DEVELOPMENT OF A TWO-DIMENSIONAL WAVE MODEL
BASED ON THE EXTENDED BOUSSINESQ EQUATIONS

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ABSTRACT

The extended Boussinesq equations by Nwogu (1993) are suitable to simulate wave propagation from relative deep to shallow water. The equations contain all shoaling, refraction, diffraction and reflection effects in a variable depth. As a compromise between the theoretical accuracy of the mathematical model and the efficiency of solving the equation system numerically, Nwogu’s extended Boussinesq equations was selected in this study.

A high-order staggered grid numerical model based on Nwogu’s two-dimensional Boussinesq equations is developed in this thesis. The equations are solved numerically by using a fourth order accurate predictor-corrector method. This scheme has the conservative form for both mass and momentum. An internal wave generation method has been incorporated into the model. Sponge layers are placed in front of the open boundaries to absorb wave energy and minimize wave reflections.

The model can be used to simulate the evolution of relatively short and weakly nonlinear waves in water of constant or variable depth. The performance of the numerical model is demonstrated by comparing theoretical results, numerical results of previously published numerical model and experimental data. The agreements are very good. In particular, excellent results in terms of mass and energy conservation were obtained in the numerical experiment of water sloshing in confined containers.
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Chapter 1

Introduction

Every coastal or ocean engineering study such as a beach nourishment project or a harbor design planning, requires the information of wave conditions in the region of interest. Usually, wave characteristics are collected offshore and it is necessary to transfer these offshore data on wave heights and wave propagation direction to the project site. The increasing demands for accurate design wave conditions and for input data for the investigation of sediment transport and surf zone circulation have resulted in significant advancement of wave transformation models during the last two decades (Mei and Liu 1993).

When wind waves are generated by a distance storm, they usually consist of a wide range of wave frequencies. The wave component with a higher wave frequency propagates at a slower speed than those with lower wave frequencies. As they propagate across the continental shelf towards the coast, long waves lead the wave group and are followed by short waves. In the deep water, wind generated waves are not affected by the bathymetry. Upon entering shoaling waters, however, they are either refracted by bathymetry or current, or diffracted around abrupt bathymetric features such as submarine ridges or canyons. A part of wave energy is reflected back to the deep sea. Continuing their shoreward propagation, waves lose some of their
energy through dissipation near the bottom. Nevertheless, each wave profile becomes steeper with increasing wave amplitude and decreasing wavelength. Because the wave speed is proportional to the square root of the water depth in very shallow water, the front face of a wave moves at a slower speed than the wave crest, causing the overturning motion of the wave crest. Such an overturning motion usually creates a jet of water, which falls near the base of the wave and generates a large splash. Turbulence associated with breaking waves is responsible not only for the energy dissipation, but also for the sediment movement in the surfzone.

In the early of 1960’s, the wave ray tracing method was a common tool for estimating wave characteristics at a design site. Today, powerful computers have provided coastal engineers with the opportunity to employ more sophisticated numerical models for wave environment assessment. However, these numerical models are still based on simplified governing equations, boundary conditions and numerical schemes, imposing different restrictions to practical applications. The computational effort required for solving a wave propagation problem exactly by taking all physical processes, which involve many different temporal and spatial scales, into account is still too large. To date, two basic kinds of numerical wave models can be distinguished: phase-resolving models, which are based on vertically integrated, time-dependent mass and momentum balance equations, and phase-averaged models, which are based on a spectral energy balance equation. The application of phase-resolving models, which require 10 ~ 100 time steps for each wave period, is still limited to relatively small areas, o (1 ~ 10 km), while phase averaged models are more relax in the spatial resolution and can be used in much larger regions. Moreover, none of the existing models, phase-resolving or not, considers all physical processes involved.
The more recent research efforts have been focused on the development of unified phase-resolving models, which can describe transient fully nonlinear wave propagation from deep water to shallow water over a large area. In the meantime, significant progress has also been made in simulating the wave-breaking process by solving the Reynolds Averaged Navier-Stokes (RANS) equations with a turbulence closure model. These RANS models have also been employed in the studies of wave and structure interactions.

The purpose of this introduction is to provide a short summary of the evolution of phase-resolving wave propagation models during the last two decades, especially focusing on the development of Boussinesq-type equation; and the numerical schemes of these Boussinesq equations.
1.1 Background of Boussinesq equations

The earliest model which is independent of the vertical coordinate, but includes weakly dispersive and nonlinear effects, was derived by Boussinesq (1872). The model was derived for horizontal bottom only. And this model assumes irrotationality of the flow, parabolic vertical dependence of the horizontal velocity (or velocity potential) and linear vertical dependence of the vertical velocity. This model, also, has the depth averaged horizontal velocity and the free surface elevation as the dependent variables. As an aside, in this work, we refer to Boussinesq-type (or simply Boussinesq) models as those which assume irrotationality and that the velocity potential (or horizontal velocity) has a polynomial vertical dependence, leading to a set of equations governing the free surface elevation and a vertically independent horizontal velocity-related variable (e.g. depth-averaged velocity, total mass flux, velocity potential at the bottom, etc).

To overcome the horizontal bottom limitation in the original Boussinesq model, which prevents the model from being a very useful tool in coastal engineering, Mei and Mehaute (1966) and Peregrine (1967) derived Boussinesq models for variable depth. The two models are similar in the sense that both use the same asymptotic assumptions which are weak nonlinearity and dispersiveness. Mei and Mehaute, however, used the velocity at the bottom as a dependent variable, whereas Peregrine used the depth-averaged velocity. Due to the wide popularity in the coastal engineering community, the model by Peregrine (1967) is often called the standard Boussinesq model. The
standard Boussinesq model is able to describe the nonlinear transformation of irregular, multidirectional waves in shallow water. Also, the standard Boussinesq model represents the depth-integrated equations for the conservation of mass and momentum for an incompressible and inviscid fluid. The vertical velocity is assumed to vary linearly over the depth to reduce the three-dimensional problem to a two-dimensional one. The model, in addition, includes the lowest-order effects of frequency dispersion and nonlinearity. The standard Boussinesq equations can thus account for the transfer of energy between different frequency components, changes in the shape of the individual waves, and the evolution of the wave groups, in the shoaling of an irregular wave train (e.g., Freilich and Guza 1984).

Although standard Boussinesq models can predict wave transformation in coastal regions with relative accuracy, its range of validity is limited to fairly shallow water (McCowan, 1987), since its linear dispersion relationship is only a polynomial approximation of the exact one. To keep errors in the phase velocity less than 5%, the water depth has to be less than about one-fifth of the equivalent deep-water wavelength (McCowan 1987). In order to be applicable in deeper water, many authors have suggested extending the validity of Boussinesq models. The extended Boussinesq models have adjustable polynomial approximations for the dispersion relationship, a major improvement over the approximation resulting from the standard Boussinesq models. Madsen et al. (1991) and Madsen and Sorensen (1992) included higher order terms with adjustable coefficients into standard Boussinesq equations for constant and variable water depth, respectively. The addition of these terms resulted in a rational polynomial expansions as the linear dispersion relationship. The coefficients were then adjusted to give better linear shoaling coefficient and a more accurate approximation
to the exact linear dispersion relationship, namely, the (2, 2) Pade approximant (Witting, 1984). Like Peregrine (1967), Madsen used the depth-averaged velocity. Beji and Nadaoka (1996) have produced an extended Boussinesq system, valid in a variable depth environment, by a simple algebraic manipulation of Peregrine’s original system. The consistency of the various equation systems is still being debated by Schaffer, Madsen (1998) and Beji and Nadaoka (1998) although most recently Schaffer and Madsen (1999) have suggested that these equation systems are all equivalent and can be derived from a further set of extended Boussinesq equations involving an additional free parameter (Schaffer and Madsen, 1995). In contrast, by defining the dependent variable as the velocity at an arbitrary depth, Nwogu (1993) achieved a rational polynomial approximation to the exact linear dispersion relationship without the need to add higher order terms to the equations. Although the arbitrary location could be chosen to give a (2,2) Pade approximation as the linear dispersion relationship, Nwogu (1993) chose an alternative value which minimized the error in the linear phase speed over some depth range. Both expressions are much closer to the exact solution in intermediate water depths than are the standard Boussinesq equations. And, Madsen et al. and Nwogu have shown by example that the extended equations are able to simulate wave propagation from relative deep to shallow water.

Despite their improved dispersion relation, the extended Boussinesq equations are still restricted to situations with weakly nonlinear interactions. In many practical cases, however, the effects of nonlinearity are too large to be treated as a weak perturbation to a primarily linear problem. As waves approach shore, wave height increases due to the effect of shoaling, and wave breaking occurs on most gentle natural slopes. The wave height to water depth ratios accompanying this physical process are
inappropriate for weakly nonlinear Boussinesq models, and thus extensions to the model are required in order to obtain a computational tool which is locally valid in the vicinity of a steep, almost breaking or breaking wave crest. Fully nonlinear models for moderately long waves have been developed and studied for several decades. Using a power series expansion, Su and Gardner (1969) investigated a fully nonlinear formulation from which they are able to derive KdV and Burgers equations for a class of nonlinear Galilean-invariant systems. Using standard perturbation methods and depth-averaged velocity, Mei (1989) derived Boussinesq equations for constant depth with no assumption of small nonlinearity. Perturbation methods and the theory of fluid sheets reduce the dimensionality of the original water wave problem by one. Chen and Liu (1995) derived a model analogous to Nwogu’s but used a velocity potential at an arbitrary depth as the dependent variable. For regular waves propagating over a slowly varying topography, the governing equations for the velocity potentials of each harmonic are a set of weakly nonlinear coupled fourth-order elliptic equations with variable coefficients. The accuracy of the parabolic approximation to a fourth-order ordinary differential equation with a weak forcing term (which may involve the other independent variable) depends on the difference between the wave number of the forcing term and the characteristic wave number of the equation. Wei et al. (1995) used Nwogu’s approach to derive a Boussinesq model (referred to henceforth as the WKGS model) without the weak nonlinearity restriction, which is called fully nonlinear Boussinesq model. In strongly nonlinear situations (such as wave evolution just prior to wave breaking), the fully nonlinear Boussinesq model provides significantly improved predictions of wave heights and internal kinematics relative to the weakly nonlinear Boussinesq model. Numerical computations showed a major improvement over the weakly nonlinear model of Nwogu, and compared well with
solitary wave solutions of the full potential problem obtained with the boundary elements method by Grilli et al. (1989). By retaining terms of higher order in both the frequency dispersion and the amplitude dispersion, Madsen et al. (1996) derived a new set of Boussinesq-type equations. In order to enhance the linear dispersion characteristics in deeper water the equations were modified using a technique described in Schaffer and Madsen (1995). Madsen and Schaffer (1999) derived higher order Boussinesq-type equations to improve dispersion and nonlinearity. Recently, Gobbi et al. (2000) presented a higher-order formulation based on a linear combination of the velocities at two arbitrary z-levels. Madsen et al. (2001) derived higher-order equations by using arbitrary z-level. The arbitrary z-level is determined by minimizing the depth-integrated error of the linear velocity profile. In comparison with the standard equations all these equations involve numerous additional derivatives, hence making a numerical solution elaborate.
1.2 Background of Boussinesq numerical Model

Most of the numerical schemes for the Boussinesq equations are based on finite difference methods. Peregrine (1966) presented probably the first finite difference method for a Boussinesq-type equation system, using second order spatial and temporal approximations. The reported results were dissipative and accuracy was only maintained by choosing sufficiently small mesh spacing. However, Abbot et al (1978, 1984) pioneered the finite difference solution of the original Boussinesq system for practical engineering problems and analyzed the methods carefully to ensure accurate solutions. Madsen et al (1991, 1992) applied extensions of these techniques in their finite difference methods for the extended Boussinesq equation systems. The differential equations are discretized by time-centered implicit scheme with variables defined on a space-staggered rectangular grid. The method is based on the ADI (Alternating Direction Implicit) algorithm, and the resulting system of finite difference equations is reduced to a tri-diagonal system, which is solved by the Double Sweep algorithm. The finite difference approximation of the spatial derivatives is a straightforward mid-centering except for the nonlinear convective terms. And Nwogu (1993), an iterative Crank-Nicolson method is employed, with predictor-corrector scheme used to provide the initial estimate. The partial derivatives are approximated using a forward difference scheme for the time variable and a central difference scheme for the space variable. The numerical solution procedure involves solving an explicit expression for $\eta$ and a tridiagonal matrix for $u_x$ in the predictor and corrector stages, and tridiagonal matrices for both variables in the iterative stage. Tridiagonal matrices
contain only diagonal and adjacent off-diagonal terms and can be solved efficiently using a Gaussian elimination process.

More recently Wei and Kirby (1995) presented an alternative technique for deriving finite difference schemes for Boussinesq-type systems and demonstrated that accurate solutions could be obtained. Noting that the non-physical dispersion present in a second order finite difference spatial discretisation was due only to the first order derivative, they differenced these terms to fourth order and the dispersive terms to second order. The system of equations is written in a form that makes the application of a higher-order time-stepping procedure convenient. They integrated in time with a fourth order predictor-corrector method to ensure that no more dispersive errors were introduced. All errors involved in solving the underlying nonlinear shallow water equations are thus reduced to 4th order in grid spacing and time step size. Spatial and temporal differencing of the higher-order dispersion terms are done to second-order accuracy, which again reduces the truncation errors to a size smaller than those terms themselves. No further back-substitution of apparent truncation error terms is performed. These finite difference methods have been used to predict wave elevations inside harbours (A. Schroter et al. 1995) and wave interactions in the nearshore zone (Nwogu 1993).

Unlike finite difference methods, finite element methods are rarely used to solve extended Boussinesq equations. Recently, Walkley and Berzins (1999, 2002) presented a finite element method for one-dimensional and two-dimensional extended Boussinesq equations.
1.3 Outline of the thesis

The Boussinesq wave equations, accounting for both weakly nonlinear and weakly dispersive wave processes, have been identified as a suitable mathematical model, although their form is not unique and the precise form of the system will have to be chosen. Considering a compromise between the theoretical accuracy of the mathematical model and the practicalities of solving the equation system numerically, Nwogu’s extended Boussinesq equations are selected. The aim of this work is to develop a higher-order numerical model based on Nwogu’s Boussinesq equations. We use a fourth-order predictor-corrector scheme for time stepping and staggered grid systems for spatial derivatives.

In Chapter 2, the derivation of Boussinesq equation systems is considered and the properties of various alternative systems are compared. In particular, it is shown that extended Boussinesq equation systems significantly increase the range of depths over which the model is applicable. Therefore, such model can provide an accurate description of wave evolution in coastal regions.

The numerical modeling of the one-dimensional form of the extended Boussinesq equations is considered in Chapter 3. It is shown that the numerical methods developed in the study can be directly applied to this equation system. Particular attention is paid to the initial conditions and boundary conditions. And techniques are
described that enable efficient use of the time integration software. Some numerical experiments of both linear and weakly nonlinear waves are carried out in this chapter. The accuracy of the numerical method is analyzed and compared to experiment data. The performance of the numerical method is also demonstrated by comparison with theoretical results.

The numerical method used must be suitable for arbitrary two-dimensional spatial domains and also must be accurate over the long time periods required for the computations. The numerical approximation of the two-dimensional extended Boussinesq equations, representing the most general problem to be considered in this work, is described in Chapter 4. The performance of the numerical method is evaluated by tests involving simple theoretical solutions, experimental data.

In Chapter 5, some conclusions were obtained.
Chapter 2

The Boussinesq Equations

2.1 Introduction

Boussinesq equation systems model water waves in shallow water, including the first order effects of nonlinearity and dispersion. Various ways of deriving Boussinesq-type systems have been discussed in the literatures. These can generally be split into so-called engineering derivations and more rigorous derivations based on small parameter expansions. Here the rigorous derivation of weakly nonlinear, dispersive equations is considered, as the techniques and approximations involved provide important insights into the numerical methods proposed in the following chapters. The mathematical framework for the analysis of nonlinear, dispersive shallow water waves is outlined in Section 2.2 based on the initial assumptions of an inviscid, incompressible Newtonian fluid. The appropriate non-dimensionalisation and scaling of the equations is described in details. It is the aim of this chapter to evaluate the possible models and to select the most suitable one for numerical computation. The Boussinesq models can be evaluated theoretically through their linear dispersion characteristics, but it is also important to consider the practicalities of applying a numerical method. The classical form of the Boussinesq equations is not derived here since it can be found in many literatures. More recently, the original Boussinesq equation system has been extended to include the higher order effects of both nonlinearity and dispersion. Here the detailed
derivation of so-called extended Boussinesq systems, which improve the dispersive properties of the equation, is presented in Section 2.2.2. And these variable depth equations are derived using the non-dimensionalisation and scaling of Peregrine. The dispersion properties of these extended systems are compared to the classical equations and to the linear theory. Detailed derivations are given for these dispersive properties with comparison of the results made to those stated in the literature, and to those for alternative extended Boussinesq systems.
2.2 The mathematical formulation

2.2.1 Euler’s Equation

The starting point for modeling water waves is the incompressible Navier-Stokes equations. However, the numerical solution of these equations, a three-dimensional problem with a free-surface boundary, is extremely complex. If the water is initially at rest and the waves propagate into the domain from the boundary, the flow can be assumed to be inviscid and irrotational. Vorticity can diffuse in from the boundary once the fluid is in motion. However, for a finite time, the effects will not be felt throughout the fluid. Therefore, in this study, the fluid is assumed to be incompressible and inviscid. And the rotational effect of the Earth is not taken into consideration. The $x$- and $y$-axes represent two horizontal coordinates, while the $z$-axis is the vertical coordinate. Then, the governing equations are the Euler equations:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2.2)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2.3)
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (2.4)
\]
in which \( u \), \( v \) and \( w \) are velocity components in the \( x \)-, \( y \)- and \( z \)-directions, respectively, and \( p \) is the dynamic pressure, \( \rho \) is the constant density of the fluid, \( g \) is gravitational acceleration.

Along the solid bottom the boundary condition can be expressed as

\[
\frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + w = 0, \quad z = -h(x,y)
\]  
(2.5)

The kinematic free surface boundary condition, in three dimensions, is

\[
\frac{\partial \eta}{\partial t} + u(x,y,\eta) \frac{\partial \eta}{\partial x} + v(x,y,\eta) \frac{\partial \eta}{\partial y} + w(x,y,\eta) \frac{\partial \eta}{\partial y} = w(x,y,\eta), \quad z = \eta(x,y,t)
\]  
(2.6)

\[
p = 0, \quad z = \eta(x,y,t)
\]  
(2.7)

Two important length scales are the characteristic water depth \( h_0 \) for the vertical direction and a typical wavelength \( l \) for the horizontal direction. The variables associated with the different length scales are considered to be of different orders of magnitude. The following independent, non-dimensional variables can be defined as:

\[
x = \frac{x'}{l}, \quad y = \frac{y'}{l}, \quad z = \frac{z'}{h_0}, \quad t = \frac{\sqrt{gh_0}}{l}t',
\]  
(2.8)

where \( g \) = gravitational acceleration; and primes are used to denote dimensional variables. For effects due to the motion of the free surface, the typical wave amplitude \( a_0 \) is also important. The following dependent, non-dimensional variables can also be defined as:

\[
u = \frac{h_0}{a_0 \sqrt{gh_0}}u', \quad v = \frac{h_0}{a_0 \sqrt{gh_0}}v', \quad w = \frac{h_0^2}{a_0 l \sqrt{gh_0}}w',
\]  
(2.9)

\[
\eta = \frac{\eta'}{a_0}, \quad h = \frac{h'}{h_0}, \quad p = \frac{p'}{\rho g a_0}
\]  
(2.10)

Where \( (u,v,w) \) = the water particle velocity vector; \( p \) = the pressure; and \( \rho \) is the fluid density.
And two important parameters are defined as followed:

\[ \varepsilon = \frac{a_0}{h_0}, \mu^2 = \left( \frac{h_0}{l} \right)^2 \]  

(2.11)

in which \( \varepsilon \) represents the nonlinearity and \( \mu^2 \) the frequency dispersion.

