GENERALIZED JACOBI THETA FUNCTIONS, MACDONALD'S IDENTITIES AND POWERS OF DEDEKIND'S ETA FUNCTION

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Summary

Ramanujan (1919) studied expansions of the form $\eta^d E(P,Q,R)$ for $d = 1$ or 3 where η is Dedekind's eta function and $E(P,Q,R)$ is some polynomial in terms of the Eisenstein series, P, Q and R. In another direction, Newman (1955) used the theory of modular forms to prove that the fourier coefficients of η^d satisfy some special arithmetic properties whenever $d = 2, 4, 6, 8, 10, 14$ and 26. Subsequently, Serre (1985) proved that for even d, η^d is lacunary if and only if d belongs to the same set of integers.

In this thesis, we generalize the results of Ramanujan, Newman and Serre by constructing infinitely many expansions of $\eta^d E(P,Q,R)$ where $d = 2, 4, 6, 8, 10$ and 14, of which the last 3 cases are new. We first use invariance properties of generalized Jacobi theta functions to construct identities involving two variables which are equivalent to the Macdonald identities for A_2, B_2 and G_2. Applying appropriate differential operators, we establish the cases for $d = 8, 10$ and 14. The problem can also be studied in a more uniform manner by using modular forms. In this case, we obtain infinitely many identities for $\eta^d F(Q,R)$ where $d =$

$2, 4, 6, 8, 10, 14, 26$ and $F(Q, R)$ is a certain polynomial in terms of Q and R. Most of the results described here are original and appears in [CCT07].

In the second part of this thesis, we will describe an original construction of the Macdonald identities for all the infinite families. Again by applying appropriate differential operators, we deduce new formulas for higher powers of η. For example, the formulas for η^{n^2+2}, η^{2n^2+n} and η^{2n^2-n} for all positive n are given. The results described here will appear in [Toh].

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Chapter 1

Jacobi theta functions

Theta functions first appeared in Jakob Bernoulli's *Ars Conjectandi* (1713) and subsequently in the works of Euler and Gauss but the first systematic study was published by C. G. J. Jacobi. Jacobi's analysis was so complete that most of the important properties of the theta functions appeared in his *Fundamenta Nova* (1829) [Jac29]. One aspect of theta functions that was not studied by Jacobi is the connection with modular forms. These came into prominence in the 1900s and were pioneered by E. Hecke.

In Section 1.1, we will define the four theta functions introduced by Jacobi and study some of their key properties. In Section 1.2, we will consider a generalization of the Jacobi theta functions and establish some useful results. These functions play a key role in the subsequent chapters of this thesis.

In Section A.2, we will review some key properties of modular forms.

1.1 Classical Jacobi theta functions

Definition 1.1.1. *Let $q = e^{\pi i t}$ where $Im(t) > 0$. The classical Jacobi theta functions are:*

$$\begin{aligned}
\theta_1(z|q) &= -iq^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2+k} e^{(2k+1)iz} \\
&= 2q^{\frac{1}{4}} \sum_{k=0}^{\infty} (-1)^k q^{k^2+k} \sin(2k+1)z, \\
\theta_2(z|q) &= q^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} q^{k^2+k} e^{(2k+1)iz} \\
&= 2q^{\frac{1}{4}} \sum_{k=0}^{\infty} q^{k^2+k} \cos(2k+1)z, \\
\theta_3(z|q) &= \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kiz} \\
&= 1 + 2\sum_{k=1}^{\infty} q^{k^2} \cos 2kz \\
\text{and} \qquad \theta_4(z|q) &= \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kiz} \\
&= 1 + 2\sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos 2kz.
\end{aligned}$$

They are analytic functions of a complex variable z and a parameter t. We shall state some important facts about the Jacobi theta functions. These are well known and can be found in [WW27, Chpt. 21] or [MM97, Chpt. 3].

Remark 1.1.2. *In the classical theory of theta functions, it is customary to set $q = e^{\pi i \tau}$ but in the theory of modular forms, the standard notation is $q = e^{2\pi i \tau}$. Throughout this thesis, we let $t = 2\tau$ and use the parameter t for theta functions and τ for modular forms.*

Proposition 1.1.3. *The Jacobi theta functions satisfy the following transformation formulas:*

$$
\begin{aligned}
\theta_1(z+\pi|q) &= -\theta_1(z|q); & \theta_1(z+\pi t|q) &= -q^{-1}e^{-2iz}\theta_1(z|q);\\
\theta_2(z+\pi|q) &= -\theta_2(z|q); & \theta_2(z+\pi t|q) &= q^{-1}e^{-2iz}\theta_2(z|q);\\
\theta_3(z+\pi|q) &= \theta_3(z|q); & \theta_3(z+\pi t|q) &= q^{-1}e^{-2iz}\theta_3(z|q);\\
\theta_4(z+\pi|q) &= \theta_4(z|q); & \theta_4(z+\pi t|q) &= -q^{-1}e^{-2iz}\theta_4(z|q).
\end{aligned}
$$

Hence they are quasi-elliptic with (quasi) periods π and πt and it suffices to study their values in the fundamental parallelogram,

$$\Pi = \{a\pi + b\pi t \mid 0 \le a < 1,\ 0 \le b < 1\}.$$

The four theta functions are all related to each other via a half-period transformation. For example,

$$\theta_2\Big(z+\frac{\pi}{2}\Big|q\Big) = -\theta_1(z|q).$$

Table 1.1 gives all the half-period transforms satisfied by the four theta functions.

	$x = z + \frac{\pi}{2}$	$x = z + \frac{\pi+\pi t}{2}$	$x = z + \frac{\pi t}{2}$
$\theta_1(x\|q)$	$\theta_2(z\|q)$	$q^{-\frac{1}{4}}e^{-iz}\theta_3(z\|q)$	$iq^{-\frac{1}{4}}e^{-iz}\theta_4(z\|q)$
$\theta_2(x\|q)$	$-\theta_1(z\|q)$	$-iq^{-\frac{1}{4}}e^{-iz}\theta_4(z\|q)$	$q^{-\frac{1}{4}}e^{-iz}\theta_3(z\|q)$
$\theta_3(x\|q)$	$\theta_4(z\|q)$	$iq^{-\frac{1}{4}}e^{-iz}\theta_1(z\|q)$	$q^{-\frac{1}{4}}e^{-iz}\theta_2(z\|q)$
$\theta_4(x\|q)$	$\theta_3(z\|q)$	$q^{-\frac{1}{4}}e^{-iz}\theta_2(z\|q)$	$iq^{-\frac{1}{4}}e^{-iz}\theta_1(z\|q)$

Table 1.1: Half-period transforms of Jacobi theta functions

Proposition 1.1.4. *Each $\theta_i(z|q)$ vanishes at exactly one point in Π. Specifically, we have*

$$\theta_1(0|q) = \theta_2\Big(\frac{\pi}{2}\Big|q\Big) = \theta_3\Big(\frac{\pi+\pi t}{2}\Big|q\Big) = \theta_4\Big(\frac{\pi t}{2}\Big|q\Big) = 0.$$

Proof. We first note the following consequences of Proposition 1.1.3,

$$\frac{\theta_1'(z+\pi t|q)}{\theta_1(z+\pi t|q)} = \frac{\theta_1'(z|q)}{\theta_1(z|q)} - 2i \qquad \text{and} \qquad \frac{\theta_1'(z+\pi|q)}{\theta_1(z+\pi|q)} = \frac{\theta_1'(z|q)}{\theta_1(z|q)}.$$

Next, let m be a constant such that, C, the boundary of the parallelogram $m+\Pi$ does not contain any zeroes of $\theta_1(z|q)$. Then the number of zeroes inside $m+\Pi$ can be computed by

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{\theta_1'(z|q)}{\theta_1(z|q)}dz &= \frac{1}{2\pi i}\int_m^{m+\pi}\left(\frac{\theta_1'(z|q)}{\theta_1(z|q)} - \frac{\theta_1'(z+\pi t|q)}{\theta_1(z+\pi t|q)}\right)dz \\
&\quad -\frac{1}{2\pi i}\int_m^{m+\pi t}\left(\frac{\theta_1'(z|q)}{\theta_1(z|q)} - \frac{\theta_1'(z+\pi|q)}{\theta_1(z+\pi|q)}\right)dz \\
&= \frac{1}{2\pi i}\int_m^{m+\pi} 2idz \\
&= 1.
\end{aligned}$$

By the principle of analytic continuation, we can let m tend to 0 and conclude that $\theta_1(z|q)$ has exactly one zero in Π. Since $\theta_1(z|q)$ is odd, it vanishes at $z=0$. The zeroes of the other theta functions can then be deduced from Table 1.1. □

We now state a fundamental lemma, the proof of which can be found in [Ahl78, Chpt. 7, Sect.2, Thm. 3 and 4].

Lemma 1.1.5. *Let $F(z,t)$ be a complex function in the variables z and t. If t is fixed and*

1. *$F(z,t)$ has at most a simple pole in Π;*

2. *$F(z+\pi,t) = F(z,t)$ and $F(z+\pi t,t) = F(z,t)$,*

then $F(z,t)$ is a constant independent of z.

The first consequence of Lemma 1.1.5 is the following.

Proposition 1.1.6 (Duplication Formula)**.**

$$\theta_1(2z|q) = 2\frac{\theta_1(z|q)\theta_2(z|q)\theta_3(z|q)\theta_4(z|q)}{\theta_2(0|q)\theta_3(0|q)\theta_4(0|q)}.$$

Proof. Using Proposition 1.1.3, we can check that both sides satisfy the same transformation formula. Moreover, it is also evident that $\theta_1(2z|q) = 0$ when $z = 0$, $\pi/2$, $(\pi + \pi t)/2$ and $\pi t/2$. Thus the quotient

$$\frac{\theta_1(2z|q)\theta_2(0|q)\theta_3(0|q)\theta_4(0|q)}{2\theta_1(z|q)\theta_2(z|q)\theta_3(z|q)\theta_4(z|q)}$$

is an entire function and equals a constant. As z approaches 0, we see that this constant equals to 1. □

We shall now look at the expressions of the theta functions as infinite products. For this purpose, we will adopt the following q-Pochhammer symbol.

Definition 1.1.7. *For $|q| < 1$, we define*

$$(a_1, \dots, a_k; q)_\infty = \prod_{n=1}^{\infty}(1 - a_1 q^{n-1}) \dots (1 - a_k q^{n-1})$$

and

$$(q)_\infty = \prod_{n=1}^{\infty}(1 - q^n).$$

The celebrated Jacobi's triple product identity is the following.

Theorem 1.1.8 (Jacobi's Triple Product Identity)**.** *For $x \neq 0$ and $|q| < 1$, we have*

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = (-xq; q^2)_\infty(-x^{-1}q; q^2)_\infty(q^2; q^2)_\infty.$$

A proof of this theorem can be found in Appendix A.1.2.

Using Theorem 1.1.8, we can easily express each $\theta_i(z|q)$ as an infinite product.

Proposition 1.1.9 (Infinite Product Formulas).

$$\theta_1(z|q) = iq^{\frac{1}{4}}e^{-iz}(q^2;q^2)_\infty(e^{2iz};q^2)_\infty(q^2e^{-2iz};q^2)_\infty \tag{1.1}$$

$$= 2q^{\frac{1}{4}}\sin z(q^2;q^2)_\infty(q^2e^{2iz};q^2)_\infty(q^2e^{-2iz};q^2)_\infty, \tag{1.2}$$

$$\theta_2(z|q) = 2q^{\frac{1}{4}}\cos z(q^2;q^2)_\infty(-q^2e^{2iz};q^2)_\infty(-q^2e^{-2iz};q^2)_\infty, \tag{1.3}$$

$$\theta_3(z|q) = (q^2;q^2)_\infty(-qe^{2iz};q^2)_\infty(-qe^{-2iz};q^2)_\infty, \tag{1.4}$$

$$\theta_4(z|q) = (q^2;q^2)_\infty(qe^{2iz};q^2)_\infty(qe^{-2iz};q^2)_\infty. \tag{1.5}$$

Corollary 1.1.10. *We have the following identities at* $z = 0$*.*

$$\theta_1'(0|q) = \theta_2(0|q)\theta_3(0|q)\theta_4(0|q) \tag{1.6}$$

$$= 2q^{\frac{1}{4}}(q^2)_\infty^3, \tag{1.7}$$

$$\theta_2(0|q) = 2q^{\frac{1}{4}}(q^2)_\infty(-q^2;q^2)_\infty^2 = 2q^{\frac{1}{4}}\frac{(q^4)_\infty^2}{(q^2)_\infty}, \tag{1.8}$$

$$\theta_3(0|q) = (q^2)_\infty(-q;q^2)_\infty^2 = \frac{(q^2)_\infty^5}{(q^4)_\infty^2(q)_\infty^2}, \tag{1.9}$$

$$\theta_4(0|q) = (q^2)_\infty(q;q^2)_\infty^2 = \frac{(q)_\infty^2}{(q^2)_\infty}. \tag{1.10}$$

Remark 1.1.11. *In Jacobi's original work, he did not prove the triple product identity directly. Instead, he first used logarithmic differentiation of the duplication formula (Proposition 1.1.6) to obtain* (1.6)*. Thereafter, he proved the product formulas via Lemma 1.1.5 and calculated the constant with* (1.6)*.*

1.2 Generalized Jacobi theta functions

We now construct a m-th order generalization of Jacobi theta functions and study the complex vector space spanned by these functions.

