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ROBUST ADAPTIVE CONTROL OF UNCERTAIN NONLINEAR SYSTEMS

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Summary

In this thesis, robust adaptive control is investigated for uncertain nonlinear systems. The main purpose of the thesis is to develop adaptive control strategies for several classes of general nonlinear systems in strict-feedback form with uncertainties including unknown parameters, unknown nonlinear systems functions, unknown disturbances, and unknown time delays. Systematic controller designs are presented using backstepping methodology, neural network parametrization and robust adaptive control. The results in the thesis are derived based on rigorous Lyapunov stability analysis. The control performance of the closed-loop systems is explicitly analyzed.

The traditional backstepping design is cancellation-based as the coupling term remaining in each design step will be cancelled in the next step. In this thesis, the coupling term in each step is decoupled by elegantly using the Young's inequality rather than leaving to it to be cancelled in the next step, which is referred to as the decoupled backstepping method. In this method, the virtual control in each step is only designed to stabilize the corresponding subsystems rather than previous subsystems and the stability result of each step obtained by seeking the boundedness of the state rather than cancelling the coupling term so that the residual set of each state can be determined individually. Two classes of nonlinear systems in strict-feedback form are considered as illustrative examples to show the design method. It is also applied throughout the thesis for practical controller design.

For nonlinear systems with unknown time delays, the main difficulty lies in the

terms with unknown time delays. In this thesis, by using appropriate Lyapunov-Krasovskii functional candidate, the uncertainties from unknown time delays are compensated for such that the design of the stabilizing control law is free from unknown time delays. In this way, the iterative backstepping design procedure can be carried out directly. Controller singularities are effectively avoided by employing practical robust control. It is first applied to a type of nonlinear strict-feedback systems with unknown time delay using neural networks approximation. Two different NN control schemes are developed and semi-global uniform ultimate boundedness of the closed-loop signals is achieved. It is then extended to a kind of nonlinear time-delay systems in parametric-strict-feedback form and global uniform ultimate boundedness of the closed-loop signals is obtained. In the latter design, a novel continuous function is introduced to construct differentiable control functions.

When there is no a priori knowledge on the signs of virtual control coefficients or high-frequency gain, adaptive control of such systems becomes much more difficult. In this thesis, controller design incorporated by the Nussbaum-type gains is presented for a class of perturbed strict-feedback nonlinear systems and a class of nonlinear time-delay systems with unknown virtual control coefficients/functions. The behavior of this class of control laws can be interpreted as the controller tries to sweep through all possible control gains and stops when a stabilizing gain is found. To cope with uncertainties and achieve global boundedness, an exponential term has to be incorporated into the stability analysis. Thus, novel technical lemmas are introduced. The proof of the key technical lemmas are given for different Nussbaum functions being chosen.

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Chapter 1

Introduction

Recent years have witnessed great progress in adaptive control of nonlinear systems due to great demands from industrial applications. In this thesis, robust adaptive control of uncertain nonlinear systems has been investigated. The main purpose of the thesis is to develop adaptive control strategies for several types of general nonlinear systems with uncertainties from unknown systems functions, unknown time delays, unknown control directions. Using backstepping technique, an iterative controller design procedure is presented for these uncertain nonlinear systems in strict-feedback form.

The traditional backstepping design is cancellation-based as the coupling term remaining in each design step will be cancelled in the next step. In this thesis, the coupling term in each step is decoupled by elegantly using the Young's inequality rather than leaving to it to be cancelled in the next step, which is referred to as the decoupled backstepping method. In this method, the virtual control in each step is only designed to stabilize the corresponding subsystems rather than previous subsystems and the stability result of each step obtained by seeking the boundedness of the state rather than cancelling the coupling term so that the residual set of each state can be determined individually. Two classes of nonlinear systems in strict-feedback form are considered as illustrative examples to show the design method. It is also applied throughout the thesis for practical controller design.

For nonlinear systems with unknown time delays, the main difficulty lies in the terms with unknown time delays. In this thesis, by using appropriate Lyapunov-Krasovskii functionals candidate, the uncertainties from unknown time delays are compensated for such that the design of the stabilizing control law is free from unknown time delays. In this way, the iterative backstepping design procedure can be carried out directly. Controller singularities are effectively avoided by employing practical robust control. It is first applied to a kind of nonlinear strict-feedback systems with unknown time delay using neural networks (NNs) approximation. Two different NN control schemes are developed and semi-global uniform ultimate boundedness of the closed-loop signals is achieved. It is then extended to a type of nonlinear time-delay systems in parametric-strict-feedback form and global uniform ultimate boundedness of the closed-loop signals is obtained. In the latter design, a novel continuous function is introduced to construct differentiable control functions.

When there is no a priori knowledge on the signs of virtual control coefficients or high-frequency gain, adaptive control of such systems becomes much more difficult. In this thesis, controller design incorporated by Nussbaum-type gains is presented for a class of perturbed strict-feedback nonlinear systems and a class of nonlinear time-delay systems with unknown virtual control coefficients/functions. The behavior of this class of control laws can be interpreted as the controller tries to sweep through all possible control gains and stops when a stabilizing gain is found. To cope with uncertainties and achieve global boundedness, an exponential term has to be incorporated in the stability analysis. Thus, novel technical lemmas are introduced. The proof of the key technical lemmas are shown to be function-dependent and much involved. Two different Nussbaum functions are chosen with distinct proofs being given.

The rest of the chapter is organized as follows. In section 1.1, the background of (i) backstepping design and neural network control, (ii) universal adaptive control using Nussbaum functions, (iii) stabilization of time-delay systems is briefly reviewed. The main topics and objectives of the thesis are discussed in Section 1.2. The organization of the thesis is summarized in Section 1.3 with a description of the purposes, contents, and methodologies used in each chapter.

1.1 Background and Motivation

1.1.1 Backstepping Design and Neural Network Control

Adaptive control plays an important role due to its ability to compensate for parametric uncertainties. In order to obtain global stability, some restrictions have to be made to nonlinearities such as matching conditions [1], extended matching conditions [2], or growth conditions [3][4]. To overcome these restrictions, a recursive design procedure called adaptive backstepping design was developed in [5] for a class of nonlinear systems transformable to a parametric-pure-feedback form or a parametric-strict-feedback form. The overall system's stability was guaranteed via Lyapunov stability analysis, by which it was shown that the stability result was local for the systems in the former form and global in the latter form. The technique of "adding an integrator" was first initiated in [6][7][8][9], and further developed in [10][11][12][13]. The advantage of adaptive backstepping design is that not only global stability and asymptotic stability can be achieved, but also the transient performance can be explicitly analyzed and guaranteed. However, the backstepping design in [5] requires multiple estimates of the same parameters. This overparametrization problem was then removed in [14] by introducing the concept of tuning function. Several extensions of adaptive backstepping design have been reported for nonlinear systems with triangular structures [15], for a class of large-scale systems transformable to the decentralized strict-feedback form [16], and for a class of nonholonomic systems [17]. For systems with unknown nonlinearities which cannot be represented in linear-in-parameter form, robust modifications were considered, including σ-modification in [18], nonlinear damping technique [19][20] and smooth projection algorithm [21]. Robust adaptive design was proposed in [22] for the systems' uncertainties satisfying an input-to-state stability property. For uncertain systems in a strict-feedback form and with disturbances, a robust adaptive backstepping scheme was presented in [23][24][25][26](to name just a few).

For nonlinear, imperfectly or partially known, and complicated systems, NNs offer some of the most effective control techniques. There are various approaches that are being proposed in the literature. The paper [27] gives a good survey for earlier achievements. Recent developments can be seen in [28][29][30][31][32] [33][34][35]

[36][37][38][39] [40][18][41] [42]. Since the pioneering works [43][44][45] on controlling nonlinear dynamical systems using NNs, there have been tremendous interests in the study of adaptive neural control of uncertain nonlinear systems with unknown nonlinearities, and a great deal of progress has been made both in theory and practical applications.

The idea of employing NN in nonlinear system identification and control was motivated by the distinguished features of NN, including a highly parallel structure, learning ability, nonlinear function approximation, fault tolerance, and efficient analog VLSI implementation for real-time applications (see [46] and the references therein). In most of the NN control approaches, neural networks are used as function approximators. The unknown nonlinearities are parametrized by linearly or nonlinearly parameterized NNs, such as radial basis functions (RBF) neural networks and multilayer neural networks (MNNs). It is notable that when applying NNs in closed-loop feedback systems, even a static NN becomes a dynamical one and it might take on some new and unexpected behaviors [47]. In the earlier NN control schemes, optimization techniques were mainly used to derive parameter adaptation laws. The neural control design was mostly demonstrated through simulation or by particaular experimental examples. The disadvantage of optimization-based neurocotrollers is that it is generally difficult to derive analytical results for stability analysis and performance evaluation of the closed-loop system. To overcome these problems, some elegant adaptive NN control approaches have been proposed for uncertain nonlinear systems [44][45][48][49][50] [51][29][31] [52][53][54][55][56] [57]. Specifically, Sanner and Slotine [45] have done in-depth treatment in the approximation of Gaussian radial basis function (RBF) networks and the stability theory to adaptive control using sliding mode control design. Lewis *at al.* [51] developed multilayer NN-based control methods and successfully applied them to robotic control for achieving stable adaptive NN systems. The features of adaptive neural control include: (i) it is based on the Lyapunov stability theory; (ii) the stability and performance of the closed-loop control system can be readily determined; (iii) the NN weights are tuned on-line, using a Lyapunov synthesis method, rather than optimization techniques. It has been found that adaptive neural control is particularly suitable for controlling highly uncertain, nonlinear, and complex systems (see [47][58] and the references therein).

By combing adaptive neural network design with backstepping methodology, some new results have begun to emerge for solving certain classes of complicated nonlinear systems. However, there are still several fundamental problems about stability, robustness, and other issues yet to be further investigated.

1.1.2 Adaptive Control Using Nussbaum Functions

Adaptive control plays an important role due to its ability to compensate for parametric uncertainties. It is characterized by a combination of identification or estimation mechanisms of the plant parameters together with a feedback controller. For a survey see [4] and [59]. An area of non-identifier-based adaptive control was initiated in [60][61][62][63], etc., in which the adaptation strategy did not invoke any identification or estimation mechanism of the unknown parameters. The adaptive controllers involving a switching strategy in the feedback were proposed. The switching strategy was mainly tuned by system information from states or output. The system under consideration were either minimum phase or, more generally, only stabilizable and observable. No assumptions were made on the upper bound of the high-frequency gain nor even on the sign of the high-frequency gain. The switching strategies could be constructed with the introduction of Nussbaum functions [62] and several control algorithm was developed based on the Nussbaum function in [63][60][64][61] [65][66][67][68]. Most results are developed for linear systems, among which, the results in [63] were for single-input-single-output linear systems with relative degree $\rho = 2$, the results in [60][64][61][67] were for single-input-single-output linear systems with any relative degree, the results in [65] for multi-input-multi-output linear systems with relative degree $\rho = 2$, the results in [66] for multi-input-multi-output linear systems with any relative degree. Later control algorithms based on Nussbaum functions were proposed for first-order nonlinear systems in [69], for nonlinearly perturbed linear systems with relative degree one or two in [70][68][71][72] to counteract the lack of a priori knowledge of the high-frequency gain. An alternative method called correction vector approach was proposed in [73] and has been extended to design adaptive control of first-order nonlinear systems with unknown high-frequency gain in [74][75]. A nonlinear robust control scheme has been proposed in [76], which can identify online the unknown high-frequency gain and can guarantee global stability of the closed-loop system. Among these works, the systems have to be restricted as second-order (vector) systems [69], [74] and [75], or the unmatched nonlinearities in [70][68][71][72] and the additive nonlinearities in [74] have to satisfy the global Lipschitz or sectoricity condition. In addition, the adaptive control law formulated in [74] and [75] are discontinuous.

As stated in Section 1.1.1, global adaptive control of nonlinear systems without any restrictions on the growth rate of nonlinearities or matching conditions has been intensively investigated in [77][78][19][79]. However, the proposed design procedure was carried out based on the assumption of the knowledge of high-frequency gain sign, which is quite restrictive for the general case. The results were first obtained for output feedback adaptive control of nonlinear systems with unknown high-frequency gain (or alternatively called "virtual control coefficients" or "control directions") in [80] with restrictions in the growth rates of nonlinear terms. The growth restrictions condition on system nonlinearities was later removed in [81], in which, however, a so-called augmented parameter vector has to be introduced, which would double the number of parameters to be updated. Another global adaptive output-feedback control scheme was developed in [82], which did not require *a priori* knowledge of the high-frequency gain sign at the price of making any restrictions on the growth rate of the system nonlinearities, and only the minimal number of parameters needed to be updated. For nonlinear systems in parametric-strict-feedback form, the technique of Nussbaum function gain was incorporated into the adaptive backstepping design in [83]. The robust control scheme was first developed in [76] for a class of nonlinear systems without *a priori* knowledge of control directions. However, the design scheme could be applied to second-order (vector) systems at most. In addition, both the bounds of the uncertainties and the bounds of their partial derivatives need to be known. The robust tracking control for more general classes of uncertain nonlinear systems was proposed in [84] and later a flat-zone modification for the scheme was introduced in [85].

While the earlier works such as [15][18][86] assumed the virtual control coefficients to be 1, adaptive control has been extended to parametric strict-feedback systems with unknown constant virtual control coefficients but with known signs (either positive or negative) [19] based on the cancellation backstepping design as stated in [87] by seeking the cancellation of the coupling terms related to $z_i z_{i+1}$ in the next step of Lyapunov design. With the aid of neural network parametrization, adaptive control schemes have been further extended to certain classes of strict-feedback in which virtual control coefficients are unknown functions of states with known signs [88][51]. For the system $\dot{x} = f(x) + g(x)u$, the unknown virtual control function $g(x)$ causess great design difficulty in adaptive control. Based on feedback linearization, certainty equivalent control $u = [-\hat{f}(x)+v]/\hat{g}(x)$ is usually taken, where $\hat{f}(x)$ and $\hat{g}(x)$ are estimates of $f(x)$ and $g(x)$, and measures have to be taken to avoid controller singularity when $\hat{g}(x) = 0$. To avoid this problem, integral Lyapunov functions have been developed in [88], and semi-globally stable adaptive controllers are developed, which do not require the estimate of the unknown function $g(x)$. Although the system's virtual control coefficients are assumed to be unknown nonlinear functions of states, their signs are assumed to be known as strictly either positive or negative. Under this assumption, stable neural network controllers have been constructed in [51] by augmenting a robustifying portion, and in [89],[90] by estimating the derivation of the control Lyapunov function.

1.1.3 Stabilization of Time-Delay Systems

Time-delay systems are also called systems with aftereffect or dead-time, hereditary systems, etc. Time delays are important phenomena in industrial processes, economical and biological systems. The monographs [91][92] give quite a lot good examples. In addition, actuators, sensors, field networks that are involved in feedback loops usually introduce delays. Thus, time delays are strongly involved in challenging areas of communication and information technologies [93]. For instance, they appear as transportation and communication lags and also arise as feedback delays in control loops. As time delays have a major influence on the stability of such dynamical systems, it is important to include them in the mathematical description. There have been a great number of papers and monographs devoted to this field of active research [94][95][96]. For survey papers see [97][98][99].

The existence of time delays may make the stabilization problem become more difficult. Useful tools such as linear matrix inequalities (LMIs) is hard to apply to nonlinear systems with time delays. Lyapunov design has been proven to be an effective tool in controller design for nonlinear systems. However, one major difficulty lies in the control of time-delayed nonlinear systems is that the delays are usually not perfectly known. A feasible approach is the preliminary compensation of delays such that the control techniques developed for systems without delays can be applied. The delay can be partially compensated through prediction, or, in some cases, can be exactly cancelled. The delay is compensated through prediction in [100][101] such that classical tools of differential geometry can be applied. In some works, the compensation is avoided with extensions of differential geometry being applied. The disturbances decoupling is concerned in [102], while the classical input-output linearization technique is extended in [103][104]. A necessary and sufficient condition for which delay systems do not admit state internal dynamics is given in [105]. For sliding mode control for delay systems, the results can be found in [106][107][108]. The unknown time delays are the main issue to be dealt with for the extension of backstepping design to such kinds of systems. A stabilizing controller design based on the Lyapunov-Krasovskii functionals is presented in [109] for a class of nonlinear time-delay systems with a so-called "triangular structure". However, few attempts have been made towards the systems with unknown parameters or unknown nonlinear functions.

1.2 Objectives of the Thesis

The objective of the thesis is to develop adaptive controllers for general uncertain nonlinear systems with uncertainties from unknown parameters, unknown nonlinearity, unknown control directions and unknown time delays.

For nonlinear systems with various uncertainties, ultimately uniformly bounded stability is often the best result achievable. The first objective is to develop a decoupling backstepping method, which is different from the traditional cancellation-based backstepping design. The intermediate control in each intermediate step is designed to guarantee the boundedness of the corresponding state of each subsystems. The decoupling backstepping design is useful for the development of smooth switching scheme in the later design.

The second objective is to utilize backstepping technique for a class of nonlinear systems with unknown time delays. Adaptive control is developed for systems in parametric-strict-feedback form and NN parametrization is used for systems with nonlinear unknown systems function. To avoid singularity problems, integral Lyapunov functions are used and practical backstepping control is introduced. As the practical controller design is applied, the compact set, over which the NNs approximation is carried out, shall be re-constructed with its feasibility to be guaranteed. To satisfy the differentiability of the intermediate control functions in the backstepping design, certain smooth functions are introduced to tackle the problem.

The third objective is to develop a global stabilizing control for systems with unknown control direction. Nussbaum-type gain is used to construct the controller and exponential term is introduced to achieve global boundedness.

1.3 Organization of the Thesis

The thesis is organized as follows.

Chapter 2 gives the mathematical preliminaries which is utilized throughout the thesis. It contains basic definitions in Lyapunov stability analysis, and useful stability results used throughout the thesis, introduction of universal adaptive control and various Nussbaum functions, and the stability result related to Nussbaum functions.

In Chapter 3, the concept of decoupled backstepping design is introduced as a general tool for control systems design where the coupling terms are decoupled by elegantly using Young's inequality, and it is first applied to a class of parametric-strict-feedback nonlinear systems with unknown disturbances which satisfies triangular bounded conditions. The design example with NN approximation is given later using the design method.

In Chapter 4, adaptive neural control is presented for a class of strict-feedback nonlinear systems with unknown time delays using a Lyapunov-Krasovskii functional to compensate for the unknown time delays and integral Lyapunov function to tackle the singular problems. In addition, a direct NN control using quadratic Lyapunov functions is proposed for the same problem.

In Chapter 5, an adaptive control is proposed for a class of parameter-strict-feedback nonlinear systems with unknown time delays. Differentiable control functions are presented.

Chapter 6, concerns with robust adaptive control for a class of perturbed strict-feedback nonlinear systems with both completely unknown control coefficients and parametric uncertainties. The proposed design method does not require the *a priori* knowledge of the signs of the unknown control coefficients. Another design example for systems with unknown control coefficients is given for nonlinear time-delay systems.

Chapter 7 concludes the contributions of the thesis and makes recommendation on the future research works.

Chapter 2

Mathematical Preliminaries

2.1 Introduction

Stability analysis is the one of the fundamental topics being discussed in the control engineering. Among the various analysis methodologies, Lyapunov stability theory plays a critial role in both design and analysis of the controlled systems. It is well known that the analysis of properties of the closed-loop signals is based on properties of the solution to the differential equation of the system. For nonlinear systems, it is generally very difficult to find a analytic solution and becomes almost impossible for uncertain systems. The only general way of pursuing stability analysis and control design for uncertain systems is the Lyapunov direct method which determines stability without explicitly solving the differential equations. Therefore, the Lyapunov direct method provides a mathematical foundation for analysis and can be used as the means of designing robust control, which is chosen as the main approach taken in this thesis.

In this chapter, some basic definitions of Lyapunov stability are presented followed by several useful technical lemmas related to the stability analysis and invoked throughout the thesis. To tackle the unknown high-frequency gain (or unknown control directions, unknown virtual control coefficients), universal adaptive control is carried out using Nussbaum functions. The basic idea of universal adaptive control is presented. Nussbaum functions are introduced with detailed analysis of their properties. In addition, several useful technical lemmas related to the stability analysis for systems using Nussbaum functions to construct control law are developed.

2.2 Lyapunov Stability Analysis

The definitions for stability, uniform stability, asymptotic stability, uniformly asymptotic stability, uniform boundedness, uniform ultimate boundedness are given as follows [110].

Definition 1 *The equilibrium point $x = 0$ is said to be Lyapunov stable (LS) (or, in short, stable), at time t_0 if, for each $\epsilon > 0$, there exists a constant $\delta(t_0, \epsilon) > 0$ such that*

$$\|x(t_0)\| < \delta(t_0, \epsilon) \Longrightarrow \|x(t)\| \leq \epsilon, \quad \forall t \geq t_0.$$

It is said to be uniformly Lyapunov stable (ULS) or, in short, uniformly stable (US) over $[t_0, \infty)$ if, for each $\epsilon > 0$, the constant $\delta(t_0, \epsilon) = \delta(\epsilon) > 0$ can be chosen as independent of initial time t_0.

Definition 2 *The equilibrium point $x = 0$ is said to be attractive at time t_0 if, for some $\delta > 0$ and each $\epsilon > 0$, there exists a finite time interval $T(t_0, \delta, \epsilon)$ such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \leq \epsilon, \quad \forall t \geq t_0 + T(t_0, \delta, \epsilon).$$

It is said to be uniformly attractive (UA) over $[t_0, \infty)$ if for all ϵ satisfying $0 < \epsilon < \delta$, the finite time interval $T(t_0, \delta, \epsilon) = T(\delta, \epsilon)$ is independent of initial time t_0.

Definition 3 *The equilibrium point $x = 0$ is asymptotically stable (AS) at time t_0 if it is Lyapunov stable at time t_0 and if it is attractive, or equivalently, there exists $\delta > 0$ such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \to \epsilon \text{ as } t \to \infty.$$

it is uniformly asymptotically stable (UAS) over $[t_0, \infty)$ if it is uniformly Lyapunov stable over $[t_0, \infty)$, and if $x = 0$ is uniformly attractive.

Definition 4 *The equilibrium point $x = 0$ at time t_0 is exponentially attractive (EA) if, for some $\delta > 0$, there exist constants $\alpha(\delta) > 0$ and $\beta > 0$ such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \leq \alpha(\delta) \exp[-\beta(t - t_0)].$$

It is said to be exponentially stable (ES) if, for some $\delta > 0$, there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \leq \alpha \exp[-\beta(t - t_0)].$$

Definition 5 *A solution $x : R^+ \to R^n$, $x(t_0) = x_0$, is said to be uniformly bounded (UB) if, for some $\delta > 0$, there is a positive constant $d(\delta) < \infty$, possibly dependent on δ (or x_0) but not on t_0, such that, for all $t \geq t_0$,*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| \leq d(\delta).$$

Definition 6 *A solution $x : R^+ \to R^n$, $x(t_0) = x_0$, is said to be uniformly ultimately bounded (UUB) with respect to a set $W \subset R^n$ containing the origin if there is a nonnegative constant $T(x_0, W) < \infty$, possibly dependent on x_0 and W but not on t_0, such that $\|x(t_0)\| < \delta$ implies $x(t) \in W$ for all $t \geq t_0 + T(x_0, W)$.*

The set W, called residue set, is usually characterized by a hyper-ball $W = B(0, \epsilon)$ centered at the origin and of radius ϵ. If ϵ is chosen such that $\epsilon \geq d(\delta)$, UUB stability reduces to UB stability. Although not explicitly stated in the definition, UUB stability is used mainly for the case that ϵ is small, which presents a better stability result than UB stability.

If both $d(\delta)$ and W can be made arbitrarily small, UB and UUB approach uniform asymptotic stability in the limit. In some literature, UB and UUB approach is called practical stability.

The UUB stability is less restrictive than UAS or ES, but, as will be shown later, it can be made arbitrarily close to UAS in many cases through making the set W

small enough as a result of a properly designed robust control. Also, UUB stability provides a measure on convergence speed by offering the time interval $T(x_0, W)$. In fact, the UUB stability is often the best result achievable in controlling uncertain systems.

The following lemmas are useful for the stability analysis throughout the thesis and are presented here for easy references.

Lemma 2.2.1 *Let $V(t)$ be continuously differentiable function defined on $[0, +\infty)$ with $V(t) \geq 0$, $\forall t \in R^+$ and finite $V(0)$, and $c_1, c_2 > 0$ be real constants. If the following inequality holds*

$$\dot{V}(t) \leq -c_1 x^2(t) + c_2 y^2(t) \tag{2.1}$$

and $y(t) \in L_2$, we can conclude that $x(t) \in L_2$. [87]

Proof: Integrating (2.1) over $[0, t]$, we have

$$V(t) - V(0) \leq -\int_0^t c_1 x^2(\tau) d\tau + \int_0^t c_2 y^2(\tau) d\tau$$

i.e.

$$0 \leq V(t) + \int_0^t c_1 x^2(\tau) d\tau \leq V(0) + \int_0^t c_2 y^2(\tau) d\tau$$

Since $V(0)$ is finite and $y(t) \in L_2$, i.e., $\int_0^t c_2 y^2(\tau) d\tau$ is finite, we can conclude that $V(t)$ is bounded and $\int_0^t c_1 x^2(\tau) d\tau$ is finite, i.e. $x(t) \in L_2$. $\diamondsuit$

Lemma 2.2.2 *Let $V(t)$ be continuously differentiable function defined on $[0, +\infty)$ with $V(t) \geq 0$, $\forall t \in R^+$ and finite $V(0)$, $\rho(t)$ be a real-valued function, and $c_1, c_2 > 0$ be real constants. If the following inequality holds*

$$\dot{V}(t) \leq -c_1 V(t) + c_2 \rho(t) \tag{2.2}$$

and $\rho(t) \in L_\infty$, we can conclude that $V(t)$ is bounded.

Proof: Upon multiplying both sides of (2.2) by $e^{c_1 t}$, it becomes

$$\frac{d}{dt}(V(t) e^{c_1 t}) \leq c_2 \rho(t) e^{c_1 t} \tag{2.3}$$

Integrating (2.3) over $[0, t]$ yields

$$V(t) \leq V(0)e^{-c_1 t} + c_2 \int_0^t e^{-c_1(t-\tau)}\rho(\tau)d\tau \tag{2.4}$$

Note the following inequality

$$\begin{aligned} c_2 \int_0^t e^{-c_1(t-\tau)}\rho(\tau)d\tau &\leq c_2 e^{-c_1 t} \int_0^t |\rho(\tau)| e^{c_1\tau} d\tau \\ &\leq c_2 e^{-c_1 t} \sup_{\tau\in[0,t]} [|\rho(\tau)|] \int_0^t e^{c_1\tau} d\tau \leq \frac{c_2}{c_1} \sup_{\tau\in[0,t]} [|\rho(\tau)|] \end{aligned} \tag{2.5}$$

Since $\rho(t) \in L_\infty$, i.e. $\rho(t)$ is finite, we know from (2.5) that $c_2 \int_0^t e^{-c_1(t-\tau)}\rho(\tau)d\tau$ is bounded. Let c_0 be the upper bound of $c_2 \int_0^t e^{-c_1(t-\tau)}\rho(\tau)d\tau$, (2.4) becomes

$$V(t) \leq c_0 + V(0)e^{-c_1 t} \leq c_0 + V(0) \tag{2.6}$$

Since $V(0)$ is finite, we can readily conclude that $V(t)$ is bounded. In addition, from (2.6), we can conclude that given any $\mu > \mu^*$ with $\mu^* = c_0$, there exists T such that for any $t > T$, we have $V(t) \leq \mu$, while T can be calculated by $c_0 + V(0)e^{-c_1 T} = \mu$ with $T = -\frac{1}{c_1} \ln\left(\frac{\mu - c_0}{V(0)}\right)$. $\diamondsuit$

Lemma 2.2.3 *Let $V(t)$ be continuously differentiable function defined on $[0, +\infty)$ with $V(t) \geq 0$, $\forall t \in R^+$ and finite $V(0)$, $\rho(t)$ be a real-valued function, and $c_1, c_2 > 0$ be real constants. If the following inequality holds*

$$\dot{V}(t) \leq -c_1 x^2(t) + c_2 x(t)\rho(t) \tag{2.7}$$

and $\rho(t) \in L_2$, we can conclude that $V(t)$ is bounded and $x(t) \in L_2$.

Proof: Applying Young's inequality to (2.7), we have

$$\dot{V}(t) \leq -c_1 x^2(t) + c_2\Big[\frac{1}{4k_1} x^2(t) + k_1\rho^2(t)\Big] \tag{2.8}$$

where positive constant k_1 is a sufficiently large such that $c_1^* \triangleq c_1 - \frac{c_2}{4k_1} > 0$. Then, (2.8) becomes

$$\dot{V}(t) \leq -c_1^* x^2(t) + c_2 k_1 \rho^2(t) \tag{2.9}$$

Invoking Lemma 2.2.1, we can conclude that $V(t)$ is bounded and $\int_0^t x^2(\tau)d\tau$ is finite, i.e., $x(t) \in L_2$. $\diamondsuit$

Lemma 2.2.4 *Let $V(t)$ be positive definite function with finite $V(0)$, $\rho(\cdot)$ be real-valued function and and $c_1, c_2 > 0$ be real constants. If the following inequality holds*

$$\dot{V}(t) \leq -c_1 x^2(t) + c_2 \rho(y(t)) \tag{2.10}$$

and $\rho(y) \in L^1$, then we can conclude that $x(t) \in L_2$.

Proof: Integrating (2.10) over $[0, t]$ yields

$$V(t) - V(0) \leq -\int_0^t c_1 x^2(\tau) d\tau + \int_0^t c_2 \rho(y(\tau)) d\tau$$

i.e.

$$V(t) + \int_0^t c_1 x^2(\tau) d\tau \leq V(0) + \int_0^t c_2 \rho(y(\tau)) d\tau$$

Since $\rho(y) \in L^1$, i.e. $\int_0^t c_2 \rho(y(\tau)) d\tau$ is bounded, we can conclude that $V(t)$ is bounded and $x(t)$ is square integrable. $\diamondsuit$

The following lemma is crucial for deriving uniformly ultimately bounded stability of closed-loop systems and gives an explicit and quantified analysis for the initial condition, transient performance and the final convergence of the closed-loop signals, and the relationship among them.

Lemma 2.2.5 *Let $V(t) \geq 0$ be smooth functions defined on $[0, +\infty)$, $\forall t \in R^+$ and $V(0)$ is finite. Suppose $V(t)$ takes the following form*

$$V(t) = \frac{1}{2} e^T(t) Q e(t) + \frac{1}{2} \tilde{W}^T(t) \Gamma^{-1} \tilde{W}(t) \tag{2.11}$$

where $e(t) = x(t) - x_d(t)$ is tracking error and $\tilde{W}(t) = \hat{W}(t) - W^*$ is parameter estimation error with $x(t) \in R^n$, $x_d(t) \in \Omega_d \subset R^n$, $\hat{W}(t) \in R^m$, $W^* \in R^m$ being constant vector, $Q = Q^T > 0 \in R^{n \times n}$, and $\Gamma = \Gamma^T > 0 \in R^{m \times m}$.

If the following inequality holds

$$\dot{V}(t) \leq -c_1 V(t) + c_2, \quad c_1 > 0, c_2 > 0 \tag{2.12}$$

for the system initiated from the following compact sets defined by

$$\Omega_0 = \left\{ x(0), x_d(0), \hat{W}(0) \;\middle|\; x(0), \hat{W}(0) \text{ finite}, \;\; x_d(0) \in \Omega_d \right\} \tag{2.13}$$

we can conclude that

(i) the states in the closed-loop system will remain in the compact set defined by

$$\begin{aligned}\Omega \;=\; & \Big\{x(t), \hat{W}(t) \;\Big|\; \|x(t)\| \le c_{e\max} + \max_{\tau\in[0,t]}\{\|x_d(\tau)\|\}, \\ & \|\hat{W}\| \le c_{\tilde{W}\max} + \max\{\|W^*\|\}\Big\}\end{aligned}$$

(ii) the closed-loop states will eventually converge to the compact sets defined by

$$\Omega_s = \Big\{x(t), \hat{W}(t) \;\Big|\; \lim_{t\to\infty}\|e(t)\| = \mu_e^*, \quad \lim_{t\to\infty}\|\tilde{W}\| = \mu_{\tilde{W}}^*\Big\}$$

where constants

$$c_{e\max} = \sqrt{\frac{2V(0)+2c_2/c_1}{\lambda_{Q\min}}}, \quad c_{\tilde{W}\max} = \sqrt{\frac{2V(0)+2c_2/c_1}{\lambda_{\Gamma\min}}} \tag{2.14}$$

$$\mu_e^* = \sqrt{\frac{2c_2}{c_1\lambda_{Q\min}}}, \quad \mu_{\tilde{W}}^* = \sqrt{\frac{2c_2}{c_1\lambda_{\Gamma\min}}} \tag{2.15}$$

with $\lambda_{Q\min} = \min_{\tau\in[0,t]}\lambda_{\min}(Q(\tau))$, *and* $\lambda_{\Gamma\min} = \min_{\tau\in[0,t]}\lambda_{\min}(\Gamma^{-1}(\tau))$.

Proof: Multiplies (2.12) by $e^{c_1 t}$ yields

$$\frac{d}{dt}(V(t)e^{c_1 t}) \le \lambda_1 e^{c_1 t} \tag{2.16}$$

Integrating (2.16) over $[0, t]$ leads to

$$0 \le V(t) \le [V(0) - c_2/c_1]e^{-c_1 t} + c_2/c_1 \tag{2.17}$$

where $V(0) = \frac{1}{2}e^T(0)Qe(0) + \frac{1}{2}\tilde{W}^T(0)\Gamma^{-1}\tilde{W}(0)$.

(i) *Uniform Boundedness (UB):*

From (2.17), we have

$$0 \le V(t) \le [V(0) - c_2/c_1]e^{-c_1 t} + c_2/c_1 \le V(0) + c_2/c_1 \tag{2.18}$$

From (2.11), we have

$$\frac{1}{2}\lambda_{Q\min}\|e(t)\|^2 \le \frac{1}{2}\lambda_{\min}(Q(t))\|e(t)\|^2 \le \frac{1}{2}e^T(t)Q(t)e(t) \le V(t) \tag{2.19}$$

$$\frac{1}{2}\lambda_{\Gamma\min}\|\tilde{W}(t)\|^2 \le \frac{1}{2}\lambda_{\min}(\Gamma^{-1}(t))\|\tilde{W}(t)\|^2 \le \frac{1}{2}\tilde{W}^T(t)\Gamma^{-1}(t)\tilde{W}(t) \le V(t) \tag{2.20}$$

then, by combining with equation (2.18), we have

$$\|e(t)\| \le c_{e\max}, \quad \|\tilde{W}(t)\| \le c_{\tilde{W}\max}$$

where $c_{e\max}$ and $c_{\tilde{W}\max}$ are given in (2.14). Since $e(t) = x(t) - x_d(t)$ and $\tilde{W}(t) = \hat{W}(t) - W^*$, we have

$$\|x(t)\| - \|x_d(t)\| \le \|x(t) - x_d(t)\| \le c_{e\max}$$
$$\|\hat{W}(t)\| - \|W^*\| \le \|\hat{W}(t) - W^*\| \le c_{\tilde{W}\max}$$

i.e.,

$$\|x(t)\| \le c_{e\max} + \|x_d(t)\| \le c_{e\max} + \max_{\tau\in[0,t]}\{\|x_d(\tau)\|\}$$
$$\|\hat{W}(t)\| \le c_{\tilde{W}\max} + \|W^*\| \tag{2.21}$$

(ii) *Uniform Ultimate Boundedness (UUB):*

From (2.17), (2.19) and (2.20), we have

$$\|e(t)\| \le \sqrt{\frac{2[V(0) - c_2/c_1]e^{-c_1 t} + 2c_2/c_1}{\lambda_{Q\min}}} \tag{2.22}$$

$$\|\tilde{W}(t)\| \le \sqrt{\frac{2[V(0) - c_2/c_1]e^{-c_1 t} + 2c_2/c_1}{\lambda_{\Gamma\min}}} \tag{2.23}$$

If it so happens that $V(0) = c_2/c_1$, then $\|e(t)\| \le \mu_e^*$, $\forall t \ge 0$. If $V(0) \ne c_2/c_1$, from (2.22), we can conclude that given any $\mu_e > \mu_e^*$, there exists T_e, such that for any $t > T_e$, we have $\|e(t)\| \le \mu_e$. Specifically, given any μ_e,

$$\mu_e = \sqrt{\frac{2[V(0) - c_2/c_1]e^{-c_1 T} + 2c_2/c_1}{\lambda_{Q\min}}}$$

then

$$T_e = T_e(\mu_e, V(0)) = -\frac{1}{c_1}\ln\left(\frac{\mu_e^2\lambda_{Q\min} - 2c_2/c_1}{2[V(0) - c_2/c_1]}\right)$$

and

$$\lim_{t\to\infty}\|e(t)\| = \mu_e^*$$

$\diamondsuit$

Remark 2.2.1 Ω *is related to* Ω_0 *while* Ω_s *is not.*

The relationship among the three compacts is illustrated in Fig. 2.1.

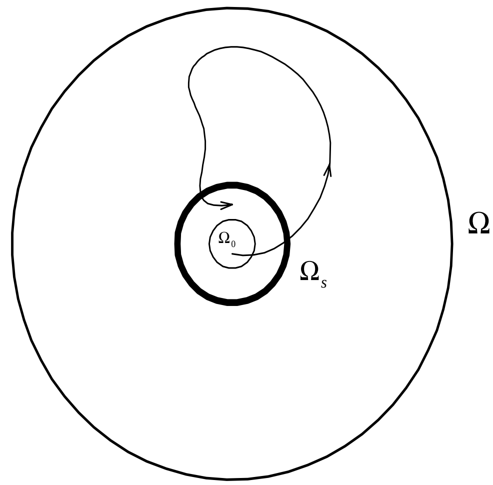

Figure 2.1: Relationship among compact Sets Ω, Ω_0 and Ω_s.

2.3 Universal Adaptive Control

To illustrate the idea, consider the following linear time-invariant scalar system

$$\begin{cases} \dot{x}(t) &= ax(t) + bu(t), \quad x(0) = x_0 \\ y(t) &= cx(t) \end{cases} \tag{2.24}$$

where $a, b, c, x_0 \in R$ are unknown and the only structural assumption is $cb \neq 0$, i.e., the system is controllable and observable.

If the feedback control law $u(t) = -ky(t)$ is chosen, the closed-loop system has the form

$$\dot{x}(t) = (a - kcb)x(t), \quad x(0) = x_0 \tag{2.25}$$

If $a/|cb| < |k|$ and sgn(k) = sgn(cb), then (2.25) is exponentially stable. However, a, b, c are not known and thus the problem is to find adaptively an appropriate k so that the motion of the feedback system tends to zero.

Choose the following time-varying feedback law

$$u(t) = -k(t)y(t) \tag{2.26}$$

where $k(t)$ need to be adjusted so that it gets large enough to ensure stability but also remains bounded, which can be achieved by the following adaptive law

$$\dot{k}(t) = y^2(t), \quad k(0) \in R \tag{2.27}$$

The nonlinear closed-loop system (2.24), (2.26), (2.27), i.e.,

$$\dot{x}(t) = [a - k(t)cb]x(t), \quad k(t) = c^2 \int_0^t x^2(s)dx + k(0) \tag{2.28}$$

which has at least a solution on a small interval $[0, \omega)$, and the non-trivial solution

$$x(t) = e^{\int_0^t [a-k(s)cb]ds} x(0), \quad x(0) > 0$$

is monotonically increasing as long as $a - k(t)cb > 0$. Hence $k(t) \geq t(cx(0))^2 + k(0)$ increases as well. Therefore, there exists a $t^* \geq 0$ such that $a - k(t^*)cb = 0$ and (2.28) yields $a - k(t)cb < 0$ for all $t > t^*$. Hence the solution $x(t)$ decays exponentially for $t > t^*$ and $\lim_{t\to\infty} k(t) = k_\infty \in R$ exists. This is a special example for the following concept of universal adaptive control.

Suppose Σ denotes a certain class of linear time-invariant systems of the form

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t) \end{cases} \tag{2.29}$$

where $(A, B, C, D) \in R^{n\times n} \times R^{n\times m} \times R^{m\times n} \times R^{m\times m}$ are unknown, m is usually fixed, the state dimension n is an arbitrary and unknown number. The aim is to design a single adaptive output feedback mechanism $u(t) = \mathcal{F}(y(\cdot)|_{[0,t]})$ which is a universal stabilizer for the class Σ, i.e. if $u(t) = \mathcal{F}(y(\cdot)|_{[0,t]})$ is applied to any system (2.29) belong to Σ, then the output $y(t)$ of the closed-loop system tends to zero as $t \to \infty$ and the internal variables are bounded.

The adaptive stabilizers are of the following simple form: A "tuning"parameter $k(t)$, generated by an adaptation law

$$\dot{k}(t) = g(y(t)), \quad k(0) = k_0, \tag{2.30}$$

where $g : R^m \to R$ is continuous and locally Lipschitz, is implemented into the feedback law via

$$u(t) = F(k(t))y(t), \tag{2.31}$$

where $F : R \to R^{m\times m}$ is piecewise continuous and locally Lipschitz.

Definition 7 *A controller, consisting of the adaptive law (2.30) and the feedback rule (2.31), is called a universal adaptive stabilizer for the class of systems* Σ*, if for arbitrary initial condition* $x_0 \in R^n$ *and any system (2.29) belonging to* Σ*, the closed-loop system (2.29)-(2.31) has a solution the properties*

(i) there exists a unique solution $(x(\cdot), k(\cdot)) : [0, \infty) \to R^{n+1}$,

(ii) $x(\cdot), y(\cdot), u(\cdot), k(\cdot)$ *are bounded,*

(iii) $\lim_{t\to\infty} y(t) = 0$,

(iv) $\lim_{t\to\infty} k(t) = k_\infty \in R$ *exists.*

The concept of adaptive tracking is similar. Suppose a class $\mathcal{Y}_{\text{ref}}$ of reference signals is given. It is desired that the error between the output $y(t)$ of (2.29) and the reference signal $y_{\text{ref}}(t)$

$$e(t) := y(t) - y_{\text{ref}}(t)$$

is forced, via a simple adaptive feedback mechanism, either to zero or towards a ball around zero of arbitrary small prespecified radius $\lambda > 0$. The latter is called λ-tracking. To achieve asymptotic tracking, an internal model

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & A^*\xi(t) + B^*v(t), \quad \xi(0) = \xi_0 \\ u(t) & = & C^*\xi(t) + D^*v(t) \end{array} \right. \tag{2.32}$$

where $(A^*, B^*, C^*, D^*) \in R^{n' \times n'} \times R^{n' \times m} \times R^{m \times n'} \times R^{m \times m}$, is implemented in series interconnection with a universal adaptive stabilizer. The precompensator resp. internal model (2.32) contains the dynamics of the reference signals. An internal model is not necessary if λ-tracking is desired.

Definition 8 *A controller, consisting of an adaptation law (2.30), a feedback law (2.31), and an internal model (2.32) is called a universal adaptive tracking controller for the class of systems* Σ *and reference signals* $\mathcal{Y}_{\text{ref}}$*, if for every* $y_{\text{ref}}(\cdot) \in \mathcal{Y}_{\text{ref}}$*,* $x_0 \in R^n$*,* $\xi_0 \in R^{n'}$*, and every system (2.29) belongs to* Σ*, the closed-loop system (2.29)-(2.32) satisfies*

(i) there exists a unique solution $(x(\cdot), \xi(\cdot), k(\cdot)) : [0, \infty) \to R^{n+n'+1}$,

(ii) *the variables* $x(t), y(t), u(t), \xi(t)$ *grow no faster than* $y_{\text{ref}}(t)$,

(iii) $\lim_{t\to\infty}[y(t) - y_{\text{ref}}(t)] = 0$,

(iv) $\lim_{t\to\infty} k(t) = k_\infty \in R$ *exists.*

2.4 Nussbaum Functions and Related Stability Results

2.4.1 Nussbaum Functions

Any continuous function $N(s) : R \to R$ is a function of Nussbaum type if it has the following properties

$$\lim_{s\to+\infty} \sup \int_{s_0}^{s} N(\zeta)d\zeta = +\infty, \tag{2.33}$$

$$\lim_{s\to+\infty} \inf \int_{s_0}^{s} N(\zeta)d\zeta = -\infty \tag{2.34}$$

with $s_0 \leq s$. For example, the continuous functions $\zeta^2\cos(\zeta)$, $\zeta^2\sin(\zeta)$, and $e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$ are functions of Nussbaum type [111].

Lemma 2.4.1 *The function* $N(\zeta) = e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$ *satisfies the conditions (2.33) and (2.34).* [62]

Proof: Define

$$N_I(s_1, s_2) = \int_{s_1}^{s_2} N(\zeta)d\zeta$$

with $s_1 \leq s_2$. Using integral inequality $(b-a)m_{f1} \leq \int_a^b f(x)dx \leq (b-a)m_{f2}$ with $m_{f1} = \inf_{a\leq x\leq b} f(x)$ and $m_{f2} = \sup_{a\leq x\leq b} f(x)$, and noting that $|\cos(\frac{\pi}{2}\zeta)| \leq 1$, we have

$$|N_I(s_1, s_2)| \leq (s_2 - s_1) \sup_{\zeta\in[s_1,s_2]} |N(\zeta)| = (s_2 - s_1)e^{s_2^2} \tag{2.35}$$

It is clear that $N(\zeta)$ is positive on interval $(4m-1, 4m+1)$ and negative on interval $(4m+1, 4m+3)$ with m an integer. To show that $N(\zeta)$ satisfies the conditions (2.33) and (2.34), it suffices to prove that $\lim_{m\to+\infty} N_I(s_0, 4m+1) = +\infty$ and $\lim_{m\to+\infty} N_I(s_0, 4m+3) = -\infty$.

Let us first observe the interval $[s_0, 4m-1]$ (assuming that $4m-1 \geq |s_0|$) and accordingly

$$N_I(s_0, 4m-1) = \int_{s_0}^{4m-1} N(\zeta)d\zeta$$

Applying (2.35), we have

$$|N_I(s_0, 4m-1)| \leq (4m-1-s_0)e^{(4m-1)^2} \tag{2.36}$$

Next, let us observe the interval $[4m-1, 4m+1]$. Noting that $N(\zeta) \geq 0$, $\forall \zeta \in [4m-1, 4m+1]$, we have the following inequality

$$N_I(4m-1, 4m+1) \geq \int_{4m-\epsilon_1}^{4m+\epsilon_1} N(\zeta)d\zeta$$

with $\epsilon_1 \in (0,1)$. Using the integral inequality, we have

$$N_I(4m-1, 4m+1) \geq 2\epsilon_1 \cos(\frac{\pi}{2}\epsilon_1)e^{(4m-\epsilon_1)^2} \tag{2.37}$$

It is known that if $|f_1(x)| \leq a_1$ and $f_2(x) \geq a_2$, then $f_1(x) + f_2(x) \geq a_2 - a_1$. Using this property, from (2.36) and (2.37), we have

$$\begin{aligned} N_I(s_0, 4m+1) &= N_I(s_0, 4m-1) + N_I(4m-1, 4m+1) \\ &\geq e^{(4m-1)^2}\Big[2\epsilon_1 \cos(\frac{\pi}{2}\epsilon_1)e^{[2(4m-1)(1-\epsilon_1)+(1-\epsilon_1)^2]} \\ &\quad -(4m-1-s_0)\Big] \end{aligned} \tag{2.38}$$

Note that the following property holds for $b_0, b_1, b_2 > 0$

$$\lim_{x\to+\infty} b_0 e^{x^2}(e^{b_1 x} - b_2 x + b_3) = +\infty, \quad \forall x \in R \tag{2.39}$$

Applying (2.39) by noting $(1-\epsilon_1) \in (0,1)$, from (2.38), we have

$$\lim_{m\to+\infty} N_I(s_0, 4m+1) = +\infty$$

In what follows, we would like to show that $\lim_{m\to+\infty} N_I(s_0, 4m+3) = -\infty$. To this end, let us first observe the interval $[s_0, 4m+1]$. Similarly, applying (2.35), we obtain

$$|N_I(s_0, 4m+1)| \leq (4m+1-s_0)e^{(4m+1)^2} \tag{2.40}$$

Then, let us observe the next immediate interval $[4m+1, 4m+3]$. Noting that $N(\zeta) \le 0$, $\forall \zeta \in [4m+1, 4m+3]$, we have the following inequality

$$\begin{aligned} N_I(4m+1, 4m+3) &\le \int_{4m+2-\epsilon_2}^{4m+2+\epsilon_2} N(\zeta) d\zeta \\ &\le -2\epsilon_2 \cos(\frac{\pi}{2}\epsilon_2) e^{(4m+2-\epsilon_2)^2} \end{aligned} \tag{2.41}$$

with $\epsilon_2 \in (0, 1)$.

It is also known that if $|f_1(x)| \le a_1$ and $f_2(x) \le a_2$, then $f_1(x) + f_2(x) \le a_2 + a_1$. Accordingly, from (2.40) and (2.41), we have

$$N_I(s_0, 4m+3) \le -e^{(4m+1)^2} \Big[2\epsilon_2 \cos(\frac{\pi}{2}\epsilon_2) e^{[2(4m+1)(1-\epsilon_2)+(1-\epsilon_2)^2]} - (4m+1-s_0) \Big] \tag{2.42}$$

Applying (2.39) by noting that $(1-\epsilon_2) \in (0, 1)$, from (2.42), we have

$$\lim_{m \to +\infty} N_I(s_0, 4m+3) = -\infty$$

which ends the proof. $\diamondsuit$

Lemma 2.4.2 *The function $N(\zeta) = \zeta^2 \cos(\zeta)$ satisfies the conditions (2.33) and (2.34).*

Proof: Consider the following integration

$$\int_{s_0}^{s} N(\zeta) d\zeta = \int_{s_0}^{s} \zeta^2 \cos(\zeta) d\zeta$$

Integrating by parts, we have

$$\begin{aligned} \int_{s_0}^{s} N(\zeta) d\zeta &= \zeta^2 \sin(\zeta)\Big|_{s_0}^{s} + 2\zeta \cos(\zeta)\Big|_{s_0}^{s} - 2\sin(\zeta)\Big|_{s_0}^{s} \\ &= s^2 \sin(s) + 2s\cos(s) - 2\sin(s) - s_0^2 \sin(s_0) \\ &\quad -2s_0 \cos(s_0) + 2\sin(s_0) \end{aligned} \tag{2.43}$$

Taking the limit as $s \to +\infty$, from (2.43), we have

$$\begin{aligned} \lim_{s \to +\infty} \int_{s_0}^{s} N(\zeta) d\zeta &= \lim_{s \to +\infty} s^2 \Big\{ \sin(s) + \frac{2\cos(s)}{s} + \frac{1}{s^2} \Big[-2\sin(s) - s_0^2 \sin(s_0) \\ &\quad -2s_0 \cos(s_0) + 2\sin(s_0) \Big] \Big\} \\ &= \lim_{s \to +\infty} [s^2 \sin(s)] \end{aligned}$$

from which it is known that as $s \to +\infty$, $\sin(s)$ changes it sign an infinite number of times, further, $\lim_{s\to+\infty} \sup[s^2 \sin(s)] = +\infty$, and $\lim_{s\to+\infty} \inf[s^2 \sin(s)] = -\infty$. Therefore, we can conclude that $N(\zeta) = \zeta^2 \cos(\zeta)$ satisfies the conditions (2.33) and (2.34). $\diamondsuit$

Functions $\sin(x)$ or $\cos(x)$ are referred as "transcendental functions", whose sign changes an infinite number of times as their arguments x increases in magnitude and tends to infinity. [4], p.363) Transcendental functions play an essential role in constructing Nussbaum functions, whose choices are not unique. The conditions (2.33) and (2.34) are the key features of the Nussbaum functions, beside which, some choices of Nussbaum functions, e.g., $e^{\zeta^2} \cos(\frac{\pi}{2}\zeta)$, $\zeta^2 \cos(\zeta)$, etc., also satisfy the following conditions

$$\lim_{s\to+\infty} \sup \frac{1}{s} \int_{s_0}^{s} N(\zeta) d\zeta = +\infty \tag{2.44}$$

$$\lim_{s\to+\infty} \inf \frac{1}{s} \int_{s_0}^{s} N(\zeta) d\zeta = -\infty \tag{2.45}$$

Corollary 1 *The function $N(\zeta) = e^{\zeta^2} \cos(\frac{\pi}{2}\zeta)$ satisfies the conditions (2.44) and (2.45).*

Outline of the proof:

Following the same procedure in proof of Lemma 2.4.1, to prove (2.44) and (2.45), it suffices to prove that $\lim_{m\to+\infty} \frac{1}{4m+1} N_I(s_0, 4m+1) = +\infty$ and $\lim_{m\to+\infty} \frac{1}{4m+3} N_I(s_0, 4m+3) = -\infty$.

The following property holds for $x \in R$, $x + a_0 \neq 0$, $b_0, b_1, b_2 > 0$

$$\lim_{x\to+\infty} \frac{b_0 e^{x^2}(e^{b_1 x} - b_2 x + b_3)}{x + a_0} = +\infty \tag{2.46}$$

which can be easily proven by applying the L'Hopital's Rule [112] as

$$\lim_{x\to+\infty} \frac{b_0 e^{x^2}(e^{b_1 x} - b_2 x + b_3)}{x + a_0} = \lim_{x\to+\infty} \frac{\frac{\partial}{\partial x}\left[b_0 e^{x^2}(e^{b_1 x} - b_2 x + b_3)\right]}{\frac{\partial}{\partial x}(x + a_0)} = +\infty$$

Using the property (2.46), from (2.38), we have

$$\lim_{m\to+\infty} \frac{1}{4m+1} N_I(s_0, 4m+1) = +\infty$$

and from (2.42), we have

$$\lim_{m\to+\infty}\frac{1}{4m+3}N_I(s_0,4m+3)=-\infty$$

$\diamondsuit$

Corollary 2 *The function* $N(\zeta) = \zeta^2\cos(\zeta)$ *satisfies the conditions (2.44) and (2.45).*

Proof: It directly follows from the equation after (2.43) and is omitted. $\diamondsuit$

Definition 9 *Suppose* $N(\zeta)$ *is a Nussbaum function which satisfies (2.44) and (2.45). A Nussbaum function is called scaling-invariant if, for arbitrary* $\alpha, \beta > 0$,

$$\tilde{N}(\zeta) := \begin{cases} \alpha N(\zeta) & \text{if} \quad N(\zeta)\geq 0 \\ \beta N(\zeta) & \text{if} \quad N(\zeta)<0 \end{cases}$$

is a Nussbaum function as well.

Example 1 *[111] The following functions are Nussbaum function:*

$$\begin{aligned}
N_1(\zeta) &= \zeta\cos\sqrt{|\zeta|}, && \zeta\in R\\
N_2(\zeta) &= \ln\zeta\cos\sqrt{\ln\zeta}, && \zeta>1\\
N_3(\zeta) &= \begin{cases} \zeta & \text{if} \quad n^2\leq|\zeta|<(n+1)^2, \quad n \ \textit{even} \\ -\zeta & \text{if} \quad n^2\leq|\zeta|<(n+1)^2, \quad n \ \textit{odd} \end{cases}, && \zeta\in R\\
N_4(\zeta) &= \begin{cases} \zeta & \text{if} \quad 0\leq|\zeta|<\tau_0 \\ \zeta & \text{if} \quad \tau_n\leq|\zeta|<\tau_{n+1}, \quad n \ \textit{even} \\ -\zeta & \text{if} \quad \tau_n\leq|\zeta|<\tau_{n+1}, \quad n \ \textit{odd} \end{cases}\\
& \textit{with } \tau_0>1,\ \tau_{n+1}:=\tau_n^2,\ \zeta\in R
\end{aligned} \tag{2.47}$$

Of course, the cosine in the above examples can be replace by sine, and similar modifications.

Logemann and Owens (1988) have proved that $N(\zeta) = e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$ is scaling-invariant. This property is important if the nominal system is subjected to certain nonlinear perturbations and/or for some universal controllers of multivariable systems.

It is easy to see that $N_1(\zeta), N_3(\zeta), N_4(\zeta)$ are in fact Nussbaum functions, whereas to prove the properties (2.44) and (2.45) for $N_2(\zeta)$ is more subtle and a proof is given below. The function $N_2(\zeta)$ has the property that the periods where the sign is kept constant compared to the increase of the gain is larger than for $N_1(\zeta)$, this will become important for relative degree two systems. Note also that $\lim_{\zeta\to\infty} \frac{d}{d\zeta} N_3(\zeta) = 0$.

Lemma 2.4.3 *[111] The function*

$$N(\zeta) : [\zeta_0, \infty] \to R, \zeta \mapsto \ln\zeta \cos\sqrt{\ln\zeta}$$

is a Nussbaum function for every $\zeta_0 > 1$.

Proof: See [111].

2.4.2 Stability Results

In this section, the Nussbaum functions are chosen to satisfy both the conditions (2.33), (2.34) and (2.44) and (2.45).

Lemma 2.4.4 *[70] Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0$ and $\zeta(t)$ monotone increasing, $\forall t \in [0, t_f)$, and $N(\zeta)$ be smooth Nussbaum function. If the following inequality holds*

$$V(t) \leq c_0 + \int_0^t (g_0 N(\zeta) + 1)\dot{\zeta} d\tau, \quad \forall t \in [0, t_f) \tag{2.48}$$

where g_0 is a nonzero constant and c_0 represents some suitable constant related to the control parameters, then $V(t)$, $\zeta(t)$ and $\int_0^t (g_0 N(\zeta) + 1)\dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

Proof: Seeking a contradiction, suppose that monotone increasing function $\zeta(t)$ is unbounded, i.e., $\zeta(t) \to +\infty$ as $t \to t_f$. Dividing (2.48) by $\zeta(t)$ yields

$$0 \leq \frac{V(t)}{\zeta(t)} \leq \frac{c_0}{\zeta(t)} + \frac{g_0}{\zeta(t)} \int_{\zeta(0)}^{\zeta(t)} N(\zeta(\tau)) d\zeta(\tau) + \frac{\zeta(t) - \zeta(0)}{\zeta(t)} \tag{2.49}$$

Taking the limit as $t \to t_f$, hence $\zeta(t) \to +\infty$, from (2.49), we have

$$0 \leq \lim_{t\to t_f} \frac{V(t)}{\zeta(t)} \leq \lim_{\zeta(t)\to+\infty} \frac{g_0}{\zeta(t)} \int_{\zeta(0)}^{\zeta(t)} N(\zeta(\tau)) d\zeta(\tau) + 1$$

which, if $g_0 > 0$, contradicts (2.45) or, if $g_0 < 0$, contradicts (2.44). Therefore, $\zeta(\cdot)$ is bounded. Hence, $\int_0^t g_0 N(\zeta)\dot{\zeta}d\tau$ is also bounded. From (2.48), it follows that $V(\cdot)$ is bounded. ♢

Lemma 2.4.5 *[83] Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0$, $\forall t \in [0, t_f)$, and $N(\cdot)$ be an even smooth Nussbaum-type function. If the following inequality holds*

$$V(t) \leq c_0 + \int_0^t (g_0 N(\zeta) + 1)\dot{\zeta}d\tau, \quad \forall t \in [0, t_f) \tag{2.50}$$

where g_0 is a nonzero constant and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t (g_0 N(\zeta) + 1)\dot{\zeta}d\tau$ must be bounded on $[0, t_f)$.

Proof: We first show that $\zeta(t)$ is bounded on $[0, t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: (i) $\zeta(t)$ has no upper bound, and (ii) $\zeta(t)$ has no lower bound, $\forall t \in [0, t_f)$.

Case (i): $\zeta(t)$ has no upper bound. In this case, there must exist a monotone increasing sequence $\{t_i\}$, $i = 1, 2, \cdots$, such that $\{\omega_i = \zeta(t_i)\}$ is monotone increasing with $\omega_1 = \zeta(t_1) > 0$, $\lim_{i\to+\infty} t_i = t_f$, and $\lim_{i\to+\infty} \omega_i = +\infty$.

Dividing (2.50) by $\omega_i = \zeta(t_i) > 0$ yields

$$\begin{aligned} 0 \leq \frac{V(t_i)}{\zeta(t_i)} &\leq \frac{c_0}{\zeta(t_i)} + \frac{1}{\zeta(t_i)} \int_{\zeta(0)}^{\zeta(t_i)} (g_0 N(\zeta(\tau)) + 1) d\zeta(\tau) \\ &= \frac{c_0}{\omega_i} + \frac{g_0}{\omega_i} \int_{\zeta(0)}^{\omega_i} N(\zeta(\tau) d\zeta(\tau) + \left(1 - \frac{\zeta(0)}{\omega_i}\right) \end{aligned} \tag{2.51}$$

On taking the limit as $i \to +\infty$, hence $t_i \to t_f$, $\omega_i \to +\infty$, from (2.51), we have

$$0 \leq \lim_{i\to+\infty} \frac{V(t_i)}{\zeta(t_i)} \leq 1 + \lim_{i\to+\infty} \frac{g_0}{\omega_i} \int_{\zeta(0)}^{\omega_i} N(\zeta(\tau) d\zeta(\tau) \tag{2.52}$$

which, if $g_0 > 0$, contradicts (2.45) or, if $g_0 < 0$, contradicts (2.44). Therefore, $\zeta(t)$ is upper bounded on $[0, t_f)$.

Case (ii): $\zeta(t)$ has no lower bound. There must exist a monotone increasing sequence $\{\underline{t}_i\}$, $i = 1, 2, \cdots$, such that $\{\underline{\omega}_i = -\zeta(\underline{t}_i)\}$ is monotone increasing with $\underline{\omega}_1 > 0$, $\lim_{i\to+\infty} \underline{t}_i = t_f$, and $\lim_{i\to+\infty} \underline{\omega}_i = +\infty$.

Dividing (2.50) by $\underline{\omega}_i = -\zeta(\underline{t}_i) > 0$ yields

$$0 \le \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} \le \frac{c_0}{-\zeta(\underline{t}_i)} - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} (g_0 N(\zeta(\tau)) + 1) d[-\zeta(\tau)] \quad (2.53)$$

Noting that $N(\cdot)$ is an even function, i.e., $N(\zeta) = N(-\zeta)$, and letting $\chi(t) = -\zeta(t)$, (2.53) becomes

$$\begin{aligned} 0 \le \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} &\le \frac{c_0}{-\zeta(\underline{t}_i)} - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} (g_0 N(\chi(\tau)) + 1) d\chi(\tau) \\ &= \frac{c_0}{\underline{\omega}_i} - \frac{g_0}{\underline{\omega}_i} \int_{\zeta(0)}^{\underline{\omega}_i} N(\chi(\tau)) d\chi(\tau) - \left(1 - \frac{\zeta(0)}{\underline{\omega}_i}\right) \end{aligned} \quad (2.54)$$

On taking the limit as $i \to +\infty$, hence $t_i \to t_f$, $\underline{\omega}_i \to +\infty$, from (2.54), we have

$$0 \le \lim_{i \to +\infty} \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} \le -1 - \lim_{i \to +\infty} \frac{g_0}{\underline{\omega}_i} \int_{\zeta(0)}^{\underline{\omega}_i} N(\chi(\tau)) d\chi(\tau)$$

which, if $g_0 > 0$, contradicts (2.44) or, if $g_0 < 0$, contradicts (2.45). Therefore, $\zeta(t)$ is lower bounded on $[0, t_f)$.

We thus conclude the boundedness of $\zeta(t)$ on $[0, t_f)$. As an immediate result, $V(t)$ and $\int_0^t g_0 N(\zeta)\dot{\zeta} d\tau$ are also bounded on $[0, t_f)$. $\diamondsuit$

Lemma 2.4.6 *Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \ge 0$, $\forall t \in [0, t_f)$, and $N(\zeta)$ be an even smooth Nussbaum-type function. If the following inequality holds:*

$$V(t) \le c_0 + e^{-c_1 t} \int_0^t g_0(x(\tau)) N(\zeta) \dot{\zeta} e^{c_1 \tau} d\tau + e^{-c_1 t} \int_0^t \dot{\zeta} e^{c_1 \tau} d\tau, \quad \forall t \in [0, t_f) \quad (2.55)$$

where constant $c_1 > 0$, $g_0(x(t))$ is a time-varying parameter which takes values in the unknown closed intervals $I := [l^-, l^+]$ with $0 \notin I$, and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0(x(\tau)) N(\zeta) \dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

Proof: See Appendix 7.2 or [113][114].

Remark 2.4.1 *Note that $N(\cdot)$ is an even function. In fact, the stability results in Lemma 2.4.5 and 2.4.6 still holds if $N(\cdot)$ is an odd function, which can be easily proven by following the same procedure.*

Lemma 2.4.7 *Let $V(\cdot)$ and $\zeta(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0$, $\forall t \in [0, t_f)$, and smooth Nussbaum-type function $N(\zeta) = \zeta^2 \cos(\zeta)$. If the following inequality holds:*

$$V(t) \leq c_0 + e^{-c_1 t} \int_0^t g_0 N(\zeta)\dot{\zeta} e^{c_1 \tau} d\tau + e^{-c_1 t} \int_0^t \dot{\zeta} e^{c_1 \tau} d\tau, \quad \forall t \in [0, t_f) \tag{2.56}$$

where constant $c_1 > 0$, g_0 is a nonzero constant, and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0 N(\zeta)\dot{\zeta} d\tau$ must be bounded on $[0, t_f)$.

Proof: See Appendix 7.2.

2.4.3 An Illustration Example

For illustration purpose, let us consider the first-order system

$$\dot{x}_1 = g_1 u + \theta_1^T \psi_1(x_1) + \Delta_1(t, x_1)$$

where g_1 is a unknown nonzero constant, θ_1^T is unknown constant vector, $\psi(x_1)$ is known smooth function, and the unknown disturbance satisfies: $|\Delta(t, x_1)| \leq p_1 \phi_1(x_1)$ with p_1 unknown constant and $\phi(x_1)$ known smooth function.

Consider the Lyapunov function candidate

$$V_1(t) = \frac{1}{2} x_1^2 + \frac{1}{2} (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} (\hat{\theta}_1 - \theta_1) + \frac{1}{2} \gamma_{p1}^{-1} (\hat{p}_1 - p_1)^2$$

Its time derivative is

$$\begin{aligned}
\dot{V}_1 &= x_1 \dot{x}_1 + (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} \dot{\hat{\theta}}_1 + \gamma_{p1}^{-1} (\hat{p}_1 - p_1) \dot{\hat{p}}_1 \\
&= x_1 [g_1 u + \theta_1^T \psi_1(x_1) + \Delta_1(t, x_1)] + (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} \dot{\hat{\theta}}_1 + \gamma_{p1}^{-1} (\hat{p}_1 - p_1) \dot{\hat{p}}_1 \\
&\leq x_1 [g_1 u + \theta_1^T \psi_1(x_1)] + p_1 \phi_1(x_1) |x_1| \\
&\quad + (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} \dot{\hat{\theta}}_1 + \gamma_{p1}^{-1} (\hat{p}_1 - p_1) \dot{\hat{p}}_1
\end{aligned} \tag{2.57}$$

The control law is chosen as

$$u = N(\zeta_1) \Big[k_1 x_1 + \hat{\theta}_1^T \psi_1 + \hat{p}_1 \phi_1 \tanh(\frac{x_1 \phi_1}{\epsilon_1}) \Big] \tag{2.58}$$

$$\dot{\zeta}_1 = k_1 x_1^2 + \hat{\theta}_1^T \psi_1 x_1 + \hat{p}_1 \phi_1 x_1 \tanh(\frac{x_1 \phi_1}{\epsilon_1}) \tag{2.59}$$

Substituting (2.58) and (2.59) into (2.57)

$$\begin{aligned}\dot{V}_1 \ &\le \ g_1 N(\zeta_1)\dot{\zeta}_1 + \theta_1^T \psi_1(x_1) x_1 + p_1 \phi_1(x_1)|x_1| \\ &\quad + (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} \dot{\hat{\theta}}_1 + \gamma_{p1}^{-1}(\hat{p}_1 - p_1)\dot{\hat{p}}_1 \end{aligned} \tag{2.60}$$

Adding and subtracting $\dot{\zeta}_1$ on the right hand side of (2.60), we have

$$\begin{aligned}\dot{V}_1 \ &\le \ -k_1 x_1^2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \\ &\quad -(\hat{\theta}_1 - \theta_1)^T \psi_1 x_1 - (\hat{p}_1 - p_1)\phi_1 x_1 \tanh(\frac{x_1\phi_1}{\epsilon_1}) \\ &\quad + p_1\Big[\phi_1|x_1| - \phi_1 x_1 \tanh(\frac{x_1\phi_1}{\epsilon_1})\Big] \\ &\quad + (\hat{\theta}_1 - \theta_1)^T \Gamma_{\theta 1}^{-1} \dot{\hat{\theta}}_1 + \gamma_{p1}^{-1}(\hat{p}_1 - p_1)\dot{\hat{p}}_1 \end{aligned} \tag{2.61}$$

Choosing the parameter adaptation laws as

$$\dot{\hat{\theta}}_1 = \Gamma_{\theta 1}\Big[\psi_1 x_1 - \sigma_{\theta 1}(\hat{\theta}_1 - \theta_1^0)\Big] \tag{2.62}$$

$$\dot{\hat{p}}_1 = \gamma_{p1}\Big[\phi_1 x_1 \tanh(\frac{x_1\phi_1}{\epsilon_1}) - \sigma_{p1}(\hat{p}_1 - p_1^0)\Big] \tag{2.63}$$

where $\sigma_{\theta 1}$, σ_{p1}, θ_1^0, p_1^0 are constants.

Substituting (2.62) and (2.63) into (2.61) and noting the following inequalities

$$\begin{aligned} &|x| - x\tanh(\frac{x}{\epsilon}) \le 0.2785\epsilon, \quad \epsilon > 0 \\ &-\sigma_{\theta 1}(\hat{\theta}_1 - \theta_1)^T(\hat{\theta}_1 - \theta_1^0) \le -\frac{1}{2}\sigma_{\theta 1}\|\hat{\theta}_1 - \theta_1\|^2 + \frac{1}{2}\sigma_{\theta 1}\|\theta_1 - \theta_1^0\|^2 \\ &-\sigma_{p1}(\hat{p}_1 - p_1)^T(\hat{p}_1 - p_1^0) \le -\frac{1}{2}\sigma_{p1}(\hat{p}_1 - p_1)^2 + \frac{1}{2}\sigma_{p1}(p_1 - p_1^0)^2 \end{aligned}$$

we have

$$\begin{aligned}\dot{V}_1 \ &\le \ -k_1 x_1^2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 - \frac{1}{2}\sigma_{\theta 1}\|\hat{\theta}_1 - \theta_1\|^2 - \frac{1}{2}\sigma_{p1}(\hat{p}_1 - p_1)^2 \\ &\quad + 0.2785 p_1 \epsilon_1 + \frac{1}{2}\sigma_{\theta 1}\|\theta_1 - \theta_1^0\|^2 + \frac{1}{2}\sigma_{p1}(p_1 - p_1^0)^2 \\ &\le \ -c_1 V_1 + c_2 + g_1 N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \end{aligned} \tag{2.64}$$

where

$$\begin{aligned} c_1 &= \min\left\{2k_1, \frac{\sigma_{\theta 1}}{\lambda_{\min}(\Gamma_{\theta 1}^{-1})}, \sigma_{p1}\gamma_{p1}\right\} \\ c_2 &= 0.2785 p_1 \epsilon_1 + \frac{1}{2}\sigma_{\theta 1}\|\theta_1 - \theta_1^0\|^2 + \frac{1}{2}\sigma_{p1}(p_1 - p_1^0)^2 \end{aligned}$$

Multiplying (2.64) by $e^{c_1 t}$, we obtain

$$\frac{d}{dt}(V_1 e^{c_1 t}) \le c_2 e^{c_1 t} + g_1 N(\zeta_1)\dot{\zeta}_1 e^{c_1 t} + \dot{\zeta}_1 e^{c_1 t} \tag{2.65}$$

Integrating (2.65) over $[0, t]$ yields

$$0 \le V_1(t) \le \frac{c_2}{c_1} + V_1(0) + e^{-c_1 t}\int_0^t g_1 N(\zeta_1)\dot{\zeta}_1 e^{c_1 \tau} d\tau + e^{-c_1 t}\int_0^t \dot{\zeta}_1 e^{c_1 \tau} d\tau$$

For simulation purpose, we consider the following first-order uncertain nonlinear system

$$\dot{x}_1 = u + 0.1x_1^2 + 0.6e^{x_1}\sin^3 t$$

Accordingly, $g_1 = 1$, $\theta_1 = 0.1$, $\psi(x_1) = x_1^2$, and $\Delta_1 = 0.6e^{x_1}\sin^3 t$, i.e., $p_1 = 0.6$, $\phi_1(x_1) = e^{x_1}$.