So the continuity equation can be expressed in non-dimensional form as:

\[ \mu^2 \frac{\partial u}{\partial x} + \mu^2 \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  

(2.12)

And Euler's equations of motion can be expressed in non-dimensional form as:

\[ \mu^2 \frac{\partial u}{\partial t} + \varepsilon \mu^2 u \frac{\partial u}{\partial x} + \varepsilon \mu^2 v \frac{\partial u}{\partial y} + \varepsilon \mu^2 w \frac{\partial u}{\partial z} = -\mu^2 \frac{\partial p}{\partial x} \]  

(2.13)

\[ \mu^2 \frac{\partial v}{\partial t} + \varepsilon \mu^2 u \frac{\partial v}{\partial x} + \varepsilon \mu^2 v \frac{\partial v}{\partial y} + \varepsilon \mu^2 w \frac{\partial v}{\partial z} = -\mu^2 \frac{\partial p}{\partial y} \]  

(2.14)

\[ \varepsilon \frac{\partial w}{\partial t} + \varepsilon^2 u \frac{\partial w}{\partial x} + \varepsilon^2 v \frac{\partial w}{\partial y} + \varepsilon^2 w \frac{\partial w}{\partial z} = -\varepsilon \frac{\partial p}{\partial z} - 1 \]  

(2.15)

The irrotational conditions are given by

\[ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0, \quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0 \]  

(2.16)

The bottom boundary condition (Dean and Dalrymple, 1984) becomes:

\[ w = -\mu^2 u \frac{\partial h}{\partial x} - \mu^2 v \frac{\partial h}{\partial y}, \text{ at } z = -h(x,y) \]  

(2.17)

The kinematic free surface boundary conditions (Dean and Dalrymple, 1984), in non-
dimensions, is

\[ w = \mu^2 \frac{\partial \eta}{\partial t} + \varepsilon \mu^2 u \frac{\partial \eta}{\partial x} + \varepsilon \mu^2 v \frac{\partial \eta}{\partial y}, \text{ at } z = \eta(x,y,t) \]  

(2.18)

\[ p = 0, \quad \text{at } z = \eta(x,y,t) \]  

(2.19)
2.2.2 Derivation of Nwogu’s extended Boussinesq equation

The following section describes the derivation of Nwogu’s extended Boussinesq equations. This derivation expands on what outlined by Nwogu for the three-dimensional equation system, and use the non-dimensionalised, scaled equation system of Euler’s equations (2.12)-(2.19) as the starting point. Walkley (1999) used the same method to derive the Boussinesq equations.

Integrating the continuity equation (2.12) from the seabed to the free surface through the whole total depth:

\[
\int_{-h}^{e_\eta} \frac{\partial W}{\partial z} \, dz = -\int_{-h}^{e_\eta} \left( \mu^2 \frac{\partial u}{\partial x} + \mu^2 \frac{\partial v}{\partial y} \right) \, dz
\]  

(2.20)

Vertically integrating every term of equation (2.20) using Leibniz’s Rule, we can get:

\[
\int_{-h}^{e_\eta} \frac{\partial W}{\partial z} \, dz = w|_{e_\eta} - w|_{-h}
\]  

(2.21)

\[
\int_{-h}^{e_\eta} \left( \mu^2 \frac{\partial u}{\partial x} + \mu^2 \frac{\partial v}{\partial y} \right) \, dz = \frac{\partial}{\partial x} \left( \int_{-h}^{e} \mu^2 u \, dz \right) + \frac{\partial}{\partial y} \left( \int_{-h}^{e} \mu^2 v \, dz \right) - \mu^2 u\big|_{-h} \frac{\partial h}{\partial x} - \mu^2 v\big|_{-h} \frac{\partial h}{\partial x} \\
- \mu^2 |_{e_\eta} \frac{\partial \eta}{\partial x} - \mu^2 |_{e_\eta} \frac{\partial \eta}{\partial x}
\]  

(2.22)

Substituting equations (2.21) and (2.22) into equation (2.20) and applying bottom boundary condition (2.17) and kinematic boundary conditions (2.18) and (2.19), equation (2.20) becomes:

\[
\mu^2 \frac{\partial}{\partial x} \left( \int_{-h}^{e} u \, dz \right) + \mu^2 \frac{\partial}{\partial y} \left( \int_{-h}^{e} v \, dz \right) + \frac{\partial \eta}{\partial t} = 0
\]  

(2.23)

Similarly, the horizontal momentum equations (2.13) and (2.14) can be integrated over the depth to give:
\[ \frac{\partial}{\partial t} \int_{-h}^{z_h} udz + \frac{\partial}{\partial x} \int_{-h}^{z_h} u^2 dz + \frac{\partial}{\partial y} \int_{-h}^{z_h} uvdz + \frac{\partial}{\partial x} \int_{-h}^{z_h} pdz - p(x,y,-h,t) \frac{\partial h}{\partial x} = 0 \quad (2.24) \]

\[ \frac{\partial}{\partial t} \int_{-h}^{z_h} vdz + \frac{\partial}{\partial x} \int_{-h}^{z_h} uvdz + \frac{\partial}{\partial y} \int_{-h}^{z_h} v^2 dz + \frac{\partial}{\partial y} \int_{-h}^{z_h} pdz - p(x,y,-h,t) \frac{\partial h}{\partial y} = 0 \quad (2.25) \]

Where equations (2.12) and (2.17) – (2.19) have been used. The pressure field is obtained by integrating the vertical momentum equation (2.15) with respect to \( z \) and applying the boundary conditions in (2.18) and (2.19) at free surface:

\[ p = \eta - \frac{z}{\varepsilon} + \frac{\partial}{\partial t} \int_{-h}^{z_h} wdz + \frac{\partial}{\partial x} \int_{-h}^{z_h} uvdz + \frac{\partial}{\partial y} \int_{-h}^{z_h} vwdz - \frac{\varepsilon}{\mu^2} w^2 = 0 \quad (2.26) \]

Finally, integrating the continuity equation (2.12) from the seabed to an arbitrary water depth \( z, (-h < z < \varepsilon \eta) \):

\[ \int_{-h}^{z} \frac{\partial w}{\partial z} dz = -\int_{-h}^{z} \left( \mu^2 \frac{\partial u}{\partial x} + \mu^2 \frac{\partial v}{\partial y} \right) dz \quad (2.27) \]

Vertically integrating every term of equation (2.20) using Leibnez’z Rule, we can get:

\[ \int_{-h}^{z} \frac{\partial w}{\partial z} dz = w\bigg|_{z} - w\bigg|_{-h} \quad (2.28) \]

\[ \int_{-h}^{z} \left( \mu^2 \frac{\partial u}{\partial x} + \mu^2 \frac{\partial v}{\partial y} \right) dz = \frac{\partial}{\partial x} \left( \int_{-h}^{z} u^2 udz \right) + \frac{\partial}{\partial y} \left( \int_{-h}^{z} v^2 vdz \right) - \mu^2 u\bigg|_{-h} \frac{\partial h}{\partial x} - \mu^2 v\bigg|_{-h} \frac{\partial h}{\partial y} \quad (2.29) \]

Substituting equations (2.21) and (2.22) into equation (2.20) and applying bottom boundary condition (2.17), equation (2.20) becomes:

\[ w\bigg|_{z} = -\mu^2 \frac{\partial}{\partial x} \left( \int_{-h}^{z} udz \right) - \mu^2 \frac{\partial}{\partial y} \left( \int_{-h}^{z} vdz \right) \quad (2.30) \]

All the foregoing equations are exact, and are valid for all orders of \( \varepsilon \) and \( \mu \). To integrate these equations, the depth dependence of the variables must be known. The horizontal velocities are initially expanded as a Taylor series at certain water depth \( z_a \).

\[ u = u_a + (z - z_a) \frac{\partial u_a}{\partial z} + \frac{(z - z_a)^2}{2} \frac{\partial^2 u_a}{\partial z^2} + \ldots \quad (2.31) \]
\[ v = v_a + (z - z_a) \frac{\partial v_a}{\partial z} + \frac{(z - z_a)^2}{2} \frac{\partial^2 v_a}{\partial z^2} + \ldots \] (2.32)

\[ \int_{-h}^0 udz = (z + h)u_a + \left( \frac{(z - z_a)^2}{2} - \frac{(h + z_a)^2}{2} \right) \frac{\partial u_a}{\partial z} + \ldots \] (2.33)

\[ \int_{-h}^0 vdz = (z + h)v_a + \left( \frac{(z - z_a)^2}{2} - \frac{(h + z_a)^2}{2} \right) \frac{\partial v_a}{\partial z} + \ldots \] (2.34)

Substituting equation (2.30) into equation (2.16), we can get:

\[ \frac{\partial u}{\partial z} = -\mu^2 \frac{\partial^2}{\partial x^2} \left( \int_{-h}^0 udz \right) - \mu^2 \frac{\partial^2}{\partial y^2} \left( \int_{-h}^0 vdz \right) \] (2.35)

Substituting equation (2.33) and (2.34) into equation (2.35), we can get:

\[ \frac{\partial u}{\partial z} = -\mu^2 \frac{\partial^2}{\partial x^2} \left( (z + h)u_a + \left( \frac{(z - z_a)^2}{2} - \frac{(h + z_a)^2}{2} \right) \frac{\partial u_a}{\partial z} + \ldots \right) + \ldots \]

\[ -\mu^2 \frac{\partial^2}{\partial y^2} \left( (z + h)v_a + \left( \frac{(z - z_a)^2}{2} - \frac{(h + z_a)^2}{2} \right) \frac{\partial v_a}{\partial z} + \ldots \right) \] (2.36)

Differentiating equation (2.36) with respect to \( z \) and noting from equations (2.35) that the first derivative is \( O(\mu^3) \):

\[ \frac{\partial^2 u}{\partial z^2} = -\mu^2 \frac{\partial^2}{\partial x^2} \left( u_a + (z - z_a) \frac{\partial u_a}{\partial z} + \frac{(z - z_a)^2}{2} \frac{\partial^2 u_a}{\partial z^2} + \ldots \right) + \ldots \]

\[ -\mu^2 \frac{\partial^2}{\partial y^2} \left( v_a + (z - z_a) \frac{\partial v_a}{\partial z} + \frac{(z - z_a)^2}{2} \frac{\partial^2 v_a}{\partial z^2} + \ldots \right) \]

\[ = -\mu^2 \frac{\partial^2 u_a}{\partial x^2} - \mu^2 \frac{\partial^2 v_a}{\partial y^2} + O(\mu^4) \] (2.37)
Differentiating equation (2.37) with respect to $z$ and noting from equations (2.35) and (2.36) that the first and second derivatives are $O(\mu^2)$:

$$
\frac{\partial^3 u}{\partial z^3} = -\mu^2 \frac{\partial^2}{\partial x^2} \left( \frac{\partial u_a}{\partial z} + (z - z_a) \frac{\partial^2 u_a}{\partial z^2} + \cdots \right) \\
- \mu^2 \frac{\partial^2}{\partial y \partial x} \left( \frac{\partial v_a}{\partial z} + (z - z_a) \frac{\partial^2 v_a}{\partial z^2} + \cdots \right) \\
= O(\mu^4)
$$

(2.38)

Repeating differentiation of this expression will produce expressions for the higher derivatives of $u$ with respect to $z$ and show them to be of $O(\mu^4)$ or greater.

Substituting equations (2.37) and (2.38) back in equation (2.36):

$$
\frac{\partial u}{\partial z} = -\mu^2 \frac{\partial^2}{\partial x^2} ((z + h)u_a) - \mu^2 \frac{\partial^2}{\partial y \partial x} ((z + h)v_a) + O(\mu^4)
$$

(2.39)

Substituting equations (2.37), (2.38) and (2.39) into the Taylor series expansion (2.31), produces an expression for the horizontal velocity component:

$$
u = u_a - \mu^2 (z - z_a) \left[ \frac{\partial^2}{\partial x^2} ((z_a + h)u_a) + \frac{\partial^2}{\partial y \partial x} ((z_a + h)v_a) \right] \\
- \mu^2 \frac{(z - z_a)^2}{2} \left( \frac{\partial^2 u_a}{\partial x^2} + \frac{\partial^2 v_a}{\partial y \partial x} \right) + O(\mu^4)
$$

(2.40)

After the same procedure, we can get:

$$
\nu = v_a + \mu^2 (z_a - z) \left[ \frac{\partial^2}{\partial x \partial y} ((z_a + h)u_a) + \frac{\partial^2}{\partial y^2} ((z_a + h)v_a) \right] \\
- \mu^2 \frac{(z - z_a)^2}{2} \left( \frac{\partial^2 u_a}{\partial x \partial y} + \frac{\partial^2 v_a}{\partial y^2} \right) + O(\mu^4)
$$

(2.41)

Substituting the horizontal velocities (2.40) and (2.41) in equation (2.30) leads to an expression for the vertical velocity component:
\[
\begin{align*}
\frac{\partial}{\partial x} ((z+h)u_a) - \frac{\partial}{\partial y} ((z+h)v_a) + O(\mu^4) \\
\end{align*}
\] (2.42)

Substituting the horizontal velocities (2.40), (2.41) and the vertical velocity (2.42) into equation (2.26) and integrating, retaining terms up to \( O(\varepsilon) \) and \( O(\mu^2) \):

\[
\begin{align*}
p &= \eta - \frac{z}{\varepsilon} + \mu^2 \frac{\partial^2}{\partial \xi \partial t} \left( \left( \frac{z^2}{2} + hz \right) u_a \right) + \mu^2 \frac{\partial^2}{\partial \eta \partial t} \left( \left( \frac{z^2}{2} + hz \right) v_a \right) + O(\varepsilon^2, \varepsilon \mu^2, \mu^4) \\
\end{align*}
\] (2.43)

Substituting the equations for the horizontal velocity components (2.40) and (2.41), the vertical velocity component (2.42) and the pressure (2.43) into the depth-integrated continuity and horizontal momentum equations [(23)–(25)] and integrating, retaining terms up to \( O(\varepsilon) \) and \( O(\mu^2) \), we can get Nwogu’s Boussinesq-type equations:

\[
\begin{align*}
\eta_x + \nabla \cdot \left[ (h + \varepsilon \eta) u_a \right] + \mu^2 \nabla \left( \frac{z_a^2}{2} - \frac{h^2}{6} \right) h \nabla \cdot (\nabla \cdot u_a) + \left( z_a + \frac{h}{2} \right) h \nabla \cdot (\nabla \cdot (hu_a)) &= 0 \\
\end{align*}
\] (2.44)

\[
\begin{align*}
u_a + \nabla \eta + \varepsilon (u_a \cdot \nabla) u_a + \mu^2 \left( \frac{z_a^2}{2} \nabla \cdot (\nabla \cdot u_a) + z_a \nabla \cdot (\nabla \cdot (hu_a)) \right) &= 0 \\
\end{align*}
\] (2.45)
2. 3 Linear dispersive wave theory

It is common to compare the linear dispersion characteristics of the Boussinesq equation systems with those of linear wave theory. Three quantities are generally used: phase velocity, \( C \), which is the velocity of a point on the free surface, group velocity, \( C_g \), which governs the propagation of energy in a wave train, and the shoaling gradient, \( s \), which relates the change in wave amplitude to the change in depth.