Definition 1.2.1 (m-th order Jacobi theta function)**.** *Let* $q = e^{\pi it}$, $\mathrm{Im}(t) > 0$*. For* *each $m, j \in \mathbb{Z}$, $m \geq 1$ and $l \in \{0,1\}$, we define*

$$\begin{aligned} T^0_{m,j}(z) &= \sum_{k=-\infty}^{\infty} q^{mk^2+jk} e^{(2mk+j)iz} \\ &= q^{\frac{m-2j}{4}} e^{(j-m)iz} \theta_2\Big(mz + (j-m)\frac{\pi t}{2}\Big| q^m\Big), \\ T^1_{m,j}(z) &= \sum_{k=-\infty}^{\infty} (-1)^k q^{mk^2+jk} e^{(2mk+j)iz} \\ &= iq^{\frac{m-2j}{4}} e^{(j-m)iz} \theta_1\Big(mz + (j-m)\frac{\pi t}{2}\Big| q^m\Big), \\ E^l_{m,j}(z) &= T^l_{m,j}(z) + T^l_{m,j}(-z) \\ \textit{and} \qquad O^l_{m,j}(z) &= T^l_{m,j}(z) - T^l_{m,j}(-z). \end{aligned}$$

These functions are equivalent to the N-th order θ-function with rational characteristic developed by Farkas and Kra.[1] Most of the results given in this section are adapted from [FK01, Chpt. 2, Sect. 7].

Proposition 1.2.2. *We have*

$$T^l_{m,j}(z) = T^l_{m,-j}(-z) \qquad \textit{and} \qquad T^l_{m,j}(z) = (-1)^l q^{m+j} T^l_{m,2m+j}(z). \tag{1.11}$$

Furthermore,

$$O^0_{m,0}(z) \equiv O^1_{m,0}(z) \equiv O^0_{m,m}(z) \equiv E^1_{m,m}(z) \equiv 0. \tag{1.12}$$

Definition 1.2.3. *For each integer $m \geq 1$ and $k, l \in \{0,1\}$, we define $V^l_{m,k}$ to be the complex vector space consisting of entire functions, $F^l_{m,k}(z)$, satisfying the following transformation formulas:*

$$F^l_{m,k}(z+\pi) = (-1)^k F^l_{m,k}(z); \quad F^l_{m,k}(z+\pi t) = (-1)^l q^{-m} e^{-2miz} F^l_{m,k}(z). \tag{1.13}$$

We can easily check that the functions $T^l_{m,j}(z)$, $E^l_{m,j}(z)$ and $O^l_{m,j}(z)$ all belong to $V^l_{m,k}$ whenever $j \equiv k \pmod 2$.

[1] See Definition A.1.1.

Proposition 1.2.4. *Let $F^l_{m,j}(z) \in V^l_{m,k}$, where $j \equiv k \pmod 2$. Then $F^l_{m,j}(z)$ has exactly m zeroes in Π, the fundamental parallelogram.*

Proof. The number of zeroes can be calculated in a similar fashion as Proposition 1.1.4 using the transformation formulas (1.13). □

Corollary 1.2.5. *Let $F^l_{m,j}(z) \in V^l_{m,k}$, where $j \equiv k \pmod 2$. Then $F^l_{m,j}(z)$ has the following special values:*

$$F^l_{m,j}\left(\frac{\pi+\pi t}{2}\right) = (-1)^{j+m+l} F^l_{m,j}\left(-\frac{\pi+\pi t}{2}\right),$$
$$F^l_{m,j}\left(\frac{\pi}{2}\right) = (-1)^{j} F^l_{m,j}\left(-\frac{\pi}{2}\right) \qquad \textit{and} \qquad F^l_{m,j}\left(\frac{\pi t}{2}\right) = (-1)^{l} F^l_{m,j}\left(-\frac{\pi t}{2}\right).$$

Corollary 1.2.5 is a direct consequence of (1.13) and is useful for locating the zeroes of $E^l_{m,j}(z)$ and $O^l_{m,j}(z)$. For example, the even function $E^1_{2n+1,2j-1}(z)$ is necessarily zero at $z = \pi/2, \pi t/2$ and $(\pi+\pi t)/2$ since it satisfies

$$E^1_{2n+1,2j-1}\left(\frac{\pi+\pi t}{2}\right) = (-1)^{2j+2n+l} E^1_{2n+1,2j-1}\left(-\frac{\pi+\pi t}{2}\right) = -E^1_{2n+1,2j-1}\left(\frac{\pi+\pi t}{2}\right),$$
$$E^1_{2n+1,2j-1}\left(\frac{\pi}{2}\right) = (-1)^{2j-1} E^1_{2n+1,2j-1}\left(-\frac{\pi}{2}\right) = -E^1_{2n+1,2j-1}\left(\frac{\pi}{2}\right),$$
$$E^1_{2n+1,2j-1}\left(\frac{\pi t}{2}\right) = (-1) E^1_{2n+1,2j-1}\left(-\frac{\pi t}{2}\right) = -E^1_{2n+1,2j-1}\left(\frac{\pi t}{2}\right).$$

Chapter 2

Powers of Dedekind's eta function

Let $q = e^{2\pi i\tau}$ where $\mathrm{Im}(\tau) > 0$. Dedekind's eta-function is defined as

$$\eta = \eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1 - q^k) = q^{\frac{1}{24}}(q)_\infty.$$

For brevity, we sometimes omit the dependence on τ and just write η.

Certain powers of η possess very remarkable properties. M. Newman (1955) and J. P. Serre (1985) proved two interesting theorems for some even powers of η using the theory of modular forms. On the other hand, S. Ramanujan (1919) used elementary methods to generalize the classical results of Euler and Jacobi for η and η^3. We shall describe their work and present our generalizations in Section 2.1. Explanations and proofs of these results will occupy the rest of the chapter. Most of the results presented in this chapter are original and appear in [CCT07].

2.1 Theorems of Ramanujan, Newman and Serre

Let r denote an even integer and define the coefficients $a_r(n)$ by

$$\eta^r(\tau) = q^{\frac{r}{24}} \sum_{n=0}^{\infty} a_r(n) q^n.$$

Then Newman [New55] proved the following interesting theorem.

Theorem 2.1.1 (Newman). *Let r be a positive even integer and p be a prime > 3 such that $r(p+1) \equiv 0 \pmod{24}$. Defining $a_r(\alpha) = 0$ whenever α is not a non-negative integer, we have*

$$a_r\left(np + \frac{r}{24}\left(p^2 - 1\right)\right) = (-p)^{\frac{r}{2}-1} a_r\left(\frac{n}{p}\right) \tag{2.1}$$

if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$.

To describe another remarkable theorem for η^r, we need the following definition.

Definition 2.1.2. *A power series is lacunary if the arithmetic density of its non-zero coefficients is zero. More precisely, $\sum a(n)q^n$ is lacunary if*

$$\lim_{x\to\infty} \frac{\left|\{n | n \le x, a(n) \ne 0\}\right|}{x} = 0.$$

Serre [Ser85] proved that:

Theorem 2.1.3 (Serre). *If r is a positive even integer, then $\eta^r(\tau)$ is lacunary if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$.*

Both Serre and Newman used the theory of modular forms. On the other hand, using elementary methods, Ramanujan [Ram88, Pg. 369] gave the following.

Theorem 2.1.4 (Ramanujan). *Let m be a non-negative integer and define*

$$S_1(m) = \sum_{\alpha \equiv 1 \pmod 6} (-1)^{(\alpha-1)/6} \alpha^m q^{\alpha^2/24},$$

$$S_3(m) = \sum_{\alpha \equiv 1 \pmod 4} \alpha^m q^{\alpha^2/8}.$$

Then

$$S_1(2m) = \eta(\tau) \sum_{j+2k+3\ell=m} a_{jk\ell} P^j Q^k R^\ell, \tag{2.2}$$

$$S_3(2m+1) = \eta^3(\tau) \sum_{j+2k+3\ell=m} b_{jk\ell} P^j Q^k R^\ell, \tag{2.3}$$

where $a_{jk\ell}$ and $b_{jk\ell}$ are rational numbers, j, k and ℓ are non-negative integers, and P, Q and R are Ramanujan's Eisenstein series defined as

$$P = P(q) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k},$$
$$Q = Q(q) = 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k},$$
$$R = R(q) = 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}.$$

When $m = 0$, (2.2) and (2.3) reduces to the classical formulas of Euler and Jacobi for $\prod_{k\geq 1}(1-q^k)$ and $\prod_{k\geq 1}(1-q^k)^3$ respectively.

The main aim of this chapter is to prove analogues of Theorem 2.1.4 involving η^r when r belongs to $\{2, 4, 6, 8, 10, 14, 26\}$. The cases when $r = 2, 4$ and 6 are direct consequences of Ramanujan's theorem.

Theorem 2.1.5. *Let m and n be non-negative integers and define*

$$\begin{aligned} S_2(m,n) &= S_1(m)S_1(n), \\ S_4(m,n) &= S_1(m)S_3(n), \\ S_6(m,n) &= S_3(m)S_3(n). \end{aligned}$$

Then

$$S_2(2m, 2n) = \eta^2(\tau) \sum_{j+2k+3\ell=m+n} a_{jk\ell}P^jQ^kR^\ell, \tag{2.4}$$
$$S_4(2m, 2n+1) = \eta^4(\tau) \sum_{j+2k+3\ell=m+n} a_{jk\ell}P^jQ^kR^\ell, \tag{2.5}$$
$$S_6(2m+1, 2n+1) = \eta^6(\tau) \sum_{j+2k+3\ell=m+n} a_{jk\ell}P^jQ^kR^\ell. \tag{2.6}$$

In each case, $a_{jk\ell}$ are rational numbers, and j, k and ℓ are non-negative integers.

The cases for $r = 8$ and $r = 10$ are as follows:

Theorem 2.1.6. *Let m and n be non-negative integers and define*

$$S_8(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod 3 \\ \alpha+\beta\equiv 0 \pmod 2}} \alpha^m \beta^n q^{(\alpha^2+3\beta^2)/12}.$$

Then $S_8(1,0) = 0$ and

$$S_8(2m+1,2n) = \eta^8(\tau) \sum_{j+2k+3\ell=m+n-1} a_{jk\ell} P^j Q^k R^\ell, \tag{2.7}$$

provided $m+n \geq 1$. Here $a_{jk\ell}$ are rational numbers, and j, k and ℓ are non-negative integers.

Theorem 2.1.7. *Let m and n be non-negative integers and define*

$$S_{10}(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod 6 \\ \beta\equiv 3 \pmod 6}} (-1)^{(\alpha+\beta-4)/6} (\alpha^m\beta^n - \alpha^n\beta^m) q^{(\alpha^2+\beta^2)/24}.$$

Then

$$S_{10}(2m+1,2n+1) = \eta^{10}(\tau) \sum_{j+2k+3\ell=m+n-1} a_{jk\ell} P^j Q^k R^\ell, \tag{2.8}$$

where $a_{jk\ell}$ are rational numbers, and j, k and ℓ are non-negative integers.

The proofs of Theorem 2.1.6 and 2.1.7 will be presented in Sections 2.2 and 2.3 respectively.

The case for $r = 14$ is more complicated and will be discussed in Section 2.4. All these identities involve the function P which is not a modular form on $\mathrm{SL}_2(\mathbb{Z})$. However, by restricting to modular forms on $\mathrm{SL}_2(\mathbb{Z})$, we can obtain more uniform results. In particular, we have the following theorems.

Theorem 2.1.8. *Let n be a positive integer and define*

$$C_{14}(n|\tau) = \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6} \operatorname{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/12}.$$

Then

$$C_{14}(6n|\tau) = \eta^{14}(\tau) \sum_{4j+6k=6n-6} a_{jk} Q^j R^k. \tag{2.9}$$

Theorem 2.1.9. *Let n be a positive integer and define*

$$\begin{aligned} C_2(n|\tau) &= \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6}(\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/24}, \\ C_2^*(n|\tau) &= \sum_{\substack{\alpha\equiv 0 \pmod 6\\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-1)/6}(\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/36}. \end{aligned}$$

Then

$$\frac{1}{3^{6n}}C_2^*(12n|\tau) - \frac{(-1)^n}{2^{6n}}C_2(12n|\tau) = \eta^{26}(\tau)\sum_{4j+6k=12n-12} a_{jk}Q^jR^k. \tag{2.10}$$

These modular form identities will be discussed in Sections 2.5 and 2.6.

Finally we remark that all these identities can be viewed as extensions of the results of Newman and Serre. By a theorem of Landau [BD04, Pg. 244], each of the series given in (2.4) to (2.10) is lacunary. Moreover, the coefficients of each series satisfy an arithmetic relation analogous to (2.1). We will give explicit examples in Theorems 2.2.4, 2.3.2 and 2.4.3.

2.2 The eighth power of $\eta(\tau)$

In this section we prove Theorem 2.1.6, the analogue of Ramanujan's result for $\eta^8(\tau)$. We first prove two lemmas and a theorem. Let $f^{(\ell)}(z|q)$ denote the ℓ-th derivative of $f(z|q)$ with respect to z.

Lemma 2.2.1.

$$\begin{aligned} &\theta_1^{(2\ell_1+1)}(0|q^{\frac{1}{2}})\theta_1^{(2\ell_2+1)}(0|q^{\frac{1}{2}})\cdots\theta_1^{(2\ell_m+1)}(0|q^{\frac{1}{2}}) \\ &= \big(\eta(\tau)\big)^{3m}\sum_{j+2k+3\ell=\ell_1+\ell_2+\cdots+\ell_m} a_{jk\ell}P^jQ^kR^\ell, \end{aligned}$$

for some rational numbers $a_{jk\ell}$, where j,k and ℓ are non-negative integers.