The simulation results are shown in the following figures. When $N(\zeta) = e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$, the figures are plotted by solid lines. When $N(\zeta) = \zeta^2\cos(\frac{\pi}{2}\zeta)$, the figures are plotted by dashed lines. The closed-loop signals x_1, u, ζ_1, $N_1(\zeta_1)$, and the norms of the parameter estimations are plotted in Fig. 2.2-2.6.

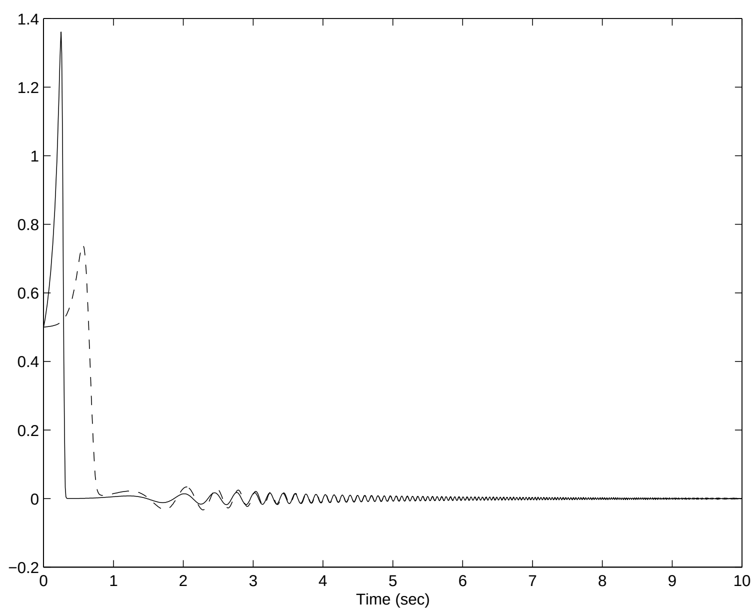

Figure 2.2: State $x_1(t)$.

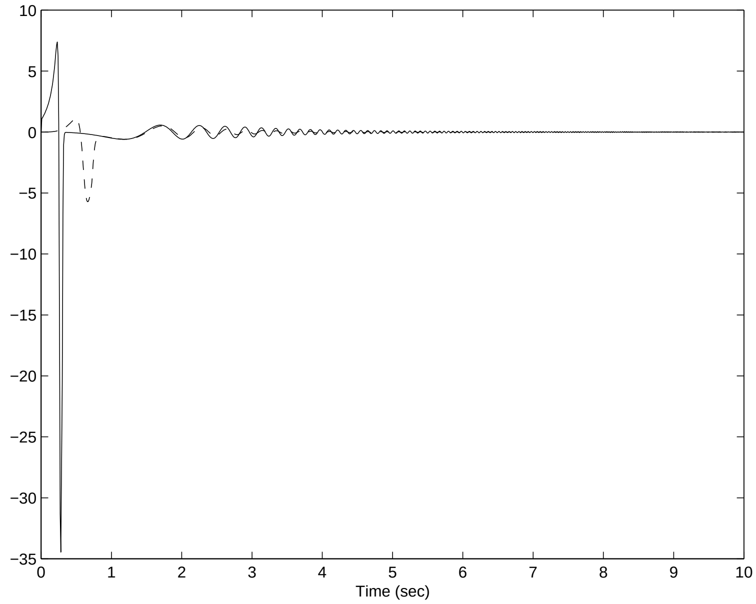

Figure 2.3: Control input $u(t)$.

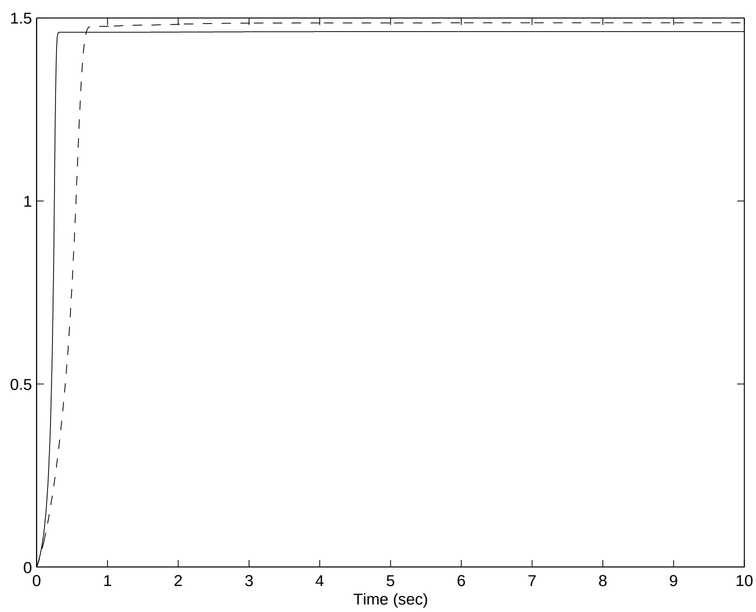

Figure 2.4: Variable $\zeta_1(t)$.

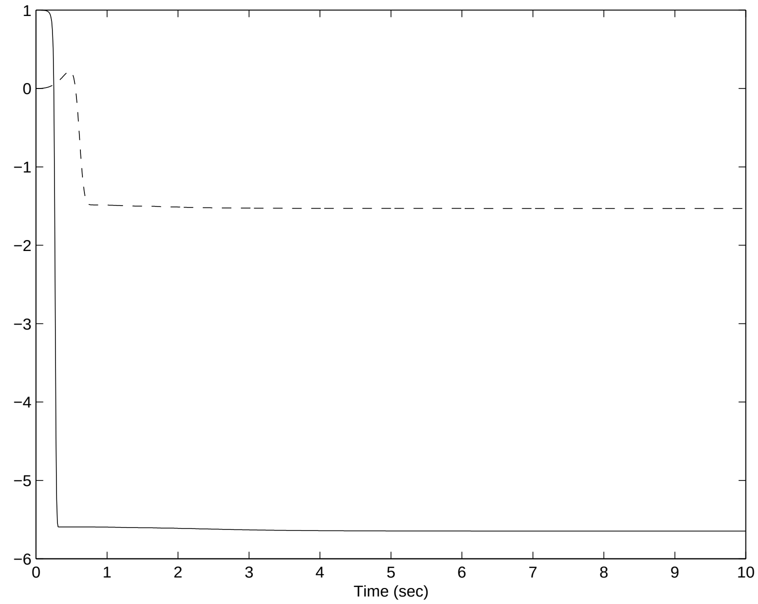

Figure 2.5: Nussbaum function $N_1(\zeta_1)$.

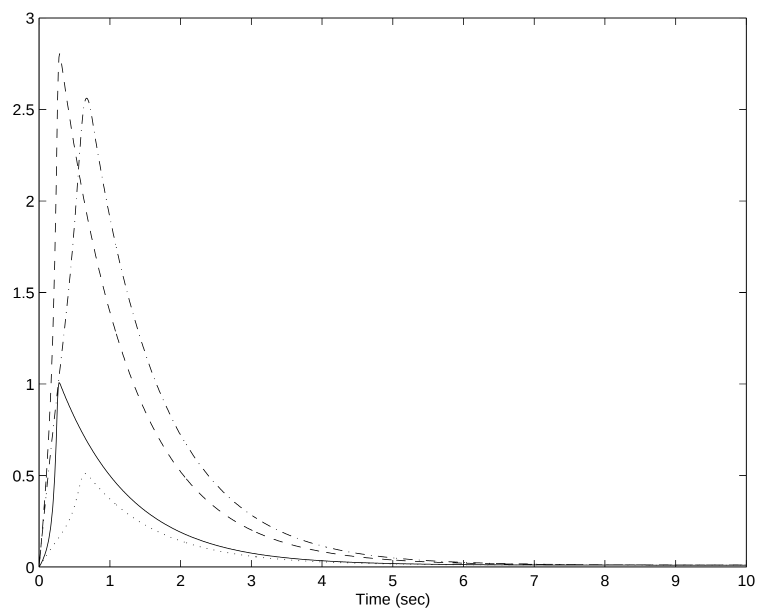

Figure 2.6: Norm of parameter estimates $\hat{\theta}_1$("−") and $\hat{p}_1$("- -").

Chapter 3

Decoupled Backstepping Design

3.1 Introduction

The traditional backstepping design is composed of n steps of iterative design for nth-order systems. Based on a coordinate transformation, a virtual control law is designed in each intermediate step for the corresponding subsystem, while the actual control $u(t)$ is designed in the final step. Specifically, there will be a coupling term $z_i z_{i+1}$ based on the new z-coordinate remaining in the Lyapunov function of Step i, which shall be and only can be dealt with/cancelled in Step $i+1$. Therefore, the corresponding Lyapunov function $V_{i+1}(t)$ of Step $i+1$ shall be constructed to include $V_i(t)$ – the Lyapunov function of Step i. Apparently, $V_i(t)$ must contain the summation of all the previous ones from $V_1(t)$ to $V_{i-1}(t)$. Usually, the boundedness of all the signals in the closed-loop can be guaranteed and the states in z-coordinate can be confined in a compact residual set, which is given for the norm of vector $z(t)$ rather than each individual $z_i(t)$ for $i = 1, \cdots, n$. For convenience of differentiation, it is referred to as the cancellation backstepping design method.

Another class of backstepping design appeared in [83][115], where the stability result was proven iteratively by showing the stability of individual state z_i in z-coordinate of each subsystem backwards through the analysis of the integral of, rather than the pure negativeness of, the differentiation of the Lyapunov function candidate. The coupling term $z_i z_{i+1}$ in each step is decoupled by elegantly using the

Young's inequality rather than leaving to it to be cancelled in the next step. Thus, it is referred to as the decoupled backstepping method. The design method was originally used to handle the completely unknown virtual control coefficients and high-frequency gain, through the aid of Nussbaum functions and time integration, where the standard backstepping design could not solve the problem. In addition, the two design methods are also different in the following aspects:

(i) the Lyapunov function $V_i(t)$ of Step i is constructed independently from $V_{i-1}(t)$ of Step $i-1$ as the coupling term $z_{i-1}z_i$ of Step $i-1$ is decoupled using Young's inequality and the exact cancellation of this term in Step i is no longer necessary;

(ii) the virtual control α_i is only designed to stabilize the ith subsystems rather than the subsystems from the 1st to the ith in z coordinate;

(iii) the stability result of Step $i-1$ is obtained by seeking the boundedness of z_i rather than cancelling the coupling term $z_{i-1}z_i$ so that the residual set of each state in z coordinate can be determined individually.

(iv) the cancellation backstepping design utilize the state interconnections, while the decoupled backstepping design tries to decouple the interconnections.

The decoupled backstepping design offers another control system design tool in handling a large class of nonlinear systems. The main contributions of the Chapter are

(i) the explicit introduction of the decoupled backstepping as a general tool for control system design, and

(ii) control system design for two classes of strict-feedback systems to show the concept clearly.

It is proved that the proposed systematic design method is able to guarantee global uniformly ultimately boundedness of all the signals in the closed-loop system in Section 3.2 and global uniformly ultimately boundedness of all the signals in the closed-loop system in Section 3.3, and the tracking error is proven to converge to a small neighborhood of the origin. In addition, the residual set of each state based on new coordinate in the closed-loop can be determined respectively. Simulation results are provided to show the effectiveness of the proposed approach.

The rest of the Chapter is organized as follows. The decoupled adaptive backstepping design and the decoupled NN backstepping design are presented in Section 3.2 and Section 3.3 respectively, with detailed problem formulation, controller design, simulation studies and conclusion in each embedded subsections.

3.2 Adaptive Decoupled Backstepping Design

3.2.1 Problem Formulation and Preliminaries

Consider a class of single-input-single-output (SISO) nonlinear systems

$$\begin{aligned} \dot{x}_i &= g_i x_{i+1} + \theta_i^T F_i(\bar{x}_i) + f_i(\bar{x}_i) + \Delta_i(t,x), \quad 1 \le i \le n-1 \\ \dot{x}_n &= g_n u + \theta_n^T F_n(x) + f_n(x) + \Delta_n(t,x), \\ y &= x_1 \end{aligned} \tag{3.1}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T \in R^i$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output respectively, g_i are unknown constants, $\theta_i \in R^{n_i}$ are unknown constant vectors, $F_i(\cdot) \in R^{n_i}$ are known smooth function vectors, $f_i(\cdot)$ are known smooth functions, and Δ_i are unknown Lipschitz continuous functions, $i = 1, \cdots, n$. The control objective is to design an adaptive controller for system (3.1) such that the output $y(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are globally uniformly ultimately bounded (GUUB). Define the desired trajectory vector $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $i = 1, \cdots, n-1$, which is the combination of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for unknown constants, unknown disturbances and reference signals.

Assumption 3.2.1 *The signs of g_i are known and assumed to be positive without loss of generality.*

Assumption 3.2.2 *There exist unknown positive constants* p_i*,* $1 \leq i \leq n$*, such that* $\forall (t, x) \in R_+ \times R^n$*,* $|\Delta_i(t, x)| \leq p_i \phi_i(\bar{x}_i)$*, where* ϕ_i *is a known nonnegative smooth function.*

Remark 3.2.1 *Assumption 3.2.2 implies that in this thesis we only consider such class of uncertainties* Δ_i *that have a triangular bound in terms of* x *for the ease of controller design. Similar assumptions to Assumption 3.2.2 have been used in [86, 116, 21]. As* p_i *is not unique, we make a similar assumption that* p_i *is the smallest value among all the values satisfying the triangular condition. In this thesis, we do not need the exact expression of* $\Delta_i(t, x) = \phi_i(\bar{x}_i) p_i$ *as investigated in [19], where it showed that the existence of disturbance terms* $\phi_i(\bar{x}_i) p_i$ *might drive the system states escape to infinity in finite time, even in case that* Δ_i *is an exponentially decaying disturbance.*

Assumption 3.2.3 *The desired trajectory vectors* $\bar{x}_{di} \in R^i$*,* $i = 1, \cdots, n-1$ *are continuous, bounded and available.*

The following lemma is used in the controller in solving the problem of chattering [86, 116].

Lemma 3.2.1 *The following inequality holds for any* $\epsilon > 0$ *and* $\eta \in R$

$$0 \leq |\eta| - \eta \tanh\left(\frac{\eta}{\epsilon}\right) \leq k\epsilon$$

where k *is a constant that satisfies* $k = e^{-(k+1)}$*, i.e.,* $k = 0.2785$*.*

Lemma 3.2.2 *Let* $V(\cdot)$ *and* $f(\cdot)$ *be continuous functions defined on* $[0, \infty)$ *with* $V(t) \geq 0$*,* $\forall t \in [0, \infty)$ *and* $V(0)$ *being bounded. If the following inequality holds*

$$\dot{V}(t) \leq -c_1 V(t) + c_2 + f(t), \quad \text{constants } c_1, c_2 > 0 \tag{3.2}$$

and $f(t)$ *is bounded, then* $V(t)$ *is also bounded.*

Proof: Multiplying (3.2) by $e^{c_1 t}$, it becomes

$$\frac{d}{dt}\Big(V(t)e^{c_1 t}\Big) \;\leq\; c_2 e^{c_1 t} + e^{c_1 t} f(t) \tag{3.3}$$

Integrating (3.3) over $[0, t]$, we have

$$V(t) \leq [V(0) - \frac{c_2}{c_1}]e^{-c_1 t} + \frac{c_2}{c_1} + e^{-c_1 t}\int_0^t e^{c_1 \tau} f(\tau) d\tau$$

From the following inequality

$$\begin{aligned} e^{-c_1 t}\int_0^t e^{c_1 \tau} f(\tau) d\tau &\leq e^{-c_1 t} \sup_{\tau \in [0,t]} [f(\tau)] \int_0^t e^{c_1 \tau} d\tau \\ & \Big(= \frac{1}{c_1} \sup_{\tau \in [0,t]} [f(\tau)](1 - e^{-c_1 t}) \Big) \\ &\leq \frac{1}{c_1} \sup_{\tau \in [0,t]} [f(\tau)] \end{aligned}$$

we have

$$V(t) \leq [V(0) - \frac{c_2}{c_1}]e^{-c_1 t} + \frac{c_2}{c_1} + \frac{1}{c_1} \sup_{\tau \in [0,t]} [f(\tau)]$$

Therefore, if $f(t)$ is bounded, i.e., $\sup_{\tau \in [0,t]}[f(\tau)]$ is finite and $V(0)$ is bounded, we can conclude that $V(t)$ is bounded. ♢

3.2.2 Adaptive Controller Design

In this section, the adaptive Lyapunov controller design is proposed for system (3.1) and the stability results of the closed-loop system are presented.

The design procedure contains n steps. At step i, an intermediate control function $\alpha_i(t)$ shall be developed using an appropriate Lyapunov function $V_i(t)$, $i = 1, \cdots, n-1$. The control law $u(t)$ is designed in the last step to stabilize the whole closed-loop system using the Lyapunov function $V_n(t)$. Different from the backstepping design investigated intensively in the literature, where the Lyapunov function of i step, i.e., $V_i(t)$ is partially composed of the Lyapunov function of the previous step, i.e., $V_{i-1}(t)$ for $i = 2, \cdots, n$. In this paper, the Lyapunov function of each step is decoupled in the sense that it does not contain the Lyapunov function of the previous step.

The design of both the control laws and the adaptive laws are based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \cdots, n$.

Step 1: Let us firstly consider the equation in (3.1) when $i = 1$, i.e.,

$$\dot{x}_1 = g_1 x_2 + \theta_1^T F_1(x_1) + f_1(x_1) + \Delta_1(t, x)$$

From the definition for new states z_1 and z_2, i.e. $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, we have

$$\dot{z}_1 = g_1(z_2 + \alpha_1) + \theta_1^T F_1(x_1) + f_1(x_1) + \Delta_1(t, x) - \dot{y}_d \tag{3.4}$$

Consider the scalar smooth function be $V_{z_1} = \frac{1}{2g_1} z_1^2$, whose time derivative along (3.4) is

$$\dot{V}_{z_1} = z_1 z_2 + z_1[\alpha_1 + \frac{1}{g_1}(\theta_1^T F_1 + f_1 + \Delta_1 - \dot{y}_d)]$$

Since the inequality $z_1 z_2 \leq \frac{1}{4k_1} z_1^2 + k_1 z_2^2$, $\forall k_1 > 0$ holds, noting Assumption 3.2.2, we have

$$\begin{aligned} \dot{V}_{z_1} &\leq \frac{1}{4k_1} z_1^2 + k_1 z_2^2 + z_1[\alpha_1 + \frac{1}{g_1}(\theta_1^T F_1 + f_1 - \dot{y}_d)] + \frac{p_1}{g_1}|z_1|\phi_1 \\ &\triangleq \frac{1}{4k_1} z_1^2 + k_1 z_2^2 + z_1(\alpha_1 + \theta_{a,1}^T F_{a,1}) + p_{a,1}|z_1|\phi_{a,1} \end{aligned} \tag{3.5}$$

where $p_{a,1}$ is an unknown constant, $\theta_{a,1}$ is an unknown constant vector, $\phi_{a,1}(\cdot)$ is a known function, and $F_{a,1}(\cdot)$ is a known function vector defined as

$$\begin{aligned} p_{a,1} &:= \frac{p_1}{g_1}, \quad \theta_{a,1} := [\frac{\theta_1^T}{g_1}, \frac{1}{g_1}]^T \in R^{n_1+1}, \\ \phi_{a,1} &:= \phi_1, \quad F_{a,1} := [F_1^T, f_1 - \dot{y}_d]^T \in R^{n_1+1}, \end{aligned}$$

Remark 3.2.2 *The introduction of $p_{a,1}$ and $\theta_{a,1}$ is to avoid possible singularity problems. We estimate $\frac{1}{g_1}$ rather than g_1 to avoid the possibility of $\hat{g}_1 = 0$.*

Consider the following Lyapunov function candidate as

$$V_1 = \frac{1}{2g_1} z_1^2 + \frac{1}{2}\tilde{\theta}_{a,1}^T \Gamma_{\theta 1}^{-1} \tilde{\theta}_{a,1} + \frac{1}{2\gamma_{p1}} \tilde{p}_{a,1}^2$$

where $\Gamma_{\theta 1} = \Gamma_{\theta 1}^T > 0$, $\lambda_{p1} > 0$, $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$, and $\hat{\theta}_{a,1}$ and $\hat{p}_{a,1}$ are the estimates of $\theta_{a,1}$ and $p_{a,1}$ respectively.

Choose the following intermediate control law and parameter adaptation law as

$$\alpha_1 = -c_1 z_1 - \frac{1}{4k_1} z_1 - \hat{\theta}_{a,1}^T F_{a,1} - \hat{p}_{a,1}\phi_{a,1} \tanh(\frac{z_1 \phi_{a,1}}{\epsilon_1}) \tag{3.6}$$

$$\dot{\hat{\theta}}_{a,1} = \Gamma_{\theta 1}(F_{a,1} z_1 - \sigma_{\theta 1}\hat{\theta}_{a,1}) \tag{3.7}$$

$$\dot{\hat{p}}_{a,1} = \gamma_{p1}\Big[z_1 \phi_{a,1} \tanh(\frac{z_1 \phi_{a,1}}{\epsilon_1}) - \sigma_{p1}\hat{p}_{a,1}\Big] \tag{3.8}$$

The time derivative of V_1 along (3.5) and (3.6)-(3.8) is

$$\begin{aligned}\dot{V}_1 &\leq -c_1 z_1^2 + k_1 z_2^2 + p_{a,1}[|z_1|\phi_{a,1} - z_1\phi_{a,1}\tanh(\frac{z_1\phi_{a,1}}{\epsilon_1})] \\ &\quad -\sigma_{p1}\tilde{p}_{a,1}\hat{p}_{a,1} - \sigma_{\theta 1}\tilde{\theta}_{a,1}^T\hat{\theta}_{a,1} \qquad (3.9)\end{aligned}$$

Applying Lemma 3.2.1 and noting the following inequalities

$$-\sigma_{p1}\tilde{p}_{a,1}\hat{p}_{a,1} \leq -\frac{1}{2}\sigma_{p1}\tilde{p}_{a,1}^2 + \frac{1}{2}\sigma_{p1}p_{a,1}^2$$

$$-\sigma_{\theta 1}\tilde{\theta}_{a,1}^T\hat{\theta}_{a,1} \leq -\frac{1}{2}\sigma_{\theta 1}\|\tilde{\theta}_{a,1}\|^2 + \frac{1}{2}\sigma_{\theta 1}\|\theta_{a,1}\|^2$$

we have

$$\begin{aligned}\dot{V}_1 &\leq -c_1 z_1^2 + k_1 z_2^2 - \frac{1}{2}\sigma_{p1}\tilde{p}_{a,1}^2 - \frac{1}{2}\sigma_{\theta 1}\|\tilde{\theta}_{a,1}\|^2 \\ &\quad +\frac{1}{2}\sigma_{p1}p_{a,1}^2 + \frac{1}{2}\sigma_{\theta 1}\|\theta_{a,1}\|^2 + 0.2785\epsilon_1 p_{a,1} \\ &\leq -\lambda_1 V_1 + \rho_1 + k_1 z_2^2 \qquad (3.10)\end{aligned}$$

where

$$\lambda_1 := \min\Big\{2c_i,\ \sigma_{p1}\gamma_{p1},\ \frac{\sigma_{\theta 1}}{\lambda_{\max}(\Gamma_{\theta 1}^{-1})}\Big\}$$

$$\rho_1 := \frac{1}{2}\sigma_{p1}p_{a,1}^2 + \frac{1}{2}\sigma_{\theta 1}\|\theta_{a,1}\|^2 + 0.2785\epsilon_1 p_{a,1}$$

Multiplying (3.10) by $e^{\lambda_1 t}$, it becomes

$$\frac{d}{dt}\Big(V_1(t)e^{\lambda_1 t}\Big) \leq \rho_1 e^{\lambda_1 t} + k_1 e^{\lambda_1 t} z_2^2 \qquad (3.11)$$

Integrating (3.11) over $[0, t]$, we have

$$V_1(t) \leq [V_1(0) - \frac{\rho_1}{\lambda_1}]e^{-\lambda_1 t} + \frac{\rho_1}{\lambda_1} + e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau \qquad (3.12)$$

In (3.12), if there is no extra term $e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau$ within the inequality, we can conclude that $V_1(t)$, z_1, $\hat{p}_{a,1}$, $\hat{\theta}_{a,1}$ are all GUUB. Noting the following inequality

$$\begin{aligned}e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau &\leq e^{-\lambda_1 t}\sup_{\tau\in[0,t]}[z_2^2(\tau)]\int_0^t k_1 e^{\lambda_1 \tau}d\tau \\ &\quad \Big(= \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)](1 - e^{-\lambda_1 t})\Big) \\ &\leq \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)] \qquad (3.13)\end{aligned}$$

we have

$$V_1(t) \leq [V_1(0) - \frac{\rho_1}{\lambda_1}]e^{-\lambda_1 t} + \frac{\rho_1}{\lambda_1} + \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)] \tag{3.14}$$

Therefore, if z_2 can be regulated as bounded, we can obtain the boundedness of the term $e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1\tau}z_2^2(\tau)d\tau$. From (3.14), we can then claim that $V_1(t)$, z_1, $\hat{p}_{a,1}$, $\hat{\theta}_{a,1}$ are GUUB.

Remark 3.2.3 *Note that the Young's inequality is used to decouple the coupling term z_1z_2, which is traditionally left to be dealt with/cancelled in the next step. If the coupling term is left intact and the intermediate control law is constructed as*

$$\alpha_1 = -c_1z_1 - \hat{\theta}_{a,1}^T F_{a,1} - \hat{p}_{a,1}\phi_{a,1}\tanh(\frac{z_1\phi_{a,1}}{\epsilon_1})$$

then we obtain

$$\dot{V}_1 \leq -\lambda_1 V_1 + \rho_1 + z_1z_2$$

Similar derivation yields

$$V_1(t) \leq [V_1(0) - \frac{\rho_1}{\lambda_1}]e^{-\lambda_1 t} + \frac{\rho_1}{\lambda_1} + e^{-\lambda_1 t}\int_0^t e^{\lambda_1\tau}z_1(\tau)z_2(\tau)d\tau \tag{3.15}$$

From (3.15), we known that it is impossible to obtain the GUUB of $V_1(t)$, z_1, $\hat{p}_{a,1}$, and $\hat{\theta}_{a,1}$ even if z_2 can be regulated as bounded. In other words, we can only obtain this property by assuming that z_1z_2 can be guaranteed to be bounded in the next step, which is actually hard to achieve. In the standard backstepping design, z_1z_2 will be cancelled in the next step, while another coupling term z_2z_3 will appear and be dealt with later, till the final step. The cancellation-based iterative backstepping design utilize the states interconnection, while the decoupled backstepping design tries to decouple the interconnection.

Step 2: Since $z_2 = x_2 - \alpha_1$, the time derivative of z_2 is given by

$$\begin{aligned}\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 \\ &= g_2x_3 + \theta_2^T F_2(\bar{x}_2) + f_2(\bar{x}_2) + \Delta_2(t,x) - \dot{\alpha}_1 \end{aligned} \tag{3.16}$$

Again, by viewing $x_3(t)$ as a virtual control, we may design a control input α_2 for (3.16). Since $z_3(t) = x_3(t) - \alpha_2(t)$, we have

$$\dot{z}_2 = g_2(z_3 + \alpha_2) + \theta_2^T F_2(\bar{x}_2) + f_2(\bar{x}_2) + \Delta_2(t,x) - \dot{\alpha}_1$$

Since α_1 is a function of x_1, y_d, $\dot{y}_d$ and $\hat{\theta}_{a,1}$, $\dot{\alpha}_1$ can be expressed as

$$\begin{aligned}\dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{\theta}_{a,1}}\dot{\hat{\theta}}_{a,1} + \frac{\partial \alpha_1}{\partial \hat{p}_{a,1}}\dot{\hat{p}}_{a,1} \\ &= \frac{\partial \alpha_1}{\partial x_1}(g_1 x_2 + \theta_1^T F_1 + f_1 + \Delta_1) + \omega_1\end{aligned}$$

where

$$\omega_1 = \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{\theta}_{a,1}}\dot{\hat{\theta}}_{a,1} + \frac{\partial \alpha_1}{\partial \hat{p}_{a,1}}\dot{\hat{p}}_{a,1}$$

then we have

$$\begin{aligned}\dot{z}_2 &= g_2(z_3 + \alpha_2) + \theta_2^T F_2(\bar{x}_2) + f_2(\bar{x}_2) + \Delta_2(t,x) \\ &\quad -\frac{\partial \alpha_1}{\partial x_1}(g_1 x_2 + \theta_1^T F_1 + f_1 + \Delta_2) - \omega_1 \\ &= g_2\Big[z_3 + \alpha_2 + \frac{1}{g_2}(\theta_2^T F_2 + f_2 + \Delta_2 \\ &\quad -g_1\frac{\partial \alpha_1}{\partial x_1}x_2 - \theta_1^T F_1 \frac{\partial \alpha_1}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_1} f_1 - \frac{\partial \alpha_1}{\partial x_1}\Delta_1 - \omega_1)\Big] \end{aligned} \tag{3.17}$$

Consider the scalar smooth function $V_{z_2} = \frac{1}{2g_2}z_2^2$, whose time derivative along (3.17) is

$$\dot{V}_{z_2} = z_2 z_3 + z_2\Big[\alpha_2 + \frac{1}{g_2}(\theta_2^T F_2 + f_2 + \Delta_2 - g_1\frac{\partial \alpha_1}{\partial x_1}x_2 - \theta_1^T F_1\frac{\partial \alpha_1}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_1}f_1 - \frac{\partial \alpha_1}{\partial x_1}\Delta_1 - \omega_1)\Big]$$

Since $z_2 z_3 \leq \frac{1}{4k_2}z_2^2 + k_2 z_3^2$, $\forall k_2 > 0$ and from Assumption 3.2.2, we have

$$\begin{aligned}\dot{V}_{z_2} &\leq \frac{1}{4k_2}z_2^2 + k_2 z_3^2 + z_2\Big[\alpha_2 + \frac{1}{g_2}(\theta_2^T F_2 + f_2 + \Delta_2 - g_1\frac{\partial \alpha_1}{\partial x_1}x_2 \\ &\quad -\theta_1^T F_1\frac{\partial \alpha_1}{\partial x_1} - \frac{\partial \alpha_1}{\partial x_1}f_1 - \frac{\partial \alpha_1}{\partial x_1}\Delta_1 - \omega_1)\Big] \\ &\leq \frac{1}{4k_2}z_2^2 + k_2 z_3^2 + z_2(\alpha_2 + \theta_{a,2}^T F_{a,2}) + |z_2| p_{a,2}^T \phi_{a,2}\end{aligned} \tag{3.18}$$

where $p_{a,2}$ and $\theta_{a,2}$ are unknown constant vectors, $\phi_{a,2}(\cdot)$ and $F_{a,2}(\cdot)$ are known function vectors defined as

$$\begin{aligned} p_{a,2} &:= [\frac{p_2}{g_2},\ \frac{p_1}{g_2}]^T \in R^2, \\ \phi_{a,2} &:= [\phi_2,\ |\frac{\partial \alpha_1}{\partial x_1}|\phi_1]^T \in R^2, \\ \theta_{a,2} &:= [\frac{\theta_2^T}{g_2}, \frac{g_1}{g_2}, \frac{\theta_1^T}{g_2}, \frac{1}{g_2}]^T \in R^{n_1+n_2+2} \\ F_{a,2} &:= [F_2^T, -\frac{\partial \alpha_1}{\partial x_1}x_2, -\frac{\partial \alpha_1}{\partial x_1}F_1^T, f_2 - \frac{\partial \alpha_1}{\partial x_1}f_1 - \omega_1]^T \in R^{n_1+n_2+2}\end{aligned}$$

Consider the following Lyapunov function candidate

$$V_2 = \frac{1}{2g_2} z_2^2 + \frac{1}{2}\tilde{\theta}_{a,2}^T \Gamma_{\theta 2}^{-1} \tilde{\theta}_{a,2} + \frac{1}{2}\tilde{p}_{a,2}^T \Gamma_{p2}^{-1} \tilde{p}_{a,2}$$

Choose the following intermediate control law and parameter adaptation law as

$$\alpha_2 = -c_2 z_2 - \frac{1}{4k_2} z_2 - \hat{\theta}_{a,2}^T F_{a,2} - \hat{p}_{a,2}^T \Phi_{a,2} \tanh_1(\frac{z_2 \phi_{a,2}}{\epsilon_2}) \tag{3.19}$$

$$\dot{\hat{\theta}}_{a,2} = \Gamma_{\theta 2}(F_{a,2} z_2 - \sigma_{\theta 2} \hat{\theta}_{a,2}) \tag{3.20}$$

$$\dot{\hat{p}}_{a,2} = \Gamma_{p2}[z_2 \Phi_{a,2} \tanh_1(\frac{z_2 \phi_{a,2}}{\epsilon_2}) - \sigma_{p2} \hat{p}_{a,2}] \tag{3.21}$$

where

$$\Phi_{a,2} := \mathrm{diag}\{\phi_2,\ |\frac{\partial \alpha_1}{\partial x_1}|\phi_1\} \in R^{2\times 2}$$

$$\tanh_1(v) := [\tanh(v_1), \tanh(v_2), \cdots, \tanh(v_n)]^T, \quad v = [v_1, v_2, \cdots, v_n]^T$$

Remark 3.2.4 *The introduction of notation $\Phi_{a,2}$ and $\Phi_{a,i}$, $i = 3, \cdots, n$ in the next steps is for the ease of applying Lemma 3.2.1 in its vector version. An alternative is to define $p_{a,2}$ and $\phi_{a,2}$ respectively as*

$$p_{a,2} := \max\{\frac{p_2}{g_2},\ \frac{p_1}{g_2}\}, \qquad \phi_{a,2} := \phi_2 + |\frac{\partial \alpha_1}{\partial x_1}|\phi_1$$

then $p_{a,2}$ is a unknown scalar constant and $\phi_{a,2}$ is a known scalar function and Lemma 3.2.1 can be applied directly.

The time derivative of V_2 along (3.18) and (3.19)-(3.21) is

$$\begin{aligned} \dot{V}_2 \;\le\; & -c_2 z_2^2 + k_2 z_3^2 + p_{a,2}^T[|z_2|\phi_{a,2} - z_2 \Phi_{a,2} \tanh_1(\frac{z_2 \phi_{a,2}}{\epsilon_2})] \\ & -\sigma_{p2} \tilde{p}_{a,2}^T \hat{p}_{a,2} - \sigma_{\theta 2} \tilde{\theta}_{a,2}^T \hat{\theta}_{a,2} \end{aligned} \tag{3.22}$$

To complete the squares and noting Lemma 3.2.1, we obtain

$$\begin{aligned} \dot{V}_2 \;\le\; & -c_2 z_2^2 + k_2 z_3^2 - \frac{1}{2}\sigma_{p2}\|\tilde{p}_{a,2}\|^2 - \frac{1}{2}\sigma_{\theta 2}\|\tilde{\theta}_{a,2}\|^2 \\ & +\frac{1}{2}\sigma_{p2}\|p_{a,2}\|^2 + \frac{1}{2}\sigma_{\theta 2}\|\theta_{a,2}\|^2 + 0.2785\epsilon_2 p_{s,2} \\ \le\; & -\lambda_2 V_2 + \rho_2 + k_2 z_3^2 \end{aligned}$$

where

$$p_{s,2} := \frac{p_2}{g_2} + \frac{p_1}{g_2}$$
$$\lambda_2 := \min\left\{2c_2,\ \frac{\sigma_{p2}}{\lambda_{\max}(\Gamma_{p2}^{-1})},\ \frac{\sigma_{\theta 2}}{\lambda_{\max}(\Gamma_{\theta 2}^{-1})}\right\}$$
$$\rho_2 := \frac{1}{2}\sigma_{p2}\|p_{a,2}\|^2 + \frac{1}{2}\sigma_{\theta 2}\|\theta_{a,2}\|^2 + 0.2785\epsilon_2 p_{s,2}$$

Similarly, if z_3 can be regulated as bounded, we can conclude that z_2 is bounded, and so is z_1.

Remark 3.2.5 *Since the coupling term $z_1 z_2$ in Step 1 has been decoupled by $\frac{1}{4k_1}z_1^2$ and $k_1 z_2^2$ so that it does not need to be cancelled in Step 2. The Lyapunov function candidate $V_2(t)$ in Step 2 is constructed independently rather than adding into the previous $V_1(t)$. Accordingly, the intermediate control of this step does not need to cancel the coupling term.*

Step i ($3 \leq i \leq n-1$): Similar procedures are taken for each steps when $i = 3, \cdots, n-1$ as in Steps 1 and 2.

The time derivative of $z_i(t)$ is given by

$$\dot{z}_i = g_i(z_{i+1} + \alpha_i) + \theta_i^T F_i(\bar{x}_i) + f_i(\bar{x}_i) + \Delta_i(t, x) - \dot{\alpha}_{i-1} \tag{3.23}$$

Since α_{i-1} is a function of $\bar{x}_{i-1}$, $\bar{x}_{di}$, $\hat{\theta}_{a,1}$, $\hat{\theta}_{a,2}$, ..., $\hat{\theta}_{a,i-1}$, $\dot{\alpha}_{i-1}$ can be expressed as

$$\begin{aligned}\dot{\alpha}_{i-1} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \frac{\partial\alpha_{i-1}}{\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{p}_{a,j}}\dot{\hat{p}}_{a,j} \\ &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1}\end{aligned}$$

where

$$\omega_{i-1} = \frac{\partial\alpha_{i-1}}{\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{p}_{a,j}}\dot{\hat{p}}_{a,j}$$

then (3.23) becomes

$$\begin{aligned}\dot{z}_i &= g_i(z_{i+1} + \alpha_i) + \theta_i^T F_i(\bar{x}_i) + f_i(\bar{x}_i) + \Delta_i(t, x) \\ &\quad - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{i-1}\end{aligned}$$

$$= g_i\Big\{z_{i+1} + \alpha_i + \frac{1}{g_i}\Big[\theta_i^T F_i + f_i + \Delta_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{i-1}\Big]\Big\} \tag{3.24}$$

Consider the scalar smooth function $V_{z_i} = \frac{1}{2g_i} z_i^2$, whose time derivative along (3.24) is

$$\dot{V}_{z_i} = z_i z_{i+1} + z_i\Big\{\alpha_i + \frac{1}{g_i}\Big[\theta_i^T F_i + f_i + \Delta_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{i-1}\Big]\Big\}$$

Since $z_i z_{i+1} \le \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2$, $\forall k_i > 0$ and from Assumption 3.2.2, we have

$$\begin{aligned}
\dot{V}_{z_i} &\le \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2 + z_i\Big\{\alpha_i + \frac{1}{g_i}\Big[\theta_i^T F_i + f_i + \Delta_i \\
&\quad - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{i-1}\Big]\Big\} \\
&\le \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2 + z_i(\alpha_i + \theta_{a,i}^T F_{a,i}) + |z_i| p_{a,i}^T \phi_{a,2}
\end{aligned} \tag{3.25}$$

where $p_{a,i}$ and $\theta_{a,i}$ are unknown constant vectors, $\phi_{a,i}(\cdot)$ and $F_{a,i}(\cdot)$ are known function vectors defined as

$$\begin{aligned}
p_{a,i} &:= [\frac{p_i}{g_i},\ \frac{g_{i-1}}{g_i} p_{a,i-1}^T]^T \in R^i \\
\phi_{a,i} &:= [\phi_i, |\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}|\phi_{i-1},\ |\frac{\partial \alpha_{i-1}}{\partial x_{i-2}}|\phi_{i-2},\ \cdots,\ |\frac{\partial \alpha_{i-1}}{\partial x_1}|\phi_1] \in R^i; \\
\theta_{a,i} &:= [\frac{\theta_i^T}{g_i},\ \frac{g_{i-1}}{g_i},\ \frac{g_{i-1}}{g_i}\theta_{a,i-1}^T]^T \in R^{\bar{n}_i} \\
F_{a,i} &:= [F_i^T,\ -\frac{\partial \alpha_{i-1}}{\partial x_{i-1}} x_i,\ -\frac{\partial \alpha_{i-1}}{\partial x_{i-1}} F_{i-1}^T, -\frac{\partial \alpha_{i-1}}{\partial x_{i-2}} x_{i-1},\ -\frac{\partial \alpha_{i-1}}{\partial x_{i-2}} F_{i-2}^T,\ \cdots, \\
&\quad -\frac{\partial \alpha_{i-1}}{\partial x_1} x_2,\ -\frac{\partial \alpha_{i-1}}{\partial x_1} F_1^T,\ f_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j} f_j - \omega_{i-1}]^T \in R^{\bar{n}_i}, \bar{n}_i = \sum_{j=1}^{i} n_j + 2i
\end{aligned}$$

Similarly, we consider the following Lyapunov function candidate

$$V_i = V_{z_i} + \frac{1}{2}\tilde{\theta}_{a,i}^T \Gamma_{\theta i}^{-1} \tilde{\theta}_{a,i} + \frac{1}{2}\tilde{p}_{a,i}^T \Gamma_{pi}^{-1} \tilde{p}_{a,i}$$

together with the following adaptive intermediate control law

$$\alpha_i = -c_i z_i - \frac{1}{4k_i} z_i - \hat{\theta}_{a,i}^T F_{a,i} - \hat{p}_{a,i}^T \Phi_{a,i} \tanh_1(\frac{z_i \phi_{a,i}}{\epsilon_i}) \tag{3.26}$$

$$\dot{\hat{\theta}}_{a,i} = \Gamma_{\theta i}(F_{a,i} z_i - \sigma_{\theta i}\hat{\theta}_{a,i}) \tag{3.27}$$

$$\dot{\hat{p}}_{a,i} = \Gamma_{pi}[z_i \Phi_{a,i} \tanh_1(\frac{z_i \phi_{a,i}}{\epsilon_i}) - \sigma_{pi}\hat{p}_{a,i}] \tag{3.28}$$

where

$$\Phi_{a,i} = \text{diag}\{\phi_i, |\frac{\partial \alpha_{i-1}}{\partial x_{i-1}}|\phi_{i-1},\ |\frac{\partial \alpha_{i-1}}{\partial x_{i-2}}|\phi_{i-2},\ \cdots,\ |\frac{\partial \alpha_{i-1}}{\partial x_1}|\phi_1\}$$

The time derivative of V_i along (3.25) and (3.26)-(3.28) is

$$\begin{aligned}\dot{V}_i &\le -c_i z_i^2 + k_i z_{i+1}^2 + p_{a,i}^T\Big[|z_i|\phi_{a,i} - z_i\Phi_{a,i}\tanh_1(\frac{z_i\phi_{a,i}}{\epsilon_i})\Big] \\ &\quad -\sigma_{pi}\tilde{p}_{a,i}^T\hat{p}_{a,2} - \sigma_{\theta i}\tilde{\theta}_{a,i}^T\hat{\theta}_{a,2}\end{aligned}$$

To complete the squares and noting Lemma 3.2.1

$$\begin{aligned}\dot{V}_i &\le -c_i z_i^2 + k_i z_{i+1}^2 - \frac{1}{2}\sigma_{pi}\|\tilde{p}_{a,i}\|^2 - \frac{1}{2}\sigma_{\theta i}\|\tilde{\theta}_{a,i}\|^2 \\ &\quad +\frac{1}{2}\sigma_{pi}\|p_{a,i}\|^2 + \frac{1}{2}\sigma_{\theta i}\|\theta_{a,i}\|^2 + 0.2785\epsilon_i p_{s,i} \\ &\le -\lambda_i V_i + \rho_i + k_i z_{i+1}^2\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}p_{s,i} &:= \sum_{j=1}^{i} p_{a,i,j} \\ \lambda_i &:= \min\Big\{2c_i,\ \frac{\sigma_{pi}}{\lambda_{\max}(\Gamma_{pi}^{-1})},\ \frac{\sigma_{\theta i}}{\lambda_{\max}(\Gamma_{\theta i}^{-1})}\Big\} \\ \rho_i &:= +\frac{1}{2}\sigma_{pi}\|p_{a,i}\|^2 + \frac{1}{2}\sigma_{\theta i}\|\theta_{a,i}\|^2 + 0.2785\epsilon_i p_{s,i}\end{aligned}$$

Similarly, if z_{i+1} can be regulated as bounded, we can conclude that z_i is also bounded.

Step n: This is the final step, since the actual control u appears in the derivative of z_n as given by

$$\dot{z}_n = g_n u + \theta_n^T F_n(x) + f_n(x) + \Delta_n(t,x) - \dot{\alpha}_{n-1} \tag{3.30}$$

Since $\dot{\alpha}_{n-1}$ can be expressed as

$$\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}$$

where

$$\omega_{n-1} = \frac{\partial \alpha_{n-1}}{\partial \bar{x}_n}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} + \sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{p}_{a,j}}\dot{\hat{p}}_{a,j}$$

then (3.30) becomes

$$\begin{aligned}\dot{z}_n &= g_n u + \theta_n^T F_n(x) + f_n(x) + \Delta_n(t,x) \\ &\quad - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{n-1} \\ &= g_n \Big\{ u + \frac{1}{g_n}\Big[\theta_n^T F_n + f_n + \Delta_n \\ &\quad - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{n-1}\Big]\Big\} \end{aligned} \tag{3.31}$$

Consider the scalar smooth function $V_{z_n} = \frac{1}{2g_n} z_n^2$, whose time derivative along (3.31) is

$$\dot{V}_{z_n} = z_n \Big\{ u + \frac{1}{g_n}\Big[\theta_n^T F_n + f_n + \Delta_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T F_j + f_j + \Delta_j) - \omega_{n-1}\Big]\Big\}$$

Noting Assumption 3.2.2, we have

$$\dot{V}_{z_n} \le z_n(u + \theta_{a,n}^T F_{a,n}) + |z_n| p_{a,n}^T \phi_{a,n} \tag{3.32}$$

where $p_{a,n}$ and $\theta_{a,n}$ are unknown constant vectors, $\phi_{a,n}(\cdot)$ and $F_{a,n}(\cdot)$ are known function vectors defined as

$$\begin{aligned} p_{a,n} &:= [\frac{p_n}{g_n},\ \frac{g_{n-1}}{g_n} p_{a,n-1}^T]^T \in R^n \\ \phi_{a,n} &:= [\phi_n, |\frac{\partial \alpha_{n-1}}{\partial x_{n-1}}|\phi_{n-1},\ |\frac{\partial \alpha_{n-1}}{\partial x_{n-2}}|\phi_{n-2},\ \cdots,\ |\frac{\partial \alpha_{n-1}}{\partial x_1}|\phi_1]^T \in R^i; \\ \theta_{a,n} &:= [\frac{\theta_n^T}{g_n},\ \frac{g_{n-1}}{g_n},\ \frac{g_{n-1}}{g_n}\theta_{a,n-1}^T]^T \in R^{\bar{n}_n} \\ F_{a,n} &:= [F_n^T,\ -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} x_n,\ -\frac{\partial \alpha_{n-1}}{\partial x_{n-1}} F_{n-1}^T,\ -\frac{\partial \alpha_{n-1}}{\partial x_{n-2}} x_{n-1},\ -\frac{\partial \alpha_{n-1}}{\partial x_{n-2}} F_{n-2}^T,\ \cdots, \\ &\quad -\frac{\partial \alpha_{n-1}}{\partial x_1} x_2,\ -\frac{\partial \alpha_{n-1}}{\partial x_1} F_1^T,\ f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} f_j - \omega_{n-1}]^T \in R^{\bar{n}_n}, \\ &\quad \bar{n}_n = \sum_{j=1}^{n} n_j + 2n \end{aligned}$$

Similarly, we consider the following Lyapunov function candidate

$$V_n = V_{z_n} + \frac{1}{2}\tilde{\theta}_{a,n}^T \Gamma_{\theta n}^{-1} \tilde{\theta}_{a,n} + \frac{1}{2}\tilde{p}_{a,n}^T \Gamma_{pn}^{-1} \tilde{p}_{a,n}$$

and the following adaptive control law

$$u = -c_n z_n - \hat{\theta}_{a,n}^T F_{a,n} - \hat{p}_{a,n}^T \Phi_{a,n} \tanh_1(\frac{z_n \phi_{a,n}}{\epsilon_n}) \tag{3.33}$$

$$\dot{\hat{\theta}}_{a,n} = \Gamma_{\theta n}(F_{a,n} z_n - \sigma_{\theta n}\hat{\theta}_{a,n}) \tag{3.34}$$

$$\dot{\hat{p}}_{a,n} = \Gamma_{pn}[z_n \Phi_{a,n} \tanh_1(\frac{z_n \phi_{a,n}}{\epsilon_n}) - \sigma_{pn}\hat{p}_{a,n}] \tag{3.35}$$

where

$$\Phi_{a,n} := \mathrm{diag}\{\phi_n, |\frac{\partial \alpha_{n-1}}{\partial x_{n-1}}|\phi_{n-1},\ |\frac{\partial \alpha_{n-1}}{\partial x_{n-2}}|\phi_{n-2},\ \cdots,\ |\frac{\partial \alpha_{n-1}}{\partial x_1}|\phi_1\}$$

The time derivative of V_n along(3.32) and (3.33)-(3.35) is

$$\begin{aligned} \dot{V}_n \quad \leq \quad & -c_i z_n^2 + p_{a,n}^T\Big[|z_n|\phi_{a,n} - z_n \Phi_{a,n} \tanh_1(\frac{z_n \phi_{a,n}}{\epsilon_n})\Big] \\ & -\sigma_{pn}\tilde{p}_{a,n}^T \hat{p}_{a,n} - \sigma_{\theta n}\tilde{\theta}_{a,n}^T \hat{\theta}_{a,n} \end{aligned} \tag{3.36}$$

To complete the squares and noting Lemma 3.2.1

$$\begin{aligned} \dot{V}_n \quad \leq \quad & -c_n z_n^2 - \frac{1}{2}\sigma_{pn}\|\tilde{p}_{a,n}\|^2 - \frac{1}{2}\sigma_{\theta n}\|\tilde{\theta}_{a,n}\|^2 \\ & +\frac{1}{2}\sigma_{pn}\|p_{a,n}\|^2 + \frac{1}{2}\sigma_{\theta n}\|\theta_{a,n}\|^2 + 0.2785\epsilon_n p_{s,n} \\ \leq \quad & -\lambda_n V_n + \rho_n \end{aligned}$$

where

$$\begin{aligned} p_{s,n} &:= \sum_{j=1}^{n} p_{s,n,j} \\ \lambda_n &:= \min\Big\{2c_n,\ \frac{\sigma_{pn}}{\lambda_{\max}(\Gamma_{pn}^{-1})},\ \frac{\sigma_{\theta n}}{\lambda_{\max}(\Gamma_{\theta n}^{-1})}\Big\} \\ \rho_n &:= \frac{1}{2}\sigma_{pn}\|p_{a,n}\|^2 + \frac{1}{2}\sigma_{\theta n}\|\theta_{a,n}\|^2 + 0.2785\epsilon_n p_{s,n} \end{aligned}$$

Theorem 3.2.1 shows the stability and control performance of the closed-loop adaptive systems.

Theorem 3.2.1 *Consider the closed-loop system consisting of the plant (3.1) under Assumptions 3.2.1-3.2.3. If we apply the controller (3.33) with parameters updating law (3.34) and (3.35), we can guarantee the following properties under bounded initial conditions* *(i)* $z_i(t)$, $\hat{\theta}_{a,i}$, $\hat{p}_{a,i}$, $i = 1, \cdots, n$, *and* $x(t)$ *are globally uniformly ultimately bounded;*

(ii) Given any $\mu_i^* > \mu_i$, *there exists* T *such that, for all* $t \geq T$, $z_i(t)$ *will remain in a compact set defined by*

$$\Omega_{z_i} := \left\{ z_i \in R \,\middle|\, |z_i| \leq \mu_i^* \right\}, \qquad i = 1, \cdots, n$$

which can be made as small as desired by an appropriate choice of the design parameters.

Proof: Consider the following Lyapunov function candidate

$$V_n = V_{z_n} + \frac{1}{2}\tilde{\theta}_{a,n}^T \Gamma_{\theta i}^{-1} \tilde{\theta}_{a,n} + \frac{1}{2}\tilde{p}_{a,n}^T \Gamma_{pn}^{-1} \tilde{p}_{a,n} \tag{3.37}$$

where $V_{z_n} = \frac{1}{2g_n} z_n^2$, and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. From the previous derivation, we have

$$\dot{V}_n(t) \leq -\lambda_n V_n(t) + \rho_n \tag{3.38}$$

it follows that

$$0 \leq V_n(t) \leq [V_n(0) - \frac{\rho_n}{\lambda_n}]e^{-\lambda_n t} + \frac{\rho_n}{\lambda_n} \leq V_n(0)e^{-\lambda_n t} + \frac{\rho_n}{\lambda_n} \tag{3.39}$$

where the constant

$$V_n(0) = \frac{1}{2g_n} z_n^2(0) + \frac{1}{2}\tilde{\theta}_{a,n}^T(0) \Gamma_{\theta n}^{-1} \tilde{\theta}_{a,n}(0) + \frac{1}{2}\tilde{p}_{a,n}^T(0) \Gamma_{pn}^{-1} \tilde{p}_{a,n}(0) \tag{3.40}$$

Considering (3.37), we know that

$$\|\tilde{\theta}_{a,n}\|^2 \leq \frac{2V_n(t)}{\lambda_{\min}(\Gamma_{\theta n}^{-1})} \tag{3.41}$$

$$\|\tilde{p}_{a,n}\|^2 \leq \frac{2V_n(t)}{\lambda_{\min}(\Gamma_{pn}^{-1})} \tag{3.42}$$

$$V_{z_n} = \frac{1}{2g_n} z_n^2 \leq V_n(t) \tag{3.43}$$

According to Lemma 2.2.5 in Chapter 2, we know from (3.39) that $V_n(t)$, z_n, $\hat{\theta}_{a,n}$ and $\hat{p}_{a,n}$ are GUUB. Thus, $V_i(t)$, z_i, $\hat{\theta}_{a,i}$ and $\hat{p}_{a,i}$ are also global uniformly ultimately bounded for $i = 1, \cdots, n-1$. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $x_2 = z_2 + \alpha_1$, since α_1 is function of bounded signals x_1, $\bar{x}_{d2}$, $\hat{\theta}_{10}$, $\hat{\theta}_1$, α_1 is thus bounded, which in turn leads to the boundedness of x_2. Following the same way, we can prove one by one that all α_{i-1} and x_i, $i = 3, \cdots, n$ are bounded. Therefore, the states of the system $x = [x_1, \cdots, x_n]^T$ remain bounded. If we let $\mu_n = \sqrt{2g_n\rho_n/\lambda_n}$, then from (3.43), we know that given any $\mu_n^* > \mu_n$, there exists T such that $z_n \leq \mu_n^*$, $\forall t \geq T$. Similarly, from(3.29), we know that given any $\mu_i^* > \mu_i$, there exists T such that $z_i \leq \mu_i^*$, $\forall t \geq T$, where $\mu_i = \sqrt{2g_i\bar{\rho}_i/\lambda_i}$ and $\bar{\rho}_i = \rho_i + k_i\mu_{i+1}^2$. Therefore, we can readily conclude that there do exist a compact set Ω_{z_i} such that $z_i \in \Omega_{z_i}$, $\forall t \geq T$. This completes the proof. $\diamondsuit$

Remark 3.2.6 *Note that Ω_z can be made arbitrarily small, which means that $z_i(t)$ can stay arbitrarily close to zero.*

Remark 3.2.7 *Different from the traditional backstepping design, the Lyapunov function candidate of the overall system $V_n(t)$ is not the sum of all previous $V_i(t)$, $i = 1, \cdots, n-1$. As a result, the residual set of each state $z_i(t)$, $i = 1, \cdots, n$ can be determined individually in an iterative way.*

3.2.3 Simulation Studies

To illustrate the proposed adaptive control algorithms, we consider the following second-order plant

$$\begin{cases} \dot{x}_1 &= g_1x_2 + \theta_1x_1^2 + x_1e^{-0.5x_1} + \Delta_1(t,x) \\ \dot{x}_2 &= g_2u + x_1x_2^2 + \Delta_2(t,x) \\ y &= x_1 \end{cases}$$

where $x = [x_1, x_2]^T$, θ_1 is unknown parameter, $\Delta_1(t,x)$ and $\Delta_2(t,x)$ are unknown disturbances. In our simulation, we assume that $g_1 = g_2 = 1$, $\theta_1 = 0.1$, $\Delta_1(t,x) = 0.6\sin x_2$ and $\Delta_2(t,x) = 0.5(x_1^2 + x_2^2)\sin^3 t$. The initial condition $[x_1(0), x_2(0)]^T = [0,0]^T$. The upper bounds of Δ_1 and Δ_2 are $|\Delta_1(t,x)| \leq p_1\phi_1(x_1)$ and $|\Delta_2(t,x)| \leq p_2\phi_2(x)$, where $p_1 = 0.6$, $\phi_1(x_1) = 1.0$, $p_2 = 0.5$, $\phi_2(x) = x_1^2 + x_2^2$. The control objective is to track the desired reference signal $y_d = 0.5[\sin(t) + \sin(0.5t)]$. For the design of adaptive controller, let $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$, and $\hat{\theta}_{a,1}$, $\hat{p}_{a,1}$, $\hat{\theta}_{a,2}$, $\hat{p}_{a,2}$ be the estimates of unknown parameters $\theta_{a,1} = [\frac{\theta_1}{g_1}, \frac{1}{g_1}]^T$, $p_{a,1} = \frac{p_1}{g_1}$, $\theta_{a,2} = [\frac{g_1}{g_2}, \frac{\theta_1}{g_2}, \frac{1}{g_2}]^T$,

$p_{a,2} = [\frac{p_2}{g_2}, \frac{p_1}{g_2}]^T$, the proposed controller is

$$\alpha_1 = -c_1 z_1 - \frac{1}{4k_1} z_1 - \hat{\theta}_{a,1} F_{a,1} - \hat{p}_{a,1}\phi_{a,1} \tanh(\frac{z_1\phi_{a,1}}{\epsilon_1})$$
$$\dot{\hat{\theta}}_{a,1} = \Gamma_{\theta 1}(F_{a,1} z_1 - \sigma_{\theta 1}\hat{\theta}_{a,1})$$
$$\dot{\hat{p}}_{a,1} = \gamma_{p1}[z_1\phi_{a,1} \tanh(\frac{z_1\phi_{a,1}}{\epsilon_1}) - \sigma_{p1}\hat{p}_{a,1}]$$
$$u = -c_2 z_2 - \hat{\theta}_{a,2} F_{a,2} - \hat{p}_{a,2}^T \Phi_{a,2} \tanh_1(\frac{z_2\phi_{a,2}}{\epsilon_2})$$
$$\dot{\hat{\theta}}_{a,2} = \Gamma_{\theta 2}(F_{a,2} z_2 - \sigma_{\theta 2}\hat{\theta}_{a,2})$$
$$\dot{\hat{p}}_{a,2} = \gamma_{p2}[z_2\Phi_{a,2} \tanh_1(\frac{z_2\phi_{a,2}}{\epsilon_2}) - \sigma_{p2}\hat{p}_{a,2}]$$

where

$$\phi_{a,1} = \phi_1(x_1), \quad F_{a,1} = [F_1, f_1 - y_d]^T,$$
$$\phi_{a,2} = [\phi_2, |\frac{\partial\alpha_1}{\partial x_1}|\phi_1]^T, \quad F_{a,2} = [-\frac{\partial\alpha_1}{\partial x_1} x_2, -\frac{\partial\alpha_1}{\partial x_1} F_1, f_2 - \frac{\partial\alpha_1}{\partial x_1} f_1 - \omega_1]^T$$

The following controller design parameters are adopted in the simulation: $\Gamma_{\theta 1} = \text{diag}\{1.5\}$, $\gamma_{p1} = 1.0$, $\Gamma_{\theta 2} = \text{diag}\{3.0\}$, $\Gamma_{p2} = \text{diag}\{5.0\}$, $\sigma_{\theta 1} = \sigma_{p1} = \sigma_{\theta 2} = \sigma_{p2} = 0.05$, $c_1 = c_2 = 2.0$, $k_1 = 1.0$, $\epsilon_1 = \epsilon_2 = 0.05$.

From Fig. 3.1, it was seen that satisfactory transient tracking performance was obtained after 10 seconds of adaptation periods. Fig. 3.2 shows that the system state is bounded. Figs. 3.3 and 3.4 show the boundedness of the control input and the estimates of the parameters in the control loop.

3.2.4 Conclusion

In this Section, adaptive decoupled backstepping has been presented as a general tool for control system design, and it has been successfully applied to a class of parametric-strict-feedback nonlinear systems with unknown disturbances which satisfies triangular bounded conditions. It has been proved that the proposed systematic design method is able to guarantee global uniformly ultimately boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin. In addition, the residual set of each state based on new coordinate in the closed-loop can be determined respectively.

Simulation results have been provided to show the effectiveness of the proposed approach.

3.3 Adaptive Neural Network Design

3.3.1 Problem Formulation and Preliminaries

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{aligned} \dot{x}_i(t) &= g_i x_{i+1}(t) + f_i(\bar{x}_i(t)), \quad 1 \leq i \leq n-1 \\ \dot{x}_n(t) &= g_n u(t) + f_n(x(t)), \\ y(t) &= x_1(t) \end{aligned} \tag{3.44}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output respectively, $f_i(\cdot)$ are unknown smooth functions, and g_i are unknown constants, $i = 1, \cdots, n$. The control objective is to design an adaptive controller for system (3.44) such that the output $y(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $i = 1, \cdots, n$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for the system functions and reference signals.

Assumption 3.3.1 *The signs of g_i are known, and there exist constants $g_{\max} \geq g_{\min} > 0$ such that $g_{\min} \leq |g_i| \leq g_{\max}$.*

Assumption 3.3.2 *The desired trajectory vectors $\bar{x}_{di}$, $i = 2, ..., n$ are continuous and available, and $\bar{x}_{di} \in \Omega_{di} \subset R^i$ with Ω_{di} known compact sets.*

3.3.2 Neural Network Control

In this section, the adaptive NN controller design is proposed for system (3.44) and the stability results of the closed-loop system are presented.

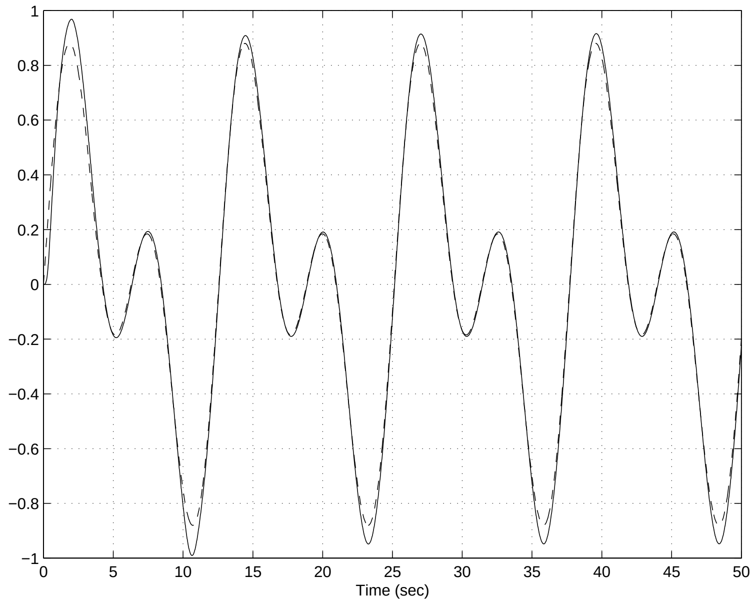

Figure 3.1: Responses of output $y(t)$(“—”), and reference y_d(“- -”)

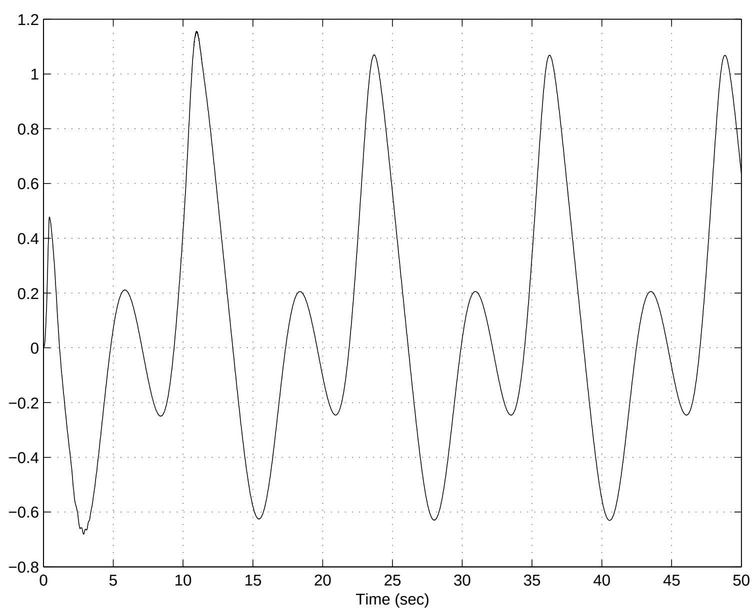

Figure 3.2: Responses of State x_2

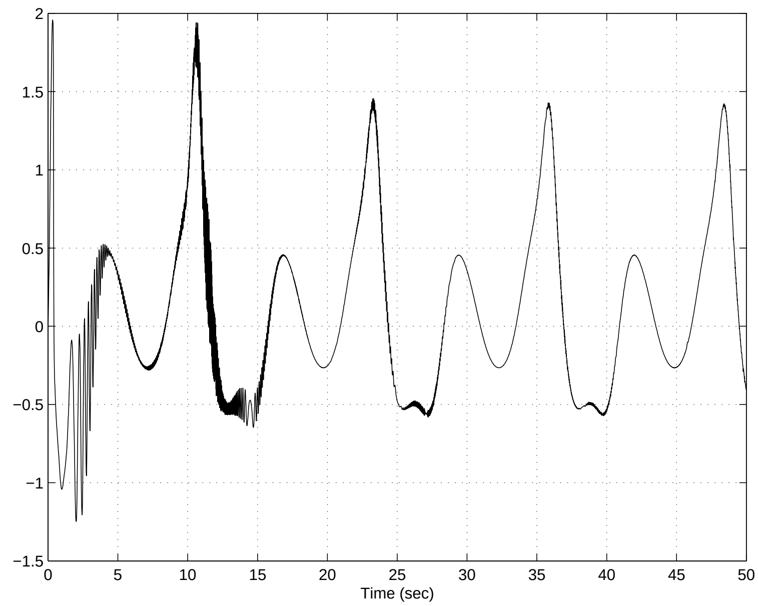

Figure 3.3: Variations of control input $u(t)$

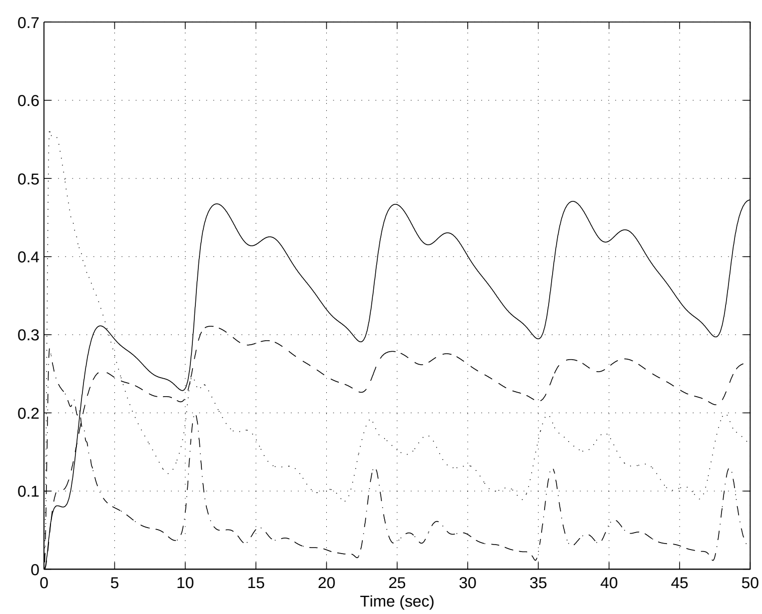

Figure 3.4: Variations of parameter estimates: $\|\hat{\theta}_{a,1}\|^2$("—"), $\hat{p}_{a,1}$("- -"), $\|\hat{\theta}_{a,2}\|^2$("···"), $\|\hat{p}_{a,2}\|^2$("-·").

The design procedure contains n steps. At step i, an intermediate control function $\alpha_i(t)$ shall be developed using an appropriate Lyapunov function $V_i(t)$, $i = 1, \cdots, n-1$. The control law $u(t)$ is designed in the last step to stabilized the whole closed-loop system using the Lyapunov function $V_n(t)$. Different from the backstepping design investigated intensively in the literature, where the Lyapunov function of i step, i.e., $V_i(t)$ is partially composed of the Lyapunov function of the previous step, i.e., $V_{i-1}(t)$ for $i = 2, \cdots, n$. In this section, the Lyapunov function of each step is decoupled in the sense that it does not contain the Lyapunov function of the previous step.

The design of both the control laws and the adaptive laws are based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \cdots, n$.

Step 1: Let us firstly consider the equation in (3.44) when $i = 1$, i.e.,

$$\dot{x}_1 = g_1 x_2 + f_1(x_1)$$

From the definition for new states z_1 and z_2, i.e. $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, we have

$$\dot{z}_1 = g_1(z_2 + \alpha_1) + f_1(x_1) - \dot{y}_d \tag{3.45}$$

Consider the scalar smooth function be $V_{z_1} = \frac{1}{2g_1} z_1^2$, whose time derivative along (3.45) is

$$\dot{V}_{z_1} = z_1 z_2 + z_1[\alpha_1 + \frac{1}{g_1}(f_1(x_1) - \dot{y}_d)] \tag{3.46}$$

Since g_1 is a unknown constant and $f_1(x_1)$ is an unknown smooth function, Let $Q_1(Z_1) = \frac{1}{g_1}(f_1(x_1) - \dot{y}_d)$ denote the unknown function with $Z_1 = [x_1, y_d, \dot{y}_d]^T \in \Omega_{Z_1} \subset R^3$ and $\bar{x}_{d2} \in \Omega_{d2}\}$. A RBF neural network is employed to approximate $Q_1(Z_1)$, i.e.,

$$Q_1(Z_1) = W_1^{*^T} S(Z_1) + \epsilon(Z_1) \tag{3.47}$$

where $\epsilon_1(Z_1)$ is the approximation error and W_1^* is the ideal weight. As W_1^* is unknown, we shall use its estimate $\hat{W}_1$ instead, which forms the intermediate control α_1 as

$$\alpha_1 = -c_1 z_1 - \hat{W}_1^T S(Z_1) \tag{3.48}$$

with constant $c_1 > 0$.

Remark 3.3.1 *The introduction of $Q_1(Z_1)$ is to avoid possible singularity problems by estimating $Q_1(Z_1)$ as a whole rather than g_1 to avoid the possibility of $\hat{g}_1 = 0$.*

Consider the following Lyapunov function candidate as

$$V_1 = \frac{1}{2g_1} z_1^2 + \frac{1}{2}(\hat{W}_1 - W_1^*)^T \Gamma_1^{-1} (\hat{W}_1 - W_1^*)$$

where matrix $\Gamma_1 = \Gamma_1^T > 0$.