The phase velocity of a wave train with frequency \( \omega \) and wave number \( k \) is defined as:

\[
C = \frac{\omega}{k} \quad (2.46)
\]

For linear wave theory:

\[
C^2 = \frac{\omega^2}{k^2} = gh \frac{\tanh(kh)}{kh} \quad (2.47)
\]

The group velocity of a wave train with frequency \( \omega \) and wave number \( k \) is defined as (Whitham G. B., *Linear and Nonlinear Waves*. Wiley-Interscience, 1974):

\[
C_g = \frac{\partial \omega}{\partial k} \quad (2.48)
\]

And

\[
\omega = \frac{2\pi}{T}, \quad T = \frac{L}{C}, \quad k = \frac{2\pi}{L} \quad (2.49)
\]

Where \( L \) is wave length and \( T \) is wave period, so we can get from equations (2.49) \( \omega = kC \)

Using the definition of the phase velocity \( C_g \) to eliminate \( \omega \):

\[
C_g = \frac{\partial kC}{\partial k} = C + k \frac{\partial C}{\partial k} \quad (2.50)
\]
Using the phase velocity from linear wave theory (2.47) one arrives at the form stated by Madsen (1992):

\[ C_g = C \cdot \frac{1}{2} \left( 1 + \frac{2kh}{\sinh(2kh)} \right) \]  

(2.51)

To simplify the following algebra in this section the group velocity is written in a more convenient form. Let:

\[ C_g = C \cdot n(kh) \]  

(2.52)

Where

\[ n(kh) = \frac{1}{2} \left( 1 + \frac{2kh}{\sinh(2kh)} \right) \]  

(2.53)

To derive the shoaling gradient we consider a wave train described by a constant frequency \( \omega \) and spatially varying wave number \( k(x) \) and wave amplitude \( a(x) \) over a varying depth \( h(x) \). The statement of the constancy of energy flux is:

\[ \frac{\partial a^2 C_g}{\partial x} = 0 \]

\[ 2a \frac{\partial a}{\partial x} C_g + a^2 \frac{\partial C_g}{\partial x} = 0 \]

\[ \frac{1}{a} \frac{\partial a}{\partial x} + \frac{1}{2C_g} \frac{\partial C_g}{\partial x} = 0 \]  

(2.54)

From equation (2.52):

\[ \frac{1}{C_g} \frac{\partial C_g}{\partial x} = \frac{1}{C} \frac{\partial C}{\partial x} + \frac{1}{n} \frac{\partial n}{\partial x} \]  

(2.55)

This equation can be simplified by differentiating the phase velocity (2.46) with respect to \( x \):

\[ \frac{\partial C}{\partial x} = -\frac{\omega}{k^2} \frac{\partial C}{\partial x} \]
\[
\frac{1}{C} \frac{\partial C}{\partial x} = - \frac{1}{k} \frac{\partial k}{\partial x} \quad (2.56)
\]

Substituting (2.56) into the group velocity expression (2.55), we can get:

\[
\frac{1}{C_g} \frac{\partial C_g}{\partial x} = \frac{1}{n} \frac{\partial n}{\partial x} - \frac{1}{k} \frac{\partial k}{\partial x} \quad (2.57)
\]

Substituting this equation (2.57) into the shoaling gradient relation (2.54) results in:

\[
\frac{1}{a} \frac{\partial a}{\partial x} + \frac{1}{2} \left( \frac{1}{n} \frac{\partial n}{\partial x} - \frac{1}{k} \frac{\partial k}{\partial x} \right) = 0 \quad (2.58)
\]

Terms involving \( n \) are eliminated from equation (2.58) by differentiating equation (2.53) with respect to \( x \):

\[
\frac{\partial n}{\partial x} = \frac{\partial n}{\partial k h} \frac{\partial k h}{\partial x}
\]

\[
= \frac{\partial k h}{\partial x} \left( \frac{\sinh 2k h - 2k h \cosh 2k h}{\sinh^2 2k h} \right) \quad (2.59)
\]

From equation (2.59), we can get:

\[
\frac{1}{n} \frac{\partial n}{\partial x} = \frac{1}{k h} \frac{\partial k h}{\partial x} \left( \frac{2k h (\sinh (2k h) - 2k h \cosh (2k h))}{\sinh (2k h) (\sinh (2k h) + 2k h)} \right) \quad (2.60)
\]

And

\[
\frac{\partial k h}{\partial x} = k \frac{\partial h}{\partial x} + h \frac{\partial k}{\partial x}
\]

\[
= \frac{1}{k h} \frac{\partial h}{\partial x} + \frac{1}{k} \frac{\partial k}{\partial x} \quad (2.61)
\]

A relationship between the wave number gradient and the depth gradient is derived from the phase velocity. Rearranging equation (2.47):

\[
\frac{\omega^2 h}{g} = k h \tanh (k h) \quad (2.62)
\]

Differentiating equation (2.62) with respect to \( x \):
\[
\frac{\omega^2}{g} \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \left( kh \tanh(kh) \right) \frac{\partial kh}{\partial x}
\]  \hspace{1cm} (2.63)

Using (2.62) to eliminate the \( \omega \) terms,

\[
\frac{1}{h} \frac{\partial h}{\partial x} \tanh(kh) = \frac{1}{kh} \frac{\partial kh}{\partial x} \left( \tanh(kh) + kh \left( 1 - \tanh^2(kh) \right) \right)
\]  \hspace{1cm} (2.64)

Substituting equation (2.61) into equation (2.64) and after some algebra calculation to derive a relationship between the wave number gradient and the depth gradient:

\[
\frac{1}{k} \frac{\partial k}{\partial x} = \left( \frac{-2kh}{2kh + \sinh(2kh)} \right) \frac{1}{h} \frac{\partial h}{\partial x}
\]  \hspace{1cm} (2.65)

Substituting equation (2.60) into the shoaling gradient relation (2.58) and using equation (2.65) to substitute for terms involving \( k \), the shoaling gradient can be written as:

\[
\frac{1}{a} \frac{\partial a}{\partial x} + \frac{1}{s} \frac{1}{k} \frac{\partial h}{\partial x} = 0
\]  \hspace{1cm} (2.66)

In which:

\[
s = \frac{2kh(\sinh(2kh) + kh(1 - \cosh(2kh)))}{(2kh + \sinh(2kh))^2}
\]  \hspace{1cm} (2.67)

Some manipulation of this expression shows that it is identical to the form stated by Madsen.
2. 4 dispersion characteristics of the extended Boussinesq Equations

The new set of equations [(2.44] and (2.45)] may be regarded as a class of equations containing most known forms of Boussinesq-type equations, with the elevation of the velocity variables, \( z_\alpha \), as a free parameter. The different possible velocity variables include the velocity at the seabed and the velocity at the still-water level. Since the equations are an approximation of the fully dispersive and nonlinear problem, one can select a velocity variable to minimize the errors introduced by the approximations. Here, we consider the linear limit and choose \( z_\alpha \) to obtain the best fit between the linear dispersion relation of the model and the exact dispersion relation for a wide range water depth. The linearized version of the equations for one horizontal dimension and constant depth can be expressed in dimensional form as:

\[
\frac{\partial \eta}{\partial t} + h \frac{\partial u_a}{\partial x} + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_a}{\partial x^3} = 0 \quad (2.68)
\]

\[
\frac{\partial u_a}{\partial t} + g \frac{\partial \eta}{\partial x} + \alpha \cdot h^2 \frac{\partial^3 u_a}{\partial x^2 \partial t} = 0 \quad (2.69)
\]

Where \( \alpha = \frac{1}{2} \left( \frac{z_\alpha}{h} \right)^2 + \frac{z_\alpha}{h} \); and the primes have been dropped. Consider a small amplitude of periodic wave with frequency \( \omega \) and wave number \( k \):

\[
\eta = a_0 \exp[i(kx - \omega t)] \quad (2.70)
\]

\[
u_a = u_0 \exp[i(kx - \omega t)] \quad (2.71)
\]
From equation (2.70) and (2.71), we can get:

$$\frac{\partial \eta}{\partial t} = -i \omega \eta, \quad \frac{\partial \eta}{\partial x} = ik \eta$$

(2.72)

$$\frac{\partial u_x}{\partial x} = ik u_x, \quad \frac{\partial^2 u_x}{\partial x^2} = -k^2 u_x, \quad \frac{\partial^3 u_x}{\partial x^3} = -i k^3 u_x,$$

(2.73)

$$\frac{\partial^3 u_x}{\partial t \partial x^3} = \frac{\partial^3 u_x}{\partial t \partial x^3} = i \omega k^2 u_x \quad (2.74)$$

Substituting equations [(2.72)-(2.74)] into equation (2.68) and (2.69), the two constant depth Boussinesq equations become:

$$\omega a_0 - kh \left[ 1 - \left( \alpha + \frac{1}{3} \right) (kh)^2 \right] u_0 = 0$$

(2.75)

$$g k a_0 - \omega \left[ 1 - \alpha (kh)^2 \right] u_0 = 0$$

(2.76)

Letting the discriminant vanish for a nontrivial solution gives the dispersion relation as:

$$C^2 = \frac{\omega^2}{k^2} = g h \left[ 1 - \left( \alpha + \frac{1}{3} \right) (kh)^2 \right] \frac{1}{1 - \alpha (kh)^2}$$

(2.77)

In which C = phase velocity. This relation is similar to Witting’s (1984) second order dispersion relation, which was also used by Madsen (1991). However, the new form of Nowgu’s Boussinesq-type equations presented here are quite different from those of Madsen (1991). The previous authors start off with a desired linear dispersion relation and simply add an extra term to the momentum equations to produce the desired characteristics. The depth-averaged velocity is still used as velocity variable and their equations are applicable in constant depth only. Nwogu’s equations are derived from the classical Euler equations without assuming any dispersion relation a priori. The velocity at an arbitrary distance from the still water level is used as velocity variable and the equations are applicable in water of varying depth. The exact linear dispersion
relation for linear waves is given by equation (2.47). The phase speeds for different values of \( \alpha \), normalized with respect to the linear-theory phase speed [(2.47)], are plotted as a function of relative depth in Fig. 2.1.

![Figure 2.1](image)

**Figure 2.1** Comparison of Normalized Phase Speeds for Different Values of \( \alpha \)

And from equation (2.75), one can derive a velocity magnitude:

\[
    u_0 = \frac{\omega a_0}{k h \left[ 1 - \left( \alpha + \frac{1}{3} \right)(k h)^2 \right]} 
\]  

(2.78)

Some more notation is introduced to simplify the following algebra:

\[
    P = 1 - \left( \alpha + \frac{1}{3} \right)(k h)^2 
\]  

(2.79)
\[ Q = 1 - \alpha (kh)^2 \]  

Hence:

\[ C^2 = gh \frac{P}{Q} \]  

\[ \frac{\partial P}{\partial k} = -2 \left( \alpha + \frac{1}{3} \right) (kh)h \]  

\[ \frac{\partial Q}{\partial k} = -2 \alpha (kh)h \]  

Differentiating equation (2.81) with respect to k:

\[ 2C \frac{\partial C}{\partial k} = gh \frac{\partial P}{\partial k} - \frac{P}{Q^2} \frac{\partial Q}{\partial k} \]  

\[ 2C^2 \frac{\partial C}{\partial k} = C gh \frac{\partial P}{\partial k} - \frac{P}{Q^2} \frac{\partial Q}{\partial k} \]  

\[ gh \frac{P}{Q} \frac{\partial C}{\partial k} = C gh \frac{-Q \left( \alpha + \frac{1}{3} \right) (kh)h + p \alpha (kh)h}{Q^2} \]  

\[ k \frac{\partial C}{\partial k} = C \frac{-\left( 1 - \alpha (kh)^2 \right) \left( \alpha + \frac{1}{3} \right) (kh)^2 + \left( 1 - \left( \alpha + \frac{1}{3} \right) (kh)^2 \right) \alpha (kh)^2}{PQ} \]  

\[ k \frac{\partial C}{\partial k} = C \frac{-\left( 1 - \alpha (kh)^2 \right) \left( \alpha + \frac{1}{3} \right) (kh)^2 + \left( 1 - \left( \alpha + \frac{1}{3} \right) (kh)^2 \right) \alpha (kh)^2}{PQ} \]  

\[ k \frac{\partial C}{\partial k} = C \frac{\left( kh \right)^2}{3PQ} \]  

The group velocity is derived from the definition (2.50) and the phase velocity (2.81)

\[ C_g = C \left( 1 - \frac{\left( kh \right)^2}{3PQ} \right) \]  

(2.86)
The normalized group velocities for different values $\alpha$ are plotted as a function of $kh$ in Figure 2. The group velocities are observed to deviate more rapidly from the exact relation than the phase velocity.

**Figure 2.2** Comparison of Normalized Group Velocities for Different Values of $\alpha$

In intermediate water depth with $kh<2.0$, the differences between the phase velocity and group velocity of the Nwogu’s Boussinesq model and Airy theory become negligible.

Following a similar argument to that used for the linear wave theory in Section (2.2.3), the shoaling gradient coefficient can be derived for the extended Boussinesq equations. Key points are the formation of the derivative of the group velocity component $n(kh)$ with respect to $x$:
\[
\frac{1}{n} \frac{\partial n}{\partial x} = T_i \frac{1}{kh} \frac{\partial kh}{\partial x}
\]

(2.87)

Where:

\[
T_i = -2 \left( \frac{(kh)^3}{PQ} \left( 1 - \alpha \left( \alpha + \frac{1}{3} \right) (kh)^4 \right) \right)
\]

(2.88)

Where:

\[
P = 1 - \left( \alpha + \frac{1}{3} \right) (kh)^2
\]

(2.89)

\[
Q = 1 - \alpha (kh)^2
\]

(2.90)

And the relationship between the wave number gradient and the depth gradient:

\[
\frac{1}{k} \frac{\partial k}{\partial x} = \left( \frac{1}{T_2} - 1 \right) \frac{1}{h} \frac{\partial h}{\partial x}
\]

(2.91)

In which

\[
T_2 = -\frac{2}{PQ} \left[ 1 + (\alpha + 1/3)(\alpha (kh)^2 - 2(kh)^2) \right]
\]

(2.92)

This finally leads to:

\[
s = \frac{1}{2} \left( 1 + \frac{T_1}{T_2} - 1 \right)
\]

(2.93)
From Figure 2.3, we can find that when \( \text{kh} < 1.6 \), extended Boussinesq equations can simulate linear shoaling problem correctly. The shoaling gradient difference between linear wave theory and extended Boussinesq equation will become larger and larger as the value of \( \text{kh} \) increases.

**Figure 2.3** Comparison of shoaling gradient between linear wave theory and extended Boussinesq equation.

From Figure 2.3, we can find that when \( \text{kh} < 1.6 \), extended Boussinesq equations can simulate linear shoaling problem correctly. The shoaling gradient difference between linear wave theory and extended Boussinesq equation will become larger and larger as the value of \( \text{kh} \) increases.
Chapter 3

An Algorithm for the One-dimensional Extended Boussinesq Equations

3. 1 Introduction

The aim of this chapter is to investigate the implementation of staggered scheme. We will demonstrate clearly finite difference spatial discretisation methods and the adaptive time integration scheme for the one-dimensional extended Boussinesq equations.

The general time integration method will be described clearly in section 3.2. Nwogu’s extended Boussinesq equation system will be reviewed in Section 3.3. In this section it will be demonstrated that Nwogu’s extended Boussinesq equations can be rewritten in a form structurally similar to the general form that can be solved by the higher-order Adams predictor-corrector schemes. The spatial approximation of the extended Boussinesq equations will be considered in Section 3.4. Appropriate boundary conditions for the equations and their numerical formulations will be presented in Section 3.5. Based on the novel form of the differential equations adopted in this work, the new formulations of the inflow, outflow and wall boundary conditions will be derived. Numerical experiments will be presented in Section 3.6, comparing the proposed staggered method with theoretical results, experimental results and the non-
staggered finite difference method. These results are used to validate the mathematical model as well as the numerical model and the boundary conditions.

3.2 Time integration

The higher-order Adams predictor-corrector schemes are derived here. A predictor-corrector method consists of a predictor step and a corrector step in each interval. The predictor estimates the solution for the new point, and then the corrector improves its accuracy. Predictor-corrector methods use the solutions for previous points instead of using intermediate points in each interval.

To explain the methods, let us consider an equispaced time interval and assume that the solution has been calculated up to time point n so that the values of $y$ and $y'$ on the previous time points may be used for the calculation of $y_{n+1}$.

Consider an ordinary differential equation

$$y' = f(y, t)$$  \hspace{1cm} (3.1)
To calculate \( y_{n+1} \) at \( t_{n+1} = t_n + h \) with a known value of \( y_n \), we integrate equation (3.1) over the interval \([t_n, t_{n+1}]\) as

\[
y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y, t) dt
\]  
(3.2)

Let us derive a third-order predictor by approximating \( y' = f(y, t) \) with a quadratic Lagrange interpolation polynomial fitted to \( f_n, f_{n-1}, f_{n-2} \):

\[
y'(z) = \frac{1}{2h^2} \left[ (z+h)(z+2h)y_n' - 2z(z+2h)y_{n-1} + z(z+h)y_{n-2} \right] + E(z)
\]  
(3.3)

Where \( z \) is a local coordinate defined by \( z = t - t_n \), and \( E(z) \) is the error of Lagrange interpolation. The error of the polynomial is

\[
E(z) = \frac{1}{3!} z(z+h)(z+2h)y^{(4)}(\xi), \quad t_{n-2} < \xi < t_{n+1}
\]  
(3.4)

Equation (3.2) can be rewritten in terms of the local coordinate \( z = t - t_n \) as

\[
y_{n+1} = y_n + \int_0^h y'(z) dz
\]  
(3.5)

Substituting equation (3.3) into equation (3.5) yields

\[
y_{n+1} = y_n + \frac{h}{12} \left[ 23y_n' - 16y_{n-1}' + 5y_{n-2}' \right] + O(h^4)
\]  
(3.6)

Equation (3.6) is called the third-order Adams-Bashforth predictor formula. The error of the equation (3.6) is attributable to equation (3.4) and is evaluated by integrating equation (3.4) in \([0, h]\), as follows

\[
O(h^4) = \frac{3}{8} h^4 y^{(4)}(\xi), \quad t_{n-2} < \xi < t_{n+1}
\]  
(3.7)

In deriving equation (3.6), notice that equation (3.3) has been used as an extrapolation. Extrapolation is less accurate than interpolation. Therefore equation (3.6) is used only
as a predictor. To derive a corrector formula, the interpolation Lagrange polynomial fitted to \( y' \) at points, \( n+1, n, n-1 \) and \( n-2 \) is

\[
y'(z) = \frac{1}{6h^3} \left[ -(z+h)z(z-h) \cdot y'_{n+2} + 3(z+2h)z(z-h) \cdot y'_{n+1} \\
- 3(z+2h)(z+h)(z-h) \cdot y'_{n} + (z+2h)(z+h)z \cdot y'_{n-1} \right] \quad (3.8)
\]

Where \( z \) is a local coordinate defined by \( z = t - t_n \), and \( E(z) \) is the error of Lagrange interpolation. The error of the polynomial is

\[
E(z) = \frac{1}{4!} (z-h)(z+h)(z+2h), \quad t_{n-2} < \xi < t_{n+1} \quad (3.9)
\]

Substituting equation (3.8) into equation (3.5) yields

\[
y_{n+1} = y_n + \frac{h}{24} \left( 9 y'_{n+1} + 19 y'_n - 5 y'_{n-1} + y'_{n-2} \right) + O(h^5) \quad (3.10)
\]

Equation (3.10) is called the fourth-order Adams-Moulton corrector formula. The error of the equation (3.10) is attributable to equation (3.9) and is evaluated by integrating equation (3.9) in \([0, h]\), as follows

\[
O(h^5) = -\frac{19}{720} h^5 y^{(5)}(\xi), \quad t_{n-2} < \xi < t_{n+1} \quad (3.11)
\]

Here in this thesis, all the time terms are calculated by equation (3.6) and equation (3.10).
3.3 The one-dimensional extended Boussinesq equation system

The Nwogu’s Boussinesq equation form in one-dimensional version,

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[ (h + \varepsilon \eta) u_a \right] + \mu^2 \frac{\partial}{\partial x} \left[ \left( \frac{z_a^2}{2} - \frac{h^2}{6} \right) h \frac{\partial^2 u_a}{\partial x^2} + \left( z_a + \frac{h}{2} \right) h \frac{\partial^2 (hu_a)}{\partial x^2} \right] = 0 \quad (3.12)
\]

\[
\frac{\partial u_a}{\partial t} + \frac{\partial}{\partial x} u_a + c \frac{\partial^2 u_a}{\partial x^2} + \mu^2 \left( \frac{z_a^2}{2} \frac{\partial^3 u_a}{\partial x^2 \partial t} + z_a \frac{\partial^2}{\partial x^2} \left( h \frac{\partial u_a}{\partial t} \right) \right) = 0 \quad (3.13)
\]

We can rewrite the above Boussinesq equation in the dimensional form

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[ (h + \eta) u_a \right] + \frac{\partial}{\partial x} \left[ \left( \frac{z_a^2}{2} - \frac{h^2}{6} \right) h \frac{\partial^2 u_a}{\partial x^2} + \left( z_a + \frac{h}{2} \right) h \frac{\partial^2 (hu_a)}{\partial x^2} \right] = 0 \quad (3.14)
\]

\[
\frac{\partial u_a}{\partial t} + g \frac{\partial}{\partial x} u_a + u_a \frac{\partial^2 u_a}{\partial x^2} + \frac{z_a^2}{2} \frac{\partial^3 u_a}{\partial x^2 \partial t} + z_a \frac{\partial^2}{\partial x^2} \left( h \frac{\partial u_a}{\partial t} \right) = 0 \quad (3.15)
\]