Proof. We have

$$\theta_1^{(2\ell_1+1)}(z|q) = 2(-1)^{\ell_1} q^{\frac{1}{4}} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2\ell_1+1} q^{k^2+k} \cos(2k+1)z.$$

Therefore

$$\begin{aligned}
\theta_1^{(2\ell_1+1)}(0|q^{\frac{1}{2}}) &= 2(-1)^{\ell_1} q^{\frac{1}{8}} \sum_{k=0}^{\infty} (-1)^k (2k+1)^{2\ell_1+1} q^{(k^2+k)/2} \\
&= 2(-1)^{\ell_1} \sum_{k=-\infty}^{\infty} (4k+1)^{2\ell_1+1} q^{(4k+1)^2/8} \\
&= 2(-1)^{\ell_1} S_3(2\ell_1+1) \\
&= \eta^3(\tau) \sum_{j+2k+3\ell=\ell_1} a_{jk\ell} P^j Q^k R^\ell,
\end{aligned}$$

by Theorem 2.1.4. The general case $m \geq 1$ now follows by multiplying m copies of this result together. $\square$

The second lemma consists of two equivalent forms of the quintuple product identity which appeared in [She99]. More information on this identity can be found in [Coo06].

Lemma 2.2.2.

$$\begin{aligned}
O^0_{3,1}(z) &= \frac{iq^{-1/4}}{(q^2)^2_\infty} \theta_1(z|q)\theta_3(z|q)\theta_4(z|q), \\
O^0_{3,2}(z) &= \frac{iq^{-1/2}}{(q^2)^2_\infty} \theta_1(z|q)\theta_2(z|q)\theta_4(z|q).
\end{aligned}$$

Proof. We prove only the first identity. Since $O^0_{3,1}(z)$ is an odd function, we observe from Corollary 1.2.5 that $O^0_{3,1}(z)$ has three zeroes in the fundamental parallelogram Π, namely $z = 0$, $\pi t/2$ and $(\pi + \pi t)/2$. Furthermore it satisfies the following transformation formulas (see (1.13)),

$$F(z+\pi, t) = -F(z,t) \qquad \text{and} \qquad F(z+\pi t, t) = q^{-3} e^{-6iz} F(z,t).$$

By Propositions 1.1.3 and 1.1.4, the product $\theta_1(z|q)\theta_3(z|q)\theta_4(z|q)$ has the same zeroes and the same transformation formulas as $O^0_{3,1}(z)$. Lemma 1.1.5 then allows us to conclude that the quotient

$$\frac{O^0_{3,1}(z)}{\theta_1(z|q)\theta_3(z|q)\theta_4(z|q)}$$

is independent of z. Now we let $z = \pi/2$ and use Table 1.1, (1.7) and Jacobi's triple product identity to simplify the required constant. We have

$$\begin{aligned}
&\frac{O^0_{3,1}(z)}{\theta_1(z|q)\theta_3(z|q)\theta_4(z|q)} \\
&= \frac{1}{\theta_2(0|q)\theta_4(0|q)\theta_3(0|q)} \sum_{k=-\infty}^{\infty} q^{3k^2+k} \left(e^{(6k+1)\pi i/2} - e^{-(6k+1)\pi i/2}\right) \\
&= \frac{1}{2q^{1/4}(q^2)^3_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2+k} (2i) \\
&= \frac{i(q^2, q^4, q^6; q^6)_\infty}{q^{1/4}(q^2)^3_\infty} \\
&= \frac{iq^{-1/4}}{(q^2)^2_\infty}.
\end{aligned}$$

□

The next theorem is equivalent to Macdonald's identity for A_2 [Mac72]. (See Chapter 3 for an introduction to the Macdonald identities.)

Theorem 2.2.3 (Identity for A_2).

$$\begin{aligned}
O^0_{3,1}(x)\theta_2(x+2y|q) &- q^{1/4}O^0_{3,2}(x)\theta_3(x+2y|q) \\
&= \frac{-iq^{-1/8}}{(q)_\infty}\theta_1(x|q^{\frac{1}{2}})\theta_1(y|q^{\frac{1}{2}})\theta_1(x+y|q^{\frac{1}{2}}).
\end{aligned}$$

Proof. Let

$$M(x,y,t) = O^0_{3,1}(x)\theta_2(x+2y|q) - q^{1/4}O^0_{3,2}(x)\theta_3(x+2y|q)$$

and

$$\begin{aligned} N(x,y,t) &= \theta_1(x|q)\theta_1(y|q)\theta_1(x+y|q)\theta_4(x|q)\theta_4(y|q)\theta_4(x+y|q) \\ &= \frac{1}{8}\theta_1(x|q^{\frac{1}{2}})\theta_1(y|q^{\frac{1}{2}})\theta_1(x+y|q^{\frac{1}{2}})\theta_2(0|q^{\frac{1}{2}})^3. \end{aligned}$$

The second equality follows from the infinite product formulas in Proposition 1.1.9. The transformation formulas in Proposition 1.1.3 and (1.13) imply that $M(x,y,t)$ and $N(x,y,t)$ satisfy the following:

$$\begin{aligned} F(x+\pi,y,t) &= F(x,y,t); & F(x+\pi t,y,t) &= q^{-4}e^{-8ix-4iy}F(x,y,t); \\ F(x,y+\pi,t) &= F(x,y,t); & F(x,y+\pi t,t) &= q^{-4}e^{-4ix-8iy}F(x,y,t). \end{aligned}$$

Fix y and consider $M(x,y,t)$ and $N(x,y,t)$ as functions of x. $N(x,y,t)$ has four simple zeroes in Π, namely $x=0$, $x=-y$, $x=\pi t/2$ and $x=-y+\pi t/2$. Now, $O^0_{3,1}(x)$ and $O^0_{3,2}(x)$ are both zero at $x=0$ and $x=\pi t/2$.

$$M(-y,y,t) = -O^0_{3,1}(y)\theta_2(y|q) + q^{1/4}O^0_{3,2}(y)\theta_3(y|q)$$

is also zero by Lemma 2.2.2. Therefore $M(x,y,t)/N(x,y,t)$ has at most a simple pole in Π and satisfy the hypothesis of Lemma 1.1.5.

Now fix x and consider $M(x,y,t)$ and $N(x,y,t)$ as functions of y. We can check that $M(x,y,t)/N(x,y,t)$ also has at most a simple pole and thus is independent of y. It follows that

$$\frac{M(x,y,t)}{N(x,y,t)} = C(t)$$

for some constant $C(t)$ independent of x and y. To calculate $C(t)$, we let $x=\pi/2$ and $y=\pi/4$.

$$\begin{aligned} \frac{M\left(\frac{\pi}{2},\frac{\pi}{4},t\right)}{N\left(\frac{\pi}{2},\frac{\pi}{4},t\right)} &= \frac{O^0_{3,1}(\frac{\pi}{2})\theta_2(\pi|q) - 0}{\frac{1}{8}\theta_1\left(\frac{\pi}{2}\middle| q^{\frac{1}{2}}\right)\theta_1\left(\frac{\pi}{4}\middle| q^{\frac{1}{2}}\right)\theta_1\left(\frac{3\pi}{4}\middle| q^{\frac{1}{2}}\right)\theta_2(0|q^{\frac{1}{2}})^3} \\ &= \frac{-2i(q^2)_\infty \cdot 2q^{1/4}(q^2)_\infty(-q^2;q^2)^2_\infty}{2q^{1/8}(q)_\infty(-q;q)^2_\infty\left(\sqrt{2}q^{1/8}(q)_\infty(-q^2;q^2)_\infty\right)^2 \cdot \frac{1}{8}\theta_2(0|q^{\frac{1}{2}})^3} \\ &= \frac{-i}{q^{1/8}(q)_\infty \cdot \frac{1}{8}\theta_2(0|q^{\frac{1}{2}})^3}. \end{aligned}$$

Hence

$$\begin{aligned} M(x,y,t) &= O^0_{3,1}(x)\theta_2(x+2y|q) - q^{1/4}O^0_{3,2}(x)\theta_3(x+2y|q) \\ &= \frac{-iq^{-1/8}}{(q)_\infty}\theta_1(x|q^{\frac{1}{2}})\theta_1(y|q^{\frac{1}{2}})\theta_1(x+y|q^{\frac{1}{2}}). \end{aligned}$$

□

We shall use the A_2 identity to prove Theorem 2.1.6.

Proof of Theorem 2.1.6. We change variables by setting x as u and $x+2y$ as v. Next, apply $\dfrac{\partial^{2m+2n+1}}{\partial u^{2m+1}\partial v^{2n}}$ to the identity in Theorem 2.2.3 and let $u=v=0$. The left hand side is

$$\begin{aligned} &\sum_{j=-\infty}^{\infty}(-1)^m(6j+1)^{2m+1}q^{3j^2+j}(2i)\sum_{k=-\infty}^{\infty}(-1)^n(2k+1)^{2n}q^{k^2+k+1/4} \\ &\quad - q^{1/4}\sum_{j=-\infty}^{\infty}(-1)^m(6j+2)^{2m+1}q^{3j^2+2j}(2i)\sum_{k=-\infty}^{\infty}(-1)^n(2k)^{2n}q^{k^2} \\ &= 2(-1)^{m+n}iq^{-1/12}\sum_{\alpha\equiv 1 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 1 \pmod 2}\beta^{2n}q^{\beta^2/4} \\ &\quad + 2(-1)^{m+n}iq^{-1/12}\sum_{\alpha\equiv 4 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 0 \pmod 2}\beta^{2n}q^{\beta^2/4} \\ &= 2(-1)^{m+n}iq^{-1/12}\sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}}\alpha^{2m+1}\beta^{2n}q^{(\alpha^2+3\beta^2)/12}. \end{aligned} \tag{2.11}$$

Since $\theta_1(z|q^{\frac{1}{2}})$ is an odd function, the right hand side is a linear combination of terms of the form

$$\frac{-iq^{-1/12}}{\eta(\tau)}\theta_1^{(2\ell_1+1)}(0|q^{\frac{1}{2}})\theta_1^{(2\ell_2+1)}(0|q^{\frac{1}{2}})\theta_1^{(2\ell_3+1)}(0|q^{\frac{1}{2}})$$

where $(2\ell_1+1)+(2\ell_2+1)+(2\ell_3+1)=2m+2n+1$. By Lemma 2.2.1, the right hand side is therefore of the form

$$\eta^8(\tau)\sum_{j+2k+3\ell=m+n-1}a_{jk\ell}P^jQ^kR^\ell. \tag{2.12}$$

If we combine (2.11) and (2.12), we complete the proof of the theorem for the case $m+n \geq 1$. The fact that $S_8(1,0) = 0$ can be deduced easily from Lemma 2.2.2. $\square$

The following identities are consequences of Theorem 2.1.6.

$$\begin{aligned}
S_8(1,0) &= 0, \\
S_8(3,0) &= -6\eta^8(\tau), \\
S_8(5,0) &= -30\eta^8(\tau)P, \\
S_8(7,0) &= -\frac{63}{2}\eta^8(\tau)(5P^2 - Q), \\
S_8(7,2) &= 2\eta^8(\tau)R, \\
S_8(5,4) &= \eta^8(\tau)(5P^3 - 3PQ).
\end{aligned}$$

An identity equivalent to $S_8(1,2) = 2\eta^8(\tau)$ was stated without proof by L. Winquist [Win69]. The formula for $\eta^8(\tau)$ given by F. Klein and R. Fricke [KF92, Pg. 373] can be shown to be equivalent to $S_8(3,0) + 27S_8(1,2) = 48\eta^8(\tau)$. We end this section with an extension of (2.1) (Newman's theorem) for the coefficients of $S_8(m,n)$.

Theorem 2.2.4. *Let m be odd and n be even and*

$$S_8(m,n) = q^{1/3}\sum_{k=0}^{\infty} a(k)q^k.$$

Then the coefficients satisfy

$$a\left(pk + \frac{p^2-1}{3}\right) = -p^{m+n}a\left(\frac{k}{p}\right),$$

where p is prime and $p \equiv 5 \pmod 6$.

Proof.

$$S_8(m,n) = \sum_{\substack{u\equiv 1 \pmod 6\\ v\equiv 1 \pmod 2}} u^m v^n q^{(u^2+3v^2)/12} + \sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 0 \pmod 2}} x^m y^n q^{(x^2+3y^2)/12}.$$

Hence

$$a(k) = \sum_{\substack{u\equiv 1 \pmod 6\\ v\equiv 1 \pmod 2\\ 12k=u^2+3v^2-4}} u^m v^n + \sum_{\substack{x\equiv 4 \pmod 6\\ y\equiv 0 \pmod 2\\ 12k=x^2+3y^2-4}} x^m y^n.$$

When

$$\frac{u^2+3v^2-4}{12} = pk + \frac{(p^2-1)}{3}, \tag{2.13}$$

we have $u^2+3v^2 \equiv 0 \pmod p$. Since -3 is a quadratic non-residue for primes $p \equiv 5 \pmod 6$, we conclude that $u \equiv v \equiv 0 \pmod p$. Let $u = -u_1 p$ and $v = v_1 p$, where $u_1 \equiv 1 \pmod 6$ and $v_1 \equiv 1 \pmod 2$. Then (2.13) reduces to

$$u_1^2 + 3v_1^2 = 12\frac{k}{p} + 4.$$

Similarly, we let $x = -x_1 p$ and $y = y_1 p$, where $x_1 \equiv 4 \pmod 6$ and $y_1 \equiv 0 \pmod 2$. Then we have

$$\begin{aligned} & a\left(pk + \frac{p^2-1}{3}\right) \\ = & \sum_{\substack{u_1\equiv 1 \pmod 6\\ v_1\equiv 1 \pmod 2\\ 12(k/p)=u_1^2+3v_1^2-4}} (-1)^m p^{m+n} u_1^m v_1^n + \sum_{\substack{x_1\equiv 4 \pmod 6\\ y_1\equiv 0 \pmod 2\\ 12(k/p)=x_1^2+3y_1^2-4}} (-1)^m p^{m+n} x_1^m y_1^n \\ = & -p^{m+n} a\left(\frac{k}{p}\right). \end{aligned}$$

$\square$

2.3 The tenth power of $\eta(\tau)$

The first few examples of Theorem 2.1.7 are:

$$\begin{aligned}
S_{10}(3,1) &= -48\eta^{10}(\tau),\\
S_{10}(5,1) &= -480\eta^{10}(\tau)P,\\
S_{10}(7,1) &= -336\eta^{10}(\tau)(15P^2-2Q),\\
S_{10}(5,3) &= -144\eta^{10}(\tau)(5P^2-2Q),\\
S_{10}(9,1) &= -192\eta^{10}(\tau)(315P^3-126PQ+16R),\\
S_{10}(7,3) &= -288\eta^{10}(\tau)(35P^3-28PQ+8R).
\end{aligned}$$

To prove Theorem 2.1.7, we construct an equivalent form of Winquist's identity [Win69], which is also the Macdonald identity for B_2 [Mac72]. See [BCLY04, CS72, CLN05, CCT07, Hir87, Kan97, KL03, Liu05] for more information about this identity.

