_1, x_2, \ldots, x_q) dx_1 dx_2 \cdots dx_q
\]  
(5.26)

\(\eta_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q)\) is the average value of \(\eta_{M_1M_2\cdots M_q}^{(k)}(x_1, x_2, \ldots, x_q, \theta_m)\). This integral can be evaluated efficiently using \(q\)-dimensional FFT.

5. Standardize \(\xi_{i_1i_2\cdots i_q}^{(k+1)}(\theta)\) to unit-variance. Note that \(\xi_{i_1i_2\cdots i_q}^{(k+1)}(\theta)\) is a zero-mean

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vector by virtue of Equation (5.26). A modified Latin hypercube orthogonalization technique (Phoon et al., 2005) is applied to reduce the product-moment correlations between $\xi_{i,t_1 \ldots t_q}^{(k+1)}(\theta)$.

6. Repeat steps (1) through (5) until the sample functions achieved the target marginal distribution.

5.4 Numerical Examples

Consider a two-dimensional zero-mean stochastic field with marginal distribution function $F$ and spectral density function $S_{\omega \omega}(\omega_1, \omega_2)$. In this section, the same target marginal distribution functions as in Chapter 3 are selected:

1. Beta distribution

$$F(y; p, q) = \frac{\Gamma(p+q)\Gamma(q)}{\Gamma(p)\Gamma(q)} z^{p-1}(1-z)^{q-1} \, dz$$

(5.27)

2. Shifted exponential distribution

$$F(y; \mu, \lambda) = 1 - e^{-\lambda(y-\mu)}$$

(5.28)

All the distribution parameters take the same values as in Chapter 3.

The following target spectral density function is selected:

$$S_{\omega \omega}(\omega_1, \omega_2) = \frac{\sigma_\omega^2 b_1 b_2}{4\pi} e^{-\left[ \frac{b_0}{2} \right] \left( \frac{b_0}{2} \right)} , \quad -\infty < \omega_1 < \infty \quad \text{and} \quad -\infty < \omega_2 < \infty$$

(5.29)

The parameter values are selected as $\sigma_\omega = 1$ and $b_1 = b_2 = 4$. Figure 5.1 shows this target spectral density function.

The values used in this example for $\omega_{j,\text{max}}$ (maximum cut-off frequency), $M_j$
(number of frequency intervals), $T_{x,0}$ (length of each realization), $N_j$ (number of sampled points in each realization), and $n$ (number of realizations) are:

$$\omega_{j,\max} = 1.25$$  
(5.30)

$$M_j = 16$$  
(5.31)

$$T_{x,0} = 80.4$$  
(5.32)

$$N_j = 32$$  
(5.33)

$$n = 5000$$  
(5.34)

The sampling interval along $x_j$ axis $= T_0/N = 80.4/32$. Therefore, the Nyquist frequency is 0.199 Hz, which is exactly $\omega_{j,\max}/2\pi$. Note that $j = 1, 2$ in this section.

Non-Gaussian K-L expansion outlined in Section 5.3 can be used to simulate the target two-dimensional stochastic field with Beta marginal distribution function as defined by Equation (5.27). Figure 5.2 shows the spectral density function and marginal probability density function of the K-L simulated field from Gaussian initial spectral random variables. Figure 5.3 shows the corresponding results for the K-L simulated field from exponential initial spectral random variables. It can be seen that marginal probability density functions of the K-L simulated fields match their targets very well (see Figure 5.2b and 5.3b). Figure 5.4 shows the relative error of the spectral density function as defined by

$$\frac{|S_{\text{sim}}(\omega_1, \omega_2) - \hat{S}_{\text{sim}}(\omega_1, \omega_2)|}{S_{\text{sim}}(\omega_1, \omega_2)}$$

(where $\hat{S}_{\text{sim}}(\omega_1, \omega_2)$ and $S_{\text{sim}}(\omega_1, \omega_2)$ are the spectral density function of the simulated and target fields, respectively) for K-L simulated fields from different initial spectral
random variables. Results in Figure 5.4 show that the spectral density functions and marginal probability density functions of the K-L simulated fields from different initial spectral random variables coincide with their respective targets.