Let \( \frac{z_a}{h} = \beta \), we can get:

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \left[ (h + \eta) u_a \right] + \frac{\partial}{\partial x} \left[ \left( \frac{\beta^2}{2} - \frac{1}{6} \right) h \frac{\partial^2 u_a}{\partial x^2} + \left( \beta + \frac{1}{2} \right) h \frac{\partial^2 (hu_a)}{\partial x^2} \right] = 0 \quad (3.16)
\]

\[
\frac{\partial u_a}{\partial t} + g \frac{\partial}{\partial x} u_a + u_a \frac{\partial^2 u_a}{\partial x^2} + \frac{\beta^2}{2} h^2 \frac{\partial^3 u_a}{\partial x^2 \partial t} + \beta h \frac{\partial^2}{\partial x^2} \left( h \frac{\partial u_a}{\partial t} \right) = 0 \quad (3.17)
\]

We can define new functions: \( U, E, F \), which involve spatial derivatives of \( \eta, u, \) and \( u_t \) as followed:

\[
E(\eta, u) = -\left[ (h + \eta) u \right] + [\alpha_1 h^3 u_{xx} + \alpha_2 h^2 (hu)_{xx}] \quad (3.18)
\]
\[ U(u) = u + h[b_1 hu_{xx} + b_2 (hu)_{xx}] \]  \hspace{1cm} (3.19)

\[ F(\eta,u) = -g \eta_x - uu_x \]  \hspace{1cm} (3.20)

After some change, we can get:

\[ F(\eta,u) = -g \eta_x - \frac{1}{2} (u^2)_x \]  \hspace{1cm} (3.21)

Where

\[ \alpha_1 = \beta^2 / 2 - 1/6; \quad \alpha_2 = \beta + 1/2; \quad b_1 = \beta^2 / 2; \quad b_2 = \beta; \]

After the definition, equations (3.16) and (3.17) become:

\[ \eta_x = E(\eta,u) \]  \hspace{1cm} (3.22)

\[ [U(u)]_x = F(\eta,u) \]  \hspace{1cm} (3.23)
3. 4 Spatial discretization

![Figure 3.2 Staggered grid notation in one-dimensional](image)

The total water depth at the points of velocities, \((h + \eta)_{i+1/2}\), can get by using interpolating:

\[
(h + \eta)_{i+1/2} = \frac{1}{2}(h_i + h_{i+1}) + \frac{1}{2}(\eta_i + \eta_{i+1})
\]  
(3.24)

\((i = 1, 2, \cdots, m - 2, m - 1)\)

So we can get

\[
\left\{ \frac{\partial [(h + \eta)u]}{\partial x} \right\}_{2} = \frac{1}{24\Delta x} \left[ -23(h + \eta)_{3/2} u_{3/2} + 21(h + \eta)_{5/2} u_{5/2} + 3(h + \eta)_{7/2} u_{7/2} - (h + \eta)_{9/2} u_{9/2} \right]
\]  
(3.25)

\[
\left\{ \frac{\partial [(h + \eta)u]}{\partial x} \right\}_{i} = \frac{1}{24\Delta x} \left[ (h + \eta)_{i-3/2} u_{i-3/2} - 27(h + \eta)_{i-1/2} u_{i-1/2} + 27(h + \eta)_{i+1/2} u_{i+1/2} - (h + \eta)_{i+3/2} u_{i+3/2} \right]
\]  
(3.26)

\((i = 3, 4, \cdots, m - 2, m - 1)\)
For second order derivatives, we use three-point difference schemes

\[
(w_{xx})_i = \frac{1}{\Delta x^2} (2w_i - 5w_{i-1} + 4w_{i-2} - w_{i-3}) \tag{3.27}
\]

\[
(w_{xx})_i = \frac{1}{\Delta x^2} (w_{i-1} - 2w_i + w_{i+1}) \tag{3.28}
\]

\[(i = 2,3,4, \cdots, m-2, m-1)\]

\[
(w_{xx})_m = \frac{1}{\Delta x^2} (2w_{m-1} - 5w_{m-2} + 4w_{m-3} - w_{m-4}) \tag{3.29}
\]

Where \(w_{xx}\) is \(hu\) or \(u\).

So we can get

\[
E_i^n = -\frac{\partial}{\partial x} \left[\alpha_i h^3 u_{xx} + \alpha_2 h^2 (hu)_{xx}\right]_{i+1/2}^n - \frac{\partial}{\partial x} \left[\alpha_i h^3 u_{xx} + \alpha_2 h^2 (hu)_{xx}\right]_{i-1/2}^n \tag{3.30}
\]

We can get the first derivatives of \(\eta\) as followed

For the points of wave wavemaker

\[
\frac{\partial \eta}{\partial x}_{i+1/2} = \frac{-\eta_{i-1} - 9\eta_i + 9\eta_{i+1} + \eta_{i+2}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial x^3} \tag{3.31}
\]

For inside points

\[
\frac{\partial \eta}{\partial x}_{i+1/2} = \frac{\eta_{i-1} - 27\eta_i + 27\eta_{i+1} - \eta_{i+2}}{24} + \frac{9}{80} \frac{\partial^7 \eta}{\partial x^7}(\Delta x)^4 \tag{3.32}
\]

\[(i = 3,4, \cdots, m-2)\]
For first order derivatives of $u^2$, we use five-point difference schemes

$$\left( \frac{\partial u^2}{\partial x} \right)_{\text{s/2}} = -\frac{3u_{1/2}^2 - 10u_{3/2}^2 + 18u_{7/2}^2 - 6u_{9/2}^2 + u^2_{11/2}}{12\Delta x}$$ (3.33)

$$\left( \frac{\partial u^2}{\partial x} \right)_{i+1/2} = \frac{u_{i-3/2}^2 - 8u_{i-1/2}^2 + 8u_{i+3/2}^2 - u_{i+5/2}^2}{12\Delta x}$$ (3.34)

$$\left( \frac{\partial u^2}{\partial x} \right)_{m-1/2} = \frac{3u_{m+1/2}^2 + 10u_{m-1/2}^2 - 18u_{m-3/2}^2 + 6u_{m-5/2}^2 - u_{m-7/2}^2}{12\Delta x}$$ (3.35)

So we can get

$$F^n_{i+1/2} = -g \left( \frac{\partial \eta^n}{\partial x} \right)_{i+1/2} - \left( \frac{\partial u^2}{\partial x} \right)_{i+1/2}$$ (3.36)

According to the equation (3.6) we can get:

$$U^{n+1}_{i+1/2} = U^n_{i+1/2} + \frac{\Delta t}{12} \left[ 23F^n_{i+1/2} - 16F^{n-1}_{i+1/2} + 5F^{n-2}_{i+1/2} \right]$$ (3.37)

According to the equation (3.19) we can get:

$$U^{n+1}_{i+1/2} = u^{n+1}_{i+1/2}$$

$$+ h_{i+1/2} \left[ b_1 h_{i+3/2} + \frac{2u_{i+3/2}^2 - 3u_{i+1/2}^2 + u_{i-1/2}^2 + (hu)_{i+1/2}^2 - 2(hu)_{i+3/2}^2 + (hu)_{i-1/2}^2}{\Delta x^2} \right]$$ (3.38)

After some algebraic calculation, equation (3.38) becomes:

$$\left( \frac{b_1 h_{i+3/2}^2}{\Delta x^2} + \frac{b_2 h_{i+1/2}^2}{\Delta x^2} \right) u_{i+1/2}^{n+1} + \left( 1 - 2 \frac{b_1 h_{i+1/2}^2}{\Delta x^2} - 2 \frac{b_2 h_{i+3/2}^2}{\Delta x^2} \right) u_{i+1/2}^{n+1}$$

$$+ \left( \frac{b_1 h_{i+3/2}^2}{\Delta x^2} + \frac{b_2 h_{i+3/2}^2}{\Delta x^2} \right) u_{i+3/2}^{n+1} = U^{n+1}_{i+1/2}$$ (3.39)
We can rewrite equation (3.39) in matrix form
\[
\begin{bmatrix}
1 - 2\frac{b_1 h_1^2}{\Delta x^2} - 2\frac{b_1 h_2^2}{\Delta x^2} & \frac{b_1 h_1^2}{\Delta x^2} + \frac{b_2 h_1 h_2}{\Delta x^2} & \vdots & \frac{b_1 h_1^2}{\Delta x^2} + \frac{b_n h_{n-1} h_n}{\Delta x^2} \\
\frac{b_1 h_2^2}{\Delta x^2} + \frac{b_2 h_1 h_2}{\Delta x^2} & 1 - 2\frac{b_2 h_2^2}{\Delta x^2} - 2\frac{b_2 h_1 h_2}{\Delta x^2} & \vdots & \frac{b_2 h_2^2}{\Delta x^2} + \frac{b_{n-1} h_{n-1} h_n}{\Delta x^2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \frac{b_1 h_{n-1}^2}{\Delta x^2} + \frac{b_{n-1} h_{n-2} h_{n-1}}{\Delta x^2} & \vdots & 1 - 2\frac{b_{n-1} h_{n-1}^2}{\Delta x^2} - 2\frac{b_{n-1} h_{n-1} h_{n-2}}{\Delta x^2} \\
\end{bmatrix}
\begin{bmatrix}
\Delta \eta_1 \\
\Delta \eta_2 \\
\vdots \\
\Delta \eta_n \\
\end{bmatrix}
= 
\begin{bmatrix}
\eta_{n+1}^+ \\
\eta_{n+1}^- \\
\vdots \\
\eta_{n+1}^{(n+1)} \\
\end{bmatrix}
\]
\[(3.40)\]

After the predicted values of \(\{\eta, u\}_{i}^{n+1}\) are evaluated, we obtain the corresponding quantities of \(\{E, F\}_{i}\) at time levels \((n+1), (n), (n-1), (n-2)\), and apply the fourth-order Adams-Moulton corrector method.

\[
\eta_{i}^{n+1} = \eta_{i}^{n} + \frac{\Delta t}{24} \left[9E_{i}^{n+1} + 19E_{i}^{n} - 5E_{i}^{n-1} + E_{i}^{n-2}\right]
\]
\[(3.41)\]

\[
U_{i+1/2}^{n+1} = U_{i+1/2}^{n} + \frac{\Delta t}{24} \left[9F_{i+1/2}^{n+1} + 19F_{i+1/2}^{n} - 5F_{i+1/2}^{n-1} + F_{i+1/2}^{n-2}\right]
\]
\[(3.42)\]

The correct procedure is the same as the predict procedure.

The correct step is iterated until the error between two successive results reaches a required limit. The error is computed for each of the three dependent variables \(\eta, u, v\) and is defined as

\[
\Delta f = \frac{\sum_{i} \left| f_{i}^{n+1} - f_{i}^{(n+1)}\right|}{\sum_{i} \left| f_{i}^{n+1}\right|}
\]
\[(3.43)\]
where $f$ denotes any of the variables and $(\cdot)^*$ denotes the previous estimate. The corrector step is iterated if any of the $\Delta f$'s exceeds 0.001. The scheme typically requires no iteration unless problems arise at boundaries. Then the same procedure is applied to the next step.
3.5 Boundary conditions

At any boundary, certain physical conditions must be satisfied by the fluid velocities. Appropriate boundary conditions are also needed for the numerical model to run correctly. The examples shown in the following involve three types of lateral boundaries, which will be discussed here in sequence. These are (1) Wavemaker boundaries; (2) reflective vertical walls; (3) transmitting or absorbing boundaries.

3.5.1 Wavemaker Boundaries

In Section 2.4, it was shown that the extended Boussinesq equations could be applied at depths right up to the deep water limit \( kh=3.0 \). This allows waves to be introduced into the domain sufficiently far from the region of interest that boundary effects can be considered insignificant. If nonlinearity is small at these boundaries, i.e., the amplitude is small compared to the depth, the linearised equations will be a sufficiently accurate approximation to the problem and a linear wave profile can be introduced at the inflow boundary. For the numerical experiments considered in Section 3.6.1, a regular periodic wave is input at the boundary. For example a simple periodic wave

\[
\eta(x,t) = a \sin (kx - \omega t)
\]

(3.44)

with amplitude \( a \) wave number \( k \) and frequency \( \omega \), where \( k \) and \( \omega \) satisfy the linear dispersion relation for Nwogu’s extended Boussinesq equations. At these boundaries the velocity profile can be derived from the equation (2.75) by using the linearised equation system, equation (2.71).
This linear wave approximation will become inaccurate if the amplitude is significant compared to the depth at the inflow boundary. This can be accounted for by introducing a first order correction to the linear wave profile for nonlinearity. Considering a nonlinear perturbation of the free surface and velocity from the linearised solution,

\[ \eta = \eta_0 + \epsilon \eta_1 + O(\epsilon^2) \]  

\[ u = u_0 + \epsilon u_1 + O(\epsilon^2) \]

where the nonlinearity is parameterised by \( \epsilon \). The constant depth extended Boussinesq equations can be written as

\[ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [u_a (h + \eta)] + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_a}{\partial x^3} = 0 \]  

\[ \frac{\partial u_a}{\partial t} + u_a \frac{\partial u_a}{\partial x} + g \frac{\partial \eta}{\partial x} + \alpha h^2 \frac{\partial^3 u_a}{\partial x^2 \partial t} = 0 \]

At lowest order,

\[ \frac{\partial \eta_0}{\partial t} + h \frac{\partial u_0}{\partial x} + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_0}{\partial x^3} = 0 \]  

\[ \frac{\partial u_0}{\partial t} + g \frac{\partial \eta_0}{\partial x} + \alpha h^2 \frac{\partial^3 u_0}{\partial x^2 \partial t} = 0 \]

There is a linearised solution with \( \eta_0 \) and \( u_0 \) given by equations (3.44) and (3.45) respectively. To first order in \( \epsilon \),
\[
\frac{\partial \eta_i}{\partial t} + h \frac{\partial u_i}{\partial x} + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_i}{\partial x^3} = - \frac{\partial}{\partial x} \left( \eta_i u_0 \right) \tag{3.52}
\]

\[
\frac{\partial u_i}{\partial t} + g \frac{\partial \eta_i}{\partial x} + \alpha \cdot h^2 \frac{\partial^3 u_i}{\partial x^2 \partial t} = - \frac{\partial}{\partial x} \left( u_0^2 \right) \tag{3.53}
\]

Substituting the solutions \( \eta_0 \) and \( u_0 \) into the right hand sides of these equations gives,

\[
\frac{\partial \eta_i}{\partial t} + h \frac{\partial u_i}{\partial x} + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_i}{\partial x^3} = - \frac{\partial}{\partial x} \left( ab \sin^2 (kx - \omega t) \right) \tag{3.54}
\]

\[
\frac{\partial u_i}{\partial t} + g \frac{\partial \eta_i}{\partial x} + \alpha \cdot h^2 \frac{\partial^3 u_i}{\partial x^2 \partial t} = - \frac{\partial}{\partial x} \left( \frac{b^2}{2} \sin^2 (kx - \omega t) \right) \tag{3.55}
\]

Where, \( b = \frac{\alpha a}{kh\left[1 - (\alpha + 1/3)(kh)^2\right]} \)

After some algebra calculation, equations (3.55) and (3.56) become

\[
\frac{\partial \eta_i}{\partial t} + h \frac{\partial u_i}{\partial x} + \left( \alpha + \frac{1}{3} \right) h^3 \frac{\partial^3 u_i}{\partial x^3} = -abk \sin (2(kx - \omega t)) \tag{3.56}
\]

\[
\frac{\partial u_i}{\partial t} + g \frac{\partial \eta_i}{\partial x} + \alpha \cdot h^2 \frac{\partial^3 u_i}{\partial x^2 \partial t} = -\frac{b^2 k}{2} \sin (2(kx - \omega t)) \tag{3.57}
\]

A solution of the form,

\[
\eta_i(x,t) = a_i \cos (2(kx - \omega t)) \tag{3.58}
\]

\[
u_i(x,t) = b_i \cos (2(kx - \omega t)) \tag{3.59}
\]

will satisfy this equation system. Substituting these solutions into equations (3.58) and (3.59) leads to the following expressions for \( a_i \) and \( b_i \).
\[ a_i = \frac{1}{g} \left( \frac{\omega}{k} b_i \left( 1 - 4\alpha (kh)^2 \right) - \frac{b_i^2}{4} \right) \]  

(3.60)

\[ b_i = \frac{\frac{ab}{2} g + \frac{b_i^2 \omega}{4}}{\frac{\omega^2}{k^2} \left( 1 - 4\alpha (kh)^2 \right) - gh \left( 1 - 4 \left( \alpha + 1/3 \right) (kh)^2 \right)} \]  

(3.61)

The wave profile (3.46) will have a slightly sharper peak and a broader, slightly raised trough, and is the first approximation to a more general periodic wave form known as a Cnoidal wave. Note that this modified wave involves a wavelength of half the primary wavelength and hence accurate modeling will require that this is properly resolved, leading in general to twice the resolution of the original wave.

### 3.5.2 Reflective Boundaries

When the wave arrives at a solid wall where the physical boundary condition is that of impermeability, the wave will be reflected. For a general reflective boundary with an outward normal vector \( n \), we would anticipate on physical grounds that the kinematic boundary condition would be completely specified by the statement

\[ u \cdot n = 0; \quad x \in \partial \Omega \]  

(3.62)

Where \( n \) is the normal to the wall at that point; \( \Omega \) = the fluid domain; \( \partial \Omega \) = the boundary; and \( x \) = a position in the domain; which in one dimension reduces to

\[ u = 0 \]  

(3.63)

A boundary condition on the free surface can be derived from a conservation argument (Ge Wei and T. Kirby 1995). Integrating the free surface equation (3.14) over the spatial domain, and using the divergence theorem on the spatial derivatives;
\[ \int_{\Omega} \frac{\partial \eta}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial m}{\partial x} \, d\Omega = 0 \]  
\quad \text{(3.64)}

\[ \frac{\partial}{\partial t} \left( \int_{\Omega} \eta \, d\Omega \right) + \int_{\Gamma} \left( \frac{\partial P}{\partial x} + \frac{\partial m}{\partial x} \right) \cdot n \, d\Gamma = 0 \]  
\quad \text{(3.65)}

Where, the first term is the total excess volume in the domain. If the domain is completely enclosed by impermeable walls, we require that the rate of change of the excess volume be zero. And \[ P = (h + \eta)u \]  
\quad \text{(3.66)}

And, where \( \Gamma \) is the domain boundary, and \( n \) is the outward normal vector. At a solid boundary, the amount of mass in the system is conserved and hence the time derivative term in expression (3.65) is zero. For the boundary integral to be equal to zero, we require \( p + m = 0 \) on the boundary. This is satisfied by the wall boundary condition on the velocity (3.63) from the definition of \( p \) (3.66) and the additional requirement on the auxiliary variable \( m \),

\[ m = 0 \]  
\quad \text{(3.67)}

at the wall. In constant water depth, we can get \[ u_{xx} = 0 \]  
\quad \text{(3.68)}
3.5.3 Absorbing Boundaries

The absorbing boundary should absorb all energy arriving at the boundary from the fluid domain. Treatment of this boundary is a problem of major interest in modeling community, and we use some fairly well-established techniques, which have been used by Wei et al. (1999), for the cases considered here.