Theorem 2.3.1 (Identity for B_2)**.**

$$O^1_{3,1}(x)O^1_{3,3}(y) - O^1_{3,1}(y)O^1_{3,3}(x) = \frac{-2}{q(q^2)^2_\infty}\theta_1(x|q)\theta_1(y|q)\theta_1(x+y|q)\theta_1(x-y|q).$$

Proof. Let $F(x,y,t)$ denote the function on the left hand side of the above identity. It satisfies the transformation formulas:

$$\begin{aligned}
F(x+\pi,y,t) &= -F(x,y,t); & F(x+\pi t,y,t) &= -q^{-3}e^{-6ix}F(x,y,t);\\
F(x,y+\pi,t) &= -F(x,y,t); & F(x,y+\pi t,t) &= -q^{-3}e^{-6iy}F(x,y,t).
\end{aligned}$$

If y is fixed, $F(x,y,t)$ has zeroes at $x=0$ and $x=\pm y$ in Π. Thus, the quotient

$$\frac{F(x,y,t)}{\theta_1(x|q)\theta_1(y|q)\theta_1(x+y|q)\theta_1(x-y|q)}$$

satisfies the hypothesis of Lemma 1.1.5 and equals a constant independent of x. The same conclusion holds when we exchange the roles of x and y. Hence

$$F(x,y,t) = C(t)\theta_1(x|q)\theta_1(y|q)\theta_1(x+y|q)\theta_1(x-y|q),$$

for some constant $C(t)$. To determine the value of this constant, we let $x = \pi/2$ and $y = \pi/3$. Since $O^1_{3,3}(\pi/3) = 0$,

$$\begin{aligned}
F\left(\frac{\pi}{2}, \frac{\pi}{3}, t\right) &= -O^1_{3,1}\left(\frac{\pi}{3}\right)O^1_{3,3}\left(\frac{\pi}{2}\right) \\
&= -\sum_{j=-\infty}^{\infty}(-1)^j q^{3j^2+j}(i\sqrt{3})\sum_{k=-\infty}^{\infty}(-1)^k q^{3k^2+3k}\left(e^{(6k+3)\frac{\pi i}{2}} - e^{-(6k+3)\frac{\pi i}{2}}\right) \\
&= -2\sqrt{3}(q^2, q^4, q^6; q^6)_\infty \cdot 2(-q^6, -q^6, q^6; q^6)_\infty \\
&= -4\sqrt{3}(q^2)_\infty(-q^6, -q^6, q^6; q^6)_\infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
F\left(\frac{\pi}{2}, \frac{\pi}{3}, t\right) &= C(t)\theta_1\left(\frac{\pi}{2}\middle| q\right)\theta_1\left(\frac{\pi}{3}\middle| q\right)\theta_1\left(\frac{\pi}{6}\middle| q\right)\theta_1\left(\frac{5\pi}{6}\middle| q\right) \\
&= C(t)\theta_2(0|q)\cdot\sqrt{3}q^{\frac{1}{4}}(q^6)_\infty\left(q^{\frac{1}{4}}(q^2)_\infty(q^2e^{\frac{\pi}{3}}, q^2e^{-\frac{\pi}{3}}; q^2)_\infty\right)^2 \\
&= C(t)\theta_2(0|q)\cdot\sqrt{3}q^{\frac{1}{4}}(q^6)_\infty\left(q^{\frac{1}{4}}\frac{(q^2)_\infty}{(-q^2;q^2)_\infty}(-q^6;q^6)_\infty\right)^2 \\
&= 2\sqrt{3}C(t)q(q^2)^3_\infty(-q^6, -q^6, q^6; q^6)_\infty.
\end{aligned}$$

So

$$C(t) = \frac{-2}{q(q^2)^2_\infty}.$$

$\square$

Proof of Theorem 2.1.7. Set q as $q^{1/2}$ and apply $\dfrac{\partial^{2m+2n+2}}{\partial x^{2m+1}\partial y^{2n+1}}$ to both sides of Theorem 2.3.1. Next, let $x = y = 0$, the left hand side simplifies to

$$\begin{aligned}
&4i^2(-1)^{m+n}\sum_{j=-\infty}^{\infty}(-1)^j(6j+1)^{2m+1}q^{(3j^2+j)/2}\sum_{k=-\infty}^{\infty}(-1)^k(6k+3)^{2n+1}q^{(3k^2+3k)/2} \\
&\quad - 4i^2(-1)^{m+n}\sum_{j=-\infty}^{\infty}(-1)^j(6j+1)^{2n+1}q^{(3j^2+j)/2}\sum_{k=-\infty}^{\infty}(-1)^k(6k+3)^{2m+1}q^{(3k^2+3k)/2}
\end{aligned}$$

$$= -4(-1)^{m+n}q^{-10/24}\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 3 \pmod 6}}(-1)^{(\alpha+\beta-4)/6}\alpha^{2m+1}\beta^{2n+1}q^{(\alpha^2+\beta^2)/24}$$

$$+ 4(-1)^{m+n}q^{-10/24}\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 3 \pmod 6}}(-1)^{(\alpha+\beta-4)/6}\alpha^{2n+1}\beta^{2m+1}q^{(\alpha^2+\beta^2)/24}$$

$$= 4(-1)^{m+n+1}q^{-10/24}\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 3 \pmod 6}}(-1)^{(\alpha+\beta-4)/6}\big(\alpha^{2m+1}\beta^{2n+1}-\alpha^{2n+1}\beta^{2m+1}\big)q^{(\alpha^2+\beta^2)/24}.$$

We omit the remaining details as they are similar to those in the proof of Theorem 2.1.6. □

We also have the following result for the coefficients of $S_{10}(m,n)$.

Theorem 2.3.2. *Let m and n be odd and*

$$S_{10}(m,n) = q^{10/24}\sum_{k=0}^{\infty}a(k)q^k.$$

Then the coefficients satisfy

$$a\left(pk+\frac{5}{12}(p^2-1)\right) = p^{m+n}a\left(\frac{k}{p}\right),$$

where $p>3$ is prime and $p\equiv 3 \pmod 4$.

Proof. We define

$$\epsilon = \begin{cases} 1 & \text{if} \quad p\equiv 7 \pmod{12}\\ -1 & \text{if} \quad p\equiv 11 \pmod{12}.\end{cases}$$

Since

$$S_{10}(m,n) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 3 \pmod 6}}(-1)^{(\alpha+\beta-4)/6}\left(\alpha^m\beta^n-\beta^m\alpha^n\right)q^{(\alpha^2+\beta^2)/24},$$

we have

$$a(k) = \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 3 \pmod 6\\ 24k=\alpha^2+\beta^2-10}}(-1)^{(\alpha+\beta-4)/6}\left(\alpha^m\beta^n-\beta^m\alpha^n\right).$$

When

$$\frac{\alpha^2+\beta^2-10}{24} = pk + \frac{5}{12}(p^2-1), \tag{2.14}$$

it follows that $\alpha^2+\beta^2 \equiv 0 \pmod{p}$. Since $p \equiv 3 \pmod{4}$, we must have $u \equiv v \equiv 0 \pmod{p}$. Let $\alpha = u\epsilon p$ and $\beta = v\epsilon p$, where $u \equiv 1 \pmod{6}$ and $v \equiv 3 \pmod{6}$. Then (2.14) reduces to

$$u^2+v^2 = 24\frac{k}{p} + 10.$$

We can also check that

$$(-1)^{(\alpha+\beta-4)/6} = (-1)^{(u+v-4)/6}.$$

Thus

$$\begin{aligned} a\left(pk + \frac{5(p^2-1)}{12}\right) &= \sum_{\substack{u\equiv 1 \pmod{6} \\ v\equiv 3 \pmod{6} \\ 24(k/p)=u^2+v^2-10}} (\epsilon p)^{m+n}(-1)^{(u+v-4)/6}\left(u^m v^n - v^m u^n\right) \\ &= p^{m+n} a\left(\frac{k}{p}\right). \end{aligned}$$

□

2.4 The fourteenth power of $\eta(\tau)$

Similar to the cases of the eighth and tenth powers, the analogue of Ramanujan's theorem for $\eta^{14}(\tau)$ is deduced from a theta function identity, which in this case is equivalent to the Macdonald identity for G_2 [Mac72, Coo97a].

Theorem 2.4.1 (Identity for G_2)**.**

$$\begin{aligned} O^1_{6,4}(x+y)O^1_{2,2}(x-y) + O^1_{6,4}(y)O^1_{2,2}(2x+y) - O^1_{6,4}(x)O^1_{2,2}(x+2y) \\ = \frac{2q^{-\frac{3}{2}}}{(q^2)^4_\infty}\theta_1(x|q)\theta_1(y|q)\theta_1(x+y|q) \\ \times\, \theta_1(x-y|q)\theta_1(2x+y|q)\theta_1(x+2y|q). \end{aligned} \tag{2.15}$$

Proof. Once again, we shall use Lemma 1.1.5. Exploiting the symmetry, it suffices to let y be fixed and consider both sides of the identity as functions of x, satisfying the transformations

$$F(x+\pi,y,t)=F(x,y,t) \quad \text{and} \quad F(x+\pi t,y,t)=q^{-8}e^{-16ix-8iy}F(x,y,t).$$

We can next check that both sides are zero at the following points:

$$\begin{array}{llll} x=0, & x=\pm y, & x=-2y, & \\ x=-\frac{y}{2}, & x=-\frac{y}{2}+\frac{\pi}{2}, & x=-\frac{y}{2}+\frac{\pi t}{2}, & x=-\frac{y}{2}+\frac{\pi+\pi t}{2}. \end{array}$$

What remains is the constant which can be calculated by setting $y=\pi/2$ and $x=\pi t/2$. □

Now replace q by $q^{1/2}$ and apply $\left.\dfrac{\partial}{\partial x}\right|_{x=0}$ followed by $\left.\dfrac{\partial^5}{\partial y^5}\right|_{y=0}$ to Theorem 2.4.1 to obtain

$$\mathcal{T}(5,1)-10\mathcal{T}(3,3)+9\mathcal{T}(1,5)=-30\eta^{14}(\tau), \tag{2.16}$$

where

$$\mathcal{T}(m,n) = \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\alpha^m\beta^n q^{(\alpha^2+3\beta^2)/12}.$$

Applying appropriate differential operators, we get additional identities:

$$\begin{aligned} -210\eta^{14}(\tau)P &= \mathcal{T}(7,1)-7\mathcal{T}(5,3)-21\mathcal{T}(3,5)+27\mathcal{T}(1,7) \\ -210\eta^{14}(\tau)(8P^2-Q) &= \mathcal{T}(9,1)-4\mathcal{T}(7,3)-42\mathcal{T}(5,5)-36\mathcal{T}(3,7)+81\mathcal{T}(1,9) \\ 4290\eta^{14}(\tau)R &= \mathcal{T}(11,1)-55\mathcal{T}(9,3)+594\mathcal{T}(7,5) \\ &\quad -1782\mathcal{T}(5,7)+1485\mathcal{T}(3,9)-243\mathcal{T}(1,11). \end{aligned}$$

From the above examples, it is clear that the formulas involving $\eta^{14}(\tau)$ are much more complicated. The derivation of the following theorem from the G_2 identity can be found in [CCT07, Sect. 5].

Theorem 2.4.2. *Let*

$$S_{14}(m,n,p) = \sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6} \left(\beta(\alpha^2-\beta^2)\right)^m \left(\alpha(\alpha^2-9\beta^2)\right)^n \left(\alpha^2+3\beta^2\right)^p q^{(\alpha^2+3\beta^2)/12}.$$

Then

$$S_{14}(2m+1,2n+1,p) = \eta^{14}(\tau) \sum_{j+2k+3\ell=3m+3n+p} a_{jk\ell} P^j Q^k R^\ell, \tag{2.17}$$

where $a_{jk\ell}$ are rational numbers, and j,k and ℓ are non-negative integers.

Theorem 2.4.3. *Let m and n be odd and*

$$\mathcal{T}(m,n) = q^{14/24} \sum_{k=0}^{\infty} a(k) q^k.$$

Then the coefficients satisfy

$$a\left(pk + \frac{7}{12}(p^2-1)\right) = p^{m+n} a\left(\frac{k}{p}\right),$$

where p is prime and $p \equiv 11 \pmod{12}$.

2.5 Modular form identities

In previous sections, we used equivalent forms of Macdonald's identities to deduce analogues of Ramanujan's theorem. In this section, we will reconsider the same problem using the theory of modular forms. The advantage is that this approach works uniformly for all cases of $r \in \{2,4,6,8,10,14\}$. A minor modification is required when $r = 26$ and this will be explained in Section 2.6. Relevant facts about modular forms can be found in Appendix A.2.