Noting the inequality $z_1 z_2 \leq \frac{1}{4k_1} z_1^2 + k_1 z_2^2$, $\forall k_1 > 0$, the time derivative of V_1 along (3.46), (3.47) and (3.48) is

$$\begin{aligned} \dot{V}_1 \ \leq \ & \frac{1}{4k_1} z_1^2 + k_1 z_2^2 - c_1 z_1^2 - \hat{W}_1^T S(Z_1) z_1 + {W_1^*}^T S(Z_1) z_1 + \epsilon(Z_1) z_1 \\ & + (\hat{W}_1 - W_1^*)^T \Gamma_1^{-1} \dot{\hat{W}}_1 \end{aligned} \quad (3.49)$$

Letting $c_1 = c_{10} + c_{11}$ with $c_{10}^* \triangleq c_{10} - \frac{1}{4k_1}$ and noting that $-c_{11} z_1^2 + \epsilon(Z_1) z_1 \leq -c_{11} z_1^2 + \epsilon_{z_1}^* |z_1| \leq \frac{{\epsilon_{z_1}^*}^2}{4c_{11}}$, (3.49) becomes

$$\dot{V}_1 \leq -c_{10}^* z_1^2 - (\hat{W}_1 - W_1^*)^T S(Z_1) z_1 + (\hat{W}_1 - W_1^*)^T \Gamma_1^{-1} \dot{\hat{W}}_1 + \frac{{\epsilon_{z_1}^*}^2}{4c_{11}} + k_1 z_2^2 \quad (3.50)$$

The following practical adaptive law is given for on-line tuning the NN weights

$$\dot{\hat{W}}_1 = \Gamma_1 [S(Z_1) z_1 - \sigma_1 \hat{W}_1] \quad (3.51)$$

where σ_1 is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Substituting (3.51) into (3.50) yields

$$\dot{V}_1 \leq -c_{10}^* z_1^2 - \sigma_1 (\hat{W}_1 - W_1^*)^T \hat{W}_1 + \frac{{\epsilon_{z_1}^*}^2}{4c_{11}} + k_1 z_2^2 \quad (3.52)$$

Noting the following inequalities

$$-\sigma_1 (\hat{W}_1 - W_1^*)^T \hat{W}_1 \leq -\frac{1}{2}\sigma_1 \|\hat{W}_1 - W_1^*\|^2 + \frac{1}{2}\sigma_1 \|W_1^*\|^2$$

equation (3.52) becomes

$$\begin{aligned} \dot{V}_1 \ \leq \ & -c_{10}^* z_1^2 - \frac{1}{2}\sigma_1 \|\hat{W}_1 - W_1^*\|^2 + \frac{1}{2}\sigma_1 \|W_1^*\|^2 + \frac{{\epsilon_{z_1}^*}^2}{4c_{11}} + k_1 z_2^2 \\ \leq \ & -\lambda_1 V_1 + \rho_1 + k_1 z_2^2 \end{aligned} \quad (3.53)$$

where

$$\lambda_1 := \min\Big\{2g_{\min}c_{10}^*,\ \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\Big\}$$
$$\rho_1 := \frac{1}{2}\sigma_1\|W_1^*\|^2 + \frac{\epsilon_{z_1}^{*^2}}{4c_{11}}$$

Multiplying (3.53) by $e^{\lambda_1 t}$, it becomes

$$\frac{d}{dt}\Big(V_1(t)e^{\lambda_1 t}\Big) \ \leq \ \rho_1 e^{\lambda_1 t} + k_1 e^{\lambda_1 t} z_2^2 \tag{3.54}$$

Integrating (3.54) over $[0, t]$, we have

$$V_1(t) \leq [V_1(0) - \frac{\rho_1}{\lambda_1}]e^{-\lambda_1 t} + \frac{\rho_1}{\lambda_1} + e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau \tag{3.55}$$

In (3.55), if there is no extra term $e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau$ within the inequality, we can conclude that $V_1(t)$, z_1, $\hat{p}_{a,1}$, $\hat{\theta}_{a,1}$ are all GUUB. Noting the following inequality

$$\begin{aligned} e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau \ &\leq \ e^{-\lambda_1 t}\sup_{\tau\in[0,t]}[z_2^2(\tau)]\int_0^t k_1 e^{\lambda_1 \tau}d\tau \\ &\quad \Big(= \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)](1 - e^{-\lambda_1 t})\Big) \\ &\leq \ \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)] \end{aligned} \tag{3.56}$$

we have

$$V_1(t) \leq [V_1(0) - \frac{\rho_1}{\lambda_1}]e^{-\lambda_1 t} + \frac{\rho_1}{\lambda_1} + \frac{k_1}{\lambda_1}\sup_{\tau\in[0,t]}[z_2^2(\tau)] \tag{3.57}$$

Therefore, if z_2 can be regulated as bounded, we can obtain the boundedness of the term $e^{-\lambda_1 t}\int_0^t k_1 e^{\lambda_1 \tau} z_2^2(\tau)d\tau$. From (3.57), we can then claim that $V_1(t)$, z_1, $\hat{p}_{a,1}$, $\hat{\theta}_{a,1}$ are SGUUB.

Step 2: Since $z_2 = x_2 - \alpha_1$, the time derivative of z_2 is given by

$$\begin{aligned} \dot{z}_2 \ &= \ \dot{x}_2 - \dot{\alpha}_1 \\ &= \ g_2 x_3 + f_2(\bar{x}_2) - \dot{\alpha}_1 \end{aligned} \tag{3.58}$$

Again, by viewing $x_3(t)$ as a virtual control, we may design a control input α_2 for (3.58). Since $z_3(t) = x_3(t) - \alpha_2(t)$, we have

$$\dot{z}_2 = g_2(z_3 + \alpha_2) + f_2(\bar{x}_2) - \dot{\alpha}_1$$

Since α_1 is a function of x_1, y_d, $\dot{y}_d$ and $\hat{W}_1$, $\dot{\alpha}_1$ can be expressed as

$$\begin{aligned}\dot{\alpha}_1 &= \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{W}_1}\dot{\hat{W}}_1 \\ &= \frac{\partial \alpha_1}{\partial x_1}(g_1 x_2 + f_1) + \omega_1\end{aligned}$$

where

$$\omega_1 = \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{W}_1}\dot{\hat{W}}_1$$

then we have

$$\begin{aligned}\dot{z}_2 &= g_2(z_3 + \alpha_2) + f_2(\bar{x}_2) - \frac{\partial \alpha_1}{\partial x_1}(g_1 x_2 + f_1) - \omega_1 \\ &= g_2\Big[z_3 + \alpha_2 + \frac{1}{g_2}(f_2 - g_1\frac{\partial \alpha_1}{\partial x_1}x_2 - \frac{\partial \alpha_1}{\partial x_1}f_1 - \omega_1)\Big] \qquad (3.59)\end{aligned}$$

Consider the scalar smooth function $V_{z_2} = \frac{1}{2g_2}z_2^2$. Noting that $z_2 z_3 \leq \frac{1}{4k_2}z_2^2 + k_2 z_3^2$, $\forall k_2 > 0$, the time derivative of V_{z_2} along (3.59) is

$$\begin{aligned}\dot{V}_{z_2} &\leq \frac{1}{4k_2}z_2^2 + k_2 z_3^2 + z_2\Big[\alpha_2 + \frac{1}{g_2}(f_2 - g_1\frac{\partial \alpha_1}{\partial x_1}x_2 - \frac{\partial \alpha_1}{\partial x_1}f_1 - \omega_1)\Big] \\ &= z_2 z_3 + z_2\Big[\alpha_2 + Q_2(Z_2)\Big]\end{aligned} \qquad (3.60)$$

where

$$Q_2(Z_2) = \frac{1}{g_2}(f_2 - g_1\frac{\partial \alpha_1}{\partial x_1}x_2 - \frac{\partial \alpha_1}{\partial x_1}f_1 - \omega_1)$$

with $Z_2 = [\bar{x}_2, \alpha_1, \partial\alpha_1/\partial x_1, \omega_1]^T \in \Omega_{Z_2} \subset R^5$.

Consider the following Lyapunov function candidate

$$V_2 = \frac{1}{2g_2}z_2^2 + \frac{1}{2}(\hat{W}_2 - W_2^*)^T\Gamma_2^{-1}(\hat{W}_2 - W_2^*)$$

with matrix $\Gamma_2 = \Gamma_2^T > 0$.

Choose the following adaptive intermediate control law as

$$\alpha_2 = -c_2 z_2 - \hat{W}_2^T S(Z_2) \qquad (3.61)$$
$$\dot{\hat{W}}_2 = \Gamma_2[S(Z_2)z_2 - \sigma_2\hat{W}_2] \qquad (3.62)$$

where constant $c_2 = c_{20} + c_{21}$ with $c_{20}, c_{21} > 0$ and $c_{20}^* = c_{20} - \frac{1}{4k_2}$, and σ_2 is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Noting the following inequalities

$$-c_{21}z_2^2+\epsilon(Z_2)z_2 \le -c_{21}z_2^2+\epsilon_{z_2}^*|z_2| \le \frac{\epsilon_{z_2}^{*^2}}{4c_{21}}$$
$$-\sigma_2(\hat{W}_2-W_2^*)^T\hat{W}_2 \le -\frac{1}{2}\sigma_2\|\hat{W}_2-W_2^*\|^2+\frac{1}{2}\sigma_2\|W_2^*\|^2$$

we obtain

$$\begin{aligned}\dot{V}_2 &\le -c_{20}^*z_2^2-\frac{1}{2}\sigma_2\|\hat{W}_2-W_2^*\|^2+\frac{1}{2}\sigma_2\|W_2^*\|^2+\frac{\epsilon_{z_2}^{*^2}}{4c_{21}}+k_2z_3^2\\ &\le -\lambda_2V_2+\rho_2+k_2z_3^2\end{aligned}$$

where

$$\lambda_2 := \min\Big\{2g_{\min}c_{20}^*,\ \frac{\sigma_2}{\lambda_{\max}(\Gamma_2^{-1})}\Big\}$$
$$\rho_2 := \frac{1}{2}\sigma_2\|W_2^*\|^2+\frac{\epsilon_{z_2}^{*^2}}{4c_{21}}$$

Similarly, if z_3 can be regulated as bounded, we can conclude that z_2 is bounded, and so is z_1.

Step i ($3 \le i \le n-1$*):* Similar procedures are taken for each steps when $i = 3, \cdots, n-1$ as in Steps 1 and 2.

The time derivative of $z_i(t)$ is given by

$$\dot{z}_i = g_i(z_{i+1}+\alpha_i)+f_i(\bar{x}_i)-\dot{\alpha}_{i-1} \tag{3.63}$$

Since α_{i-1} is a function of $\bar{x}_{i-1}$, $\bar{x}_{di}$, $\hat{W}_1$, $\hat{W}_2$, ..., $\hat{W}_{i-1}$, $\dot{\alpha}_{i-1}$ can be expressed as

$$\begin{aligned}\dot{\alpha}_{i-1} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j+\frac{\partial\alpha_{i-1}}{\bar{x}_{di}}\dot{\bar{x}}_{di}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j\\ &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j+\omega_{i-1}\end{aligned}$$

where

$$\omega_{i-1} = \frac{\partial\alpha_{i-1}}{\bar{x}_{di}}\dot{\bar{x}}_{di}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j$$

then (3.63) becomes

$$\begin{aligned}\dot{z}_i &= g_i(z_{i+1}+\alpha_i)+f_i(\bar{x}_i)+\Delta_i(t,x)-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(g_jx_{j+1}+f_j)-\omega_{i-1}\\ &= g_i\Big\{z_{i+1}+\alpha_i+\frac{1}{g_i}\Big[f_i-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(g_jx_{j+1}+f_j)-\omega_{i-1}\Big]\Big\}\end{aligned} \tag{3.64}$$

Consider the scalar smooth function $V_{z_i} = \frac{1}{2g_i} z_i^2$. Noting that $z_i z_{i+1} \leq \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2$, $\forall k_i > 0$, the time derivative of V_{z_i} along (3.64) is

$$\begin{aligned} \dot{V}_{z_i} &\leq \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2 + z_i \Big\{ \alpha_i + \frac{1}{g_i} \Big[f_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} (g_j x_{j+1} + f_j) - \omega_{i-1} \Big] \Big\} \\ &= \frac{1}{4k_i} z_i^2 + k_i z_{i+1}^2 + z_i [\alpha_i + Q_i(Z_i)] \end{aligned} \tag{3.65}$$

where $Z_i = [\bar{x}_i, \alpha_{i-1}, \frac{\partial \alpha_{i-1}}{\partial x_1}, \frac{\partial \alpha_{i-1}}{\partial x_2}, ..., \frac{\partial \alpha_{i-1}}{\partial x_{i-1}}, \omega_{i-1}] \in \Omega_{Z_i} \subset R^{2i+1}$

Consider the following Lyapunov function candidate

$$V_i = \frac{1}{2g_i} z_i^2 + \frac{1}{2} (\hat{W}_i - W_i^*)^T \Gamma_i^{-1} (\hat{W}_i - W_i^*) \tag{3.66}$$

with matrix $\Gamma_i = \Gamma_i^T > 0$.

Choose the following adaptive intermediate control law as

$$\alpha_i = -c_i z_i - \hat{W}_i^T S(Z_i) \tag{3.67}$$

$$\dot{\hat{W}}_i = \Gamma_i [S(Z_i) z_i - \sigma_i \hat{W}_i] \tag{3.68}$$

where constant $c_i = c_{i0} + c_{i1}$ with $c_{i0}, c_{i1} > 0$ and $c_{i0}^* = c_{i0} - \frac{1}{4k_i}$, and σ_i is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Noting the following inequalities

$$-c_{i1} z_i^2 + \epsilon(Z_i) z_i \leq -c_{i1} z_i^2 + \epsilon_{z_i}^* |z_i| \leq \frac{\epsilon_{z_i}^{*^2}}{4c_{i1}}$$

$$-\sigma_i (\hat{W}_i - W_i^*)^T \hat{W}_i \leq -\frac{1}{2} \sigma_i \|\hat{W}_i - W_i^*\|^2 + \frac{1}{2} \sigma_i \|W_i^*\|^2$$

we obtain

$$\begin{aligned} \dot{V}_i &\leq -c_{i0}^* z_i^2 - \frac{1}{2} \sigma_i \|\hat{W}_i - W_i^*\|^2 + \frac{1}{2} \sigma_i \|W_i^*\|^2 + \frac{\epsilon_{z_i}^{*^2}}{4c_{i1}} + k_i z_{i+1}^2 \\ &\leq -\lambda_i V_i + \rho_i + k_i z_{i+1}^2 \end{aligned}$$

where

$$\lambda_i := \min \Big\{ 2g_{\min} c_{i0}^*, \ \frac{\sigma_i}{\lambda_{\max}(\Gamma_i^{-1})} \Big\}$$

$$\rho_i := \frac{1}{2} \sigma_i \|W_i^*\|^2 + \frac{\epsilon_{z_i}^{*^2}}{4c_{i1}}$$

Similarly, if z_{i+1} can be regulated as bounded, we can conclude that z_i is also bounded.

Step n: This is the final step, since the actual control u appears in the derivative of z_n as given by

$$\dot{z}_n = g_n u + f_n(x) - \dot{\alpha}_{n-1} \tag{3.69}$$

Since $\dot{\alpha}_{n-1}$ can be expressed as

$$\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} \dot{x}_j + \omega_{n-1}$$

where

$$\omega_{n-1} = \frac{\partial \alpha_{n-1}}{\partial \bar{x}_n} \dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{W}_j} \dot{\hat{W}}_j$$

then (3.69) becomes

$$\begin{aligned} \dot{z}_n &= g_n u + f_n(x) - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + f_j) - \omega_{n-1} \\ &= g_n \Big\{ u + \frac{1}{g_n} \Big[f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + f_j) - \omega_{n-1} \Big] \Big\} \end{aligned} \tag{3.70}$$

Consider the scalar smooth function $V_{z_n} = \frac{1}{2g_n} z_n^2$, whose time derivative along (3.70) is

$$\begin{aligned} \dot{V}_{z_n} &= z_n \Big\{ u + \frac{1}{g_n} \Big[f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + f_j) - \omega_{n-1} \Big] \Big\} \\ &= z_n [u + Q_n(Z_n)] \end{aligned} \tag{3.71}$$

where

$$Q_n(Z_n) = \frac{1}{g_n} \Big[f_n - \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + f_j) - \omega_{n-1} \Big]$$

with $Z_n = [x, \alpha_{n-1}, \frac{\partial \alpha_{n-1}}{\partial x_1}, \frac{\partial \alpha_{n-1}}{\partial x_2}, ..., \frac{\partial \alpha_{n-1}}{\partial x_{n-1}}, \omega_{n-1}] \in \Omega_{Z_n} \subset R^{2n+1}$.

Similarly, we consider the following Lyapunov function candidate

$$V_n = \frac{1}{2g_i} z_n^2 + \frac{1}{2} (\hat{W}_n - W_n^*)^T \Gamma_n^{-1} (\hat{W}_n - W_n^*) \tag{3.72}$$

with matrix $\Gamma_n = \Gamma_n^T > 0$.

Choose the following adaptive intermediate control law as

$$\alpha_n = -c_n z_n - \hat{W}_n^T S(Z_n) \tag{3.73}$$
$$\dot{\hat{W}}_i = \Gamma_i[S(Z_i)z_i - \sigma_i \hat{W}_i] \tag{3.74}$$

where constant $c_n = c_{n0} + c_{n1}$ with $c_{n0}, c_{i1} > 0$, and σ_i is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Noting the following inequalities

$$-c_{n1}z_n^2 + \epsilon(Z_n)z_n \le -c_{n1}z_n^2 + \epsilon_{z_n}^*|z_n| \le \frac{\epsilon_{z_n}^{*^2}}{4c_{n1}}$$
$$-\sigma_n(\hat{W}_n - W_n^*)^T\hat{W}_n \le -\frac{1}{2}\sigma_n\|\hat{W}_n - W_n^*\|^2 + \frac{1}{2}\sigma_n\|W_n^*\|^2$$

we obtain

$$\begin{aligned}\dot{V}_n &\le -c_{n0}z_n^2 - \frac{1}{2}\sigma_n\|\hat{W}_n - W_n^*\|^2 + \frac{1}{2}\sigma_n\|W_n^*\|^2 + \frac{\epsilon_{z_n}^{*^2}}{4c_{n1}} \\ &\le -\lambda_n V_n + \rho_n \end{aligned} \tag{3.75}$$

where

$$\lambda_n := \min\Big\{2g_{\min}c_{n0},\ \frac{\sigma_n}{\lambda_{\max}(\Gamma_n^{-1})}\Big\}$$
$$\rho_n := \frac{1}{2}\sigma_n\|W_n^*\|^2 + \frac{\epsilon_{z_n}^{*^2}}{4c_{n1}}$$

Theorem 3.3.1 shows the stability of control performance of the closed-loop adaptive systems.

Theorem 3.3.1 *Consider the closed-loop system consisting of the plant (3.44) under Assumptions 3.3.1 and 3.3.2. If we apply the controller (3.73) with NN weights updating law (3.74), we can guarantee the following properties under bounded initial conditions*

(i) z_i, $\hat{W}_i$, $i = 1, \cdots, n$, *and* $x(t)$ *are semi-globally uniformly ultimately bounded and the vector* $Z = [Z_1^T, ..., Z_n^T]^T$ *remains in a compact set* Ω_Z *specified as*

$$\begin{aligned}\Omega_Z = &\Big\{Z|\sum_{i=1}^n z_i^2 \le 2g_{\max}C_0, \sum_{i=1}^n \|\tilde{W}_i\|^2 \le \frac{2C_0}{\lambda_{\min}(\Gamma_i^{-1})}, \\ &\bar{x}_{di} \in \Omega_{di}, i = 2, ..., n\Big\}\end{aligned} \tag{3.76}$$

where $C_0 > 0$ is a constant whose size depends on the initial conditions (as will be defined later in the proof); and

(ii) the closed-loop signal $z = [z_1, ..., z_n]^T \in R^n$ will eventually converge to a compact set defined by

$$\Omega_S := \{z \mid \|z\|^2 \leq \mu\} \tag{3.77}$$

with $\mu > 0$ is a constant related to the design parameters and will be defined later in the proof, and Ω_S can be made as small as desired by an appropriate choice of the design parameters.

Proof: Consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^{n} \frac{1}{2g_i} z_i^2 + \frac{1}{2}(\hat{W}_i - W_i^*)^T \Gamma_i^{-1} (\hat{W}_i - W_i^*) \tag{3.78}$$

From the previous derivation, we have

$$\dot{V}_n(t) \leq -\lambda_n V_n(t) + \rho_n \tag{3.79}$$

it follows that

$$0 \leq V_n(t) \leq [V_n(0) - \frac{\rho_n}{\lambda_n}]e^{-\lambda_n t} + \frac{\rho_n}{\lambda_n} \leq V_n(0) + \bar{\rho}_n \tag{3.80}$$

with constants $\bar{\rho}_n = \rho_n/\lambda_n$ and $V_n(0) = \frac{1}{2g_n} z_n^2(0) + \frac{1}{2}\tilde{W}_n^T(0)\Gamma_n^{-1}\tilde{W}_a(0)$. From (3.72), we have $z_n^2 \leq 2g_{\max}V_n(t)$, and $\|\tilde{W}_n\|^2 \leq 2V_n(t)/\lambda_{\min}(\Gamma_n^{-1})$.

In Step $n-1$, we have obtained

$$\dot{V}_{n-1}(t) \leq -\lambda_{n-1}V_{n-1}(t) + \rho_{n-1} + k_{n-1}z_n^2 \tag{3.81}$$

As $z_n^2 \leq 2g_{\max}V_n(t)$ and $V_n(t) \leq V_n(0) + \bar{\rho}_n$, we have

$$\dot{V}_{n-1}(t) \leq -\lambda_{n-1}V_{n-1}(t) + \rho_{n-1} + 2k_{n-1}g_{\max}(V_n(0) + \bar{\rho}_n) \tag{3.82}$$

Letting $\bar{\rho}_{n-1} = [\rho_{n-1} + 2k_{n-1}g_{\max}(V_n(0) + \bar{\rho}_n)]/\lambda_{n-1}$, from (3.82), we have

$$V_{n-1}(t) \leq [V_{n-1}(0) - \bar{\rho}_{n-1}]e^{-\lambda_{n-1}t} + \bar{\rho}_{n-1} \leq V_{n-1}(0) + \bar{\rho}_{n-1} \tag{3.83}$$

Noting (3.66), it follows

$$z_{n-1}^2 \leq 2g_{\max}V_{n-1}(t) \leq 2g_{\max}(V_{n-1}(0) + \bar{\rho}_{n-1})$$

Similarly, we can conclude that for $i = 1, \cdots, n$

$$z_i^2 \leq 2g_{\max}(V_i(0) + \bar{\rho}_i), \quad \|\tilde{W}_i\|^2 \leq \frac{2(V_i(0) + \bar{\rho}_i)}{\lambda_{\min}(\Gamma_i^{-1})}$$

with $\bar{\rho}_i = [\rho_i + 2k_i g_{\max}(V_{i+1}(0) + \bar{\rho}_{i+1})]/\lambda_i$.

Considering (3.78), we know that

$$V(t) \leq C_0 \tag{3.84}$$

with $C_0 = \sum_{i=1}^n V_i(0) + \bar{\rho}_i$, from which we can conclude that z_i and $\hat{W}_i$ are bounded, $i = 1, \cdots, n$. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $x_2 = z_2 + \alpha_1$, since α_1 is function of bounded signals z_1, Z_1, $\hat{W}_1$, α_1 is thus bounded, which in turn leads to the boundedness of x_2. Following the same way, we can prove one by one that all α_{i-1} and x_i, $i = 3, ..., n$ are bounded. Therefore, the systems' states x_i, $i = 1, ..., n$ are bounded.

Considering (3.78), we know that

$$\sum_{i=1}^n z_i^2 \leq 2g_{\max}V(t), \quad \sum_{i=1}^n \|\tilde{W}_i\|^2 \leq \frac{2V(t)}{\lambda_{\min}(\Gamma_1^{-1}, ..., \Gamma_n^{-1})} \tag{3.85}$$

From (3.84) and (3.85), we readily have the compact set Ω_Z defined in (3.76) over which the NN approximation is carried out.

In addition, from (3.80) and (3.83), we have that $\lim_{t\to\infty} \|z\|^2 = 2g_{\max}\sum_{i=1}^n \bar{\rho}_i$. Let $\mu = 2g_{\max}\sum_{i=1}^n \bar{\rho}_i$. We can conclude that the vector z will eventually converge to the compact set Ω_S defined in (3.77). This completes the proof. $\diamondsuit$

3.3.3 Conclusion

In this section, decoupled adaptive neural network backstepping control has been presented as a general tool for control system design, and it has been successfully applied to a class of strict-feedback nonlinear systems with unknown system functions. It has been proved that the proposed systematic design method is able to guarantee semi-globally uniformly ultimately boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin. In addition, the residual set of each state based on new coordinate in the closed-loop can be determined respectively.

Chapter 4

Adaptive NN Control of Nonlinear Systems with Unknown Time Delays

4.1 Introduction

Adaptive control has proven its great capability in compensating for linearly parameterized uncertainties. To obtain global stability, some restrictions have to be made to system nonlinearities such as matching conditions [1], extended matching conditions [2], or growth conditions [3]. To overcome these restrictions, a recursive design procedure called adaptive backstepping design was developed in [5]. The overparametrization problem was then removed in [14] by introducing the concept of tuning function. Several adaptive approaches for nonlinear systems with triangular structures have been proposed in [15][117]. Robust adaptive backstepping control has been studied for certain class of nonlinear systems whose uncertainties are not only from parametric ones but also from unknown nonlinear functions [15][86][24] and among others.

For system $\dot{x} = f(x) + g(x)u$, the unknown function $g(x)$ causes great design difficulty in adaptive control. Based on feedback linearization, certainty equivalent control $u = [-\hat{f}(x) + v]/\hat{g}(x)$ is usually taken, where $\hat{f}(x)$ and $\hat{g}(x)$ are estimates of $f(x)$ and $g(x)$, and measures have to be taken to avoid controller singularity when $\hat{g}(x) = 0$. Although the system's virtual control coefficients are assumed to be unknown nonlinear functions of states, their signs are assumed to be known as strictly either positive or negative. Under this assumption, stable neural network controllers have been constructed in [51][118][119][31] and in [89][90] by estimating the derivation of the control Lyapunov function. To avoid the singularity problem, integral Lyapunov functions have been developed in [120][88], and semi-globally stable adaptive controllers are developed, which do not require the estimate of the unknown function $g(x)$. However, the controller design becomes quite complicated due to the introduction of the integral Lyapunov functions especially combined with backstepping design. In [121], a novel stable neural network control scheme was developed based on the simple quadratic Lyapunov function under mild assumptions on the system functions, by which the singularity problem was effectively avoided.

Practically, systems with time delays are frequently encountered (e.g., process control). Time-delayed linear systems have been intensively investigated as summarized in [122][92]. The existence of time delays may degrade the control performance and make the stabilization problem become more difficult. However, the useful tools such as linear matrix inequalities (LMIs) is hard to apply to nonlinear systems with time delays. Lyapunov design has been proven to be an effective tool in controller design for nonlinear systems. One major difficulty lies in the control of time-delayed nonlinear systems is that the delays are usually not perfectly known. One way to ensure stability robustness with respect to this uncertainty is to employ stability criteria valid for any nonnegative value of the delays, i.e., delay-independent results. A class of quadratic Lyapunov-Krasovskii functionals originated in [123] has been used early as checking criteria for time-delay systems' stability. The unknown time delays are the main issue to be dealt with for the extension of backstepping design to such kinds of systems. A stabilizing controller design based on the Lyapunov-Krasovskii functionals is presented in [109] for a class of nonlinear time-delay systems with a so-called "triangular structure". However, few attempts have been made towards the systems with unknown parameters or unknown nonlinear functions.

Motivated by previous works on the nonlinear systems with both unknown time delays and uncertainties from unknown nonlinear functions, we present in this chapter the practical adaptive controllers for a class of unknown nonlinear systems in a strict-feedback form with unknown time delays. Using appropriate Lyapunov-Krasovskii functionals in the Lyapunov function candidate, the uncertainties from unknown time delays are removed such that the design of the stabilizing control law is free from these uncertainties. In this way, the iterative backstepping design procedure can be carried out directly. Practical stability is introduced to solve the singularity problem [114][124][125] due to the appearance of $1/z_i$ or $1/z_i^2$ in the controller and the tracking error can be made to confine in a compact domain of attraction. Neural networks is utilized as an function approximator to tackle the uncertainties from unknown nonlinear functions and its feasibility of approximation is guaranteed in novelly defined compact sets. Time-varying control gains rather than fixed gains are chosen to guarantee the boundedness of all the signals in closed-loop system. Semi-globally uniformly ultimately boundedness (SGUUB) of the signals in the closed-loop system is obtained and the output of the systems is proven to converge to a small neighborhood of the desired trajectory.

To the best of our knowledge, there is little work dealing with such a kind of systems in the literature at present stage. The proposed method expands the class of nonlinear systems that can be handled using adaptive backstepping techniques. The main contributions of the chapter are:

(i) the use of integral or quadratic Lyapunov functions to avoid controller singularity problem commonly encountered in feedback linearization control;

(ii) the combination of Lyapunov-Krasovskii functional and the Young's inequality in eliminating the unknown time delay τ_i in the upper bounding function of the Lyapunov functional derivative, which makes neural network parametrization with known inputs possible;

(iii) the introduction of practical robust control to avoid possible singularity problem due to the appearance of $1/z_i$ or $1/z_i^2$ in the controller, by which it is guaranteed that the tracking error will be confined in a compact domain of attraction;

(iv) the use of neural networks as function approximators with its feasibility being guaranteed over the practical compact sets for the first time in the literature;

(vi) the choice of the time-varying control gains instead of fixed gains to guarantee the boundedness of all the signals in closed-loop systems.

The rest of the chapter is organized as follows.

In Section 4.2, the neural network control for a class of nonlinear time-delay system in strict-feedback form is presented. The problem formulation and preliminaries is given in Section 4.2.1. Section 4.2.2 gives a brief introduction of linearly parametrized neural networks. A robust adaptive controller design and its stability analysis are presented in Section 4.2.3. A simulation example is given in Section 4.2.4 followed by Section 4.2.5, which concludes this section.

In Section 4.3, the problem studied in Section 4.2 is revisited with quadratic Lyapunov function being used rather than the integral Lyapunov function chosen in Section 4.2. The problem is formulated in Section 4.3.1 followed by the controller design for first-order system, the controller design for nth-order system and the conclusion in Sections 4.3.2, 4.3.3 and 4.3.4 respectively.

4.2 Adaptive Neural Network Control

4.2.1 Problem Formulation and Preliminaries

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{cases} \dot{x}_i(t) = g_i(\bar{x}_i(t))x_{i+1}(t) + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)), \quad 1 \le i \le n-1 \\ \dot{x}_n(t) = g_n(x(t))u + f_n(x(t)) + h_n(x(t-\tau_n)) \end{cases} \tag{4.1}$$

where $\bar{x}_i = [x_1, x_2, ..., x_i]^T$, $x = [x_1, x_2, ..., x_n]^T \in R^n$ and $u \in R$ are the state variables and system input respectively, $g_i(\cdot)$, $f_i(\cdot)$ and $h_i(\cdot)$ are unknown smooth functions, and τ_i are unknown time delays of the states, $i = 1, ..., n$. The control objective is to design an adaptive controller for system (4.1) such that the state $x_1(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, ..., y_d^{(i)}]^T$, $i =$

$1, ..., n-1$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for the system's signals, unknown functions and reference signals.

Assumption 4.2.1 *The system states $x(t)$ and their partial time derivatives, $\dot{\bar{x}}_{n-1}(t)$, are all available for feedback.*

Remark 4.2.1 *Note that the requirement for $\dot{\bar{x}}_{n-1}(t)$ is a constraint but realistic for many physical systems as we are not requiring $\dot{x}_n$ which is directly influenced by the control.*

Assumption 4.2.2 *The signs of $g_i(\bar{x}_i)$ are known, and there exist constants g_{i0} and known smooth functions $\bar{g}_i(\bar{x}_i)$ such that $g_{i0} \leq |g_i(\bar{x}_i)| \leq \bar{g}_i(\bar{x}_i) < \infty$, $\forall \bar{x}_i \in R^i$.*

Remark 4.2.2 *Assumption 4.2.2 implies that smooth functions $g_i(\bar{x}_i)$ are strictly either positive or negative. In the following, we only consider the case when $g_{i0} \leq g_i(\bar{x}_i) \leq \bar{g}_i(\bar{x}_i)$, $\forall \bar{x}_i \in R^i$. Assumption 4.2.2 is reasonable because $g_i(\bar{x}_i)$ being away from zero are controllable conditions of system (4.1), which is made in most of control schemes [19]. For a given practical system, the upper bounds of $g_i(\bar{x}_i)$ are not difficult to determine by choosing $\bar{g}_i(\bar{x}_i)$ large enough. It should be emphasized that the low bounds g_{i0} are only required for analytical purposes, their true values are not necessarily known.*

Assumption 4.2.3 *The desired trajectory vectors $\bar{x}_{di}$, $i = 2, ..., n$ are continuous and available, and $\bar{x}_{di} \in \Omega_{di} \subset R^i$ with Ω_{di} known compact sets.*

Assumption 4.2.4 *The unknown smooth functions $h_i(\bar{x}_i(t))$ satisfy the following inequality $|h_i(\bar{x}_i(t))| \leq \sum_{j=1}^{i} |x_j(t)| \varrho_{ij}(\bar{x}_i(t))$ where $\varrho_{ij}(\cdot)$ are known smooth functions.*

Assumption 4.2.5 *The size of the unknown time delays is bounded by a known constant, i.e., $\tau_i \leq \tau_{\max}$, $i = 1, ..., n$.*

Remark 4.2.3 *There are many physical processes which are governed by nonlinear differential equations of the form (4.1). Examples are recycled reactors, recycled storage tanks and cold rolling mills [92]. In general, most of the recycling processes inherit delays in their state equations.*

4.2.2 Linearly Parametrized Neural Networks

A function approximator shall be used to approximate the unknown nonlinear functions. There are two basic types of artificial neural networks, (i) linearly parametrized neural networks (LPNNs) and (ii) multilayer neural networks (MNNs). In control engineering, the Radial Basis Function (RBF) neural network (NN) as a kind of LPNNs is usually used as a tool for modeling nonlinear functions because of its nice approximation properties. The RBF NN can be considered as a two-layer network in which the hidden layer performs a fixed nonlinear transformation with no adjustable parameters, i.e., the input space is mapped into a new space. The output layer then combines the outputs in the latter space linearly. Therefore, it belongs to a class of linearly parameterized networks. In this section, the following RBF NN [46] is used to approximate the continuous function $h(Z) : R^q \rightarrow R$,

$$h_{nn}(Z) = W^T S(Z) \tag{4.2}$$

where the input vector $Z \in \Omega_Z \subset R^q$, weight vector $W = [w_1, w_2, \cdots, w_l]^T \in R^l$, the NN node number $l > 1$; and $S(Z) = [s_1(Z), \cdots, s_l(Z)]^T$, with $s_i(Z)$ being chosen as the commonly used Gaussian functions, which have the form

$$s_i(Z) = \exp\left[\frac{-(Z-\mu_i)^T(Z-\mu_i)}{\eta_i^2}\right], \quad i = 1, 2, \cdots, l$$

where $\mu_i = [\mu_{i1}, \mu_{i2}, \cdots, \mu_{iq}]^T$ is the center of the receptive field and η_i is the width of the Gaussian function. Universal approximation results in [45, 126] indicate that, if l is chosen sufficiently large, $W^T S(Z)$ can approximate any continuous function, $h(Z)$, to any desired accuracy over a compact set $\Omega_Z \subset R^q$ to arbitrary accuracy in the form of

$$h(Z) = W^{*T} S(Z) + \epsilon(Z), \quad \forall Z \in \Omega_Z \subset R^q \tag{4.3}$$

where W^* is the ideal constant weight vector, and $\epsilon(Z)$ is the approximation error which is bounded over a compact set, i.e., $|\epsilon(Z)| \leq \epsilon^*$, $\forall Z \in \Omega_Z$ where $\epsilon^* > 0$ is an unknown constant. The ideal weight vector W^* is an "artificial" quantity required for analytical purposes. W^* is defined as the value of W that minimizes $\epsilon(Z_1)$ for all $Z \in \Omega_Z \subset R^q$, i.e.,

$$W^* := \arg\min_{W \in R^l} \{ \sup_{Z \in \Omega_Z} |h(Z) - W^T S(Z)| \}.$$

The stability results obtained in NN control literature are semi-global in the sense that, as long as the input variables Z of the NNs remain within some pre-fixed compact set, $\Omega_Z \subset R^q$, where the compact set Ω_Z can be made as large as desired, there exists controller(s) with sufficiently large number of NN nodes such that all the signals in the closed-loop remain bounded.

It should be noted that RBF neural networks can be replaced by any linearly parameterized networks without any technical difficulty such as fuzzy systems, polynomial, splines and wavelet networks.

4.2.3 Adaptive NN Controller Design

In this section, adaptive neural control is proposed for system (4.1) and the stability results of the closed-loop system are presented. The backstepping design procedure contains n steps. The design of adaptive control laws is based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$, where $\alpha_i(t)$ is an intermediate control which shall be developed for the corresponding i-th subsystem based on an appropriate Lyapunov function $V_i(t)$. The control law $u(t)$ is designed in the last step to stabilize the whole closed-loop system based on the overall Lyapunov function V_n, which is the sum of the previous $V_i(t)$, $i = 1, ..., n-1$.

Define $g_{i\gamma}^{-1}(\bar{x}_i) = \frac{\gamma_i(\bar{x}_i)}{g_i(\bar{x}_i)}$, where $\gamma_i(\bar{x}_i) : R^i \to R_+$ is a smooth weighting function to be defined later. For notation $g_\gamma^{-1}(x)$, $g^{-1}(x)$ indicates $\frac{1}{g(x)}$, and the subscript $(*)_\gamma$ denotes the multiplication operation, then $(g_{i\gamma}^{-1})^2 = \frac{\gamma_i^2}{g_i^2}$. Based on the definition of new coordinates z_i, $i = 1, \cdots, n$, the following integral scalar function will be used in the controller design [52, 88]

$$V_{z_i} = \int_0^{z_i} \sigma g_{i\gamma}^{-1}(\bar{x}_{i-1}, \sigma + \alpha_{i-1}) d\sigma, \quad i = 1, \cdots, n \tag{4.4}$$

The choice of weighting function $\gamma_i(\cdot)$ is a key step in the controller design. The resulting controller is not unique and the control performance also varies with the different choice of $\gamma_i(\cdot)$. The apparent and convenient choices for $\gamma_i(\cdot)$ are 1 and $\bar{g}_i(\bar{x}_i)$ for general nonlinear systems. Detailed explanation will be given based on weighing function $\gamma_i(\bar{x}_i) = \bar{g}_i(\bar{x}_i)$ in the following, while a remark will be given directly addressing the controller design, relevant stability and performance analysis for $\gamma_i(\bar{x}_i) = 1$ without derivation for conciseness.

By choosing $\gamma_i(\bar{x}_i) = \bar{g}_i(\bar{x}_i)$, we have $g_{i\gamma}^{-1}(\bar{x}_i) = \frac{\bar{g}_i(\bar{x}_i)}{g_i(\bar{x}_i)}$. From Assumption 4.2.2, we know that $g_{i\gamma}^{-1}(\bar{x}_i)$ are bounded by known functions as $1 < g_{i\gamma}^{-1}(\bar{x}_i) \le \frac{\bar{g}_i(\bar{x}_i)}{g_{i0}}$. Clearly, V_{z_i} are positive definite functions, $i = 1, ..., n$.

In this Section, the following inequalities play an important role, $i = 1, ..., n$

$$\sigma_i(\hat{W}_i - W_i^*)^T(\hat{W}_i - W_i^0) \le \frac{1}{2}\sigma_i\|\hat{W}_i - W_i^*\|^2 - \frac{1}{2}\sigma_i\|W_i^* - W_i^0\|^2 \tag{4.5}$$

$$\epsilon_i(Z_i)z_i(t) \le \frac{1}{2}\epsilon_{z_i}^{*2} + \frac{1}{2}z_i^2(t) \tag{4.6}$$

and the following even function $p_i(\cdot) : R \to R$ is introduced for the purpose of the practical controller design in Section 4.2.3:

$$p_i(x) = \begin{cases} 1, & |x| \ge c_{z_i} \\ 0, & |x| < c_{z_i} \end{cases} \quad \forall x \in R. \tag{4.7}$$

Step 1: Let us first consider the equation in (4.1) when $i = 1$, i.e.,

$$\dot{x}_1(t) = g_1(x_1(t))x_2(t) + f_1(x_1(t)) + h_1(x_1(t - \tau_1))$$

From the definition for new states z_1 and z_2, i.e. $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, we have

$$\dot{z}_1(t) = g_1(x_1(t))(z_2(t) + \alpha_1(t)) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) - \dot{y}_d(t) \tag{4.8}$$

According to (4.4), consider the following scalar smooth function

$$V_{z_1}(t) = \int_0^{z_1} \sigma g_{1\gamma}^{-1}(\sigma + y_d)d\sigma$$

By variable change $\sigma = \theta z_1$, we may rewrite V_{z_1} as $V_{z_1} = z_1^2 \int_0^1 \theta g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta$. Noting that $1 \le g_{1\gamma}^{-1}(\theta z_1 + y_d) \le \bar{g}_1(\theta z_1 + y_d)/g_{10}$, we have

$$\frac{z_1^2}{2} \le V_{z_1} \le \frac{z_1^2}{g_{10}} \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d)d\theta \tag{4.9}$$

The time derivative of V_{z_1} is

$$\dot{V}_{z_1}(t) = z_1(t)g_{1\gamma}^{-1}(x_1(t))\dot{z}_1(t) + \int_0^{z_1} \sigma \frac{\partial g_{1\gamma}^{-1}(\sigma + y_d)}{\partial y_d}\dot{y}_d d\sigma$$

Noting (4.8) and integration by parts, we have

$$\begin{aligned}
\dot{V}_{z_1}(t) &= z_1(t)g_{1\gamma}^{-1}(x_1(t))\Big[g_1(x_1(t))(z_2(t) + \alpha_1(t)) + f_1(x_1(t)) \\
&\quad + h_1(x_1(t-\tau_1)) - \dot{y}_d(t)\Big] + \dot{y}_d(t)\Big[\ \sigma g_{1\gamma}^{-1}(\sigma + y_d)\Big|_0^{z_1} - \int_0^{z_1} g_{1\gamma}^{-1}(\sigma + y_d)d\sigma\Big] \\
&= z_1(t)\Big[\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) \\
&\quad + g_{1\gamma}^{-1}(x_1(t))h_1(x_1(t-\tau_1)) - \dot{y}_d(t)\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta\Big]
\end{aligned}$$

Applying Assumption 4.2.4, we have

$$\begin{aligned}
\dot{V}_{z_1}(t) &\leq z_1(t)[\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) \\
&\quad - \dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta] \\
&\quad + |z_1(t)|g_{1\gamma}^{-1}(x_1(t))|x_1(t-\tau_1)|\varrho_{11}(x_1(t-\tau_1))
\end{aligned} \tag{4.10}$$

By using Young's Inequality, (4.10) becomes

$$\begin{aligned}
\dot{V}_{z_1}(t) &\leq z_1(t)[\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) \\
&\quad - \dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta] + \frac{1}{2}z_1^2(t)[g_{1\gamma}^{-1}(x_1(t))]^2 \\
&\quad + \frac{1}{2}x_1^2(t-\tau_1)\varrho_{11}^2(x_1(t-\tau_1))
\end{aligned} \tag{4.11}$$

In standard iterative backstepping design, $\alpha_1(t)$ is usually designed to stabilize the z_1-subsystem except for the coupling term $\bar{g}_1 z_1 z_2$ to be dealt with in the next step. In doing so under the assumption of known functions, one more difficulty exists in the new problem setting. Although $\varrho_{11}(\cdot)$ is a known function, it cannot be utilized in designing $\alpha_1(t)$ as $x_1(t-\tau_1)$ is undetermined because of unknown time delay τ_1.

Intuitively, approximation methods such as neural networks can be used to approximate the unknown functions. The unknown functions $g_1(\cdot)$ and $f_1(\cdot)$ can be dealt with in this way without any problem. However, due to the existence of the unknown time delay τ_1, functions of $x_1(t-\tau_1)$ are hard to be approximated using neural networks since the input $x_1(t-\tau_1)$ is undetermined because of the uncertain τ_1. To overcome the design difficulties from the unknown time delay τ_1, the following Lyapunov-Krasovskii functional is considered

$$V_{U_1}(t) = \int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau$$

where $U_1(\cdot)$ is a positive definite function chosen as

$$U_1(x_1(t)) = \frac{1}{2}x_1^2(t)\varrho_{11}^2(x_1(t))$$

The time derivative of $V_{U_1}(t)$ is

$$\dot{V}_{U_1}(t) = U_1(x_1(t)) - U_1(x_1(t-\tau_1)) \tag{4.12}$$

which can be used to cancel the time-delay term on the right hand side of (4.11) and thus eliminate the design difficulty from the unknown time delay τ_1 without introducing any uncertainties to the system. Accordingly, we obtain

$$\begin{aligned}\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) \leq\ & z_1(t)[\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) \\ & -\dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta + \frac{1}{2}z_1(t)[g_{1\gamma}^{-1}(x_1(t))]^2 \\ & +\frac{1}{2z_1(t)}x_1^2(t)\varrho_{11}^2(x_1(t))]\end{aligned} \tag{4.13}$$

Comparing (4.13) with (4.11), it is found that the difficulty from the unknown time delay τ_1 has been eliminated by introducing the Lyapunov-Krasovskii functional $V_{U_1}(t)$. By differentiating $V_{U_1}(t)$ with respect to time, the unknown time delay term $U_1(x_1(t-\tau_1)) = \frac{1}{2}x_1^2(t-\tau_1)\varrho_{11}^2(x_1(t-\tau_1))$ appeared in (4.12) can be used for exact cancellation on the right hand side of (4.11). The remaining term $U_1(x_1(t))$ from $\dot{V}_{U_1}(t)$ is a known function of known variables, which does not introduce any uncertainties to the system. Therefore, the design of intermediate control $\alpha_1(t)$ is free from unknown time delay τ_1 at present stage.

For conciseness of notation, we will omit the time variables t and $t-\tau_1$ after time-delay terms have been eliminated. Under the assumption of exact knowledge, the certainty equivalent control is

$$\alpha_1^* = \frac{1}{\bar{g}_1(x_1)}[-k_1(t)z_1 - Q_1(Z_1)] \tag{4.14}$$

where

$$Q_1(Z_1) = g_{1\gamma}^{-1}(x_1)f_1(x_1) - \dot{y}_d\int_0^1 g_{1\gamma}^{-1}(\theta z_1 + y_d)d\theta + \frac{1}{2}z_1(g_{1\gamma}^{-1})^2 + \frac{1}{2z_1}x_1^2\varrho_{11}^2(x_1)$$

with $Z_1 = [x_1, y_d, \dot{y}_d]^T \in \Omega_{Z_1} \subset R^3$ and $\Omega_{Z_1} := \{z_1, \bar{x}_{d2} | z_1 \in R, \bar{x}_{d2} \in \Omega_{d2}\}$. It is apparent that controller singularity may occur. In addition, it is certain that α_1^* is not an admissible control, since α_1^* is not well-defined when $z_1 = 0$ as $\lim_{z_1 \to 0} 2z_1 = 0$, $\lim_{z_1 \to 0} x_1^2 \varrho_{11}^2(x_1) \neq 0$ and L'Hopital's rule [112] is not applicable to obtain the limit $\lim_{z_1 \to 0} \frac{x_1^2 \varrho_{11}^2(x_1)}{2z_1}$. Therefore, care must be taken to guarantee the boundedness of the controller.

It is noted that point $z_1 = 0$ is not only an isolated point in Ω_{Z_1}, but also the case that the system reaches the origin at this point. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. For ease of discussion, let us define sets $\Omega_{c_{z_1}} \subset \Omega_{Z_1}$ and $\Omega_{Z_1}^0$ as follows

$$\Omega_{c_{z_1}} := \{z_1 \,\big|\, |z_1| < c_{z_1}\} \tag{4.15}$$

$$\Omega_{Z_1}^0 := \Omega_{Z_1} - \Omega_{c_{z_1}} \tag{4.16}$$

where c_{z_1} is a constant that can be chosen arbitrarily small and "$-$" in (4.16) is used to denote the complement of set B in set A as $A - B := \{x | x \in A \text{ and } x \notin B\}$.

Accordingly, the following practical control law is proposed

$$\alpha_1^* = \frac{p_1(z_1)}{\bar{g}_1(x_1)}[-k_1(t)z_1 - Q_1(Z_1)] \tag{4.17}$$

where $p_i(\cdot)$ is defined in (4.7).

Since $f(\cdot)$ and $g(\cdot)$ are unknown smooth functions, the desired practical control α_1^* in (4.17) cannot be implemented in practice. Neural networks can be used to approximate the unknown function $Q_1(Z_1)$. Note that control action is only activated when $z_1 \in \Omega_{Z_1}^0$, which means unknown function $Q_1(Z_1)$ is approximated by neural networks over the set $\Omega_{Z_1}^0$. According to the main result stated in [127], any real-valued continuous function can be arbitrarily closely approximated by a network of RBF type over a compact set. The compactness of set $\Omega_{Z_1}^0$ is a must to guarantee the feasibility of neural networks approximation.

The following lemma shows the compactness of set $\Omega_{z_1}^0$, which is useful to reconstruct the compact domain of neural network approximation.

Lemma 4.2.1 *Set $\Omega_{Z_1}^0$ is a compact set.*

Proof: First, we show that $\Omega^0_{Z_1}$ is a *closed* set. From(4.16) and applying De Morgan's laws, we have

$$\Omega^{0c}_{Z_1} = \Omega^c_{Z_1} \cup \Omega_{c_{z_1}} \tag{4.18}$$

where $\Omega^{0c}_{Z_1}$ and $\Omega^c_{Z_1}$ denote the complements of $\Omega^0_{Z_1}$ and Ω_{Z_1} respectively. Since Ω_{Z_1} is a *compact* set, i.e., it is *closed* and *bounded* ([128], Theorem 1.6), $\Omega^c_{Z_1}$ is an *open* set. In addition, $\Omega_{c_{z_1}}$ is also an open set from its definition. From (4.18), we know that $\Omega^{0c}_{Z_1}$ is an open set, which means that its complement $\Omega^0_{Z_1}$ is a closed set. Second, from (4.16), we know that $\Omega^0_{Z_1} \subset \Omega_{Z_1}$. Since a closed subset of a compact set is compact ([128], Remark 1.30) , we can conclude that $\Omega^0_{Z_1}$ is a compact set. $\diamondsuit$

Based on Lemma 4.2.1, it is known that $Q_1(Z_1)$ is continuous and well-defined over compact set $\Omega^0_{Z_1}$ and can be approximated by neural networks to arbitrary any accuracy as follows

$$Q_1(Z_1) = W_1^{*T} S(Z_1) + \epsilon_1(Z_1)$$

where $\epsilon_1(Z_1)$ is the approximation error. As the ideal weight W_1^* is unknown, we shall use its estimate $\hat{W}_1$ instead, which forms the intermediate control α_1 as

$$\alpha_1(t) = \frac{p_1(z_1)}{\bar{g}_1(x_1)}[-k_1(t)z_1 - \hat{W}_1^T S(Z_1)] \tag{4.19}$$

Note that $Q_1(Z_1)$ contains unknown functions as well as known ones and is approximated by NN as a whole. In doing so, although we may have lost some useful information of the system by lumping the known terms into unknown terms, the possibly controller singularity problem is effectively avoided. The scheme also applies to the following steps. To demonstrate the power of approximation-based control law, we would like to present the following arguments.

Remark 4.2.4 *In Section 4.2, we are to present an adaptive neural network controller that is well-defined and guarantee the boundedness of all the signals in the closed-loop. In fact, in order to achieve the convergence of tracking error to zero, the desired controller α_1^* in (4.14) is not well-defined when $z_1 = 0$, under the assumption of exact knowledge by following the standard derivation. Alternatively, we have to relax our control objective to a small ball of origin rather than the origin. It is really a pity for the powerful model-based control. However, we find that* *problem can be elegantly solved by using approximation-based controller design over redefined compact sets although only SGUUB can be guaranteed.*

Remark 4.2.5 *It is noted that the tracking problem is discussed throughout the section. If the regulation problem is discussed, the change of coordinates will be* $z_1 = x_1$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \cdots, n$, *in which the slight difference from tracking problems lies in the definition of* z_1. *In this case, (4.13) becomes*

$$\begin{aligned}\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) \;\le\;& z_1(t)\Big\{\bar{g}_1(x_1(t))z_2(t) + \bar{g}_1(x_1(t))\alpha_1(t) + g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) \\ &+\frac{1}{2}z_1(t)[g_{1\gamma}^{-1}(x_1(t))]^2 + \frac{1}{2}x_1(t)\varrho_{11}^2(x_1(t))\Big\} \qquad (4.20)\end{aligned}$$

Comparing (4.20) with (4.13), it is found that the term $\frac{1}{2z_1}x_1^2\varrho_{11}^2(x_1)$ *actually becomes* $\frac{1}{2}x_1\varrho_{11}^2(x_1)$. *This is due to the cancellation of* z_1 *and* x_1 *to each other. In this case, the desired intermediate control* $\alpha_1(t)$ *can be chosen as*

$$\alpha_1^*(t) = \frac{1}{\bar{g}_1(x_1)}\Big[-k_1(t)z_1(t) - Q_1(Z_1(t)) - \frac{1}{2}x_1(t)\varrho_{11}^2(x_1(t))\Big]$$

where

$$Q_1(Z_1) = g_{1\gamma}^{-1}(x_1(t))f_1(x_1(t)) + \frac{1}{2}z_1(t)[g_{1\gamma}^{-1}(x_1(t))]^2$$

Note that (i) no controller singularity problem will occur; and (ii) useful system information is utilized as much as possible as well-defined and known term $\frac{1}{2}x_1(t)\varrho_{11}^2(x_1(t))$ *is used for constructing the control* α_1^* *rather than being incorporated into* $Q_1(Z_1)$ *as unknown function. However, in the rest of steps of the iterative backstepping design for regulation problem, similar controller singularity problems from possibly singular terms will still occur. This is because that the cancellation of* z_i *to* x_i *will no longer be possible since* $z_i \neq x_i$ *for* $i = 2, \cdots, n$. *The controller singularity problem can only be solved using the techniques stated before for the rest of steps.*

For uniformity of notation, we define sets $\Omega_{c_{z_i}} \subset \Omega_{Z_i}$ and $\Omega_{Z_i}^0$, $i = 2, ..., n$ as

$$\Omega_{c_{z_i}} := \{z_i \,\big|\, |z_i| < c_{z_i}\} \qquad (4.21)$$

$$\Omega_{Z_i}^0 := \Omega_{Z_i} - \Omega_{c_{z_i}} \qquad (4.22)$$

Note that the control objective is to show that certain compact set Ω_S is domain of attraction in the sense that for all bounded initial conditions, there exists Ω_S such that all closed-loop signals will eventually converge to Ω_S. i.e., all $Z_i(t)$ starting from within $\Omega^0_{Z_i}$ will enter into Ω_S and will stay within Ω_S thereafter.

In the following steps, α_i is designed for i-th subsystem, $i = 2, \cdots, n$ and $u(t)$ is designed for n-th subsystem, and the unknown functions $Q_i(Z_i)$, $i = 2, ..., n$ will be approximated by neural networks as

$$Q_i(Z_i) = W_i^{*T} S(Z_i) + \epsilon_i(Z_i), \quad \forall Z_i \in \Omega^0_{Z_i} \tag{4.23}$$

Consider the Lyapunov function candidate $V_1(t)$ as

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \frac{1}{2}(\hat{W}_1(t) - W_1^*)^T \Gamma_1^{-1} (\hat{W}_1(t) - W_1^*) \tag{4.24}$$

Its time derivative along (4.13), (4.19) and (4.23) for $z_1 \in \Omega^0_{Z_1}$ is

$$\begin{aligned} \dot{V}_1 \;\; \leq \;\; & -k_1(t) z_1^2 + \bar{g}_1(x_1) z_1 z_2 - (\hat{W}_1 - W_1^*)^T S(Z_1) z_1 + z_1 \epsilon_{z_1} \\ & + (\hat{W}_1 - W_1^*)^T \Gamma_1^{-1} \dot{\hat{W}}_1 \end{aligned} \tag{4.25}$$

The following practical adaptive law is given for on-line tuning the NN weights

$$\dot{\hat{W}}_1 = p_1(z_1) \Gamma_1 [S(Z_1) z_1 - \sigma_1 (\hat{W}_1 - W_1^0)] \tag{4.26}$$

where σ_1 is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Substituting (4.26) into (4.25) and using (4.5) and (4.6), we have

$$\dot{V}_1 \leq -k_1(t) z_1^2 - \frac{1}{2}\sigma_1 \|\hat{W}_1 - W_1^*\|^2 + \bar{g}_1(x_1) z_1 z_2 + c_1 \tag{4.27}$$

where

$$c_1 := \frac{1}{2}\sigma_1 \|W_1^* - W_1^0\|^2 + \frac{1}{2}\epsilon_{z_1}^{*2}$$

For $z_1 \in \Omega^0_{Z_1}$, noting (4.9) and choosing

$$k_1(t) = \frac{1}{\varepsilon_{10}}[1 + \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d) d\theta + \frac{1}{z_1^2}\int_{t-\tau_{\max}}^t \frac{1}{2} x_1^2(\tau) \varrho_{11}^2(x_1(\tau)) d\tau] \tag{4.28}$$

with $0 < \varepsilon_{10} \leq 2$, we have

$$\begin{aligned} \dot{V}_1 \;\; \leq \;\; & -\frac{1}{\varepsilon_{10}} z_1^2 - \frac{g_{10}}{\varepsilon_{10}} V_{z_1} - \frac{1}{\varepsilon_{10}} \int_{t-\tau_1}^t \frac{1}{2} x_1^2(\tau) \varrho_{11}^2(x_1(\tau)) d\tau \\ & -\frac{1}{2}\sigma_1 \|\hat{W}_1 - W_1^*\|^2 + \bar{g}_1(x_1) z_1 z_2 + c_1 \end{aligned} \tag{4.29}$$

Since $[t-\tau_1, t] \subset [t-\tau_{\max}, t]$, we have the inequality

$$\int_{t-\tau_1}^{t} \frac{1}{2}x_1^2(\tau)\varrho_{11}^2(x_1(\tau))d\tau \leq \int_{t-\tau_{\max}}^{t} \frac{1}{2}x_1^2(\tau)\varrho_{11}^2(x_1(\tau))d\tau$$

Accordingly, (4.29) becomes

$$\dot{V}_1 \leq \bar{g}_1(x_1)z_1z_2 - \frac{g_{10}}{\varepsilon_{10}}V_{z_1} - \frac{1}{\varepsilon_{10}}V_{U_1} - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_1 \tag{4.30}$$

where the coupling term $\bar{g}_1(x_1)z_1z_2$ will be handled in the next step.

Remark 4.2.6 *Applying Young's inequality, we known that* $\bar{g}_1(x_1)z_1z_2 \leq \frac{1}{2}z_1^2 + \frac{1}{2}\bar{g}_1^2(x_1)z_2^2$. *The choice for* ε_{10} *is to guarantee that* $-(\frac{1}{\varepsilon_{10}} - \frac{1}{2})z_1^2 \leq 0$ *so that the undesired destabilizing term* $\frac{1}{2}z_1^2$ *can be suppressed.*

Step 2: Since $z_2 = x_2 - \alpha_1$, the time derivative of z_2 is given by

$$\begin{aligned}\dot{z}_2(t) &= \dot{x}_2(t) - \dot{\alpha}_1(t)\\ &= g_2(\bar{x}_2(t))x_3(t) + f_2(\bar{x}_2(t)) + h_2(\bar{x}_2(t-\tau_2)) - \dot{\alpha}_1(t)\end{aligned} \tag{4.31}$$

Again, by viewing $x_3(t)$ as a virtual control, we may design a control input α_2 for (4.31). Since $z_3(t) = x_3(t) - \alpha_2(t)$, we have

$$\dot{z}_2(t) = g_2(\bar{x}_2(t))(z_3(t) + \alpha_2(t)) + f_2(\bar{x}_2(t)) + h_2(\bar{x}_2(t-\tau_2)) - \dot{\alpha}_1(t)$$

Consider the following scalar function

$$V_{z_2}(t) = \int_0^{z_2} \sigma g_{2\gamma}^{-1}(x_1, \sigma + \alpha_1)d\sigma$$

Its time derivative is given by

$$\begin{aligned}\dot{V}_{z_2} &= \frac{\partial V_{z_2}}{\partial z_2}\dot{z}_2 + \frac{\partial V_{z_2}}{\partial x_1}\dot{x}_1 + \frac{\partial V_{z_2}}{\partial \alpha_1}\dot{\alpha}_1\\ &= z_2 g_{2\gamma}^{-1}(\bar{x}_2)\dot{z}_2 + \int_0^{z_2}\sigma\Big[\frac{\partial g_{2\gamma}^{-1}(x_1, \sigma+\alpha_1)}{\partial x_1}\dot{x}_1\\ &\quad + \frac{\partial g_{2\gamma}^{-1}(x_1, \sigma+\alpha_1)}{\partial \alpha_1}\dot{\alpha}_1\Big]d\sigma\end{aligned} \tag{4.32}$$

Noting that

$$\int_0^{z_2}\sigma\frac{\partial g_{2\gamma}^{-1}(x_1, \sigma+\alpha_1)}{\partial x_1}\dot{x}_1 d\sigma = z_2^2\dot{x}_1\int_0^1\theta\frac{\partial g_{2\gamma}^{-1}(x_1, \theta z_2+\alpha_1)}{\partial x_1}d\theta$$

$$\int_0^{z_2}\sigma\frac{\partial g_{2\gamma}^{-1}(x_1, \sigma+\alpha_1)}{\partial \alpha_1}\dot{\alpha}_1 d\sigma = \dot{\alpha}_1\left[z_2 g_{2\gamma}^{-1}(\bar{x}_2) - z_2\int_0^1 g_{2\gamma}^{-1}(x_1, \theta z_2+\alpha_1)d\theta\right]$$

$$\dot{\alpha}_1 = \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \omega_1, \quad \omega_1 = \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{W}_1}\dot{\hat{W}}_1$$

and using (4.31), (4.32) becomes

$$\begin{aligned}\dot{V}_{z_2}(t) &= z_2(t)\Big[\bar{g}_2(\bar{x}_2(t))(z_3(t)+\alpha_2(t)) \\ &\quad + g_{2\gamma}^{-1}(\bar{x}_2(t))f_2(\bar{x}_2(t)) + g_{2\gamma}^{-1}(\bar{x}_2(t))h_2(\bar{x}_2(t-\tau_2)) \\ &\quad + \dot{x}_1 z_2(t)\int_0^1 \theta\frac{\partial g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)}{\partial x_1}d\theta - \dot{\alpha}_1\int_0^1 g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)d\theta\Big]\end{aligned}$$

Noting Assumption 4.2.4, we have

$$\begin{aligned}\dot{V}_{z_2}(t) &= z_2(t)\Big[\bar{g}_2(\bar{x}_2(t))(z_3(t)+\alpha_2(t)) + g_{2\gamma}^{-1}(\bar{x}_2(t))f_2(\bar{x}_2(t)) + \frac{1}{2}z_2(t)(g_{2\gamma}^{-1})^2 \\ &\quad + \dot{x}_1 z_2(t)\int_0^1 \theta\frac{\partial g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)}{\partial x_1}d\theta - \dot{\alpha}_1\int_0^1 g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)d\theta\Big] \\ &\quad + \frac{1}{2}\sum_{j=1}^{2}x_j^2(t-\tau_2)\varrho_{2j}^2(\bar{x}_2(t-\tau_2))\end{aligned} \tag{4.33}$$

Consider the following Lyapunov function candidate $V_2(t)$ as

$$V_2(t) = V_1(t) + V_{z_2}(t) + V_{U_2}(t) + \frac{1}{2}(\hat{W}_2(t)-W_2^*)^T\Gamma_2^{-1}(\hat{W}_2(t)-W_2^*)$$

where

$$V_{U_2}(t) = \int_{t-\tau_2}^{t} U_2(\bar{x}_2(\tau))d\tau$$

with $U_2(\cdot)$ being a positive definite function which is defined by

$$U_2(\bar{x}_2(t)) = \frac{1}{2}\sum_{j=1}^{2}x_j^2(t)\varrho_{2j}^2(\bar{x}_2(t))$$

Its time derivative along (4.30) and (4.33) for $z_2 \in \Omega_{Z_2}^0$ is

$$\begin{aligned}\dot{V}_2 &\le \bar{g}_2(\bar{x}_2)z_2z_3 + z_2\Big[\bar{g}_1(x_1)z_1 + \bar{g}_2(\bar{x}_2)\alpha_2 + Q_2(Z_2)\Big] + (\hat{W}_2-W_2^*)^T\Gamma_2^{-1}\dot{\hat{W}}_2 \\ &\quad - \frac{g_{10}}{\varepsilon_{10}}V_{z_1} - \frac{1}{\varepsilon_{10}}V_{U_1} - \frac{1}{2}\sigma_1\|\hat{W}_1-W_1^*\|^2 + \frac{1}{2}\sigma_1\|W_1^*-W_1^0\|^2 + \frac{1}{2}\epsilon_{z_1}^2\end{aligned} \tag{4.34}$$

where $Q_2(Z_2(t))$ is used to denote all the terms related to the unknown functions $g_{2\gamma}^{-1}(\cdot)$ and $f_2(\cdot)$, which is defined by

$$\begin{aligned}Q_2(Z_2(t)) &= g_{2\gamma}^{-1}(\bar{x}_2(t))f_2(\bar{x}_2(t)) + \frac{1}{2}z_2(t)(g_{2\gamma}^{-1})^2 + \frac{1}{2z_2(t)}\sum_{j=1}^{2}x_j^2(t)\varrho_{2j}^2(\bar{x}_2(t)) \\ &\quad + \dot{x}_1 z_2(t)\int_0^1 \theta\frac{\partial g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)}{\partial x_1}d\theta - \dot{\alpha}_1\int_0^1 g_{2\gamma}^{-1}(x_1,\theta z_2+\alpha_1)d\theta\end{aligned}$$

with $Z_2(t) = [\bar{x}_2, \dot{x}_1, \alpha_1, \partial\alpha_1/\partial x_1, \omega_1]^T \in \Omega^0_{z_2} \subset R^6$.

Similarly, we have the following intermediate adaptive neural network control law

$$\alpha_2 = \frac{p_2(z_2)}{\bar{g}_2(\bar{x}_2)}[-\bar{g}_1(x_1)z_1(t) - k_2(t)z_2 - \hat{W}_2^T S(Z_2)] \quad (4.35)$$

$$\dot{\hat{W}}_2 = p_2(z_2)\Gamma_2[S(Z_2)z_2 - \sigma_2(\hat{W}_2 - W_2^0)] \quad (4.36)$$

where σ_2 is a small constant and is to introduce the $\sigma-$modification for the closed-loop system.

Substituting (4.35) and (4.36) into (4.34), and using (4.5) and (4.6), we have

$$\begin{aligned}\dot{V}_2(t) \leq\ & \bar{g}_2(\bar{x}_2(t))z_2(t)z_3(t) - (k_2(t) - \frac{1}{2})z_2^2(t) - \frac{1}{2}\sigma_2\|\hat{W}_2 - W_2^*\|^2 \\ & -\frac{g_{10}}{\varepsilon_{10}}V_{z_1} - \frac{1}{\varepsilon_{10}}V_{U_1} - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_1 + c_2\end{aligned}$$

where

$$c_2 := \frac{1}{2}\sigma_2\|W_2^* - W_2^0\|^2 + \frac{1}{2}\epsilon_{z_2}^{*2}$$

For $z_2 \in \Omega^0_{Z_2}$, noting (4.9) and choosing

$$k_2(t) = \frac{1}{\varepsilon_{20}}\Big[1 + \int_0^1 \theta\bar{g}_2(x_1, \theta z_2 + \alpha_1)d\theta + \frac{\int_0^t \frac{1}{2}\sum_{j=1}^2 x_j^2(\tau)\varrho_{2j}^2(\bar{x}_2(\tau))d\tau}{z_2^2(t)}\Big]$$

with $0 < \varepsilon_{20} \leq 2$, we have

$$\begin{aligned}\dot{V}_2(t) \leq\ & \bar{g}_2(\bar{x}_2(t))z_2(t)z_3(t) - \frac{g_{20}}{\varepsilon_{20}}V_{z_2} - \frac{1}{\varepsilon_{20}}V_{U_2} - \frac{1}{2}\sigma_2\|\hat{W}_2 - W_2^*\|^2 \\ & -\frac{g_{10}}{\varepsilon_{10}}V_{z_1} - \frac{1}{\varepsilon_{10}}V_{U_1} - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_1 + c_2\end{aligned}$$

where the coupling term $\bar{g}_2(\bar{x}_2(t))z_2(t)z_3(t)$ will be handled in the next step.

Step i ($2 \leq i \leq n-1$): Similar procedures are taken for $i = 2, ..., n-1$ as in Step 1.