Figure 5.5 compares the cumulative distribution of the extreme values in each realization of the K-L simulated non-Gaussian stochastic fields from Gaussian and exponential initial, showing that they are different. Therefore, different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function can be generated by K-L expansion. Realizations of the K-L simulated stochastic fields from Gaussian and exponential initial spectral random variables are plotted in Figure 5.6.

Figures 5.7-5.10 show the corresponding results of Figures 5.2-5.6 for the target stochastic field with shifted exponential marginal distribution. Figures 5.7 and 5.8 show the spectral density function and marginal probability density function of the K-L simulated fields from Gaussian and exponential initial spectral random variables, respectively. The relative error of the spectral density function for these K-L simulated fields is similar to Figure 5.4 with maximum relative error $=2 \times 10^{-4}$. Hence, the K-L simulated stochastic fields from different initial spectral random variables satisfy the same target spectral density function and the same target marginal distribution function. Results in Figure 5.9 show that these K-L simulated fields have different cumulative distribution of the extreme values. Therefore, the same conclusion can be drawn that different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function can be
generated by K-L expansion. Figure 5.10 plots realizations of the K-L simulated stochastic fields from Gaussian and exponential initial spectral random variables.

Figure 5.11 shows the cumulative distribution function of a typical spectral random variable $V_{s,s}$ for K-L simulated fields from Gaussian and exponential initial random variables using a Gaussian probability plot. The linear plots obtained for those from Gaussian initial spectral random variables imply that they are Gaussian random variables. It is found that spectral random variables at other frequencies are also Gaussian for the K-L simulated fields from Gaussian initial spectral random variables. For those from exponential initial spectral random variables, the spectral random variables are not Gaussian for any frequency. The same hypothesis tests as in Chapter 3 are performed for the spectral random variables from Gaussian initial and results show that they do not satisfy multivariate normality. Therefore, although they are uncorrelated Gaussian (maximum correlation between spectral random variables are $2.12 \times 10^{-4}$ and $5.27 \times 10^{-4}$ for Beta and shifted exponential distributed examples, respectively), they are not independent.

### 5.5 Comparison between Karhunen-Loeve Fields and Translation Fields

In this section, a numerical example is selected to compare the K-L fields with the translation fields. The same marginal distribution function (shifted exponential) as in section 5.5 is selected:

$$F(y, \mu, \lambda_e) = 1 - e^{-\lambda_e (y - \mu)}$$  \hspace{1cm} (5.35)
where $\lambda_v = 1$, and $\mu_v = -1$.

To ensure the existence of the translation field, the spectral density function of the underlying Gaussian field is prescribed, rather than the spectral density function of the non-Gaussian translation field:

$$S_{\text{gaussian}}(\omega_1, \omega_2) = \sigma^2 \frac{b_1 b_2}{4\pi} \left( \frac{h_{\omega_1}}{2} \right)^2 \left( \frac{h_{\omega_2}}{2} \right)^2, \quad -\infty < \omega_1 < \infty \quad \text{and} \quad -\infty < \omega_2 < \infty \quad (5.36)$$

The parameter values are also selected as $\sigma = 1$ and $b_1 = b_2 = 4$.

In this section, all the parameters take the same values as in section 5.4, as defined by Equations (5.30)-(5.34).

The corresponding translation field (denoted by $T$) can be first generated. Then, the spectral density function of the target non-Gaussian translation field can be obtained, as shown in Figure 5.12.