Theorem 2.5.1. *Let*

$$C_2(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/24},$$

then $C_2(4n|\tau)/\eta^2(\tau)$ is a weight $4n$ modular form on $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 2.5.2. *Let*

$$C_2^*(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 0 \pmod 6\\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-1)/6}(\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/36},$$

then $C_2^*(6n|\tau)/\eta^2(\tau)$ *is a weight* $6n$ *modular form on* $\mathrm{SL}_2(\mathbb{Z})$*.*

Theorem 2.5.3. *Let*

$$C_4(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-1)/6}\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/24},$$

then $C_4(2n+1|\tau)/\eta^4(\tau)$ *is a weight* $2n$ *modular form on* $\mathrm{SL}_2(\mathbb{Z})$*.*

Theorem 2.5.4. *Let*

$$C_6(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 4\\ \beta\equiv 1 \pmod 4}} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/8},$$

then $C_6(4n+2|\tau)/\eta^6(\tau)$ *is a weight* $4n$ *modular form on* $\mathrm{SL}_2(\mathbb{Z})$*.*

Theorem 2.5.5. *Let*

$$C_8(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}} (\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/12},$$

then $C_8(6n+3|\tau)/\eta^8(\tau)$ *is a modular form of weight* $6n$ *on* $\mathrm{SL}_2(\mathbb{Z})$*.*

Theorem 2.5.6. *Let*

$$C_{10}(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} \mathrm{Im}\left((\alpha+i\beta)^n\right) q^{(\alpha^2+\beta^2)/12},$$

then $C_{10}(4n+4|\tau)/\eta^{10}(\tau)$ *is a modular form of weight* $4n$ *on* $\mathrm{SL}_2(\mathbb{Z})$*.*

Theorem 2.5.7. *Let*

$$C_{14}(n|\tau) \;=\; \sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/12},$$

then $C_{14}(6n+6|\tau)/\eta^{14}(\tau)$ *is a modular form of weight* $6n$ *on* $\mathrm{SL}_2(\mathbb{Z})$*.*

A few special cases of the above theorems appeared in Ramanujan's Lost Notebook, for example [Ram88, p.249]. Some of these identities have been examined by S. S. Rangachari [Ran82, Ran88].

We shall give a detailed proof of Theorem 2.5.5. The details for the other theorems are similar, and a detailed proof of Theorem 2.5.6 can be found in [CCT07]. We first recall some properties of a class of theta functions studied by B. Schoeneberg [Sch74].

Let f be an even positive integer and $A = (a_{\mu,\nu})$ be a *symmetric* $f \times f$ matrix such that

1. $a_{\mu,\nu} \in \mathbb{Z}$;
2. $a_{\mu,\mu}$ is even;
3. $\mathbf{x}^t A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^f$ such that $\mathbf{x} \neq \mathbf{0}$.

Let N be the smallest positive integer such that NA^{-1} also satisfies conditions 1—3. Let

$$P_k^A(\mathbf{x}) := \sum_{\mathbf{y}} c_{\mathbf{y}}(\mathbf{y}^t A\mathbf{x})^k,$$

where the sum is over finitely many $\mathbf{y} \in \mathbb{C}^f$ with the property $\mathbf{y}^t A\mathbf{y} = 0$, and $c_{\mathbf{y}}$ are arbitrary complex numbers.

When $A\mathbf{h} \equiv \mathbf{0} \pmod N$ and $\mathrm{Im}(\tau) > 0$, we define

$$\vartheta_{A,\mathbf{h},P_k^A}(\tau) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^f \\ \mathbf{n}\equiv\mathbf{h} \pmod N}} P_k^A(\mathbf{n}) e^{\frac{2\pi i\tau}{N}\frac{1}{2}\frac{\mathbf{n}^t A\mathbf{n}}{N}}.$$

The result which we need is the following [Sch74, Pg. 210, Theorem 2]:

Theorem 2.5.8. *The function $\vartheta_{A,\mathbf{h},P_k^A}$ satisfies the following transformation formulas.*

$$\vartheta_{A,\mathbf{h},P_k^A}(\tau+1) = e^{\frac{2\pi i}{N}\frac{1}{2}\frac{\mathbf{h}^t A\mathbf{h}}{N}} \vartheta_{A,\mathbf{h},P_k^A}(\tau)$$

and

$$\vartheta_{A,\mathbf{h},P_k^A}\left(-\frac{1}{\tau}\right) = \frac{(-i)^{\frac{f}{2}+2k}\tau^{\frac{f}{2}+k}}{\sqrt{|\det A|}} \sum_{\substack{\mathbf{g} \pmod N \\ A\mathbf{g}\equiv \mathbf{0} \pmod N}} e^{\frac{2\pi i}{N}\frac{\mathbf{g}^t A\mathbf{h}}{N}} \vartheta_{A,\mathbf{g},P_k^A}(\tau).$$

We will also need the following.

Lemma 2.5.9. *Let*

$$\varphi_{r,s}(\tau) = \varphi_{r,s}(6n+3;\tau) = \sum_{\substack{\alpha\equiv r \pmod{12} \\ \beta\equiv s \pmod{12}}} (3\alpha + i\sqrt{3}\beta)^{6n+3} e^{\frac{2\pi i\tau}{12}\frac{1}{2}\frac{6\alpha^2+2\beta^2}{12}},$$

where we suppressed the dependency on n. Then

$$\varphi_{r,s}(\tau+1) = e^{2\pi i(3r^2+s^2)/12^2}\varphi_{r,s}(\tau) \tag{2.18}$$

and

$$\varphi_{r,s}\left(-\frac{1}{\tau}\right) = \frac{(-i)^7\tau^{6n+4}}{\sqrt{12}} \sum_{\substack{(u,v) \pmod{12} \\ (6u,2v)\equiv(0,0) \pmod{12}}} e^{2\pi i(6ru+2sv)/12^2}\varphi_{u,v}(\tau). \tag{2.19}$$

Proof. These follow from Theorem 2.5.8 on taking

$$A = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} r \\ s \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 1 \\ i\sqrt{3} \end{pmatrix},$$

$N = 12$ and $k = 6n+3$. □

Proof of Theorem 2.5.5. We observe that

$$\begin{aligned}
&6^{6n+3}C_8(6n+3|\tau) \qquad (2.20)\\
&= \sum_{\substack{\alpha\equiv 1 \pmod 6 \\ \beta\equiv 1 \pmod 2}} (6\alpha + i6\sqrt{3}\beta)^{6n+3} q^{(\alpha^2+3\beta^2)/12} + \sum_{\substack{\alpha\equiv 4 \pmod 6 \\ \beta\equiv 0 \pmod 2}} (6\alpha + i6\sqrt{3}\beta)^{6n+3} q^{(\alpha^2+3\beta^2)/12} \\
&= \sum_{\substack{\alpha\equiv 2 \pmod{12} \\ \beta\equiv 6 \pmod{12}}} (3\alpha + i\sqrt{3}\beta)^{6n+3} q^{(3\alpha^2+\beta^2)/12^2} + \sum_{\substack{\alpha\equiv 8 \pmod{12} \\ \beta\equiv 0 \pmod{12}}} (3\alpha + i\sqrt{3}\beta)^{6n+3} q^{(3\alpha^2+\beta^2)/12^2} \\
&= \varphi_{2,6}(\tau) + \varphi_{8,0}(\tau).
\end{aligned}$$

(2.18) implies

$$\varphi_{2,6}(\tau+1)+\varphi_{8,0}(\tau+1)=e^{2\pi i/3}\big(\varphi_{2,6}(\tau)+\varphi_{8,0}(\tau)\big), \tag{2.21}$$

and (2.19) gives

$$\begin{aligned}
&\varphi_{2,6}\left(-\frac{1}{\tau}\right)+\varphi_{8,0}\left(-\frac{1}{\tau}\right)\\
&= \frac{(-i)^7\tau^{6n+4}}{2\sqrt{3}}\sum_{j=0}^{5}\sum_{k=0}^{1}\left(e^{2\pi i(2j+6k)/12}+e^{2\pi i(8j/12)}\right)\varphi_{2j,6k}(\tau)\\
&= \frac{(-i)^7\tau^{6n+4}}{2\sqrt{3}}\Bigg(2\varphi_{0,0}+2\varphi_{6,6}+(-1+i\sqrt{3})\varphi_{4,0}+(-1-i\sqrt{3})\varphi_{8,0}\\
&\qquad +(-1-i\sqrt{3})\varphi_{2,6}+(-1+i\sqrt{3})\varphi_{10,6}\Bigg).
\end{aligned}$$

If we use the relation $\varphi_{r,s}(\tau)=-\varphi_{12-r,12-s}(\tau)$ and simplify, we find that

$$\varphi_{2,6}\left(-\frac{1}{\tau}\right)+\varphi_{8,0}\left(-\frac{1}{\tau}\right)=(-i)^8\tau^{6n+4}\Big(\varphi_{2,6}(\tau)+\varphi_{8,0}(\tau)\Big). \tag{2.22}$$

(2.20), (2.21), (2.22) and (A.4) imply that the function

$$F(\tau):=\frac{C_8(6n+3|\tau)}{\eta^8(\tau)}$$

satisfies the transformation properties

$$F(\tau+1)=F(\tau)\quad\text{and}\quad F\left(-\frac{1}{\tau}\right)=\tau^{6n}F(\tau).$$

Since $F(\tau)$ is holomorphic, it is a modular form of weight $6n$ on $\mathrm{SL}_2(\mathbb{Z})$. This completes the proof of Theorem 2.5.5. □

2.6 The twenty-sixth power of $\eta(\tau)$

The analogue of Ramanujan's theorem for the 26th power of $\eta(\tau)$ is:

Theorem 2.6.1. *Let $n \geq 1$, then*

$$\frac{1}{\eta^{26}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n \frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a weight $12n-12$ modular form on $\mathrm{SL}_2(\mathbb{Z})$

Proof. Calculations using Theorems 2.5.1 and 2.5.2 imply that the first few terms in the q-expansions are

$$\begin{aligned}
C_2(12n|\tau) &= (-64)^n q^{1/12}\Big(1 - \left((2+3i)^{12n} + (2-3i)^{12n}\right) q \\
&\qquad + \left(5^{12n} - (4+3i)^{12n} - (4-3i)^{12n}\right) q^2 + \cdots\Big), \\
C_2^*(12n|\tau) &= (729)^n q^{1/12}\Big(1 - \left((1+2i\sqrt{3})^{12n} + (1-2i\sqrt{3})^{12n}\right) q \\
&\qquad - 5^{12n} q^2 + \cdots\Big).
\end{aligned}$$

The q^2 terms in the two expansions are different because $\mathrm{Re}((4+3i)/5)^{12n} \neq 1$ for any integer n [Niv56, Cor. 3.12]. Therefore $C_2(12n|\tau)$ and $C_2^*(12n|\tau)$ are linearly independent. It follows that

$$\frac{1}{\eta^{2}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n \frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a cusp form of weight $12n$ on $\mathrm{SL}_2(\mathbb{Z})$, and so must be of the form $\eta^{24}(\tau)F$, where F is a modular form of weight $12n-12$. $\square$

Corollary 2.6.2.

$$\eta^{26}(\tau) = \frac{1}{16308864}\left(\frac{C_2(12|\tau)}{64} + \frac{C_2^*(12|\tau)}{729}\right).$$

Proof. Take $n = 1$ in Theorem 2.6.1 and observe that

$$(2+3i)^{12} + (2-3i)^{12} - (1+2i\sqrt{3})^{12} - (1-2i\sqrt{3})^{12} = 16308864.$$

$\square$

The twenty-sixth power of $\eta(\tau)$ is interesting because it cannot be explained by any of the Macdonald identities. The first known formula for $\eta^{26}(\tau)$ was discovered by A.O.L. Atkin [Atk] and his formula was stated without proof in [Dys72]. (See also Sect. 3.1.) The first published proof of an identity for $\eta^{26}(\tau)$ appeared in [Ser85]. A different proof of Corollary 2.6.2 using elliptic parameter methods is given in [CCT06].

Chapter 3

Macdonald's Identities

The Macdonald identities are generalizations of the Weyl denominator formula. We will give a brief background to these identities in Section 3.1. In Section 3.2, we will describe an original construction for the infinite families. Finally in Section 3.3, we will discuss an application of these identities to obtain new formulas for q-products. These results will appear in [Toh].

3.1 Background

F. J. Dyson in his Josiah Willard Gibbs lecture [Dys72], gave the following formula.

$$\eta^{24} = \sum \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{1!\,2!\,3!\,4!} q^n, \qquad (3.1)$$

where the summation is over all sets of integers, a, b, c, d, e, with

$$\begin{aligned} a, b, c, d, e &\equiv 1, 2, 3, 4, 5 \pmod 5 \text{ respectively}, \\ a + b + c + d + e &= 0, \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 10n. \end{aligned}$$

A proof of Dyson's formula can be found on page 46. He went on to say "I found that there exists a formula of the same degree of elegance as (3.1) for the dth power of η whenever d belongs to the following sequence of integers:

$$d = 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots.$$

In fact the case $d = 3$ was discovered by Jacobi (1.1.8) [Jac29], the case $d = 8$ by Klein and Fricke [KF92], and the cases $d = 14, 26$ by Atkin [Atk]."

It turns out that these numbers with the exception of 26[1], correspond to the dimensions of finite-dimensional simple Lie algebras. This important connection was established by I. G. Macdonald. In his landmark paper [Mac72], Macdonald introduced and completely classified affine root systems. The list of all irreducible affine root systems are:

• Infinite families:

$$\begin{gathered} A_{n-1},\ n \geq 2; \quad B_n,\ n \geq 3; \quad B_n^\vee,\ n \geq 3; \quad C_n,\ n \geq 2; \\ C_n^\vee,\ n \geq 2; \quad BC_n,\ n \geq 1; \quad D_n,\ n \geq 4; \end{gathered}$$

• Exceptional cases:

$$G_2,\ G_2^\vee,\ F_4,\ F_4^\vee,\ E_6,\ E_7,\ E_8.$$

To each root system, he associated a multi-variate infinite product, and computed the corresponding Laurent series. These resulted in a list of identities that equate a series to an infinite product, now commonly known as the Macdonald identities. When the variables in the identities are specialized, we obtain the formulas that were alluded to by Dyson.