The dynamics of z_i-subsystem is given by

$$\dot{z}_i(t) = g_i(\bar{x}_i(t))[z_{i+1}(t) + \alpha_i(t)] + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t - \tau_i)) - \dot{\alpha}_{i-1}(t)$$

Consider the following scalar function

$$V_{z_i}(t) = \int_0^{z_i} \sigma g_{i\gamma}^{-1}(\bar{x}_{i-1}, \sigma + \alpha_{i-1})d\sigma \quad (4.37)$$

The time derivative of $V_{z_i}(t)$ is given by

$$\begin{aligned}\dot{V}_{z_i}(t) &= z_i(t)[\bar{g}_i(\bar{x}_i(t))(z_{i+1}(t)+\alpha_i(t)) + g_{i\gamma}^{-1}(\bar{x}_i(t))f_i(\bar{x}_i(t)) \\ &\quad + g_{i\gamma}^{-1}(\bar{x}_i(t))h_i(\bar{x}_i(t-\tau_i)) + \dot{\bar{x}}_{i-1}z_i(t)\int_0^1 \theta\frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\theta \\ &\quad -\dot{\alpha}_{i-1}\int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta]\end{aligned}$$

Noting Assumption 4.2.4, we have

$$\begin{aligned}\dot{V}_{z_i}(t) &= z_i(t)[\bar{g}_i(\bar{x}_i(t))(z_{i+1}(t)+\alpha_i(t)) + g_{i\gamma}^{-1}(\bar{x}_i(t))f_i(\bar{x}_i(t)) + \frac{1}{2}z_i(t)[g_{i\gamma}^{-1}(\bar{x}_i(t))]^2 \\ &\quad + \dot{\bar{x}}_{i-1}z_i(t)\int_0^1 \theta\frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\theta \\ &\quad -\dot{\alpha}_{i-1}\int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta] + \frac{1}{2}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i)) \quad (4.38)\end{aligned}$$

Consider the following Lyapunov function candidate $V_i(t)$ as

$$V_i(t) = V_{i-1}(t) + V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}(\hat{W}_i(t) - W_i^*)^T\Gamma_i^{-1}(\hat{W}_i(t) - W_i^*)$$

where

$$V_{U_i}(t) = \int_{t-\tau_i}^{t} U_i(\bar{x}_i(\tau))ds\tau \quad (4.39)$$

with $U_i(\cdot)$ being a positive definite function which is defined by

$$U_i(\bar{x}_i(t)) = \frac{1}{2}\sum_{j=1}^{i}x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t))$$

In Step $i-1$, for $z_j \in \Omega_{Z_j}^0$, $j = 1, ..., i-1$, it has been obtained that

$$\dot{V}_{i-1} \le \bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1}z_i + \sum_{j=1}^{i-1}\Big(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j\Big) \quad (4.40)$$

where

$$c_j := \frac{1}{2}\sigma_2\|W_j^* - W_j^0\|^2 + \frac{1}{2}\epsilon_{z_j}^{*2}$$

For $z_j \in \Omega_{Z_j}^0$, $j = 1, ..., i$, the time derivative of $V_i(t)$ along (4.38) and (4.40) is

$$\begin{aligned}\dot{V}_i &\le \bar{g}_i(\bar{x}_i)z_iz_{i+1} + z_i\left[\bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1} + \bar{g}_i(\bar{x}_i)\alpha_i + Q_i(Z_i)\right] + (\hat{W}_i - W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i \\ &\quad + \sum_{j=1}^{i-1}\Big(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j\Big) \quad (4.41)\end{aligned}$$

where

$$
\begin{aligned}
Q_i(Z_i) &= g_{i\gamma}^{-1}(\bar{x}_i)f_i(\bar{x}_i) + \frac{1}{2}z_i(g_{i\gamma}^{-1})^2 + \frac{1}{2z_i(t)}\sum_{j=1}^{i} x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t)) \\
&+\dot{\bar{x}}_{i-1}z_i\int_0^1 \theta\frac{\partial g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\theta \\
&-\dot{\alpha}_{i-1}\int_0^1 g_{i\gamma}^{-1}(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta
\end{aligned}
$$

with $Z_i(t) = [\bar{x}_i, \dot{\bar{x}}_{i-1}, \alpha_{i-1}, \frac{\partial \alpha_{i-1}}{\partial x_1}, \frac{\partial \alpha_{i-1}}{\partial x_2}, ..., \frac{\partial \alpha_{i-1}}{\partial x_{i-1}}, \omega_{i-1}] \in \Omega_{Z_i}^0 \subset R^{3i}$, where

$$
\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1}, \omega_{i-1} = \frac{\partial \alpha_{i-1}}{\partial \bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{W}_j}\dot{\hat{W}}_j
$$

Similarly, we have the following intermediate control law

$$
\alpha_i = \frac{p_i(z_i)}{\bar{g}_i(\bar{x}_i)}[-\bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1} - k_i(t)z_i - \hat{W}_i^T S(Z_i)] \tag{4.42}
$$

$$
\dot{\hat{W}}_i = p_i(z_i)\Gamma_i[S(Z_i)z_i - \sigma_i(\hat{W}_i - W_i^0)] \tag{4.43}
$$

$$
\begin{aligned}
k_i(t) &= \frac{1}{\varepsilon_{i0}}[1 + \int_0^1 \theta\bar{g}_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta \\
&+\frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{i}x_j^2(\tau)\varrho_{ij}^2(\bar{x}_i(\tau))d\tau]
\end{aligned} \tag{4.44}
$$

with $0 < \varepsilon_{i0} \leq 2$. Substituting (4.42)-(4.44) into (4.41), and using (4.5), (4.6) and (4.9), we have

$$
\dot{V}_i \leq \bar{g}_i(\bar{x}_i)z_i z_{i+1} + \sum_{j=1}^{i}(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j)
$$

where the coupling term $\bar{g}_i(\bar{x}_i)z_i z_{i+1}$ will be handled in the next step.

Step n: This is the final step, since the actual control u appears in the dynamics of z_n-subsystem as given by

$$
\dot{z}_n = g_n(x(t))u + f_n(x(t)) + h_n(x(t-\tau_n)) - \dot{\alpha}_{n-1}(t)
$$

Consider the following scalar function

$$
V_{z_n}(t) = \int_0^{z_n}\sigma g_{n\gamma}^{-1}(\bar{x}_{n-1},\sigma+\alpha_{n-1})d\sigma
$$

Its time derivative is given by

$$\begin{aligned}\dot{V}_{z_n}(t) &= z_n(t)\Big[\bar{g}_n(x(t))u(t) + g_{n\gamma}^{-1}(x(t))f_n(x(t)) + g_{n\gamma}^{-1}(x(t))h_n(x(t-\tau_n))\\&+\dot{\bar{x}}_{n-1}z_n(t)\int_0^1 \theta\frac{\partial g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial \bar{x}_{n-1}}d\theta\\&-\dot{\alpha}_{n-1}\int_0^1 g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta\Big]\end{aligned}$$

Noting Assumption 4.2.4, we have

$$\begin{aligned}\dot{V}_{z_n}(t) &= z_n(t)[\bar{g}_n(x(t))u(t) + g_{n\gamma}^{-1}(x(t))f_n(x(t)) + \frac{1}{2}z_n(t)[g_{n\gamma}^{-1}(x(t))]^2\\&+\dot{\bar{x}}_{n-1}z_n(t)\int_0^1 \theta\frac{\partial g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial \bar{x}_{n-1}}d\theta\\&-\dot{\alpha}_{n-1}\int_0^1 g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta]\\&+\frac{1}{2}\sum_{j=1}^{n}x_j^2(t-\tau_n)\varrho_{nj}^2(x(t-\tau_n))\end{aligned}\qquad(4.45)$$

Consider the following Lyapunov function candidate V_n as

$$V_n(t) = V_{n-1}(t) + V_{z_n}(t) + V_{U_n}(t) + \frac{1}{2}(\hat{W}_n(t)-W_n^*)^T\Gamma_n^{-1}(\hat{W}_n(t)-W_n^*)$$

where

$$V_{U_n}(t) = \int_{t-\tau_n}^{t} U_n(x(\tau))d\tau$$

with $U_n(\cdot)$ being a positive definite function which is defined by

$$U_n(x(t)) = \frac{1}{2}\sum_{j=1}^{n}x_j^2(t)\varrho_{nj}^2(x(t))$$

In Step $n-1$, for $z_i \in \Omega_{Z_i}^0$, $i=1,...,n-1$, it has been obtained that

$$\dot{V}_{n-1} \le \bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1}z_n + \sum_{j=1}^{n-1}\Big(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j\Big)\qquad(4.46)$$

For $z_i \in \Omega_{Z_i}^0$, $i=1,...,n$, the time derivative of $V_n(t)$ along (4.45) and (4.46) is

$$\begin{aligned}\dot{V}_n &\le z_n[\bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1} + \bar{g}_n(x)u + Q_n(Z_n)] + (\hat{W}_n - W_n^*)^T\Gamma_n^{-1}\dot{\hat{W}}_n\\&+\sum_{j=1}^{n-1}\Big(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j\Big)\end{aligned}\qquad(4.47)$$

where

$$
\begin{aligned}
Q_n(Z_n) \;=\;& g_{n\gamma}^{-1}(x)f_n(x) + \frac{1}{2}z_n(g_{n\gamma}^{-1})^2 + \frac{1}{2z_n(t)}\sum_{j=1}^{n} x_j^2(t)\varrho_{nj}^2(x(t)) \\
&+\dot{\bar{x}}_{n-1}z_n\int_0^1 \theta\frac{\partial g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial \bar{x}_{n-1}}d\theta \\
&-\dot{\alpha}_{n-1}\int_0^1 g_{n\gamma}^{-1}(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta
\end{aligned}
$$

with $Z_n(t) = [x, \dot{\bar{x}}_{n-1}, \alpha_{n-1}, \frac{\partial\alpha_{n-1}}{\partial x_1}, \frac{\partial\alpha_{n-1}}{\partial x_2}, ..., \frac{\partial\alpha_{n-1}}{\partial x_{n-1}}, \omega_{n-1}] \in \Omega_{Z_n}^0 \subset R^{3n}$, where

$$
\begin{aligned}
\dot{\alpha}_{n-1} &= \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1} \\
\omega_{n-1} &= \frac{\partial\alpha_{n-1}}{\partial\bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j
\end{aligned}
$$

We construct the following adaptive neural control law

$$
\begin{aligned}
u(t) &= \frac{p_n(z_n)}{\bar{g}_n(x)}[-\bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1} - k_n(t)z_n - \hat{W}_n^T S(Z_n)] && (4.48)\\
\dot{\hat{W}}_n &= p_n(z_n)\Gamma_i[S(Z_n)z_n - \sigma_i(\hat{W}_n - W_n^0)] && (4.49)\\
k_n(t) &= \frac{1}{\varepsilon_{n0}}[1 + \int_0^1 \theta\bar{g}_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta \\
&\quad + \frac{1}{z_n^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{n}x_j^2(\tau)\varrho_{nj}^2(x(\tau))d\tau] && (4.50)
\end{aligned}
$$

with $0 < \varepsilon_{n0} \leq 2$. Substituting (4.48)-(4.50) into (4.47), and using (4.5), (4.6) and (4.9), we have

$$
\dot{V}_n(t) \leq \sum_{j=1}^{n}(-\frac{g_{j0}}{\varepsilon_{j0}}V_{z_j} - \frac{1}{\varepsilon_{j0}}V_{U_j} - \frac{1}{2}\sigma_j\|\hat{W}_j - W_j^*\|^2 + c_j) \qquad (4.51)
$$

where

$$
c_n := \frac{1}{2}\sigma_2\|W_n^* - W_n^0\|^2 + \frac{1}{2}\epsilon_{z_n}^{*^2}
$$

The following theorem shows the stability of the closed-loop adaptive system.

Theorem 4.2.1 *Consider the closed-loop system consisting of the plant (4.1) under Assumptions 4.2.1-4.2.5, the controller (4.48) and the NN weight updating law (4.49). For bounded initial conditions, the following properties hold:*

(i) *all signals in the closed-loop system remain semi-globally uniformly ultimately bounded and the vector* $Z = [Z_1^T, ..., Z_n^T]^T$ *remains in a compact set* $\Omega_Z^0 := \Omega_{Z_1}^0 \cup ... \cup \Omega_{Z_n}^0$ *specified as*

$$\Omega_Z^0 = \{Z | \sum_{i=1}^{n} z_i^2 \le 2C_0, \sum_{i=1}^{n} \|\tilde{W}_i\|^2 \le \frac{2C_0}{\lambda_{\min}(\Gamma_i^{-1})}, \bar{x}_{di} \in \Omega_{di}, i = 2, ..., n, z_i \notin \Omega_{c_{z_i}}, i = 1, ..., n\} \tag{4.52}$$

where $C_0 > 0$ *is a constant whose size depends on the initial conditions (as will be defined later in the proof);*

(ii) *the closed-loop signal* $z(t) = [z_1, ..., z_n]^T \in R^n$ *will eventually converge to a compact set defined by*

$$\Omega_S := \{z | \|z\|^2 \le \mu\} \tag{4.53}$$

where $\mu > 0$ *is a constant related to the design parameters and will be defined later in the proof, and* Ω_S *can be made as small as desired by an appropriate choice of the design parameters.*

Proof: Consider the following Lyapunov function candidate

$$V_n(t) = \sum_{i=1}^{n} [V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T \Gamma_i^{-1} \tilde{W}_i] \tag{4.54}$$

where $V_{z_i}(t)$ and $V_{U_i}(t)$ are defined in (4.37) and (4.39) respectively, and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The following three cases are considered.

Case 1): $z_i \in \Omega_{c_{z_i}}$, $i = 1, ..., n$. In this case, the controls $\alpha_i = 0$, $i = 1, ..., n-1$, $u = 0$ and $\dot{\hat{W}}_i = 0$, $i = 1, ..., n$. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $i = 2, ..., n$, x_i is bounded as $x_i = z_i + \alpha_{i-1}$ and $\alpha_{i-1} = 0$. In addition, $\hat{W}_i$ is kept unchanged in a bounded value, $i = 1, ..., n$. Observing the definition for $V_{z_i}(t)$ and $V_{U_i}(t)$ and noting that $g_{i\gamma}(\cdot)$, $\varrho_{ij}(\cdot)$ are smooth functions, we know that for bounded x_i, z_i and $\hat{W}_i$, $V_{z_i}(t)$ and $V_{U_i}(t)$ are bounded, i.e., there exists a finite C_B such that

$$V_n(t) \le C_B \tag{4.55}$$

Case 2): $z_i \in \Omega_{Z_i}^0$, $i = 1, ..., n$. From (4.51), we have $\dot{V}_n(t) \le -C_1 V_n(t) + C_2$ where $C_1 = \sum_{i=1}^{n} c_i$ and

$$C_2 = \min\{\frac{g_{10}}{\varepsilon_{10}}, ..., \frac{g_{n0}}{\varepsilon_{n0}}, \frac{1}{\varepsilon_{10}}, ..., \frac{1}{\varepsilon_{n0}}, \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}, ..., \frac{\sigma_n}{\lambda_{\max}(\Gamma_n^{-1})}\}$$

Let $\rho = C_2/C_1$, it follows that

$$0 \leq V_n(t) \leq [V_n(0) - \rho]e^{-C_1 t} + \rho \leq V_n(0) + \rho \tag{4.56}$$

where constant

$$V_n(0) = \sum_{i=1}^{n} [\int_0^{z_i(0)} \sigma g_{i\gamma}^{-1}(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0)) d\sigma + \frac{1}{2}\tilde{W}_i^T(0)\Gamma_i^{-1}\tilde{W}_i(0)]$$

with $g_{i\gamma}^{-1}(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0)) = g_{1\gamma}^{-1}(\sigma)$ for $i = 1$.

Case 3): Some $z_i \in \Omega_{Z_i}^0$ and some $z_j \in \Omega_{c_{z_j}}$. In this case, the corresponding α_i or u and the adaptation law for $\hat{W}_i$ will be invoked for $z_i \in \Omega_{Z_i}^0$ while $\alpha_j = 0$ or $u = 0$ and $\dot{\hat{W}}_j = 0$ for $z_j \in \Omega_{c_{z_j}}$. Let us define $V_I(t) = \sum_i (V_{z_i} + V_{U_i} + \frac{1}{2}\tilde{W}_i^T\Gamma_i^{-1}\tilde{W}_i)$ and $V_J(t) = \sum_j (V_{z_j} + V_{U_j} + \frac{1}{2}\tilde{W}_j^T\Gamma_j^{-1}\tilde{W}_j)$. For $z_j \in \Omega_{c_{z_j}}$, we obtain that $V_J(t)$ is bounded, i.e., $V_J(t) \leq C_J$ with C_J being finite, and $z_i \in \Omega_{Z_i}^0$, we have that $\dot{V}_I(t) \leq -C_1^I V_I(t) + C_2^I$, i.e.,

$$V_I(t) \leq [V_I(0) - \rho_I]e^{-C_1^I t} + \rho_I \leq V_I(0) + \rho_I \tag{4.57}$$

where $\rho_I = C_2^I/C_1^I$ with $C_1^I = \sum_i c_i$ and $C_2^I = \min_i\{g_{i0}/\varepsilon_{i0}, 1/\varepsilon_{i0}, \sigma_i/\lambda_{\max}(\Gamma_i^{-1})\}$. Therefore, it can be obtained that

$$V_n(t) = V_I(t) + V_J(t) \leq V_I(0) + \rho_I + C_J \tag{4.58}$$

Thus, from (4.55), (4.56) and (4.58) for Cases 1), 2) and 3), we can conclude that

$$V_n(t) \leq C_0 \tag{4.59}$$

where $C_0 = \max\{C_B, V_n(0) + \rho, V_I(0) + \rho_I + C_J\}$. From (4.59), we know that $V_n(t)$, z_i and $\hat{W}_i$, $i = 1, ..., n$, are bounded. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $x_2 = z_2 + \alpha_1$, since α_1 is function of bounded signals z_1, Z_1, $\hat{W}_1$, α_1 is thus bounded, which in turn leads to the boundedness of x_2. Following the same way, we can prove one by one that all α_{i-1} and x_i, $i = 3, ..., n$ are bounded. Therefore, the systems' states x_i, $i = 1, ..., n$ are bounded.

Considering (4.54) and the property for $V_{z_i}(t)$ that

$$\frac{1}{2}z_i^2 \leq V_{z_i}(t) \leq \frac{z_i^2}{g_{i0}} \int_0^1 \theta \bar{g}_i(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1}) d\theta$$

we know that

$$\sum_{i=1}^{n} z_i^2 \le 2\sum_{i=1}^{n} V_{z_i}(t) \le 2V_n(t), \quad \sum_{i=1}^{n} \|\tilde{W}_i\|^2 \le \frac{2V_n(t)}{\lambda_{\min}(\Gamma_i^{-1})} \tag{4.60}$$

From (4.59) and (4.60), we readily have the compact set Ω_Z^0 defined in (4.52) over which the NN approximation is carried out with its feasibility being guaranteed.

In addition, in Case 1), as $z_i \in \Omega_{c_{z_i}}$, $i = 1, ...n$, we know that $\|z\|^2 = \sum_{i=1}^{n} z_i^2 \le \sum_{i=1}^{n} c_{z_i}^2$. In Case 2), from (4.56) and (4.60), we have that $\lim_{t\to\infty} \|z\|^2 = 2\rho$. In Case 3), from (4.57) and (4.60), we have that $\lim_{t\to\infty} \sum_i z_i^2 = 2\rho_I$ and $\sum_j z_j^2 \le \sum_j c_{z_j}^2$. Therefore as $t \to \infty$, we can conclude that $\|z\|^2 \le \mu$ where $\mu = \max\{2\rho, 2\rho_I, \sum_{i=1}^{n} c_{z_i}^2\}$, i.e., the vector z will eventually converge to the compact set Ω_S defined in (4.53). This completes the proof. $\diamondsuit$

Remark 4.2.7 *Note that the choices of* $\gamma_i(\bar{x}_i)$ *are not unique. By choosing* $\gamma_i(\bar{x}_i) = 1$*, we have* $g_{i\gamma}^{-1}(\bar{x}_i) = \frac{1}{g_i(\bar{x}_i)}$ *[52] and* $V_{z_i} = \int_0^{z_i} \frac{\sigma}{g_i(\bar{x}_{i-1}, \sigma + \alpha_{i-1})} d\sigma$*,* $i = 1, ..., n$*. By Mean Value Theorem,* V_{z_i} *can be rewritten as* $V_{z_i} = \frac{\lambda_s z_i^2}{g_i(\bar{x}_{i-1}, \lambda_s z_i + \alpha_{i-1})}$*,* $\lambda_s \in (0,1)$*. From Assumption 4.2.2,* $0 \le g_{i0} \le g_i(\bar{x}_i)$*, we know that* $0 < V_{z_i} \le \frac{\lambda_s}{g_{10}} z_i^2$*. The adaptive control laws are given by*

$$\begin{aligned}
\alpha_i = \quad & \tfrac{p_i(z_i)}{\bar{g}_i(\bar{x}_i)}[- \bar{g}_{i-1}(\bar{x}_{i-1})z_{i-1} - k_i(t)z_i - \hat{W}_i^T S(Z_i) \\
& -\tfrac{1}{2z_i}\textstyle\sum_{j=1}^{i} x_j^2 \varrho_{ij}^2(\bar{x}_i)] \\
u = \quad & \tfrac{p_n(z_n)}{\bar{g}_n(x)}[- \bar{g}_{n-1}(\bar{x}_{n-1})z_{n-1} - k_n(t)z_n \\
& -\hat{W}_n^T S(Z_n) - \tfrac{1}{2z_n}\textstyle\sum_{j=1}^{n} x_j^2 \varrho_{nj}^2(x)] \\
\dot{\hat{W}}_i = {} & p_i(z_i)\Gamma_i[S(Z_i)z_i - \sigma_i(\hat{W}_i - W_i^0)] \\
k_i(t) = {} & \frac{1}{\varepsilon_{i0}}[1 + \lambda_s + \frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}\sum_{j=1}^{i} x_j^2(\tau)\varrho_{ij}^2(\bar{x}_i(\tau))d\tau]
\end{aligned}$$

where $0 < \varepsilon_{i0} \le 2$*. For bounded initial conditions, all closed-loop signals remain bounded and the tracking error converges to a small neighborhood around zero by appropriately choosing design parameters.*

Remark 4.2.8 *Note that the size of the compact set* Ω_Z^0 *is characterized by* C_0*, which depends on system initial conditions* $x_i(0)$ *and* $\hat{W}_i(0)$ *as well as the design parameters* σ_i*,* Γ_i*,* W_i^0 *and* ε_{i0}*,* $i = 1, ..., n$*. For the compact set* Ω_S *to which the* *closed-loop signals eventually converge, its size only depends on the design parameters. Therefore, it can be seen that large initial errors $z_i(0)$ and $\tilde{W}_i(0)$, $i = 1, ..., n$ may lead to a large transient tracking error during the initial period of adaptation, but will not affect the final convergence of the closed-loop signals.*

Remark 4.2.9 *Since the function approximation property (4.3) of neural networks is only guaranteed within a compact set, the stability result proposed is semi-global in the following sense: Given any bounded initial compact set such that $z_i(0), \tilde{W}_i(0) \in \Omega_I$, the proposed NN controller with sufficiently large number of nodes guarantees that all the closed-loop signals will stay within the compact set, i.e., Ω_Z^0 in the section, if the compact set Ω_{Zc}^0, over which the neural network approximation is constructed, satisfies that $\Omega_Z^0 \subseteq \Omega_{Zc}^0$, and eventually all the closed-loop signals will converge to the steady state compact set, i.e., Ω_S in the section. The relationships among the sets are as: $\Omega_I, \Omega_S \subseteq \Omega_Z^0 \subseteq \Omega_{Zc}^0$. It is apparent that the larger the compact set Ω_{Zc}^0 over which the NN controller is built upon, the more relaxed the initial compact set Ω_I is.*

4.2.4 Simulation Studies

To illustrate the proposed adaptive neural control algorithms, we consider the following second-order plant

$$\begin{cases} \dot{x}_1(t) &= [1 + x_1^2(t)]x_2(t) + x_1(t)e^{-0.5x_1(t)} + 2x_1^2(t - \tau_1) \\ \dot{x}_2(t) &= [3 + \cos(x_1(t)x_2(t))]u(t) + x_1(t)x_2^2(t) + 0.2x_2(t - \tau_2)\sin(x_2(t - \tau_2)) \end{cases}$$

with the output $y_1 = x_1$, the initial condition $[x_1(0), x_2(0)]^T = [0, 0]^T$, and the time delays $\tau_1 = 2$sec, $\tau_2 = 2$sec. The unknown virtual control coefficients are $g_1(x_1) = 1 + x_1^2$, $g_2(\bar{x}_2) = 3 + \cos(x_1 x_2)$. The time delay terms are: $h_1(x_1) = 2x_1^2$, $h_2(x_2) = 0.2x_2\sin(x_2)$, which means that $\varrho_{11}(x_1) = 2|x_1|$, $\varrho_{12}(x) = \varrho_{21}(x) = 0$, and $\varrho_{22}(x) = 0.2$. The control objective is to track the desired reference signal $y_d = 0.5[\sin(t) + \sin(0.5t)]$. For the design of adaptive neural controller, let $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$. For simplicity, simulation is carried out based on Remark 4.2.7 for the case $\gamma_i(\bar{x}_i) = 1$ as follows

$$\alpha_1(t) = \begin{cases} -k_1(t)z_1(t) - \hat{W}_1^T S(Z_1) - \frac{1}{2z_1(t)}x_1^2(t)\varrho_{11}^2, & z_1 \in \Omega_{z_1}^0 \\ 0, & z_1 \in \Omega_{c_{z_1}} \end{cases}$$

$$u(t) = \begin{cases} -z_1(t) - k_2(t)z_2(t) - \hat{W}_2^T S(Z_2) - \frac{1}{2z_2(t)} x_2^2 \varrho_{22}^2, & z_2 \in \Omega_{z_2}^0 \\ 0, & z_2 \in \Omega_{c_{z_2}} \end{cases}$$

$$\dot{\hat{W}}_i = \Gamma_i[S(Z_i)z_i(t) - \sigma_i(\hat{W}_i - W_i^0)], \quad i = 1, 2$$

where $Z_1 = [x_1, y_d, \dot{y}_d]^T$, $Z_2 = [x_1, x_2, \alpha_1, \frac{\partial \alpha_1}{\partial x_1}, \frac{\partial \alpha_1}{\partial \hat{W}_1} \dot{\hat{W}}_1]^T$, and $k_i(t)$, $i = 1, 2$ can be calculated by

$$k_i(t) = \begin{cases} \frac{1}{\varepsilon_{i0}} \Big[1 + \lambda_s + \frac{\int_0^t \frac{1}{2} \sum_{j=1}^i x_j^2(\tau) \varrho_{ij}^2(\bar{x}_i(\tau)) d\tau}{z_i^2(t)}\Big], & z_i \in \Omega_{z_i}^0 \\ 0, & z_i \in \Omega_{c_{z_i}} \end{cases}$$

The following controller design parameters are adopted in the simulation:
$\Gamma_1 = \text{diag}\{4\}$, $\Gamma_2 = \text{diag}\{3\}$, $\sigma_1 = \sigma_2 = 0.1$, $W_1^0 = W_2^0 = 0.01$, $\varepsilon_{10} = \varepsilon_{20} = 1$, $\lambda_s = 0.5$, and $c_{z_1} = c_{z_2} = 1.0e^{-7}$.

In practice, the selection of the centers and widths of RBF has a great influence on the performance of the designed controller. According to [45], Gaussian RBF NNs arranged on a regular lattice on R^n can uniformly approximate sufficiently smooth functions on closed, bounded subsets. Accordingly, in the following simulation studies, the centers and widths are chosen on a regular lattice in the respective compact sets. Specifically, neural networks $\hat{W}_1^T S_1(Z_1)$ contains 27 nodes (i.e., $l_1 = 27$) with centers $\mu_l (l = 1, \cdots, l_1)$ evenly spaced in $[-1, +1] \times [-1, +1] \times [-1, +1]$, and widths $\eta_l^2 = 2 (l = 1, \cdots, l_1)$. Neural networks $\hat{W}_2^T S_2(Z_2)$ contains 243 nodes (i.e., $l_2 = 243$) with centers $\mu_l (l = 1, \cdots, l_2)$ evenly spaced in $[-1, +1] \times [-1, +1] \times [-1, +1] \times [-1, +1] \times [-1, +1]$, and widths $\eta_l^2 = 3 (l = 1, \cdots, l_2)$. The initial weights are set as $\hat{W}_1(0) = 0.0$, $\hat{W}_2(0) = 0.0$.

From the theorems, we know that the integral term in control gain k_i is used to provide robustness against the uncertainties from the unknown time delays. To illustrate this point, simulations are conducted with and without this term. Fig. 4.1 shows that the output actually blows up in a short time (less than 6 sec) without the integral term, while satisfactory transient performance is obtained in Fig. 4.2 once the integral term was added in k_i and good tracking performance is achieved after 10 seconds learning periods. Figs. 4.3 and 4.4 show the boundedness of the control input and the NN weights with the integral term in the control loop.

We would like to point out that the choice of c_{z_i} for control gain k_i plays an important role in achieving the desired performance. Through extensive simulation study, it was found that larger c_{z_i} causes chattering in control signals as shown in Fig. 4.5 and poor control performance as shown in Fig. 4.6, smaller c_{z_i} leads to smoother control signals as seen from Fig. 4.7 and better tracking performance as can be seen from Fig. 4.8. Note that in all the simulations, it was found that the weights of the neural networks are bounded, they are omitted here for clarity. Actually, c_{z_i} can be chosen arbitrarily small but equals zero, then the control signals generated are almost continuous, and the control performance is much more improved.

Remark 4.2.10 *As stated in [88], the integrals in control gain (4.44) might not be solved analytically for some functions* $\bar{g}_i(\bar{x}_i)$*, and may make the controller implementation difficult. This problem can be dealt with by suitably choosing the design functions* $\bar{g}_i(\bar{x}_i)$*. Since the choices of* $\bar{g}_i(\bar{x}_i)$ *are only required to be larger than* $g_i(\bar{x}_i)$*, the designer has the freedom to find suitable* $\bar{g}_i(\bar{x}_i)$ *such that the integrals are analytically solvable. As an alternative scheme, one can also use on-line numerical approximation to calculated the integral, which however requires more computational power in practical applications.*

4.2.5 Conclusion

An adaptive neural-based control has been addressed for a class of strict-feedback nonlinear systems with unknown time delays. The unknown time delays has been compensated for through the use of appropriate Lyapunov-Krasovskii functionals. As a result, the iterative backstepping design can be carried out. In addition, the controller is free from singularity problem by using the integral Lyapunov function and practical robust neural network control. The proposed systematic backstepping design method has been proven to be able to guarantee semi-global uniformly ultimately boundedness of closed-loop signals and the output of the system has been proven to converge to an arbitrarily small neighborhood of the origin.

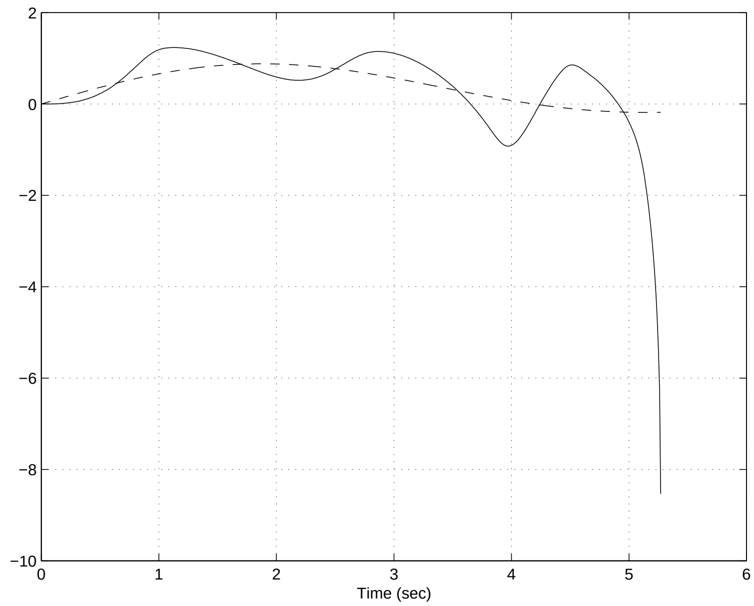

Figure 4.1: Output $y(t)$(“–”) and reference y_d(“- -”) without integral term.

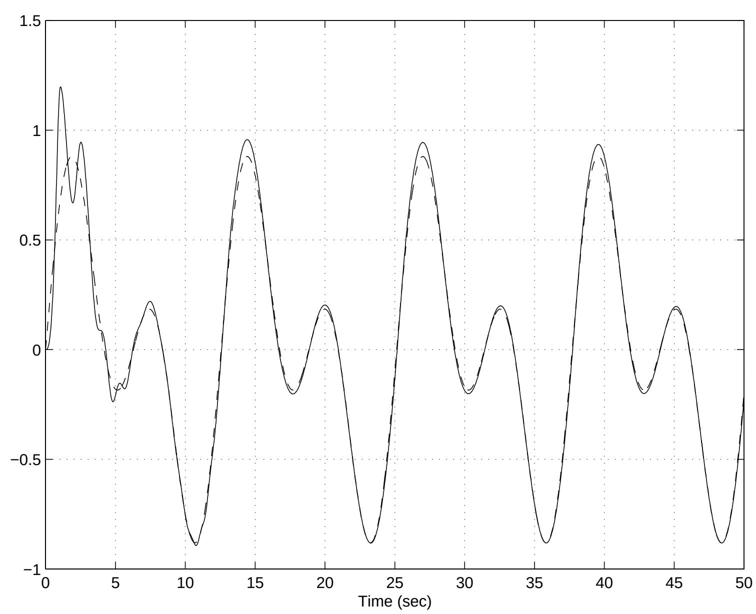

Figure 4.2: Output $y(t)$(“–”) and reference y_d(“- -”) with integral term.

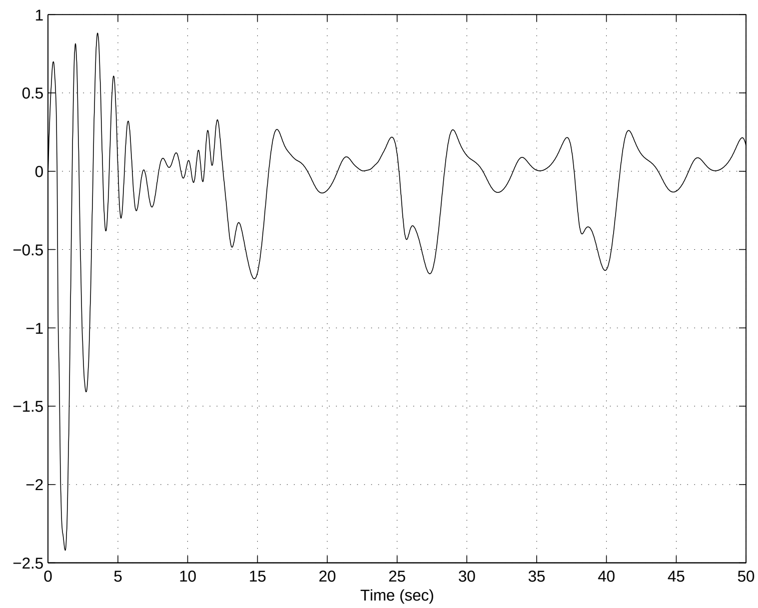

Figure 4.3: Control input $u(t)$ with integral term.

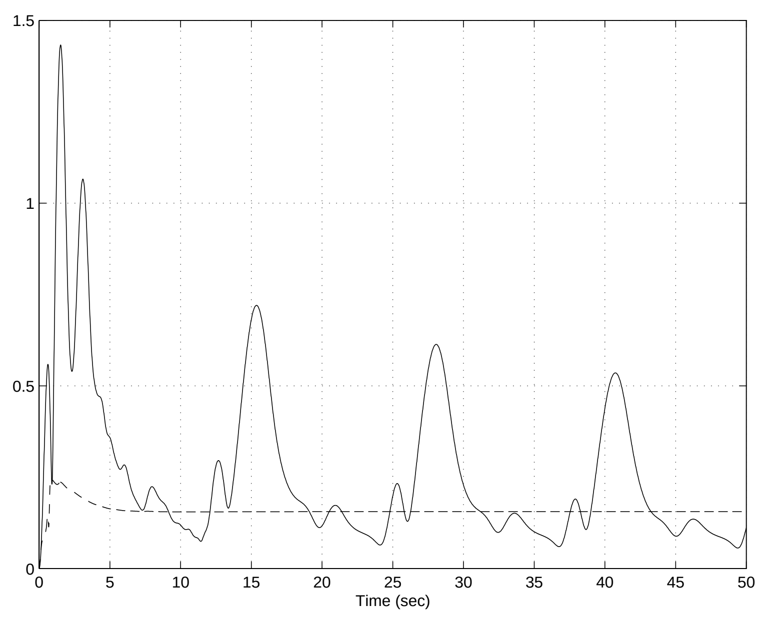

Figure 4.4: $\|\hat{W}_1\|$("—") and $\|\hat{W}_2\|$("- -") with integral term.

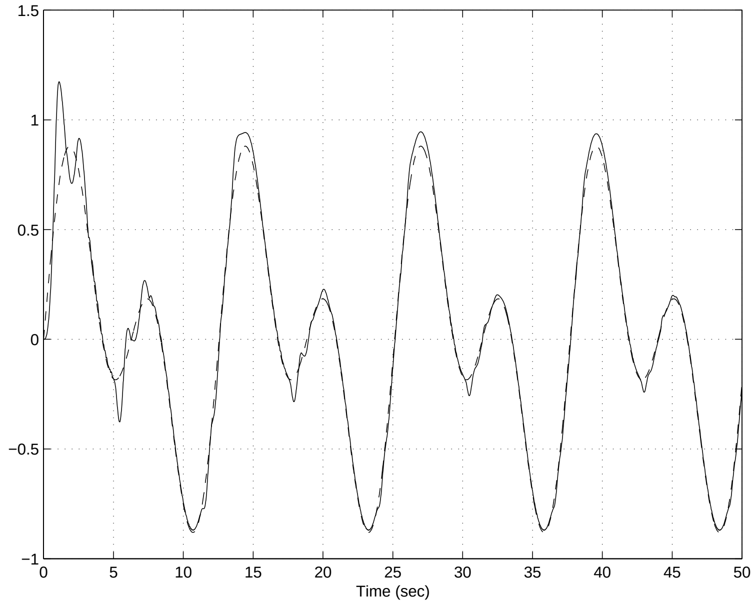

Figure 4.5: $y(t)$("–") and y_d("- -") with $c_{z_i} = 0.01$.

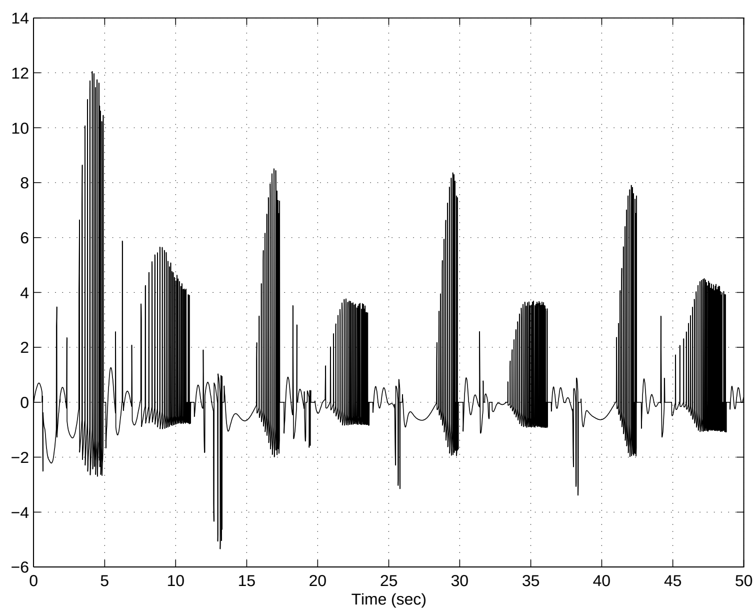

Figure 4.6: Control input $u(t)$ with $c_{z_i} = 0.01$.

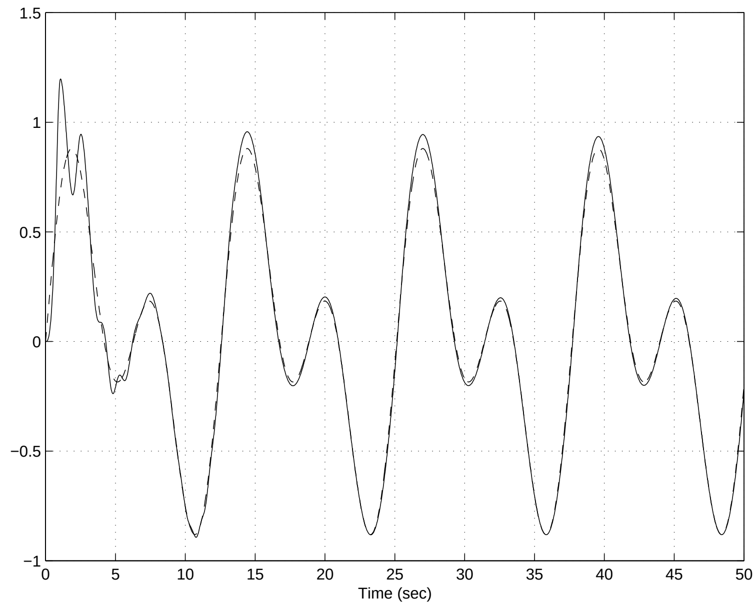

Figure 4.7: $y(t)$(“—”) and y_d(“- -”) with $c_{z_i} = 1.0e^{-10}$.

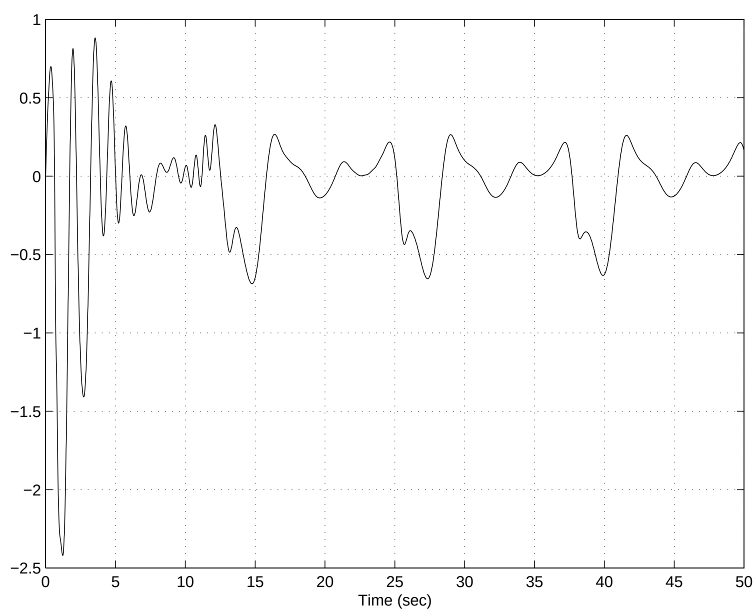

Figure 4.8: Control input $u(t)$ with $c_{z_i} = 1.0e^{-10}$.

4.3 Direct Neural Network Control

4.3.1 Problem Formulation

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{cases} \dot{x}_i(t) = g_i(\bar{x}_i(t))x_{i+1}(t) + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)), \quad 1 \leq i \leq n-1 \\ \dot{x}_n(t) = g_n(x(t))u + f_n(x(t)) + h_n(x(t-\tau_n)) \end{cases} \tag{4.61}$$

where $\bar{x}_i = [x_1, x_2, ..., x_i]^T$, $x = [x_1, x_2, ..., x_n]^T \in R^n$ and $u \in R$ are the state variables and system input respectively, $g_i(\cdot)$, $f_i(\cdot)$ and $h_i(\cdot)$ are unknown smooth functions, and τ_i are unknown time delays of the states, $i = 1, ..., n$. The control objective is to design an adaptive controller for system (4.61) such that the state $x_1(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, ..., y_d^{(i)}]^T$, $i = 1, ..., n-1$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$.

We have the following assumptions for the system's signals, unknown functions and reference signals.

A1). The system states $x(t)$ and part of their time derivatives, $\dot{\bar{x}}_{n-1}(t)$, are all available for feedback.

A2). The signs of g_i are known, and there exist constants $g_{\max} \geq g_{\min} > 0$ such that $g_{\min} \leq |g_i| \leq g_{\max}$. There exist constants $g_{id} > 0$ such that $|\dot{g}_i(\cdot)| \leq g_{id}$, $\forall \bar{x}_i \in R^i$.

A3). The desired trajectory vectors $\bar{x}_{di}$, $i = 2, ..., n$ are continuous and available, and $\bar{x}_{di} \in \Omega_{di} \subset R^i$ with Ω_{di} known compact sets.

A4). The unknown smooth functions $h_i(\bar{x}_i(t))$ satisfy the following inequality $|h_i(\bar{x}_i(t))| \leq \sum_{j=1}^{i} |x_j(t)| \varrho_{ij}(\bar{x}_i(t))$ where $\varrho_{ij}(\cdot)$ are known smooth functions.

A5). The size of the unknown time delays is bounded by a known constant, i.e., $\tau_i \leq \tau_{\max}$, $i = 1, ..., n$.

The Assumption A1) implies that unknown constants g_i are strictly either positive or negative. Without losing generality, we shall only consider the case when $g_i >$

0. It should be emphasized that the bounds $g_{\min}$ and $g_{\max}$ are only required for analytical purposes, their true values are not necessarily known since they are not used for controller design. Note that the requirement for $\dot{\bar{x}}_{n-1}(t)$ is a constraint but realistic for many physical systems as we are not requiring $\dot{x}_n$ which is directly influenced by the control.

There are many physical processes which are governed by nonlinear differential equations of the form (4.61). Examples are recycled reactors, recycled storage tanks and cold rolling mills [92]. In general, most of the recycling processes inherit delays in their state equations.

The even function $p_i(\cdot,\cdot) : R \times R \to R$ is introduced for the purpose of practical controller design later.

$$p_i(x, c_{ai}) = \begin{cases} 1, & |x| \geq c_{ai} \\ 0, & |x| < c_{ai} \end{cases}, \quad \forall x \in R. \tag{4.62}$$

4.3.2 Direct NN Control for First-order System

To illustrate the design methodology clearly, we first consider the tracking problem of a first-order system

$$\dot{x}_1(t) = g_1(x_1(t))u(t) + f_1(x_1(t)) + h_1(x_1(t-\tau_1)) \tag{4.63}$$

where $u(t)$ is the control input. Define the tracking error $z_1 = x_1 - y_d$, we have

$$\dot{z}_1(t) = g_1(x_1(t))u(t) + f_1(x_1(t)) + h_1(x_1(t-\tau_1)) - \dot{y}_d(t) \tag{4.64}$$

Based on feedback linearization, the certainty equivalent control is usually taken the form $u(t) = \frac{1}{g_1(x_1)}[-f_1(x_1)+v(t)]$. In the case that $g_1(\cdot)$ and $f_1(\cdot)$ are unknown, their estimates $\hat{g}_1$ and $\hat{f}_1$ shall be used instead to construct the controller and singularity problem may occur when $\hat{g}_1(x_1) = 0$. To avoid the singularity problem, we shall estimate the unknown term, e.g., $\frac{f_1(x_1)}{g_1(x_1)}$ as a whole rather than estimate the function $g_1(\cdot)$ and $f_1(\cdot)$ individually.

Another design difficulty comes from the unknown time-delay τ_1, which can be compensated for by introducing the Lyapunov-Krasovskii functional in the form of

$$V_U(t) = \int_{t-\tau_1}^{t} U(x(t))d\tau \tag{4.65}$$

with $U(\cdot) \geq 0$ being a properly chosen function. The time derivative of $V_U(t)$ is

$$\dot{V}_U(t) = U(x(t)) - U(x(t-\tau_1))$$

among which the term $U(x(t-\tau_1))$ can be used to compensate for the unknown time-delay terms related to τ_1, while the remaining term $U(x(t))$ does not introduce any uncertainties to the system.

Consider the scalar smooth function $V_{z_1} = \frac{1}{2g_1(x_1)} z_1^2(t)$ and Lyapunov-Krasovskii functional V_{U_1} as

$$V_{U_1}(t) = \frac{1}{2g_{\min}} \int_{t-\tau_1}^{t} U_1(x_1(t)) d\tau \tag{4.66}$$

with $U_1(x_1(t)) = \frac{1}{2} x_1^2(t) \varrho_1(x_1(t)) \geq 0$. Accordingly, we have

$$\begin{aligned}
\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) &= \frac{z_1(t)\dot{z}_1(t)}{g_1(x_1)} - \frac{\dot{g}_1(x_1)}{2g_1^2(x_1)} z_1^2 + \frac{1}{2g_{\min}} U_1(x_1(t)) - \frac{1}{2g_{\min}} U_1(x_1(t-\tau_1)) \\
&= z_1(t)\Big\{u(t) + \frac{1}{g_1(x_1)}[f_1(x_1(t)) + h_1(x_1(t-\tau_1)) - \dot{y}_d(t)]\Big\} \\
&\quad - \frac{\dot{g}_1(x_1)}{2g_1^2(x_1)} z_1^2(t) + \frac{1}{2g_{\min}} U_1(x_1(t)) - \frac{1}{2g_{\min}} U_1(x_1(t-\tau_1))
\end{aligned}$$

Noting Assumption A4), we have

$$\begin{aligned}
\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) &\leq z_1(t)\Big\{u(t) + \frac{1}{g_1(x_1)}[f_1(x_1(t)) - \dot{y}_d(t)]\Big\} - \frac{\dot{g}_1(x_1)}{2g_1^2(x_1)} z_1^2(t) \\
&\quad + \frac{1}{g_1(x_1)} |z_1(t)||x_1(t-\tau_1)|\varrho_1(t-\tau_1) \\
&\quad + \frac{1}{2g_{\min}} U_1(x_1(t)) - \frac{1}{2g_{\min}} U_1(x_1(t-\tau_1))
\end{aligned} \tag{4.67}$$

The terms $z_1(t)$ and $|x_1(t-\tau_1)|\varrho_1(x_1(t-\tau_1))$, which are entangled in their present form, shall be separated such that the terms with unknown time delay can be dealt with separately. Using Young's inequality, (4.67) becomes

$$\begin{aligned}
\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) &\leq z_1(t)\Big\{u(t) + \frac{1}{g_1(x_1)}[f_1(x_1(t)) - \dot{y}_d(t) + \frac{1}{2} z_1(t)]\Big\} - \frac{\dot{g}_1(x_1)}{2g_1^2(x_1)} z_1^2(t) \\
&\quad + \frac{1}{2g_1(x_1(t))} x_1^2(t-\tau_1)\varrho_1^2(x_1(t-\tau_1)) \\
&\quad + \frac{1}{2g_{\min}} x_1^2(t)\varrho_1^2(x_1(t)) - \frac{1}{2g_{\min}} x_1^2(t-\tau_1)\varrho_1^2(x_1(t-\tau_1))
\end{aligned} \tag{4.68}$$

As $g_1(x_1(t)) \geq g_{\min}$, it follows that $\frac{1}{2g_1} x_1^2(t-\tau_1)\varrho_1^2(x_1(t-\tau_1)) \leq \frac{1}{2g_{\min}} x_1^2(t-\tau_1)\varrho_1^2(x_1(t-\tau_1))$. In addition, from Assumption A2), we have $-\frac{\dot{g}_1(x_1) z_1^2}{2g_1^2(x_1)} \leq \frac{|\dot{g}_1(x_1)| z_1^2}{2g_1^2(x_1)} \leq$

$\frac{g_{1d}}{2g_{\min}}z_1^2$. Thus, (4.68) becomes

$$\dot{V}_{z_1}(t) + \dot{V}_{U_1}(t) \le z_1(t)[u(t) + Q_1(Z_1(t))] + \frac{g_{1d}}{2g_{\min}}z_1^2 \tag{4.69}$$

where

$$Q_1(Z_1(t)) = \frac{1}{g_1(x_1)}[f_1(x_1(t)) - \dot{y}_d(t) + \frac{1}{2}z_1(t)] + \frac{1}{2g_{\min}z_1(t)}x_1^2(t)\varrho_1^2(x_1(t))$$

with $Z_1 = [x_1, y_d, \dot{y}_d]^T \in \Omega_{Z_1} \subset R^3$ and $\Omega_{Z_1} := \{z_1, \bar{x}_{d2} | z_1 \in R, \bar{x}_{d2} \in \Omega_{d2}\}$.

From (4.69), it is found that the controller design is free from unknown time-delay τ_1 at present stage. For notation conciseness, we will omit the time variables t and after time-delay terms have been eliminated.

Since $f_1(\cdot)$ and $g_1(\cdot)$ are unknown smooth function, neural networks shall be used to approximate the function $Q_1(Z_1)$. According to the main result stated in [127], any real-valued continuous function can be arbitrarily closely approximated by a network of RBF type over a compact set. However, it is apparent that $Q_1(Z_1)$ is not continuous over the compact set Ω_{Z_1} as it is not well-defined at $z_1(t) = 0$. Therefore, we shall re-construct the compact set over which the neural network approximation is feasible and valid. To this end, let us define sets $\Omega_{c_{z_1}} \subset \Omega_{Z_1}$ and $\Omega_{Z_1}^0$ as follows

$$\Omega_{c_{z_1}} := \{z_1 \,\big|\, |z_1| < c_{z_1}\} \tag{4.70}$$

$$\Omega_{Z_1}^0 := \Omega_{Z_1} - \Omega_{c_{z_1}} \tag{4.71}$$

From Lemma 4.2.1, we know that $\Omega_{Z_1}^0$ is a compact set, over which function $Q_1(Z_1)$ is continuous and well-defined and can be approximated by neural networks to an arbitrary accuracy as follows

$$Q_1(Z_1) = W_1^{*T}S(Z_1) + \epsilon_1(Z_1) \tag{4.72}$$

where $\epsilon_1(Z_1)$ is the approximation error. Note that as the ideal weight W_1^* is unknown, we shall use its estimate $\hat{W}_1$ instead in the later controller design.

As can be seen from the previous discussion, the control effort will be activated only in the compact set $\Omega_{Z_1}^0$ so that we would like to relax our control objective to boundedness of states around the origin rather than the asymptotic convergence to origin. Accordingly, the following practical adaptive control is proposed

$$u(t) = p_1(z_1, c_{z_1})[-k_1(t)z_1 - \hat{W}_1^T S(Z_1)] \tag{4.73}$$
$$\dot{\hat{W}}_1 = p_1(z_1, c_{z_1})\Gamma_1[S(Z_1)z_1 - \sigma_1(\hat{W}_1 - W_1^0)] \tag{4.74}$$

where $p_1(\cdot,\cdot)$ is defined in (4.62), matrix $\Gamma_1 = \Gamma_1^T > 0$, σ_1 is a small constant to introduce the σ-modification for the closed-loop system, and $k_1(t) > 0$ will be specified later.

The following theorem gives the stability analysis of the proposed controller design.

Theorem 4.3.1 *Consider the closed-loop systems consisting of the first-order plant (4.63), the controller (4.73), if the gain* $k_1(t) = k_{10} + k_{11} + k_{12}(t)$ *is chosen with constants* $k_{10}^* \triangleq k_{10} - \frac{g_{1d}}{2g_{\min}} > 0$, $k_{11} > 0$, *and*

$$k_{12}(t) = \frac{\varepsilon_{10}}{z_1^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}x_1^2(\tau)\varrho_1^2(x_1(\tau))d\tau \tag{4.75}$$

with constant $\varepsilon_{10} > 0$, *and the NN weights are updated by (4.74), then for bounded initial conditions* $x_1(0)$ *and* $\hat{W}_1(0)$, *all signals in the closed-loop systems are SGUUB, and the vector* Z_1 *remains in a compact set* $\Omega_{Z_1}^0$ *specified by*

$$\Omega_{Z_1}^0 = \left\{ Z_1 \left| z_1^2 \le 2g_{\max}C_{01},\ \|\hat{W}_1\|^2 \le \frac{2C_{01}}{\lambda_{\min}(\Gamma_1^{-1})},\ \bar{x}_{d2} \in \Omega_{d2},\ z_1 \notin \Omega_{c_{z_1}} \right. \right\} \tag{4.76}$$

whose size, $C_{01} > 0$, *can be adjusted by appropriately choosing the design parameters.*

Proof: Consider the Lyapunov function candidate $V_1(t)$ as

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \frac{1}{2}(\hat{W}_1(t) - W_1^*)^T\Gamma_1^{-1}(\hat{W}_1(t) - W_1^*) \tag{4.77}$$

Its time derivative along (4.69) is

$$\dot{V}_1(t) \le z_1(t)[u(t) + Q_1(Z_1(t))] + \frac{g_{1d}}{2g_{\min}}z_1^2 + (\hat{W}_1 - W_1^*)^T\Gamma_1^{-1}\dot{\hat{W}}_1 \tag{4.78}$$

Substituting (4.72), (4.73) and (4.74) into (4.78) yields

$$\begin{aligned}\dot{V}_1(t) \le\ & -(p_1k_{10} - \frac{g_{1d}}{2g_{\min}})z_1^2 - p_1k_{12}(t)z_1^2 - p_1k_{11}z_1^2 + \epsilon(Z_1)z_1 \\ & +(1-p_1)W_1^{*T}S(Z_1)z_1 - p_1\sigma_1(\hat{W}_1 - W_1^*)^T(\hat{W}_1 - W_1^0)\end{aligned} \tag{4.79}$$

Now, the stability analysis will be carried out in the following two Regions: (i) $z_1 \in \Omega^0_{Z_1}$, and (ii) $z_1 \in \Omega_{c_{z_1}}$.

Region (i) $z_1 \in \Omega^0_{Z_1}$*:* In this region, $p_1(z_1, c_{z_1}) = 1$, eq. (4.79) becomes

$$\begin{aligned} \dot{V}_1(t) &\leq -(k_{10} - \frac{g_{1d}}{2g_{\min}})z_1^2 - k_{12}(t)z_1^2 - k_{11}z_1^2 + \epsilon(Z_1)z_1 \\ &\quad -\sigma_1(\hat{W}_1 - W_1^*)^T(\hat{W}_1 - W_1^0) \end{aligned} \tag{4.80}$$

Noting the following inequalities

$$-k_{11}z_1^2 + z_1\epsilon_1(Z_1) \leq -k_{11}z_1^2 + |z_1|\epsilon^*_{z_1} \leq \frac{{\epsilon^*_{z_1}}^2}{4k_{11}}$$
$$-\sigma_1(\hat{W}_1 - W_1^*)^T(\hat{W}_1 - W_1^0) \leq -\frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + \frac{1}{2}\sigma_1\|W_1^* - W_1^0\|^2$$

and substituting (4.75) into (4.80), we have

$$\dot{V}_1 \leq -k^*_{10}z_1^2 - \varepsilon_{10}\int_{t-\tau_{\max}}^t \frac{1}{2}x_1^2(\tau)\varrho_1^2(x_1(\tau))d\tau - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_{\epsilon 1} \tag{4.81}$$

with $c_{\epsilon 1} := \frac{1}{2}\sigma_1\|W_1^* - W_1^0\|^2 + \frac{{\epsilon^*_{z_1}}^2}{4k_{11}}$. Since $[t - \tau_1, t] \subset [t - \tau_{\max}, t]$, we have the inequality

$$\int_{t-\tau_1}^t \frac{1}{2}x_1^2(\tau)\varrho_1^2(x_1(\tau))d\tau \leq \int_{t-\tau_{\max}}^t \frac{1}{2}x_1^2(\tau)\varrho_1^2(x_1(\tau))d\tau$$

Accordingly, (4.81) becomes

$$\begin{aligned} \dot{V}_1(t) &\leq -2k^*_{10}g_{\min}V_{z_1}(t) - \varepsilon_{10}g_{\min}V_{U_1}(t) - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_{\epsilon 1} \\ &\leq -c_1V_1(t) + c_{\epsilon 1} \end{aligned} \tag{4.82}$$

where constant $c_1 > 0$ is defined by

$$c_1 := \min\left\{2k^*_{10}g_{\min}, \varepsilon_{10}g_{\min}, \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\right\}$$

Let $\rho_1 := c_{\epsilon 1}/c_1$, it follows that

$$0 \leq V_1(t) \leq \rho_1 + [V_1(0) - \rho_1]e^{-c_1t} \leq \rho_1 + V_1(0) \tag{4.83}$$

Region (ii) $z_1 \in \Omega_{c_{z_1}}$*:* In this region, $|z_1| < c_{z_1}$, i.e., z_1 is already bounded, and $p_1 = 0$, hence $\dot{\hat{W}}_1 = 0$. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. In addition, the adaptation for $\hat{W}_1$ has stopped and $\hat{W}_1$ is kept unchanged in bounded value. Therefore, there exists a finite C_{B1} such that

$$V_1(t) \leq C_{B1} \tag{4.84}$$

From (4.83) and (4.84) for Region (i) and Region (ii), we can conclude that

$$V_1(t) \leq C_{01} \tag{4.85}$$

where $C_0 = \max\{C_{B1}, \rho_1 + V_1(0)\}$. From (4.85), we know that $V_1(t)$ is bounded, hence z_1, x_1, $\hat{W}_1$ are bounded.

In addition, from (4.77), we have

$$z_1^2 \leq 2g_{\max}V_1(t), \quad \|\tilde{W}_1\|^2 \leq \frac{2V_1(t)}{\lambda_{\min}(\Gamma_1^{-1})} \tag{4.86}$$

From (4.85) and (4.86), we readily have the compact set $\Omega_{Z_1}^0$ specified in (4.76), over which the NN approximation is carried out with its feasibility being guaranteed. $\diamondsuit$

Now we are ready to extend the above design methodology to higher-order system using backstepping design.

4.3.3 Direct NN Control for Nth-Order System

In this section, adaptive neural control is proposed for system (4.61) and the stability results of the closed-loop system are presented. The backstepping design procedure contains n steps. The design of adaptive control laws is based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, ..., n$, where $\alpha_i(t)$ is an intermediate control functions designed for the corresponding i-th subsystem based on an appropriate Lyapunov function $V_i(t)$. The control law $u(t)$ is designed in the last step to stabilized the whole closed-loop system based on the overall Lyapunov function V_n, which is partially composed of the sum of the previous $V_i(t)$, $i = 1, ..., n-1$. Note that the controller design based on such compact sets $\Omega_{Z_i}^0$ will render α_i not differentiable at points $|z_i| = c_{z_i}$. This problem can be easily fixed by simply setting the differentiation at these points to be any finite value, say 0, and then every signal in the closed-loop system can be shown to be bounded. Theoretically speaking, by doing so, there is no much loss either as these points are isolated with finite energy and can be ignored. For ease and clarity of presentation, we assume that all the control functions are differentiable throughout this Section.

For uniformity of notation, throughout this section, define estimation errors $\tilde{W}_i = \hat{W}_i - W_i^*$, compact sets $\Omega_{c_{z_i}}$ and $\Omega_{Z_i}^0$ as

$$\begin{aligned} \Omega_{c_{z_i}} &:= \{z_i \mid |z_i| < c_{z_i}\} \\ \Omega_{Z_i}^0 &:= \Omega_{Z_i} - \Omega_{c_{z_i}} \end{aligned}$$

where constants $c_{z_i} > 0$, $\hat{W}_i \in R^{l_i}$ are the estimates of ideal NN weights $W_i^* \in R^{l_i}$, and the following integral Lyapunov functions $V_{z_i}(t)$, the Lyapunov-Krasovskii functionals $V_{U_i}(t)$, and the Lyapunov function candidates $V_i(t)$ as

$$V_{z_i}(t) = \frac{1}{2g_i(\bar{x}_i)} z_i^2 \tag{4.87}$$

$$V_{U_i}(t) = \frac{1}{2g_{\min}} \int_{t-\tau_i}^{t} U_i(\bar{x}_i(\tau)) d\tau \tag{4.88}$$

$$V_i(t) = V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T(t)\Gamma_i^{-1}\tilde{W}_i(t), \tag{4.89}$$

where positive functions $U_i(\bar{x}_i(t)) = \sum_{j=1}^{i} x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t))$.

In the following steps, the unknown functions $Q_i(Z_i)$, $i = 2, ..., n$, which will be defined later, will be approximated by neural networks as

$$Q_i(Z_i) = W_i^{*T} S(Z_i) + \epsilon_i(Z_i), \forall Z_i \in \Omega_{Z_i}^0 \tag{4.90}$$

$\epsilon_{z_i}^*$ are the upper bounds of the NN approximation errors, i.e., $|\epsilon_i(Z_i)| \le \epsilon_{z_i}^*$ with Z_i being the corresponding inputs to be defined later,

Step 1*:* Let us firstly consider the z_1-subsystem as $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$

$$\dot{z}_1(t) = g_1(x_1(t))[z_2(t) + \alpha_1(t)] + f_1(x_1(t)) + h_1(x_1(t-\tau_1)) - \dot{y}_d(t) \tag{4.91}$$

Consider the Lyapunov function candidate in (4.89). Following the same procedure as in Section 4.3.2 by applying Assumption A4) and Young's inequality, we obtain

$$\dot{V}_1 \le z_1[\alpha_1 + Q_1(Z_1)] + \frac{g_{1d}}{2g_{\min}} z_1^2 + z_1 z_2 + (\hat{W}_1 - W_1^*)^T \Gamma_1^{-1} \dot{\hat{W}}_1 \tag{4.92}$$

Applying Young's inequality again for z_1z_2, i.e., $z_1z_2 \le \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$, (4.92) becomes

$$\dot{V}_1 \le (\frac{g_{1d}}{2g_{\min}} + \frac{1}{2})z_1^2 + \frac{1}{2}z_2^2 + z_1[\alpha_1 + Q_1(Z_1)] + (\hat{W}_1 - W_1^*)^T\Gamma_1^{-1}\dot{\hat{W}}_1 \qquad (4.93)$$

where

$$Q_1(Z_1) = \frac{1}{g_1(x_1)}[f_1(x_1) - \dot{y}_d + \frac{1}{2}z_1] + \frac{1}{2g_{\min}z_1}x_1^2\varrho_1^2(x_1)$$

The following practical adaptive control is proposed

$$\alpha_1 = p_1(z_1, c_{z_1})[-k_1(t)z_1 - \hat{W}_1^T S(Z_1)] \qquad (4.94)$$

$$\dot{\hat{W}}_1 = p_1(z_1, c_{z_1})\Gamma_1[S(Z_1)z_1 - \sigma_1(\hat{W}_1 - W_1^0)] \qquad (4.95)$$

Substituting (4.94) and (4.95) into (4.93) yields

$$\dot{V}_1 \le -[k_1(t) - \frac{g_{1d}}{2g_{\min}} - \frac{1}{2}]z_1^2 + \frac{1}{2}z_2^2 + z_1\epsilon(Z_1) - \sigma_1(\hat{W}_1 - W_1^*)^T(\hat{W}_1 - W_1^0)$$

Letting $k_1(t) = k_{10} + k_{11} + k_{12}(t)$ with constant $k_{10}, k_{11} > 0$ such that $k_{10}^* \triangleq k_{10} - \frac{g_{1d}}{2g_{\min}} - \frac{1}{2} > 0$ and

$$k_{12}(t) = \frac{\varepsilon_{10}}{z_1^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}x_1^2(\tau)\varrho_1^2(x_1(\tau))d\tau, \quad \varepsilon_{10} > 0 \qquad (4.96)$$

For $z_1 \in \Omega_{Z_1}^0$, substituting (4.90), (4.94), (4.95), and (4.96) into (4.93) yields

$$\begin{aligned} \dot{V}_1(t) &\le -2k_{10}^* g_{\min}V_{z_1}(t) - \varepsilon_{10}g_{\min}V_{U_1} - \frac{1}{2}\sigma_1\|\hat{W}_1 - W_1^*\|^2 + c_{\epsilon 1} + \frac{1}{2}z_2^2 \\ &\le -c_1V_1(t) + c_{\epsilon 1} + \frac{1}{2}z_2^2 \qquad (4.97) \end{aligned}$$

where constants c_1 and $c_{\epsilon 1}$ are defined as

$$c_1 := \min\left\{2k_{10}^* g_{\min}, \varepsilon_{10}g_{\min}, \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\right\} \qquad (4.98)$$

$$c_{\epsilon 1} := \frac{1}{2}\sigma_1\|W_1^* - W_1^0\|^2 + \frac{{\epsilon_{z_1}^*}^2}{4k_{11}} \qquad (4.99)$$

From (4.97), we know that if z_2 can be regulated as bounded, the boundedness of $V_1(t)$, z_1, x_1 and $\hat{W}_1$ can be obtained as can be seen from Theorem 4.3.1.

The regulation of z_2 will be left to the next step.

Step i ($2 \le i \le n-1$): Similar procedures are taken for $i = 2, \cdots, n-1$ as in Step 1.

The dynamics of z_i-subsystem is given by

$$\dot{z}_i(t) = g_i(\bar{x}_i(t))[z_{i+1}(t) + \alpha_i(t)] + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)) - \dot{\alpha}_{i-1}(t)$$

Consider the Lyapunov function candidate $V_i(t)$ in (4.89). The time derivative of $V_i(t)$ is

$$\begin{aligned}\dot{V}_i(t) &= z_i(t)\Big\{z_{i+1}(t) + \alpha_i(t) + \frac{1}{g_i(\bar{x}_i(t))}[f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)) - \dot{\alpha}_{i-1}(t)]\Big\} \\ &\quad -\frac{\dot{g}_i(\bar{x}_i(t))}{2g_i^2(\bar{x}_i(t))}z_i^2(t) + \frac{1}{2g_{\min}}U_i(\bar{x}_i(t)) - \frac{1}{2g_{\min}}U_i(\bar{x}_i(t-\tau_i)) \\ &\quad +(\hat{W}_i(t) - W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i(t) \end{aligned} \tag{4.100}$$

Using Young's inequality and noting Assumption A4), we have

$$\begin{aligned}\dot{V}_i(t) &\le -[\frac{\dot{g}_i(\bar{x}_i)}{2g_i^2(\bar{x}_i)} - \frac{1}{2}]z_i^2(t) + \frac{1}{2}z_{i+1}^2(t) \\ &\quad +z_i(t)\Big\{\alpha_i(t) + \frac{1}{g_i(\bar{x}_i(t))}[f_i(\bar{x}_i(t)) - \dot{\alpha}_{i-1}(t) + \frac{1}{2}z_i(t)]\Big\} \\ &\quad +\frac{1}{2g_i(\bar{x}_i(t))}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i)) \\ &\quad +\frac{1}{2g_{\min}}\sum_{j=1}^{i}x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t)) - \frac{1}{2g_{\min}}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i)) \\ &\quad +(\hat{W}_i - W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i \end{aligned} \tag{4.101}$$

As $g_i(\bar{x}_i(t)) \ge g_{\min}$, it follows that

$$\frac{1}{2g_i(\bar{x}_i(t))}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i)) - \frac{1}{2g_{\min}}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i)) \le 0$$

Thus, (4.101) becomes

$$\begin{aligned}\dot{V}_i &\le -[\frac{\dot{g}_i(\bar{x}_i)}{2g_i^2(\bar{x}_i)} - \frac{1}{2}]z_i^2(t) + \frac{1}{2}z_{i+1}^2(t) + z_i[\alpha_i + Q_i(Z_i)] \\ &\quad +(\hat{W}_i - W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i \end{aligned} \tag{4.102}$$

where

$$Q_i(Z_i) = \frac{1}{g_i(\bar{x}_i)}[f_i(\bar{x}_i) - \dot{\alpha}_{i-1} + \frac{1}{2}z_i] + \frac{1}{2g_{\min}z_i}\sum_{j=1}^{i}x_j^2(t)\varrho_{ij}^2(\bar{x}_i(t))$$

with $Z_i(t) = [\bar{x}_i, \dot{\bar{x}}_{i-1}, \alpha_{i-1}, \frac{\partial\alpha_{i-1}}{\partial x_1}, \frac{\partial\alpha_{i-1}}{\partial x_2}, ..., \frac{\partial\alpha_{i-1}}{\partial x_{i-1}}, \omega_{i-1}] \in \Omega^0_{Z_i} \subset R^{3i}$, where

$$\dot{\alpha}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1}, \quad \omega_{i-1} = \frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j$$

Similarly, we have the following intermediate control law

$$\alpha_i = q_i(z_i, c_{z_i})[-k_i(t)z_i - \hat{W}_i^T S(Z_i)] \tag{4.103}$$

$$\dot{\hat{W}}_i = q_i(z_i, c_{z_i})\Gamma_i[S(Z_i)z_i - \sigma_i(\hat{W}_i - W_i^0)] \tag{4.104}$$

$$k_i(t) = k_{i0} + k_{i1} + k_{i2}(t), k_{i0}, k_{i1} > 0, k^*_{i0} \triangleq k_{i0} - \frac{g_{id}}{2g_{\min}} - \frac{1}{2} > 0 \tag{4.105}$$

$$k_{i2}(t) = \frac{\varepsilon_{i0}}{z_i^2}\int_{t-\tau_{\max}}^t \frac{1}{2}\sum_{j=1}^{i} x_j^2(\tau)\varrho_{ij}^2(\bar{x}_i(\tau))d\tau, \quad \varepsilon_{i0} > 0 \tag{4.106}$$

For $z_i \in \Omega^0_{Z_i}$, substituting (4.103)-(4.106) into (4.102), and using (4.90), we have

$$\begin{aligned}\dot{V}_i(t) &\leq -2k^*_{i0}g_{\min}V_{z_i}(t) - \varepsilon_{i0}g_{\min}V_{U_i}(t) - \frac{1}{2}\sigma_i\|\hat{W}_i - W_i^*\|^2 + c_{\epsilon i} + \frac{1}{2}z_{i+1}^2 \\ &\leq -c_iV_i(t) + c_{\epsilon i} + \frac{1}{2}z_{i+1}^2 \end{aligned} \tag{4.107}$$

where

$$c_i := \min\left\{2k^*_{i0}g_{\min}, \varepsilon_{i0}g_{\min}, \frac{\sigma_i}{\lambda_{\max}(\Gamma_i^{-1})}\right\} \tag{4.108}$$

$$c_{\epsilon i} := \frac{1}{2}\sigma_i\|W_i^* - W_i^0\|^2 + \frac{{\epsilon^*_{z_i}}^2}{4k_{i1}} \tag{4.109}$$

The effect of z_{i+1} will be handled in the next step.