Non-Gaussian stochastic fields following Equation (5.35) as the target marginal distribution function and Figure 5.12 as the target spectral density function can be simulated using the non-Gaussian K-L simulation algorithm. The spectral random variables are first initialized as independent Gaussian and exponential spectral random variables, and the generated non-Gaussian fields are denoted by $G$ and $E$, respectively. Results in Figures 5.13 and 5.14 show that the marginal probability density functions of $G$ and $E$ match their targets very well. The relative error of the spectral density function for $G$ and $E$ is similar to Figure 5.4 with maximum relative error $= 2 \times 10^{-4}$. Hence, both the spectral density functions and marginal probability density functions of $G$ and $E$ coincide with their respective targets.
Since T, G and E satisfy the same spectral density function and the same marginal distribution function, comparisons among these three fields (T, G and E) are performed. Figure 5.15a compares the cumulative distribution of the maximum value in each realization of T, G and E. It can be seen that E is different from both T and G, whereas T and G are very similar, indicating that E is a non-translation field. Figure 5.15b shows the corresponding results for minimum value in each realization of the non-Gaussian fields. Results in Figure 5.15b show that both G and E are non-translation fields.

Figure 5.16a plots a realization of the non-Gaussian translation field T, and the corresponding realization of the underlying Gaussian field is also plotted in Figure 5.16b for comparison. Figure 5.17 plots realizations of the K-L simulated non-Gaussian stochastic field G and E, respectively.

Spectral random variables for G and E are defined in Equation (5.17). Once the translation field T is computed numerically, it is also possible to back-calculate the equivalent spectral random variables using Equation (5.26). Figure 5.18 shows the cumulative distribution function of a typical spectral random variable $V_{s,8}$ for T, G and E using a Gaussian probability plot. Results imply that the spectral random variables of T and G are Gaussian random variables, and that those of E are not Gaussian. It is found that spectral random variables at other frequencies are also Gaussian for T and G, and not Gaussian for E. The same hypothesis tests are performed for the spectral random variables for T and G, and results show that they are not independent although they are uncorrelated Gaussian (maximum correlation
between spectral random variables are $6.33 \times 10^{-2}$ and $1.66 \times 10^{-4}$ for T and G, respectively).

Hence, K-L expansion can simulate different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function using initial spectral random variables with different distributions. Numerical results show that the K-L fields generated from independent Gaussian and exponential initial spectral random variables are non-translation fields. The reason also can be explained as follows.

In the non-Gaussian K-L scheme, if the K-L random variables are initialized as the spectral random variables of the translation field T, the same translation field T will be obtained in the first iterative step, as defined by Equation (5.23). Since the generated sample functions are exactly shifted exponential distributed, they will remain unchanged after the second and third iterative steps (see Equations (5.24) and (5.25)). Consequently, the back-calculated random variables will be the same as the initial random variables as defined by Equations (5.26). Since the spectral random variables of the translation field T are uncorrelated, they will also remain the same after the modified Latin hypercube orthogonalization step. Therefore, non-Gaussian K-L scheme will produce translation fields if the K-L random variables are initialized as the spectral random variables of the target translation fields. In this case, the initial spectral random variables are uncorrelated Gaussian but not independent. Since the specified initial spectral random variables (independent Gaussian and exponential random variables) do not satisfy these conditions, the generated K-L
fields are therefore non-translational.

Next, the spectral random variables are initialized as correlated Gaussian spectral random variables with exponential and square exponential covariance function, and the generated non-Gaussian fields are denoted by $G_1$ and $G_2$, respectively. Results in Figures 5.19 and 5.20 show that the marginal probability density functions of $G_1$ and $G_2$ match their targets. The relative error of the spectral density function for $G_1$ and $G_2$ is also similar to Figure 5.4 with maximum relative error $= 2 \times 10^{-4}$. Hence, both the spectral density functions and marginal probability density functions of $G_1$ and $G_2$ coincide with their respective targets. Figure 5.21 compares the cumulative distribution of the maximum value in each realization of $T$, $G_1$ and $G_2$. Results show that $G_1$ and $G_2$ are different, and that they are both non-translation fields. Figure 5.22 plots realizations of the K-L simulated non-Gaussian stochastic field $G_1$ and $G_2$, respectively. Thus, K-L expansion can also simulate different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function using different correlated Gaussian initial spectral random variables, and the generated K-L fields are non-translation fields. Therefore, in the non-Gaussian K-L scheme, the initial spectral random variables are not necessarily independent. The K-L scheme still works when the spectral random variables are initialized as correlated random variables.