For example, the A_{n-1} identity is

$$\begin{aligned} \prod_{1\leq j<k\leq n} \left(x_j x_k^{-1}; q\right)_\infty \left(q x_j^{-1} x_k; q\right)_\infty &= \frac{1}{(q)_\infty^{n-1}} \sum_{\mu\in M} x_1^{nm_1} \ldots x_n^{nm_n} q^{\sum_{j=1}^n \frac{n}{2} m_j^2 + j m_j} \\ &\quad \times \prod_{1\leq j<k\leq n} \left(1 - \frac{x_j q^{m_j}}{x_k q^{m_k}}; q\right)_\infty , \end{aligned} \tag{3.2}$$

[1]See Section 2.6 for more on η^{26}

where

$$M = \Big\{\mu = (m_1, \ldots, m_n) \,\Big|\, m_j \in \mathbb{Z},\ \sum_{j=1}^{n} m_j = 0\Big\}.$$

In the previous chapter, we had also seen equivalent forms of the Macdonald identities for A_2, B_2 and G_2.

Several elementary proofs of the Macdonald identities for the infinite families have appeared in the literature, notably by D. Stanton [Sta89]. The A_{n-1} case was studied separately by S. Cooper [Coo97b] and S. C. Milne [Mil85]. Recently, H. Rosengren and M. Schlosser [RS06] used elliptic determinant evaluations to give new proofs for all the infinite families. With the exception of [Mil85] which involves a basic hypergeometric series generalization of the q-binomial theorem, all of the above proofs are based on the affine root systems. In Section 3.2, we will present a construction for all the infinite families that is independent of root systems. (This work will appear in [Toh].)

On the other hand, very little has been done for the exceptional cases since Macdonald. Cooper [Coo97a] gave elementary proofs for G_2 and $G_2^\vee$ and an elliptic function proof for G_2 can be found in [CCT07]. (See Section 2.4.)

We should also mention that there are non elementary approaches to the Macdonald identities. In particular, they can be interpreted in terms of Kac-Moody algebras. (See the introductions in [Kac90] and [Mil85].)

3.2 An original construction for the infinite families

In this section, we will construct the equivalent forms of the Macdonald identities for the infinite families. These identities were discovered in January 2006. We found out later that they appeared in the work of Rosengren and Schlosser [RS06].

However, our construction differs from theirs and is independent of the affine root systems. For ease of comparison with the existing literature, we will label our identities according to the affine root systems.

Previously in Definition 1.2.3, we defined the vector space $V^l_{m,k}$. We shall use the structure of these spaces to construct the Macdonald identities.

Method of Construction

Fix a subspace of $V^l_{m,k}$ spanned by functions $\{F_1, \ldots, F_n\}$, where each F_i is of the type $T^l_{m,j}(z)$, $E^l_{m,j}(z)$ or $O^l_{m,j}(z)$ and $j \equiv k \pmod 2$.

1. Form a determinant function in n independent variables,

$$\det_{1\le j,k\le n}\big(F_j(z_k)\big).$$

2. Locate the zeroes of this function. Some of these are intrinsic zeroes of all the basis functions, i.e. $F_j(a) = 0$, for all $1 \le j \le n$. These can be calculated from Corollary 1.2.5. The remaining zeroes result from the determinant function. For example, if $z_j = z_k$, we will have two identical columns in the determinant.

3. Form the infinite product on the right hand side of the identity with one Jacobi theta function for each zero.

$$\det_{1\le j,k\le n}\big(F_j(z_k)\big) = \text{constant} \times \prod_{\text{intrinsic}} \theta(z_l) \prod \theta_1(z_j - z_k).$$

4. Check that both sides satisfy the same transformation formulas, giving an elliptic function identity.

5. Calculate the constant.

There are altogether twenty possible cases depending on the parities of m, k and l, giving us twenty families of identities.

Theorem 3.2.1 (Identities for $\tilde{A}_{n-1}$). *For $l \equiv n \pmod 2$,*

$$\det_{1\le j,k\le n}\left(T^l_{n,2j-n}(z_k)\right) = \frac{(-i)^{\frac{n^2-n-2}{2}} q^{\frac{-n^2+n-2}{8}}}{(q^2)_\infty^{\frac{n^2-3n+2}{2}}}\,\theta_1\left(z_1+z_2+\ldots+z_n|q\right) \times \prod_{1\le j<k\le n}\theta_1(z_j-z_k|q). \tag{3.3}$$

$$\det_{1\le j,k\le n}\left(T^l_{n,2j-n-1}(z_k)\right) = \frac{(-i)^{\frac{n^2-n}{2}} q^{\frac{-n^2+n}{8}}}{(q^2)_\infty^{\frac{n^2-3n+2}{2}}}\,\theta_4\left(z_1+z_2+\ldots+z_n|q\right) \times \prod_{1\le j<k\le n}\theta_1(z_j-z_k|q). \tag{3.4}$$

For $l \not\equiv n \pmod 2$,

$$\det_{1\le j,k\le n}\left(T^l_{n,2j-n}(z_k)\right) = \frac{(-i)^{\frac{n^2-n}{2}} q^{\frac{-n^2+n-2}{8}}}{(q^2)_\infty^{\frac{n^2-3n+2}{2}}}\,\theta_2\left(z_1+z_2+\ldots+z_n|q\right) \times \prod_{1\le j<k\le n}\theta_1(z_j-z_k|q). \tag{3.5}$$

$$\det_{1\le j,k\le n}\left(T^l_{n,2j-n-1}(z_k)\right) = \frac{(-i)^{\frac{n^2-n}{2}} q^{\frac{-n^2+n}{8}}}{(q^2)_\infty^{\frac{n^2-3n+2}{2}}}\,\theta_3\left(z_1+z_2+\ldots+z_n|q\right) \times \prod_{1\le j<k\le n}\theta_1(z_j-z_k|q). \tag{3.6}$$

Theorem 3.2.2 (Identities for B_n).

$$\det_{1\le j,k\le n}\left(O^1_{2n-1,2j-1}(z_k)\right) = \frac{i^n 2q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\prod_{l=1}^n\theta_1(z_l|q). \tag{3.7}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n-1,2j-1}(z_k)\right) = \frac{2q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\prod_{l=1}^n\theta_2(z_l|q). \tag{3.8}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n-1,2j-2}(z_k)\right) = \frac{2q^{-\frac{n^2-n}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\prod_{l=1}^n\theta_3(z_l|q). \tag{3.9}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n-1,2j-2}(z_k)\right) = \frac{2q^{-\frac{n^2-n}{4}}}{(q^2)_\infty^{n^2-n}}\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\prod_{l=1}^n\theta_4(z_l|q). \tag{3.10}$$

Theorem 3.2.3 (Identities for $B_n^\vee$)**.**

$$\det_{1\le j,k\le n}\left(O^1_{2n,2j}(z_k)\right) = \frac{i^n 2q^{-\frac{n^2+n}{4}}(q^4)_\infty}{(q^2)_\infty^{n^2+1}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_2(z_l|q). \tag{3.11}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n,2j-2}(z_k)\right) = \frac{2q^{-\frac{n^2-n}{4}}(q^4)_\infty}{(q^2)_\infty^{n^2+1}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_3(z_l|q)\theta_4(z_l|q). \tag{3.12}$$

Theorem 3.2.4 (Identities for BC_n)**.**

$$\det_{1\le j,k\le n}\left(E^1_{2n+1,2j-1}(z_k)\right) = \frac{q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2+n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_2(z_l|q)\theta_3(z_l|q)\theta_4(z_l|q). \tag{3.13}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n+1,2j-1}(z_k)\right) = \frac{i^n q^{-\frac{n^2}{4}}}{(q^2)_\infty^{n^2+n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_3(z_l|q)\theta_4(z_l|q). \tag{3.14}$$

$$\det_{1\le j,k\le n}\left(O^0_{2n+1,2j}(z_k)\right) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_2(z_l|q)\theta_4(z_l|q). \tag{3.15}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n+1,2j}(z_k)\right) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q)$$
$$\times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_2(z_l|q)\theta_3(z_l|q). \tag{3.16}$$

Theorem 3.2.5 (Identities for C_n).

$$\det_{1\le j,k\le n}\left(O^0_{2n+2,2j}(z_k)\right) = \frac{i^n q^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+2n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q) \times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_2(z_l|q)\theta_3(z_l|q)\theta_4(z_l|q). \tag{3.17}$$

Theorem 3.2.6 (Identities for $C_n^\vee$).

$$\det_{1\le j,k\le n}\left(O^0_{2n,2j-1}(z_k)\right) = \frac{i^n q^{-\frac{n^2}{4}}}{(-q;q^2)_\infty (q^2)_\infty^{n^2-1}(q^4)_\infty} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q) \times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_4(z_l|q). \tag{3.18}$$

$$\det_{1\le j,k\le n}\left(O^1_{2n,2j-1}(z_k)\right) = \frac{i^n q^{-\frac{n^2}{4}}}{(q)_\infty (q^2)_\infty^{n^2-2}(q^4)_\infty} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q) \times \prod_{l=1}^{n} \theta_1(z_l|q)\theta_3(z_l|q). \tag{3.19}$$

$$\det_{1\le j,k\le n}\left(E^0_{2n,2j-1}(z_k)\right) = \frac{q^{-\frac{n^2}{4}}}{(-q;q^2)_\infty (q^2)_\infty^{n^2-1}(q^4)_\infty} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q) \times \prod_{l=1}^{n} \theta_2(z_l|q)\theta_3(z_l|q). \tag{3.20}$$

$$\det_{1\le j,k\le n}\left(E^1_{2n,2j-1}(z_k)\right) = \frac{q^{-\frac{n^2}{4}}}{(q)_\infty (q^2)_\infty^{n^2-2}(q^4)_\infty} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q) \times \prod_{l=1}^{n} \theta_2(z_l|q)\theta_4(z_l|q). \tag{3.21}$$

Theorem 3.2.7 (Identities for D_n, $n > 1$).

$$\det_{1\le j,k\le n}\left(E^0_{2n-2,2j-2}(z_k)\right) = \frac{4q^{-\frac{n^2-n}{4}}}{(q^2)_\infty^{n^2-2n}} \prod_{1\le j<k\le n} \theta_1(z_j \pm z_k|q). \tag{3.22}$$

Remark 3.2.8. *1. The identities associated to the same root system are equivalent up to a half period transform. (Table 1.1.) For example* (3.7) *can be obtained from* (3.10) *by replacing* z_j *with* $z_j + \pi t/2$ *for all* j*.*

2. Identities (3.19) *and* (3.21) *can be obtained from* (3.18) *and* (3.20) *respectively, by replacing* q *with* $-q$*.*

3. The infinite product on the right hand side of Theorem 3.2.1 has an extra theta factor when compared to the Macdonald identity for A_{n-1}*, thus we label them as* $\tilde{A}_{n-1}$*. Nevertheless the two identities are equivalent.*

We shall now prove the C_n case in detail.

Proof of Identity (3.17)*.* We consider the subspace of $V^0_{m,0}$ with m even, spanned by the functions $\{O^0_{m,j} | j \text{ even}\}$. From the fact that both $O^0_{m,0}(z)$ and $O^0_{m,m}(z)$ are identically zero, and (1.11), we can conclude that there are at most $\frac{m-2}{2}$ linearly independent functions. If we let n denote this number, then the possible candidates for a basis would be

$$\left\{ O^0_{2n+2,2}, \ldots, O^0_{2n+2,2n} \right\}.$$

1. Let $F(z_1, \ldots, z_n)$ denote the determinant expression in (3.17). We first assume that all $z_j, j \neq 1$ are fixed, distinct complex numbers in the fundamental parallelogram Π, that are different from $0, \pi/2$, $\pi t/2$ and $(\pi + \pi t)/2$. Then, $F(z_1, \ldots, z_n)$ can be considered as a function of z_1, i.e $F(z_1, \ldots, z_n) = F(z_1)$.

2. As a function of z_1, $F(z_1)$ is a linear combination of odd functions and is also odd. Corollary 1.2.5 allows us to conclude that the intrinsic zeroes of $F(z_1)$ are the four values $0, \pi/2$, $\pi t/2$ and $(\pi + \pi t)/2$. It is also evident that $F(\pm z_j) = 0$, $2 \leq j \leq n$. This accounts for all the $2n + 2$ zeroes of $F(z_1)$ in Π. (The points $-z_j$ are not in Π but their equivalent points $-z_j + \pi + \pi t$ are.)

3. Now, we let

$$P(z_1,\ldots,z_n)=\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\prod_{l=1}^{n}\theta_1(z_l|q)\theta_2(z_l|q)\theta_3(z_l|q)\theta_4(z_l|q).$$

As a function of z_1, i.e. $P(z_1,\ldots,z_n)=P(z_1)$, it has the same zeroes as $F(z_1)$.

4. Since $F(z_1)$ is a linear combination of $O^0_{2n+2,2j}(z_1)$ for $j=1$ to n, it satisfies the transformation formula (1.13). From Proposition (1.1.3), we check that $P(z_1)$ also satisfies the same transformation formula. Thus the quotient $F(z_1)/P(z_1)$ is elliptic and entire. By Lemma 1.1.5, $F(z_1)/P(z_1)$ is a "constant" expression $c(z_2,\ldots,z_n,q)$ that is independent of z_1.