A perfect radiation boundary should not allow wave reflection to occur. For the case where the wave phase speed \( C \) and the propagation direction \( \theta \) at the boundary are known, the radiation condition is

\[
\eta_t + c \eta_x \cos \theta = 0
\]

(3.69)

However, in most cases there is no single phase velocity \( C \). Furthermore, in two-dimensional applications, the wave direction \( \theta \) is generally not known a priori. To solve the second problem, approximations to the perfect radiation condition are made. For wave propagation with the principal direction close to the x-axis, the approximate radiation boundary can be written (Engquist and Majda 1977)

\[
\eta_{tt} + c \eta_{tt} - \frac{c^2}{2} \eta_{yy} = 0
\]

(3.70)

which corresponds to the imposition of a parabolic approximation on the outgoing wave. To treat the first problem, phase speed \( c \) is specified by long-wave limit

\[
c = \sqrt{gh}
\]
The aforementioned approximate radiation condition inevitably introduces wave reflection along the boundaries, and can eventually cause the model to blow up. To reduce the reflection, a damping layer is applied to the computing domain. Damping terms are added to the momentum equations as

\[ U_i = F(\eta, u) - w_1(x)u - w_2(x)u_{xx} \quad (3.71) \]

where the damping terms with \( u \) is called “Newtonian cooling” and this with second-order derivative is analogous to linear viscous terms in Navier-Stokes equation (Israeli and Orszag 1981). The damping coefficients \( w_1(x) \) and \( w_2(x) \) are defined as

\[
w_1(x) = \begin{cases} 
0; & x < x_s \\
\alpha_1 \omega f(x); & x > x_s 
\end{cases} \quad (3.72)
\]

\[
w_2(x) = \begin{cases} 
0; & x < x_s \\
\alpha_2 \omega f(x); & x > x_s 
\end{cases} \quad (3.73)
\]

Where \( \alpha_1 \) and \( \alpha_2 \) = constants to be determined for the specific running; \( \omega \) = frequency of wave to be damped; \( x_s \) = starting coordinate of damping layer (the computing domain is from \( x = 0 \) to \( x = x_s \)); \( \nu \) = viscous coefficient; and \( f(x) \) is expressed as

\[
f(x) = \frac{\exp\left(\frac{x - x_s}{x_i - x_s}\right)^n - 1}{\exp(1) - 1} \quad (3.74)
\]

The width of the damping layer (i.e., \( x_i - x_s \)) is usually taken to be two or three times the wavelength. Numerical experiments show that the addition of damping layer combined with radiation boundary conditions works much better than radiation conditions alone.
3.6 Numerical experiments

3.6.1 A periodic wave with constant depth

The first test case is a periodic wave of wavelength 4.96m, wave period $T=2.0s$ and amplitude $a=0.002$ m propagating into an initially undisturbed region of constant depth $h=0.86m$. The incident wave has the form,

$$
\eta(x,t) = a \sin( kx - \omega t )
$$

The time step is chosen to be $\Delta t = 0.04s$. The simple linear inflow boundary condition described in Section 3.5.1 is applied at $x=0.0m$. A 60 point wide absorbing sponge layer cover the region from $x=88$ m to $x =100m$. The spatial domain $x \in [0,100m]$ is uniformly discretised with 500 grids, e.g. $\Delta x = 0.2m$. The Courant number is about 0.5 in this case.

![Figure 3.3 Linear wave propagates in constant water depth](image-url)
The second test case is a periodic wave with wave period $T=2.0\text{s}$ and amplitude $a = 0.02 \text{ m}$ propagating into an initially undisturbed region of constant depth $h=0.86\text{m}$. This wave has some nonlinear properties. We will test it by using two types inflow wave form: one is linear wave; another is nonlinear wave simulating by equations (3.58) and (3.59).

The time step is chosen to be $\Delta t = 0.04\text{s}$. The simple linear inflow boundary condition described in Section 3.5.1 is applied at $x=0.0\text{m}$. The spatial domain $x \in [0,140\text{m}]$ is uniformly discretised with 700 grids, e.g. $\Delta x = 0.2\text{m}$. The Courant number is 0.5 in this case. Figure 3.4 shows the free surface profiles in space. The nonlinear corrections appear to have eliminated the secondary oscillation from the solution. And there a steady wave profile with constant peak amplitude when the inflow wave boundary is given the weakly nonlinear wave, whereas if the input wave is linear wave, the profile shows the wave is unsteady.

Figure 3.4 Weakly nonlinear wave propagates in constant water depth
3.6.2 Linear shoaling

In order to verify the linear shoaling properties of the new extended Boussinesq model and the accuracy of the staggered grid system, the following test case has been studied: At the seaward boundary the water depth is 13.0 m. The bottom is flat for the first 10 m distance from the boundary while it has a constant slope of 1/50 from 10 m to 650 m distance. Finally from 650 m to 730 m the bottom is flat again with a water depth of 0.2 m. All nonlinear terms in the Boussinesq equation are included, which is different with Madsen (1992). Sponge boundary is used in left side which is from 690 to 730. The numerical simulation is performed with time step $\Delta t = 0.04s$ and grid size $\Delta x = 0.5m$, and we can get Courant number is about 0.78.

![Simulation of linear shoaling from deep water to shallow water](image)

**Figure 3.5** Simulation of linear shoaling from deep water to shallow water

The propagation of solitary wave over a long distance provides a good test of the stability and conservative properties of the basic numerical scheme. To apply this test,

For convenience, we rewrite Nwogu’s equations in dimensionless form for one-dimensional horizontal flow in constant depth as

\[
\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \delta \frac{\partial}{\partial x} (\eta u) + \mu^2 \left( \alpha + \frac{1}{3} \right) \frac{\partial^3 u}{\partial x^3} = 0
\]

\[
\frac{\partial u}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial x} + \mu^2 \alpha \frac{\partial^3 u}{\partial x^3 \partial t} = 0
\]

Where \( \mu = k h \) and \( \delta = H / h \) (k is wave number, h the water depth, H the amplitude) are dimensionless parameters that represent the effects of dispersion and nonlinearity, respectively. \( \alpha \) is defined by

\[
\alpha = \frac{1}{2} z_a^2 + z_a
\]

Instead of the velocity \( u \), we use the velocity potential \( \phi \) (such that \( u = \phi_x \)) as a variable. Then the forgoing two equations become

\[
\eta_x + \phi_{xx} + \delta (\eta \phi_x)_x + \mu^2 \left( \alpha + \frac{1}{3} \right) \phi_{xxxx} = 0 \tag{3.76}
\]

\[
\phi_t + \eta + \frac{\delta}{2} \phi_x + \mu^2 \alpha \phi_{xx} = 0 \tag{3.77}
\]

Substituting \( \eta \) from (3.77) into (3.76) and retaining terms that are consistent with the ordering in the Boussinesq equations, we have

\[
- \phi_t + \phi_{xx} - \delta (2 \phi_x \phi_{xx} + \phi \phi_{xx}) + \mu^2 \left[ (\alpha + 1/3) \phi_{xxxx} - \alpha \phi_{xxxxx} \right] = O(\delta^2, \delta \mu^2) \tag{3.78}
\]
The truncated terms on the right-hand side of (3.78) are

\[
\frac{3}{2} \delta^2 \phi_x \phi_{xx} + \phi \mu^2 \alpha (\phi_{xxx} \phi_x)_x
\]  
(3.79)

And truncated terms are responsible for the fact that the analytic solitary waves differ by a small amount from their numerical counterparts, even in the limit of small step size. Equation (3.78) is transformed into an ordinary differential equation by introducing \( \xi = x - Ct \). Then \( \phi_x = \phi' \) and \( \phi_i = -C \phi' \), where primes denote differentiation.

\[
(1 - C^2) \phi'' + 3 \delta C \phi' \phi'' + \mu^2 (\alpha + 1/3 - \alpha C^2) \phi''' = 0
\]  
(3.80)

Integrating the above equation once results in

\[
(1 - C^2) \phi' + \frac{3}{2} \delta C (\phi')^2 + \mu^2 (\alpha + 1/3 - \alpha C^2) \phi'' = G_i
\]  
(3.81)

Multiplying (3.81) by \( 2 \phi'' \) and integrating again yields

\[
(1 - C^2) (\phi')^2 + \delta C (\phi')^3 + \mu^2 (\alpha + 1/3 - \alpha C^2) (\phi'')^2 = 2G_i \phi' + G_2
\]  
(3.82)

The integration constants \( G_1 \) and \( G_2 \) are zero for solitary waves since \( \phi' = \phi'' = \phi''' = 0 \) as \( |\xi| \to \infty \). Assuming the solution form of \( \phi' \) to be

\[
\phi' = A \sec h^2 (B \xi)
\]  
(3.83)

Substituting (3.83) into ((3.82) yields

\[
A = \frac{C^2 - 1}{\delta C}
\]  
(3.84)

\[
B = \left[ \frac{C^2 - 1}{4 \mu^2 (\alpha + 1/3 - \alpha C^2)} \right]
\]  
(3.85)

From equation (3.77), we have

\[
\eta = A_1 \sec h (B \xi) + A_2 \sec h^4 (B \xi)
\]  
(3.86)
Where

\[
A_1 = \frac{C^2 - 1}{3\delta(\alpha + 1/3 - \alpha C^2)} \tag{3.87}
\]

\[
A_2 = \frac{\left(C^2 - 1\right)(\alpha + 1/3 + 2\alpha C^2)}{2\delta C^2(\alpha + 1/3 - \alpha C^2)} \tag{3.88}
\]

The corresponding dimensional expressions for A, B, A1, and A2 are

\[
A = \frac{C^2 - gh}{C} \tag{3.89}
\]

\[
B = \left[\frac{C^2 - gh}{4(\alpha + 1/3)gh^3 - \alpha h^2 C^2}\right] \tag{3.90}
\]

\[
A_1 = \frac{C^2 - gh}{3(\alpha + 1/3)gh - \alpha C^2}h \tag{3.91}
\]

\[
A_2 = \frac{(C^2 - gh)^2((\alpha + 1/3)gh + 2\alpha C^2)}{2gh C^2((\alpha + 1/3)gh - \alpha C^2)}h \tag{3.92}
\]

The horizontal velocity \(u\) and surface elevation \(\eta\) of a solitary wave are

\[
u = A \sec h^2\left[B(x - Ct)\right] \tag{3.93}
\]

\[
\eta = A_1 \sec h^2\left[B(x - Ct)\right] + A_2 \sec h^4\left[B(x - Ct)\right] \tag{3.94}
\]

The analytical solution is used to specify the incident-wave boundary condition for the numerical model. The model was used to investigate solitary-wave propagation in constant water depth of 0.5 m over a horizontal distance of 400 with \(dx = 0.2m\), and \(dt = 0.05\) (s). The Courant number is about 0.55. The wave is initially at the origin and is propagated for 160 (s) over an undisturbed depth of 0.5m. Figure 3.6 shows the spatial variation of a solitary wave with amplitude 0.05 m (\(\delta = 0.1\)) at various time steps. It was found that the solitary wave exhibited a slight phase error from Figure 3.7, with the numerical solutions slightly behind the theoretical solution. Figure 3.8 shows the
spatial variation of a solitary wave with amplitude 0.05 m ($\delta = 0.1$) at various time steps.

![Spatial profile of solitary wave](image)

**Figure 3.6** Spatial profile of solitary wave evolving in water of constant depth ($h=0.5$ m) with $\delta = 0.1$

The results indicate that the initial wave pulses specified according to the theory in (3.52) and (3.53) undergo evolution at the start of the wave channel, with the result that a slightly higher solitary wave is formed together with a small dispersive tail. The amplitude of the tail and the initial deviation in solitary-wave height both increase with increasing initial wave height. This result is partially because the fourth-order equation used to develop the analytical solution is only asymptotically equivalent to the model being solved numerically, so that the wave being input at the boundary of the numerical model does not correspond exactly to a solitary waveform as predicted by the model.
Figure 3.7 Spatial profile of solitary wave evolving in water of constant depth (h=0.5 m) with $\delta = 0.1$

Figure 3.8 Spatial profile of solitary wave evolving in water of constant depth (h=0.5 m) with $\delta = 0.3$
3.6.4 Periodic wave train passing a submerged breakwater

Submerged breakwaters are usually deployed in the coastal area to reduce the wave energy transmission. When a wave train passes through a submerged breakwater, it normally experiences dramatic changes of waveform and significant nonlinear energy transfer among different wave modes. Flow separations can be induced if the breakwater slope is steep. Under such circumstance, the hydrostatic pressure assumption becomes invalid. Beji and Battjes (1993, 1994) investigated this problem experimentally and numerically. They found that very poor prediction could result if the dispersion terms, which reflect the influence of non-hydrostatic pressure, are not properly modeled in the Boussinesq equations.

In this section, we attempt to simulate a similar problem. A regular wave train that has the wave height of 0.04m and wave period of 2.86 s was sent from the left boundary (x=0) by using the inflow boundary introduced in section 3.5.1. The absorbing boundary was installed on the right side of wave flume in the experiment but was replaced by a radiation boundary in constant water depth in the numerical simulation. The grid size $\Delta x$ is 0.03m, and time step $\Delta t$ is 0.01 s. The Courant number is about 0.89. The topograph is shown in figure 3.9. Totally 8 wave gauges were deployed at x=20.44m, 24.04m, 26.04m, 28.04m, 30.44m, 33.64m, and 41.04m, respectively.
Figure 3.9 Experimental setup of a periodic wave train passing a submerged breakwater.

Figure 3.10 shows comparisons between numerical results and gauge data for free surface displacement at the wave gauge locations. From $x=20.04$ m to $x=24.04$ m where wave starts to climb to slope, the gradually increasing wave sleeping due to shoaling effect is observed. At $x=26.04$, 28.04 m and 30.44 m, where the wave rides over the top of the breakwater, the growth of secondary wave becomes apparent. The numerical model captures this process accurately, though some details of wave signature do not completely agree between numerical results and experimental data.
Figure 3.10 Comparisons of free surface displacement at the last eight wave gauge locations between numerical results (solid lines) and experimental data (circles).
Behind the breakwater, the secondary wave mode gains energy from the main wave mode and the effective wavelength becomes shorter. This can be seen at the last three gauge location. Generally, the prediction of wave transformation in this region is of the most difficulty because of the complicated flow separation and nonlinear wave energy transfer. Nevertheless, the overall agreements between the numerical results and experimental data are still very encouraging. This demonstrates that the model has the capability of simulating complicated problems of wave-structure interaction.

3.6.5 Water sloshing in a confined container

In this test, water sloshing in a confined container with infinite length in the y-direction is simulated. The length of the container in the x-direction is \( W = 10 \text{m} \). Thus, the problem is basically two-dimensional. The free surface of the fluid in the container has the initial slope of \( s = \tan \theta = 0.0002 \) with the still water depth of \( h = 0.2 \text{m} \). Once the fluid begins to move under gravity, there exist an infinite number of standing wave modes in the container. If we neglect the viscous and nonlinear effect, the motion of the fluid could be approximated by the linear wave theory, which gives the free surface displacement as the function of \( x, t \) and \( s \) as follows,

\[
\eta = \sum_{n=1}^{\infty} A_n \sin(k_n x) \cos(\bar{\omega}_n t) \tag{3.95}
\]

where \( k_n = (n\pi)/W \) (wave number of the nth mode) and \( \bar{\omega}_n = \sqrt{gk_n \tanh(k_n h)} \) (frequency of the nth mode), and \( A_n = sW/(n^2\pi^2)\left[4\sin(n\pi/2) - 2\sin(n\pi)\right] \) (wave amplitude of the nth mode).
Figure 3.11 Comparisons of numerical results and analytical solution for water sloshing in a confined container in the first period
Figure 3.12 Comparisons of numerical results and analytical solution for water sloshing in a confined container in different periods
In the numerical simulation, the domain is discretized by 200 uniform grids in x-direction and \( \Delta x = 0.05m \). A constant \( \Delta t = 0.02 \) s is used to carry out the computation. The Courant number is about 0.56 in this case. Since the leading mode \((n=1)\) of the standing wave has the wave period \( T = 2\pi / \bar{w}_1 = 14.2 \) s, the entire simulation covers about five wave periods for the leading mode. In Figure 3.11, comparisons between numerical results and analytical solutions based on equation (3.95) are presented at \( t/T \sim 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 3.0, 3.1, 5.0 \). It is noted that the analytical curves are based on the inclusion of the first 100 modes and the further increase of \( n \) will change little of the solution. The first six frames in Figure 3.12 correspond to the free surface variation during one wave period for leading mode and the last two are present to demonstrate how longer time computation behaves. From the comparisons shown in Figure 3.11, the numerical results agree very well with the analytical solution during the first wave. A little larger discrepancy appears in the comparisons for the longer time computation. This might be caused by the accumulated errors in the numerical model. The overall comparisons, however, are fairly good, indicating that the mode can predict the free surface location accurately.

The model is further tested for the conservation of total mass and energy. At \( t/T = 0.0 \), the total fluid in the tank as \( M_0 \). In figure 3.13, the time histories of mass \( M \) normalized by \( M_0 \) are shown. It is observed that the mass is perfectly conserved within the entire computation. The potential and kinetic energy alternate during the computation with the total energy nearly conserved. The total energy decay is less than 2 \%, which is acceptable for most numerical studies.
Figure 3.13 Comparisons of time series of relative error of water volume $M$
Chapter 4

An Algorithm for the Two-dimensional Extended Boussinesq Equations

4.1 Introduction

In this chapter, the staggered scheme developed for Nwogu’s one-dimensional Boussinesq equation system is extended to the two-dimensional form of the equations, representing the most general form of the equations to be considered here. The extended Boussinesq equations as derived by Nwogu are presented in section 5.2.