Step n: This is the final step, since the actual control u appears in the dynamics of z_n-subsystem as given by

$$\dot{z}_n = g_n(x(t))u + f_n(x(t)) + h_n(x(t-\tau_n)) - \dot{\alpha}_{n-1}(t)$$

Consider the Lyapunov function candidate $V_n(t)$ given in (4.89). The time derivative of $V_n(t)$ is

$$\begin{aligned}\dot{V}_n(t) \;=\; & z_n(t)\Big\{u(t) + \frac{1}{g_n(x(t))}[f_n(x(t)) + h_n(x(t-\tau_n)) - \dot{\alpha}_{n-1}(t)]\Big\} \\ & -\frac{\dot{g}_n(x(t))}{2g_n^2(x_n(t))}z_n^2(t) + \frac{1}{2g_{\min}}U_n(x(t)) - \frac{1}{2g_{\min}}U_n(x(t-\tau_n)) \\ & +(\hat{W}_n(t) - W_n^*)^T\Gamma_n^{-1}\dot{\hat{W}}_n(t)\end{aligned}$$

Using Young's inequality and noting Assumption A4), we have

$$\begin{aligned}\dot{V}_n(t) &\le -\frac{\dot{g}_n(x)}{2g_n^2(x)}z_n^2(t) + z_n(t)\Big\{u(t) + \frac{1}{g_n(x(t))}[f_n(x(t)) - \dot{\alpha}_{n-1}(t) + \frac{1}{2}z_n(t)]\Big\}\\ &\quad + \frac{1}{2g_n(x(t))}\sum_{j=1}^{n} x_j^2(t-\tau_n)\varrho_{nj}^2(x(t-\tau_n))\\ &\quad + \frac{1}{2g_{\min}}\sum_{j=1}^{n} x_j^2(t)\varrho_{nj}^2(x(t)) - \frac{1}{2g_{\min}}\sum_{j=1}^{n} x_j^2(t-\tau_n)\varrho_{nj}^2(x(t-\tau_n))\\ &\quad + (\hat{W}_n - W_n^*)^T\Gamma_n^{-1}\dot{\hat{W}}_n \end{aligned} \tag{4.110}$$

As $g_n(x(t)) \ge g_{\min}$, it follows that

$$\frac{1}{2g_n(x(t))}\sum_{j=1}^{n} x_j^2(t-\tau_n)\varrho_{nj}^2(x(t-\tau_n)) - \frac{1}{2g_{\min}}\sum_{j=1}^{n} x_j^2(t-\tau_n)\varrho_{nj}^2(x(t-\tau_n)) \le 0$$

Thus, (4.110) becomes

$$\dot{V}_n \le -\frac{\dot{g}_n(x)}{2g_n^2(x)}z_n^2(t) + z_n[\alpha_n + Q_n(Z_n)] + (\hat{W}_n - W_n^*)^T\Gamma_n^{-1}\dot{\hat{W}}_n \tag{4.111}$$

where

$$Q_n(Z_n) = \frac{1}{g_n(x)}[f_n(x) - \dot{\alpha}_{n-1} + \frac{1}{2}z_n] + \frac{1}{2g_{\min}z_n}\sum_{j=1}^{n} x_j^2(t)\varrho_{nj}^2(x(t))$$

with $Z_n(t) = [x, \dot{\bar{x}}_{n-1}, \alpha_{n-1}, \frac{\partial\alpha_{n-1}}{\partial x_1}, \frac{\partial\alpha_{n-1}}{\partial x_2}, ..., \frac{\partial\alpha_{n-1}}{\partial x_{n-1}}, \omega_{n-1}] \in \Omega_{Z_n}^0 \subset R^{3n}$, where

$$\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}, \quad \omega_{n-1} = \frac{\partial\alpha_{n-1}}{\partial\bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j$$

Similarly, we have the following intermediate control law

$$u = q_n(z_n, c_{z_n})[-k_n(t)z_n - \hat{W}_n^T S(Z_n)] \tag{4.112}$$

$$\dot{\hat{W}}_n = q_n(z_n, c_{z_n})\Gamma_i[S(Z_n)z_n - \sigma_i(\hat{W}_n - W_n^0)] \tag{4.113}$$

$$k_n(t) = k_{n0} + k_{n1} + k_{n2}(t), k_{n0}, k_{n1} > 0, k_{n0}^* \triangleq k_{n0} - \frac{g_{nd}}{2g_{\min}} > 0 \tag{4.114}$$

$$k_{n2}(t) = \frac{\varepsilon_{n0}}{z_n^2}\int_{t-\tau_{\max}}^{t}\frac{1}{2}\sum_{j=1}^{n} x_j^2(\tau)\varrho_{nj}^2(x(\tau))d\tau, \quad \varepsilon_{n0} > 0 \tag{4.115}$$

For $z_n \in \Omega_{Z_n}^0$, substituting (4.112)-(4.115) into (4.111), and using (4.90), we have

$$\begin{aligned}\dot{V}_n(t) &\le -2k_{i0}^* g_{\min}V_{z_n}(t) - \varepsilon_{n0}g_{\min}V_{U_n}(t) - \frac{1}{2}\sigma_n\|\hat{W}_n - W_n^*\|^2 + c_{\epsilon n}\\ &\le -c_nV_n(t) + c_{\epsilon n}\end{aligned} \tag{4.116}$$

where

$$c_n := \min\left\{2k_{n0}^* g_{\min}, \varepsilon_{n0} g_{\min}, \frac{\sigma_n}{\lambda_{\max}(\Gamma_n^{-1})}\right\} \tag{4.117}$$

$$c_{\epsilon n} := \frac{1}{2}\sigma_n \|W_n^* - W_n^0\|^2 + \frac{\epsilon_{z_n}^{*^2}}{4k_{n1}} \tag{4.118}$$

The following theorem shows the stability of the closed-loop adaptive system.

Theorem 4.3.2 *Consider the closed-loop system consisting of the plant (4.61) under Assumptions A1)-A5), the controller (4.112) and the NN weight updating law (4.113). For bounded initial conditions, the following properties hold:*

(i) all signals in the closed-loop system remain semi-globally uniformly ultimately bounded and the vector $Z = [Z_1^T, ..., Z_n^T]^T$ *remains in a compact set* $\Omega_Z^0 := \Omega_{Z_1}^0 \cup ... \cup \Omega_{Z_n}^0$ *specified as*

$$\Omega_Z^0 = \Big\{Z| \sum_{i=1}^n z_i^2 \le 2g_{\max}C_0, \ \sum_{i=1}^n \|\tilde{W}_i\|^2 \le \frac{2C_0}{\lambda_{\min}(\Gamma_i^{-1})}, \ \bar{x}_{di} \in \Omega_{di}, i = 2, ..., n, \\ z_i \notin \Omega_{c_{z_i}}, i = 1, ..., n\Big\} \tag{4.119}$$

where $C_0 > 0$ *is a constant whose size depends on the initial conditions (as will be defined later in the proof);*

(ii) the closed-loop signal $z(t) = [z_1, ..., z_n]^T \in R^n$ *will eventually converge to a compact set defined by*

$$\Omega_S := \{z \;\big|\; \|z\|^2 \le \mu\} \tag{4.120}$$

with $\mu > 0$ *is a constant related to the design parameters and will be defined later in the proof, and* Ω_S *can be made as small as desired by an appropriate choice of the design parameters.*

Proof: Consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^n [V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T \Gamma_i^{-1} \tilde{W}_i] \tag{4.121}$$

where $V_{z_i}(t)$ and $V_{U_i}(t)$ are defined in (4.87) and (4.88) respectively, and $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The following three cases are considered.

Case 1): $z_i \in \Omega_{c_{z_i}}$, $i = 1, ..., n$. In this case, the controls $\alpha_i = 0$, $i = 1, ..., n-1$, $u = 0$ and $\dot{\hat{W}}_i = 0$, $i = 1, ..., n$. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $i = 2, ..., n$, x_i is bounded as $x_i = z_i + \alpha_{i-1}$ and $\alpha_{i-1} = 0$. In addition, $\hat{W}_i$ is kept unchanged in a bounded value, $i = 1, ..., n$. Observing the definition for $V_{z_i}(t)$ and $V_{U_i}(t)$ and noting that $g_i(\cdot)$, $\varrho_{ij}(\cdot)$ are smooth functions, we know that for bounded x_i, z_i and $\hat{W}_i$, $V_{z_i}(t)$ and $V_{U_i}(t)$ are bounded, i.e., there exists a finite C_B such that

$$V(t) \le C_B \tag{4.122}$$

Case 2): $z_i \in \Omega^0_{Z_i}$, $i = 1, ..., n$. From (4.116), we have $\dot{V}_n(t) \le -c_n V_n(t) + c_{\epsilon n}$ where c_n and $c_{\epsilon n}$ are define in (4.117) and (4.118) respectively. Let $\rho_n = c_{\epsilon n}/c_n$, it follows that

$$0 \le V_n(t) \le [V_n(0) - \rho_n]e^{-c_n t} + \rho_n \le V_n(0) + \rho_n \tag{4.123}$$

where constant $V_n(0) = \frac{1}{2g_n(x(0))} z_n^2(0) + \frac{1}{2}\tilde{W}_n^T(0)\Gamma_n^{-1}\tilde{W}_n(0)]$. From (4.89), we have $z_n^2 \le 2g_{\max}V_n(t)$, and $\|\tilde{W}_n\|^2 \le 2V_n(t)/\lambda_{\min}(\Gamma_n^{-1})$.

In Step $n-1$, we have obtained

$$\dot{V}_{n-1}(t) \le -c_{n-1}V_{n-1}(t) + c_{\epsilon,n-1} + \frac{1}{2}z_n^2 \tag{4.124}$$

As $z_n^2 \le 2g_{\max}V_n(t)$ and $V_n(t) \le V_n(0) + \rho_n$, we have

$$\dot{V}_{n-1}(t) \le -c_{n-1}V_{n-1}(t) + c_{\epsilon,n-1} + g_{\max}(V_n(0) + \rho_n) \tag{4.125}$$

Letting $\rho_{n-1} = [c_{\epsilon,n-1} + g_{\max}(V_n(0) + \rho_n)]/c_{n-1}$, from (4.125), we have

$$V_{n-1}(t) \le [V_{n-1}(0) - \rho_{n-1}]e^{-c_{n-1}t} + \rho_{n-1} \le V_{n-1}(0) + \rho_{n-1} \tag{4.126}$$

Noting (4.89), it follows

$$z_{n-1}^2 \le 2g_{\max}V_{n-1}(t) \le 2g_{\max}(V_{n-1}(0) + \rho_{n-1})$$

Similarly, we can conclude that for $i = 1, \cdots, n$

$$z_i^2 \le 2g_{\max}(V_i(0) + \rho_i), \quad \|\tilde{W}_i\|^2 \le \frac{2(V_i(0) + \rho_i)}{\lambda_{\min}(\Gamma_i^{-1})}$$

with $\rho_i = [c_{\epsilon i} + g_{\max}(V_{i-1}(0) + \rho_{i-1})]$.

Case 3): Some $z_i \in \Omega^0_{Z_i}$ and some $z_j \in \Omega_{c_{z_j}}$. In this case, the corresponding α_i or u and the adaptation law for $\hat{W}_i$ will be invoked for $z_i \in \Omega^0_{Z_i}$ while $\alpha_j = 0$ or $u = 0$ and $\dot{\hat{W}}_j = 0$ for $z_j \in \Omega_{c_{z_j}}$. Let us define $V_I(t) = \sum_i (V_{z_i} + V_{U_i} + \frac{1}{2}\tilde{W}_i^T \Gamma_i^{-1} \tilde{W}_i)$ and $V_J(t) = \sum_j (V_{z_j} + V_{U_j} + \frac{1}{2}\tilde{W}_j^T \Gamma_j^{-1} \tilde{W}_j)$. For $z_j \in \Omega_{c_{z_j}}$, we know that $V_J(t)$ is bounded, i.e., $V_J(t) \le C_J$ with C_J being finite, and for $z_i \in \Omega^0_{Z_i}$, we obtain that $\dot{V}_i(t) \le -c_i^I V_i(t) + c^I_{\epsilon,i} + \frac{1}{2} z^2_{i+1}$. Let us define $\rho_i^I = [c^I_{\epsilon,i} + \frac{1}{2}\max\{z^2_{i+1}\}]/c_i^I$, we have

$$V_i(t) \le [V_i(0) - \rho_i^I] e^{-c_i^I t} + \rho_i^I \le V_i(0) + \rho_i^I \tag{4.127}$$

Thus, $V_I \le V_I(0) + \rho_I$ with $V_I(0) = \sum_i V_i(0)$ and $\rho_I = \sum_i \rho_i^I$. Therefore, it can be obtained that

$$V(t) = V_I(t) + V_J(t) \le V_I(0) + \rho_I + C_J \tag{4.128}$$

Thus, from Cases 1), 2) and 3), we can conclude that

$$V(t) \le C_0 \tag{4.129}$$

where $C_0 = \max\{C_B, \sum_{i=1}^n (V_i(0) + \rho_i), V_I(0) + \rho_I + C_J\}$. From (4.129), we know that $V_i(t)$, z_i and $\hat{W}_i$, $i = 1, ..., n$, are bounded. Since $z_1 = x_1 - y_d$ and y_d is bounded, x_1 is bounded. For $x_2 = z_2 + \alpha_1$, since α_1 is function of bounded signals z_1, Z_1, $\hat{W}_1$, α_1 is thus bounded, which in turn leads to the boundedness of x_2. Following the same way, we can prove one by one that all α_{i-1} and x_i, $i = 3, ..., n$ are bounded. Therefore, the systems' states x_i, $i = 1, ..., n$ are bounded.

Considering (4.121), we know that

$$\sum_{i=1}^n z_i^2 \le 2 g_{\max} V(t), \quad \sum_{i=1}^n \|\tilde{W}_i\|^2 \le \frac{2V(t)}{\lambda_{\min}(\Gamma_1^{-1}, ..., \Gamma_n^{-1})} \tag{4.130}$$

From (4.129) and (4.130), we readily have the compact set Ω^0_Z defined in (4.119) over which the NN approximation is carried out with its feasibility being guaranteed.

In addition, in Case 1), as $z_i \in \Omega_{c_{z_i}}$, $i = 1, ...n$, we know that $\|z\|^2 = \sum_{i=1}^n z_i^2 \le \sum_{i=1}^n c^2_{z_i}$. In Case 2), from (4.123) and (4.126), we have that $\lim_{t\to\infty} \|z\|^2 = 2 g_{\max} \sum_{i=1}^n \rho_i$. In Case 3), from (4.127) and (4.130), we have that $\lim_{t\to\infty} \sum_i z_i^2 = 2 g_{\max} \rho_I$ and $\sum_j z_j^2 \le \sum_j c^2_{z_j}$. Therefore as $t \to \infty$, we can conclude that $\|z\|^2 \le \mu$ where $\mu = \max\{2 g_{\max} \sum_{i=1}^n \rho_i, 2 g_{\max}\rho_I, \sum_{i=1}^n c^2_{z_i}\}$, i.e., the vector z will eventually converge to the compact set Ω_S defined in (4.120). This completes the proof. $\diamondsuit$

The practical decoupled backstepping design procedure is illustrated in Fig. 4.9.

Remark 4.3.1 *Note that the proposed design requires the information of $\dot{\bar{x}}_{n-1}(t)$. In fact, the requirement could be removed and similar yet much more involved design can be developed as can be shown in Step i.*

In Step i, $\dot{\alpha}_{i-1}$ can be expressed as

$$\begin{aligned}\dot{\alpha}_{i-1} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j+\omega_{i-1},\quad \omega_{i-1}=\frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j\\ &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}[g_j(\bar{x}_j(t))x_{j+1}(t)+f_j(\bar{x}_j(t))+h_j(\bar{x}_j(t-\tau_j))]+\omega_{i-1}\end{aligned}$$

Consider the quadratic function $V_{z_i}(t)$ given in (4.87). Its time derivative is

$$\begin{aligned}\dot{V}_{z_i}(t) &\le -[\frac{\dot{g}_i(\bar{x}_i)}{2g_i^2(\bar{x}_i)}-\frac{1}{2}]z_i^2(t)+\frac{1}{2}z_{i+1}^2(t)\\ &\quad +z_i(t)\Big\{\alpha_i(t)+\frac{1}{g_i(\bar{x}_i(t))}[f_i(\bar{x}_i(t))+\frac{1}{2}z_i(t)\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^2+\frac{1}{2}z_i(t)]\Big\}\\ &\quad +\frac{1}{2g_i(\bar{x}_i(t))}\sum_{j=1}^{i}x_j^2(t-\tau_i)\varrho_{ij}^2(\bar{x}_i(t-\tau_i))\\ &\quad +\frac{1}{2g_i(\bar{x}_i(t))}\sum_{j=1}^{i-1}\sum_{k=1}^{j}x_k^2(t-\tau_j)\varrho_{jk}^2(\bar{x}_j(t-\tau_j))\\ &\quad +(\hat{W}_i-W_i^*)^T\Gamma_i^{-1}\dot{\hat{W}}_i\end{aligned}$$

The Lyapunov-Krasovskii functional $V_{U_i}(t)$ is given as

$$V_{U_i}(t)=\frac{1}{2g_{\min}}\sum_{j=1}^{i}\int_{t-\tau_j}^{t}U_j(\bar{x}_j(\tau))d\tau \tag{4.131}$$

with positive function $U_j(\bar{x}_j(t))=\sum_{k=1}^{j}x_k^2(t)\varrho_{jk}^2(\bar{x}_j(t))$. Considering the Lyapunov function candidate $V_i(t)$ given in (4.89), we can obtain (4.102) with

$$Q_i(Z_i)=\frac{1}{g_i(\bar{x}_i)}[f_i(\bar{x}_i)+\frac{1}{2}z_i(t)\sum_{j=1}^{i-1}(\frac{\partial\alpha_{i-1}}{\partial x_j})^2+\frac{1}{2}z_i]+\frac{1}{2g_{\min}z_i}\sum_{j=1}^{i}\sum_{k=1}^{j}x_k^2(t)\varrho_{jk}^2(\bar{x}_j(t))$$

with $Z_i(t)=[\bar{x}_i,\frac{\partial\alpha_{i-1}}{\partial x_1},\frac{\partial\alpha_{i-1}}{\partial x_2},...,\frac{\partial\alpha_{i-1}}{\partial x_{i-1}},\omega_{i-1}]\in\Omega_{Z_i}^0\subset R^{2i}$. It can be seen that requirement of $\dot{\bar{x}}_{i-1}$ has been removed and hence the number of the NN input $Z_i(t)$ has been dramatically reduced from $(3i-1)$ to $2i$.

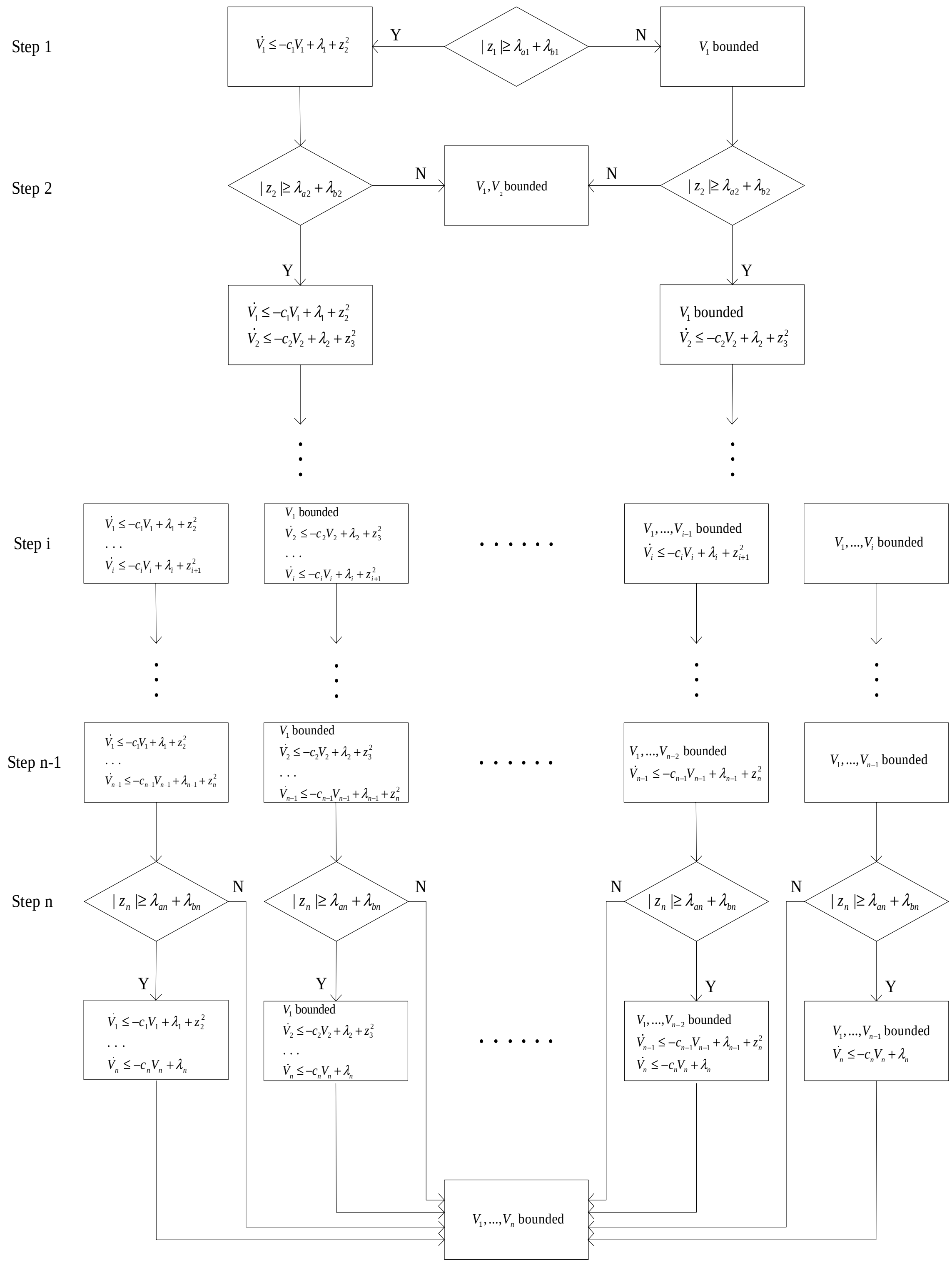

Figure 4.9: Practical decoupled backstepping design procedure.

4.3.4 Conclusion

Practical adaptive neural control has been addressed for a class of nonlinear systems with unknown time delays in strict-feedback form. The unknown time delays has been compensated for through the use of appropriate Lyapunov-Krasovskii functionals. Controller singularity problems have been solved by employing practical neural network control based on decoupled backstepping design. The proposed design has been proven to be able to guarantee semi-globally uniformly ultimate boundedness of all the signals in the closed-loop system and the tracking error is proven to converge to a small neighborhood of the origin. In addition, the residual set of each states in the closed-loop systems has been determined respectively.

Chapter 5

Robust Adaptive Control of Nonlinear Systems with Unknown Time Delays

5.1 Introduction

Motivated by previous works on the nonlinear systems with both unknown time delays and uncertainties from unknown parameters and nonlinear functions, we present in this chapter a practical robust adaptive controller for a class of unknown nonlinear systems in a parametric-strict-feedback form [129]. Using appropriate Lyapunov-Krasovskii functionals in the Lyapunov function candidate, the uncertainties from unknown time delays are removed such that the design of the stabilizing control law is free from these uncertainties. In this way, the iterative backstepping design procedure can be carried out directly. In addition, controller singularities are effectively avoided by employing practical robust control. Time-varying control gains rather than fixed gains are chosen to guarantee the boundedness of all the signals in closed-loop system. The global uniformly ultimately boundedness (GUUB) of the signals in the closed-loop system is achieved and the output of the systems is proven to converge to a small neighborhood of the desired trajectory.

To the best of our knowledge, there is little work dealing with such a kind of systems in the literature at present stage. The proposed method expands the class of nonlinear systems that can be handled using adaptive control techniques. The main contributions of the chapter lie in:

(i) the first employment of robust adaptive backstepping controller design to a class of unknown nonlinear time-delay systems in strict-feedback form, in which the unknown time delays are compensated for by using appropriate Lyaponov-Krasovskii functionals,

(ii) the introduction of differentiable practical control in solving the controller singularity problem, which can be carried out in backstepping design and guarantee that the tracking error will be confined in a compact domain of attraction,

(iii) the elegant re-grouping of unknown parameters, by which the controller singularity problem is effectively avoided, and the lumping of unknown parameter vectors as scalars, by which the number of parameters being estimated is dramatically reduced and the order and complexity of the controller are greatly reduced, and

(iv) the choice of time-varying control gains instead of fixed gains to guarantee the boundedness of all the signals in closed-loop systems.

The rest of the chapter is organized as follows.

The problem formulation and preliminaries are given in Section 5.2. A robust adaptive controller design is illustrated for a first-order system in Section 5.3. The design scheme is extended to a general nth-order system with its stability proof in Section 5.4. A simulation example is given in Section 5.5 followed by Section 5.6, which concludes the work.

5.2 Problem Formulation and Preliminaries

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{aligned} \dot{x}_i(t) &= g_i x_{i+1}(t) + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)), \quad 1 \le i \le n-1 \\ \dot{x}_n(t) &= g_n u(t) + f_n(x(t)) + h_n(x(t-\tau_n)), \\ y(t) &= x_1(t) \end{aligned} \tag{5.1}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output respectively, $f_i(\cdot)$ and $h_i(\cdot)$ are unknown smooth functions, g_i are unknown constants, and τ_i are unknown time delays of the states, $i = 1, \cdots, n$. The control objective is to design an adaptive controller for system (5.1) such that the output $y(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $i = 1, \cdots, n$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$. We have the following assumptions for the system functions, unknown time delays and reference signals.

Assumption 5.2.1 *The signs of g_i are known, and there exist constants $g_{\max} \ge g_{\min} > 0$ such that $g_{\min} \le |g_i| \le g_{\max}$.*

The above assumption implies that unknown constants g_i are either strictly positive or strictly negative. Without losing generality, we shall only consider the case when $g_i > 0$. It should be emphasized that the bounds $g_{\min}$ and $g_{\max}$ are only required for analytical purposes, their true values are not necessarily known since they are not used for controller design.

Assumption 5.2.2 *The unknown functions $f_i(\cdot)$ and $h_i(\cdot)$ can be expressed as*

$$\begin{aligned} f_i(\bar{x}_i(t)) &= \theta_{fi}^T F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) \\ h_i(\bar{x}_i(t)) &= \theta_{hi}^T H_i(\bar{x}_i(t)) + \delta_{hi}(\bar{x}_i(t)) \end{aligned}$$

where $F_i(\cdot)$, $H_i(\cdot)$ are known smooth function vectors, $\theta_{fi} \in R^{n_i}$, $\theta_{hi} \in R^{m_i}$ are unknown constant parameter vectors, n_i, m_i are positive integers, $\delta_{fi}(\cdot)$, $\delta_{hi}(\cdot)$ are *unknown smooth functions, which satisfy the so-called triangular bounds conditions*

$$\begin{aligned} |\delta_{fi}(\bar{x}_i(t))| &\leq c_{fi}\phi_i(\bar{x}_i(t)) \\ |\delta_{hi}(\bar{x}_i(t))| &\leq c_{hi}\psi_i(\bar{x}_i(t)) \end{aligned}$$

where c_{fi}, c_{hi} are constant parameters, which are not necessarily known, and $\phi_i(\cdot)$, $\psi_i(\cdot)$ are known nonnegative smooth functions.

Assumption 5.2.2 is rather weak as only a rough form of $f_i(\cdot)$ and $h_i(\cdot)$ need to be known.

Assumption 5.2.3 *The size of the unknown time delays is bounded by a known constants, i.e., $\tau_i \leq \tau_{\max}$, $i = 1, \cdots, n$.*

There are many physical processes which are governed by nonlinear differential equations of the form (5.1). Examples are recycled reactors, recycled storage tanks and cold rolling mills [130]. In general, most of the recycling processes inherit delays in their state equations. Compared with the systems in [109], the system we consider in this section is more general in the sense that the uncertainty is due to both parametric uncertainty and unknown nonlinear functions. These unknown functions might come from inaccurate modeling or modeling reduction.

To make the problem formulation precisely, the system is presented again as follows

$$\begin{aligned} \dot{x}_i(t) &= g_i x_{i+1}(t) + \theta_{fi}^T F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) + \theta_{hi}^T H_i(\bar{x}_i(t-\tau_i)) + \delta_{hi}(\bar{x}_i(t-\tau_i)), \\ &\quad 1 \leq i \leq n-1 \\ \dot{x}_n(t) &= g_n u(t) + \theta_{fn}^T F_n(x(t)) + \delta_{fn}(x(t)) + \theta_{hn}^T H_n(x(t-\tau_n)) + \delta_{hn}(x(t-\tau_n)), \\ y(t) &= x_1(t) \end{aligned} \tag{5.2}$$

Assumption 5.2.4 *The desired trajectory vectors $\bar{x}_{di} \in \Omega_{di} \subset R^i$, $i = 2, \cdots, n$ are continuous and available with Ω_{di} known compact set.*

The following lemma is used in the controller in solving the problem of chattering.

Lemma 5.2.1 *The following inequality holds for any $\epsilon_1 > 0$ and for any $\eta \in R$*

$$0 \leq |\eta| - \eta \tanh\left(\frac{\eta}{\epsilon_1}\right) \leq k\epsilon_1$$

where k is a constant that satisfies $k = e^{-(k+1)}$, i.e., $k = 0.2785$.

The following two functions are introduced for the purpose of the practical controller design in the next section, and differentiable backstepping design in Section 5.4.

F1). Even function $p_i(\cdot) : R \to R$

$$p_i(x) = \begin{cases} 1, & |x| \geq \lambda_{ai} \\ 0, & |x| < \lambda_{ai} \end{cases}, \quad \forall x \in R. \tag{5.3}$$

F2). Even function $q_i(x) : R \to R$

$$q_i(x) = \begin{cases} 1, & |x| \geq \lambda_{ai} + \lambda_{bi} \\ c_{qi} \int_{\lambda_{ai}}^{x} [(\frac{\lambda_{bi}}{2})^2 - (\sigma - \lambda_{ai} - \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & \lambda_{ai} < x < \lambda_{ai} + \lambda_{bi} \\ c_{qi} \int_{x}^{-\lambda_{ai}} [(\frac{\lambda_{bi}}{2})^2 - (\sigma + \lambda_{ai} + \frac{\lambda_{bi}}{2})^2]^{n-i} d\sigma, & -(\lambda_{ai} + \lambda_{bi}) < x < -\lambda_{ai} \\ 0, & |x| \leq \lambda_{ai} \end{cases} \tag{5.4}$$

where $c_{qi} = \frac{[2(n-i)+1]!}{\lambda_{bi}^{2(n-i)+1}[(n-i)!]^2}$, $\lambda_{ai}, \lambda_{bi} > 0$ and integer $i \in R^+$, is $(n-i)$th differentiable, i.e., $q_i(x) \in C^{n-i}$ and bounded by 1.

5.3 Robust Design for First-order Systems

To illustrate the design methodology clearly, let us consider the tracking problem of a first-order system first

$$\dot{x}_1(t) = g_1 u(t) + \theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) + \theta_{h1}^T H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) \tag{5.5}$$

with $u(t)$ being the control input. Define $z_1 = x_1 - y_d$, we have

$$\begin{aligned} \dot{z}_1(t) &= g_1 u(t) + \theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\ &\quad + \theta_{h1}^T H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t) \end{aligned} \tag{5.6}$$

Consider the scalar function $V_{z_1}(t) = \frac{1}{2g_1} z_1^2(t)$, whose time derivative along (5.6) is

$$\begin{aligned} \dot{V}_{z_1}(t) &= z_1(t)\Big\{u(t) + \frac{1}{g_1}\Big[\theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\ &\quad + \theta_{h1}^T H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t)\Big]\Big\} \end{aligned}$$

Since $\delta_{f1}(\cdot)$ and $\delta_{h1}(\cdot)$ are partially known according to Assumption 5.2.2, we have

$$\begin{aligned}\dot{V}_{z_1}(t) \;\; \leq \;\; & z_1(t)u(t) + \frac{1}{g_1}\Big[z_1(t)\theta_{f1}^T F_1(x_1(t)) \\ & +|z_1(t)|c_{f1}\phi_1(x_1(t)) + z_1(t)\theta_{h1}^T H_1(x_1(t-\tau_1)) \\ & +|z_1(t)|c_{h1}\psi_1(x_1(t-\tau_1)) - z_1(t)\dot{y}_d(t)\Big] \qquad (5.7)\end{aligned}$$

Remark 5.3.1 *It can be seen from (5.7) that the design difficulties come from two system uncertainties: unknown parameters and unknown time delay τ_1. Although $H_1(\cdot)$ and $\psi_1(\cdot)$ are known, they are functions of delayed state $x_1(t-\tau_1)$, which is undetermined due to the unknown time delay τ_1. Thus, functions $H_1(x_1(t-\tau_1))$ and $\psi_1(x_1(t-\tau_1))$ cannot be used in the controller design. In addition, the unknown time delay τ_1 and unknown parameters θ_{h1}^T and c_{h1} are entangled together in a nonlinear fashion, which makes the problem even more complex to solve. Therefore, we have to convert these related terms into such a form that the uncertainties from τ_1, θ_{h1}^T and c_{h1} can be dealt with separately.*

Using Young's Inequality [131], we have

$$\begin{aligned}\dot{V}_{z_1}(t) \;\; \leq \;\; & z_1(t)u(t) + \frac{1}{g_1}\Big[z_1(t)\theta_{f1}^T F_1(x_1(t)) + |z_1(t)|c_{f1}\phi_1(x_1(t)) \\ & +\frac{1}{2}z_1^2(t)\theta_{h1}^T\theta_{h1} + \frac{1}{2}H_1^T(x_1(t-\tau_1))H_1(x_1(t-\tau_1)) \\ & +\frac{1}{2}z_1^2(t)c_{h1}^2 + \frac{1}{2}\psi_1^2(x_1(t-\tau_1)) - z_1(t)\dot{y}_d(t)\Big] \qquad (5.8)\end{aligned}$$

where θ_{h1} and $H_1(x_1(t-\tau_1))$, and c_{h1} and $\psi_1(x_1(t-\tau_1))$ are separated respectively. In fact, parameter vector θ_{h1} and function vector $H_1(x_1(t-\tau_1))$ have been lumped as scalars by applying Young's Inequality, for which they can be dealt with separately as detailed later.

To overcome the design difficulties from the unknown time delay τ_1, the following Lyapunov-Krasovskii functional can be considered

$$V_{U_1}(t) = \frac{1}{2g_1}\int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau \qquad (5.9)$$

where $U_1(\cdot)$ is a positive definite function chosen as

$$U_1(x_1(t)) = H_1^T(x_1(t))H_1(x_1(t)) + \psi_1^2(x_1(t)) \qquad (5.10)$$

The time derivative of $V_{U_1}(t)$ is

$$\begin{aligned}\dot{V}_{U_1}(t) &= \frac{1}{2g_1}\Big[U_1(x_1(t)) - U_1(x_1(t-\tau_1))\Big]\\ &= \frac{1}{2g_1}\Big[H_1^T(x_1)H_1(x_1) + \psi_1^2(x_1) - H_1^T(x_1(t-\tau_1))H_1(x_1(t-\tau_1))\\ &\quad -\psi_1^2(x_1(t-\tau_1))\Big]\end{aligned}$$

which can be used to cancel the time-delay terms on the right hand side of (5.8) and thus eliminate the design difficulty from the unknown time delay τ_1 without introducing any uncertainties to the system. For notation conciseness, we will omit the time variable after time-delay terms have been eliminated. Accordingly, we obtain

$$\begin{aligned}\dot{V}_{z_1} + \dot{V}_{U_1} &\leq z_1 u + \frac{1}{g_1}\Big[z_1\theta_{f1}^T F_1(x_1) + |z_1| c_{f1}\phi_1(x_1)\\ &\quad +\frac{1}{2}z_1^2\theta_{h1}^T\theta_{h1} + \frac{1}{2}H_1^T(x_1)H_1(x_1) + \frac{1}{2}z_1^2 c_{h1}^2 + \frac{1}{2}\psi_1^2(x_1) - z_1\dot{y}_d\Big]\\ &\triangleq z_1(u + \theta_1^T F_{\theta 1}) + \theta_{10}|z_1|\phi_{10}\end{aligned} \quad (5.11)$$

where θ_{10} is an unknown constant, θ_1 is an unknown constant vector, $\phi_{10}(\cdot)$ is a known function, and $F_{\theta 1}(\cdot)$ is a known function vector defined below

$$\begin{aligned}&\theta_{10} := \frac{c_{f1}}{g_1}, \quad \theta_1 := [\frac{\theta_{f1}^T}{g_1}, \frac{\theta_{h1}^T\theta_{h1} + c_{h1}^2}{g_1}, \frac{1}{g_1}]^T \in R^{n_1+2},\\ &\phi_{10} := \phi_1, \quad F_{\theta 1} = \Big[F_1^T,\ \frac{1}{2}z_1,\ \frac{1}{2z_1}(H_1^T H_1 + \psi_1^2) - \dot{y}_d\Big]^T \in R^{n_1+2}\end{aligned}$$

Note that the design of $u(t)$ is free from unknown time delay τ_1 at present stage. To stabilize $z_1(t)$, the following desired certainty equivalent control [59] under the assumption of exact knowledge could be proposed as

$$u^* = -k_1 z_1 - \theta_1^T F_{\theta 1} - \beta_1(z_1) \quad (5.12)$$

where $k_1 > 0$ and $\beta_1(z_1) = \mathrm{sgn}(z_1)\theta_{10}\phi_{10}$.

Remark 5.3.2 *The introduction of θ_1 has two advantages. Firstly, we only need to estimate $\frac{1}{g_1}$ rather than g_1 such that the possible controller singularity due to $\hat{g}_1 = 0$ is avoided. Secondly, after applying Young's inequality, unknown constant vector $\theta_{h1} \in R^{m_1}$ is lumped as a scalar $\theta_{h1}^T\theta_{h1}$. By doing so, the number of parameters being* *estimated is dramatically reduced, which greatly reduces the order and complexity of the controller.*

However, controller singularity may occur since the proposed desired control (5.12) is not well-defined at $z_1 = 0$. Therefore, care must be taken to guarantee the boundedness of the control. It is noted that the controller singularity takes place at the point $z_1 = 0$, where the control objective is supposed to be achieved. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. As $z_1 = 0$ is hard to detect owing to the existence of measurement noises, it is more practical to relax our control objective of convergence to a bounded region rather than the origin. Next, let us show that certain bounded region is a domain of attraction in the sense that all z_1 will enter into this region and will stay within thereafter. In the case that the parameters are unknown, we propose the practical robust adaptive control law to guarantee the systems stability as detailed in Lemma 5.3.1.

Lemma 5.3.1 *For the first-order system (5.5), if the practical robust control law is chosen as*

$$u = p_1(z_1)\left[-k_1(t)z_1 - \hat{\theta}_1^T F_{\theta 1} - \beta_1(z_1, \hat{\theta}_{10})\right] \tag{5.13}$$

$$\beta_1(z_1, \hat{\theta}_{10}) = \mathrm{sgn(z_1)}\hat{\theta}_{10}\phi_{10} \tag{5.14}$$

where $p_1(\cdot)$ *is defined in (5.3),* $\hat{\theta}_{10}$ *and* $\hat{\theta}_1$ *are the estimates of* θ_{10} *and* θ_1 *respectively,* $k_1(t) \geq k^* > 0$ *with* k^* *being any positive constant, and the parameters are updated by*

$$\dot{\hat{\theta}}_{10} = p_1(z_1)\gamma_1|z_1|\phi_{10} \tag{5.15}$$

$$\dot{\hat{\theta}}_1 = p_1(z_1)\Gamma_1 F_{\theta 1} z_1 \tag{5.16}$$

with $\gamma_1 > 0$ *and* $\Gamma_1 = \Gamma_1^T > 0$*, then for bounded initial conditions* $x_1(0)$*,* $\hat{\theta}_{10}(0)$ *and* $\hat{\theta}_1(0)$*, all signals in the closed-loop system are bounded, and the tracking error* $z_1(t)$ *will finally stay in a compact set defined by* $\Omega_{z_1} = \{z_1 \in R \mid |z_1| \leq \lambda_{a1}\}$*.*

Proof: To show Ω_{z_1} to be a domain of attraction, we first find a Lyapunov function candidate $V_1(t) > 0$ such that $\dot{V}_1(t) \leq 0$, $\forall z_1 \notin \Omega_{z_1}$. For $|z_1| \geq \lambda_{a1}$, let us consider the following Lyapunov function candidate

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \frac{1}{2}\gamma_1^{-1}\tilde{\theta}_{10}^2(t) + \frac{1}{2}\tilde{\theta}_1^T(t)\Gamma_1^{-1}\tilde{\theta}_1(t)$$

where $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)$. The time derivative of $V_1(t)$ along (5.11) is

$$\dot{V}_1(t) \le z_1(u + \theta_1^T F_{\theta 1}) + \theta_{10}|z_1|\phi_{10} + \gamma_1^{-1}\tilde{\theta}_{10}\dot{\hat{\theta}}_{10} + \tilde{\theta}_1^T\Gamma_1^{-1}\dot{\hat{\theta}}_1 \tag{5.17}$$

Substituting (5.13), (5.14), (5.15) and (5.16) into (5.17), we obtain $\dot{V}_1 \le -k_1(t)z_1^2 \le -k^* z_1^2 \le 0$. Hence, $V_1(t)$ is a Lyapunov function and $z_1(t)$, $x_1(t)$, $\hat{\theta}_{10}(t)$, $\hat{\theta}_1(t)$ are bounded. In addition, z_1 is square integrable since $\int_0^t k^* z_1^2(\tau)d\tau \le V_1(0)$ and $u(t)$ is bounded due to the boundedness of x_1, $\hat{\theta}_{10}$ and $\hat{\theta}_1$. Thus, $\dot{z}_1$ is bounded. From Barbalat's Lemma, we know that $\lim_{t\to\infty} z_1(t) = 0$. Note that the control effort is only activated when $|z_1| \ge \lambda_{a1}$, we can conclude that for $t \to \infty$, $|z_1(t)| \le \lambda_{a1}$. For $|z_1| < \lambda_{a1}$, since $z_1 = x_1 - x_d$, $\dot{\hat{\theta}}_{10} = 0$ and $\dot{\hat{\theta}}_1 = 0$, x_1 is bounded, $\hat{\theta}_{10}$ and $\hat{\theta}_1$ are kept unchanged in bounded values. We can readily conclude that the tracking error $|z_1(t)| \le \lambda_{a1}$ while all the other closed-loop signals are bounded. $\diamondsuit$

The key point of the proposed design lies in two aspects. Firstly, the Lyapunov-Krasovskii functional is utilized such that the design difficulties from unknown time delay has been removed. Secondly, the practical robust control scheme has employed to avoid possible controller singularity. It is well known in [132][133] that the above discontinuous control scheme should be avoided as it will cause chattering phenomena and excite high-frequency unmodeled dynamics. Furthermore, we would like to extend the methodology described in this section from first-order systems to more general nth-order systems. To achieve this objective, the iterative backstepping design can be used, which requires the differentiation of the control u and the control component β_1 at each step. Therefore, appropriate smooth control functions shall be used, and at the same time the controller should guarantee the boundedness of all the signals in the closed-loop and z_1 will still stay in certain domain of attraction.

5.4 Robust Design for Nth-order Systems

In this section, the adaptive design will be extended to nth-order systems (5.2) and the stability results of the closed-loop system are presented.

Note that the extension requires the smoothness of control functions to certain degree, which is not straightforward but very much involved. In the recursive backstepping design, the computation of the control function $\alpha_i(t)$ in each step requires that of $\dot{\alpha}_{i-1}(t)$, $\ddot{\alpha}_{i-2}(t)$, ..., $\alpha_1^{(i-1)}(t)$. As a result, $\alpha_i(t)$ need to be at least $(n-i)$th differentiable. On the other hand, the unknown time delay terms of all the previous subsystems will appear in Step i, which have to be compensated for one by one. In the following controller design, function $q_i(\cdot)$ is utilized to construct the differentiable control function. For ease of notation, the following compact sets are defined

$$\begin{aligned}
\Omega_{Z_i} &:= \{z_i \in R \mid |z_i| \le \lambda_{ai}\} \\
\Omega_{Z_i}^{0_I} &:= \{z_i \in R \mid \lambda_{ai} < |z_i| < \lambda_{ai} + \lambda_{bi}\} \\
\Omega_{Z_i}^{0_O} &:= \{z_i \in R \mid |z_i| \ge \lambda_{ai} + \lambda_{bi}\}
\end{aligned}$$

The backstepping design procedure contains n steps. At each step, an intermediate control function $\alpha_i(t)$ shall be developed using an appropriate Lyapunov function $V_i(t)$. The design of both the control laws and the adaptive laws are based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \cdots, n$.

Step 1: Let us firstly consider the z_1-subsystem as

$$\begin{aligned}
\dot{z}_1(t) &= g_1(z_2(t) + \alpha_1(t)) + \theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\
&\quad + \theta_{h1}^T H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t)
\end{aligned} \tag{5.18}$$

The time derivative of the scalar function $V_{z_1}(t) = \frac{1}{2g_1} z_1^2(t)$ along (5.18) is

$$\begin{aligned}
\dot{V}_{z_1}(t) &= z_1(t)z_2(t) + z_1(t)\Big\{\alpha_1(t) + \frac{1}{g_1}\Big[\theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) \\
&\quad + \theta_{h1}^T H_1(x_1(t-\tau_1)) + \delta_{h1}(x_1(t-\tau_1)) - \dot{y}_d(t)\Big]\Big\}
\end{aligned}$$

Following the same procedure as in section 5.3 by choosing V_{U_1} in (5.9) and applying Assumption 5.2.2 and Young's inequality, we obtain

$$\dot{V}_{z_1} + \dot{V}_{U_1} \le z_1 z_2 + z_1(\alpha_1 + \theta_1^T F_{\theta 1}) + \theta_{10}|z_1|\phi_{10} \tag{5.19}$$

As stated in section 5.3, the control objective now is to show that z_1 will converge to certain domain of attraction rather than the origin. At the same time, the control functions shall be smooth or at least differentiable to certain degree.

Let us consider the following smooth adaptive scheme

$$\alpha_1 = q_1(z_1)\left[-k_1(t)z_1 - \hat{\theta}_1^T F_{\theta 1} - \beta_1\right] \quad (5.20)$$

$$k_1(t) = k_{10} + \frac{1}{z_1^2}\int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))d\tau \quad (5.21)$$

$$\beta_1 = \hat{\theta}_{10}\xi_1 \quad (5.22)$$

$$\xi_1 = \phi_{10}\tanh\left(\frac{z_1\phi_{10}}{\epsilon_1}\right) \quad (5.23)$$

$$\dot{\hat{\theta}}_{10} = q_1(z_1)\gamma_1(z_1\xi_1 - \sigma_{10}\hat{\theta}_{10}) \quad (5.24)$$

$$\dot{\hat{\theta}}_1 = q_1(z_1)\Gamma_1(F_{\theta 1}z_1 - \sigma_1\hat{\theta}_1) \quad (5.25)$$

where $k_{10} > 0$ is a design constant, $\epsilon_1 > 0$ is a small constant, σ_{10}, $\sigma_1 > 0$ are small constants to introduce the $\sigma-$modification for the closed-loop system.

Consider the following Lyapunov function candidate

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \frac{1}{2}\gamma_1^{-1}\tilde{\theta}_{10}^2(t) + \frac{1}{2}\tilde{\theta}_1^T(t)\Gamma_1^{-1}\tilde{\theta}_1(t) \quad (5.26)$$

Let us first show the time derivative of $V_1(t)$ along (5.20)-(5.25) for $z_1 \in \Omega_{Z_1}^{0O}$. As $q_1(z_1) = 1$ as $z_1 \in \Omega_{Z_1}^{0O}$, we have

$$\dot{V}_1(t) \leq -k_{10}z_1^2 - \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))d\tau + z_1z_2 + \theta_{10}\left[|z_1|\phi_{10} - z_1\phi_{10}\tanh\left(\frac{z_1\phi_{10}}{\epsilon_1}\right)\right]$$
$$-\sigma_{10}\tilde{\theta}_{10}\hat{\theta}_{10} - \sigma_1\tilde{\theta}_1^T\hat{\theta}_1$$

Using the inequalities

$$-\frac{1}{4}k_{10}z_1^2 + z_1z_2 \leq \frac{1}{k_{10}}z_2^2$$

$$-\sigma_{10}\tilde{\theta}_{10}\hat{\theta}_{10} \leq -\frac{1}{2}\sigma_{10}\tilde{\theta}_{10}^2 + \frac{1}{2}\sigma_{10}\theta_{10}^2 \quad (5.27)$$

$$-\sigma_1\tilde{\theta}_1^T\hat{\theta}_1 \leq -\frac{1}{2}\sigma_1\|\tilde{\theta}_1\|^2 + \frac{1}{2}\sigma_1\|\theta_1\|^2 \quad (5.28)$$

and applying Lemma 5.2.1, we have

$$\dot{V}_1(t) \leq -\frac{3}{4}k_{10}z_1^2 - \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))d\tau + \frac{1}{k_{10}}z_2^2 - \frac{1}{2}\sigma_{10}\tilde{\theta}_{10}^2 - \frac{1}{2}\sigma_1\|\tilde{\theta}_1\|^2 + \lambda_1 \quad (5.29)$$

where constant $\lambda_1 > 0$ is defined by

$$\lambda_1 := \frac{1}{2}\sigma_{10}\theta_{10}^2 + \frac{1}{2}\sigma_1\|\theta_1\|^2 + 0.2785\epsilon_1\theta_{10}$$

Since $\tau_1 \leq \tau_{\max}$ according to Assumption 5.2.3, the following inequality holds

$$\int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau \leq \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))d\tau$$

Accordingly, (5.29) becomes

$$\begin{aligned}\dot{V}_1(t) &\leq -\frac{3}{2}g_{\min}k_{10}V_{z_1} - 2g_{\min}V_{U_1} - \frac{1}{2}\sigma_{10}\tilde{\theta}_{10}^2 - \frac{1}{2}\sigma_1\|\tilde{\theta}_1\|^2 + \lambda_1 + \frac{1}{k_{10}}z_2^2 \\ &\leq -c_1V_1(t) + \lambda_1 + \frac{1}{k_{10}}z_2^2 \end{aligned} \tag{5.30}$$

where constant $c_1 > 0$ is defined by

$$c_1 := \min\Big\{\frac{3}{2}g_{\min}k_{10},\ 2g_{\min},\ \sigma_{10}\gamma_1,\ \frac{\sigma_1}{\lambda_{\max}(\Gamma_1^{-1})}\Big\}$$

Remark 5.4.1 *For $z_1 \in \Omega_{Z_1}^{0_O}$, if there is no extra term z_2^2 within the inequality (5.30), we can conclude that $V_1(t)$ is bounded, and thus z_1, $\hat{\theta}_{10}$ and $\hat{\theta}_1$ are bounded. However, it may not be the case due to the presence of the extra term z_2^2. It is found that if z_2 can be regulated as bounded, say, $|z_2| \leq z_{2\max}$ with $z_{2\max}$ being finite, we have*

$$\dot{V}_1(t) \leq -c_1V_1(t) + \bar{\lambda}_1$$

with $\bar{\lambda}_1 = \lambda_1 + \frac{1}{k_{10}}z_{2\max}^2$. The stability analysis for this case will be conducted later.

Next, let us consider $z_1 \in \Omega_{Z_1}^{0_I}$, i.e., $\lambda_{a1} < |z_1| < \lambda_{a1} + \lambda_{b1}$. As z_1 is bounded, $x_1 = z_1 + y_d$ is also bounded. Considering the smooth positive functions $V_{z_1}(t)$ and $V_{U_1}(t)$, we know that $V_{z_1}(t)$ and $V_{U_1}(t)$ are bounded. Let us define positive function $V_{\theta 1}(t) := \frac{1}{2}\tilde{\theta}_1^T(t)\Gamma_1^{-1}\tilde{\theta}_1(t)$. Its time derivation along (5.25) is

$$\dot{V}_{\theta 1}(t) = q_1(z_1)\tilde{\theta}_1^T(F_{\theta 1}z_1 - \sigma_1\hat{\theta}_1) \tag{5.31}$$

Applying the inequalities

$$\begin{aligned} q_1(z_1)\tilde{\theta}_1^TF_{\theta 1}z_1 &\leq \frac{1}{2k_{\theta 1}}q_1(z_1)\|\tilde{\theta}_1\|^2 + \frac{k_{\theta 1}}{2}q_1(z_1)F_{\theta 1}^TF_{\theta 1}z_1^2, \quad k_{\theta 1} > 0 \\ -q_1(z_1)\sigma_1\tilde{\theta}_1^T\hat{\theta}_1 &\leq -\frac{1}{2}q_1(z_1)\sigma_1\|\tilde{\theta}_1\|^2 + \frac{1}{2}q_1(z_1)\sigma_1\|\theta_1\|^2 \end{aligned}$$

eq. (5.31) becomes

$$\dot{V}_{\theta 1}(t) \leq -\frac{1}{2}q_1(z_1)(\sigma_1 - \frac{1}{k_{\theta 1}})\|\tilde{\theta}_1\|^2 + \frac{1}{2}q_1(z_1)(\sigma_1\|\theta_1\|^2 + k_{\theta 1}F_{\theta 1}^TF_{\theta 1}z_1^2)$$

For $z_1 \in \Omega_{Z_1}^{0_I}$, we know that $q_1(z_1) \in (0,1)$, and $F_{\theta 1}$ is smooth and bounded. Choosing $k_{\theta 1}$ such that $\sigma_1^* := \sigma_1 - \frac{1}{k_{\theta 1}} > 0$, and letting $\lambda_{\theta 1} := \sup_{z_1 \in \Omega_{Z_1}^{0_I}} \{\sigma_1 \|\theta_1\|^2 + k_{\theta 1} F_{\theta 1}^T F_{\theta 1} z_1^2\}$, we have

$$\begin{aligned} \dot{V}_{\theta 1}(t) &\leq -\frac{1}{2} q_1(z_1) \sigma_1^* \|\tilde{\theta}_1\|^2 + \frac{1}{2} q_1(z_1) \lambda_{\theta 1} \\ &\leq -q_1(z_1) \frac{\sigma_1^*}{\lambda_{\max}(\Gamma_1^{-1})} V_{\theta 1}(t) + \frac{1}{2} q_1(z_1) \lambda_{\theta 1} \end{aligned} \tag{5.32}$$

Letting $c_{\theta 1}^q := q_1(z_1) \frac{\sigma_1^*}{\lambda_{\max}(\Gamma_1^{-1})}$, $\lambda_{\theta 1}^q := \frac{1}{2} q_1(z_1) \lambda_{\theta 1}$, and

$$\rho_{\theta 1}^q := \lambda_{\theta 1}^q / c_{\theta 1}^q = \frac{1}{2} \lambda_{\theta 1} \lambda_{\max}(\Gamma_1^{-1}) / \sigma_1^*$$

it follows from (5.32) that

$$0 \leq V_{\theta 1}(t) \leq [V_{\theta 1}(0) - \rho_{\theta 1}^q] e^{-c_{\theta 1}^q t} + \rho_{\theta 1}^q \leq V_{\theta 1}(0) + \rho_{\theta 1}^q$$

from which, we can conclude that $V_{\theta 1}(t)$ is bounded, and hence $\tilde{\theta}_1$ is bounded. Similarly, it can be shown that $\tilde{\theta}_{10}$ is bounded as well. Consider the Lyapunov function candidate $V_1(t)$ defined in (5.26). As it has been already shown that $V_{z_1}(t)$, $V_{U_1}(t)$, $\tilde{\theta}_{10}$ and $\tilde{\theta}_1$ are bounded, we can conclude that $V_1(t)$ is bounded for $z_1 \in \Omega_{Z_1}^{0_I}$.

For $z_1 \in \Omega_{Z_1}$, i.e., $|z_1| \leq \lambda_{a1}$ is bounded, we know that $q_1(z_1) = 0$, $\dot{\hat{\theta}}_{10} = 0$ and $\dot{\hat{\theta}}_1 = 0$. Hence, $x_1 = z_1 + y_d$ is bounded, and $\hat{\theta}_{10}$ and $\hat{\theta}_1$ are kept unchanged in bounded values. As $V_{z_1}(t)$ and $V_{U_1}(t)$ are smooth functions, we know that for bounded x_1 and z_1, $V_{z_1}(t)$ and $V_{U_1}(t)$ are bounded, and $V_1(t)$ is bounded.

Remark 5.4.2 *Note that the boundedness of the closed-loop signals as x_1, z_1, $\hat{\theta}_{10}$, $\hat{\theta}_1$ for $z_1 \in \Omega_{Z_1}^{0_O}$ and $z_1 \in \Omega_{Z_1}$ is independent of the signal z_2.*

Remark 5.4.3 *Note that both the intermediate control function (5.20) and the updating laws (5.24), (5.25) are differentiable, which makes it possible to carry out the backstepping design in the next steps.*

The regulation of z_2 will be shown in the next steps.

Step 2: Since $z_2 = x_2 - \alpha_1$ and $z_3 = x_3 - \alpha_2$, the time derivative of z_2 is given by

$$\begin{aligned}\dot{z}_2(t) &= g_2(z_3(t) + \alpha_2(t)) + \theta_{f2}^T F_2(\bar{x}_2(t)) + \delta_{f2}(\bar{x}_2(t)) \\ &\quad +\theta_{h2}^T H_2(\bar{x}_2(t-\tau_2)) + \delta_{h2}(\bar{x}_2(t-\tau_2)) - \dot{\alpha}_1(t) \qquad (5.33)\end{aligned}$$

By viewing $x_3(t)$ as a virtual control, we may design a control input $\alpha_2(t)$ for (5.33).

Since $\alpha_1(t)$ is a function of $x_1(t)$, y_d, $\dot{y}_d$, $\hat{\theta}_{10}$ and $\hat{\theta}_1$, $\dot{\alpha}_1(t)$ can be expressed as

$$\begin{aligned}\dot{\alpha}_1(t) &= \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 + \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{\theta}_{10}}\dot{\hat{\theta}}_{10} + \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1 \\ &= \frac{\partial \alpha_1}{\partial x_1}\Big[g_1 x_2(t) + \theta_{f1}^T F_1(x_1(t)) + \delta_{f1}(x_1(t)) + \theta_{h1}^T H_1(x_1(t-\tau_1)) \\ &\quad +\delta_{h1}(x_1(t-\tau_1))\Big] + \omega_1(t) \qquad (5.34)\end{aligned}$$

where

$$\omega_1(t) = \frac{\partial \alpha_1}{\partial \bar{x}_{d2}}\dot{\bar{x}}_{d2} + \frac{\partial \alpha_1}{\partial \hat{\theta}_{10}}\dot{\hat{\theta}}_{10} + \frac{\partial \alpha_1}{\partial \hat{\theta}_1}\dot{\hat{\theta}}_1$$

Similarly, let us consider scalar function $V_{z_2}(t) = \frac{1}{2g_2}z_2^2(t)$. By applying Assumption 5.2.2 and Using Young's Inequality, its time derivative along (5.33) and (5.34) is given by

$$\begin{aligned}\dot{V}_{z_2} \leq\ & z_2 z_3 + z_2\alpha_2 + \frac{1}{g_2}\Big\{z_2\theta_{f2}^T F_2(\bar{x}_2) + |z_2| c_{f2}\phi_2(\bar{x}_2) + \frac{1}{2}z_2^2\theta_{h2}^T\theta_{h2} \\ & +\frac{1}{2}H_2^T(\bar{x}_2(t-\tau_2))H_2(\bar{x}_2(t-\tau_2)) + \frac{1}{2}z_2^2 c_{h2}^2 + \frac{1}{2}\psi_2^2(\bar{x}_2(t-\tau_2)) \\ & -g_1 z_2\frac{\partial \alpha_1}{\partial x_1}x_2 - z_2\frac{\partial \alpha_1}{\partial x_1}\theta_{f1}^T F_1(x_1) + |z_2||\frac{\partial \alpha_1}{\partial x_1}|c_{f1}\phi_1(x_1) \\ & +\frac{1}{2}z_2^2(\frac{\partial \alpha_1}{\partial x_1})^2\theta_{h1}^T\theta_{h1} + \frac{1}{2}H_1^T(x_1(t-\tau_1))H_1(x_1(t-\tau_1)) \\ & +\frac{1}{2}z_2^2(\frac{\partial \alpha_1}{\partial x_1})^2 c_{h1}^2 + \frac{1}{2}\psi_1^2(x_1(t-\tau_1)) - z_2\omega_1\Big\}\end{aligned}$$

Note that due to the differentiating of $\alpha_1(t)$, both the unknown time delay τ_1 from the first subsystem and τ_2 from the current subsystem have appeared. The Lyapunov-Krasovskii functional used earlier to compensate for τ_1 shall be utilized repeatedly in this step to construct the following functional

$$V_{U_2}(t) = \frac{1}{2g_2}\Big[\int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau + \int_{t-\tau_2}^{t} U_2(\bar{x}_2(\tau))d\tau\Big]$$

where $U_2(\cdot)$ is a positive definite function defined by

$$U_2(\bar{x}_2(t)) = H_2^T(\bar{x}_2(t))H_2(\bar{x}_2(t)) + \psi_2^2(\bar{x}_2(t))$$

and $U_1(\cdot)$ is defined in (5.10), we have

$$
\begin{aligned}
\dot{V}_{z_2} + \dot{V}_{U_2} \ \leq\ & z_2 z_3 + z_2\alpha_2 + \frac{1}{g_2}\Big\{ z_2\theta_{f2}^T F_2(\bar{x}_2) + |z_2| c_{f2}\phi_2(\bar{x}_2) + \frac{1}{2} z_2^2 \theta_{h2}^T\theta_{h2} \\
& + \frac{1}{2} H_2^T(\bar{x}_2) H_2(\bar{x}_2) + \frac{1}{2} z_2^2 c_{h2}^2 + \frac{1}{2}\psi_2^2(\bar{x}_2) \\
& - g_1 z_2 \frac{\partial\alpha_1}{\partial x_1} x_2 - z_2 \frac{\partial\alpha_1}{\partial x_1}\theta_{f1}^T F_1(x_1) + |z_2|\left|\frac{\partial\alpha_1}{\partial x_1}\right| c_{f1}\phi_1(x_1) \\
& + \frac{1}{2} z_2^2 \left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 \theta_{h1}^T\theta_{h1} + \frac{1}{2} H_1^T(x_1) H_1(x_1) + \frac{1}{2} z_2^2 \left(\frac{\partial\alpha_1}{\partial x_1}\right)^2 c_{h1}^2 + \frac{1}{2}\psi_1^2(x_1) \\
& - z_2\omega_1 \Big\} \\
\triangleq\ & z_2 z_3 + z_2(\alpha_2 + \theta_2^T F_{\theta 2}) + \theta_{20}|z_2|\phi_{20} \qquad (5.35)
\end{aligned}
$$

where θ_{20} is an unknown constant, θ_2 is an unknown constant vector, $\phi_{20}(\cdot)$ is a known function, and $F_{\theta 2}(\cdot)$ is a known function vector defined below

$$
\begin{aligned}
\theta_{20} &:= \max\{c_{f1}, c_{f2}\}, \\
\theta_2 &:= \left[\frac{\theta_{f2}^T}{g_2},\ \frac{\theta_{h2}^T\theta_{h2} + c_{h2}^2}{g_2},\ \frac{g_1}{g_2},\ \frac{\theta_{f1}^T}{g_2},\ \frac{\theta_{h1}^T\theta_{h1} + c_{h1}^2}{g_2},\ \frac{1}{g_2}\right]^T \in R^{n_1+n_2+4}, \\
\phi_{20} &:= \phi_2 + \left|\frac{\partial\alpha_1}{\partial x_1}\right|\phi_1, \\
F_{\theta 2} &:= \Big[F_2^T,\ \frac{1}{2} z_2,\ -\frac{\partial\alpha_1}{\partial x_1} x_2,\ -\frac{\partial\alpha_1}{\partial x_1} F_1, \frac{1}{2} z_2 \left(\frac{\partial\alpha_1}{\partial x_1}\right)^2,\ \frac{1}{2z_2}\sum_{j=1}^{2}(H_j^T H_j + \psi_j^2) - \omega_1\Big]^T \\
& \quad \in R^{n_1+n_2+4}
\end{aligned}
$$

Similarly, the following robust adaptive intermediate control law is proposed

$$
\alpha_2 = q_2(z_2)[-k_2(t) z_2 - \hat{\theta}_2^T F_{\theta 2} - \beta_2] \qquad (5.36)
$$

$$
k_2(t) = k_{20} + \frac{1}{z_2^2}\int_{t-\tau_{\max}}^{t} [U_1(x_1(\tau)) + U_2(\bar{x}_2(\tau))] d\tau \qquad (5.37)
$$

$$
\beta_2 = \hat{\theta}_{20}\xi_2 \qquad (5.38)
$$

$$
\xi_2 = \phi_{20}\tanh\left(\frac{z_2\phi_{20}}{\epsilon_2}\right) \qquad (5.39)
$$

where $k_{20} > 0$ is a design constant, $\epsilon_2 > 0$ is a small constant.

The adaptive laws are given for online tuning the unknown parameters

$$
\dot{\hat{\theta}}_{20} = q_2(z_2)\gamma_2(z_2\xi_2 - \sigma_{20}\hat{\theta}_{20}) \qquad (5.40)
$$

$$
\dot{\hat{\theta}}_2 = q_2(z_2)\Gamma_2(F_{\theta 2} z_2 - \sigma_2\hat{\theta}_2) \qquad (5.41)
$$

where $\gamma_2 > 0$, $\Gamma_2 = \Gamma_2^{-1} > 0$, and $\sigma_{20}, \sigma_2 > 0$ are small constants to introduce the $\sigma-$modification for the closed-loop system.

Consider the following Lyapunov function candidate

$$V_2(t) = V_{z_2}(t) + V_{U_2}(t) + \frac{1}{2}\gamma_2^{-1}\tilde{\theta}_{20}^2(t) + \frac{1}{2}\tilde{\theta}_2^T(t)\Gamma_2^{-1}\tilde{\theta}_2(t)$$

For $z_2 \in \Omega_{Z_2}^{0_O}$, the control effort α_2 is invoked, and the time derivative of $V_2(t)$ along (5.35) and (5.36)-(5.41) is

$$\dot{V}_2(t) \leq -c_2 V_2(t) + \lambda_2 + \frac{1}{k_{20}} z_3^2$$

where

$$c_2 := \min\Big\{\frac{3}{2} g_{\min} k_{20},\ 2g_{\min},\ \sigma_{20}\gamma_2,\ \frac{\sigma_2}{\lambda_{\max}(\Gamma_2^{-1})}\Big\}$$
$$\lambda_2 := \frac{1}{2}\sigma_{20}\theta_{20}^2 + \frac{1}{2}\sigma_2\|\theta_2\|^2 + 0.2785\epsilon_2\theta_{20}$$

For $z_2 \in \Omega_{Z_1}^{0_I}$, the following two cases are considered: (i) if $z_1 \in \Omega_{Z_1}^{0_I}$ or $z_1 \in \Omega_{Z_1}$, i.e., $|z_1| \leq \lambda_{a1} + \lambda_{b1}$, $V_1(t)$ and $V_2(t)$ are bounded, hence, z_1, z_2, $\hat{\theta}_{10}$, $\hat{\theta}_1$, $\hat{\theta}_{20}$ and $\hat{\theta}_2$ are bounded; (ii) if $z_1 \in \Omega_{Z_1}^{0_O}$, i.e., $|z_1| \geq \lambda_{a1} + \lambda_{b1}$, we know from Remark 5.4.1 that $\dot{V}_1(t) \leq -c_1 V_1(t) + \bar{\lambda}_1$ with $\bar{\lambda}_1 = \lambda_1 + \frac{1}{k_{10}}(\lambda_{a2} + \lambda_{b2})^2$, for which the stability analysis will be conducted later.

For $z_2 \in \Omega_{Z_1}$, the analysis is similar as for $z_2 \in \Omega_{Z_1}^{0_I}$. The effect of z_3 will be dealt with in the next step.

Step i ($3 \leq i \leq n-1$): Similar procedures are taken for each steps when $i = 3, \cdots, n-1$ as in Steps 1 and 2.