We can repeat the same argument for each of the z_j and conclude that the quotient $F(z_1,\ldots,z_n)/P(z_1,\ldots,z_n)$ equals a constant $c(q)$ that is dependent only on q. The principle of analytic continuation then allows us to conclude that the identity holds for all z_j.

5. We now calculate explicitly $c(q)$, following the method of [RS06]. Let $z_k=\pi k/(2n+2)$ and let w denote the primitive $(2n+2)$-th root of unity, i.e.

$$w^k=e^{2iz_k}=e^{\frac{2\pi i}{2n+2}k}.$$

Thus,

$$\begin{aligned}
&\det_{1\le j,k\le n}\left(O^0_{2n+2,2j}(z_k)\right)\\
&=\det_{1\le j,k\le n}\left(\sum_{l=-\infty}^{\infty}q^{(2n+2)l^2+2jl}(w^{kj}-w^{-kj})\right)\\
&=\left(\prod_{j=1}^{n}(q^{4n+4},-q^{2n+2+2j},-q^{2n+2-2j};q^{4n+4})_\infty\right)\det_{1\le j,k\le n}\left(w^{kj}-w^{-kj}\right)\\
&=\left((q^{4n+4})^n_\infty\prod_{j=1}^{2n+2}(-q^{2j};q^{4n+4})_\infty\right)(-q^{2n+2};q^{2n+2})^{-1}_\infty\det_{1\le j,k\le n}\left(w^{kj}-w^{-kj}\right)\\
&=(q^{4n+4})^{n-1}_\infty(q^{2n+2};q^{2n+2})_\infty(-q^2;q^2)_\infty\det_{1\le j,k\le n}\left(w^{kj}-w^{-kj}\right),
\end{aligned}\qquad(3.23)$$

where we have used Jacobi's triple product identity to convert the series into infinite products.

To evaluate the determinant expression explicitly, we use [Kra99, Identity (2.3)] to obtain

$$\begin{aligned}
&\det_{1\le j,k\le n}\left(w^{kj}-w^{-kj}\right)\\
&= (w^1\dots w^n)^{-n}\prod_{1\le j<k\le n}(w^j-w^k)(1-w^jw^k)\prod_{k=1}^{n}(w^{2k}-1)\\
&= (w^1\dots w^n)^{-n+1}\prod_{1\le j<k\le n}(w^j-w^k)(1-w^jw^k)\prod_{k=1}^{n}(w^{k}-w^{-k})\\
&= \prod_{1\le j<k\le n}(w^{\frac{j-k}{2}}-w^{\frac{k-j}{2}})(w^{\frac{-j-k}{2}}-w^{\frac{j+k}{2}})\prod_{k=1}^{n}(w^{k}-w^{-k}). \qquad (3.24)
\end{aligned}$$

Next, we evaluate the right hand side of (3.17) for the same values of z_k. From the duplication formula (Proposition 1.1.6) and the infinite product formulas (Proposition (1.1.9)), we have

$$\begin{aligned}
&\prod_{l=1}^{n}\theta_1(z_l|q)\theta_2(z_l|q)\theta_3(z_l|q)\theta_4(z_l|q)\\
&= (-i)^n q^{\frac{n}{2}}(q^2)_\infty^{4n}\prod_{l=1}^{n}(w^l-w^{-l})(q^2w^{2l},q^2w^{-2l};q^2)_\infty\\
&= (-i)^n q^{\frac{n}{2}}(q^2)_\infty^{4n-2}(q^{2n+2};q^{2n+2})_\infty^2\prod_{l=1}^{n}(w^l-w^{-l}), \qquad (3.25)
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{1\le j<k\le n}\theta_1(z_j\pm z_k|q)\\
&= (-i)^{n^2-n}q^{\frac{n^2-n}{4}}(q^2)_\infty^{n^2-n}\prod_{1\le j<k\le n}(w^{\frac{j-k}{2}}-w^{\frac{k-j}{2}})(w^{\frac{j+k}{2}}-w^{\frac{-j-k}{2}})\\
&\quad\times\prod_{1\le j<k\le n}(q^2w^{j-k},q^2w^{k-j},q^2w^{j+k},q^2w^{-j-k};q^2)_\infty. \qquad (3.26)
\end{aligned}$$

We first observe that the products involving only powers of w is equal to the determinant evaluation in (3.24) up to a factor of $(-1)^{\frac{n^2-n}{2}}$. To simplify the four infinite products in the last line of (3.26), we set k as j and j as k for the second product, k as $k+1$ for the third product. For the last product set j as $n-j$ and k as $n-k+1$ to get

$$\begin{aligned}
&\prod_{1\le j<k\le n}(q^2w^{j-k},q^2w^{k-j},q^2w^{j+k},q^2w^{-j-k};q^2)_\infty \\
&= \frac{\prod_{j=1}^{n}\prod_{k=1}^{n}(q^2w^{j-k};q^2)_\infty}{\prod_{k=1}^{n}(q^2)_\infty}\times\prod_{j=1}^{n-1}\prod_{k=1}^{n-1}(q^2w^{j+k+1};q^2)_\infty\prod_{k=1}^{n-1}(q^2w^{2k+1};q^2)_\infty \\
&= \frac{\prod_{j=1}^{n-1}\prod_{k=1}^{2n+2}(q^2w^{j+k};q^2)_\infty}{(q^2)_\infty^{n-1}\prod_{k=1}^{n-1}(q^2w^{k+1},q^2w^{n+k+1};q^2)_\infty}\times\prod_{k=1}^{n-1}(q^2w^{2k+1};q^2)_\infty \\
&= \frac{(q^{4n+4})_\infty^{n-1}(q^2w,q^2w^{n+1},q^2w^{2n+1};q^2)_\infty}{(q^2)_\infty^{n-2}(q^{4n+4})_\infty}\times\prod_{k=1}^{n-1}(q^2w^{2k+1};q^2)_\infty \\
&= \frac{(q^{4n+4})_\infty^{n-2}(-q^2;q^2)_\infty}{(q^2)_\infty^{n-2}}\times\prod_{k=0}^{n}(q^2w^{2k+1};q^2)_\infty. \qquad (3.27)
\end{aligned}$$

Substituting (3.27) into (3.26) and combining with (3.25), we have a simplified expression for the right hand side of identity (3.17). Comparing with the expression (3.23), we can conclude that the constant

$$c(q)=\frac{i^nq^{-\frac{n^2+n}{4}}}{(q^2)_\infty^{n^2+2n}}.$$

□

Theorems 3.2.1 to 3.2.7, provide a clearer picture of the structure of the vector spaces $V^l_{m,k}$.

Theorem 3.2.9. *$V^l_{m,k}$ has dimension m and*

$$\left\{T^l_{m,j}(z)\,\middle|\,j\equiv k \pmod 2,-m<j\le m\right\}$$

forms a basis.

Proof. Since the infinite product in each identity of Theorem 3.2.1 is non-trivial, the determinant is also non-trivial. Hence the set of m functions $T^l_{m,j}$ that appears in the determinant are linearly independent. To complete the proof, we will show in Lemma A.1.4 that the dimension of each $V^l_{m,k}$ is at most m. $\square$

Each $V^l_{m,k}$ can be decomposed into odd or even subspaces spanned respectively by $\left\{O^l_{m,j}(z)\right\}$ or $\left\{E^l_{m,j}(z)\right\}$. The dimensions of the odd and even subspaces depend on the parities of m, l and k. These are tabulated in Table 3.1.

Basis	m odd		m even	
$j \equiv k \pmod 2$	$k=1$	$k=0$	$k=1$	$k=0$
$O^0_{m,j}(z)$	$\frac{m-1}{2}$	$\frac{m-1}{2}$	$\frac{m}{2}$	$\frac{m}{2}-1$
$E^0_{m,j}(z)$	$\frac{m+1}{2}$	$\frac{m+1}{2}$	$\frac{m}{2}$	$\frac{m}{2}+1$
$O^1_{m,j}(z)$	$\frac{m+1}{2}$	$\frac{m-1}{2}$	$\frac{m}{2}$	$\frac{m}{2}$
$E^1_{m,j}(z)$	$\frac{m-1}{2}$	$\frac{m+1}{2}$	$\frac{m}{2}$	$\frac{m}{2}$

Table 3.1: Dimensions of subspaces of $V^l_{m,k}$

3.3 Formulas for q-products

The main application of the Macdonald identities is to obtain formulas for η products in terms of some power series. We shall ignore the fractional powers and consider a general a q-product, i.e. an infinite product of the following form,

$$(q^a)^\alpha_\infty (q^b)^\beta_\infty \cdots (q^k)^\kappa_\infty.$$

In this section, we give new formulas for q-products in terms of determinants, or more precisely Wronskians, as applications of the identities proven in Section 3.2. The method of proof is uniform in all cases. For each variable z_k, we apply an appropriate power of the differential operator $\frac{\partial}{\partial z_k}$, followed by setting $z_k = 0$. Making use of the fact that all even derivatives of $\theta_1(z_k|q)$ is zero at $z_k = 0$, we can obtain a simple expression in terms of $\theta_1'(0|q)$ which equals $2q^{\frac{1}{4}}(q^2)^3_\infty$. (See (1.7).) Replacing q^2 by q gives the required formula. We will illustrate in more detail for the identities associated to $\tilde{A}_{n-1}$, and simply list the differential operator used for the other identities.

Finally, we remark that in the appendix of [Mac72], Macdonald gave one or more specializations of each identity to obtain various formulas. These specializations all correspond to the half-period transforms that gave us the various identities in our list. Besides [Mac72], other representations for powers of $(q)_\infty$ were also given in [LM99a] and [LM99b].

Formula 3.3.1 ($\tilde{A}_{n-1}$: Representations of $(q)_\infty^{n^2+2}$)**.**

$$(q)_\infty^{n^2+2} = \left(\frac{i^{\frac{n^2-n-2}{2}} 2^{\frac{-n^2+n-2}{2}}}{n\prod_{j=1}^{n-1}(-1)^j(j)!}\right) \det_{1\le j,k\le n} f(j,k,n), \tag{3.28}$$

where for $k<n$,

$$\begin{aligned} f(j,k,n) &= \sum_{\ell=-\infty}^{\infty}(-1)^{n\ell}\left((2n\ell+2j-n)i\right)^{k-1} q^{(n\ell^2+(2j-n)\ell)/2} \\ \text{and} \qquad f(j,n,n) &= \sum_{\ell=-\infty}^{\infty}(-1)^{n\ell}\left((2n\ell+2j-n)i\right)^{n} q^{(n\ell^2+(2j-n)\ell)/2}. \end{aligned}$$

Proof. For each k, from 1 to $n-1$, we shall apply successively, the operator $\left(\frac{\partial}{\partial z_k}\right)^{k-1}\Big|_{z_k=0}$. All four identities for $\tilde{A}_{n-1}$ can be treated uniformly at this point. After applying $\left(\frac{\partial}{\partial z_1}\right)^{0}\Big|_{z_1=0}$, the right hand side equals

$$C_\mu\theta_\mu(z_2+\ldots+z_n|\tau)\prod_{j=2}^{n}\theta_1(-z_j|q)\prod_{2\le j,k\le n}\theta_1(z_j-z_k|q).$$

Next apply $\left(\frac{\partial}{\partial z_2}\right)^{1}\Big|_{z_2=0}$ and the right hand side equals

$$C_\mu\theta_\mu(z_3+\ldots+z_n|\tau)\big(-\theta_1'(0|q)\big)\prod_{j=3}^{n}\theta_1(-z_j|q)^2\prod_{3\le j,k\le n}\theta_1(z_j-z_k|q).$$

Inductively, after applying $\left(\frac{\partial}{\partial z_{n-1}}\right)^{n-2}\Big|_{z_{n-1}=0}$, the right hand side equals

$$C_\mu \theta_\mu(z_n|\tau)\left(\prod_{j=1}^{n-1}(-1)^j j!\right)\theta_1'(0|q)^{\frac{(n-2)(n-1)}{2}}\theta_1(-z_n|q)^{n-1}.$$

At this point, if $\mu = 1$, we apply $\left(\frac{\partial}{\partial z_n}\right)^{n}\Big|_{z_n=0}$ to obtain formula 3.3.1 after simplification. Otherwise, we apply $\left(\frac{\partial}{\partial z_n}\right)^{n-1}\Big|_{z_n=0}$ to obtain the following three formulas. □

Formula 3.3.2 ($\tilde{A}_{n-1}$: Consequence of (3.4))**.**

$$(q)_\infty^{n^2-2}(q^{\frac{1}{2}})_\infty^2 = \frac{i^{\frac{n^2-n}{2}}2^{\frac{-n^2+n}{2}}}{\prod_{j=1}^{n-1}(-1)^j(j)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{n\ell}\left((2n\ell+2j-n-1)i\right)^{k-1}q^{(n\ell^2+(2j-n-1)\ell)/2}\right). \tag{3.29}$$

Formula 3.3.3 ($\tilde{A}_{n-1}$: Consequence of (3.5))**.**

$$(q)_\infty^{n^2-2}(q^2)_\infty^2 = \frac{i^{\frac{n^2-n}{2}}2^{\frac{-n^2+n-2}{2}}}{\prod_{j=1}^{n-1}(-1)^j(j)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{(n+1)\ell}\left((2n\ell+2j-n)i\right)^{k-1}q^{(n\ell^2+(2j-n)\ell)/2}\right). \tag{3.30}$$

Formula 3.3.4 ($\tilde{A}_{n-1}$: Consequence of (3.6))**.**

$$\frac{(q)_\infty^{n^2+4}}{(q^2)_\infty^2(q^{\frac{1}{2}})_\infty^2} = \frac{i^{\frac{n^2-n}{2}}2^{\frac{-n^2+n}{2}}}{\prod_{j=1}^{n-1}(-1)^j(j)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{(n+1)\ell}\left((2n\ell+2j-n-1)i\right)^{k-1}q^{(n\ell^2+(2j-n-1)\ell)/2}\right). \tag{3.31}$$

Due to the extra theta factor in identity (3.3), we obtain a representation of $(q)_\infty^{n^2+2}$ instead of $(q)_\infty^{(n-1)^2+2(n-1)}$ given in [Mac72]. This can be overcome with a slight modification. We illustrate this modification by giving a proof of Dyson's formula (3.1). The case for general n can be proven similarly.