There are many two-dimensional time domain simulations for finite difference models of the original Boussinesq equation system. Abbot et al (1978) developed a finite difference method for the original Boussinesq equations, where spatial terms were differenced to second order accuracy and corrections added in to account for any non-physical dispersion or diffusion produced by the scheme. This scheme was probably the first Boussinesq model to be used as an engineering tool and an application to a real harbour geometry was reported in this work. Hauguel (1980) presented a finite difference scheme using a high order characteristic type method for the convection terms coupled with a three stage difference scheme. An example of this model was presented to apply to a harbour geometry. Smallman and Cooper (1989) used a predictor-corrector time integration scheme and second order spatial differences to investigate set-down in harbours.
More recently, there have been several finite difference methods presented for extended Boussinesq equation systems. Madsen et al (1991) presented a finite difference scheme for their constant depth extended Boussinesq equations. They extended the difference scheme of Abbot et al (1984) to these equations and to the variable depth equation system in a subsequent publication (1992). Nwogu (1993) applied a similar difference scheme to his extended equations. This was used to investigate nonlinear interactions of irregular multidirectional waves. Wei and Kirby (1995) extended their finite difference model of Nwogu’s extended equations and compared their results with some standard experimental test cases. The accuracy of this model was confirmed in further work by Zang et al (1996), and more recently in the work of Skotner and Apelt (1999) who applied this model to a fictitious harbour geometry. A. Schroter et al (1994) applied a finite difference method, which was similar in principle to that of Abbot et al (1978), to another set of extended Boussinesq equations. This model was used to simulate the wave disturbances in a real harbour geometry, and reported in a separate publication (A. Schroter et al,1994). Beji and Nadaoka (1996) applied a three-level time centered scheme and centered finite differences for their extended equations. This method was validated with some standard test problems.
4.2 The two-dimensional extended Boussinesq equation system

Nwogu’s two-dimensional extended Boussinesq equations (Nwogu, 1993) can be derived from the three-dimensional incompressible, irrotational Euler equations in a similar manner to that presented in Section 2.2.2.

\[
\eta_t + \nabla \cdot \left[ (h + \eta) u_\alpha \right] + \nabla \left( \frac{z_\alpha^2}{2} - \frac{h^2}{6} \right) h \nabla \cdot u_\alpha + \left( \frac{z_\alpha}{2} + \frac{h}{2} \right) h \nabla \left[ \nabla \cdot (h u_\alpha) \right] = 0 \tag{4.94}
\]

\[
u_{tt} + g \eta + (u_\alpha \cdot \nabla) u_\alpha + \frac{\mu^2}{2} \left( \frac{z_\alpha^2}{2} \nabla \cdot u_\alpha \right) + z_\alpha \nabla \left[ \nabla \cdot (h u_\alpha) \right] = 0 \tag{4.95}
\]

Where \( \eta \) = surface elevation; \( h \) = local water depth; \( u_\alpha = (u, v) \) = the horizontal velocity at an arbitrary depth \( z_\alpha \); The two-dimensional vector differential operator \( \nabla \) is defined by

\[
\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \tag{4.96}
\]

and \( g \) = the gravitational acceleration. These equations are statements of conservation of mass and momentum, respectively. Compared to the Boussinesq equations based on depth-averaged velocity (referred to hereafter as standard Boussinesq equations) derived by Peregrine (1967), there is an additional dispersive term in the continuity equation, and the coefficients of dispersive terms in the momentum equations are different. These differences arise due to the choice of using the velocity. As was shown by Nwogu (1993), it is these differences that improve the linear dispersive properties of the model and make new form of equations usable in regions with
relatively deep water. And these characters make it more difficult to solve these equations.

We can make some change so as to make them easy to solve. Therefore, the equations (4.1) and (4.2) become

\[
\eta + \frac{\partial}{\partial x} \left[ (h + \eta)u \right] + \frac{\partial}{\partial y} \left[ (h + \eta)v \right] \\
+ \frac{\partial}{\partial x} \left[ a_1 h \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + a_2 h^2 \left( \frac{\partial^2 (hu)}{\partial x^2} + \frac{\partial^2 (hv)}{\partial x \partial y} \right) \right] \\
+ \frac{\partial}{\partial y} \left[ a_1 h \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + a_2 h^2 \left( \frac{\partial^2 (hu)}{\partial x \partial y} + \frac{\partial^2 (hv)}{\partial y^2} \right) \right] = 0
\]  

(4.97)

\[
\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + b_1 h \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \\
+ b_2 h \left( \frac{\partial^2 (hu)}{\partial x \partial y} + \frac{\partial^2 (hv)}{\partial y^2} \right) = 0
\]

(4.98)

\[
\frac{\partial v}{\partial t} + g \frac{\partial \eta}{\partial y} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + b_1 h \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) \\
+ b_2 h \left( \frac{\partial^2 (hu)}{\partial x \partial y} + \frac{\partial^2 (hv)}{\partial y^2} \right) = 0
\]

(4.99)

The constants \( a_1, a_2, b_1 \) and \( b_2 \) are given by

\[ a_1 = \beta^2 / 2 - 1/6; \quad a_2 = \beta + 1/2; \quad b_1 = 1/6; \quad b_2 = \beta \]

where \( \beta = z_a / h \). Equation (4.4) is the continuity equation and equation (4.5) is the momentum equation in x-direction and equation (4.6) is the momentum equation in y-direction.

The two-dimensional nonlinear convection terms in the velocity equations (4.5) and (4.6) are expressed in a form that can not be directly rewritten in a conservative form as in one-dimension. However, it is possible to approximate these terms within the
order of accuracy of the Boussinesq equations to arrive at a compact form.

Considering the nonlinear convection term from the first velocity component $u$ of equation (4.5)

$$u_a \frac{\partial u_a}{\partial x} + v_a \frac{\partial u_a}{\partial y} = u_a \frac{\partial u_a}{\partial x} + v_a \frac{\partial v_a}{\partial x} - v_a \frac{\partial v_a}{\partial x} + v_a \frac{\partial u_a}{\partial y}$$

$$= \frac{1}{2} \frac{\partial}{\partial x} \left( u_a^2 + v_a^2 \right) + v_a \left( \frac{\partial u_a}{\partial y} - \frac{\partial v_a}{\partial x} \right) \quad (4.100)$$

In Section 2.2.2 Nwogu’s scaled, non-dimensionalised velocity variable $u(x,t)$ was shown to be an $O(\sigma^2)$ perturbation of the true velocity. So we can get

$$u_a \frac{\partial u_a}{\partial x} + v_a \frac{\partial u_a}{\partial y} = \frac{1}{2} \frac{\partial}{\partial x} \left( u_a^2 + v_a^2 \right) + O(\sigma^2) \quad (4.101)$$

Similarly, the nonlinear convection term from the second velocity component $v_a$ can be rewritten as

$$u_a \frac{\partial v_a}{\partial x} + v_a \frac{\partial v_a}{\partial y} = \frac{1}{2} \frac{\partial}{\partial y} \left( u_a^2 + v_a^2 \right) + O(\sigma^2) \quad (4.102)$$

This rewriting of the convection terms was used by Beji and Nadaoka (1996) in the development of a finite difference method for their extended equation system, although they stated it without any explanation.
4.3 Numerical Model

Here, we rewrite the system of equations (4.4) ~ (4.6) in a form that makes the application of a higher-order time-stepping procedure convenient (Ge and Kirby, 1995). We use a fourth-order Adams Predictor-corrector scheme which is the same as one-dimensional to perform this updating. Equations (4.4) ~ (4.6) become

\[ \eta_t = E(\eta, u, v) \]  
\[ U_t = F(\eta, u, v) + \left[ F_1(v) \right]_t \]  
\[ V_t = G(\eta, u, v) + \left[ G_1(u) \right]_t \]

Where

\[ U(u) = u + h[b_1 h u_{xx} + b_2 (hu)_{xx}] \]  
\[ V(v) = v + h[b_1 h v_{yy} + b_2 (hv)_{yy}] \]

are treated as simple variables in the time-stepping scheme. The remaining quantities, \( E, F, G, F_1 \) and \( G_1 \) are functions of \( \eta, u \) and \( v \) that are defined as (Ge wei and T. Kirby, 1995)

\[ E(\eta, u, v) = - \left[ (h + \eta)u \right]_x - \left[ (h + \eta)v \right]_y \]

\[ - \left\{ \alpha h^3 \left( u_{xx} + v_{xy} \right) + \alpha h^2 \left[ (hu)_{xx} + (hv)_{xy} \right] \right\}_x \]

\[ - \left\{ \alpha h^3 \left( u_{xy} + v_{yy} \right) + \alpha h^2 \left[ (hu)_{xy} + (hv)_{yy} \right] \right\}_y \]

\[ F(\eta, u, v) = -g \eta_x - \frac{1}{2} \left( u^2 \right)_x - vu_y \]

\[ G(\eta, u, v) = -g \eta_y - \frac{1}{2} \left( v^2 \right)_y - uv_x \]

\[ F_1(v) = -h[b_1 hv_{yy} + b_2 (hv)_{yy}] \]
The total water depth at the points of velocities \( u \) and \( v \), i.e. \((h + \eta)_{i+1/2,j}\) and \((h + \eta)_{i,j+1/2}\) can be obtained by using interpolating:

\[
(h + \eta)_{i+1/2,j} = \frac{1}{2}(h_{i,j} + h_{i+1,j}) + \frac{1}{2}(\eta_{i,j} + \eta_{i+1,j}) \tag{4.113}
\]

\[
(h + \eta)_{i,j+1/2} = \frac{1}{2}(h_{i,j} + h_{i,j+1}) + \frac{1}{2}(\eta_{i,j} + \eta_{i,j+1}) \tag{4.114}
\]
So we can obtain

$$\frac{\partial((h + \eta)u)}{\partial x}_{2, j} = \frac{1}{24\Delta x} \left[ -23(h + \eta)_{3/2,j}u_{3/2,j} + 21(h + \eta)_{5/2,j}u_{5/2,j} 
+ 3(h + \eta)_{7/2,j}u_{7/2,j} - (h + \eta)_{9/2,j}u_{9/2,j} \right]$$

(4.115)

\((i = 2), \ (j = 1, 2, \cdots, n-1, n)\)
\[
\left\{ \frac{\partial[(h + \eta)u]}{\partial x} \right\}_{i,j} = \frac{1}{24\Delta x} \left[ (h + \eta)_{i-3/2,j} u_{i-3/2,j} - 27(h + \eta)_{i-1/2,j} u_{i-1/2,j} 
+ 27(h + \eta)_{i+1/2,j} u_{i+1/2,j} - (h + \eta)_{i+3/2,j} u_{i+3/2,j} \right] 
\]
\[ (i = 3, 4, \ldots, m - 2, m - 2), \ (j = 1, 2, \ldots, n - 1, n) \]

\[
\left\{ \frac{\partial[(h + \eta)v]}{\partial x} \right\}_{i,j} = \frac{1}{24\Delta x} \left[ 23(h + \eta)_{i+1/2,j} u_{i+1/2,j} - 21(h + \eta)_{i-1/2,j} u_{i-1/2,j} 
- 3(h + \eta)_{i-3/2,j} u_{i-3/2,j} + (h + \eta)_{i-5/2,j} u_{i-5/2,j} \right] 
\]
\[ (i = m - 1), \ (j = 1, 2, \ldots, n - 1, n) \]

\[
\left\{ \frac{\partial[(h + \eta)v]}{\partial y} \right\}_{i,j} = \frac{1}{24\Delta x} \left[ -23(h + \eta)_{i,j-1/2} v_{i,j-1/2} + 21(h + \eta)_{i,j+1/2} v_{i,j+1/2} 
+ 3(h + \eta)_{i,j+3/2} v_{i,j+3/2} - (h + \eta)_{i,j+5/2} v_{i,j+5/2} \right] 
\]
\[ (i = 1, 2, \ldots, m - 1, m), \ (j = 2) \]

\[
\left\{ \frac{\partial[(h + \eta)v]}{\partial y} \right\}_{i,j} = \frac{1}{24\Delta x} \left[ (h + \eta)_{i,j-3/2} v_{i,j-3/2} - 27(h + \eta)_{i,j-1/2} v_{i,j-1/2} 
+ 27(h + \eta)_{i,j+1/2} v_{i,j+1/2} - (h + \eta)_{i,j+3/2} v_{i,j+3/2} \right] 
\]
\[ (i = 1, 2, \ldots, m - 1, m), \ (j = 3, 4, \ldots, n - 2, n - 2) \]

\[
\left\{ \frac{\partial[(h + \eta)v]}{\partial y} \right\}_{i,j} = \frac{1}{24\Delta x} \left[ 23(h + \eta)_{i,j+1/2} v_{i,j+1/2} + 21(h + \eta)_{i,j+3/2} v_{i,j+3/2} 
+ 3(h + \eta)_{i,j-3/2} v_{i,j-3/2} - (h + \eta)_{i,j-5/2} v_{i,j-5/2} \right] 
\]
\[ (i = 1, 2, \ldots, n - 1, n), \ (j = n - 1) \]

For second order derivatives, we use three-point difference schemes
\[
(w_{xy})_{i,3/2} = \frac{1}{\Delta x} \left( 2w_{i,3/2} - 5w_{i,5/2} + 4w_{i,7/2} - w_{i,9/2} \right) 
\]
\[ (i = 1, 2, \ldots, m - 1, m) \]
\[
(w_{yy})_{i,j+1/2} = \frac{1}{\Delta x^2} \left( w_{i,j-1/2} - 2w_{i,j+1/2} + w_{i,j+3/2} \right) \\
\quad \quad (i = 1, 2, \ldots, m - 1, m), (j = 2, \ldots, n - 1, n)
\]

\[
(w_{yy})_{n+1/2} = \frac{1}{\Delta x^2} \left( 2w_{i,n+1/2} - 5w_{i,n-1/2} + 4w_{i,n-3/2} - w_{i,n-5/2} \right) \\
\quad \quad (i = 1, 2, \ldots, m - 1, m)
\]

Where \( w_{yy} \) is \( h^2 \) or \( v \).

\[
(w_{xx})_{3/2,j} = \frac{1}{\Delta x^2} \left( 2w_{3/2,j} - 5w_{5/2,j} + 4w_{7/2,j} - w_{9/2,j} \right) \\
\quad \quad (j = 1, 2, \ldots, n - 1, n)
\]

\[
(w_{xx})_{i,j} = \frac{1}{\Delta x^2} \left( w_{i-1/2,j} - 2w_{i+1/2,j} + w_{i+3/2,j} \right) \\
\quad \quad (i = 2, 3, 4, \ldots, m - 2, m - 1), (j = 1, 2, \ldots, n - 1, n)
\]

\[
(w_{xx})_{m+1/2} = \frac{1}{\Delta x^2} \left( 2w_{m+1/2,j} - 5w_{m-1/2,j} + 4w_{m-3/2,j} - w_{m-5/2,j} \right) \\
\quad \quad (j = 1, 2, \ldots, n - 1, n)
\]

Where \( w_{xx} \) is \( h^2 \) or \( u \).

\[
(v_{xy})_{l-1/2,j} = \frac{v_{l,j+1/2} + v_{l-1,j+1/2} - v_{l,j-1/2} - v_{l-1,j-1/2}}{\Delta x \Delta y} \\
\quad \quad (i = 2, \ldots, m - 1, m), (j = 2, \ldots, n - 1, n)
\]

\[
[ (hv) ]_{l-1/2,j} = \frac{(hv)_{l,j+1/2} + (hv)_{l-1,j+1/2} - (hv)_{l,j-1/2} - (hv)_{l-1,j-1/2}}{\Delta x \Delta y} \\
\quad \quad (i = 2, \ldots, m - 1, m), (j = 2, \ldots, n - 1, n)
\]

\[
(u_{xy})_{l,j-1/2} = \frac{u_{l+1/2,j} + u_{l+1/2,j-1} - u_{l-1/2,j} - u_{l+1/2,j-1}}{\Delta x \Delta y} \\
\quad \quad (i = 2, \ldots, m - 1, m), (j = 2, \ldots, n - 1, n)
\]

\[
[ (hu) ]_{l,j-1/2} = \frac{(hu)_{l+1/2,j} + (hu)_{l+1/2,j-1} - (hu)_{l-1/2,j} - (hu)_{l-1/2,j-1}}{\Delta x \Delta y} \\
\quad \quad (i = 2, \ldots, m - 1, m), (j = 2, \ldots, n - 1, n)
\]
So we can obtain

\[
E_{i,j}^n = \left\{ \frac{\partial}{\partial x} \left[ \frac{\partial h}{\partial x} \right] \right\}_{i,j}^{(i+1/2,j)} - \left\{ \frac{\partial}{\partial y} \left[ \frac{\partial h}{\partial y} \right] \right\}_{i,j}^{(i+1/2,j)} + \frac{1}{\Delta x} \left[ \alpha_i h^3 (u_{xx} + v_{xy}) + \alpha_j h^2 ((hu)_{xx} + (hv)_{xy}) \right]_{i,j}^{(i+1/2,j)} - \left[ \alpha_i h^3 (u_{xx} + v_{xy}) + \alpha_j h^2 ((hu)_{xx} + (hv)_{xy}) \right]_{i,j}^{(i-1/2,j)} + \frac{1}{\Delta y} \left[ \alpha_i h^3 (u_{xy} + v_{yy}) + \alpha_j h^2 ((hu)_{xy} + (hv)_{yy}) \right]_{i,j+1/2}^{(i,j+1/2)} - \left[ \alpha_i h^3 (u_{xy} + v_{yy}) + \alpha_j h^2 ((hu)_{xy} + (hv)_{yy}) \right]_{i,j-1/2}^{(i,j-1/2)} \right) \]

(4.131)

We can get the first derivatives of \( \eta \) as followed

\[
\left( \frac{\partial \eta}{\partial x} \right)_{i,j+1/2}^{(i+1/2,j)} = -\frac{\eta_{i-1,j} - 9\eta_{i,j} + 9\eta_{i+1,j} + \eta_{i+2,j}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial x^3} (i = 3, \ j = 2, \cdots, n - 2, n - 1) \tag{4.132}
\]

\[
\left( \frac{\partial \eta}{\partial x} \right)_{i,j+1/2}^{(i+1/2,j)} = \frac{\eta_{i-1,j} - 27\eta_{i,j} + 27\eta_{i+1,j} - \eta_{i+2,j}}{12\Delta x} + \frac{9}{80} \Delta x^4 \frac{\partial^5 \eta}{\partial x^5} (i = 3, \cdots, m - 3, m - 2, \ j = 2, \cdots, n - 1) \tag{4.133}
\]

\[
\left( \frac{\partial \eta}{\partial x} \right)_{i,j+1/2}^{(i+1/2,j)} = -\frac{\eta_{i-1,j} - 9\eta_{i,j} + 9\eta_{i+1,j} + \eta_{i+2,j}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial x^3} (i = m - 1, \ j = 2, \cdots, n - 1) \tag{4.134}
\]

\[
\left( \frac{\partial \eta}{\partial y} \right)_{i+1/2,j}^{(i,j+1/2)} = -\frac{\eta_{i,j-1} - 9\eta_{i,j} + 9\eta_{i,j+1} + \eta_{i,j+2}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial y^3} (i = 2, \cdots, m - 1, m - 1, \ j = 2) \tag{4.135}
\]

\[
\left( \frac{\partial \eta}{\partial y} \right)_{i+1/2,j}^{(i,j+1/2)} = \frac{\eta_{i,j-1} - 27\eta_{i,j} + 27\eta_{i,j+1} - \eta_{i,j+2}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial y^3} (i = 2, \cdots, m - 2, m - 1, \ j = 3, \cdots, n - 2) \tag{4.136}
\]

\[
\left( \frac{\partial \eta}{\partial y} \right)_{i+1/2,j}^{(i,j+1/2)} = -\frac{\eta_{i,j-1} - 9\eta_{i,j} + 9\eta_{i,j+1} + \eta_{i,j+2}}{12\Delta x} - \frac{1}{8} \Delta x^2 \frac{\partial^3 \eta}{\partial y^3} (i = 2, \cdots, m - 2, m - 1, \ j = n - 1) \tag{4.137}
\]
For first order derivatives of $u^2\,v^2$, we use five-point difference schemes

\[
\frac{\partial u^2}{\partial x}\bigg|_{5/2,j} = \frac{-3u^2_{1/2,j} - 10u^2_{3/2,j} + 18u^2_{7/2,j} - 6u^2_{9/2,j} + u^2_{11/2,j}}{12\Delta x}
\] (4.138)

\[(j = 1,2,\ldots, n-1, n)\]

\[
\frac{\partial u^2}{\partial x}\bigg|_{i+1/2,j} = \frac{u^2_{i-3/2,j} - 8u^2_{i-1/2,j} + 8u^2_{i+3/2,j} - u^2_{i+5/2,j}}{12\Delta x}
\] (4.139)

\[(i = 3,4,\ldots, m-2), \ (j = 1,2,\ldots, n-1, n)\]

\[
\frac{\partial u^2}{\partial x}\bigg|_{m-1/2} = \frac{3u^2_{m+1/2,j} + 10u^2_{m-1/2,j} - 18u^2_{m-3/2,j} + 6u^2_{m-5/2,j} - u^2_{m-7/2,j}}{12\Delta x}
\] (4.140)

\[(j = 1,2,\ldots, n-1, n)\]

\[
\frac{\partial v^2}{\partial y}\bigg|_{i,5/2} = \frac{-3v^2_{i,1/2} - 10v^2_{i,3/2} + 18v^2_{i,7/2} - 6v^2_{i,9/2} + v^2_{i,11/2}}{12\Delta y}
\] (4.141)

\[(i = 1,2,\ldots, m-1, m)\]

\[
\frac{\partial v^2}{\partial y}\bigg|_{i,j+1/2} = \frac{v^2_{i,j-3/2} - 8v^2_{i,j-1/2} + 8v^2_{i,j+3/2} - v^2_{i,j+5/2}}{12\Delta y}
\] (4.142)

\[(i = 1,2,\ldots, m-1, m), \ (j = 2,\ldots, n-2, n-1)\]

\[
\frac{\partial v^2}{\partial y}\bigg|_{i,n-1/2} = \frac{3v^2_{i,n+1/2} + 10v^2_{i,n-1/2} - 18v^2_{i,n-3/2} + 6v^2_{i,n-5/2} - v^2_{i,n-7/2}}{12\Delta y}
\] (4.143)

\[(i = 1,2,\ldots, m-1, m)\]

Similar expressions can be obtained for first order derivatives of $u$ and $v$.