The time derivative of $z_i(t)$ is given by

$$\begin{aligned} \dot{z}_i(t) &= g_i[z_{i+1}(t) + \alpha_i(t)] + \theta_{fi}^T F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) \\ &\quad + \theta_{hi}^T H_i(\bar{x}_i(t-\tau_i)) + \delta_{hi}(\bar{x}_i(t-\tau_i)) - \dot{\alpha}_{i-1}(t) \end{aligned} \tag{5.42}$$

Since $\alpha_{i-1}(t)$ is a function of $\bar{x}_{i-1}$, $\bar{x}_{di}$, $\hat{\theta}_{10}, \cdots, \hat{\theta}_{i-1,0}$, $\hat{\theta}_1, \cdots, \hat{\theta}_{i-1}$, $\dot{\alpha}_{i-1}(t)$ can be expressed as

$$\begin{aligned} \dot{\alpha}_{i-1}(t) &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{j0}}\dot{\hat{\theta}}_{j0} + \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_j}\dot{\hat{\theta}}_j \\ &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j + \omega_{i-1} \end{aligned}$$

where

$$\omega_{i-1} = \frac{\partial \alpha_{i-1}}{\partial \bar{x}_{di}} \dot{\bar{x}}_{di} + \sum_{j=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_{j0}} \dot{\hat{\theta}}_{j0} + \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_j} \dot{\hat{\theta}}_j \right)$$

then (5.42) becomes

$$\begin{aligned} \dot{z}_i(t) &= g_i[z_{i+1}(t) + \alpha_i(t)] + \theta_{fi}^T F_i(\bar{x}_i(t)) + \delta_{fi}(\bar{x}_i(t)) \\ &\quad + \theta_{hi}^T H_i(\bar{x}_i(t-\tau_i)) + \delta_{hi}(\bar{x}_i(t-\tau_i)) \\ &\quad - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \Big[g_j x_{j+1} + \theta_{fj}^T F_j(\bar{x}_j) + \delta_{fj}(\bar{x}_j) \\ &\quad + \theta_{hj}^T H_j(\bar{x}_j(t-\tau_j)) + \delta_{hj}(\bar{x}_j(t-\tau_j)) \Big] - \omega_{i-1}(t) \end{aligned}$$

Consider the scalar functions $V_{z_i}(t) = \frac{1}{2g_i} z_i^2(t)$. By applying Assumption 5.2.2 and using Young's Inequality, its time derivative is

$$\begin{aligned} \dot{V}_{z_i} &\le z_i z_{i+1} + z_i \alpha_i + \frac{1}{g_i} \Big\{ z_i \theta_{fi}^T F_i(\bar{x}_i) + |z_i| c_{fi} \phi_i(\bar{x}_i) + \frac{1}{2} z_i^2 \theta_{hi}^T \theta_{hi} \\ &\quad + \frac{1}{2} H_i^T(\bar{x}_i(t-\tau_i)) H_i(\bar{x}_i(t-\tau_i)) + \frac{1}{2} z_i^2 c_{hi}^2 + \frac{1}{2} \psi_i^2(\bar{x}_i(t-\tau_i)) \\ &\quad + \sum_{j=1}^{i-1} \Big[- z_i \frac{\partial \alpha_{i-1}}{\partial x_j} g_j x_{j+1} - z_i \frac{\partial \alpha_{i-1}}{\partial x_j} \theta_{fj}^T F_j(\bar{x}_j) + |z_i| \Big| \frac{\partial \alpha_{i-1}}{\partial x_j} \Big| c_{fj} \phi_j(\bar{x}_j) \\ &\quad + \frac{1}{2} z_i^2 \Big(\frac{\partial \alpha_{i-1}}{\partial x_j} \Big)^2 \theta_{hj}^T \theta_{hj} + \frac{1}{2} H_j^T(\bar{x}_j(t-\tau_j)) H_j(\bar{x}_j(t-\tau_j)) \\ &\quad + \frac{1}{2} z_i^2 \Big(\frac{\partial \alpha_{i-1}}{\partial x_j} \Big)^2 c_{hj}^2 + \frac{1}{2} \psi_j^2(\bar{x}_j(t-\tau_j)) \Big] - z_i \omega_{i-1} \Big\} \end{aligned}$$

Considering the following Lyapunov-Krasovskii functional

$$V_{U_i}(t) = \frac{1}{2g_i} \sum_{j=1}^{i} \int_{t-\tau_i}^{t} U_i(\bar{x}_i(\tau)) d\tau$$

where $U_1(\cdot), \cdots, U_{i-1}(\cdot)$ are defined in the previous steps and $U_i(\cdot)$ is a positive definite function defined by

$$U_i(\bar{x}_i(t)) = H_i^T(\bar{x}_i(t)) H_i(\bar{x}_i(t)) + \psi_i^2(\bar{x}_i(t))$$

we have

$$\begin{aligned} \dot{V}_{z_i} + \dot{V}_{U_i} &\le z_i z_{i+1} + z_i \alpha_i + \frac{1}{g_i} \Big\{ z_i \theta_{fi}^T F_i(\bar{x}_i) + |z_i| c_{fi} \phi_i(\bar{x}_i) + \frac{1}{2} z_i^2 \theta_{hi}^T \theta_{hi} \\ &\quad + \frac{1}{2} H_i^T(\bar{x}_i) H_i(\bar{x}_i) + \frac{1}{2} z_i^2 c_{hi}^2 + \frac{1}{2} \psi_i^2(\bar{x}_i) \end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{i-1}\Big[-z_i\frac{\partial\alpha_{i-1}}{\partial x_j}g_jx_{j+1}-z_i\frac{\partial\alpha_{i-1}}{\partial x_j}\theta_{fj}^TF_j(\bar{x}_j)+|z_i||\frac{\partial\alpha_{i-1}}{\partial x_j}|c_{fj}\phi_j(\bar{x}_j) \\
& +\frac{1}{2}z_i^2(\frac{\partial\alpha_{i-1}}{\partial x_j})^2\theta_{hj}^T\theta_{hj}+\frac{1}{2}H_j^T(\bar{x}_j)H_j(\bar{x}_j)+\frac{1}{2}z_i^2(\frac{\partial\alpha_{i-1}}{\partial x_j})^2c_{hj}^2+\frac{1}{2}\psi_j^2(\bar{x}_j)\Big] \\
& -z_i\omega_{i-1}\Big\} \\
\triangleq\ & z_iz_{i+1}+z_i(\alpha_i+\theta_i^TF_{\theta i})+\theta_{i0}|z_i|\phi_{i0} \qquad (5.43)
\end{aligned}
$$

where θ_{i0} is an unknown constant, θ_i is an unknown constant vector, $\phi_{i0}(\cdot)$ is a known function, and $F_{\theta i}(\cdot)$ is a known function vector defined below

$$
\begin{aligned}
\theta_{i0} &:= \max\{c_{f1},\cdots,c_{fi}\}, \\
\theta_i &:= \Big[\frac{\theta_{fi}^T}{g_i},\ \frac{\theta_{hi}^T\theta_{hi}+c_{hi}^2}{g_i},\ \frac{g_{i-1}}{g_i},\ \frac{g_{i-1}}{g_i}\theta_{i-1}^T\Big]^T\in R^{\bar{n}_i}, \\
\phi_{i0} &:= \phi_i+\sum_{j=1}^{i-1}|\frac{\partial\alpha_{i-1}}{\partial x_j}|\phi_j, \\
F_{\theta i} &:= [F_i^T,\ \frac{1}{2}z_i,\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i,\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}F_{i-1}^T,\ \frac{1}{2}z_i(\frac{\partial\alpha_{i-1}}{\partial x_{i-1}})^2, \\
&\quad -\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}x_{i-1},\ -\frac{\partial\alpha_{i-1}}{\partial x_{i-2}}F_{i-2}^T,\ \frac{1}{2}z_i(\frac{\partial\alpha_{i-1}}{\partial x_{i-2}})^2,\ \cdots, \\
&\quad -\frac{\partial\alpha_{i-1}}{\partial x_1}x_2,\ -\frac{\partial\alpha_{i-1}}{\partial x_1}F_1^T,\ \frac{1}{2}z_i(\frac{\partial\alpha_{i-1}}{\partial x_1})^2, \\
&\quad \frac{1}{2z_i}\sum_{j=1}^{i}H_j^TH_j+\psi_j^2-\omega_{i-1}]^T\in R^{\bar{n}_i},\quad \bar{n}_i=\sum_{j=1}^{i}n_j+2i
\end{aligned}
$$

Similarly, the following robust adaptive intermediate control law is proposed

$$
\begin{aligned}
\alpha_i &= q_i(z_i)\Big[-k_i(t)z_i-\hat{\theta}_i^TF_{\theta i}-\beta_i\Big] && (5.44) \\
k_i(t) &= k_{i0}+\frac{1}{z_i^2}\sum_{j=1}^{i}\int_{t-\tau_{\max}}^{t}U_j(\bar{x}_j(\tau))d\tau && (5.45) \\
\beta_i &= \hat{\theta}_{i0}\xi_i && (5.46) \\
\xi_i &= \phi_{i0}\tanh\Big(\frac{z_i\phi_{i0}}{\epsilon_i}\Big) && (5.47)
\end{aligned}
$$

where $k_{i0}>0$ is a design constant and $\epsilon_i>0$ is a small constant.

The adaptive laws are given for online tuning the unknown parameters

$$
\begin{aligned}
\dot{\hat{\theta}}_{i0} &= q_i(z_i)\gamma_i(z_i\xi_i-\sigma_{i0}\hat{\theta}_{i0}) && (5.48) \\
\dot{\hat{\theta}}_i &= q_i(z_i)\Gamma_i(F_{\theta i}z_i-\sigma_i\hat{\theta}_i) && (5.49)
\end{aligned}
$$

where $\gamma_i > 0$, $\Gamma_i = \Gamma_i^{-1} > 0$, and $\sigma_{i0}, \sigma_i > 0$ are small constants to introduce the $\sigma-$modification for the closed-loop system.

Consider the following Lyapunov function candidate

$$V_i(t) = V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(t) + \frac{1}{2}\tilde{\theta}_i^T(t)\Gamma_i^{-1}\tilde{\theta}_i(t)$$

For $z_i \in \Omega_{Z_i}^{0_O}$, the control effort α_i is invoked and the time derivative of $V_i(t)$ along (5.43) and (5.44)-(5.49) is

$$\dot{V}_i(t) \leq -c_i V_i(t) + \lambda_i + \frac{1}{k_{i0}} z_{i+1}^2 \tag{5.50}$$

where

$$\begin{aligned} c_i &:= \min\Big\{\frac{3}{2}g_{\min}k_{i0}, 2g_{\min}, \sigma_{i0}\gamma_i, \frac{\sigma_i}{\lambda_{\max}(\Gamma_i^{-1})}\Big\} \\ \lambda_i &:= \frac{1}{2}\sigma_{i0}\theta_{i0}^2 + \frac{1}{2}\sigma_i\|\theta_i\|^2 + 0.2785\epsilon_i\theta_{i0} \end{aligned}$$

If z_{i+1} can be regulated as bounded, say, $|z_{i+1}| \leq z_{i+1,\max}$ with $z_{i+1,\max}$ being finite, from (5.50), we have that $\dot{V}_i(t) \leq -c_i V_i(t) + \bar{\lambda}_i$ with $\bar{\lambda}_i = \lambda_i + \frac{1}{k_{i0}} z_{i+1,\max}^2$. The stability analysis for this case will be shown later and the effect of z_{i+1} will be handled in the next steps.

For $z_i \in \Omega_{Z_i}^{0_I}$ or $z_i \in \Omega_{Z_i}$, similarly as in Step 2, the following two cases are considered: (i) if $z_{i-1} \in \Omega_{Z_{i-1}}^{0_I}$ or $z_{i-1} \in \Omega_{Z_{i-1}}$, and (ii) if $z_{i-1} \in \Omega_{Z_{i-1}}^{0_O}$.

Step n: This is the final step, since the actual control u appears in the derivative of $z_n(t)$ as given in

$$\begin{aligned} \dot{z}_n(t) &= g_n u(t) + \theta_{fn}^T F_n(x(t)) + \delta_{fn}(x(t)) \\ &\quad + \theta_{hn}^T H_n(x(t-\tau_n)) + \delta_{hn}(x(t-\tau_n)) - \dot{\alpha}_{n-1}(t) \end{aligned} \tag{5.51}$$

Since $\alpha_{n-1}(t)$ is a function of $\bar{x}_{n-1}$, $\bar{x}_{dn}$, $\hat{\theta}_{10}, \cdots, \hat{\theta}_{n-1,0}$, $\hat{\theta}_1, \cdots, \hat{\theta}_{n-1}$, $\dot{\alpha}_{n-1}(t)$ can be expressed as

$$\begin{aligned} \dot{\alpha}_{n-1}(t) &= \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\dot{x}_j + \frac{\partial \alpha_{n-1}}{\partial \bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_{j0}}\dot{\hat{\theta}}_{j0} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j}\dot{\hat{\theta}}_j\right) \\ &= \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}(t) \end{aligned}$$

where

$$\omega_{n-1}(t) = \frac{\partial \alpha_{n-1}}{\partial \bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\left(\frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_{j0}}\dot{\hat{\theta}}_{j0} + \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}_j}\dot{\hat{\theta}}_j\right)$$

then (5.51) becomes

$$\begin{aligned}\dot{z}_n(t) &= g_n u(t) + \theta_{fn}^T F_n(x(t)) + \delta_{fn}(x(t)) + \theta_{hn}^T H_n(x(t-\tau_n)) + \delta_{hn}(x(t-\tau_n))\\ &\quad -\sum_{j=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial x_j}\Big[g_j x_{j+1} + \theta_{fj}^T F_j(\bar{x}_j) + \delta_{fj}(\bar{x}_j)\\ &\quad +\theta_{hj}^T H_j(\bar{x}_j(t-\tau_j)) + \delta_{hj}(\bar{x}_j(t-\tau_j))\Big] - \omega_{n-1}(t)\end{aligned}$$

Consider the scalar functions $V_{z_n}(t) = \frac{1}{2g_n}z_n^2(t)$. By applying Assumption 5.2.2 and using Young's Inequality, its time derivative is

$$\begin{aligned}\dot{V}_{z_i}(t) &\le z_n u(t) + \frac{1}{g_n}\Big\{z_n\theta_{fn}^T F_n(x) + |z_n|c_{fn}\phi_n(x) + \frac{1}{2}z_n^2\theta_{hn}^T\theta_{hn}\\ &\quad +\frac{1}{2}H_n^T(x(t-\tau_n))H_n(x(t-\tau_n)) + \frac{1}{2}z_n^2 c_{hn}^2 + \frac{1}{2}\psi_n^2(x(t-\tau_n))\\ &\quad +\sum_{j=1}^{n-1}\Big[-z_n\frac{\partial \alpha_{n-1}}{\partial x_j}g_j x_{j+1} - z_n\frac{\partial \alpha_{n-1}}{\partial x_j}\theta_{fj}^T F_j(\bar{x}_j) + |z_n||\frac{\partial \alpha_{n-1}}{\partial x_j}|c_{fj}\phi_j(\bar{x}_j)\\ &\quad +\frac{1}{2}z_n^2(\frac{\partial \alpha_{n-1}}{\partial x_j})^2\theta_{hj}^T\theta_{hj} + \frac{1}{2}H_j^T(\bar{x}_j(t-\tau_j))H_j(\bar{x}_j(t-\tau_j))\\ &\quad +\frac{1}{2}z_n^2(\frac{\partial \alpha_{n-1}}{\partial x_j})^2 c_{hj}^2 + \frac{1}{2}\psi_j^2(\bar{x}_j(t-\tau_j))\Big] - z_n\omega_{n-1}\Big\}\end{aligned}$$

Consider the Lyapunov-Krasovskii functional

$$V_{U_n}(t) = \frac{1}{2g_n}\sum_{j=1}^{n}\int_{t-\tau_n}^{t} U_n(x(\tau))d\tau$$

where $U_1(\cdot), \cdots, U_{n-1}(\cdot)$ are defined before and $U_n(\cdot)$ is a positive definite function defined by

$$U_n(x(t)) = H_n^T(x(t))H_n(x(t)) + \psi_n^2(x(t))$$

we have

$$\begin{aligned}\dot{V}_{z_n} + \dot{V}_{U_n} &\le z_n u(t) + \frac{1}{g_n}\Big\{z_n\theta_{fn}^T F_n(x) + |z_n|c_{fn}\phi_n(x) + \frac{1}{2}z_n^2\theta_{hn}^T\theta_{hn}\\ &\quad +\frac{1}{2}H_n^T(x)H_n(x) + \frac{1}{2}z_n^2 c_{hn}^2 + \frac{1}{2}\psi_n^2(x)\\ &\quad +\sum_{j=1}^{n-1}\Big[-z_n\frac{\partial \alpha_{n-1}}{\partial x_j}g_j x_{j+1} - z_n\frac{\partial \alpha_{n-1}}{\partial x_j}\theta_{fj}^T F_j(\bar{x}_j)\end{aligned}$$

$$
\begin{aligned}
& +|z_n||\frac{\partial\alpha_{n-1}}{\partial x_j}|c_{fj}\phi_j(\bar{x}_j) + \frac{1}{2}z_n^2(\frac{\partial\alpha_{n-1}}{\partial x_j})^2\theta_{hj}^T\theta_{hj} + \frac{1}{2}H_j^T(\bar{x}_j)H_j(\bar{x}_j) \\
& +\frac{1}{2}z_n^2(\frac{\partial\alpha_{n-1}}{\partial x_j})^2c_{hj}^2 + \frac{1}{2}\psi_j^2(\bar{x}_j)\Big] - z_n\omega_{n-1}\Big\} \\
\triangleq\ & z_n(u + \theta_n^T F_{\theta n}) + \theta_{n0}|z_n|\phi_{n0} \qquad (5.52)
\end{aligned}
$$

where θ_{n0} is an unknown constant, θ_n is an unknown constant vector, $\phi_{n0}(\cdot)$ is a unknown parameter vector, and $F_{\theta n}(\cdot)$ is a known function vector defined below

$$
\begin{aligned}
\theta_{n0} &:= \max\{c_{f1}, \cdots, c_{fn}\}, \\
\theta_n &= \Big[\frac{\theta_{fn}^T}{g_n},\ \frac{\theta_{hn}^T\theta_{hn} + c_{hn}^2}{g_n},\ \frac{g_{n-1}}{g_n},\ \frac{g_{n-1}}{g_n}\theta_{n-1}^T\Big]^T \in R^{\bar{n}_n}, \\
\phi_{n0} &:= \phi_n + \sum_{j=1}^{n-1}|\frac{\partial\alpha_{n-1}}{\partial x_j}|\phi_j, \\
F_{\theta n} &= [F_n^T,\ \frac{1}{2}z_n,\ -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}x_n,\ -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}F_{n-1}^T,\ \frac{1}{2}z_n(\frac{\partial\alpha_{n-1}}{\partial x_{n-1}})^2, \\
&\quad -\frac{\partial\alpha_{n-1}}{\partial x_{n-2}}x_{n-1},\ -\frac{\partial\alpha_{n-1}}{\partial x_{n-2}}F_{n-2}^T,\ \frac{1}{2}z_n(\frac{\partial\alpha_{n-1}}{\partial x_{n-2}})^2,\ \cdots, \\
&\quad -\frac{\partial\alpha_{n-1}}{\partial x_1}x_2,\ -\frac{\partial\alpha_{n-1}}{\partial x_1}F_1^T,\ \frac{1}{2}z_n(\frac{\partial\alpha_{n-1}}{\partial x_1})^2, \\
&\quad \frac{1}{2z_n}\sum_{j=1}^{n}H_j^TH_j + \psi_j^2 - \omega_{n-1}]^T \in R^{\bar{n}_n},\quad \bar{n}_n = \sum_{j=1}^{n}n_j + 2n
\end{aligned}
$$

Similarly, the following robust adaptive control law is proposed

$$
\begin{aligned}
u &= q_n(z_n)[-k_n(t)z_n - \hat{\theta}_n^T F_{\theta n} - \beta_n] && (5.53) \\
k_n(t) &= k_{n0} + \frac{1}{z_n^2}\sum_{j=1}^{n}\int_{t-\tau_{\max}}^{t}U_j(\bar{x}_j(\tau))d\tau && (5.54) \\
\beta_n &= \hat{\theta}_{n0}\xi_n && (5.55) \\
\xi_n &= \phi_{n0}\tanh\Big(\frac{z_n\phi_{n0}}{\epsilon_n}\Big) && (5.56)
\end{aligned}
$$

where $k_{n0} > 0$ is a design constant and $\epsilon_n > 0$ is a small constant.

The adaptive laws are given for online tuning the unknown parameters

$$
\begin{aligned}
\dot{\hat{\theta}}_{n0} &= q_n(z_n)\gamma_n(z_n\xi_n - \sigma_{n0}\hat{\theta}_{n0}) && (5.57) \\
\dot{\hat{\theta}}_n &= q_n(z_n)\Gamma_n(F_{\theta n}z_n - \sigma_n\hat{\theta}_n) && (5.58)
\end{aligned}
$$

where $\gamma_n > 0$, $\Gamma_n = \Gamma_n^{-1} > 0$, and $\sigma_{n0}, \sigma_n > 0$ are small constants to introduce the $\sigma-$modification for the closed-loop system.

Consider the following Lyapunov function candidate

$$V_n(t) = V_{z_n}(t) + V_{U_n}(t) + \frac{1}{2}\gamma_n^{-1}\tilde{\theta}_{n0}^2(t) + \frac{1}{2}\tilde{\theta}_n^T(t)\Gamma_n^{-1}\tilde{\theta}_n(t)$$

For $z_n \in \Omega_{Z_n}^{0_O}$, the final control $u(t)$ is invoked and the time derivative of $V(t)$ along (5.52) and (5.53)-(5.58) is

$$\dot{V}_n(t) \le -c_n V_n(t) + \lambda_n \tag{5.59}$$

where

$$c_n := \min\left\{2g_{\min}k_{n0}, 2g_{\min}, \sigma_{n0}\gamma_n, \frac{\sigma_n}{\lambda_{\max}(\Gamma_n^{-1})}\right\}$$
$$\lambda_n := \frac{1}{2}\sigma_{n0}\theta_{n0}^2 + \frac{1}{2}\sigma_n\|\theta_n\|^2 + 0.2785\epsilon_n\theta_{n0}$$

It is known from (5.59) that $V_n(t)$ is bounded, hence z_n, $\hat{\theta}_{n0}$ and $\hat{\theta}_n$ are bounded.

For $z_n \in \Omega_{Z_n}^{0_I}$ or $z_n \in \Omega_{Z_n}$, two cases are considered: (i) if $z_{n-1} \in \Omega_{Z_{n-1}}^{0_I}$ or $z_{n-1} \in \Omega_{Z_{n-1}}$, and (ii) $z_{n-1} \in \Omega_{Z_{n-1}}^{0_O}$.

Theorem 5.4.1 shows the stability and control performance of the closed-loop adaptive system.

Theorem 5.4.1 *Consider the closed-loop system consisting of the plant (5.2) under Assumptions 5.2.1-5.2.4. If we apply the controller (5.53)-(5.56) with parameters updating law (5.57) and (5.58), we can guarantee the following properties under bounded initial conditions*

(i) z_i, $\hat{\theta}_{i0}$, $\hat{\theta}_i$ *and* x_i, $i = 1, \cdots, n$, *are globally uniformly ultimately bounded;*

(ii) the signal $z(t) = [z_1, \cdots, z_n]^T \in R^n$ *will eventually converge to the compact set defined by*

$$\Omega_z := \left\{ z \;\middle|\; \|z\| \le \mu \right\}$$

with $\mu = \max\{\sqrt{2g_{\max}\rho}, \sqrt{\sum_{j=1}^n(\lambda_{ai} + \lambda_{bi})^2}\}$ *and the compact set* Ω_z *can be made as small as desired by an appropriate choice of the design parameters.*

Proof: Consider the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^n \left[V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(t) + \frac{1}{2}\tilde{\theta}_i^T(t)\Gamma_i^{-1}\tilde{\theta}_i(t)\right] \tag{5.60}$$

where $V_{z_i}(t)$, $V_{U_i}(t)$, $i = 1, \cdots, n$ are defined as before, and $(\tilde{\cdot}) = (\hat{\cdot}) - (\cdot)$. The following three cases are considered.

Case 1): $z_i \in \Omega^{0_O}_{Z_i}$, $i = 1, ..., n$.

From the previous derivation, we have the following inequality for $z_i \in \Omega^{0_O}_{Z_i}$, $i = 1, ..., n$

$$\dot{V}(t) \leq -cV(t) + \lambda$$

where $c := \min\{c_1, \cdots, c_n\}$ and $\lambda := \sum_{i=1}^n \lambda_i$. Let $\rho := \lambda/c$, it follows that

$$0 \leq V(t) \leq [V(0) - \rho]e^{-ct} + \rho \leq V(0) + \rho \tag{5.61}$$

where the constant

$$V(0) = \sum_{i=1}^n \Big[\frac{1}{2g_i} z_i^2(0) + \frac{1}{2}\gamma_i^{-1}\tilde{\theta}_{i0}^2(0) + \frac{1}{2}\tilde{\theta}_i^T(0)\Gamma_i^{-1}\tilde{\theta}_i(0)\Big]$$

Considering (5.60), we know that

$$\sum_{i=1}^n z_i^2 \leq 2g_{\max}[V(0) + \rho] \tag{5.62}$$

$$\sum_{i=1}^n \tilde{\theta}_{i0}^2 \leq 2\max\{\gamma_i\}[V(0) + \rho], \quad \sum_{i=1}^n \|\tilde{\theta}_i\|^2 \leq \frac{2[V(0) + \rho]}{\lambda_{\min}\{\Gamma_i^{-1}\}} \tag{5.63}$$

It can be seen from (5.61), (5.62) and (5.63) that $V(t)$ is bounded, hence z_i, $\hat{\theta}_{i0}$ and $\hat{\theta}_i$ are uniformly bounded for $z_i \in \Omega^{0_O}_{Z_i}$, $i = 1, ..., n$.

In addition, from (5.60) and (5.61), we have

$$\|z\| \leq \sqrt{2g_{\max}[(V(0) - \rho)e^{-ct} + \rho]}$$

i.e., $\lim_{t\to\infty} \|z\| = \sqrt{2g_{\max}\rho}$. Since the above analysis is carried out for $|z_i| \geq \lambda_{ai} + \lambda_{bi}$, $i = 1, ..., n$, we have that $\lim_{t\to\infty} \|z\| = \max\{\sqrt{2g_{\max}\rho}, \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}\}$.

Case 2): $z_i \in \Omega^{0_I}_{Z_i}$ or $z_i \in \Omega_{Z_i}$, $i = 1, ..., n$.

In this case, $V_n(t)$ is bounded, hence z_i, x_i, $\hat{\theta}_{i0}$ and $\hat{\theta}_i$, $i = 1, ..., n$ are all bounded. In addition, $\|z\| \leq \sqrt{\sum_{j=1}^n (\lambda_{ai} + \lambda_{bi})^2}$.

Case 3): Some z_i's are satisfying $z_i \in \Omega^{0_O}_{Z_i}$, while some z_j's are satisfying $z_j \in \Omega^{0_I}_{Z_j}$ or $z_j \in \Omega_{Z_j}$.

For $z_i \in \Omega^{0_O}_{Z_i}$, the control effort α_i will render $\dot{V}_i \le -c_i V_i + \lambda_i + \frac{1}{k_{i0}} z^2_{i+1}$. If z_{i+1} is bounded, the boundedness of z_i can be guaranteed. Otherwise, the control effort α_{i+1} will be invoked, which yields $\dot{V}_{i+1} \le -c_{i+1} V_{i+1} + \lambda_{i+1} + z^2_{i+2}$. Similarly, regulation of z_{i+2} will be left to the next steps till the final step where z_n will be regulated as bounded. Therefore, those z_i's will be regulated as bounded finally. For those $z_j \in \Omega^{0_I}_{Z_j}$ or $z_j \in \Omega_{Z_j}$, their boundedness has already obtained.

Therefore, we can conclude from Cases 1), 2) and 3) that all the closed-loop signals are GUUB and there does exist a compact set Ω_z such that z will eventually converge to. This completes the proof. $\diamondsuit$

Remark 5.4.4 *Theorem 5.4.1 shows that the system tracking error converges to a domain of attraction defined by compact set Ω_z rather than the origin. This is due to the introduction of the practical control, the smooth β_i control component and the σ-modification for the parameter adaptation. Even though the size of the compact set is unknown due to the unknown parameters $g_{\min}$, $g_{\max}$, θ_{i0} and θ_i, $i = 1, ..., n$, it is possible to make it as small as possible by appropriately choosing the design parameters. However, parameters such as λ_{ai} or λ_{bi} cannot be made zero to void possibly control singularity and computational singularity. Therefore, in practical applications, the design parameters should be adjusted carefully for achieving suitable transient performance and control action.*

Remark 5.4.5 *The unknown parameters have been rearranged into a newly defined vector in each step of the iterative backstepping design. By doing so, on one hand, unknown vectors θ_{hi}, $i = 1, \cdots, n$ have been lumped as scalars, which reduces the number of parameters to be estimated in each step and finally reduces the order of the controller dramatically. On the other hand, we only need to estimate $\frac{1}{g_i}$ rather than g_i such that possible controller singularities due to $\hat{g}_i = 0$ have been avoided.*

Remark 5.4.6 *Note that the integration in computing $k_i(t)$ is conducted in the time interval $[t - \tau_{\max}, t]$. If the integration is conducted alternatively in $[0, t]$, the stability result still hold. However, the integral result will progressively tend to a large value as the time increases, which as a result may lead to instability* *of the overall system. To avoid this, the integration shall be conducted in a more conservative time interval, i.e.,* $[t-\tau_{\max}, t]$.

5.5 Simulation Studies

To illustrate the proposed robust adaptive control algorithms, we consider the following second-order plant

$$\begin{cases} \dot{x}_1(t) &=& g_1 x_2(t) + \theta_{f1} x_1^2(t) + \delta_{f1}(x_1(t)) \\ \dot{x}_2(t) &=& g_1 u(t) + \theta_{h2} x_2(t-\tau_2) + \delta_{h2}(x(t-\tau_2)) \\ y(t) &=& x_1(t) \end{cases}$$

where g_1, g_2 are unknown virtual control coefficients, θ_{f1}, θ_{h2} are unknown parameters, and $\delta_{f1}(\cdot)$, $\delta_{h2}(\cdot)$ are unknown functions. For simulation purpose, we assume that $g_1 = 2$, $g_2 = 1$, $\theta_{f1} = 0.1$, $\theta_{h2} = 0.2$, and let $\delta_{f1} = 0.6\sin(x_1)$, $\delta_{h2} = 0.5(x_1^2+x_2^2)\sin(x_2)$. The bounds on $\delta_{f1}(\cdot)$ and $\delta_{h2}(\cdot)$ are $|\delta_{f1}(x_1)| \le c_{f1}\phi_1(x_1)$, $|\delta_{h2}(x)| \le c_{h2}\psi_2(x)$, where $c_{f1} = 0.6$, $\phi_1(x_1) = 1$, $c_{h2} = 0.5$, $\psi_2(x) = x_1^2 + x_2^2$. The unknown time delays are $\tau_1 = 0$, $\tau_2 = 3$sec. The control objective is to track the desired reference signal $y_d(t) = 0.5[\sin(t) + \sin(0.5t)]$. For the design of robust adaptive controller, let $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$ and $\hat{\theta}_1$, $\hat{\theta}_2$ be the estimates of unknown parameter vectors $\theta_1 = [\frac{\theta_{f1}}{g_1}, \frac{1}{g_1}]^T$, $\theta_2 = [\frac{\theta_{h2}^2+c_{h2}^2}{g_2}, \frac{g_1}{g_2}, \frac{\theta_{f1}}{g_2}, \frac{1}{g_2}]^T$ respectively, we have

$$\begin{aligned} &\alpha_1(t) = q_1(z_1)[-k_1(t)z_1 - \hat{\theta}_1^T F_{\theta 1} - \beta_1] \\ &u(t) = q_2(z_2)[-k_2(t)z_2 - \hat{\theta}_2^T F_{\theta 2} - \beta_2] \\ &\beta_i = \hat{\theta}_{i0}\xi_i, \quad \xi_i = \phi_{i0}\tanh(\frac{z_i\phi_{i0}}{\epsilon_i}) \\ &\dot{\hat{\theta}}_{i0} = q_i(z_i)\gamma_i(z_i\xi_i - \sigma_{i0}\hat{\theta}_{i0}), \quad \dot{\hat{\theta}}_i = q_i(z_i)\Gamma_i(F_{\theta i}z_i - \sigma_i\hat{\theta}_i), \quad i = 1, 2 \end{aligned}$$

where $k_i(t)$ is calculated by

$$k_i(t) = k_{i0} + \frac{1}{z_i^2}\sum_{j=1}^{i}\int_{t-\tau_{jmax}}^{t} U_j(\bar{x}_j(\tau))d\tau, \quad k_{i0} > 0$$

The following design parameters are adopted in the simulation: $[x_1(0), x_2(0)]^T = [0.1, 0.1]^T$, $\gamma_1 = \gamma_2 = 1$, $\Gamma_1 = \Gamma_2 = \mathrm{diag}\{1\}$, $\sigma_{10} = \sigma_{20} = \sigma_1 = \sigma_2 = 0.05$,

$\theta_{10}^0 = \theta_{20}^0 = 0$, $\theta_1^0 = \theta_2^0 = 0$, $k_{10} = k_{20} = 0.8$, $\epsilon_1 = \epsilon_2 = 0.1$, and $\lambda_{a_1} = \lambda_{a_2} = 1.0e^{-3}$, $\lambda_{b_1} = \lambda_{b_2} = 1.0e^{-5}$.

From Fig. 5.1, it was seen that satisfactory transient tracking performance was obtained after 10 seconds of adaptation periods. Figs. 5.2 and 5.3 show the boundedness of the control input and the estimates of the parameters in the control loop.

Among the design parameters, the choices of c_{z_i} are critical for achieving good control performance. Through extensive simulation study, it was found that c_{z_i} should not be chosen as too small. From analytical point of view, it is found that the known functions $F_{\theta i}$ which are used for on-line parameters tuning contain possibly singular terms. The robust design is then carried out to make sure those terms to be bounded. Although c_{z_i} can be chosen arbitrarily small theoretically, it is not the case in real implementation due to the limited actuator tolerance and computational capacity.

5.6 Conclusion

A robust adaptive control has been addressed for a class of parametric-strict-feedback nonlinear systems with varying unknown time delays. The uncertainty from unknown time delays has been compensated through the use of appropriate Lyapunov-Krasovskii functionals. The controller has been made to be free from singularity problem by employing practical robust control and regrouping unknown parameters. Backstepping design has been carried out for a class of nonlinear systems in strict feedback form by using differentiable approximation. The proposed systematic backstepping design method has been proved to be able to guarantee global uniformly ultimately boundedness of closed-loop signals. In addition, the output of the system has been proven to converge to an arbitrarily small neighborhood of the origin. Simulation results have been provided to show the effectiveness of the proposed approach.

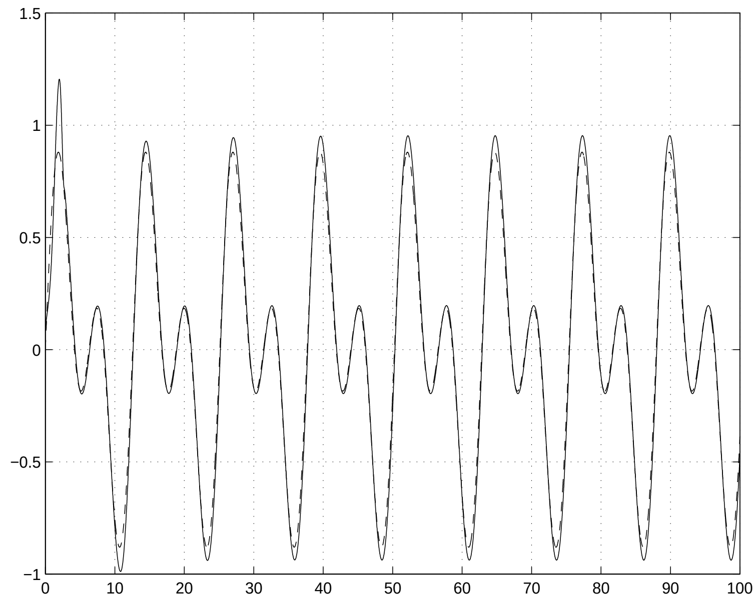

Figure 5.1: Output $y(t)$("—"), and reference y_d("- -").

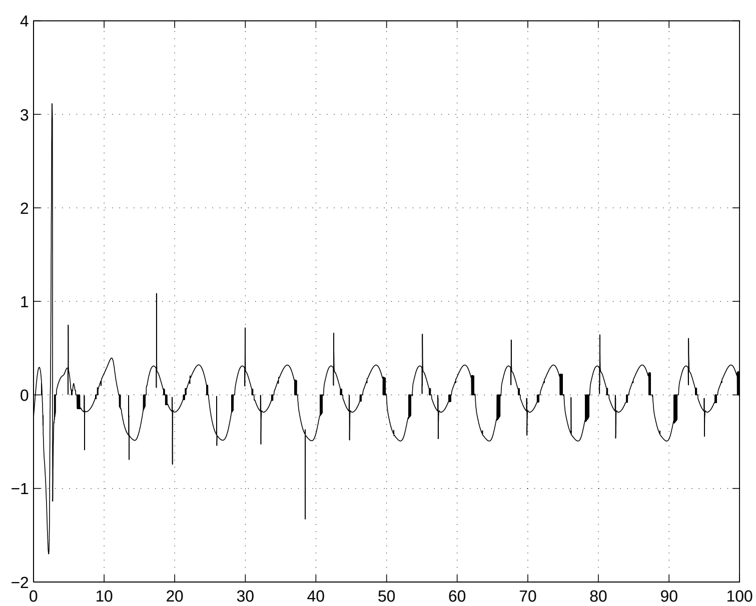

Figure 5.2: Control input $u(t)$.

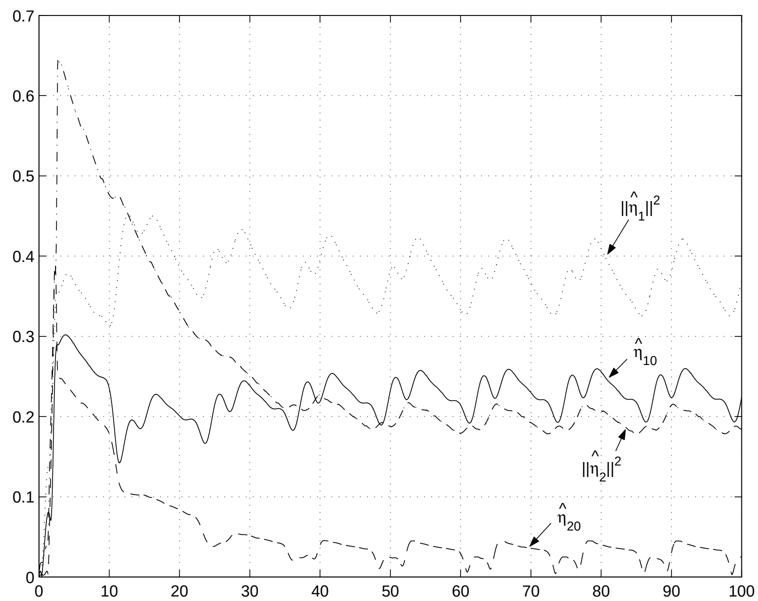

Figure 5.3: Parameter estimates: $\hat{\theta}_{10}$(“–”), $\hat{\theta}_{20}$(“- -”), $\|\hat{\theta}_1\|^2$(“· · ·”), $\|\hat{\theta}_2\|^2$(“-·”).

Chapter 6

Robust Adaptive Control Using Nussbaum Functions

6.1 Introduction

Recently, robust adaptive control has been studied for a class of strict-feedback systems by combining robust backstepping design with robust control strategy [15][134][22][23] [135][24][18][136][21], which guaranteed global uniform ultimate boundedness in the presence of parametric uncertainties or unknown functions. While the earlier works such as [15, 86, 18] assumed the virtual control coefficients to be 1, adaptive control has been extended to parametric strict-feedback systems with unknown constant virtual control coefficients but with known signs (either positive or negative) [19] based on the cancellation backstepping design as stated in [87] by seeking for a cancellation of the coupling terms related to $z_i z_{i+1}$ in the next step of Lyapunov design. With the aid of neural network parametrization, adaptive control schemes have been further extended to certain classes of strict-feedback in which virtual control coefficients are unknown functions of states with known signs [88][51]. For system $\dot{x} = f(x) + g(x)u$, the unknown virtual control function $g(x)$ causes great design difficulty in adaptive control. Based on feedback linearization, certainty equivalent control $u = [-\hat{f}(x) + v]/\hat{g}(x)$ is usually taken, where $\hat{f}(x)$ and $\hat{g}(x)$ are estimates of $f(x)$ and $g(x)$, and measures have to be taken to avoid controller singularity when $\hat{g}(x) = 0$. To avoid this problem, integral Lyapunov functions have been developed in [88], and semi-globally stable adaptive controllers are developed, which do not require the estimate of the unknown function $g(x)$. Although the system's virtual control coefficients are assumed to be unknown nonlinear functions of states, their signs are assumed to be known as strictly either positive or negative. Under this assumption, stable neural network controllers have been constructed in [51] by augmenting a robustifying portion, and in [89, 90] by estimating the derivation of the control Lyapunov function.

When there is no *a priori* knowledge about the signs of virtual control coefficients, adaptive control of such systems becomes much more difficult. The first solution was given in [62] for a class of first-order linear systems, where the Nussbaum-type gain was originally proposed. When the high-frequency control gains and their signs are unknown, gains of Nussbaum type [62] have been effectively used in controller design in solving the difficulty of unknown control directions [69, 70] in which the arguments of the constructed Nussbaum functions are required to be monotone increasing. This method was then generalized to higher-order linear systems in [64]. For nonlinear systems, some results have also been reported in the literature. Without the requirement for monotone increasing arguments for the Nussbaum functions, the same technique has extended to higher order systems for constant virtual control coefficients [83, 115] using decoupled backstepping formally stated in [87] without seeking for the cancellation of the coupling terms related to $z_i z_{i+1}$ but to decouple z_i and z_{i+1} using Young's inequality and seek for the boundedness of z_{i+1} next. Under the assumption that the virtual control coefficients are time-varying, with unknown signs and bounded in finite intervals, it has also been used to construct robust adaptive control for a class of nonlinear systems with bounded disturbances by introducing exponentially decaying terms to handle the bounded disturbances [137]. The behavior of this class of control laws can be interpreted as the controller tries to sweep all possible control gains and stops when a stabilizing gain is found.

Thus far, few results are available for the robust adaptive control of system with unknown virtual control coefficients (VCC) and bounded disturbance. In [113], a class of time-varying uncertain nonlinear systems was studied with completely unknown time-varying virtual control coefficients, uncertain time-varying parameters and unknown time-varying bounded disturbances. Due to the presence of the exponential term in the stability analysis, the proof has to be function dependent and the general properties of the Nussbaum functions are difficult to be utilized. Though a much neater proof was provided for $N(\zeta) = \exp(\zeta^2)\cos(\frac{\pi}{2}\zeta)$ in [113], it is not the case for $N(\zeta) = \zeta^2 \cos(\zeta)$ as chosen in this chapter. The proof cannot be straightforwardly extended and the specific properties of this function need to be exploited fully in the derivation throughout the proof. Due to the different problem formulation and methodology used (e.g., projection algorithm has to be utilized for on-line tuning of the time-varying unknown parameters in [113]), the proposed design in this chapter is much more tighter and the controller is composed of smooth functions, which is a must in backstepping design.

For robust control of nonlinear systems with time delays [122, 92], the existence of time delays may degrade the control performance and make the stabilization problem become more difficult. By using appropriate *Lyapunov-Krasovskii functionals* [123], uncertainties from unknown time delays can be compensated for. In [129], we studied a class of nonlinear time-delay systems, in which the virtual control coefficients are unknown constants with known sign and the system uncertainties are linearly parametrized with unknown constant parameters and known nonlinear functions. Practical stability was introduced to solve the singularity problem due to the appearance of $1/z_i$ or $1/z_i^2$ in the controller and the tracking error can be made to confine in a compact domain of attraction. When the virtual control coefficients are unknown nonlinear functions of states, the problem becomes even more complicated. Although the system's virtual control coefficients are assumed to be unknown nonlinear functions of states, their signs are assumed to be known as strictly either positive or negative. Under the same assumption, stable neural network controllers have also been constructed in [124] by compensating for the unknown time-delay terms completely under the assumption that signals $\dot{\bar{x}}_{n-1}$ are available for feedback and more strict assumption on the time delay terms.

Motivated by previous works on both systems with time-delay and unknown virtual control coefficient (VCC), two adaptive neural controllers without the requirements for $\dot{\bar{x}}_{n-1}$ are presented for a class of strict-feedback nonlinear systems with unknown time delays, and unknown nonlinear functions with unknown signs. For clarity, the first controller is developed based on distinct definitions of two separate compact sets $\Omega_{c_{z_i}} \subset \Omega_{Z_i}$ and $\Omega^0_{Z_i} = \Omega_{Z_i} - \Omega_{c_{z_i}} \subset \Omega_{Z_i}$ where "–" denotes the complement operation. However, the controller has a "technical problem" – the intermediate controls are not differentiable at isolated points $|z_i| = c_{z_i}$. To solve this problem, one practical way is to simply set the differentiation at these points to be any finite value, say 0, and then every signal in the closed-loop system can be shown to be bounded. By modifying the first controller such that the intermediate controls are differentiable, we have the second controller for the class of systems in the section – which is mathematically rigorous. To the best of our knowledge, there is little work dealing with such a kind of systems in the literature at present stage, except for some preliminary results presented in [138][124]. The main contributions of the chapter lie in:

(i) the introduction of a new technical lemma, which plays a fundamental role in solving the proposed problem;

(ii) the controller does not require the *a priori* knowledge of the signs of the unknown control coefficients,

(iii) the use of the Nussbaum-type functions in solving the problem of the completely unknown control direction;

(iv) the novel introduction of smooth functions in making the intermediate control laws continuous and differentiable to certain desired order in solving the differentiability problems at some isolated points incurred by the first practical control; and

(v) the proposed design method expands the class of nonlinear systems for which robust adaptive control approaches have been studied through the introduction of exponential decaying terms in stability analysis.

The rest of the chapter is organized as follows.

The problem formulation and preliminaries for a class of perturbed strict-feedback systems are given in Section 6.2.1. A robust adaptive control scheme using Nussbaum functions is presented in Section 6.2.2. A simulation example is given in Section 6.2.3, and followed by Section 6.2.4 which concludes the work.

The problem formulation and preliminaries are given in Section 6.3.1. An adaptive neural controller design for first-order systems is presented in Section 6.3.2. The scheme is extended to nth-order systems in Section 6.3.3. A simulation example is given in Section 6.3.4, and followed by Section 6.3.5 which concludes the work.

6.2 Robust Adaptive Control for Perturbed Nonlinear Systems

6.2.1 Problem Formulation and Preliminaries

Consider a class single-input-single-output (SISO) nonlinear systems in the presence of time-varying disturbances in the perturbed strict-feedback form

$$\begin{aligned} \dot{x}_i &= g_i x_{i+1} + \theta_i^T \psi_i(\bar{x}_i) + \Delta_i(t,x), \quad i = 1, ..., n-1 \\ \dot{x}_n &= g_n u + \theta_n^T \psi_n(x) + \Delta_n(t,x) \end{aligned} \tag{6.1}$$

where $x = [x_1, ..., x_n]^T \in R^n$, $\bar{x}_i = [x_1, ..., x_i]^T$, $i = 1, ..., n-1$ are the state vectors, $u \in R$ is the control, $\theta_i \in R^{p_i}$, $i = 1, ..., n$ are the unknown constant parameter vectors, p_i's are positive integers, $\psi_i(\bar{x}_i)$, $i = 1, ..., n$ are known nonlinear functions which are continuous and satisfy $\psi_i(0) = 0$, unknown constants g_i, $i = 1, ..., n-1$ are referred to as virtual control coefficients [19], g_n is referred to as the high-frequency gain, and Δ_i's are unknown Lipschitz continuous functions. The control objective is to construct a robust adaptive nonlinear control law so that the state x_1 of system (6.1) is driven to a small neighborhood of the origin, while keep internal Lagrange stability.

In system (6.1), the unknown nonlinear functions $\Delta_i(t,x)$ could be due to many factors [86], such as measurement noise, modeling errors, external time-varying disturbances, modeling simplifications or changes due to time variations. The occurrence of virtual control coefficients g_i's is also quite common in practice. The examples range from electric motors and robotic manipulators to flight dynamics

[19].

Assumption 6.2.1 *There exist unknown positive constants* p_i^*, $1 \leq i \leq n$, *such that* $\forall (t, x) \in R_+ \times R^n$, $|\Delta_i(t, x)| \leq p_i^* \phi_i(x_1, \cdots, x_i)$, *where* ϕ_i *is a known nonnegative smooth function.*

Remark 6.2.1 *Though the terms* $\theta_i^T \psi(\bar{x}_i)$ *can be absorbed into* $\Delta_i(t, x)$, $i = 1, ..., n$, *for a reduced order controller, the disadvantage is that the residue error will be large as can be seen from the definitions of* μ^*, ρ_i, *and* c_{i2} *later. In addition, for better control performance, knowledge of the system should be fully exploited.*

The technical Lemma 2.4.7 introduced in Chapter 2 is critical in solving the robust control problem in this chapter and and is rewritten here for easy reference.

Lemma 6.2.1 *Let* $V(\cdot)$ *and* $\zeta(\cdot)$ *be smooth functions defined on* $[0, t_f)$ *with* $V(t) \geq 0$, $\forall t \in [0, t_f)$, *and smooth Nussbaum-type function* $N(\zeta) = \zeta^2 \cos(\zeta)$. *If the following inequality holds:*

$$0 \leq V(t) \leq c_0 + e^{-c_1 t} \int_0^t g_0 N(\zeta) \dot{\zeta} e^{c_1 \tau} d\tau + e^{-c_1 t} \int_0^t \dot{\zeta} e^{c_1 \tau} d\tau, \quad \forall t \in [0, t_f) \qquad (6.2)$$

where constant $c_1 > 0$, g_0 *is a nonzero constant, and* c_0 *represents some suitable constant, then* $V(t)$, $\zeta(t)$ *and* $\int_0^t g_0 N(\zeta) \dot{\zeta} d\tau$ *must be bounded on* $[0, t_f)$.

Though the proof is not trivial even for finite t_f already, it is the case that $t_f \to \infty$ is of interest. This can be easily extended due to Proposition 1 below. Consider

$$\dot{x}(t) \in F(x(t)), \quad x(0) = x^0 \qquad (6.3)$$

where $z \mapsto F(z) \subset R^N$ is upper semicontinuous on R^n with non-empty convex and compact values. It is well known that the initial-value problem has a solution and that every solution can be maximally extended.

Proposition 1 *[70] If* $x : [0, t_f) \to R^N$ *is a bounded maximal solution of (6.3), then* $t_f = \infty$.

Remark 6.2.2 *As can be seen from Appendix 7.2, the proof of Lemma 6.2.1 is very much involved and indeed a contribution by itself. In addition, we would like to point out that* $N(\cdot)$ *is not necessarily an even function, which is only made for the convenience of proof. If* $N(\cdot)$ *is chosen as an odd function, e.g.,* $N(\zeta) = \zeta^2 \sin(\zeta)$*, the lemma can be easily proven by following the same procedure. From our understanding, we can make a conjecture that Lemma 6.2.1 is true for all the Nussbaum functions. We hope that interested reader can prove the lemma for general Nussbaum functions.*

6.2.2 Robust Adaptive Control and Main Results

In this section, the robust adaptive control design procedure for nonlinear system (6.1) is presented. The design of both the control law and the adaptive laws is based on a change of coordinates

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 - \alpha_1(x_1, \hat{\theta}_{a,1}, \hat{b}_1, \zeta_1) \\
&\vdots \\
z_i &= x_i - \alpha_{i-1}(x_1, \cdots, x_{i-1}, \hat{\theta}_{a,1}, \cdots, \hat{\theta}_{a,i-1}, \hat{b}_1, \cdots, \hat{b}_{i-1}, \zeta_{i-1}) \\
&\vdots \\
z_n &= x_n - \alpha_{n-1}(x_1, \cdots, x_{n-1}, \hat{\theta}_{a,1}, \cdots, \hat{\theta}_{a,n-1}, \hat{b}_1, \cdots, \hat{b}_{n-1}, \zeta_{n-1})
\end{aligned}
$$

where the functions $\alpha_i, i = 1, \cdots, n-1$ are referred to as intermediate control functions which will be designed using backstepping technique, $\hat{b}_i$ is the parameter estimate for b_i^* which is the grouped unknown bound for p_i^*, $\hat{\theta}_{a,i}$ represents the estimate of unknown parameter $\theta_{a,i}^*$ which is an augmented parameter and consists of $g_j, j = 1, \cdots, i-1$ and $\theta_j, j = 1, \cdots, i$ as will be clarified later, and ζ_i is the argument of the Nussbaum function. At each intermediate step i, we design the following intermediate control function α_i using an appropriate Lyapunov function V_i, and give the updating laws $\dot{\hat{b}}_i$, $\dot{\hat{\theta}}_{a,i}$ and $\dot{\zeta}_i$. At the nth step, the actual control u appears and the design is completed. For clarity and conciseness, the intermediate variables including the control functions and adaptive laws, $i = 1, ..., n-1$, are defined

$$
\eta_i = k_i z_i + \hat{\theta}_{a,i}^T \psi_{a,i} + \hat{b}_i \bar{\phi}_i \tanh\left(\frac{z_i \bar{\phi}_i}{\epsilon_i}\right) \tag{6.4}
$$

$$\begin{aligned}
\alpha_i &= N(\zeta_i)\eta_i && (6.5)\\
\dot{\zeta}_i &= z_i\eta_i && (6.6)\\
\dot{\hat{\theta}}_{a,i} &= \Gamma_i\Big[z_i\psi_{a,i} - \sigma_{\theta_i}(\hat{\theta}_{a,i} - \theta^0_{a,i})\Big] && (6.7)\\
\dot{\hat{b}}_i &= \gamma_i\Big[z_i\bar{\phi}_i \tanh\Big(\frac{z_i\bar{\phi}_i}{\epsilon_i}\Big) - \sigma_{b_i}(\hat{b}_i - b^0_i)\Big] && (6.8)
\end{aligned}$$

where the variables including $\psi_{a,i}$ and $\bar{\phi}_i$ will be defined later, $\Gamma_i = \Gamma_i^T > 0$, $\gamma_i > 0$, $\hat{\theta}_{a,i}$ and $\hat{b}_i$ are the parameter estimates of $\theta^*_{a,i}$ and b^*_i, constant $k_i > \frac{1}{4}$, ϵ_i is a small positive constant and σ_{θ_i}, σ_{b_i}, $\theta^0_{a,i}$, and b^0_i are positive design constants.

Step 1: To start, let us study the following subsystem of (6.1):

$$\dot{x}_1 = g_1x_2 + \theta_1^T\psi_1(x_1) + \Delta_1(t,x) \qquad (6.9)$$

where x_2 is taken for a virtual control input. To design a stabilizing adaptive control law for system (6.9), consider a Lyapunov function candidate $V_0(x_1) = \frac{1}{2}z_1^2$. In light of Assumption 6.2.1, the time derivative of V_0 along the solutions of (6.9) satisfies

$$\dot{V}_0 = z_1(g_1x_2 + \theta_1^T\psi_1(x_1) + \Delta_1(t,x)) \le z_1(g_1x_2 + \theta_1^T\psi_1) + b^*_1|z_1|\bar{\phi}_1 \qquad (6.10)$$

where $b^*_1 = p^*_1$, $\bar{\phi}_1 = \phi_1$. For notation consistence, let $\theta^*_{a,1} = \theta_1$, $\psi_{a,1} = \psi_1$. Consider the Lyapunov function candidate

$$V_1 = V_0 + \frac{1}{2}(\hat{\theta}_{a,1} - \theta^*_{a,1})^T\Gamma_1^{-1}(\hat{\theta}_{a,1} - \theta^*_{a,1}) + \frac{1}{2\gamma_1}(\hat{b}_1 - b^*_1)^2$$

The time derivative of V_1 along (6.10) is

$$\dot{V}_1 \le z_1(g_1x_2 + \theta^{*T}_{a,1}\psi_{a,1}) + b^*_1|x_1|\bar{\phi}_1 + (\hat{\theta}_{a,1} - \theta^*_{a,1})^T\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \frac{1}{\gamma_1}(\hat{b}_1 - b^*_1)\dot{\hat{b}}_1 \qquad (6.11)$$

Since $x_2 = z_2 + \alpha_1$, substituting (6.4)-(6.6) with $i = 1$ into (6.11) yields

$$\dot{V}_1 \le g_1z_1z_2 + g_1N(\zeta_1)\dot{\zeta}_1 + z_1\theta^{*T}_{a,1}\psi_{a,1} + b^*_1|x_1|\bar{\phi}_1 + (\hat{\theta}_{a,1} - \theta^*_{a,1})^T\Gamma_1^{-1}\dot{\hat{\theta}}_{a,1} + \frac{1}{\gamma_1}(\hat{b}_1 - b^*_1)\dot{\hat{b}}_1 \qquad (6.12)$$

Adding and subtracting $\dot{\zeta}_1$ on the right hand side of (6.12), and noting (6.7) and (6.8), we have

$$\begin{aligned}
\dot{V}_1 \le\; & -k_1z_1^2 + g_1z_1z_2 + g_1N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + b^*_1|x_1|\bar{\phi}_1 - b^*_1x_1\bar{\phi}_1\tanh\Big(\frac{x_1\bar{\phi}_1}{\epsilon_1}\Big)\\
& -\sigma_{\theta_1}(\hat{\theta}_{a,1} - \theta^*_{a,1})^T(\hat{\theta}_{a,1} - \theta^0_{a,1}) - \sigma_{b_1}(\hat{b}_1 - b^*_1)(\hat{b}_1 - b^0_1) \qquad (6.13)
\end{aligned}$$

By completing the squares

$$
\begin{aligned}
-\sigma_{\theta_1}(\hat{\theta}_{a,1}-\theta^*_{a,1})^T(\hat{\theta}_{a,1}-\theta^0_{a,1}) &\le -\frac{1}{2}\sigma_{\theta_1}\|\hat{\theta}_{a,1}-\theta^*_{a,1}\|^2+\frac{1}{2}\sigma_{\theta_1}\|\theta^*_{a,1}-\theta^0_{a,1}\|^2\\
-\sigma_{b_1}(\hat{b}_1-b^*_1)(\hat{b}_1-b^0_1) &\le -\frac{1}{2}\sigma_{b_1}(\hat{b}_1-b^*_1)^2+\frac{1}{2}\sigma_{b_1}(b^*_1-b^0_1)^2
\end{aligned}
$$

and using the following nice property with regard to function $\tanh(\cdot)$ [86]

$$
0\le |x|-x\tanh(\frac{x}{\epsilon})\le 0.2785\epsilon, \quad \text{for} \quad \epsilon>0, x\in R
$$

equation (6.13) can be further written as

$$
\begin{aligned}
\dot{V}_1 &\le -k_1z_1^2+g_1z_1z_2+g_1N(\zeta_1)\dot{\zeta}_1+\dot{\zeta}_1-\frac{1}{2}\sigma_{\theta_1}\|\hat{\theta}_{a,1}-\theta^*_{a,1}\|^2-\frac{1}{2}\sigma_{b_1}(\hat{b}_1-b^*_1)^2\\
&\quad +b^*_1 0.2785\epsilon_1+\frac{1}{2}\sigma_{\theta_1}\|\theta^*_{a,1}-\theta^0_{a,1}\|^2+\frac{1}{2}\sigma_{b_1}(b^*_1-b^0_1)^2\\
&\le -k_{10}z_1^2-\frac{1}{2}\sigma_{\theta_1}\|\hat{\theta}_{a,1}-\theta^*_{a,1}\|^2-\frac{1}{2}\sigma_{b_1}(\hat{b}_1-b^*_1)^2+g_1N(\zeta_1)\dot{\zeta}_1+\dot{\zeta}_1\\
&\quad +b^*_1 0.2785\epsilon_1+\frac{1}{2}\sigma_{\theta_1}\|\theta^*_{a,1}-\theta^0_{a,1}\|^2+\frac{1}{2}\sigma_{b_1}(b^*_1-b^0_1)^2+g_1^2z_2^2\\
&\le -c_{11}V_1+c_{12}+g_1N(\zeta_1)\dot{\zeta}_1+\dot{\zeta}_1+g_1^2z_2^2
\end{aligned}
\tag{6.14}
$$

where the constants $k_{10}=k_1-\frac{1}{4}>0$, $c_{11}>0$ and $c_{12}>0$ are defined as

$$
\begin{aligned}
c_{11}: &= \min\{2k_{10},\frac{\sigma_{\theta_1}}{\lambda_{\min}(\Gamma_1^{-1})},\sigma_{b_1}\gamma_1\}\\
c_{12}: &= b^*_1 0.2785\epsilon_1+\frac{1}{2}\sigma_{\theta_1}\|\theta^*_{a,1}-\theta^0_{a,1}\|^2+\frac{1}{2}\sigma_{b_1}(b^*_1-b^0_1)^2
\end{aligned}
$$

Let $\rho_1:=\frac{c_{12}}{c_{11}}$. Multiplying (6.14) by $e^{c_{11}t}$ leads to

$$
\frac{d}{dt}(V_1e^{c_{11}t})\le c_{12}e^{c_{11}t}+g_1N(\zeta_1)\dot{\zeta}_1e^{c_{11}t}+\dot{\zeta}_1e^{c_{11}t}+g_1^2z_2^2e^{c_{11}t} \tag{6.15}
$$

Integrating (6.15) over $[0,t]$, we have

$$
\begin{aligned}
0\le V_1(t) &\le \rho_1+V_1(0)+e^{-c_{11}t}\int_0^t g_1N(\zeta_1)\dot{\zeta}_1e^{c_{11}\tau}d\tau+e^{-c_{11}t}\int_0^t\dot{\zeta}_1e^{c_{11}\tau}d\tau\\
&\quad +\int_0^t g_1^2z_2^2e^{-c_{11}(t-\tau)}d\tau
\end{aligned}
\tag{6.16}
$$

Remark 6.2.3 *If there was no uncertain term Δ_1 as in [81][83], where the uncertainty is from unknown parameters only, adaptive control can be used to solve the problem elegantly and the asymptotic stability can be guaranteed. However, it is* *not the case here due to the presence of the uncertainty terms Δ_1 in system (6.1). For illustration, integrating (6.14) over $[0,t]$ leads to*

$$V_1(t) \le V_1(0) + c_{12}t + \int_0^t (g_1 N(\zeta_1)+1)\dot{\zeta}_1 d\tau + \int_0^t g_1^2 z_2^2 d\tau$$

from which, no conclusion on the boundedness of $V_1(t)$ or $\zeta_1(t)$ can be drawn by applying Lemma 1 in [83] due to the extra term $c_{12}t$. The problem can be successfully solved by multiplying the exponential term $e^{c_{11}t}$ to both sides of (6.14) as in this chapter. From (6.16), the stability results can be drawn by invoking Lemma 6.2.1 if $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ is upper bounded.

Remark 6.2.4 *In equation (6.16), if there is no extra term $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ within the inequality, we can conclude that $V_1(t), \zeta_1$ and $z_1, \hat{\theta}_{a,1}, \hat{b}_1$ are all bounded on $[0, t_f)$ according to Lemma 6.2.1. Thus, from Proposition 1, $t_f = \infty$, and we claim that $z_1, \hat{\theta}_{a,1}, \hat{b}_1$ are globally uniformly ultimately bounded. Due to the presence of term $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ in (6.16), Lemma 6.2.1 cannot be applied directly. By noting that*

$$e^{-c_{11}t}\int_0^t g_1^2 z_2^2 e^{c_{11}\tau} d\tau \le e^{-c_{11}t} g_1^2 \sup_{\tau\in[0,t]} z_2^2 \int_0^t e^{c_{11}\tau} d\tau \le \frac{g_1^2 \sup_{\tau\in[0,t]} z_2^2}{c_{11}}$$

we know that if z_2 can be regulated as bounded, the boundedness of $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ is obvious. Then, according to Lemma 6.2.1, the boundedness of $z_1(t)$ can be guaranteed. The effect of $\int_0^t g_1^2 z_2^2 e^{-c_{11}(t-\tau)} d\tau$ will be dealt with at the following steps.

Step i ($2 \le i \le n-1$): In view of Assumption 6.2.1, we have

$$z_i\Big(\Delta_i(t,x) - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}\Delta_j(t,x)\Big) \le |z_i|\Big(p_i^*\phi_i + \sum_{j=1}^{i-1} p_j^* |\frac{\partial \alpha_{i-1}}{\partial x_j}|\phi_j\Big) \le b_i^*|z_i|\bar{\phi}_i(\bar{x}_i)$$

where $b_i^* = \max\{p_1^*, \cdots, p_i^*\}$, $\bar{\phi}_i(\bar{x}_i) \ge \phi_i + \sum_{j=1}^{i-1}|\frac{\partial \alpha_{i-1}}{\partial x_j}|\phi_j$ is a smooth positive function. A simple example is $\bar{\phi}_i = \phi_i + \sum_{j=1}^{i-1}\Big(\frac{1}{4}(\frac{\partial \alpha_{i-1}}{\partial x_j})^2 + 1\Big)\phi_j$. The derivative of $\frac{1}{2}z_i^2$ is

$$\begin{aligned} z_i\dot{z}_i &= z_i\Big[g_i x_{i+1} + \theta_i^T\psi_i + \Delta_i - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T\psi_j + \Delta_j) \\ &\quad - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{b}_j}\dot{\hat{b}}_j - \frac{\partial \alpha_{i-1}}{\partial \zeta_{i-1}}\dot{\zeta}_{i-1}\Big] \end{aligned}$$

$$\begin{aligned} &\leq z_i\Big(g_i x_{i+1} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}g_j x_{j+1} + \theta_i^T\psi_i - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\theta_j^T\psi_j + \beta_i\Big) + b_i^*|z_i|\bar{\phi}_i \\ &\leq z_i(g_i x_{i+1} + \theta_{a,i}^{*T}\psi_{a,i}) + b_i^*|z_i|\bar{\phi}_i \end{aligned}$$

where

$$\begin{aligned} \beta_i &= -\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j - \frac{\partial\alpha_{i-1}}{\partial\zeta_{i-1}}\dot{\zeta}_{i-1} \\ \theta_{a,i}^* &= [1, g_1, \cdots, g_{i-1}, \theta_i^T, \theta_1^T, \cdots, \theta_{i-1}^T]^T \\ \psi_{a,i} &= [\beta_i, -\frac{\partial\alpha_{i-1}}{\partial x_1}x_2, \cdots, -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}x_i, \psi_i^T, -\frac{\partial\alpha_{i-1}}{\partial x_1}\psi_1^T, \cdots, -\frac{\partial\alpha_{i-1}}{\partial x_{i-1}}\psi_{i-1}^T]^T \end{aligned}$$

Consider the Lyapunov function candidate

$$V_i = \frac{1}{2}z_i^2 + \frac{1}{2}(\hat{\theta}_{a,i} - \theta_{a,i}^*)^T\Gamma_i^{-1}(\hat{\theta}_{a,i} - \theta_{a,i}^*) + \frac{1}{2\gamma_i}(\hat{b}_i - b_i^*)^2$$

Selecting α_i and parameters adaptation laws as in (6.5)-(6.8), we can similarly obtain

$$\begin{aligned} \dot{V}_i &\leq z_i(g_i x_{i+1} + \theta_{a,i}^{*T}\psi_{a,i}) + b_i^*|z_i|\bar{\phi}_i + (\hat{\theta}_{a,i} - \theta_{a,i}^*)^T\Gamma_i^{-1}\dot{\hat{\theta}}_{a,i} + \frac{1}{\gamma_i}(\hat{b}_i - b_i^*)\dot{\hat{b}}_i \\ &\leq -k_{i0}z_i^2 + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i - \frac{1}{2}\sigma_{\theta_i}\|\hat{\theta}_{a,i} - \theta_{a,i}^*\|^2 - \frac{1}{2}\sigma_{b_i}(\hat{b}_i - b_i^*)^2 \\ &\quad + b_i^* 0.2785\epsilon_i + \frac{1}{2}\sigma_{\theta_i}\|\theta_{a,i}^* - \theta_{a,i}^0\|^2 + \frac{1}{2}\sigma_{b_i}(b_i^* - b_i^0)^2 + g_i^2 z_{i+1}^2 \\ 0 \leq V_i(t) &\leq \rho_i + V_i(0) + e^{-c_{i1}t}\int_0^t g_i N(\zeta_i)\dot{\zeta}_i e^{c_{i1}\tau}d\tau + e^{-c_{i1}t}\int_0^t \dot{\zeta}_i e^{c_{i1}\tau}d\tau \\ &\quad + \int_0^t g_i^2 z_{i+1}^2 e^{-c_{i1}(t-\tau)}d\tau \end{aligned}$$

where $\rho_i := \frac{c_{i2}}{c_{i1}}$, the constants $k_{i0} = k_i - \frac{1}{4} > 0$, $c_{i1} > 0$ and $c_{i2} > 0$ are defined as

$$c_{i1} := \min\{2k_{i0}, \frac{\sigma_{\theta_i}}{\lambda_{\min}(\Gamma_i^{-1})}, \sigma_{b_i}\gamma_i\} \tag{6.17}$$

$$c_{i2} := b_i^* 0.2785\epsilon_i + \frac{1}{2}\sigma_{\theta_i}\|\theta_{a,i}^* - \theta_{a,i}^0\|^2 + \frac{1}{2}\sigma_{b_i}(b_i^* - b_i^0)^2 \tag{6.18}$$

Remark 6.2.5 *Similarly, if z_{i+1} can be regulated as bounded, and therefore $\int_0^t g_i^2 z_{i+1}^2 e^{-c_{i1}(t-\tau)}d\tau$ is bounded at the following steps, then according to Lemma 6.2.1, the boundedness of $z_i(t)$ can be guaranteed.*

Step n: In this final step, the actual control u appears. Similarly, we have

$$\begin{aligned} z_n\dot{z}_n &\leq z_n\Big[g_n u + \theta_n^T\psi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T\psi_j) \\ &\quad - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j - \frac{\partial\alpha_{n-1}}{\partial\zeta_{n-1}}\dot{\zeta}_{n-1}\Big] + b_n^*|z_n|\bar{\phi}_n \\ &= z_n\Big[g_n u + \theta_n^T\psi_n - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}(g_j x_{j+1} + \theta_j^T\psi_j) + \beta_n\Big] + b_n^*|z_n|\bar{\phi}_n \\ &\leq z_n(g_n u + \theta_{a,n}^{*T}\psi_{a,n}) + b_n^*|z_n|\bar{\phi}_n \end{aligned}$$

where

$$\begin{aligned} \beta_n &= -\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{\theta}_{a,j}}\dot{\hat{\theta}}_{a,j} - \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{b}_j}\dot{\hat{b}}_j - \frac{\partial\alpha_{n-1}}{\partial\zeta_{n-1}}\dot{\zeta}_{n-1} \\ b_n^* &= \max\{p_1^*, \cdots, p_n^*\} \\ \bar{\phi}_n(\bar{x}_n) &= \phi_n + \sum_{j=1}^{n-1}|\frac{\partial\alpha_{n-1}}{\partial x_j}|\phi_j \\ \theta_{a,n}^* &= [1, g_1, \cdots, g_{n-1}, \theta_n^T, \theta_1^T, \cdots, \theta_{n-1}^T]^T \\ \psi_{a,n} &= [\beta_n, -\frac{\partial\alpha_{n-1}}{\partial x_1}x_2, \cdots, -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}x_n, \psi_n^T, -\frac{\partial\alpha_{n-1}}{\partial x_1}\psi_1^T, \cdots, -\frac{\partial\alpha_{n-1}}{\partial x_{n-1}}\psi_{n-1}^T]^T \end{aligned}$$

For clarity, the final control law and parameter adaptation laws are given explicitly:

$$\eta_n = k_n z_n + \hat{\theta}_{a,n}^T\psi_{a,n} + \hat{b}_n\bar{\phi}_n\tanh\Big(\frac{z_n\bar{\phi}_n}{\epsilon_n}\Big) \tag{6.19}$$

$$u = N(\zeta_n)\eta_n \tag{6.20}$$

$$\dot{\zeta}_n = z_n\eta_n \tag{6.21}$$

$$\dot{\hat{\theta}}_{a,n} = \Gamma_n\Big[\psi_a z_n - \sigma_{\theta_n}(\hat{\theta}_{a,n} - \theta_{a,n}^0)\Big] \tag{6.22}$$

$$\dot{\hat{b}}_n = \gamma_n\Big[z_n\bar{\phi}_n\tanh\Big(\frac{z_n\bar{\phi}_n}{\epsilon_n}\Big) - \sigma_{b_n}(\hat{b}_n - b_n^0)\Big] \tag{6.23}$$

where constant $k_n > 0$ (different from $k_i > \frac{1}{4}$ in the intermediate steps) and ϵ_n is a small positive constant, $\Gamma_n = \Gamma_n^T > 0$, γ_n, σ_{θ_n}, σ_{b_n}, $\theta_{a,n}^0$ and b_n^0 are positive design constants.

Consider the Lyapunov function candidate

$$V_n = \frac{1}{2}z_n^2 + \frac{1}{2}(\hat{\theta}_{a,n} - \theta_{a,n}^*)^T\Gamma_n^{-1}(\hat{\theta}_{a,n} - \theta_{a,n}^*) + \frac{1}{2\gamma_n}(\hat{b}_n - b_n^*)^2$$

Similarly, we have

$$\begin{aligned}\dot{V}_n &\leq -k_n z_n^2 + g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n - \frac{1}{2}\sigma_{\theta_n}\|\hat{\theta}_{a,n} - \theta_{a,n}^*\|^2 - \frac{1}{2}\sigma_{b_n}(\hat{b}_n - b_n^*)^2 \\ &\quad + b_n^* 0.2785\epsilon_n + \frac{1}{2}\sigma_{\theta_n}\|\theta_{a,n}^* - \theta_{a,n}^0\|^2 + \frac{1}{2}\sigma_{b_n}(b_n^* - b_n^0)^2 \\ 0 \leq V_n(t) &\leq \rho_n + V_n(0) + e^{-c_{n1}t}\int_0^t g_n N(\zeta_n)\dot{\zeta}_n e^{c_{n1}\tau}d\tau + e^{-c_{n1}t}\int_0^t \dot{\zeta}_n e^{c_{n1}\tau}d\tau\end{aligned}$$

where $\rho_n := \frac{c_{n2}}{c_{n1}}$, the constants $c_{n1} > 0$ and $c_{n2} > 0$ are defined as

$$\begin{aligned}c_{n1} &:= \min\{2k_n, \frac{\sigma_{\theta_n}}{\lambda_{\min}(\Gamma_n^{-1})}, \sigma_{b_n}\gamma_n\} \\ c_{n2} &:= b_n^* 0.2785\epsilon_n + \frac{1}{2}\sigma_{\theta_n}\|\theta_{a,n}^* - \theta_{a,n}^0\|^2 + \frac{1}{2}\sigma_{b_n}(b_n^* - b_n^0)^2\end{aligned}$$

Using Lemma 6.2.1, we can conclude that $\zeta_n(t)$ and $V_n(t)$, hence $z_n(t)$, $\hat{\theta}_{a,n}(t)$, $\hat{b}_{a,n}(t)$ are bounded on $[0, t_f)$. From the boundedness of $z_n(t)$, the boundedness of the extra term $\int_0^t g_{n-1}^2 z_n^2 e^{-c_{n-1,1}(t-\tau)}d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 6.2.1 backward $(n-1)$ times, it can be seen from the above design procedures that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_{a,i}(t)$, and hence $x_i(t)$ are bounded on $[0, t_f)$.