5.6 Summary

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Non-Gaussian K-L expansion technique has been successfully applied to the spectral representation for simulation of highly skewed non-Gaussian processes within a more general K-L framework. An extension to simulation of multi-dimensional non-Gaussian stochastic fields using K-L expansion has been shown in this chapter. Numerical examples pertaining to non-Gaussian stochastic fields with different marginal distributions are presented. It is demonstrated that different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function can be generated using different initial spectral random variables, and that these K-L simulated non-Gaussian fields are in general non-translation fields. Numerical results also show that the initial spectral random variables are not necessarily independent.
Figure 5.1 Spectral density function of target non-Gaussian stochastic fields
Figure 5.2  Spectral density function and PDF of K-L simulated stochastic field with Beta PDF from Gaussian initial spectral random variables
Figure 5.3  Spectral density function and PDF of K-L simulated stochastic field with Beta PDF from exponential initial spectral random variables
Figure 5.4 Relative error of spectral density function for K-L simulated stochastic fields with Beta PDF from Gaussian and exponential initial spectral random variables.
Figure 5.5 Comparison of cumulative distribution of extreme value in each realization for K-L simulated stochastic fields with Beta PDF from Gaussian and exponential initial spectral random variables.
Figure 5.6 Realizations of K-L simulated stochastic fields with Beta PDF from Gaussian and exponential initial spectral random variables
Figure 5.7 Spectral density function and PDF of K-L simulated stochastic field with shifted exponential PDF from Gaussian initial spectral random variables
Figure 5.8 Spectral density function and PDF of K-L simulated stochastic field with shifted exponential PDF from exponential initial spectral random variables.
Figure 5.9  Comparison of cumulative distribution of extreme value in each realization for K-L simulated stochastic fields with shifted exponential PDF from Gaussian and exponential initial spectral random variables
Figure 5.10  Realizations of K-L simulated stochastic fields with shifted exponential PDF from Gaussian and exponential initial spectral random variables
Figure 5.11  Gaussian probability plot of spectral random variables for K-L simulated stochastic fields with Beta and shifted exponential PDF from Gaussian and exponential initial spectral random variables
Figure 5.12  Spectral density function of target non-Gaussian translation field
Figure 5.13  Spectral density function and PDF of K-L simulated stochastic field $G$
Figure 5.14  Spectral density function and PDF of K-L simulated stochastic field E
Figure 5.15  Comparison of cumulative distribution of extreme value in each realization for stochastic fields T, G and E
Figure 5.16  Realizations of target non-Gaussian translation field $T$ and the underlying Gaussian field.
Figure 5.17  Realizations of K-L simulated stochastic fields G and E
Figure 5.18  Gaussian probability plot of spectral random variables for stochastic fields T, G and E
Figure 5.19  Spectral density function and PDF for K-L simulated stochastic field G1
Figure 5.20  Spectral density function and PDF for K-L simulated stochastic field G2
Figure 5.21 Comparison of cumulative distribution of extreme value in each realization for stochastic fields T, G1 and G2.
Figure 5.22  Realizations of K-L simulated stochastic fields G1 and G2
6.1 Conclusions

This study is an extension of the work that was initiated by Huang (2001) and further developed by Huang (2004) for simulation of stochastic processes using K-L expansion. Topics covered in this thesis are limited to non-Gaussian simulation using K-L expansion. As the result of the studies conducted, the following conclusions can be drawn:

1. The K-L expansion with uncorrelated non-Gaussian random variables has been successfully applied to the simulation of processes with highly skewed non-Gaussian marginal distributions. When the random process is indexed over a domain that is much larger than the correlation distance, the K-L expansion approaches the spectral representation. It follows that the K-L simulation method proposed by Phoon et al. (2002a and 2005) could be applied to the spectral representation as a special case. This special case is of pragmatic interest because the power spectral density function is less costly to evaluate (via FFT of the covariance function) than the eigenvalues and eigenfunctions in the K-L expansion. Numerical examples with different spectral density functions and different marginal distribution functions show that non-Gaussian K-L expansion can generate different processes satisfying the same target spectral density function and the same target marginal distribution function, which is potentially capable of providing a better fit to observed data.

2. Theoretically, a translation process exists only when the prescribed target spectral density function and the prescribed target marginal distribution function
have a compatible relationship. The situation with incompatible functions was previously handled by correcting the original target spectral density function using a spectral preconditioning procedure. Examples with both strongly and weakly incompatible spectral density functions and marginal distribution functions show that non-Gaussian K-L expansion can generate different processes satisfying the same target spectral density function and the same target marginal distribution function which are incompatible from the point of view of translation approach. Results show that the previously used spectral preconditioning technique will change the original target process significantly for those with strongly incompatible target functions. Numerical results also show that non-Gaussian K-L expansion can be used to generate different processes satisfying both the target spectral density function and marginal distribution function which is compatible, and that the K-L generated non-Gaussian processes are generally non-translation processes. Therefore, the K-L expansion is a more robust method for simulation of non-Gaussian processes because it can generate different processes satisfying the same target spectral density function and the same target marginal distribution function regardless of their compatibility.

3. The non-Gaussian K-L expansion technique has been further extended to simulate multi-dimensional non-Gaussian stochastic fields. Since non-Gaussian K-L expansion technique has been successfully applied to the spectral representation for simulation of highly skewed non-Gaussian processes, it is straightforward to simulate multi-dimensional non-Gaussian stochastic fields using multi-dimensional spectral representation within a more general K-L framework. Numerical examples pertaining to non-Gaussian stochastic fields
with different marginal distributions are presented. It is demonstrated that different stochastic fields satisfying the same target spectral density function and the same target marginal distribution function can be generated using the non-Gaussian K-L technique, and that the K-L simulated non-Gaussian stochastic fields are in general non-translation fields. Numerical results also show that the initial spectral random variables are not necessarily independent. The ability to generate non-translation fields is clearly of significant practical interest because the observed data, in general, may be non-Gaussian and non-translational.

6.2 Recommendations for Future Research

The followings are potential areas for further research:

1. K-L expansion is particularly suited for stochastic finite element analysis for its efficiency in the representation of stochastic processes, as it involves minimum number of random variables. It has been used to represent Gaussian processes in stochastic finite element analysis (Ghanem and Spanos, 1991a and 1991b). As an extension for further study, it may be used to represent non-Gaussian processes in stochastic finite element analysis.

2. Polynomial chaos expansion is another general method for simulation of non-Gaussian stochastic processes. According to the Cameron–Martin theorem (Cameron and Martin, 1947), the Hermite polynomial chaos expansion can represent any finite second-moment stochastic processes. However, current simulation algorithms of Hermite polynomial chaos expansion are all based on the translation concept. Since both polynomial chaos expansion and non-Gaussian K-L expansion can represent non-translation processes, the relationship between them needs to be explored.
3. Non-Gaussian processes are mostly simulated by matching a prescribed marginal distribution function and a prescribed covariance function or equivalently, the spectral density function. It is well known that non-Gaussian processes cannot be defined uniquely by the first two moments. Non-Gaussian K-L expansion can generate different processes satisfying the same first two moments. Therefore, it is possible to simulate non-Gaussian processes with greater control than the first two moments using non-Gaussian K-L expansion. Further study is suggested to investigate higher-order correlation characteristics such as bispectra and trispectra.
REFERENCES

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