So we can get

\[
F^n_{(\eta_{1/2,j})} = -g(\eta_{i+1/2,j}) - \frac{1}{2} \left( \frac{\partial u^2}{\partial x}\bigg|_{i+1/2,j} \right)^n
\]
The governing equations are finite-differenced on an un-staggered grid in time. Level \( n \) refers to information at the present, known time level. The predictor step is third-order explicit Adams-Bashforth scheme (Press et al. 1989) given previously (detail procedure see section 3.2). According to equations (4.10) ~ (4.12), we can get

\[
\eta_{i,j+1}^{n+1} = \eta_{i,j}^{n} + \frac{\Delta t}{12} \left[ 23E_{i,j}^{n} - 16E_{i,j}^{n-1} + 5F_{i,j}^{n-2} \right]
\]

(4.148)

\[
U_{i+1/2,j}^{n+1} = U_{i+1/2,j}^{n} + \frac{\Delta t}{12} \left[ 23F_{i+1/2,j}^{n} - 16F_{i+1/2,j}^{n-1} + 5F_{i+1/2,j}^{n-2} \right] + \frac{\Delta t}{12} \left[ 23E_{i+1/2,j}^{n} - 16E_{i+1/2,j}^{n-1} + 5E_{i+1/2,j}^{n-2} \right]
\]

(4.149)

\[
V_{i,j+1/2}^{n+1} = V_{i,j+1/2}^{n} + \frac{\Delta t}{12} \left[ 23G_{i,j+1/2}^{n} - 16G_{i,j+1/2}^{n-1} + 5G_{i,j+1/2}^{n-2} \right] + \frac{\Delta t}{12} \left[ 23G_{i,j+1/2}^{n} - 16G_{i,j+1/2}^{n-1} + 5G_{i,j+1/2}^{n-2} \right]
\]

(4.150)

From the definition, we see that the terms \( F_{i} \) and \( G_{i} \), involve time derivatives.

Then their time derivatives for predictor stage are

\[
\left[ F_{i,j+1/2,j} \right]^{n+1} = \frac{1}{2\Delta t} \left[ 3(F_{i,j+1/2,j})^{n} - 4(F_{i,j+1/2,j})^{n-1} + (F_{i,j+1/2,j})^{n-2} \right] + O(\Delta t^2)
\]

(4.151)

\[
\left[ F_{i,j+1/2,j} \right]^{n-1} = \frac{1}{2\Delta t} \left[ (F_{i,j+1/2,j})^{n} - (F_{i,j+1/2,j})^{n-2} \right] + O(\Delta t^2)
\]

(4.152)
\[
\left[ (F_i) \right]_{i+1/2,j}^{n-1} = -\frac{1}{2\Delta t} \left[ 3(F_i)_{i+1/2,j}^{n-2} - 4(F_i)_{i+1/2,j}^{n-1} + (F_i)_{i+1/2,j}^n \right] + O(\Delta t^2) \quad (4.153)
\]

\[
\left[ (G_i) \right]_{i,j+1/2}^n = \frac{1}{2\Delta t} \left[ 3(G_i)_{i,j+1/2}^{n-1} - 4(G_i)_{i,j+1/2}^{n-2} + (G_i)_{i,j+1/2}^n \right] + O(\Delta t^2) \quad (4.154)
\]

\[
\left[ (G_i) \right]_{i,j+1/2}^n = \frac{1}{2\Delta t} \left[ (G_i)_{i,j+1/2}^{n-1} - (G_i)_{i,j+1/2}^{n-2} \right] + O(\Delta t^2) \quad (4.155)
\]

\[
\left[ (G_i) \right]_{i,j+1/2}^n = -\frac{1}{2\Delta t} \left[ 3(G_i)_{i,j+1/2}^{n-2} - 4(G_i)_{i,j+1/2}^{n-1} + (G_i)_{i,j+1/2}^n \right] + O(\Delta t^2) \quad (4.156)
\]

By substituting the \((F_i)\) and \((G_i)\) into the equations (4.55) ~ (4.57), the equations become

\[
\eta_{i,j}^{n+1} = \eta_{i,j}^n + \frac{\Delta t}{12} \left[ 23E_{i,j}^n - 16E_{i,j}^{n-1} + 5F_{i,j}^{n-2} \right] \quad (4.157)
\]

\[
U_{i+1/2,j}^{n+1} = U_{i+1/2,j}^n + \frac{\Delta t}{12} \left[ 23F_{i+1/2,j}^n - 16F_{i+1/2,j}^{n-1} + 5F_{i+1/2,j}^{n-2} \right] + 2(F_i)_{i+1/2,j}^{n-1} - 3(F_i)_{i+1/2,j}^{n-2} + (F_i)_{i+1/2,j}^n \quad (4.158)
\]

\[
V_{i+1/2,j}^{n+1} = V_{i+1/2,j}^n + \frac{\Delta t}{12} \left[ 23G_{i+1/2,j}^n - 16G_{i+1/2,j}^{n-1} + 5G_{i+1/2,j}^{n-2} \right] + 2(G_i)_{i+1/2,j}^{n-1} - 3(G_i)_{i+1/2,j}^{n-2} + (G_i)_{i+1/2,j}^n \quad (4.159)
\]

Where, all information on the right hand sides of is known from previous calculations.

The values of \(\eta_{i,j}^{n+1}\) are thus straightforward to obtain. The evaluation of horizontal velocities at the new time level, however, requires simultaneous solution of tridiagonal matrix systems which are linear in the unknowns at level \(n+1\). Specifically, for a given \(j\), \(u_{i,j}^{n+1}(i = 1, 2, \cdots, m)\) are solved by a system of tridiagonal matrix equation. Similarly, \(v_{i,j}^{n+1}(i = 1, 2, \cdots, n)\) are obtained by a system of tridiagonal matrix equation for \(i\) given.

The matrices involved are constant in time and may be pre-factored, inverted and stored for use at each time step.
After the predicted values of \( \{ \eta, \mu, v \}^{n+1}_{i,j} \) are evaluated, we obtain the corresponding quantities of \( \{ E, F, G, F, G \}^{n}_{i,j} \) at time levels \((n+1), (n), (n-1), (n-2)\), and apply the fourth-order Adams-Moulton corrector method, given by

\[
\eta^{n+1}_{i,j} = \eta^n_{i,j} + \frac{\Delta t}{24} \left[ 9E^{n+1}_{i,j} + 19E^n_{i,j} - 5E^{n-1}_{i,j} + E^{n-2}_{i,j} \right] \tag{4.160}
\]

\[
U^{n+1}_{i+1/2,j} = U^n_{i+1/2,j} + \frac{\Delta t}{24} \left[ 9F^{n+1}_{i+1/2,j} + 19F^n_{i+1/2,j} - 5F^{n-1}_{i+1/2,j} + F^{n-2}_{i+1/2,j} \right] \\
+ \frac{\Delta t}{24} \left[ 9G^{n+1}_{i+1/2,j} + 19G^n_{i+1/2,j} - 5G^{n-1}_{i+1/2,j} + G^{n-2}_{i+1/2,j} \right] \tag{4.161}
\]

\[
V^{n+1}_{i,j+1/2} = V^n_{i,j+1/2} + \frac{\Delta t}{24} \left[ 9G^{n+1}_{i,j+1/2} + 19G^n_{i,j+1/2} - 5G^{n-1}_{i,j+1/2} + G^{n-2}_{i,j+1/2} \right] \\
+ \frac{\Delta t}{24} \left[ 9G^{n+1}_{i,j+1/2} + 19G^n_{i,j+1/2} - 5G^{n-1}_{i,j+1/2} + G^{n-2}_{i,j+1/2} \right] \tag{4.162}
\]

From the definition, we see that the terms \((F_i)\), and \((G_i)\), involve time derivatives.

Then their time derivatives for predictor stage are

\[
\left[ (F_i) \right]_{i+1/2}^{n+1} = \frac{1}{6\Delta t} \left[ 11(F_i)_{i+1/2}^{n+1} - 18(F_i)_{i+1/2}^{n} + 9(F_i)_{i+1/2}^{n-1} - 2(F_i)_{i+1/2}^{n-2} \right] + O(\Delta t^3) \tag{4.163}
\]

\[
\left[ (F_i) \right]_{i+1/2}^{n} = \frac{1}{6\Delta t} \left[ 2(F_i)_{i+1/2}^{n+1} + 3(F_i)_{i+1/2}^{n} - 6(F_i)_{i+1/2}^{n-1} + (F_i)_{i+1/2}^{n-2} \right] + O(\Delta t^3) \tag{4.164}
\]

\[
\left[ (F_i) \right]_{i+1/2}^{n-1} = -\frac{1}{6\Delta t} \left[ 2(F_i)_{i+1/2}^{n+1} + 3(F_i)_{i+1/2}^{n} - 6(F_i)_{i+1/2}^{n-1} + (F_i)_{i+1/2}^{n-2} \right] + O(\Delta t^3) \tag{4.165}
\]

\[
\left[ (F_i) \right]_{i+1/2}^{n-2} = -\frac{1}{6\Delta t} \left[ 11(F_i)_{i+1/2}^{n+1} - 18(F_i)_{i+1/2}^{n} + 9(F_i)_{i+1/2}^{n-1} - 2(F_i)_{i+1/2}^{n-2} \right] + O(\Delta t^3) \tag{4.166}
\]

\[
\left[ (G_i) \right]_{i,j+1/2}^{n+1} = \frac{1}{6\Delta t} \left[ 11(G_i)_{i,j+1/2}^{n+1} - 18(G_i)_{i,j+1/2}^{n} + 9(G_i)_{i,j+1/2}^{n-1} - 2(G_i)_{i,j+1/2}^{n-2} \right] + O(\Delta t^3) \tag{4.167}
\]
\[
\begin{align*}
\left[ (G_1)_i^{\nu} \right]_{i,j} & = \frac{1}{6\Delta t} \left[ 2(G_1)_i^{\nu_{i+\frac{1}{2}}} + 3(G_1)_i^{\nu_{i+\frac{1}{2}}} - 6(G_1)_i^{\nu_{i-\frac{1}{2}}} + 2(G_1)_i^{\nu_{i-\frac{1}{2}}} \right] + O(\Delta t^3) \quad (4.168) \\
\left[ (G_1)_i^{\nu_{i+\frac{1}{2}}} \right]_{i,j} & = -\frac{1}{6\Delta t} \left[ 2(G_1)_i^{\nu_{i+\frac{1}{2}}} + 3(G_1)_i^{\nu_{i+\frac{1}{2}}} - 6(G_1)_i^{\nu_{i-\frac{1}{2}}} + 2(G_1)_i^{\nu_{i-\frac{1}{2}}} \right] + O(\Delta t^3) \quad (4.169) \\
\left[ (G_1)_i^{\nu_{i+\frac{1}{2}}} \right]_{i,j} & = -\frac{1}{6\Delta t} \left[ 11(G_1)_i^{\nu_{i+\frac{1}{2}}} - 18(G_1)_i^{\nu_{i+\frac{1}{2}}} + 9(G_1)_i^{\nu_{i+\frac{1}{2}}} - 2(G_1)_i^{\nu_{i+\frac{1}{2}}} \right] + O(\Delta t^3) \quad (4.170)
\end{align*}
\]

By substituting the \((F_1)_i\) and \((G_1)_i\) into the equations (4.67) ~ (4.69), the equations become

\[
\eta_{i,j}^{\nu+1} = \eta_{i,j}^{\nu} + \frac{\nu}{24} \left[ 9E_{i,j}^{\nu+1} + 19E_{i,j}^{\nu} - 5E_{i,j}^{\nu-1} + E_{i,j}^{\nu-2} \right] \quad (4.171)
\]

\[
U_{i,j+\frac{1}{2}}^{\nu+1} = U_{i,j+\frac{1}{2}}^{\nu} + \frac{\nu}{24} \left[ 9F_{i,j+\frac{1}{2}}^{\nu+1} + 19F_{i,j+\frac{1}{2}}^{\nu} - 5F_{i,j+\frac{1}{2}}^{\nu-1} + F_{i,j+\frac{1}{2}}^{\nu-2} \right] + \left( F_{i,j+\frac{1}{2}}^{\nu+1} - F_{i,j+\frac{1}{2}}^{\nu} \right) \quad (4.172)
\]

\[
V_{i,j+\frac{1}{2}}^{\nu+1} = V_{i,j+\frac{1}{2}}^{\nu} + \frac{\nu}{24} \left[ 9G_{i,j+\frac{1}{2}}^{\nu+1} + 19G_{i,j+\frac{1}{2}}^{\nu} - 5G_{i,j+\frac{1}{2}}^{\nu-1} + G_{i,j+\frac{1}{2}}^{\nu-2} \right] + \left( G_{i,j+\frac{1}{2}}^{\nu+1} - G_{i,j+\frac{1}{2}}^{\nu} \right) \quad (4.173)
\]
The correct step is iterated until the error between two successive results reaches a required limit. The error is computed for each of the three dependent variables \( \eta, u, v \) and is defined as

\[
\Delta f = \frac{\sum_{i,j} |f_{i,j}^{n+1} - f_{i,j}^{(n+1)^*}|}{\sum_{i,j} |f_{i,j}^{n+1}|}
\]  

(4.174)

where \( f \) denotes any of the variables and \(( )^*\) denotes the previous estimate. The corrector step is iterated if any of the \( \Delta f \)'s exceeds 0.001. The scheme typically requires no iteration unless problems arise at boundaries. Then the same procedure is applied to the next step.
4.4 Boundary conditions

At any boundary, certain physical conditions must be satisfied by the fluid velocities. Appropriate boundary conditions are also needed for the numerical model to run correctly. The examples shown in the following involve three types of lateral boundaries, which are discussed here in sequence. These are (1) Wavemaker boundaries; (2) reflective vertical walls; (3) transmitting or absorbing boundaries; (4) internal generation of waves.

4.4.1 Wavemaker Boundaries

In Section 2.4, it was shown that the extended Boussinesq equations could be applied at depths right up to the deep water limit $kh=3.0$. This allows waves to be introduced into the domain sufficiently far from the region of interest that boundary effects can be considered insignificant. If nonlinearity is small at these boundaries, i.e., the amplitude is small compared to the depth, then the linearised equations will be a sufficiently accurate approximation to the problem and a linear wave profile can be introduced at the inflow boundary. For the numerical experiments considered in Section 3.6.1 a regular periodic wave is input at the boundary. For example a simple periodic wave

$$\eta(x,t) = a \sin(kx - \omega t)$$  \hspace{1cm} (4.175)

with amplitude $a$, wave number $k$ and frequency $\omega$ where $k$ and $\omega$ satisfy the linear dispersion relation for Nwogu’s extended Boussinesq equations. At these boundaries the velocity profile can be derived from the equation (2-75) by using the linearised equation system, equation (2-71).
\[ u_a = \frac{\omega}{kh[1-(\alpha+1/3)(kh)^2]} \eta(x,t) \cos \theta \]  
\[ v_a = \frac{\omega}{kh[1-(\alpha+1/3)(kh)^2]} \eta(x,t) \sin \theta \]  

(4.176) 
(4.177) 

Where \( k \) = wave number; \( h \) = water depth; and \( \theta \) = angle of wave propagation relative to the x-axis.

This linear wave approximation will become inaccurate if the amplitude is significant compared to the depth at the inflow boundary. This can be accounted for by introducing a first order correction to the linear wave profile for nonlinearity (M. Walkley, 1999). Considering a nonlinear perturbation of the free surface and velocity from the linearised solution,

\[ \eta = \eta_0 + \varepsilon \eta_1 + O(\varepsilon^2) \]  
\[ u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \]  

(4.178) 
(4.179) 

where the nonlinearity is parameterised by \( \varepsilon \). And A solution of the form,

\[ \eta_h(x,t) = a_i \cos \left( 2(kx - \omega t) \right) \]  
\[ u_i(x,t) = b_i \cos \left( 2(kx - \omega t) \right) \]  

(4.180) 
(4.181) 

And where

\[ a_i = \frac{1}{g} \left( \frac{\omega}{k} b_i \left( 1-4\alpha(kh)^2 \right) - b_i \right) \]  
\[ b_i = \frac{ab}{2g} + \frac{b_i^2}{4k} \frac{\omega}{4} \]  

(4.182) 
(4.183) 

The wave profile (4.85) will have a slightly sharper peak and a broader, slightly raised trough, and is the first approximation to a more general periodic wave form known as
a Cnoidal wave. Note that this modified wave involves a wavelength of half the primary wavelength and hence accurate modeling will require that this is properly resolved, leading in general to twice the resolution of the original wave.