Theorem 6.2.1 *For the perturbed strict-feedback nonlinear system (6.1) with completely unknown control coefficients g_i, under Assumption 6.2.1, if we apply the controller (6.19)-(6.21) with the parameters updating laws (6.22) and (6.23), the solutions of the resulting closed-loop adaptive system are globally uniformly ultimately bounded. Furthermore, given any $\mu > \mu^* = \sqrt{\sum_{i=1}^n 2(\rho_i + c_i)}$, there exists T such that, for all $t \geq T$, we have $\|z(t)\| \leq \mu$, where $z(t) := [z_1, \cdots, z_n]^T \in R^n$, $\rho_i := \frac{c_{i2}}{c_{i1}}$, $i = 1, \cdots, n$, constants $c_{i1} > 0$ and $c_{i2} > 0$ are defined by (6.17) and (6.18) respectively, and c_i is the upper bound of $\int_0^t (g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2)e^{-c_{i1}(t-\tau)}d\tau$, $i = 1, \cdots, n-1$ and c_n is the upper bound of $\int_0^t (g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n)e^{-c_{n1}(t-\tau)}d\tau$. The compact set $\Omega_z = \{z \in R^n | \|z(t)\| \leq \mu\}$ can be made as small as desired by appropriately choosing the design constants. Furthermore, the output $y(t)$ satisfies the following property:*

$$|y(t)| \leq \sqrt{2V_1(0)e^{-c_{11}t} + 2(\rho_1 + c_1)}, \quad \forall t \geq 0. \tag{6.24}$$

Proof: The proof can be easily completed by following the above design procedures from Step 1 to Step n. According to Proposition 1, if the solution of the closed-loop system is bounded, then $t_f = \infty$. Therefore, we can obtain the globally uniformly ultimately boundedness of all the signals in the closed-loop system. Since $x_1(t) = z_1(t)$, from the definition of V_1 and (6.16), the property (6.24) can be readily obtained. Thus, by appropriately choosing the design constants, we can achieve the regulation of the state $x_1(t)$ to any prescribed accuracy while keeping the boundedness of all the signals and states of the close-loop system. ♢

Corollary 3 *Under the conditions of Theorem 6.2.1, if function ψ_i in system (6.1) and ϕ_i in Assumption 6.2.1 vanish at the origin, then we can find an adaptive controller of the form (6.20)-(6.21) with $\sigma_{\theta_i} = \sigma_{b_i} = 0, i = 1, \cdots, n$ such that all the solutions of the closed-loop system satisfy $\lim_{t\to\infty} \|x(t)\| = 0$.*

Proof: Following the same design procedure, in the present case, we have

$$\dot{V}_i \leq -k_{i0} z_i^2 + g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + g_i^2 z_{i+1}^2, \quad i = 1, \cdots, n-1 \tag{6.25}$$

$$\dot{V}_n \leq -k_n z_n^2 + g_n N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n \tag{6.26}$$

From (6.26) and using Lemma 6.2.1, it follows that $\zeta_n(t)$ and $V_n(t)$, hence $z_n(t)$, $\hat{\theta}_{a,n}(t)$, $\hat{b}_{a,n}(t)$ are globally uniformly ultimately bounded. Moreover, $z_n(t)$ is square integrable. Noting (6.25), and applying Lemma 6.2.1 backward $(n-1)$ times, it can be obtained that $V_i(t)$, $z_i(t)$, $\hat{\theta}_{a,i}(t)$, $\hat{b}_{a,i}(t)$, and hence $x_i(t)$ are globally uniformly ultimately bounded. In addition, since $\dot{x}_i$, $1 \leq i \leq n$ are bounded, functions $x_i(t)$ are uniformly continuous. Hence, a direct application of Barbalat's lemma gives that $\lim_{t\to\infty} \|x(t)\| = 0$. ♢

6.2.3 Simulation Studies

To illustrate the proposed robust adaptive control algorithms, we consider the regulation of the second-order system

$$\begin{aligned}
\dot{x}_1 &= g_1 x_2 + \theta_1 x_1^2 + \Delta_1(t, x) \\
\dot{x}_2 &= g_2 u + \Delta_2(t, x) \\
y &= x_1
\end{aligned}$$

where $x = [x_1, x_2]^T$, g_1, g_2 are unknown control coefficients, θ_1 is an unknown parameter, and $\Delta_1(t,x), \Delta_2(t,x)$ are unknown disturbances. For simulation purpose, we assume that $\theta_1 = 0.1, g_1 = 1, g_2 = 1$ and let $\Delta_1(t,x) = 0.6\sin(x_2)$, $\Delta_2(t,x) = 0.5(x_1^2 + x_2^2)\sin^3 t$. The bounds on Δ_1 and Δ_2 are $|\Delta_1(x,t)| \le p_1^*\phi_1(x_1)$, $|\Delta_2(x,t)| \le p_2^*\phi_2(x)$, where $p_1^* = 0.6$, $p_2^* = 0.5$, $\phi_1(x_1) = 1$, and $\phi_2(x) = x_1^2 + x_2^2$. $b_1^* = p_1^*$, $b_2^* = \max\{p_1^*, p_2^*\}$. For the design of robust adaptive controller, let $\hat{\theta}_{a,1}$, $\hat{\theta}_{a,2}$, $\hat{b}_1$, $\hat{b}_2$ be the estimates of unknown parameters $\theta_{a,1}^* = \theta_1$, $\theta_{a,2}^* = [1, g_1, \theta_1^T]$, b_1^*, b_2^*, and $z_1 = x_1$, $z_2 = x_2 - \alpha_1$, we have

$$
\begin{aligned}
\bar{\phi}_1 &= \phi_1 \\
\alpha_1 &= N(\zeta_1)\Big(k_1 z_1 + \hat{\theta}_{a,1}^T \psi_{a,1}(x_1) + \hat{b}_1 \bar{\phi}_1(x_1)\tanh[\frac{x_1\bar{\phi}_1(x_1)}{\epsilon_1}]\Big) \\
\bar{\phi}_2 &= \phi_2 + |\frac{\partial \alpha_1}{\partial x_1}|\phi_1 \\
u &= N(\zeta_2)\Big(k_2 z_2 + \hat{\theta}_{a,2}^T \psi_{a,2} + \hat{b}_2 \bar{\phi}_2 \tanh\big[\frac{z_2\bar{\phi}_2}{\epsilon_2}\big]\Big)
\end{aligned}
$$

where $N(\zeta_i) = \exp(\zeta_i^2)\cos(\frac{\pi}{2}\zeta_i), i = 1,2$ are the Nussbaum functions, $\psi_{a,1} = x_1^2$, $\psi_{a,2} = [-\frac{\partial\alpha_1}{\partial\hat{\theta}_{a,1}}\dot{\hat{\theta}}_{a,1} - \frac{\partial\alpha_1}{\partial\hat{b}_1}\dot{\hat{b}}_1, -\frac{\partial\alpha_1}{\partial x_1}x_2, -\frac{\partial\alpha_1}{\partial x_1}x_1^2]^T$, and ζ_1, ζ_2 are computed using (6.6). The adaptive laws are given by

$$
\begin{aligned}
\dot{\hat{\theta}}_{a,1} &= \Gamma_1 z_1 \psi_{a,1} - \Gamma_1 \sigma_{\theta_1}(\hat{\theta}_{a,1} - \theta_{a,1}^0) \\
\dot{\hat{\theta}}_{a,2} &= \Gamma_2 z_2 \psi_{a,2} - \Gamma_2 \sigma_{\theta_2}(\hat{\theta}_{a,2} - \theta_{a,2}^0) \\
\dot{\hat{b}}_1 &= \lambda_1 z_1 \bar{\phi}_1 \tanh(\frac{z_1\bar{\phi}_1}{\epsilon_1}) - \lambda_1 \sigma_{b_1}(\hat{b}_1 - b_1^0) \\
\dot{\hat{b}}_2 &= \lambda_2 z_2 \bar{\phi}_2 \tanh(\frac{z_2\bar{\phi}_2}{\epsilon_2}) - \lambda_2 \sigma_{b_2}(\hat{b}_2 - b_2^0)
\end{aligned}
$$

The following initial conditions and controller design parameters are adopted in the simulation: $x(0) = [1, 0]^T, \hat{\theta}_{a,1}(0) = 0, \hat{\theta}_{a,2}(0) = 0, \hat{b}_1(0) = 0, \hat{b}_2(0) = 0$, and $k_1 = k_2 = 1$, $\Gamma_1 = \Gamma_2 = 0.2$, $\lambda_1 = \lambda_2 = 0.1$, $\sigma_{\theta_1} = \sigma_{\theta_2} = \sigma_{b_1} = \sigma_{b_1} = 0.1$, $\epsilon_1 = \epsilon_2 = 0.05$, $\theta_{a,1}^0 = \theta_{a,2}^0 = 0$, and $b_1^0 = b_2^0 = 0.1$. Simulation results are shown in Figures 6.1-6.4. Figure 6.1 shows that the system states converges to a small neighborhood around zero. The boundedness of control input and the parameter estimates are illustrated in Figures 6.2-6.3. Figure 6.4 shows the variations of parameters ζ_1, ζ_2 and Nussbaum gains respectively.

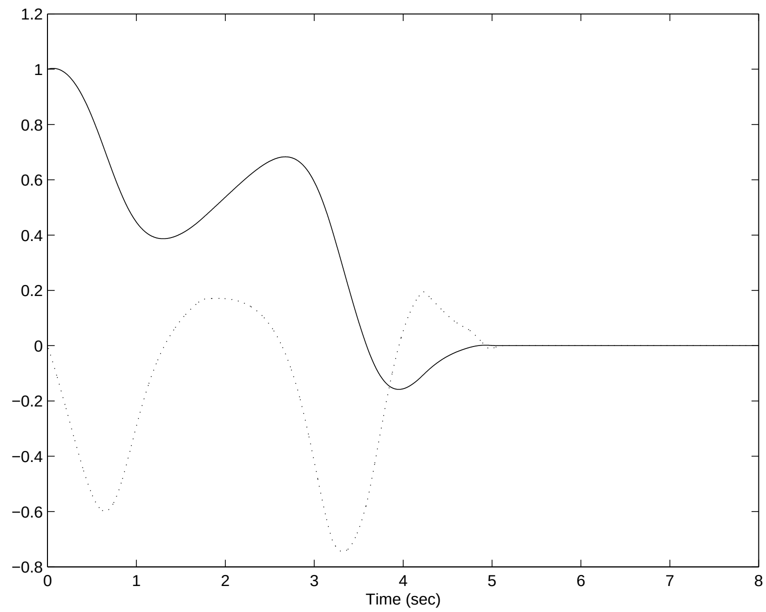

Figure 6.1: States (x_1("−") and x_2("···").

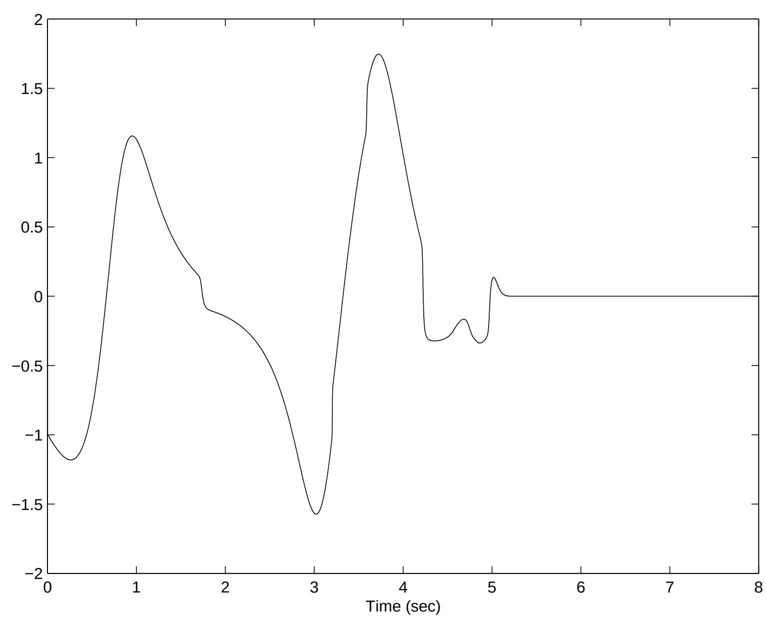

Figure 6.2: Control input u.

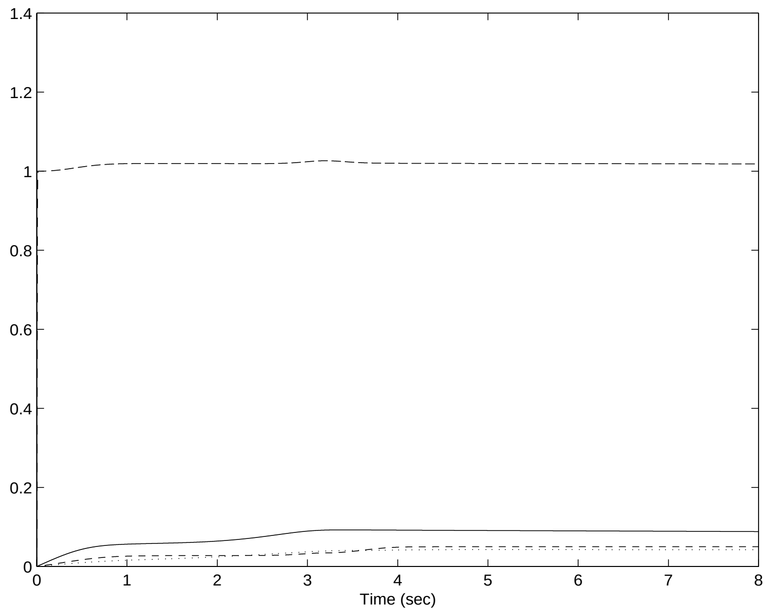

Figure 6.3: Estimation of parameters $\hat{\theta}_{a,1}$(“—”), $\|\hat{\theta}_{a,2}\|$(“- -”), $\hat{b}_1$(“$\cdots$”), $\hat{b}_2$(“-·”).

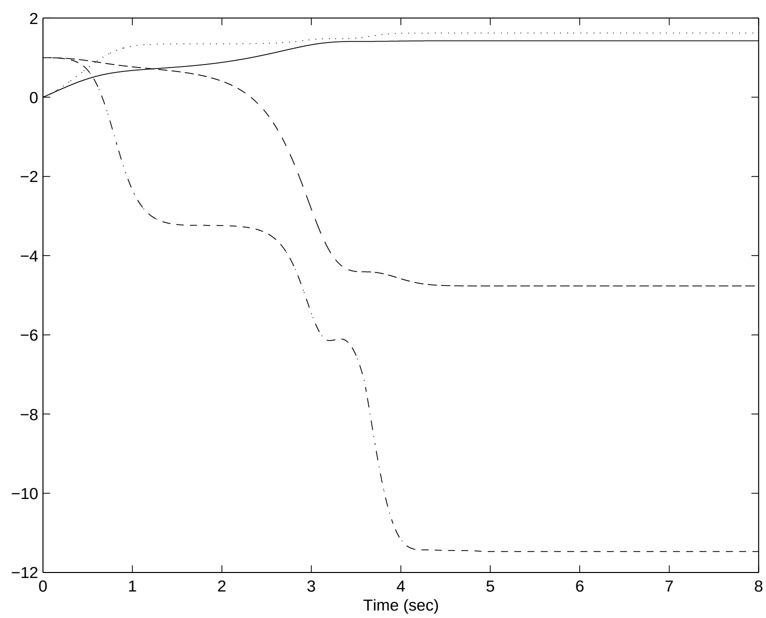

Figure 6.4: Updated variables ζ_1(“—”) and “gain” $N(\zeta_1)$(“- -”); ζ_2(“$\cdots$”) and “gain” $N(\zeta_2)$(“-·”).

6.2.4 Conclusion

In this section, a robust adaptive control approach for a class of perturbed uncertain strict-feedback nonlinear systems with unknown control coefficients has been presented. The design method does not require the *a priori* knowledge of the signs of the unknown control coefficients due to the incorporation of Nussbaum gain in the controller design. It has been proved that the proposed robust adaptive scheme can guarantee the global uniform ultimate boundedness of the closed-loop system signals.

6.3 NN Control of Time-Delay Systems with Unknown VCC

6.3.1 Problem Formulation and Preliminaries

Consider a class of single-input-single-output (SISO) nonlinear time-delay systems

$$\begin{aligned} \dot{x}_i(t) &= g_i(\bar{x}_i(t))x_{i+1}(t) + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)), \\ & \quad i = 1, \cdots, n-1 \\ \dot{x}_n(t) &= g_n(x(t))u(t) + f_n(x(t)) + h_n(x(t-\tau_n)), \\ y(t) &= x_1(t) \end{aligned} \tag{6.27}$$

where $\bar{x}_i = [x_1, x_2, \cdots, x_i]^T$, $x = [x_1, x_2, \cdots, x_n]^T \in R^n$, $u \in R$, $y \in R$ are the state variables, system input and output respectively, $g_i(\cdot)$ and $f_i(\cdot)$, $h_i(\cdot)$ are unknown smooth functions, and τ_i are unknown time delays of the states, $i = 1, \cdots, n$. The control objective is to design an adaptive controller for system (6.27) such that the output $y(t)$ follows a desired reference signal $y_d(t)$, while all signals in the closed-loop system are bounded. Define the desired trajectory $\bar{x}_{d(i+1)} = [y_d, \dot{y}_d, \cdots, y_d^{(i)}]^T$, $i = 1, \cdots, n-1$, which is a vector of y_d up to its ith time derivative $y_d^{(i)}$.

Assumption 6.3.1 *Functions $g_i(\bar{x}_i)$ and their signs are unknown, and there exist constants g_{i0} and known smooth functions $\bar{g}_i(\bar{x}_i)$ such that $0 < g_{i0} \leq |g_i(\bar{x}_i)| \leq \bar{g}_i(\bar{x}_i)$, $\forall \bar{x}_i \in R^i$.*

Assumption 6.3.2 *Known smooth functions $\bar{g}_i(\bar{x}_i)$ take value in the unknown closed intervals $I_i := [l_i^-, l_i^+] \subset [g_{i0}, +\infty)$ with $0 \notin I_i$.*

Assumption 6.3.3 *The desired trajectory vectors* $\bar{x}_{di}$, $i = 2, \cdots, n$ *are continuous and available, and* $\bar{x}_{di} \in \Omega_{di} \subset R^i$ *with* Ω_{di} *known compact sets.*

Remark 6.3.1 *Assumption 6.3.1 implies that smooth functions* $g_i(\bar{x}_i)$ *are strictly either positive or negative, which is reasonable because* $g_i(\bar{x}_i)$ *being away from zero is the controllable condition of system (6.27), which is made in most control schemes [19][139]. For a given practical system, the upper bounds of* $g_i(\bar{x}_i)$ *are not difficult to determine by choosing* $\bar{g}_i(\bar{x}_i)$ *large enough. It should be emphasized that the low bounds* g_{i0}*, the lower and upper bounds of the closed intervals* l_i^- *and* l_i^+ *are only required for analytical purposes, their true values are not necessarily known.*

Accordingly, we define positive-definite functions $\beta_i(\bar{x}_i) = \bar{g}_i(\bar{x}_i)/|g_i(\bar{x}_i)|$, $i = 1, \cdots, n$. From Assumption 6.3.1, we know that $\beta_i(\bar{x}_i)$ are bounded by known functions as $1 < \beta_i(\bar{x}_i) \leq \frac{\bar{g}_i(\bar{x}_i)}{g_{i0}}$.

Assumption 6.3.4 *The unknown smooth functions* $h_i(\bar{x}_i(t))$ *satisfy the inequality* $|h_i(\bar{x}_i(t))| \leq \varrho_i(\bar{x}_i(t))$ *where* $\varrho_i(\cdot)$ *are known positive smooth functions.*

This assumption is much more relaxed than $|h_i(\bar{x}_i(t))| \leq \sum_{j=1}^{i} |x_j(t)|\varrho_{ij}(\bar{x}_i(t))$ as has been made in [124].

Assumption 6.3.5 *The unknown time delays are bounded by a known constant, i.e.,* $\tau_i \leq \tau_{\max}$, $i = 1, \cdots, n$.

Remark 6.3.2 *There are many physical processes which are governed by nonlinear differential equations of the form (6.27). Examples are recycled reactors, recycled storage tanks and cold rolling mills [92]. In general, most of the recycling processes inherit delays in their state equations.*

The technical Lemma 2.4.6 introduced in Chapter 2 is critical in solving the robust control problem in this chapter and and is rewritten here for easy reference.

Lemma 6.3.1 *Let* $V(\cdot)$ *and* $\zeta(\cdot)$ *be smooth functions defined on* $[0, t_f)$ *with* $V(t) \geq 0$, $\forall t \in [0, t_f)$*, and* $N(\zeta)$ *be an even smooth Nussbaum-type function. If the he* *following inequality holds:*

$$V(t) \le c_0 + e^{-c_1 t}\int_0^t g_0(x(\tau))N(\zeta)\dot{\zeta}e^{c_1\tau}d\tau + e^{-c_1 t}\int_0^t \dot{\zeta}e^{c_1\tau}d\tau, \quad \forall t \in [0, t_f) \quad (6.28)$$

where constant $c_1 > 0$, $g_0(x(t))$ is a time-varying parameter which takes values in the unknown closed intervals $I := [l^-, l^+]$ with $0 \notin I$, and c_0 represents some suitable constant, then $V(t)$, $\zeta(t)$ and $\int_0^t g_0(x(\tau))N(\zeta)\dot{\zeta}d\tau$ must be bounded on $[0, t_f)$.

For clarity, the even Nussbaum function, $N(\zeta) = e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$ is used in this section.

For the construction of differentiable control laws, two continuous functions are introduced as follows.

F1). Even function $q_i(x) : R \to R$

$$q_i(x) = \begin{cases} 1, & |x| \ge \lambda_{ai} + \lambda_{bi} \\ c_{qi}\int_{\lambda_{ai}}^{x}[(\frac{\lambda_{bi}}{2})^2 - (\sigma - \lambda_{ai} - \frac{\lambda_{bi}}{2})^2]^{n-i}d\sigma, & \lambda_{ai} < x < \lambda_{ai} + \lambda_{bi} \\ c_{qi}\int_{x}^{-\lambda_{ai}}[(\frac{\lambda_{bi}}{2})^2 - (\sigma + \lambda_{ai} + \frac{\lambda_{bi}}{2})^2]^{n-i}d\sigma, & -(\lambda_{ai} + \lambda_{bi}) < x < -\lambda_{ai} \\ 0, & |x| \le \lambda_{ai} \end{cases} \quad (6.29)$$

where $c_{qi} = \frac{[2(n-i)+1]!}{\lambda_{bi}^{2(n-i)+1}[(n-i)!]^2}$, $\lambda_{ai}, \lambda_{bi} > 0$ and integer $i \in R^+$, is $(n-i)$th differentiable, i.e., $q_i(x) \in C^{n-i}$ and bounded by 1.

F2). Even function $\kappa(\cdot) : R \to R$

$$\kappa(x) = \frac{x^2\cosh(x)}{1 + x^2}, \quad \forall x \in R \quad (6.30)$$

is continuous, and monotonic, i.e., for any $|x| \ge c$, where c is a positive constant, $\kappa(x) \ge \kappa(c)$.

6.3.2 Adaptive Control for First-order System

To illustrate the design methodology clearly, we first consider the tracking problem of a first-order system

$$\dot{x}_1(t) = g_1(x_1(t))u(t) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) \quad (6.31)$$

where $u(t)$ is the control input. Define the tracking error $z_1 = x_1 - y_d$, we have

$$\dot{z}_1(t) = g_1(x_1(t))u(t) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) - \dot{y}_d(t) \quad (6.32)$$

Define $\beta_1(x_1) = \bar{g}_1(x_1)/|g_1(x_1)|$, and a smooth scalar function

$$V_{z_1}(t) = \int_0^{z_1} \sigma\beta_1(\sigma + y_d)d\sigma$$

By changing the variable $\sigma = \theta z_1$, we may rewrite V_{z_1} as $V_{z_1} = z_1^2 \int_0^1 \theta\beta_1(\theta z_1 + y_d)d\theta$. Noting that $1 \leq \beta_1(\theta z_1 + y_d) \leq \bar{g}_1(\theta z_1 + y_d)/g_{10}$, we have

$$\frac{z_1^2}{2} \leq V_{z_1} \leq \frac{z_1^2}{g_{10}} \int_0^1 \theta\bar{g}_1(\theta z_1 + y_d)d\theta \tag{6.33}$$

Its time derivative is

$$\dot{V}_{z_1}(t) = z_1(t)\beta_1(x_1(t))\dot{z}_1(t) + \int_0^{z_1} \sigma\frac{\partial\beta_1(\sigma + y_d)}{\partial y_d}\dot{y}_d d\sigma$$

Noting (6.32) and doing the integration by parts, we have

$$\begin{aligned}
\dot{V}_{z_1}(t) &= z_1(t)\beta_1(x_1(t))\Big[g_1(x_1(t))u(t) + f_1(x_1(t)) + h_1(x_1(t-\tau_1)) - \dot{y}_d(t)\Big] \\
&\quad +\dot{y}_d(t)\Big[\ \sigma\beta_1(\sigma + y_d)|_0^{z_1} - \int_0^{z_1} \beta_1(\sigma + y_d)d\sigma\Big] \\
&= z_1(t)\Big[\beta_1(x_1(t))g_1(x_1(t))u(t) + \beta_1(x_1(t))f_1(x_1(t)) \\
&\quad +\beta_1(x_1(t))h_1(x_1(t-\tau_1)) - \dot{y}_d(t)\int_0^1 \beta_1(\theta z_1 + y_d)d\theta\Big]
\end{aligned}$$

Applying Assumption 6.3.4, we have

$$\begin{aligned}
\dot{V}_{z_1}(t) &\leq z_1(t)\Big[\beta_1(x_1(t))g_1(x_1(t))u(t) + \beta_1(x_1(t))f_1(x_1(t)) \\
&\qquad -\dot{y}_d(t)\int_0^1 \beta_1(\theta z_1 + y_d)d\theta\Big] + |z_1(t)|\beta_1(x_1(t))\varrho_1(x_1(t-\tau_1))
\end{aligned} \tag{6.34}$$

Remark 6.3.3 *It can be seen from (6.34) that the design difficulties are mainly from two uncertainties: unknown functions $f_1(\cdot)$, $\beta_1(\cdot)$ (due to unknown function $g_1(\cdot)$) and unknown time delay τ_1. Although $\varrho_1(\cdot)$ is known, state $x_1(t-\tau_1)$ should not appear in the designed controller as it is undetermined due to known τ_1. In addition, the unknown time delay τ_1 and the unknown function $\beta_1(x_1(t))$ are entangled together in a nonlinear fashion, which makes the problem even more complex to solve. Therefore, we have to convert these related terms into such a form that the uncertainties from τ_1 and $\beta_1(x_1(t))$ can be dealt with separately.*

By using Young's Inequality, (6.34) becomes

$$\dot{V}_{z_1}(t) \leq z_1(t)\Big[\beta_1(x_1(t))g_1(x_1(t))u(t) + \beta_1(x_1(t))f_1(x_1(t))$$

$$
\begin{aligned}
&-\dot{y}_d(t)\int_0^1 \beta_1(\theta z_1 + y_d)d\theta\Big] \\
&+\frac{1}{2}z_1^2(t)\beta_1^2(x_1(t)) + \frac{1}{2}\varrho_1^2(x_1(t-\tau_1)) \qquad (6.35)
\end{aligned}
$$

where $\beta_1(x_1(t))$ and $\varrho_1(x_1(t-\tau_1))$ are separated and can be dealt with one by one as detailed later.

To overcome the design difficulties from the unknown time delay τ_1, the following Lyapunov-Krasovskii functional can be considered

$$
V_{U_1}(t) = \frac{1}{2}\int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau, \quad U_1(x_1(t)) = \varrho_1^2(x_1(t)) \qquad (6.36)
$$

The time derivative of $V_{U_1}(t)$ is

$$
\dot{V}_{U_1}(t) = \frac{1}{2}\varrho_1^2(x_1) - \frac{1}{2}\varrho_1^2(x_1(t-\tau_1))
$$

which can be used to cancel the time-delay term on the right hand side of (6.35) and thus eliminate the design difficulty from the unknown time delay τ_1 without introducing any uncertainties to the system. For notation conciseness, we will omit the time variables t and $t-\tau_1$ after time-delay terms have been eliminated. Accordingly, we obtain

$$
\dot{V}_{z_1} + \dot{V}_{U_1} \le z_1\beta_1(x_1)g_1(x_1)u + Q_1(Z_1)z_1 \qquad (6.37)
$$

where

$$
Q_1(Z_1) = \beta_1(x_1)f_1(x_1) - \dot{y}_d\int_0^1 \beta_1(\theta z_1 + y_d)d\theta + \frac{1}{2}z_1\beta_1^2(x_1) + \frac{1}{2z_1}\varrho_1^2(x_1) \quad (6.38)
$$

with $Z_1 = [x_1, y_d, \dot{y}_d]^T \in \Omega_{Z_1} \subset R^3$, where Ω_{Z_1} is a compact set.

At present stage, suppose the Lyapunov function candidate is chosen as $V_1(t) = V_{z_1}(t) + V_{U_1}(t)$. From (6.37), we know that we can design a stabilizing $u(t)$ which is free from unknown time delay τ_1 under the assumption of known system functions.

Note that if $Q_1(Z_1)$ is utilized to construct the controller, controller singularity may occur since $\frac{1}{2z_1}\varrho_1^2(x_1)$ is not well-defined at $z_1 = 0$. Therefore, care must be taken to guarantee the boundedness of the control. It is noted that the controller singularity takes place at the point $z_1 = 0$, where the control objective is supposed to be achieved. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. As $z_1 = 0$ is hard to detect owing to the existence of measurement noise, it is more practical to relax our control objective of convergence to a "ball" rather than the origin [129].

For ease of discussion, let us define sets $\Omega_{c_{z_1}} \subset \Omega_{Z_1}$ and $\Omega^0_{Z_1}$ as follows

$$\Omega_{c_{z_1}} := \{z_1, \bar{x}_{d2} \,| |z_1| < c_{z_1}, \bar{x}_{d2} \in \Omega_{d2}\} \quad (6.39)$$

$$\Omega^0_{Z_1} := \Omega_{Z_1} - \Omega_{c_{z_1}} \quad (6.40)$$

where c_{z_1} is a constant that can be chosen arbitrarily small and "$-$" in (6.40) is used to denote the complement of set B in set A as follows

$$A - B := \{x \,| x \in A \text{ and } x \notin B\}$$

Lemma 6.3.2 *Set $\Omega^0_{Z_1}$ is a compact set.*

Proof: See Section 4.2.3 of Chapter 4. ◇

Under the assumption of known system functions, we have the practical robust control law to guarantee the closed-loop stability as detailed in Lemma 6.3.3.

Lemma 6.3.3 *For the first-order system (6.31), if the practical robust control law is chosen as*

$$u(t) = \begin{cases} N(\zeta_1)[k_1(t)z_1 + Q_1(Z_1)], & z_1 \in \Omega^0_{Z_1} \\ 0, & z_1 \in \Omega_{c_{z_1}} \end{cases} \quad (6.41)$$

$$\dot{\zeta}_1 = k_1(t)z_1^2 + Q_1(Z_1)z_1 \quad (6.42)$$

where $k_1(t) \geq k^* > 0$ with k^* being any positive constant, then for bounded initial conditions, all the signals in the closed-loop system are globally uniformly ultimately bounded.

Proof: We first show that all the closed-loop signals are GUUB for $z_1 \in \Omega^0_{Z_1}$. Consider the following Lyapunov function candidate

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t)$$

Its time derivative along (6.37) is

$$\dot{V}_1(t) \le z_1\beta_1(x_1)g_1(x_1)u + Q_1(Z_1)z_1 \tag{6.43}$$

For $z_1 \in \Omega^0_{Z_1}$, substituting (6.41) into (6.43) yields

$$\dot{V}_1(t) \le \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + Q_1(Z_1)z_1 \tag{6.44}$$

Adding and subtracting $k_1(t)z_1^2 + Q_1(Z_1)z_1$ on the right hand side of (6.44), we have

$$\begin{aligned} \dot{V}_1(t) &\le \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 - \dot{\zeta}_1 + Q_1(Z_1)z_1 \\ &\le -k_1^* z_1^2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \end{aligned} \tag{6.45}$$

Integrating (6.45) over $[0,t]$, $\forall t \in [0,t_f)$, we have the following inequality

$$\begin{aligned} V_1(t) + \int_0^t k_1^* z_1^2(\tau)d\tau \le V_1(0) \\ + \int_0^t [\beta_1(x_1(\tau))g_1(x_1(\tau))N(\zeta_1(\tau)) + 1]\dot{\zeta}_1(\tau)d\tau \end{aligned} \tag{6.46}$$

Since $\int_0^t k_1^* z_1^2(\tau)d\tau \ge 0$, we further have

$$V_1(t) \le V_1(0) + \int_0^t [\beta_1(x_1(\tau))g_1(x_1(\tau))N(\zeta_1(\tau)) + 1]\dot{\zeta}_1(\tau)d\tau$$

Applying Lemma 1 in [83], we can conclude that $V_1(t)$, $\int_0^t(\beta_1 g_1 N(\zeta_1) + 1)\dot{\zeta}_1 d\tau$, and $\zeta_1(t)$ are bounded. Since $\frac{1}{2}z_1^2(t) \le V_{z_1}(t) \le V_1(t)$, we know that $z_1(t)$ are bounded on $[0,t_f)$. According to Proposition 2 in [70], if the solution of the closed-loop is bounded, then $t_f = +\infty$. From (6.46), $z_1(t)$ is square integrable and as an immediate result, x_1, u and $\dot{z}_1$ are also bounded on $[0,+\infty]$. Since $\dot{z}_1 \in L^\infty$, and $z_1 \in L^2 \cap L^\infty$, by Barbalat's lemma, $\lim_{t\to+\infty} z_1 = 0$. Note that the above results are obtained for $z_1 \in \Omega^0_{Z_1}$, therefore we can guarantee that $\Omega_{c_{z_1}}$ is domain of attraction. $\diamondsuit$

Remark 6.3.4 *For the first-order system, the definition of the compact set $\Omega^0_{Z_1}$ in (6.40) and the corresponding practical control law $u(t)$ in (6.41) can guarantee the stability of the closed-loop system. To extend the above design methodology to higher-order systems, modification has to be made since $u(t)$ is not differentiable* *at* $|z_1| = c_{z_1}$*. We will discuss this issue at a later stage when the problem is clearly shown.*

In the case that $f_1(\cdot)$ and $g_1(\cdot)$ are completely unknown, the proposed controller (6.41) in Lemma 6.3.3 is not feasible due to the unknown function $Q_1(Z_1)$. On the other hand, by employing the robust control in (6.41), control action is only activated when $z_1 \in \Omega^0_{Z_1}$. Apparently, $Q_1(Z_1)$ is continuous and well-defined over compact set $\Omega^0_{Z_1}$ and can be approximated by neural networks to arbitrary any accuracy as follows

$$Q_1(Z_1(t)) = W_1^{*T} S_1(Z_1) + \epsilon_1(Z_1) \tag{6.47}$$

where $|\epsilon_1(Z_1)| \le \epsilon^*_{z_1}$ is the approximation error, $W_1^* \in R^{l_1}$ are unknown ideal constant weights, and $S_1(Z_1) \in R^{l_1}$ are the basis functions. Let us use its estimate $\hat{W}_1$ instead to form the adaptive control

$$u(t) = \begin{cases} N(\zeta_1)[k_1(t)z_1 + \hat{W}_1^T S_1(Z_1)], & z_1 \in \Omega^0_{Z_1} \\ 0, & z_1 \in \Omega_{c_{z_1}} \end{cases} \tag{6.48}$$

$$\dot{\zeta}_1 = k_1(t)z_1^2 + \hat{W}_1^T S_1(Z_1)z_1 \tag{6.49}$$

$$\dot{\hat{W}}_1 = \Gamma_1[S_1(Z_1)z_1 - \sigma_1 \hat{W}_1] \tag{6.50}$$

where matrix $\Gamma_1 = \Gamma_1^T > 0$, and small constant $\sigma_1 > 0$ is to introduce the σ-modification for the closed-loop system.

Theorem 6.3.1 summarizes the stability result for the proposed adaptive scheme, and shows that certain compact set is a domain of attraction.

Theorem 6.3.1 *Consider the closed-loop systems consisting of the first-order plant (6.31) and controller (6.48), (6.49), if gain* $k_1(t) = k_{10} + k_{11}(t)$ *with* $k_{10} > 0$ *being a design constant, and* $k_{11}(t)$ *is chosen as*

$$k_{11}(t) = \frac{1}{\varepsilon_1}\Big[1 + \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d)d\theta + \frac{1}{z_1^2}\int_{t-\tau_{\max}}^t \frac{1}{2}U_1(x_1(\tau))d\tau\Big] \tag{6.51}$$

with constant $\varepsilon_1 > 0$*, and the NN weights are updated by (6.50), then for bounded initial conditions* $x_1(0)$ *and* $\hat{W}_1(0)$*, all signals in the closed-loop system are semi-globally uniformly ultimately bounded, and the vector* Z_1 *remains in a compact set* $\Omega^0_{Z_1}$ *defined by*

$$\Omega^0_{Z_1} = \Big\{Z_1 \Big| |z_1| \le \mu_1, \frac{1}{2}\|\tilde{W}_1\|^2 \le \frac{V_1(t)}{\lambda_{\min}(\Gamma_1^{-1})}, \bar{x}_{d2} \in \Omega_{d2}\Big\}$$

whose size, $\mu_1 > 0$, can be adjusted by appropriately choosing the design parameters.

Proof: The same as in the proof of Lemma 6.3.3, let us consider the following Lyapunov function candidate

$$V_1(t) = V_{z_1}(t) + V_{U_1}(t) + \frac{1}{2}\tilde{W}_1^T(t)\Gamma_1^{-1}\tilde{W}_1(t) \tag{6.52}$$

where $\tilde{(\cdot)} = \hat{(\cdot)} - (\cdot)^*$. The time derivative of $V_1(t)$ along (6.37) is

$$\dot{V}_1 \leq z_1\beta_1(x_1)g_1(x_1)u + Q_1(Z_1)z_1 + \tilde{W}_1^T\Gamma_1^{-1}\dot{\hat{W}}_1 \tag{6.53}$$

For $z_1 \in \Omega^0_{Z_1}$, substituting (6.48) and (6.50) into (6.53), we have

$$\dot{V}_1 \leq \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + Q_1(Z_1)z_1 + \tilde{W}_1^T S_1(Z_1)z_1 - \sigma_1\tilde{W}_1^T\hat{W}_1 \tag{6.54}$$

Adding and subtracting $k_1(t)z_1^2 + \hat{W}_1^T S_1(Z_1)z_1$ on the right hand side of (6.54) and noting (6.47), we have

$$\begin{aligned}
\dot{V}_1 &\leq \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 - \dot{\zeta}_1 + \hat{W}_1^T S_1(Z_1)z_1 + z_1\epsilon_{z_1} - \sigma_1\tilde{W}_1^T\hat{W}_1 \\
&= -k_1(t)z_1^2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + z_1\epsilon_{z_1} - \sigma_1\tilde{W}_1^T\hat{W}_1
\end{aligned} \tag{6.55}$$

Noting $k_1(t) = k_{10} + k_{11}(t)$, (6.55) becomes

$$\dot{V}_1 \leq -k_{11}(t)z_1^2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 - k_{10}z_1^2 + z_1\epsilon_{z_1} - \sigma_1\tilde{W}_1^T\hat{W}_1 \tag{6.56}$$

Using the inequalities

$$\begin{aligned}
-k_{10}z_1^2 + z_1\epsilon_{z_1} &\leq \frac{\epsilon_{z_1}^2}{4k_{10}} \leq \frac{\epsilon_{z_1}^{*^2}}{4k_{10}} \\
-\sigma_1\tilde{W}_1^T\hat{W}_1 &\leq -\frac{1}{2}\sigma_1\|\tilde{W}_1\|^2 + \frac{1}{2}\sigma_1\|W_1^*\|^2
\end{aligned}$$

and substituting (6.51) into (6.56), we have

$$\begin{aligned}
\dot{V}_1 \leq\ & -\frac{z_1^2}{\varepsilon_1}\Big[1 + \int_0^1 \theta\bar{g}_1(\theta z_1 + y_d)d\theta\Big] - \frac{1}{\varepsilon_1}\int_{t-\tau_{\max}}^t \frac{1}{2}U_1(x_1(\tau))d\tau \\
& -\frac{1}{2}\sigma_1\|\tilde{W}_1\|^2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + c_{\epsilon 1}
\end{aligned}$$

where

$$c_{\epsilon 1} = \frac{\epsilon_{z_1}^{*^2}}{4k_{10}} + \frac{1}{2}\sigma_1\|W_1^*\|^2 \tag{6.57}$$

Since $\tau_1 \leq \tau_{\max}$ according to Assumption 6.3.5, inequality $\int_{t-\tau_1}^{t} U_1(x_1(\tau))d\tau \leq \int_{t-\tau_{\max}}^{t} U_1(x_1(\tau))d\tau$ holds. From (6.33) and (6.36), we have

$$\begin{aligned} \dot{V}_1(t) &\leq -\frac{g_{10}}{\varepsilon_1}V_{z_1} - \frac{1}{\varepsilon_1}V_{U_1} - \frac{1}{2}\sigma_1\|\tilde{W}_1\|^2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + c_{\epsilon 1} \\ &\leq -c_1V_1(t) + c_{\epsilon 1} + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \end{aligned} \tag{6.58}$$

where positive constant c_1 is defined by

$$c_1 := \min\Big\{\frac{g_{10}}{\varepsilon_1},\ \frac{1}{\varepsilon_1},\ \frac{\sigma_1}{\lambda_{\min}(\Gamma_1^{-1})}\Big\} \tag{6.59}$$

Letting $\rho_1 := c_{\epsilon 1}/c_1$ and multiplying (6.58) by $e^{c_1 t}$, it becomes

$$\frac{d}{dt}(V_1(t)e^{c_1 t}) \leq c_{\epsilon 1}e^{c_1 t} + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 e^{c_1 t} + \dot{\zeta}_1 e^{c_1 t} \tag{6.60}$$

Integrating (6.60) over $[0, t]$, we have

$$\begin{aligned} V_1(t) &\leq \rho_1 + [V_1(0) - \rho_1]e^{-c_1 t} + e^{-c_1 t}\int_0^t \Big(\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1\Big)e^{c_1\tau}\dot{\zeta}_1 d\tau \\ &\leq \rho_1 + V_1(0)e^{-c_1 t} + e^{-c_1 t}\int_0^t \Big(\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1\Big)e^{c_1\tau}\dot{\zeta}_1 d\tau \end{aligned} \tag{6.61}$$

Applying Lemma 6.3.1, we can conclude that $V_1(t)$, $\int_0^t(\beta_1 g_1 N(\zeta_1) + 1)\dot{\zeta}_1 d\tau$, and $\zeta_1(t)$, hence $z_1(t)$, $\hat{W}_1$ are SGUUB on $[0, t_f)$. According to Proposition 2 in [70], if the solution of the closed-loop system is bounded, then $t_f = +\infty$. Let $c_{\beta 1}$ be the upper bound of $\int_0^t |(\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1)\dot{\zeta}_1| d\tau$, then we have the following inequalities

$$\begin{aligned} & e^{-c_1 t}\int_0^t (\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1)e^{c_1\tau}\dot{\zeta}_1 d\tau \\ \leq\ & \int_0^t \Big|(\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1)\dot{\zeta}_1\Big| e^{-c_1(t-\tau)} d\tau \\ \leq\ & \int_0^t \Big|(\beta_1(x_1)g_1(x_1)N(\zeta_1) + 1)\dot{\zeta}_1\Big| d\tau \leq c_{\beta 1} \end{aligned}$$

Thus, equation (6.61) becomes

$$V_1(t) \leq (\rho_1 + c_{\beta 1}) + V_1(0)e^{-c_1 t} \tag{6.62}$$

where constant

$$V_1(0) = \int_0^{z_1(0)} \sigma\beta_1(\sigma + y_d(0))d\sigma + \frac{1}{2}\tilde{W}_1^T(0)\Gamma_1^{-1}\tilde{W}_1(0)$$

It follows from (6.33), (6.52) and (6.62) that

$$\begin{aligned} \frac{1}{2}z_1^2(t) \le V_{z_1}(t) &\le V_1(t) \le (\rho_1 + c_0) + V_1(0) \\ \frac{1}{2}\|\tilde{W}_1\|^2 &\le \frac{V_1(t)}{\lambda_{\min}(\Gamma_1^{-1})} \end{aligned}$$

By letting $\mu_1 = \sqrt{2(\rho_1 + c_{\beta 1}) + 2V_1(0)}$, we know that $|z_1| \le \mu_1$. We can readily conclude that there do exist a compact set $\Omega_{Z_1}^0$ such that $Z_1 \in \Omega_{Z_1}^0$, $\forall t \ge 0$. $\diamondsuit$

Remark 6.3.5 *If system uncertainties are in the linear-in-the-parameter form as in [83], adaptive control can be used to solve the problem elegantly and the asymptotic stability can be guaranteed by applying Lemma 1 in [83]. In this section, the unknown functions are approximated by RBF NN, which has an intrinsic approximation error, therefore Lemma 1 in [83] is no longer applicable. To show the point clearly, the time derivative of* $V_1(t)$ *is re-written as follows*

$$\dot{V}_1(t) \le -c_1 V_1(t) + c_{\epsilon 1} + \beta_1(x_1) g_1(x_1) N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 \tag{6.63}$$

Integrating (6.63) over $[0, t]$, *we have*

$$V_1(t) \le V_1(0) + c_{\epsilon 1} t + \int_0^t (\beta_1(x_1) g_1(x_1) N(\zeta_1) + 1)\dot{\zeta}_1 d\tau \tag{6.64}$$

From (6.64), we cannot draw any conclusion for the boundedness of $V_1(t)$ *or* $\zeta_1(t)$ *by applying Lemma 1 in [83] due to the extra term* $c_{\epsilon 1} t$. *From the definition of* $c_{\epsilon 1}$ *in (6.57), we know that* $c_{\epsilon 1}$ *is a function of NN approximation error* $\epsilon_{z_1}^*$ *and* $\frac{1}{2}\sigma_1\|W_1^*\|^2$. *Even though we can remove the latter by setting* σ_1 *as zero, the former effect from NN approximation error* $\epsilon_{z_1}^*$ *cannot be eliminated. The problem is successfully solved by multiplying the exponential term* $e^{c_1 t}$ *to both sides of (6.63) as did in the proof of Theorem 6.3.1. Consequently, the stability results can be drawn by invoking Lemma 6.3.1.*

Remark 6.3.6 *Although the system has been proven to converge into a compact set which is actually unknown due to unknown* g_{10}, $\epsilon_{z_1}^*$, W_1^*, c_0, *and* $V_1(0)$, *it is possible to adjust the size by appropriately choosing design parameters* σ_1 *and* Γ_1.

Remark 6.3.7 *The computation of the second integral of $k_{11}(t)$ in (6.51) should be conducted in the time interval $[t-\tau_{\max}, t]$. If the integration is conducted alternatively in $[0, t]$, the stability result may seem to hold. However, the integral result will progressively tend to a large value as the time increases, which may saturate the actuator and destroy the closed-loop stability. To avoid this, a rather conservative time interval $[t-\tau_{\max}, t]$ should be chosen for conducting the integration. The same conservative measure will be taken in the later recursive backstepping design.*

Remark 6.3.8 *Though it is known that the stability of time-delay systems depends on the size of the time delay, it is not necessarily true for general nonlinear systems as is illustrated by the following example. Consider the linear time-delay system*

$$\dot{x}(t) = -bx(t-\tau)$$

with $b > 0$, $\tau > 0$. It has been proven that the linear time delay system is stable if $\tau < \frac{1}{b}$, and the system is unstable if τ is too large. However, for the forced linear time delay system given by

$$\dot{x}(t) = -bx(t-\tau) + u(t)$$

with $b > 0$, $\tau > 0$, subject to the sliding mode control

$$u(t) = -\text{sgn}(x(t))[b_1|x(t-\tau)| + \epsilon], \quad b_1 > b$$

we have the resulting nonlinear time delay closed-loop system

$$\dot{x}(t) + bx(t-\tau) + \text{sgn}(x(t))[b_1|x(t-\tau)| + \epsilon] = 0 \tag{6.65}$$

For the nonlinear time delay system (6.65), consider the Lyapunov function candidate $V(t) = \frac{1}{2}x^2(t)$, we have

$$\begin{aligned} \dot{V}(t) &= -bx(t)x(t-\tau) - b_1|x(t)||x(t-\tau)| - \epsilon|x(t)| \\ &\leq -\epsilon|x(t)| \leq 0 \end{aligned}$$

Apparently, the nonlinear time delay system (6.65) is stable for arbitrary τ. This also verifies the rich dynamic behaviors of nonlinear systems.

We have developed a practical adaptive neural control for first-order system (6.31). Now we are ready to extend the above design methodology to higher-order systems.

6.3.3 Practical Adaptive Backstepping Design

In this section, the adaptive design will be extended to nth-order systems (6.27) and the stability results of the closed-loop system are presented.

Note that the extension is not straightforward as in the classical cases of backstepping design for nonlinear systems in strict feedback form without time delays. In the proposed recursive backstepping design, the computation of $\alpha_i(t)$ requires the computation of $\alpha_{i-1}(t)$. As a result, the unknown time-delay terms of all the previous subsystems will appear in Step i, which have to be compensated for one by one. Though the idea of Lyapunov-Krasovskii functional $V_{U_i}(t)$ shall be used to handle the unknown time delays terms as in Section 6.3.2, different from the classical cases, the Lyapunov function candidate $V_i(t)$ is much more involved, in which the following terms $\int_{t-\tau_1}^t U_1(x_1(\tau))d\tau$, ..., $\int_{t-\tau_{i-1}}^t U_{i-1}(\bar{x}_{i-1}(\tau))d\tau$, and $\int_{t-\tau_i}^t U_i(\bar{x}_i(\tau))d\tau$ appeared i times, twice and once respectively rather than a simple summation of the previous ones. The derivations are very troublesome in order to see the choices of the above functionals clearly, and cannot be further simplified because of the nature of the problem.

The backstepping design procedure contains n steps. At each step, an intermediate control function $\alpha_i(t)$ shall be developed using an appropriate Lyapunov function $V_i(t)$. The design of both the control laws and the adaptive laws are based on the following change of coordinates: $z_1 = x_1 - y_d$, $z_i = x_i - \alpha_{i-1}$, $i = 2, \cdots, n$. Note that the controller design based on such compact sets $\Omega_{Z_i}^0$ will render α_i not differentiable at points $|z_i| = c_{z_i}$. This appears to be a "technical problem" as the differentiation of α_i is not defined at these isolated points. To solve this problem, one practical way is to simply set the differentiation at these points to be any finite value, say 0, and then every signal in the closed-loop system can be shown to be bounded. Theoretically speaking, by doing so, there is no much loss either as these points are isolated and can be ignored. For ease and clarity of presentation, we assume that all the control functions are differentiable throughout this section.

For uniformity of notation, throughout this section, define estimation errors $\tilde{W}_i = \hat{W}_i - W_i^*$, compact sets $\Omega_{c_{z_i}}$ and $\Omega_{Z_i}^0$ as

$$\Omega_{c_{z_i}} := \{z_i, \bar{x}_{d,i+1} \,||z_i| < c_{z_i}, \bar{x}_{d,i+1} \in \Omega_{d,i+1}\}$$

$$\Omega^0_{Z_i} \;:=\; \Omega_{Z_i} - \Omega_{c_{z_i}}$$

with constants $c_{z_i} > 0$, and positive constants c_i, $c_{\epsilon i}$, ρ_i as

$$\begin{aligned} c_i &:= \min\left\{\frac{g_{i0}}{\varepsilon_i}, \frac{1}{\varepsilon_i}, \frac{\sigma_i}{\lambda_{\min}(\Gamma_i^{-1})}\right\} \\ c_{\epsilon i} &:= \frac{\epsilon^{*^2}_{z_i}}{4k_{i0}} + \frac{1}{2}\sigma_i\|W_i^*\|^2 \\ \rho_i &:= \frac{c_{\epsilon i}}{c_i} \end{aligned}$$

where $\hat{W}_i \in R^{l_i}$ are the estimates of ideal NN weights $W_i^* \in R^{l_i}$, g_{i0} are the lower bounds of $|g_i(\bar{x}_i)|$, constants $0 < \varepsilon_i \le 4$, small constants $\sigma_i > 0$, matrices $\Gamma_i = \Gamma_i^T > 0$, constants $k_{i0} > 0$, $\epsilon^*_{z_i}$ are the upper bounds of the NN approximation errors, i.e., $|\epsilon_i(Z_i)| \le \epsilon^*_{z_i}$ with Z_i being the corresponding inputs to be defined later, and the following integral Lyapunov functions $V_{z_i}(t)$, the Lyapunov-Krasovskii functionals $V_{U_i}(t)$ with the positive scalar functions $U_i(\cdot)$, and the Lyapunov function candidates $V_i(t)$ as

$$V_{z_1}(t) = \int_0^{z_1} \sigma\beta_1(\sigma + y_d)d\sigma \tag{6.66}$$

$$V_{z_i}(t) = \int_0^{z_i} \sigma\beta_i(\bar{x}_{i-1}, \sigma + \alpha_{i-1})d\sigma, i = 2, \cdots, n \tag{6.67}$$

$$V_{U_i}(t) = \frac{1}{2}\int_{t-\tau_i}^{t} U_i(\bar{x}_i(\tau))d\tau + \sum_{j=1}^{i-1}\int_{t-\tau_j}^{t} U_j(\bar{x}_j(\tau))d\tau, \quad i = 1, \cdots, n \tag{6.68}$$

$$V_i(t) = V_{z_i}(t) + V_{U_i}(t) + \frac{1}{2}\tilde{W}_i^T(t)\Gamma_i^{-1}\tilde{W}_i(t), i = 1, \cdots, n \tag{6.69}$$

where positive functions $U_i(\bar{x}_i(t)) = \varrho_i^2(\bar{x}_i(t))$.

The adaptive neural control laws are as follows, for $i = 1, \cdots, n$

$$\alpha_i = \begin{cases} N(\zeta_i)[k_i(t)z_i + \hat{W}_i^T S_i(Z_i)], & z_i \in \Omega^0_{Z_i} \\ 0, & z_i \in \Omega_{c_{z_i}} \end{cases} \tag{6.70}$$

$$\dot{\zeta}_i = k_i(t)z_i^2 + \hat{W}_i^T S_i(Z_i)z_i \tag{6.71}$$

$$\dot{\hat{W}}_i = \Gamma_i[S_i(Z_i)z_i - \sigma_i\hat{W}_i] \tag{6.72}$$

where $k_i(t) = k_{i0} + k_{i1}(t)$, $k_{i1}(t)$ is chosen as

$$\begin{aligned} k_{i1}(t) = \frac{1}{\varepsilon_i}\Big[&1 + \int_0^1 \theta\bar{g}_i(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1})d\theta \\ &+ \frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t}\Big(\frac{1}{2}U_i(\bar{x}_i(\tau)) + \sum_{j=1}^{i-1}U_j(\bar{x}_j(\tau))\Big)d\tau\Big] \end{aligned} \tag{6.73}$$

and $S_i(Z_i) \in R^{l_i}$ are the basis functions with Z_i being the input vectors defined in (6.88) and (6.95) later.

Note that when $i = n$, α_n is actually the control input $u(t)$.

Step 1: Let us firstly consider the equation in (6.27) when $i = 1$, i.e.,

$$\dot{x}_1(t) = g_1(x_1(t))x_2(t) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) \tag{6.74}$$

From the definition for new states z_1 and z_2, i.e. $z_1 = x_1 - y_d$ and $z_2 = x_2 - \alpha_1$, we have

$$\dot{z}_1(t) = g_1(x_1(t))(z_2(t) + \alpha_1(t)) + f_1(x_1(t)) + h_1(x_1(t - \tau_1)) - \dot{y}_d(t) \tag{6.75}$$

Consider $V_{z_1}(t)$ in (6.66). Its time derivative along (6.75) is

$$\begin{aligned} \dot{V}_{z_1}(t) &= z_1(t)\Big[\beta_1(x_1(t))g_1(x_1(t))z_2(t) + \beta_1(x_1(t))g_1(x_1(t))\alpha_1(t) \\ &\quad +\beta_1(x_1(t))f_1(x_1(t)) + \beta_1(x_1(t))h_1(x_1(t - \tau_1)) \\ &\quad -\dot{y}_d(t)\int_0^1 \beta_1(\theta z_1 + y_d)d\theta\Big] \end{aligned} \tag{6.76}$$

Following the same procedure as in Section 6.3.2 by choosing V_{U_1} in (6.68) and applying Assumption 6.3.4 and Young's inequality, we obtain

$$\dot{V}_{z_1} + \dot{V}_{U_1} \le z_1\beta_1(x_1)g_1(x_1)z_2 + z_1\beta_1(x_1)g_1(x_1)\alpha_1 + Q_1(Z_1)z_1 \tag{6.77}$$

where $Q_1(Z_1)$ is defined in (6.38).

As stated in Section 6.3.2, the control objective now is to show that z_1 converges to certain domain of attraction rather than the origin. To this end, let us show the derivative of Lyapunov function candidate is non-positive when $z_1 \in \Omega^0_{Z_1}$. Consider the Lyapunov function candidate $V_1(t)$ given in (6.69). Its time derivative along (6.77) is

$$\dot{V}_1(t) = z_1\beta_1(x_1)g_1(x_1)z_2 + z_1\beta_1(x_1)g_1(x_1)\alpha_1 + Q_1(Z_1)z_1 + \tilde{W}_1^T\Gamma_1^{-1}\dot{\tilde{W}}_1$$

Choose the practical adaptive neural intermediate control law and NN weights updating law as given in (6.70)-(6.72) with $k_{11}(t)$ given in (6.73). Now, using the same procedure as in Section 6.3.2, it can be shown that

$$\begin{aligned} \dot{V}_1(t) &\le -\frac{z_1^2}{\varepsilon_1}\Big[1 + \int_0^1 \theta\bar{g}_1(\theta z_1 + y_d)d\theta\Big] - \frac{1}{\varepsilon_1}\int_{t-\tau_{\max}}^t \frac{1}{2}U_1(x_1(\tau))d\tau - \frac{1}{2}\sigma_1\|\tilde{W}_1\|^2 \\ &\quad +\beta_1(x_1)g_1(x_1)z_1z_2 + \beta_1(x_1)g_1(x_1)N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + c_{\epsilon 1} \end{aligned} \tag{6.78}$$

Noting that $\beta_1 g_1 z_1 z_2 \le \frac{1}{4} z_1^2 + \beta_1^2 g_1^2 z_2^2$, (6.78) becomes

$$
\begin{aligned}
\dot{V}_1(t) \ \le\ & -\frac{z_1^2}{\varepsilon_1}\Big[1 - \frac{\varepsilon_1}{4} + \int_0^1 \theta \bar{g}_1(\theta z_1 + y_d) d\theta\Big] \\
& -\frac{1}{\varepsilon_1}\int_{t-\tau_{\max}}^{t} \frac{1}{2} U_1(x_1(\tau)) d\tau - \frac{1}{2}\sigma_1 \|\tilde{W}_1\|^2 \\
& + \beta_1^2(x_1) g_1^2(x_1) z_2^2 + \beta_1(x_1) g_1(x_1) N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + c_{\epsilon 1} \qquad (6.79)
\end{aligned}
$$

Remark 6.3.9 *In the cancellation based backstepping design, the coupling term $\beta_1 g_1 z_1 z_2$ is left as it is and it will be cancelled in the next step by augmenting the Lyapunv candidate. In decoupled backstepping design, we will not seeking the cancellation of the coupling term $\beta_1 g_1 z_1 z_2$, but seeking the boundedness of z_2 in the next step. According to Lemma 6.3.1, if we could prove that z_2 is bounded, then the stability of z_1 is apparent and easy. This fundamental change makes control system design for this problem solvable [87].*

Since $0 < \varepsilon_1 \le 4$, we have

$$
\begin{aligned}
\dot{V}_1(t) \ \le\ & -\frac{g_{10}}{\varepsilon_1} V_{z_1} - \frac{1}{\varepsilon_1} V_{U_1} - \frac{1}{2}\sigma_1 \|\tilde{W}_1\|^2 \\
& + \beta_1^2(x_1) g_1^2(x_1) z_2^2 + \beta_1(x_1) g_1(x_1) N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + c_{\epsilon 1} \\
\le\ & -c_1 V_1(t) + c_{\epsilon 1} + \beta_1(x_1) g_1(x_1) N(\zeta_1)\dot{\zeta}_1 + \dot{\zeta}_1 + \beta_1^2(x_1) g_1^2(x_1) z_2^2 \qquad (6.80)
\end{aligned}
$$

Multiplying (6.80) by $e^{c_1 t}$, it becomes

$$
\begin{aligned}
\frac{d}{dt}(V_1(t) e^{c_1 t}) \ \le\ & c_{\epsilon 1} e^{c_1 t} + \beta_1(x_1) g_1(x_1) N(\zeta_1)\dot{\zeta}_1 e^{c_1 t} \\
& + \dot{\zeta}_1 e^{C_{11} t} + \beta_1^2(x_1) g_1^2(x_1) z_2^2 e^{c_1 t} \qquad (6.81)
\end{aligned}
$$

Integrating (6.81) over $[0, t]$, we have

$$
\begin{aligned}
V_1(t) \ \le\ & \rho_1 + [V_1(0) - \rho_1] e^{-c_1 t} + e^{-c_1 t} \int_0^t \Big((\beta_1(x_1) g_1(x_1) N(\zeta_1) + 1\Big) e^{c_1 \tau}) \dot{\zeta}_1 d\tau \\
& + e^{-c_1 t} \int_0^t \beta_1^2(x_1) g_1^2(x_1) z_2^2 e^{c_1 \tau} d\tau \\
\le\ & \rho_1 + V_1(0) e^{-c_1 t} + e^{-c_1 t} \int_0^t \Big((\beta_1(x_1) g_1(x_1) N(\zeta_1) + 1\Big) e^{c_1 \tau}) \dot{\zeta}_1 d\tau \\
& + e^{-c_1 t} \int_0^t \beta_1^2(x_1) g_1^2(x_1) z_2^2 e^{c_1 \tau} d\tau \qquad (6.82)
\end{aligned}
$$

Remark 6.3.10 *In (6.82), if there is no extra term* $e^{-c_1 t}\int_0^t \beta_1^2 g_1^2 z_2^2 e^{c_1\tau}d\tau$ *within the inequality, we can conclude that* $V_1(t)$, ζ_1, $\hat{W}_1$, *are all bounded on* $[0, t_f)$ *according to Lemma 6.3.1. According to Proposition 2 in [70],* $t_f = +\infty$ *and we can claim that* z_1, $\hat{W}_1$ *are SGUUB. Remark 2.3 in [115] also explains the problem. Due to the presence of extra term* $e^{-c_1 t}\int_0^t \beta_1^2 g_1^2 z_2^2 e^{c_1\tau}d\tau$ *in (6.82), Lemma 6.3.1 cannot be applied directly. It was supposed in [83] that if* z_2 *can be regulated such that it is square integrable, the regulation of* z_1 *can be achieved. However, the situation is different in this section. Owing to the introduction of exponential term in Lemma 6.3.1, the requirement for square integrability can be further relaxed to boundedness.*

Noting Assumption 6.3.2, we have the following inequality [115]

$$\begin{aligned} & e^{-c_1 t}\int_0^t \beta_1^2 g_1^2 z_2^2 e^{c_1\tau}d\tau = e^{-c_1 t}\int_0^t \bar{g}_1^2 z_2^2 e^{c_1\tau}d\tau \\ \le\ & e^{-c_1 t} l_1^{+2} \sup_{\tau\in[0,t]}[z_2^2(\tau)]\int_0^t e^{c_1\tau}d\tau \le \frac{1}{c_1} l_1^{+2} \sup_{\tau\in[0,t]}[z_2^2(\tau)] \end{aligned} \tag{6.83}$$

Thus if z_2 can be regulated as bounded, then from (6.83) we can readily conclude the boundedness of the extra term $e^{-c_1 t}\int_0^t \beta_1^2 g_1^2 z_2^2 e^{c_1\tau}d\tau$.

The effect of $e^{-c_1 t}\int_0^t \beta_1^2 g_1^2 z_2^2 e^{c_1\tau}d\tau$ will be dealt with in the following steps.

Step i $(2 \le i \le n-1)$*:* Similar procedures are taken recursively for each step of $i = 2, \cdots, n-1$.

The time derivative of $z_i(t)$ is given by

$$\dot{z}_i(t) = g_i(\bar{x}_i(t))[z_{i+1}(t) + \alpha_i(t)] + f_i(\bar{x}_i(t)) + h_i(\bar{x}_i(t-\tau_i)) - \dot{\alpha}_{i-1}(t) \tag{6.84}$$

Consider $V_{z_i}(t)$ given in (6.67). Its time derivative is

$$\begin{aligned} \dot{V}_{z_i}(t) &= \frac{\partial V_{z_i}}{\partial z_i}\dot{z}_i + \frac{\partial V_{z_i}}{\partial \bar{x}_{i-1}}\dot{\bar{x}}_{i-1} + \frac{\partial V_{z_i}}{\partial \alpha_{i-1}}\dot{\alpha}_{i-1} \\ &= z_i\beta_i(\bar{x}_i)\dot{z}_i + \int_0^{z_i}\sigma\Big[\dot{\bar{x}}_{i-1}^T\frac{\partial \beta_i(\bar{x}_{i-1}, \sigma+\alpha_{i-1})}{\partial \bar{x}_{i-1}} \\ &\quad + \dot{\alpha}_{i-1}\frac{\partial \beta_i(\bar{x}_{i-1}, \sigma+\alpha_{i-1})}{\partial \alpha_{i-1}}\Big]d\sigma \end{aligned} \tag{6.85}$$

Noting (6.84) and

$$\int_0^{z_i}\sigma\dot{\bar{x}}_{i-1}^T\frac{\partial \beta_i(\bar{x}_{i-1}, \sigma+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\sigma = z_i^2\dot{\bar{x}}_{i-1}^T\int_0^1\theta\frac{\partial \beta_i(\bar{x}_{i-1}, \theta z_i+\alpha_{i-1})}{\partial \bar{x}_{i-1}}d\theta$$

$$\int_0^{z_i}\sigma\dot{\alpha}_{i-1}\frac{\partial \beta_i(\bar{x}_{i-1}, \sigma+\alpha_{i-1})}{\partial \alpha_{i-1}}d\sigma = \dot{\alpha}_{i-1}\Big[z_i\beta_i(\bar{x}_i) - z_i\int_0^1\beta_i(\bar{x}_{i-1}, \theta z_i+\alpha_{i-1})d\theta\Big]$$

equation (6.85) becomes

$$\begin{aligned}\dot{V}_{z_i}(t) &= z_i(t)\Big[\beta_i(\bar{x}_i(t))g_i(\bar{x}_i(t))(z_{i+1}(t)+\alpha_i(t))+\beta_i(\bar{x}_i(t))f_i(\bar{x}_i(t)) \\ &\quad +\beta_i(\bar{x}_i(t))h_i(\bar{x}_i(t-\tau_i))+z_i(t)\dot{\bar{x}}_{i-1}^T\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta \\ &\quad -\dot{\alpha}_{i-1}\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta\Big]\end{aligned}$$

where

$$\begin{aligned}\dot{\bar{x}}_{i-1} &= [\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_{i-1}]^T \\ &= [g_1(x_1)x_2+f_1(x_1)+h_1(x_1(t-\tau_1)), \\ &\quad g_2(\bar{x}_2)x_3+f_2(\bar{x}_2)+h_2(\bar{x}_2(t-\tau_2)),\cdots, \\ &\quad g_{i-1}(\bar{x}_{i-1})x_i+f_{i-1}(\bar{x}_{i-1})+h_{i-1}(\bar{x}_{i-1}(t-\tau_{i-1}))]^T\end{aligned}$$

Since α_{i-1} is a function of $\bar{x}_{i-1}$, ζ_{i-1}, $\bar{x}_{di}$, $\hat{W}_1$,..., $\hat{W}_{i-1}$, $\dot{\alpha}_{i-1}$ can be expressed as

$$\begin{aligned}\dot{\alpha}_{i-1} &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\dot{x}_j+\omega_{i-1}(t) \\ &= \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\Big[g_j(\bar{x}_j)x_{j+1}+f_j(\bar{x}_j)+h_j(\bar{x}_j(t-\tau_j))\Big]+\omega_{i-1}(t)\end{aligned}$$

where

$$\omega_{i-1}(t)=\frac{\partial\alpha_{i-1}}{\partial\zeta_{i-1}}\dot{\zeta}_{i-1}+\frac{\partial\alpha_{i-1}}{\partial\bar{x}_{di}}\dot{\bar{x}}_{di}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j$$

Note that the computation of $\dot{\alpha}_{i-1}$, which is required by the recursive backstepping design, and the appearance of $\dot{\bar{x}}_{i-1}$ make the unknown time delays of all the previous subsystems appear, which should all be compensated for in this step. In other words, Lyapunov-Krasovskii functionals (6.68) shall be utilized to compensate for not only the unknown time delay τ_i, but also $\tau_{i-1},...,\tau_1$. This difficulty or complexity was avoided by assuming that $\dot{x}_{i-1}$ is available for feedback control in [124].

Applying Assumption 6.3.4 and using Young's Inequality, we have

$$\begin{aligned}\dot{V}_{z_i}(t) &= z_i(t)\Big[\beta_i(\bar{x}_i(t))g_i(\bar{x}_i(t))(z_{i+1}(t)+\alpha_i(t))+\beta_i(\bar{x}_i(t))f_i(\bar{x}_i(t)) \\ &\quad +z_i(t)\bar{f}_{i-1}^T\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\Big]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}z_i^4(t)\Big[\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\Big]^T\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\\
&+\frac{1}{2}z_i^2(t)\beta_i^2(\bar{x}_i(t))+\frac{1}{2}\varrho_i^2(\bar{x}_i(t-\tau_i))\\
&-z_i\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta\Big[\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(g_j(\bar{x}_j)x_{j+1}+f_j(\bar{x}_j))+\omega_{i-1}(t)\Big]\\
&+\frac{1}{2}\Big[\sum_{j=1}^{i-1}z_i^2(\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta)^2\Big(\frac{\partial\alpha_{i-1}}{\partial x_j}\Big)^2+2\varrho_j^2(\bar{x}_j(t-\tau_j))\Big] \quad (6.86)
\end{aligned}
$$

where $\bar{f}_{i-1}=[g_1(x_1)x_2+f_1(x_1),\cdots,g_{i-1}(\bar{x}_{i-1})x_i+f_{i-1}(\bar{x}_{i-1})]^T$.