4.4.2 Reflective Boundaries

When wave arrives at a solid wall where the physical boundary condition is that of impermeability, the wave will be reflected. For a general reflective boundary with an outward normal vector \( \mathbf{n} \), we would anticipate on physical grounds that the kinematic boundary condition would be completely specified by the statement

\[
\mathbf{u} \cdot \mathbf{n} = 0; \quad x \in \partial \Omega
\]

(4.184)

Where \( n \) is the normal to the wall at that point; \( \Omega \) = the fluid domain; \( \partial \Omega \) = the boundary; and \( x \) = a position in the domain.

A boundary condition on the free surface can be derived from a conservation argument (Ge Wei and T. Kirby 1995). Integrating the free surface equation (4.1) over the spatial domain, and using the divergence theorem on the spatial derivatives;

\[
\int_{\Omega} \left( \frac{\partial \eta}{\partial t} + \nabla \cdot (p, q) + \nabla \cdot m \right) d\Omega = 0
\]

(4.185)

where

\[
P = (h + \eta)u, \quad q = (h + \eta)v
\]

(4.186)

Using the divergence theorem and taking the time derivative outside the spatial integral gives,

\[
\frac{\partial}{\partial t} \left( \int_{\Omega} \eta d\Omega \right) + \int_{\Gamma} (\mathbf{p} \cdot \mathbf{r}) \cdot n d\Gamma = 0
\]

(4.187)
Where, $\Gamma$ is the domain boundary, and $n = (n_x, n_y)$ is the outward normal vector on $\Gamma$, the first term is the total excess volume in the domain. If the domain is completely enclosed by impermeable walls, we require that the rate of change of the excess volume be zero. And so the amount of mass in the system is conserved and hence the time derivative term in expression (3-66) is zero. Hence,

$$\int_\Gamma ((p, q) + m) \cdot nd\Gamma = 0$$  \hspace{1cm} (4.188)

Using the definitions of $p$ and $q$, respectively equation (4.94), this can be written as

$$\int_\Gamma ((h + \eta)u \cdot n + m \cdot n) d\Gamma = 0$$  \hspace{1cm} (4.189)

Equation 94.91) implies that the first part of this expression is zero on $\Gamma$ and hence additionally requiring,

$$m \cdot n = 0$$  \hspace{1cm} (4.190)

will completely satisfy the boundary condition (4.96).

Here, in the scheme the wall boundary conditions we applied are that the normal velocity and the normal derivative of the free surface are zero. And the tangential velocity at the wall is equal to the tangential velocity at the adjacent internal point, essentially imposing no shear at the wall.
4.4.3 Absorbing Boundaries

The absorbing boundary should absorb all energy arriving at the boundary from with the fluid domain. Treatment of this boundary is a problem of major interest in modeling community, and we use some fairly well-established techniques for the cases considered here.

A perfect radiation boundary should not allow wave reflection to occur. For the case where the wave phase speed $C$ and the propagation direction $\theta$ at the boundary are known, the radiation condition is

$$\eta_t + c \eta_x \cos \theta = 0 \quad (4.191)$$

However, in most cases there is no single phase velocity $C$. Further, in two-dimensional applications, the wave direction $\theta$ is generally not known a priori. To solve the second problem, approximations to the perfect radiation condition are made. For wave propagation with the principal direction close to the x-axis, the approximate radiation boundary can be written (Engquist and Majda 1977)

$$\eta_{tt} + c^2 \eta_{xx} - \frac{c^2}{2} \eta_{yy} = 0 \quad (4.192)$$

which corresponds to the imposition of a parabolic approximation on the outgoing wave. To treat the first problem, phase speed $c$ is specified by long-wave limit

$$c = \sqrt{gh}.$$

The aforementioned approximate radiation condition inevitably introduces wave reflection along the boundaries and can eventually cause the model to blow up. To
reduce the reflection, a damping layer is applied to the computing domain. Damping terms are added to the momentum equations as

\[ U_x = F(\eta, u) - w_1(x)u - w_2(x)u_{xx} \]  

where the damping terms with \( u \) is called “Newtonian cooling” and this with second-order derivative is analogous to linear viscous terms in Navier-Stokes equation (Israeli and Orszag 1981). The damping coefficients \( w_1(x) \) and \( w_2(x) \) are defined as

\[
w_1(x) = \begin{cases} 
0; & x < x_s \\
\alpha_1 \omega f(x); & x > x_s 
\end{cases} \]  

\[
w_2(x) = \begin{cases} 
0; & x < x_s \\
\alpha_2 \omega f(x); & x > x_s 
\end{cases} \]

(4.194)  

(4.195)

Where \( \alpha_1 \) and \( \alpha_2 \) = constants to be determined for the specific running; \( \omega = \) frequency of wave to be damped; \( x_s = \) starting coordinate of damping layer (the computing domain is from \( x = 0 \) to \( x = x_i \)); \( v = \) viscous coefficient; and \( f(x) \) is expressed as

\[ f(x) = \frac{\exp\left(\frac{x-x_s}{x_i-x_s}\right)^n - 1}{\exp(1) - 1} \]  

(4.196)

The width of the damping layer (i.e., \( x_i - x_s \)) is usually taken to be two or three times the wavelength. Numerical experiments show that the addition of damping layer combined with radiation boundary conditions works much better than radiation conditions alone.
4.4.4 Internal generation of waves

Larsen and Dancy (1983) were the first to describe such an approach for a two-dimensional finite difference model of the original Boussinesq equation system. They introduced an increment to the free surface along a generation line which varied periodically to produce a wave motion. They showed that this approach could be used on fairly general geometries, within the restrictions of the cartesian finite difference grid. Skotner and Apelt (1999) have described an alternative internal wave generation method for a two-dimensional finite difference model of Nwogu’s extended Boussinesq equations. This method is relatively complex, involving a modified stencil in the wave generation region and explicit removal of the wave in the region behind the wave generator to prevent reflection. In order to solve the boundary problems, we use the source function driven by Wei et al. (1999) in constant water depth continuity equation.

\[ \eta_t + h \nabla \cdot u + \alpha h^3 \nabla^2 (\nabla \cdot u) = f(x,y) \]  
\[ (4.197) \]

In a constant water depth of h, we want to generate a plane wave with amplitude \( a_0 \) and angular frequency \( \omega \). The angle between the propagation direction of the wave and the x-axis is \( \theta \). Without losing generality, we assume that the center of the source region is parallel to the y-axis. Then we split the source function \( f(x,y) \) into two parts as

\[ f(x,y) = g(x)s(y) \]  
\[ (4.198) \]

where \( g(x) \) is a Gaussian shape function and \( s(y,t) \) the input time series of the magnitude of source function. It is convenient to make this separation, since the
dimensionality of the input signal required to run the model, \(s(y,t)\), is reduced by one relative to \(f(x,y,t)\). The functions \(g(x)\) and \(s(y,t)\) are defined as

\[
g(x) = \exp[-\beta(x-x_c)^2] \tag{4.199}
\]

\[
s(y,t) = D \sin(k_y y - \omega t) \tag{4.200}
\]

where \(\beta\) is the shape coefficient for the source function, and \(x_c\) is the central location of the source in the \(x\) direction, for a source oriented parallel to the \(y\) axis. \(k_y = k \sin(\theta)\) is the wavenumber in the \(y\) direction, and \(k\) is the linear wavenumber. \(D\) is the magnitude of the source function, for a monochromatic wave or a single wave component of a random wave train, the magnitude \(D\) of the source function can be determined by

\[
D = \frac{2a_0 \cos(\theta)(\omega^2 - \alpha_1 gk^4 h^3)}{\omega k l [1 - \alpha(kh)^3]} \tag{4.201}
\]

where \(\alpha\) and \(\alpha_1\) are the same as before. And \(I\) is the integral given by

\[
I = \sqrt{\frac{\pi}{\beta}} \exp(-l^2 / 4\beta) \tag{4.202}
\]

where \(l = k \cos(\theta)\) is the wavenumber in the \(x\) direction. In theory, the shape coefficient can be any number. The larger the value \(\beta\), the narrower the source function becomes.

\(\beta\) is given by

\[
\beta = \frac{80}{\delta^2 L^3} \tag{4.203}
\]

Where, \(L\) is the wavelength and \(\delta\) is of order 1.
4.5 Numerical experiments

4.5.1 Wave evolution in closed rectangular basin

The complexity of the 2D model requires careful programming and objective testing. To verify the correctness of the model and to test its stability and associated boundary conditions in 2D, we first use the model to study wave evolution in a closed basin of size $(L_x = 10 \text{ m}) \times (L_y = 10 \text{ m})$. By providing initial values of $\eta$, $u$ and $v$ for the first three time steps to the model, we can obtain the subsequent variations of $\eta$, $u$ and $v$. Analyzing these data has enabled us to correct coding errors and to discover various properties of the model.

This case also gives us the opportunity to test a linearized version of the model against an analytic result, and to study the robustness of the modeled dispersion relation in a linear wave field containing a range of wave frequencies.

We consider the domain $0 \leq x \leq L_x, 0 \leq y \leq L_y$ bounded by reflective vertical walls. Within this domain, we take the initial condition to be a superelevation of the surface $\eta_0(x,y)$ above an otherwise constant depth $h_0 = 0.50 \text{ m}$. The initial values of $\eta$ are not zero. For the runs shown here, the initial surface elevation is of Gaussian shape

\[
\eta_{0,i,j} = H_0 \exp\left[-\beta \left((i - 1.5 - i_c)^2 (\Delta x)^2 + (j - 1.5 - j_c)^2 (\Delta y)^2\right)\right]
\]

$i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, N; \quad k = 0, 1, 2$
Where $H_0$ is the initial height of the Gaussian hump. $\beta$ is the shape coefficient (in this case $\beta = -0.4$), $i_c$ and $j_c$ are the grid numbers for the center of the basin in x and y directions, respectively. The Gaussian hump water is released in a rectangular basin with dimensions $10\text{m} \times 10\text{m}$ and with constant water depth $h = 0.5 \text{ m}$.

Firstly, we consider the symmetry and conservation properties of the solution. For these example, the initial maximum elevation $H_0$ is taken to be $0.1 \text{ m}$, with a corresponding height-to-depth ratio of 0.1. The model was run for 100s, by using a grid size of $\Delta x = 0.2 \text{ m}$ and time step of $\Delta t = 0.01 \text{ s}$. The corresponding Courant number is 0.11. Due to gravitational forcing waves are generated and propagate out of the center and then are reflected back in the domain by four side walls. Although there exist no analytical solutions or experiment data for this case, the symmetric characteristics of the basin and initial conditions should result in symmetric spatial profiles of $\eta$, which are shown in Figure 4.2. The symmetric property had been proved to be an efficient way for checking coding errors in 2D models. The axis-symmetry of the evolving waveform about the origin of the Gaussian hill is apparent from the contour plot of surface elevation $\eta$ at $t=0 \text{ s}$ in Figure 4.2, and verifies that cross-derivative terms are being handled correctly. Figure 4.2 also shows that the axis-symmetry of the evolving waveform about the origin of the Gaussian hill is apparent from the contour plot of surface elevation $\eta$ at $t=0$, 10, 20, 30, 40 and 50 s.
Figure 4.2 Contour plots of surface elevation at time $t = 0, 10, 20, 30, 40,$ and 50 (s).
Figure 4.3 shows the spatial profiles of surface elevation at time $t = 0, 10, 20, 30, 40,$ and $50$ (s).

Figure 4.3 shows the spatial profiles of surface elevation at time $t = 0, 10, 20, 30, 40$ and $50$ s for illustrate purposes.
Secondly, we check the mass conservation of this model. Since no water can escape from the basin, the water volume should remain constant with time, which is a good criterion to test the stability of the model and associated boundary conditions. The computational results for the percentage error in the total volume are less than 0.4%. The results are obtained by using the reflective boundary conditions, which yield results that were sufficient accurate by using zero normal velocity conditions and the governing equations alone. As shown from Figure 4.4, the relative errors of water volume from Boussinesq model is all most the same as the initial value, indicating that the conservation property of these models works quite well.

![Figure 4.4](image_url)  
**Figure 4.4** the Relative Error of Water Volume E of Time Series.
4.5.2 Wave focusing by a topographic lens

Nonlinear refraction-diffraction over a semicircular shoal was studied experimentally by Whalin (1971) for waves in deep, intermediate and shallow water. The spatial domain in our numerical experiment is \((x, y) \in [0, 30.6] \times [0, 6.096]\). There is an inflow boundary at \(x = 0.0\). And solid walls are put at \(y = 0.0\) and \(y = 6.096\) m. At left side there is an absorbing sponge layer, active for \(x \in [26.0, 30.6]\). Figures 4.5 (a)-(b) show the bathymetry on the mesh, and along the centre line of the mesh respectively. The depth variation within the domain is given by,

\[
h(x, y) = \begin{cases} 
0.4572 & 0 \leq x \leq 10.67 - G \\
0.4572 + 0.04(10.67 - G - x) & 10.67 - G \leq x \leq 18.27 - G \\
0.1524 & 18.27 - G \leq x
\end{cases}
\]

\[
G(y) = \sqrt{y(6.096 - y)}
\]

Where the length variables \(x\) and \(y\) are measured in meters. The topography is symmetric with respect to the centerline at \(y = 3.048\) m, the width is 6.096 m and the water depth varies continually from 0.4572 m to 0.1524 m. At the wave maker the waves are linear, but after the focusing on the shoal, higher harmonics become significant due to nonlinear effect. The inflow parameters are,

Amplitude: \(a_0 = 0.0195\) m, wave period \(T = 1.0\) s.

In this case, the value of \(kh\) varies from 1.9 in front of the shoal to 0.6 behind the shoal. The minimum wave length becomes approximately 1.10 m. So we choose \(dx = 0.0762\) m and \(dt = 0.001\) s. The corresponding Courant number is 0.02. And wave maker is put at the left side where \(x=0.0\) (m).
Figure 4.5 Bathymetry for wave focusing experiment.
Figure 4.6 Lengthwise free surface elevation at t = 40 (s).

Figure 4.6 (a), (b), (c) and (d), shows the free surface profile along four lines parallel to the x-axis at t = 40 s and y = 0.229 m, 1.143 m, 1.905 m and 3.048 m. It is very
clear that the wave height is amplified along the centre line and decreases near the
wall. These results are in good visual agreement with the wave envelope and
centreline free surface elevation plots of Madsen et al (1992). Figure 4.7 shows the
wave height coefficient distribution over the mesh. Figure 4.8 a perspective view of
the fully-developed wave field is depicted to give an idea about the wave patterns.

Figure 4.7 Wave height coefficient
4.5.3 Wave propagation over an elliptical shoal

Now we apply the 2D version of the numerical model to study monochromatic-wave propagation over a shoal. This problem has been used as a standard variable depth test case for dispersive wave models by G. Wei and Kirby (1995, 1999). The geometry used corresponds to the experimental arrangement of Berkhoff.

The domain is \((x, y) \in [-10,10] \times [-15,15]\) with an inflow boundary at \(y = -10\) m and solid walls at \(x = -10\) m and \(x=10\) m. The outflow boundary at \(y =15\) m has an absorbing sponge layer, active for \(y \in [12,15]\).

The depth variation within the domain is specified by a combination of a (1:50) slope at an angle of 20° to the y-axis. And the bottom bathymetry consists of an elliptic shoal resting on the plane beach.

\[
x_r = \cos(20) \cdot x - \sin(20) \cdot y
\]
\[ y_r = \sin(20) \cdot x + \cos(20) \cdot y \]

\[
h(x, y) = \begin{cases} 
0.45 & y_r \leq -5.82 \\
0.45 - 0.02(5.82 + y_r) & y_r > -5.82
\end{cases}
\]

and an elliptical bump centered on the origin. For the region.

\[
\left( \frac{x_r}{4} \right)^2 + \left( \frac{y_r}{3} \right)^2 < 1
\]

And the thickness of the shoal is

\[
d = -0.3 + 0.5 \sqrt{1 - (x_r / 5)^2 - (y_r / 3.75)^2}
\]

So the depth is modified by,

\[
h(x, y) \Rightarrow h(x, y) + 0.3 - 0.5 \sqrt{1 - (x_r / 5)^2 - (y_r / 3.75)^2}
\]

Figures 4.9 (a) and (b) show the bathymetry on the mesh, and along the centre line of the mesh respectively. The flow parameters are: the water depth at inflow place is \( h_m = 0.45 \), and amplitude is \( a_0 = 0.0232m \). The wave period is \( T = 1.0s \) and the wave length is about \( \lambda = 1.485m \). The constant water depth in left side is 0.07 m. So we choose \( dx = 0.1m \) and \( dt = 0.002 \) s. The corresponding Courant number is 0.025. Measurements of wave height coefficient are taken along the following coordinate sections

\[ \{(x, y_m), y_m = 1.0, 3.0, 5.0, 7.0, 9.0\}, \{(x_m, y), x_m = -2.0, 0.0, 2.0\} \]
Figure 4.9 Bathymetry for the elliptical shoal experiment.
Figure 4.10 Comparisons of amplitude along specified sections: solid line-numerical results, circle-experiment data
Chapter 5

Conclusions

In this thesis, the high-order numerical approximation on Nwogu’s Boussinesq wave equations was considered. Nwogu’s equations (1993) are simpler to solve numerically and are relatively accurate in simulation of wave propagation and transformation in coastal region. Considering a compromise between the theoretical accuracy of the mathematical model and the practicalities of solving the equation system numerically, Nwogu’s extended Boussinesq equations was selected. An internal wave generation method has been incorporated into the model. This wave generation method enables the incident wave field to be added along line segments inside the computational domain.

A high-order staggered grid numerical model based on Nwogu’s Boussinesq equations has been developed in this study. The variables of the surface elevation and the particle velocity in the equations of Nwogu are placed in a staggered grid system. This conservative form explicitly enforces the mass and momentum conservation. Sponge layers are placed at the boundaries in order to minimize wave reflections from the boundaries.

The numerical solution of Nwogu’s one-dimensional extended Boussinesq equation was described in Chapter 3. The new formulations of the all kinds of boundary conditions which are suitable for staggered grid systems for these equations were derived. The numerical method was tested for theoretical and experimental problems.
and excellent results were obtained. Chapter 4 described the numerical approximation of Nwogu’s two-dimensional extended Boussinesq equations. The boundary conditions developed for the one-dimensional equation system were extended to the two-dimensional system. Some classical numerical experiments were carried out in chapter 4.

The results obtained from chapter 3 and chapter 4 show that this method can produce accurate results for nonlinear, dispersive wave problems. For experiment of water sloshing in a confined container which has an analytical solution, the good agreement between analytical solution and numerical result is found. For other cases, such as periodic wave train passing a submerged breakwater and wave propagation over an elliptical shoal, whose experiment data are available, the agreement between the experimental data and numerical results are also quite good. Computational tests of wave evolution in closed rectangular basin showed greatly increased accuracy in mass conservation.

A small Courant number is, however, needed in some 2D cases when cross differential terms dominate, which results in longer calculation time than the non-staggered grid models. Therefore, to relax this excessive stability constraint is one of the future works. Other future works include extending the current model to swash zone where wave breaking and runup are important, testing the model for random wave field, simulating wave interaction with various coastal structure, and etc.
Bibliography