Considering the Lyapunov-Krasovskii functional $V_{U_i}(t)$ as given in (6.68), we have

$$
\dot{V}_{z_i}+\dot{V}_{U_i}\leq z_i\beta_i(\bar{x}_i)g_i(\bar{x}_i)z_{i+1}+z_i\beta_i(\bar{x}_i)g_i(\bar{x}_i)\alpha_i+z_iQ_i(Z_i) \quad (6.87)
$$

where

$$
\begin{aligned}
Q_i(Z_i) &= \beta_i(\bar{x}_i)f_i(\bar{x}_i)+\frac{1}{2}z_i\beta_i^2(\bar{x}_i)+\frac{1}{2z_i}\varrho_i^2(\bar{x}_i)\\
&\quad+z_i\bar{f}_{i-1}^T\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\\
&\quad+\frac{1}{2}z_i^3(t)\Big[\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\Big]^T\int_0^1\theta\frac{\partial\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})}{\partial\bar{x}_{i-1}}d\theta\\
&\quad+\sum_{j=1}^{i-1}\Big[-\frac{\partial\alpha_{i-1}}{\partial x_j}(g_j(\bar{x}_j)x_{j+1}+f_j(\bar{x}_j))\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta\\
&\quad+\frac{1}{2}z_i\Big(\frac{\partial\alpha_{i-1}}{\partial x_j}\Big)^2(\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta)^2+\frac{1}{z_i}\varrho_j^2(\bar{x}_j)\Big]\\
&\quad-\omega_{i-1}\int_0^1\beta_i(\bar{x}_{i-1},\theta z_i+\alpha_{i-1})d\theta\\
Z_i &= [\bar{x}_i,\alpha_{i-1},\frac{\partial\alpha_{i-1}}{\partial x_1},\frac{\partial\alpha_{i-1}}{\partial x_2},\cdots,\frac{\partial\alpha_{i-1}}{\partial x_{i-1}},\omega_{i-1}]\in\Omega_{z_i}\subset R^{2i+1} \quad (6.88)
\end{aligned}
$$

For the adaptive neural intermediate control law given in (6.70)-(6.72) with $k_{i1}(t)$ being given in (6.73), consider Lyapunov function candidate $V_i(t)$ given in (6.69). Its time derivative along (6.70)-(6.72) and (6.87) is

$$
\dot{V}_i(t)\leq -c_iV_i(t)+c_{\epsilon i}+\beta_i(\bar{x}_i)g_i(\bar{x}_i)N(\zeta_i)\dot{\zeta}_i+\dot{\zeta}_i+\beta_i^2(\bar{x}_i)g_i^2(\bar{x}_i)z_{i+1}^2 \quad (6.89)
$$

Multiplying (6.89) by e^{c_it}, it becomes

$$
\frac{d}{dt}(V_i(t)e^{c_it})\leq c_{\epsilon i}e^{c_it}+\beta_i(\bar{x}_i)g_i(\bar{x}_i)N(\zeta_i)\dot{\zeta}_ie^{c_it}+\dot{\zeta}_ie^{c_it}+\beta_i^2(\bar{x}_i)g_i^2(\bar{x}_i)z_{i+1}^2e^{c_it} \quad (6.90)
$$

Integrating (6.90) over $[0, t]$, we have

$$\begin{aligned} V_i(t) &\leq \rho_i + [V_i(0) - \rho_i]e^{-c_i t} + e^{-c_i t}\int_0^t (\beta_i(\bar{x}_i)g_i(\bar{x}_i)N(\zeta_i) + 1)e^{c_i\tau}\dot{\zeta}_i d\tau \\ &\quad + e^{-c_i t}\int_0^t \beta_i^2(\bar{x}_i)g_i^2(\bar{x}_i)z_{i+1}^2 e^{c_i\tau}d\tau \\ &\leq \rho_i + V_i(0) + e^{-c_i t}\int_0^t (\beta_i(\bar{x}_i)g_i(\bar{x}_i)N(\zeta_i) + 1)e^{c_i\tau}\dot{\zeta}_i d\tau \\ &\quad + e^{-c_i t}\int_0^t \beta_i^2(\bar{x}_i)g_i^2(\bar{x}_i)z_{i+1}^2 e^{c_i\tau}d\tau \end{aligned} \tag{6.91}$$

Remark 6.3.11 *Similarly as discussed in Remark 6.3.10, if z_{i+1} can be regulated as bounded, we can readily guarantee the boundedness of the extra term $e^{-c_i t}\int_0^t \beta_i^2 g_i^2 z_{i+1}^2 e^{c_i\tau}d\tau$ in (6.91). Then applying Lemma 6.3.1, the boundedness of $V_i(t)$, $z_i(t)$, $\zeta_i(t)$ and $\hat{W}_i(t)$ can be readily obtained.*

The effect of $e^{-c_i t}\int_0^t \beta_i^2 g_i^2 z_{i+1}^2 e^{c_i\tau}d\tau$ will be dealt with in the next step.

Step n. This is the final step, since the actual control $u(t)$ appears in the derivative of $z_n(t)$ as given in

$$\dot{z}_n = g_n(\bar{x}_n(t))u + f_n(\bar{x}_n(t)) + h_n(\bar{x}_n(t - \tau_n)) - \dot{\alpha}_{n-1}(t) \tag{6.92}$$

Consider the scalar function $V_{z_n}(t)$ given in (6.67). Its time derivative is

$$\begin{aligned} \dot{V}_{z_n}(t) &= z_n(t)\Big[\beta_n(x(t))g_n(x(t))u(t) + \beta_n(x(t))f_n(x(t)) \\ &\quad + \beta_n(x(t))h_n(x(t-\tau_n)) + z_n(t)\dot{\bar{x}}_{n-1}^T\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta \\ &\quad - \dot{\alpha}_{n-1}\int_0^1 \beta_n(\bar{x}_{n-1}, \theta z_n + \alpha_{n-1})d\theta\Big] \end{aligned}$$

Since α_{n-1} is a function of $\bar{x}_{n-1}$, ζ_{n-1}, $\bar{x}_{dn}$, $\hat{W}_1$,..., $\hat{W}_{n-1}$, $\dot{\alpha}_{n-1}$ can be expressed as

$$\begin{aligned} \dot{\alpha}_{n-1} &= \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\dot{x}_j + \omega_{n-1}(t) \\ &= \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}\Big[g_j(\bar{x}_j)x_{j+1} + f_j(\bar{x}_j) + h_j(\bar{x}_j(t-\tau_j))\Big] + \omega_{n-1}(t) \end{aligned}$$

where

$$\omega_{n-1}(t) = \frac{\partial\alpha_{n-1}}{\partial\zeta_{n-1}}\dot{\zeta}_{n-1} + \frac{\partial\alpha_{n-1}}{\partial\bar{x}_{dn}}\dot{\bar{x}}_{dn} + \sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{W}_j}\dot{\hat{W}}_j$$

Applying Assumption 6.3.4 and using Young's Inequality, we have

$$\dot{V}_{z_n}(t) = z_n(t)\Big[\beta_n(x(t))g_n(x(t))u(t) + \beta_n(x(t))f_n(x(t))$$
$$+z_n(t)\bar{f}_{n-1}^T\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta\Big]$$
$$+\frac{1}{2}z_n^4(t)\Big[\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta\Big]^T\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta$$
$$+\frac{1}{2}z_n^2(t)\beta_n^2(x(t)) + \frac{1}{2}\varrho_n^2(x(t-\tau_n))$$
$$-z_n\int_0^1 \beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta\Big[\sum_{j=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_j}(g_j(\bar{x}_j)x_{j+1}+f_j(\bar{x}_j))+\omega_{n-1}(t)\Big]$$
$$+\frac{1}{2}\Big[\sum_{j=1}^{n-1}z_n^2\Big(\int_0^1 \beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta\Big)^2\Big(\frac{\partial\alpha_{n-1}}{\partial x_j}\Big)^2+2\varrho_j^2(\bar{x}_j(t-\tau_j))\Big] \quad (6.93)$$

where $\bar{f}_{n-1} = [g_1(x_1)x_2 + f_1(x_1), \cdots, g_{n-1}(\bar{x}_{n-1})x_n + f_{n-1}(\bar{x}_{n-1})]^T$.

Considering the Lyapunov-Krasovskii functional $V_{U_n}(t)$ given in (6.68), we have

$$\dot{V}_{z_n} + \dot{V}_{U_n} \le z_n\beta_n(x)g_n(x)u + z_nQ_n(Z_n) \quad (6.94)$$

where

$$Q_n(Z_n) = \beta_n(x)f_n(x) + \frac{1}{2}z_n\beta_n^2(x) + \frac{1}{2z_n}\varrho_n^2(x)$$
$$+z_n\bar{f}_{n-1}^T\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta$$
$$+\frac{1}{2}z_n^3(t)\Big[\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta\Big]^T\int_0^1 \theta\frac{\partial\beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})}{\partial\bar{x}_{n-1}}d\theta$$
$$+\sum_{j=1}^{n-1}\Big\{-\frac{\partial\alpha_{n-1}}{\partial x_j}(g_j(\bar{x}_j)x_{j+1}+f_j(\bar{x}_j))\int_0^1 \beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta$$
$$+\frac{1}{2}z_n\Big(\frac{\partial\alpha_{n-1}}{\partial x_j}\Big)^2\Big[\int_0^1 \beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta\Big]^2+\frac{1}{z_n}\varrho_j^2(\bar{x}_j)\Big\}$$
$$-\omega_{n-1}\int_0^1 \beta_n(\bar{x}_{n-1},\theta z_n+\alpha_{n-1})d\theta$$
$$Z_n = [x, \alpha_{n-1}, \frac{\partial\alpha_{n-1}}{\partial x_1}, \frac{\partial\alpha_{n-1}}{\partial x_2}, \cdots, \frac{\partial\alpha_{n-1}}{\partial x_{n-1}}, \omega_{n-1}] \in \Omega_{Z_n} \subset R^{2n+1} \quad (6.95)$$

For the adaptive neural control law given in (6.70)-(6.72) with $k_{n1}(t)$ being given in (6.73), consider the Lyapunov function candidate $V_n(t)$. Its time derivative along (6.70)-(6.72) and (6.94) is

$$\dot{V}_n(t) \le -c_nV_n(t) + c_{\epsilon n} + \beta_n(x)g_i(x)N(\zeta_n)\dot{\zeta}_n + \dot{\zeta}_n \quad (6.96)$$

Multiplying (6.96) by $e^{c_n t}$, it becomes

$$\frac{d}{dt}(V_n(t)e^{c_n t}) \leq c_{\epsilon n}e^{c_n t} + \beta_n(x)g_n(x)N(\zeta_n)\dot{\zeta}_n e^{c_n t} + \dot{\zeta}_n e^{c_n t} \tag{6.97}$$

Integrating (6.97) over $[0, t]$, we have

$$\begin{aligned} V_n(t) &\leq \rho_n + [V_n(0) - \rho_n]e^{-c_n t} + e^{-c_n t}\int_0^t (\beta_n(x)g_n(x)N(\zeta_n) + 1)e^{c_n \tau}\dot{\zeta}_n d\tau \\ &\leq \rho_n + V_n(0) + e^{-c_n t}\int_0^t (\beta_n(x)g_n(x)N(\zeta_n) + 1)e^{c_n \tau}\dot{\zeta}_n d\tau \end{aligned} \tag{6.98}$$

Using Lemma 6.3.1, we can conclude that $V_n(t)$ and $\zeta_n(t)$, hence $z_n(t)$, $\hat{W}_n$ are SGUUB on $[0, t_f)$. From the boundedness of $z_n(t)$, the boundedness of the extra term $e^{-c_{n-1}t}\int_0^t \beta_{n-1}^2 g_{n-1}^2 z_n^2 e^{c_{n-1}\tau} d\tau$ at Step $(n-1)$ is readily obtained. Applying Lemma 6.3.1 for $(n-1)$ times backwards, it can be seen from the above iterative design procedures that $V_i(t)$, $z_i(t)$, $\hat{W}_i(t)$ and hence $x_i(t)$ are SGUUB, $i = 1, ..., n-1$.

The following theorem shows the stability and control performance of the closed-loop adaptive system.

Theorem 6.3.2 *Consider the closed-loop system consisting of the plant (6.27) under Assumptions 6.3.1-6.3.4, the adaptive neural control laws (6.70)-(6.73). We can guarantee the following properties under bounded initial conditions (i) all signals in the closed-loop system remain semi-globally uniformly ultimately bounded;*

(ii) the vectors Z_i remain in the compact set $\Omega_{Z_i}^0 \subset R^{2i+1}$, $i = 1, ..., n$, specified as

$$\Omega_{Z_i}^0 := \left\{ Z_i \Big| |z_i| \leq \mu_i, \|\tilde{W}_i\|^2 \leq \frac{\mu_i^2}{\lambda_{\min}(\Gamma_i^{-1})}, \bar{x}_{di} \in \Omega_{di} \right\}$$

whose sizes, $\mu_i > 0$, can be adjusted by appropriately choosing the design parameters.

Proof: Consider the Lyapunov function candidate $V_n(t)$ given in (6.69) with $V_{z_n}(t)$, $V_{U_n}(t)$ being defined in (6.67) and (6.68). From the previous derivation, we have

$$V_n(t) \leq \rho_n + V_n(0) + e^{-c_n t}\int_0^t (\beta_n(x)g_n(x)N(\zeta_n) + 1)e^{c_n \tau}\dot{\zeta}_n d\tau$$

From the above iterative design procedures from Step 1 to Step n, we can conclude $V_i(t)$, $\zeta_i(t)$, $z_i(t)$, $\hat{W}_i(t)$, $i = 1, ..., n$, and hence $x(t)$ are SGUUB.

Letting $c_{\beta n}$ be the upper bound of $e^{-c_n t}\int_0^t |\beta_n g_n N(\zeta_n) + 1| e^{c_n\tau}\dot{\zeta}_n d\tau$ and noting the definition of $V_n(t)$, we have

$$\begin{aligned} \frac{1}{2}z_n^2 \leq V_n(t) &\leq (\rho_n + c_{\beta n}) + V_n(0) \\ \|\tilde{W}_n\|^2 &\leq \frac{2V_n(t)}{\lambda_{\min}(\Gamma_n^{-1})} \end{aligned}$$

In the rest of the steps from $n-1$ to 1, we obtain

$$\begin{aligned} V_i(t) \leq\ & \rho_i + V_i(0) + e^{-c_i t}\int_0^t (\beta_i(\bar{x}_i)g_i(\bar{x}_i)N(\zeta_i) + 1)e^{c_i\tau}\dot{\zeta}_i d\tau \\ & + e^{-c_i t}\int_0^t \beta_i^2(\bar{x}_i)g_i^2(\bar{x}_i)z_{i+1}^2 e^{c_i\tau}d\tau, i = 1, ..., n-1 \end{aligned}$$

Letting $c_{\beta i}$ be the upper bound of $e^{-c_i t}\int_0^t |\beta_i g_i N(\zeta_i)\dot{\zeta}_i + \dot{\zeta}_i + \beta_i^2 g_i^2 z_i^2| e^{c_i\tau}d\tau$ and noting the definition of $V_i(t)$, we have

$$\begin{aligned} \frac{1}{2}z_i^2 \leq V_i(t) &\leq (\rho_i + c_{\beta i}) + V_i(0) \\ \|\tilde{W}_i\|^2 &\leq \frac{2V_i(t)}{\lambda_{\min}(\Gamma_i^{-1})} \end{aligned}$$

where constant

$$V_i(0) = \int_0^{z_i(0)} \sigma\beta_i(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0))d\sigma + \frac{1}{2}\tilde{W}_i^T(0)\Gamma_i^{-1}\tilde{W}_i(0)$$

with $\beta_i(\bar{x}_{i-1}(0), \sigma + \alpha_{i-1}(0)) = \beta_1(\sigma + y_d(0))$ for $i = 1$.

By letting $\mu_i = \sqrt{2(\rho_i + c_{\beta i} + V_i(0))}$, we know that $|z_i| \leq \mu_i$. We can conclude that there do exist compact sets $\Omega_{Z_i}^0$ such that $Z_i \in \Omega_{Z_i}^0$, $\forall t \geq 0$. $\diamondsuit$

Remark 6.3.12 *For the choice of $k_{i1}(t)$ in (6.73), it is found that if c_{z_i} is chosen to be very small, $k_{i1}(t)$ will take a very large value, which may saturate the control actuator. To solve this problem, we would like to find an alternative for $k_{i1}(t)$ such that it provides smooth control input, and at the same time guarantees the stability result. One such choice is*

$$\begin{aligned} k_{i1}(t) =\ & \frac{1}{\varepsilon_i}\Big[1 + \int_0^1 \theta\bar{g}_i(\bar{x}_{i-1}, \theta z_i + \alpha_{i-1})d\theta \\ & + \frac{\cosh(z_i)}{1 + z_i^2}\int_{t-\tau_{\max}}^t \Big(\frac{1}{2}U_i(\bar{x}_i(\tau)) + \sum_{j=1}^{i-1} U_j(\bar{x}_j(\tau))\Big)d\tau\Big] \end{aligned}$$

Following the same derivation procedure and using the property of function $\kappa(.)$ in (6.30), we can readily obtain 6.89 with c_i being modified/changed to

$$c_i := \min\Big\{\frac{g_{i0}}{\varepsilon_i},\ \frac{\kappa(c_{z_i})}{\varepsilon_i},\ \frac{\sigma_i}{\lambda_{\min}(\Gamma_i^{-1})}\Big\}$$

Although the bounded region may be enlarged by introducing the function $\kappa(\cdot)$, there are still design flexibility from ε_i, Γ_i and σ_i, which can help reduce the bounded region. Note that such modifications together with the choice of function $\kappa(\cdot)$ are also not unique and worth further investigation.

Remark 6.3.13 *Note that the choices of $\beta_i(\bar{x}_i)$ are not unique [88]. As an alternative, we can choose $\beta_i(\bar{x}_i) = 1/|g_i(\bar{x}_i)|$. In this case, the upper bound function of $|g_i(\bar{x}_i)|$, i.e., $\bar{g}_i(\bar{x}_i)$ are not necessarily known. The smooth integral scalar function becomes*

$$V_{z_i} = \int_0^{z_i} \frac{\sigma}{|g_i(\bar{x}_{i-1}, \sigma + \alpha_{i-1})|} d\sigma, \quad i = 1, \cdots, n$$

By Mean Value Theorem, V_{z_i} can be rewritten as

$$V_{z_i} = \frac{\lambda_s z_i^2}{|g_i(\bar{x}_{i-1}, \lambda_s z_i + \alpha_{i-1})|}, \quad \lambda_s \in (0,1)$$

From Assumption 6.3.1, $0 \leq g_{i0} \leq |g_i(\bar{x}_i)|$, we know that $V_{z_i}(t)$ is a positive definite function and $V_{z_i}(t) \leq \frac{\lambda_s}{g_{10}} z_i^2$. For conciseness of presentation, we give the control and adaptive laws directly without proof, as well as the stability results.

Theorem 6.3.3 *For system (6.27), we choose the adaptive neural control laws (6.70)-(6.72), where $k_i(t) = k_{i0} + k_{i1}(t)$ with constant $k_{i0} > 0$ and $k_{i1}(t)$ is chosen as*

$$k_{i1}(t) = \frac{1}{\varepsilon_i}\Big[1 + \lambda_s + \frac{1}{z_i^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}\Big(\frac{1}{2}U_i(\bar{x}_i(\tau)) + \sum_{j=1}^{i-1} U_j(\bar{x}_j(\tau))\Big)d\tau\Big] \tag{6.99}$$

with $0 < \varepsilon_{i0} \leq 4$, $\lambda_s \in (0,1)$. Then, under the bounded initial conditions, all signals in the closed-loop system remain bounded and the tracking error converges to a small neighborhood around zero by appropriately choosing design parameters.

Similar as the modification made to k_{i1} in Remark 6.3.12, we can modify (6.99) to

$$k_{i1}(t) = \frac{1}{\varepsilon_i}\Big[1 + \lambda_s + \frac{\cosh(z_i)}{1 + z_i^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}\Big(\frac{1}{2}U_i(\bar{x}_i(\tau)) + \sum_{j=1}^{i-1} U_j(\bar{x}_j(\tau))\Big)d\tau\Big] \tag{6.100}$$

for a relatively gentle control gain.

Though the non-differentiability of the intermediate controls can be solved in a very practical way as discussed in the previous subsection. In fact, this problem can also be solved theoretically by modifying the control laws such that they are differentiable to certain desired order as will be discussed below. It should be pointed out that the solution is not unique. For clarity, only one such a solution is presented.

It can be seen that the computation of $\alpha_i(t)$ requires that of $\dot{\alpha}_{i-1}(t)$. This is also the case for the computation of α_{i-1}, $\cdots$, and α_2, which requires to compute $\dot{\alpha}_{i-2}$, $\cdots$, and $\dot{\alpha}_1$ respectively. Therefore, we know that the computation of α_i shall include that of $\alpha_1^{(i-1)}, \alpha_2^{(i-2)}$, $\cdots$, and $\dot{\alpha}_{i-1}$. This rule applies to the rest of the steps till the last step n. We can conclude that α_i need to be at least $(n-i)$th differentiable. By using the property of (n-i)th order differentiable function $q_i(z_i)$ (6.29), the intermediate control, α_i (6.70) can be easily modified to satisfy the required $(n-i)$th order differentiability as follows

$$\alpha_i^q = q_i(z_i)N(\zeta_i)[k_i(t)z_i + \hat{W}_i^T S_i(Z_i)], \quad i = 1, \cdots, n-1 \tag{6.101}$$

where $q_i(z_i)$ is defined in (6.29). It can be easily verified by actual differentiation.

The above modification not only guarantees the differentiability of the intermediate controls, but also preserves the closed-loop stability of the practical control design by noticing that $\alpha_i^q = \alpha_i \; \forall z_i \in \Omega_{Z_i}^0$. The analysis is similar as in Section 5.4.

6.3.4 Simulation

To illustrate the proposed adaptive neural control algorithms, we consider the following second-order time-delay system

$$\begin{aligned}
\dot{x}_1(t) &= g_1(x_1)x_2(t) + f_1(x_1) + h_1(x_1(t-\tau_1)) \\
\dot{x}_2(t) &= g_2(x)u(t) + f_2(x) + h_2(x(t-\tau_2)) \\
y_1(t) &= x_1(t)
\end{aligned}$$

where $g_1(x_1) = 1 + x_1^2$, $g_2(x) = 3 + \cos(x_1 x_2)$, $f_1(x_1) = x_1(t)e^{-0.5x_1(t)}$, $f_2(x) = x_1(t)x_2^2(t)$, $h_1(x_1) = 2x_1^2$, and $h_2(x) = 0.2x_2\sin(x_2)$. Apparently, by choosing

$\varrho_1(x_1) = 2x_1^2$ and $\varrho_2(x) = 0.2|x_2|$, Assumption 6.3.4 satisfies. Choose the initial condition $[x_1(0), x_2(0)]^T = [0, 0]^T$, the time delay $\tau_1 = \tau_2 = 2$sec., and the desired reference signal $y_d = 0.5[\sin(t) + \sin(0.5t)]$. For the design of neural adaptive controller, let $z_1 = x_1 - y_d$, $z_2 = x_2 - \alpha_1$. For simplicity, simulation is carried out based on Theorem 6.3.3 for the case $\beta_i(\bar{x}_i) = 1/|g_i(\bar{x}_i)|$. The intermediate control α_i and control $u(t)$ are given by (6.101) and (6.70) respectively with $k_{i1}(t)$ being chosen in (6.100) as follows

$$\begin{aligned}
&\alpha_1(t) = q_1(z_1)N(\zeta_1)[k_1(t)z_1 + \hat{W}_1^T S_1(Z_1)],\\
&u(t) = \begin{cases} N(\zeta_2)[k_2(t)z_2 + \hat{W}_2^T S_2(Z_2)], & |z_2| \geq c_{z_2} \\ 0, & \text{otherwise} \end{cases}\\
&\dot{\zeta}_i = k_i(t)z_i^2 + \hat{W}_i^T S_i(Z_i)z_i, \qquad i = 1, 2\\
&\dot{\hat{W}}_i = \Gamma_i[S_i(Z_i)z_i - \sigma_i(\hat{W}_i - W_i^0)], \qquad i = 1, 2
\end{aligned}$$

where $N(\zeta_i) = e^{\zeta_i^2}\cos(\frac{\pi}{2}\zeta_i)$, $i = 1, 2$ are the Nussbaum functions, $Z_1 = [x_1, y_d, \dot{y}_d]^T$, $Z_2 = [x_1, x_2, \alpha_1, \frac{\partial \alpha_1}{\partial x_1}, \omega_1]^T$, and $k_i(t) = k_{i0} + k_{i1}(t)$ with constant $k_{i0} > 0$ and $k_{i1}(t)$ being chosen as

$$k_{i1}(t) = \frac{1}{\varepsilon_i}\Big[1 + \lambda_s + \frac{\cosh(z_i)}{1 + z_i^2}\int_{t-\tau_{\max}}^{t} \frac{1}{2}\Big(\frac{1}{2}U_i(\bar{x}_i(\tau)) + \sum_{j=1}^{i-1} U_j(\bar{x}_j(\tau))\Big)d\tau\Big]$$

The following design parameters are adopted in the simulation: $\Gamma_1 = \text{diag}[0.2]$, $\Gamma_2 = \text{diag}[0.4]$, $\sigma_1 = \sigma_2 = 0.5$, $W_1^0 = W_2^0 = 0.01$, $\varepsilon_1 = 4$, $\varepsilon_2 = 4$, $\lambda_s = 0.5$, and $c_{z_1} = c_{z_2} = 1.0e^{-7}$.

In practice, the selection of the centers and widths of RBF has a great influence on the performance of the designed controller. According to [45], Gaussian RBF NNs arranged on a regular lattice on R^n can uniformly approximate sufficiently smooth functions on closed, bounded subsets. Accordingly, in the following simulation studies, the centers and widths are chosen on a regular lattice in the respective compact sets. Specifically, neural networks $\hat{W}_1^T S_1(Z_1)$ contains 27 nodes (i.e., $l_1 = 27$) with centers $\mu_l (l = 1, \cdots, l_1)$ evenly spaced in $[-1, +1] \times [-1, +1] \times [-1, +1]$, and widths $\eta_l^2 = 1 (l = 1, \cdots, l_1)$. Neural networks $\hat{W}_2^T S_1(Z_2)$ contains 243 nodes (i.e., $l_2 = 243$) with centers $\mu_l (l = 1, \cdots, l_2)$ evenly spaced in $[-1, +1] \times [-1.5, +1] \times [-1.5, +1] \times [-5, +5] \times [-5, +5]$, and widths $\eta_2^2 = 8 (l = 1, \cdots, l_2)$. The initial weight estimates are assumed to me 0, i.e., $\hat{W}_1(0) = 0.0$ and $\hat{W}_2(0) = 0.0$.

Fig. 6.5 shows that good tracking performance is achieved after 10 seconds learning periods. Fig. 6.6 shows that the state x_2 in the closed-loop is also bounded. Figs. 6.7 and 6.8 show the boundedness of the control input and the NN weights in the control loop.

6.3.5 Conclusion

An adaptive neural-based control has been addressed for a class of parametric-strict-feedback nonlinear systems with unknown time delays. The proposed design method does not require *a priori* knowledge of the signs of the unknown virtual control coefficients. The unknown time delays have been compensated for by using appropriate Lyapunov-Krasovskii functionals. The proposed systematic backstepping design method has been proved to be able to guarantee semi-global uniformly ultimately boundedness of all the signals. In addition, the output of the system has been proven to converge to a small neighborhood of the origin. Simulation has been conducted to show the effectiveness of the proposed approach.

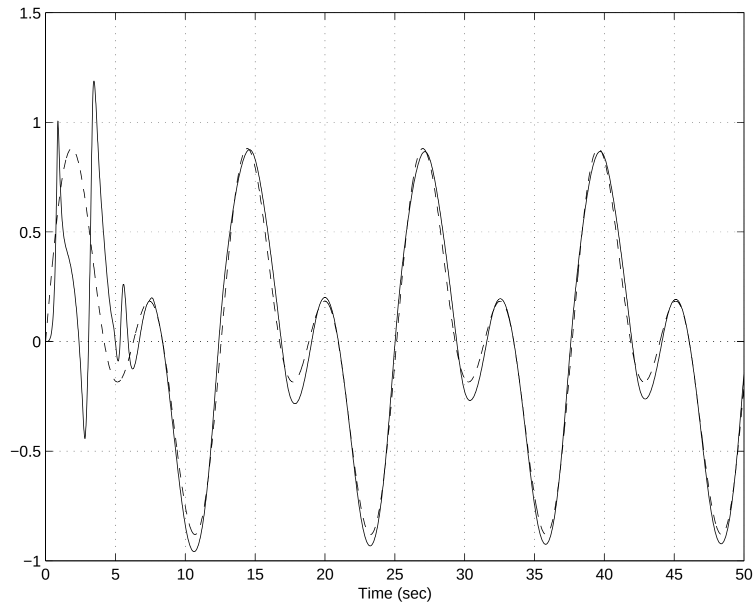

Figure 6.5: Output $y(t)$(“—”) and reference y_d(“- -”).

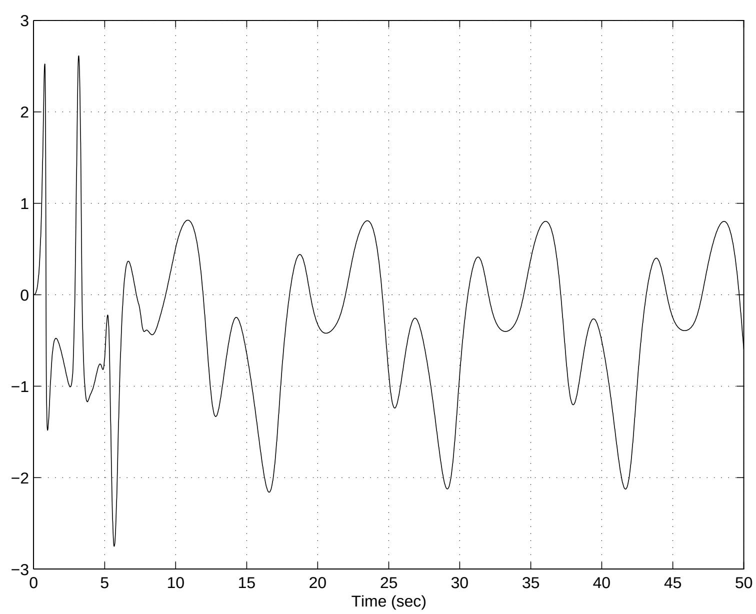

Figure 6.6: Trajectory of state $x_2(t)$.

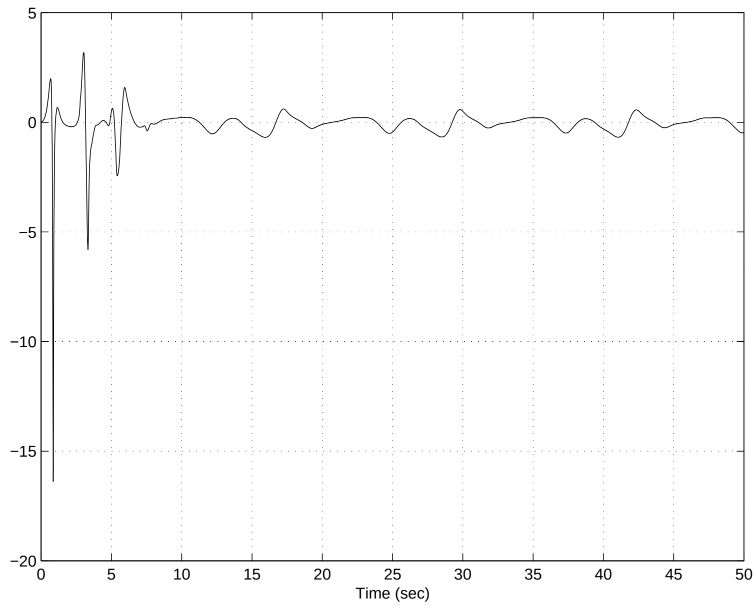

Figure 6.7: Control input $u(t)$.

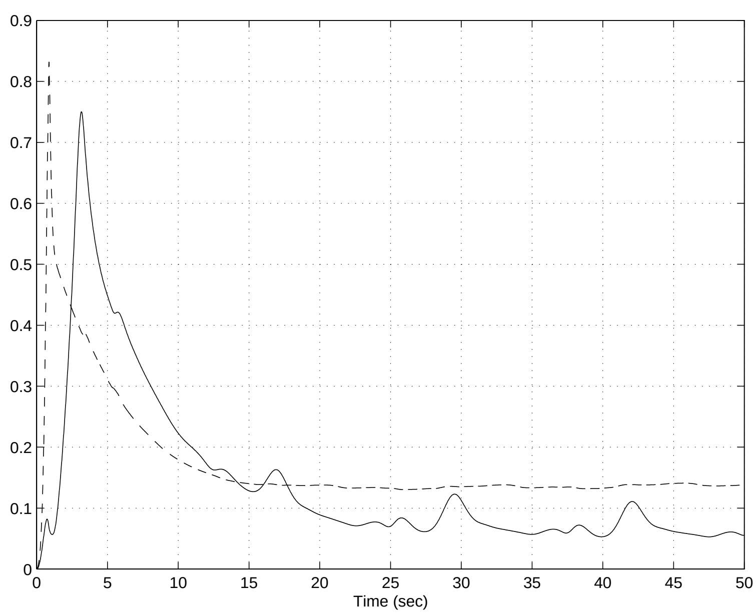

Figure 6.8: Norms of NN weights $\|\hat{W}_1\|$(“–”) and $\|\hat{W}_2\|$(“- -”).

Chapter 7

Conclusions and Future Research

7.1 Conclusions

In this thesis, robust adaptive control has been investigated for uncertain nonlinear systems. The main purpose of the thesis is to develop adaptive control strategies for several classes of general nonlinear systems in strict-feedback form with uncertainties including unknown parameters, unknown nonlinear systems functions, unknown disturbances, and unknown time delays. Systematic controller designs have been presented using backstepping methodology, neural network parametrization and robust adaptive control. The results in the thesis have been derived based on rigorous Lyapunov stability analysis. The control performance of the closed-loop systems has been explicitly analyzed.

The traditional backstepping design is cancellation-based as the coupling term remaining in each design step will be cancelled in the next step. In this thesis, the coupling term in each step has been decoupled by elegantly using the Young's inequality rather than leaving to it to be cancelled in the next step, which was referred to as the decoupled backstepping method. In this method, the virtual control in each step has been only designed to stabilize the corresponding subsystems rather than previous subsystems and the stability result of each step obtained by seeking the boundedness of the state rather than cancelling the coupling term so that the residual set of each state can be determined individually. Two classes of nonlinear systems in strict-feedback form have been considered as illustration examples to show the design method. It has been also applied throughout the thesis for practical controller design.

For nonlinear system with unknown time delays, the main difficulty lies in the terms with unknown time delays. In this thesis, by using appropriate Lyapunov-Krasovskii functionals in the Lyapunov function candidate, the uncertainties from unknown time delays have been compensated for such that the design of the stabilizing control law was free from unknown time delays. In this way, the iterative backstepping design procedure can be carried out directly. Controller singularities have been effectively avoided by employing practical robust control. It has been first applied to a kind of nonlinear strict-feedback systems with unknown time-delay using neural networks approximation. Two different NN control schemes have been developed and semi-globally uniformly ultimately boundedness of the closed-loop signals is achieved. It has been then extended to a kind of nonlinear time-delay systems in parametric-strict-feedback form and globally uniformly ultimately boundedness of the closed-loop signals has obtained. In the latter design, a novel continuous function has been introduced to construct differentiable control functions.

When there is no a priori knowledge on the signs of virtual control coefficients or high-frequency gain, adaptive control of such systems becomes much more difficult. In this thesis, controller design incorporated by Nussbaum-type gains has been presented for a class of perturbed strict-feedback nonlinear systems and a class of nonlinear time-delay systems with unknown virtual control coefficients/functions. To cope with uncertainties and achieve global boundedness, an exponential term has been incorporated into the stability analysis and novel technical lemmas have been introduced. The proof of the key technical lemmas was given for different Nussbaum functions being chosen.

In summary, Chapter 2 has given the basic definition and useful results related to stability, while the decoupled backstepping design introduced in Chapter 3 is the fundamental design tool being utilized throughout the thesis. The following three chapters have dealt with several kinds of nonlinear systems with unknown time delays. The virtual control coefficients of the systems under consideratin were unknown functions of states in Chapter 4 and unknown constants in Chapter 5, where their signs have been assumed known, while in Chapter 6, the virtual control coefficients were unknown functions of states with unknown sign. Due to the different problem formulation, the design methodology being utilized in these chapters were different. Chapter 4 and Chapter 6 have used NNs as a function approximator to deal with the unknown nonlinearity while adaptive scheme was proposed in Chapter 5 for unknown parametric uncertainties. As Chapter 6 considered the case when the signs of the virtual control coefficients were unknown, adaptive and adaptive neural control schemes using Nussbaum functions were proposed.

7.2 Further Research

In the following, some suggestions are made for further studies.

- *Sliding Mode Control of Nonlinear Time-Delay Systems:* Time-delay systems are actually infinite-dimensional systems. The extension of sliding mode control strategy to infinite-dimensional systems [140] makes the application of sliding mode control to time-delay systems possible. It has shown that for systems with state delays, the idea are essentially the same as for finite-dimensional systems, even if design and computations are much more complicated. Due to the rich dynamic behaviors of nonlinear systems, the sliding mode control of nonlinear time-delay systems is a promising and challenging future research are.

- *Systems with Input Delay:* The presence of an input delay in the systems is still an open problem [99]. Even matching additive disturbance is difficult to be rejected. It is even more challenging when considering nonlinear case or control input nonaffine case.

- *Universal Adaptive Controller:* The behavior of the universal adaptive controller using Nussbaum-gain can be interpreted as the controller tries to sweep all possible control gains and stops when a stabilizing gain is found, i.e., the switching of the control gain will finally stop when the system has "found" the unknown control directions. To determine the settling time is well worth being investigated for better control performance and computation efficiency.

- *Overparametrization Problem in Decoupled Backstepping Design:* The proposed decoupled backstepping design procedure has the disadvantage of overparameterization, which may reduce its practicality. Obviously, overparmeterization increases controller's dynamic order In addition, it may deteriorate the parameter convergence and system robustness. Future research could be done to remove this drawback.

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Appendix A

Proof of Lemma 2.4.6

Proof: Since $g_0(x(t)) \in [l^-, l^+]$, let us define $g_{\max} = \max\{|l^-|, |l^+|\}$ and $g_{\min} = \min\{|l^-|, |l^+|\}$ for convenience. We first show that $\zeta(t)$ is bounded on $[0, t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: (i) $\zeta(t)$ has no upper bound and (ii) $\zeta(t)$ has no lower bound.

Case (i): $\zeta(t)$ has no upper bound on $[0, t_f)$. In this case, there must exist a monotone increasing sequence $\{t_i\}$, $i = 1, 2, \cdots$, such that $\{\omega_i = \zeta(t_i)\}$ is monotone increasing with $\omega_1 = \zeta(t_1) > 0$, $\lim_{i\to+\infty} t_i = t_f$, and $\lim_{i\to+\infty} \omega_i = +\infty$.

For clarity, define

$$N_g(\omega_i, \omega_j) = \int_{\omega_i}^{\omega_j} g_0(x(\tau))N(\zeta(\tau))e^{-c_1(t_j-\tau)}d\zeta(\tau)$$

with an understanding that $N_g(\omega_i, \omega_j) = N_g(\omega(t_i), \omega(t_j)) = N_g(t_i, t_j)$ for notation convenience, and $\omega_i \le \omega_j$, $\tau \in [t_i, t_j]$.

Using integral inequality $(b-a)m_{f1} \le \int_a^b f(x)dx \le (b-a)m_{f2}$ with $m_{f1} = \inf_{a\le x\le b} f(x)$ and $m_{f2} = \sup_{a\le x\le b} f(x)$, and noting that $g_0(x(t)) \le g_{\max}$, $0 < e^{-c_1(t-\tau)} \le 1$ for $\tau \in [0, t]$, we have

$$|N_g(\omega_i, \omega_j)| \le g_{\max}(\omega_j - \omega_i) \sup_{\zeta\in[\omega_i,\omega_j]} |N(\zeta)| = g_{\max}(\omega_j - \omega_i)e^{\omega_j^2} \tag{A.1}$$

for the Nussbaum function $N(\zeta) = e^{\zeta^2}\cos(\frac{\pi}{2}\zeta)$, which is positive for $\zeta \in (4m-1, 4m+1)$ and negative for $\zeta \in (4m+1, 4m+3)$ with m an integer.

Let us first consider the case $g_0(x) > 0$. First, let us consider the interval $[\omega_0, \omega_{m_1}] =$

$[\omega_0, 4m-1]$, and the following expression

$$N_g(\omega_0, \omega_{m_1}) = \int_{\omega_0}^{\omega_{m_1}} g_0(x(\tau))e^{-c_1(t_{m_1}-\tau)}N(\zeta(\tau))d\zeta(\tau)$$

Applying (A.1), we have

$$|N_g(\omega_0, \omega_{m_1})| \leq g_{\max}(4m-1-\omega_0)e^{(4m-1)^2} \tag{A.2}$$

Next, let us observe variation in the interval $[\omega_{m_1}, \omega_{m_2}] = [4m-1, 4m+1]$. Noting that $N(\zeta) \geq 0$, $\forall \zeta \in [\omega_{m_1}, \omega_{m_2}]$, we have the following inequality

$$N_g(\omega_{m_1}, \omega_{m_2}) \geq \int_{4m-\epsilon_1}^{4m+\epsilon_1} g_0(x(\tau))e^{-c_1(t_{m_2}-\tau)}N(\zeta(\tau))d\zeta(\tau)$$

with $\epsilon_1 \in (0,1)$. Similarly using the integral inequality by noting that $g_0(x(t)) \geq g_{\min}$, $e^{-c_1(t-\tau)} \geq e^{-c_1(t_{m_2}-t_{m_1})}$ for $\tau \in [t_{m_1}, t_{m_2}]$, we have

$$N_g(\omega_{m_1}, \omega_{m_2}) \geq 2\epsilon_1 g_{\min} e^{-c_1(t_{m_2}-t_{m_1})} \inf_{\zeta\in[\omega_{m_1},\omega_{m_2}]} N(\zeta) = c_{b_1}e^{(4m-\epsilon_1)^2} \tag{A.3}$$

where $c_{b_1} = 2\epsilon_1 g_{\min}\cos(\frac{\pi}{2}\epsilon_1)e^{-c_1(t_{m_2}-t_{m_1})}$.

It is known that if $|f_1(x)| \leq a_1$ and $f_2(x) \geq a_2$, then $f_1(x)+f_2(x) \geq a_2 - a_1$. Using this property, from (A.2) and (A.3), we obtain

$$N_g(\omega_0, \omega_{m_2}) \geq e^{(4m-1)^2}\{c_{b_1}e^{[2(4m-1)(1-\epsilon_1)+(1-\epsilon_1)^2]} - g_{\max}(4m-1-\omega_0)\}$$

which can be further written as

$$\frac{1}{\omega_{m_2}}N_g(\omega_0, \omega_{m_2}) \geq \frac{e^{(4m-1)^2}}{4m+1}\{c_{b1}e^{[2(4m-1)(1-\epsilon_1)+(1-\epsilon_1)^2]} - g_{\max}(4m-1-\omega_0)\} \tag{A.4}$$

The following property is useful for our derivation

$$\lim_{x\to+\infty}\frac{b_0e^{x^2}(e^{b_1x}-b_2x+b_3)}{x+a_0} = +\infty, \quad x+a_0 \neq 0,\ b_0, b_1, b_2 > 0 \tag{A.5}$$

which can be easily proven by applying the L'Hopital's Rule as

$$\lim_{x\to+\infty}\frac{b_0e^{x^2}(e^{b_1x}-b_2x+b_3)}{x+a_0} = \lim_{x\to+\infty}\frac{\frac{\partial}{\partial x}\left[b_0e^{x^2}(e^{b_1x}-b_2x+b_3))\right]}{\frac{\partial}{\partial x}(x+a_0)} = +\infty$$

Using property (A.5) and noting $(1-\epsilon_1) \in (0,1)$, from (A.4), we have

$$\lim_{m\to+\infty}\frac{1}{\omega_{m_2}}N_g(\omega_0, \omega_{m_2}) = +\infty \tag{A.6}$$

We have shown that $\lim_{m\to+\infty}\frac{1}{4m+1}N_g(\omega_0,4m+1)=+\infty$, now we would like to show that $\lim_{m\to+\infty}\frac{1}{4m+3}N_g(\omega_0,4m+3)=-\infty$.

To this end, let us first observe the interval $[\omega_0,\omega_{m_2}]=[\omega_0,4m+1]$. Similarly, applying (A.1), we can obtain

$$|N_g(\omega_0,\omega_{m_2})|\le g_{\max}(4m+1-\omega_0)e^{(4m+1)^2} \tag{A.7}$$

Then, let us consider the next immediate interval $[\omega_{m_2},\omega_{m_3}]=[4m+1,4m+3]$. Noting that $N(\zeta)\le 0$, $\forall\zeta\in[\omega_{m_2},\omega_{m_3}]$, as for $\omega\in[\omega_{m_1},\omega_{m_2}]$, we have the following inequality

$$\begin{aligned}N_g(\omega_{m_2},\omega_{m_3}) &\le \int_{4m+2-\epsilon_2}^{4m+2+\epsilon_2} g_0(x(\tau))e^{-c_1(t_{m_2}-\tau)}N(\zeta(\tau))d\zeta(\tau)\\ &\le -c_{b_2}e^{(4m+2-\epsilon_2)^2}\end{aligned} \tag{A.8}$$

where $c_{b_2}=2\epsilon_2 g_{\min}\cos(\frac{\pi}{2}\epsilon_2)e^{-c_1(t_{m_3}-t_{m_2})}$ and $\epsilon_2\in(0,1)$.

It is also known that if $|f_1(x)|\le a_1$ and $f_2(x)\le a_2$, then $f_1(x)+f_2(x)\le a_2+a_1$. Accordingly, from (A.7) and (A.8), we obtain

$$N_g(\omega_0,\omega_{m_3})\le -e^{(4m+1)^2}\{c_{b_2}e^{[2(4m+1)(1-\epsilon_2)+(1-\epsilon_2)^2]}-g_{\max}(4m+1-\omega_0)\}$$

which can be further written as

$$\frac{1}{\omega_{m_3}}N_g(\omega_0,\omega_{m_3})\le -\frac{e^{(4m+1)^2}}{4m+3}\{c_{b_2}e^{[2(4m+1)(1-\epsilon_2)(1-\epsilon_2)^2]}-g_{\max}(4m+1-\omega_0)\} \tag{A.9}$$

Using property (A.5) and noting $(1-\epsilon_2)\in(0,1)$, from (A.9), we have

$$\lim_{m\to+\infty}\frac{1}{\omega_{m_3}}N_g(\omega_0,\omega_{m_3})=-\infty \tag{A.10}$$

Therefore, from (A.6) and (A.10), we can conclude that, $g_0(x)>0$,

$$\lim_{\omega_j\to+\infty}\sup\frac{1}{\omega_j}N_g(\omega_0,\omega_j)=+\infty \tag{A.11}$$

$$\lim_{\omega_j\to+\infty}\inf\frac{1}{\omega_j}N_g(\omega_0,\omega_j)=-\infty \tag{A.12}$$

In what follows, we would like to show that (A.11) and (A.12) also hold for $g_0(x)<0$. Let us observe the following intervals $[\omega_0,4m-1]$, $[4m-1,4m+1]$ and, $[\omega_0,4m+1]$

and $[4m+1, 4m+3]$, respectively for $g_0(x) < 0$. In the intervals $[\omega_0, 4m-1]$ and $[\omega_0, 4m+1]$, inequalities (A.2) and (A.7) remain. In the interval $[4m-1, 4m+1]$, noting that $g_0(x) < 0$ and $N(\zeta) \geq 0$, we can similarly obtain

$$\begin{aligned} N_g(\omega_{m_1}, \omega_{m_2}) &\leq \int_{4m-\epsilon_1}^{4m+\epsilon_1} g_0(x(\tau))e^{-c_1(t_{m_2}-\tau)}N(\zeta(\tau))d\zeta(\tau) \\ &\leq -c_{b_1}e^{(4m-\epsilon_1)^2} \end{aligned} \tag{A.13}$$

Combining (A.2) and (A.13) yields

$$\frac{1}{\omega_{m_2}}N_g(\omega_0, \omega_{m_2}) \leq -\frac{e^{(4m-1)^2}}{4m+1}\{c_{b_1}e^{[2(4m-1)(1-\epsilon_1)+(1-\epsilon_1)^2]} - g_{\max}(4m-1-\omega_0)\} \tag{A.14}$$

Using the property (A.5) and noting $(1-\epsilon_1) \in (0,1)$, from (A.14), we have

$$\lim_{m\to+\infty} \frac{1}{\omega_{m_2}}N_g(\omega_0, \omega_{m_2}) = -\infty \tag{A.15}$$

In the interval $[4m+1, 4m+3]$, noting that $g_0(x) < 0$ and $N(\zeta) \leq 0$, we have

$$\begin{aligned} N_g(\omega_{m_2}, \omega_{m_3}) &\geq \int_{4m+2-\epsilon_2}^{4m+2+\epsilon_2} g_0(x(\tau))e^{-c_1(t_{m_2}-\tau)}N(\zeta(\tau))d\zeta(\tau) \\ &\geq c_{b_2}e^{(4m+2-\epsilon_2)^2} \end{aligned} \tag{A.16}$$

Combining the inequalities (A.7) and (A.16) on the intervals $[\omega_0, 4m+1]$ and $[4m+1, 4m+3]$ respectively, we have

$$\frac{1}{\omega_{m_3}}N_g(\omega_0, \omega_{m_3}) \geq \frac{e^{(4m+1)^2}}{4m+3}\{c_{b_2}e^{[2(4m+1)(1-\epsilon_2)+(1-\epsilon_2)^2]} - g_{\max}(4m+1-\omega_0)\} \tag{A.17}$$

Similarly using the property (A.5) and noting $(1-\epsilon_2) \in (0,1)$, from (A.17), we have

$$\lim_{m\to+\infty} \frac{1}{\omega_{m_3}}N_g(\omega_0, \omega_{m_3}) = +\infty \tag{A.18}$$

From (A.15) and (A.18), we can also obtain (A.11) and (A.12). Therefore, we can conclude that (A.11) and (A.12) hold no matter $g_0(x(t)) > 0$ or $g_0(x(t)) < 0$.

Dividing (2.55) by $\omega_i = \zeta(t_i) > 0$ yields

$$\begin{aligned} 0 \leq \frac{V(t_i)}{\zeta(t_i)} &\leq \frac{c_0}{\zeta(t_i)} + \frac{\zeta(t_i)-\zeta(0)}{\zeta(t_i)} \sup_{\zeta\in[\zeta(0),\zeta(t_i)]} e^{-c_1(t_i-\tau)} \\ &\quad + \frac{1}{\zeta(t_i)}\int_{\zeta(0)}^{\zeta(t_i)} g_0(x(\tau))N(\zeta(\tau))e^{-c_1(t_i-\tau)}d\zeta(\tau) \\ &= \frac{c_0}{\omega_i} + \Big(1 - \frac{\zeta(0)}{\omega_i}\Big) \\ &\quad + \frac{1}{\omega_i}\int_{\zeta(0)}^{\omega_i} g_0(x(\tau))N(\zeta(\tau))e^{-c_1(t_i-\tau)}d\zeta(\tau) \end{aligned} \tag{A.19}$$

On taking the limit as $i \to +\infty$, hence $t_i \to t_f$, $\omega_i \to +\infty$, (A.19) becomes

$$0 \le \lim_{i\to+\infty} \frac{V(t_i)}{\zeta(t_i)} \le 1 + \lim_{i\to+\infty} \frac{1}{\omega_i} N_g(\zeta(0), \omega_i)$$

which takes a contradiction as can be seen from (A.12). Therefore, $\zeta(t)$ is upper bounded on $[0, t_f)$.

Case (ii): $\zeta(t)$ has no lower bound on $[0, t_f)$. There must exist a monotone increasing sequence $\{\underline{t}_i\}$, $i = 1, 2, \cdots$, such that $\{\underline{\omega}_i = -\zeta(\underline{t}_i)\}$ with $\omega_1 = \zeta(\underline{t}_1) > 0$, $\lim_{i\to+\infty} \underline{t}_i = t_f$, and $\lim_{i\to+\infty} \underline{\omega}_i = +\infty$.

Dividing (2.55) by $\underline{\omega}_i = -\zeta(\underline{t}_i) > 0$ yields

$$\begin{aligned} 0 \le \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} \quad &\le \quad \frac{c_0}{-\zeta(\underline{t}_i)} - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} e^{-c_1(\underline{t}_i - \tau)} d[-\zeta(\tau)] \\ &\quad - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} g_0(x(\tau)) N(\zeta(\tau)) e^{-c_1(\underline{t}_i - \tau)} d[-\zeta(\tau)] \quad \text{(A.20)} \end{aligned}$$

Noting that $N(\cdot)$ is an even function, i.e., $N(\zeta) = N(-\zeta)$, and letting $\chi(t) = -\zeta(t)$, (A.20) becomes

$$\begin{aligned} 0 \le \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} \quad &\le \quad \frac{c_0}{-\zeta(\underline{t}_i)} - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} e^{-c_1(\underline{t}_i - \tau)} d\chi(\tau) \\ &\quad - \frac{1}{-\zeta(\underline{t}_i)} \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} g_0(x(\tau)) N(\chi(\tau)) e^{-c_1(\underline{t}_i - \tau)} d\chi(\tau) \\ &\le \quad \frac{c_0}{\underline{\omega}_i} - \frac{\underline{\omega}_i - \zeta(0)}{\underline{\omega}_i} \inf_{\tau \in [0, \underline{t}_i]} e^{-c_1(\underline{t}_i - \tau)} \\ &\quad - \frac{1}{\underline{\omega}_i} \int_{\zeta(0)}^{\underline{\omega}_i} g_0(x(\tau)) N(\chi(\tau)) e^{-c_1(\underline{t}_i - \tau)} d\chi(\tau) \\ &= \quad \frac{c_0}{\underline{\omega}_i} - \left(1 - \frac{\zeta(0)}{\underline{\omega}_i}\right) e^{-c_1 \underline{t}_i} - \frac{1}{\underline{\omega}_i} \int_{\zeta(0)}^{\underline{\omega}_i} g_0(x(\tau)) N(\chi(\tau)) e^{-c_1(\underline{t}_i - \tau)} d\chi(\tau) \end{aligned}$$

Taking the limit as $i \to +\infty$, hence $\underline{t}_i \to t_f$, $\underline{\omega}_i \to +\infty$, we have

$$0 \le \lim_{i\to+\infty} \frac{V(\underline{t}_i)}{-\zeta(\underline{t}_i)} \le -e^{-c_1 t_f} - \lim_{i\to+\infty} \frac{1}{\underline{\omega}_i} N_g(\zeta(0), \underline{\omega}_i)$$

which takes a contradiction as can be seen from (A.11). Therefore, $\zeta(t)$ is lower bounded on $[0, t_f)$.

Therefore, $\zeta(t)$ must be bounded on $[0, t_f)$. In addition, $V(t)$ and $\int_0^t g_0(x(\tau)) N(\zeta) \dot{\zeta} d\tau$ are bounded on $[0, t_f)$. $\diamondsuit$

Appendix B

Proof of Lemma 2.4.7

Proof: We first show that $\zeta(t)$ is bounded on $[0, t_f)$ by seeking a contradiction. Suppose that $\zeta(t)$ is unbounded and two cases should be considered: (i) $\zeta(t)$ has no upper bound and (ii) $\zeta(t)$ has no lower bound.

Case (i): $\zeta(t)$ has no upper bound on $[0, t_f)$. In this case, there must exist a monotone increasing sequence $\{t_i\}$, $i = 1, 2, \cdots$, such that $\{\omega_i = \zeta(t_i)\}$ is monotone increasing with $\omega_1 = \zeta(t_1) > 0$, $\lim_{i\to+\infty} t_i = t_f$, and $\lim_{i\to+\infty} \omega_i = +\infty$.

For clarity, define

$$N_g(\omega_i, \omega_j) = \int_{\omega_i}^{\omega_j} g_0 N(\zeta(\tau)) e^{-c_1(t_j-\tau)} d\zeta(\tau) \tag{B.1}$$

with an understanding that $N_g(\omega_i, \omega_j) = N_g(\omega(t_i), \omega(t_j)) = N_g(t_i, t_j)$ for notation convenience, and $\omega_i \le \omega_j$, $\tau \in [t_i, t_j]$. Let $\zeta^{-1}(x)$ denote the inverse function of $\zeta(x)$, i.e., $\zeta(\zeta^{-1}(x)) = \zeta^{-1}(\zeta(x)) \equiv x$ (according to the definition of inverse function). Noting $N(\zeta) = \zeta^2 \cos(\zeta)$, (B.1) can be re-written as

$$\begin{aligned} N_g(\omega_i, \omega_j) &= \int_{\omega_i}^{\omega_j} g_0 N(\zeta(\tau)) e^{-c_1[t_j-\zeta^{-1}(\zeta(\tau))]} d\zeta(\tau) \\ &= \int_{\omega_i}^{\omega_j} g_0 \zeta^2 \cos(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]} d\zeta \end{aligned} \tag{B.2}$$

Integration by parts, we have

$$\begin{aligned} N_g(\omega_i, \omega_j) &= \int_{\omega_i}^{\omega_j} g_0 \zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]} d[\sin(\zeta)] \\ &= g_0 \zeta^2 \sin(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ &\quad - \int_{\omega_i}^{\omega_j} g_0 \sin(\zeta) d\{\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} \end{aligned} \tag{B.3}$$

Applying the following property for the derivative of inverse function

$$\frac{d\zeta^{-1}(\zeta(x))}{d\zeta(x)} = \frac{1}{\frac{d\zeta(x)}{dx}} \tag{B.4}$$

we have

$$\begin{aligned} \frac{d}{d\zeta}\left\{\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\right\} &= 2\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]} + c_1\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\frac{d\zeta^{-1}(\zeta))}{d\zeta} \\ &= 2\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]} + c_1\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\frac{d\tau}{d\zeta} \end{aligned}$$

i.e.,

$$d\left\{\zeta^2 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\right\} = 2\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta + c_1\zeta^2 e^{-c_1(t_j-\tau)}d\tau \tag{B.5}$$

then (B.3) becomes

$$\begin{aligned} N_g(\omega_i,\omega_j) &= g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} - \int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ &\quad - \int_{t_i}^{t_j} c_1 g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{B.6}$$

Integration by parts for the term $\int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta$ in (B.6), we have

$$\begin{aligned} \int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta &= -2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ &\quad + \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)d\{\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} \end{aligned} \tag{B.7}$$

Applying (B.4), we have

$$\frac{d}{d\zeta}\left\{\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\right\} = e^{-c_1[t_j-\zeta^{-1}(\zeta)]} + c_1\zeta e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\frac{d\tau}{d\zeta} \tag{B.8}$$

then (B.7) becomes

$$\begin{aligned} &\int_{\omega_i}^{\omega_j} 2g_0\zeta\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ = &-2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} + \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta \\ &+ \int_{t_i}^{t_j} 2c_1 g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{B.9}$$

Substituting (B.9) into (B.6) yields

$$\begin{aligned} N_g(\omega_i,\omega_j) &= g_0\zeta^2\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} + 2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ &\quad - \int_{\omega_i}^{\omega_j} 2g_0\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}d\zeta - \int_{t_i}^{t_j} 2c_1 g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau \\ &\quad - \int_{t_i}^{t_j} c_1 g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \end{aligned} \tag{B.10}$$

Similarly, integration by parts for the term $\int_{\omega_i}^{\omega_j} 2g_0 \cos(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]} d\zeta$ in (B.10) by noting that

$$\frac{d}{d\zeta}\{e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\} = c_1 e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\frac{d\tau}{d\zeta}$$

we have

$$\begin{aligned}\int_{\omega_i}^{\omega_j} 2g_0 \cos(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]} d\zeta &= 2g_0 \sin(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ &\quad - \int_{t_i}^{t_j} 2c_1 g_0 \sin(\zeta) e^{-c_1(t_j-\tau)} d\tau \qquad \text{(B.11)}\end{aligned}$$

Substituting (B.11) into (B.10), we have

$$\begin{aligned}N_g(\omega_i,\omega_j) &= g_0\zeta^2 \sin(\zeta) e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} + 2g_0\zeta\cos(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} \\ &\quad -2g_0\sin(\zeta)e^{-c_1[t_j-\zeta^{-1}(\zeta)]}\Big|_{\omega_i}^{\omega_j} + \int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \\ &\quad -\int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau - \int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau \qquad \text{(B.12)}\end{aligned}$$

Let us first consider the term $\int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau$ on the right side of (B.12). Using integral inequality $(b-a)m_{f1} \le \int_a^b f(x)dx \le (b-a)m_{f2}$ with $m_{f1} = \inf_{a\le x\le b} f(x)$ and $m_{f2} = \sup_{a\le x\le b} f(x)$, and noting that $0 < e^{-c_1(t_j-\tau)} \le 1$ for $\tau \in [t_i, t_j]$, we have

$$\Big|\int_{t_i}^{t_j} 2c_1g_0\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau\Big| \le (t_j - t_i)2c_1g_0 \qquad \text{(B.13)}$$

Next, for the term $\int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau$, applying integral inequality similarly by noting that $0 < e^{-c_1(t_j-\tau)} \le 1$ for $\tau \in [t_i, t_j]$, we have

$$\Big|\int_{t_i}^{t_j} 2c_1g_0\zeta\cos(\zeta)e^{-c_1(t_j-\tau)}d\tau\Big| \le (t_j - t_i)2c_1g_0\omega_j \qquad \text{(B.14)}$$

Then, let us consider the term $\int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{-c_1(t_j-\tau)}d\tau$. Using the property that if $f(x) \le g(x)$, $\forall x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$ and noting that

$$-\omega_j^2 e^{c_1\tau} \le \zeta^2(\tau)\sin(\zeta(\tau))e^{c_1\tau} \le \omega_j^2 e^{c_1\tau}, \quad \forall \tau \in [t_i, t_j]$$

we have

$$e^{-c_1t_j}\int_{t_i}^{t_j} c_1g_0\zeta^2\sin(\zeta)e^{c_1\tau}d\tau \le e^{-c_1t_j}c_1g_0\omega_j^2\int_{t_i}^{t_j} e^{c_1\tau}d\tau = g_0\omega_j^2[1 - e^{-c_1(t_j-t_i)}]$$

and

$$e^{-c_1 t_j}\int_{t_i}^{t_j} c_1 g_0 \zeta^2 \sin(\zeta) e^{c_1\tau} d\tau \geq -e^{-c_1 t_j} c_1 g_0 \omega_j^2 \int_{t_i}^{t_j} e^{c_1\tau} d\tau = -g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}]$$

i.e.,

$$\left| e^{-c_1 t_j}\int_{t_i}^{t_j} c_1 g_0 \zeta^2 \sin(\zeta) e^{c_1\tau} d\tau \right| \leq g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] \tag{B.15}$$

Noting that $\zeta^{-1}(\omega_i) = \zeta^{-1}(\zeta(t_i)) = t_i$ and $\zeta^{-1}(\omega_j) = \zeta^{-1}(\zeta(t_j)) = t_j$, from (B.13), (B.14) and (B.15), we have the following two inequalities

$$\begin{aligned} N_g(\omega_i,\omega_j) \ \leq \ & g_0\omega_j^2 \sin(\omega_j) + 2g_0\omega_j\cos(\omega_j) - 2g_0\sin\omega_j \\ & + g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] + (t_j-t_i)2c_1g_0\omega_j + (t_j-t_i)2c_1g_0 \\ & - g_0e^{-c_1(t_j-t_i)}\omega_i^2\sin(\omega_i) - 2g_0e^{-c_1(t_j-t_i)}\omega_i\cos(\omega_i) \\ & + 2g_0e^{-c_1(t_j-t_i)}\sin(\omega_i) \end{aligned} \tag{B.16}$$

and

$$\begin{aligned} N_g(\omega_i,\omega_j) \ \geq \ & g_0\omega_j^2 \sin(\omega_j) + 2g_0\omega_j\cos(\omega_j) - 2g_0\sin\omega_j \\ & - g_0\omega_j^2[1-e^{-c_1(t_j-t_i)}] - (t_j-t_i)2c_1g_0\omega_j - (t_j-t_i)2c_1g_0 \\ & - g_0e^{-c_1(t_j-t_i)}\omega_i^2\sin(\omega_i) - 2g_0e^{-c_1(t_j-t_i)}\omega_i\cos(\omega_i) \\ & + 2g_0e^{-c_1(t_j-t_i)}\sin(\omega_i) \end{aligned} \tag{B.17}$$

Re-write (2.56) as

$$0 \leq V(t_i) \ \leq \ c_0 + \int_{\zeta(0)}^{\zeta(t_i)} g_0 N(\zeta(\tau)) e^{-c_1(t_i-\tau)} d\zeta(\tau) + \int_{\zeta(0)}^{\zeta(t_i)} e^{-c_1(t_i-\tau)} d\zeta(\tau) \tag{B.18}$$

Using (B.16) by noting $\omega_i = \zeta(t_i)$, we have

$$\begin{aligned} 0 \leq V(t_i) \ \leq \ & c_0 + N_g(\zeta(0),\omega_i) + [\omega_i - \zeta(0)] \sup_{\tau\in[0,t_i]} e^{-c_1(t_i-\tau)} \\ \leq \ & c_0 + g_0\omega_i^2\sin(\omega_i) + 2g_0\omega_i\cos(\omega_i) - 2g_0\sin\omega_i \\ & + g_0\omega_i^2[1-e^{-c_1t_i}] + 2t_ic_1g_0\omega_i + 2t_ic_1g_0 + [\omega_i - \zeta(0)] \\ & - g_0e^{-c_1t_i}\zeta^2(0)\sin(\zeta(0)) - 2g_0e^{-c_1t_i}\zeta(0)\cos(\zeta(0)) + 2g_0e^{-c_1t_i}\sin(\zeta(0)) \\ = \ & \omega_i^2\left\{g_0[\sin(\omega_i) + 1 - e^{-c_1t_i}] + \frac{f(\omega_i)}{\omega_i^2}\right\} \end{aligned} \tag{B.19}$$

where

$$\begin{aligned} f(\omega_i) &= c_0 + 2g_0\omega_i\cos(\omega_i) - 2g_0\sin\omega_i + 2t_ic_1g_0\omega_i + 2t_ic_1g_0 + [\omega_i - \zeta(0)] \\ &\quad -g_0e^{-c_1t_i}\zeta^2(0)\sin(\zeta(0)) - 2g_0e^{-c_1t_i}\zeta(0)\cos(\zeta(0)) \\ &\quad +2g_0e^{-c_1t_i}\sin(\zeta(0)) \end{aligned} \tag{B.20}$$

Taking the limit as $i \to +\infty$, hence $t_i \to t_f$, $\omega_j \to +\infty$, $\frac{f(\omega_i)}{\omega_i^2} \to +\infty$, we have

$$0 \le \lim_{i\to+\infty} V(t_i) \le \lim_{i\to+\infty} \omega_i^2 g_0[\sin(\omega_i) + 1 - e^{-c_1t_i}] \tag{B.21}$$

which, if $g_0 > 0$, draws a contradiction when $[\sin(\omega_i) + 1 - e^{-c_1t_i}] < 0$, and if $g_0 < 0$, draws a contradictions when $[\sin(\omega_i) + 1 - e^{-c_1t_i}] > 0$. Therefore, $\zeta(t)$ is upper bounded on $[0, t_f)$.

Case (ii): $\zeta(t)$ has no lower bound on $[0, t_f)$. There must exist a monotone increasing sequence $\{\underline{t}_i\}$, $i = 1, 2, \cdots$, such that $\{\underline{\omega}_i = -\zeta(\underline{t}_i)\}$ with $\omega_1 = \zeta(\underline{t}_1) > 0$, $\lim_{i\to+\infty} \underline{t}_i = t_f$, and $\lim_{i\to+\infty} \underline{\omega}_i = +\infty$.

Re-write (2.56) as

$$\begin{aligned} 0 \le V(\underline{t}_i) &\le c_0 - \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} g_0N(\zeta(\tau))e^{-c_1(\underline{t}_i-\tau)}d[-\zeta(\tau)] \\ &\quad - \int_{\zeta(0)}^{-\zeta(\underline{t}_i)} e^{-c_1(\underline{t}_i-\tau)}d[-\zeta(\tau)] \end{aligned} \tag{B.22}$$

Since $N(\cdot)$ is an even function, we have $N(\zeta) = N(-\zeta)$. Letting $\chi(t) = -\zeta(t)$, (B.22) becomes

$$0 \le V(\underline{t}_i) \le c_0 - \int_{\zeta(0)}^{\omega_i} g_0N(\chi(\tau))e^{-c_1(\underline{t}_i-\tau)}d\chi(\tau) - \int_{\zeta(0)}^{\omega_i} e^{-c_1(\underline{t}_i-\tau)}d\chi(\tau) \tag{B.23}$$

Using (B.17) by noting $\underline{\omega}_i = -\zeta(\underline{t}_i)$, we further have

$$\begin{aligned} 0 \le V(\underline{t}_i) &\le c_0 - N_g(\zeta(0), \underline{\omega}_i) - [\underline{\omega}_i - \zeta(0)] \inf_{\tau\in[0,\underline{t}_i]} e^{-c_1(\underline{t}_i-\tau)} \\ &\le c_0 - g_0\underline{\omega}_i^2\sin(\underline{\omega}_i) - 2g_0\underline{\omega}_i\cos(\underline{\omega}_i) + 2g_0\sin(\underline{\omega}_i) \\ &\quad +g_0\underline{\omega}_i^2[1 - e^{-c_1\underline{t}_i}] + 2\underline{t}_ic_1g_0\underline{\omega}_i + 2\underline{t}_ic_1g_0 - [\underline{\omega}_i - \zeta(0)]e^{-c_1\underline{t}_i} \\ &\quad +g_0e^{-c_1\underline{t}_i}\zeta^2(0)\sin(\zeta(0)) + 2g_0e^{-c_1\underline{t}_i}\zeta(0)\cos(\zeta(0)) - 2g_0e^{-c_1\underline{t}_i}\sin(\zeta(0)) \\ &= \underline{\omega}_i^2\left\{g_0[-\sin(\underline{\omega}_i) + 1 - e^{-c_1\underline{t}_i}] + \frac{f(\underline{\omega}_i)}{\underline{\omega}_i^2}\right\} \end{aligned} \tag{B.24}$$

where

$$\begin{aligned}\underline{f}(\underline{\omega}_i) &= c_0 - 2g_0\underline{\omega}_i\cos(\underline{\omega}_i) - 2g_0\sin(\underline{\omega}_i) + 2\underline{t}_i c_1 g_0 \underline{\omega}_i + 2\underline{t}_i c_1 g_0 \\ &\quad -[\underline{\omega}_i - \zeta(0)]e^{-c_1\underline{t}_i} + g_0 e^{-c_1\underline{t}_i}\zeta^2(0)\sin(\zeta(0)) + 2g_0 e^{-c_1\underline{t}_i}\zeta(0)\cos(\zeta(0)) \\ &\quad -2g_0 e^{-c_1\underline{t}_i}\sin(\zeta(0)) \end{aligned} \tag{B.25}$$

Taking the limit as $i \to +\infty$, hence $\underline{t}_i \to t_f$, $\underline{\omega}_j \to +\infty$, $\frac{\underline{f}(\underline{\omega}_i)}{\underline{\omega}_i^2} \to +\infty$, we have

$$0 \leq \lim_{i\to+\infty} V(\underline{t}_i) \leq \lim_{i\to+\infty} \underline{\omega}_i^2 g_0[\sin(\underline{\omega}_i) + 1 - e^{-c_1\underline{t}_i}] \tag{B.26}$$

which, if $g_0 > 0$, draws a contradiction when $[-\sin(\underline{\omega}_i) + 1 - e^{-c_1\underline{t}_i}] < 0$, and if $g_0 < 0$, draws a contradictions when $[-\sin(\underline{\omega}_i) + 1 - e^{-c_1\underline{t}_i}] > 0$. Therefore, $\zeta(t)$ is lower bounded on $[0, t_f)$.

Therefore, $\zeta(t)$ must be bounded on $[0, t_f)$. In addition, $V(t)$ and $\int_0^t g_0(x(\tau))N(\zeta)\dot{\zeta}d\tau$ are bounded on $[0, t_f)$. $\diamondsuit$

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