Proof of Dyson's formula. We shall use the $\tilde{A}_4$ identity in (3.6). We first introduce a new variable x by replacing z_k with $z_k + x/5$. Our resulting identity is now

$$\det_{1\le j,k\le 5}\left(T^0_{5,2j-6}\left(z_k+\frac{x}{5}\right)\right) = \frac{-q^{-\frac{5}{2}}}{(q^2)^6_\infty}\,\theta_3\left(x+z_1+\ldots+z_5|q\right)\prod_{1\le j<k\le 5}\theta_1(z_j-z_k|q).$$

Now apply $\left(\frac{\partial}{\partial z_k}\right)^{k-1}\Big|_{z_k=0}$ for each k. The right hand side simplifies to

$$-\,2^{10}\left(\prod_{k=1}^{4}k!\right)(q^2)^{24}_\infty\theta_3(x|q). \tag{3.32}$$

On the other hand, we have

$$\begin{aligned}
&\det_{1\le j,k\le 5}\left(\sum_{\ell=-\infty}^{\infty}((10\ell+2j-6)i)^{k-1}\,q^{(5\ell^2+(2j-6)\ell)}e^{2\ell ix}e^{\frac{2j-6}{5}ix}\right)\\
&=\sum_{\sigma\in S_5}\operatorname{sgn}(\sigma)\sum_{(m_1,\ldots,m_5)\in\mathbb{Z}^5}(10m_1-4)^{\sigma(1)-1}(10m_2-2)^{\sigma(2)-1}(10m_3)^{\sigma(3)-1}\\
&\qquad\times(10m_4+2)^{\sigma(4)-1}(10m_5+4)^{\sigma(5)-1}\\
&\qquad\qquad\times q^{5m_1^2+\ldots+5m_5^2-4m_1-2m_2+2m_4+4m_5}e^{2i(m_1+\ldots+m_5)x}.
\end{aligned}$$

Extracting the term independent of x and setting

$$(5m_1-2, 5m_2-1, 5m_3, 5m_4+1, 5m_5+2) = (a,b,c,d,e),$$

we have

$$\begin{aligned}
&\left(\prod_{k=1}^{4}k!\right)(q^2)^{24}_\infty =\\
&\quad\sum_{\sigma\in S_5}\operatorname{sgn}(\sigma)\sum_{\substack{(a,b,c,d,e)\in\mathbb{Z}^5\\ a+b+c+d+e=0}}a^{\sigma(1)-1}b^{\sigma(2)-1}c^{\sigma(3)-1}d^{\sigma(4)-1}e^{\sigma(5)-1}q^{\frac{a^2+b^2+c^2+d^2+e^2-10}{5}}.
\end{aligned}$$

By switching the order of summation and using Vandemonde's determinant formula [Kra99, Identity 2.1], we complete the proof. □

Formula 3.3.5 (B_n: Representations of $(q)^{2n^2+n}_\infty$)**.**

$$\begin{aligned}
(q)^{2n^2+n}_\infty &= \frac{2^{-n^2+n-1}}{\prod_{k=1}^{n}(-1)^{k-1}(2k-1)!} \qquad\qquad (3.33)\\
&\times\det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left((4n-2)\ell+2j-1\right)^{2k-1}q^{((2n-1)\ell^2+(2j-1)\ell)/2}\right).
\end{aligned}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.7). This is equivalent to specialization (a) in [Mac72, p. 135]. $\square$

Formula 3.3.6 (B_n: Consequence of (3.8))**.**

$$\left((q)_\infty^{2n-3}(q^2)_\infty^2\right)^n = \frac{2^{-n^2+n-1}}{\prod_{k=1}^{n-1}(-1)^k(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((4n-2)\ell+2j-1\right)^{2(k-1)} q^{((2n-1)\ell^2+(2j-1)\ell)/2}\right). \tag{3.34}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.8). This is equivalent to specialization (b) in [Mac72, p. 135]. $\square$

Formula 3.3.7 (B_n: Consequence of (3.9))**.**

$$\left(\frac{(q)_\infty^{2n+3}}{(q^2)_\infty^2(q^{\frac{1}{2}})_\infty^2}\right)^n = \frac{2^{n-1}}{\prod_{k=1}^{n-1}(-1)^k(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((2n-1)\ell+j-1\right)^{2(k-1)} q^{((2n-1)\ell^2+(2j-2)\ell)/2}\right). \tag{3.35}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.9). This does not appear in [Mac72]. $\square$

Formula 3.3.8 (B_n: Consequence of (3.10))**.**

$$\left((q)_\infty^{2n-3}(q^{\frac{1}{2}})_\infty^2\right)^n = \frac{2^{n-1}}{\prod_{k=1}^{n-1}(-1)^k(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left((2n-1)\ell+j-1\right)^{2(k-1)} q^{((2n-1)\ell^2+(2j-2)\ell)/2}\right). \tag{3.36}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.10). This is equivalent to specialization (c) in [Mac72, p. 135]. $\square$

Formula 3.3.9 ($B_n^\vee$: Consequence of (3.11))**.**

$$\left((q)_\infty^{n-1}(q^2)_\infty\right)^{2n+1} = \frac{2^{-1}}{\prod_{k=1}^{n}(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left(2n\ell+j\right)^{2k-1}q^{n\ell^2+j\ell}\right). \tag{3.37}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.11). This is equivalent to specialization (a) in [Mac72, p. 136]. □

Formula 3.3.10 ($B_n^\vee$: Consequence of (3.12))**.**

$$\left(\frac{(q)_\infty^{n+1}}{(q^2)_\infty}\right)^{2n-1} = \frac{2^{n-1}}{\prod_{k=1}^{n-1}(-1)^{k}(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left(2n\ell+j-1\right)^{2(k-1)}q^{2n\ell^2+(j-1)\ell}\right). \tag{3.38}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.12). This is equivalent to specialization (b) in [Mac72, p. 136]. □

Formula 3.3.11 (BC_n: Representations of $(q)_\infty^{2n^2-n}$)**.**

$$(q)_\infty^{2n^2-n} = \frac{2^{-n^2+n}}{\prod_{k=1}^{n-1}(-1)^{k}(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left((4n+2)\ell+2j-1\right)^{2(k-1)}q^{((2n+1)\ell^2+(2j-1)\ell)/2}\right). \tag{3.39}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.13).This is equivalent to specialization (c) in [Mac72, p. 138]. □

Formula 3.3.12 (BC_n: Consequence of (3.14))**.**

$$\left(\frac{(q)_\infty^{2n+3}}{(q^2)_\infty^2}\right)^{n} = \frac{2^{-n^2+n}}{\prod_{k=1}^{n}(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((4n+2)\ell+2j-1\right)^{2k-1}q^{((2n+1)\ell^2+(2j-1)\ell)/2}\right). \tag{3.40}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.14). This is equivalent to specialization (a) in [Mac72, p. 138]. □

Formula 3.3.13 (BC_n: Consequence of (3.15))**.**

$$\left((q)_\infty^{2n-3}(q^{\frac{1}{2}})_\infty^2(q^2)_\infty^2\right)^n = \frac{1}{\prod_{k=1}^n(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((2n+1)\ell+j\right)^{2k-1} q^{((2n+1)\ell^2+2j\ell)/2}\right). \tag{3.41}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.15). This is equivalent to specialization (b) in [Mac72, p. 138]. □

Formula 3.3.14 (BC_n: Consequence of (3.16))**.**

$$\left(\frac{(q)_\infty^{2n+3}}{(q^{\frac{1}{2}})_\infty^2}\right)^n = \frac{1}{\prod_{k=1}^n(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{\ell+k-1}\left((2n+1)\ell+j\right)^{2k-1} q^{((2n+1)\ell^2+2j\ell)/2}\right). \tag{3.42}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.16). This is equivalent to specialization (d) in [Mac72, p. 138]. □

Formula 3.3.15 ($C_n^\vee$: Consequence of (3.18))**.**

$$\left((q)_\infty^{n-1}(q^{\frac{1}{2}})_\infty\right)^{2n+1} = \frac{2^{-n^2+n}}{\prod_{k=1}^n(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left(4n\ell+2j-1\right)^{2k-1} q^{(2n\ell^2+(2j-1)\ell)/2}\right). \tag{3.43}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.18). This is equivalent to specialization (a) in [Mac72, p. 137]. □

Formula 3.3.16 ($C_n^\vee$: Consequence of (3.19))**.**

$$\left(\frac{(q)_\infty^{n+2}}{(q^2)_\infty (q^{\frac{1}{2}})_\infty}\right)^{2n+1} = \frac{2^{-n^2+n}}{\prod_{k=1}^{n}(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty} (-1)^{\ell+k-1}\,(4n\ell+2j-1)^{2k-1}\, q^{(2n\ell^2+(2j-1)\ell)/2}\right). \tag{3.44}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.19). This does not appear in [Mac72]. □

Formula 3.3.17 ($C_n^\vee$: Consequence of (3.20))**.**

$$\left(\frac{(q)_\infty^{n+1}}{(q^{\frac{1}{2}})_\infty}\right)^{2n-1} = \frac{2^{-n^2+n}}{\prod_{k=1}^{n-1}(-1)^{k}(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty} (-1)^{k-1}\,(4n\ell+2j-1)^{2k-2}\, q^{(2n\ell^2+(2j-1)\ell)/2}\right). \tag{3.45}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.20). This is equivalent to specialization (b) in [Mac72, p. 137]. □

Formula 3.3.18 ($C_n^\vee$: Consequence of (3.21))**.**

$$\left((q)_\infty^{n-2}(q^2)_\infty (q^{\frac{1}{2}})_\infty\right)^{2n-1} = \frac{2^{-n^2+n}}{\prod_{k=1}^{n-1}(-1)^{k}(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty} (-1)^{\ell+k-1}\,(4n\ell+2j-1)^{2k-2}\, q^{(2n\ell^2+(2j-1)\ell)/2}\right). \tag{3.46}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.21). This does not appear in [Mac72]. □

Formula 3.3.19 (C_n: Representations of $(q)_\infty^{2n^2+n}$)**.**

$$(q)_\infty^{2n^2+n} = \frac{1}{\prod_{k=1}^{n}(-1)^{k-1}(2k-1)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty} (-1)^{k-1}\,((2n+2)\ell+j)^{2k-1}\, q^{(n+1)\ell^2+j\ell}\right). \tag{3.47}$$

Formula	q-product	R.S.	Formula	q-product	R.S.
3.3.1	$(q)_\infty^{n^2+2}$	$\tilde{A}_{n-1}$	3.3.11	$(q)_\infty^{2n^2-n}$	BC_n
3.3.2	$(q)_\infty^{n^2-2}(q^{\frac{1}{2}})_\infty^2$	$\tilde{A}_{n-1}$	3.3.12	$\left(\frac{(q)_\infty^{2n+3}}{(q^2)_\infty^2}\right)^n$	BC_n
3.3.3	$(q)_\infty^{n^2-2}(q^2)_\infty^2$	$\tilde{A}_{n-1}$	3.3.13	$\left((q)_\infty^{2n-3}(q^{\frac{1}{2}})_\infty^2(q^2)_\infty^2\right)^n$	BC_n
3.3.4	$\frac{(q)_\infty^{n^2+4}}{(q^2)_\infty^2(q^{\frac{1}{2}})_\infty^2}$	$\tilde{A}_{n-1}$	3.3.14	$\left(\frac{(q)_\infty^{2n+3}}{(q^{\frac{1}{2}})_\infty^2}\right)^n$	BC_n
3.3.5	$(q)_\infty^{2n^2+n}$	B_n	3.3.15	$\left((q)_\infty^{n-1}(q^{\frac{1}{2}})_\infty\right)^{2n+1}$	$C_n^\vee$
3.3.6	$\left((q)_\infty^{2n-3}(q^2)_\infty^2\right)^n$	B_n	3.3.16	$\left(\frac{(q)_\infty^{n+2}}{(q^2)_\infty(q^{\frac{1}{2}})_\infty}\right)^{2n+1}$	$C_n^\vee$
3.3.7	$\left(\frac{(q)_\infty^{2n+3}}{(q^2)_\infty^2(q^{\frac{1}{2}})_\infty^2}\right)^n$	B_n	3.3.17	$\left(\frac{(q)_\infty^{n+1}}{(q^{\frac{1}{2}})_\infty}\right)^{2n-1}$	$C_n^\vee$
3.3.8	$\left((q)_\infty^{2n-3}(q^{\frac{1}{2}})_\infty^2\right)^n$	B_n	3.3.18	$\left((q)_\infty^{n-2}(q^2)_\infty(q^{\frac{1}{2}})_\infty\right)^{2n-1}$	$C_n^\vee$
3.3.9	$\left((q)_\infty^{n-1}(q^2)_\infty\right)^{2n+1}$	$B_n^\vee$	3.3.19	$(q)_\infty^{2n^2+n}$	C_n
3.3.10	$\left(\frac{(q)_\infty^{n+1}}{(q^2)_\infty}\right)^{2n-1}$	$B_n^\vee$	3.3.20	$(q)_\infty^{2n^2-n}$	D_n

Table 3.2: Formulas for q-products

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2k-1}\Big|_{z_k=0}$ to identity (3.17). This is equivalent to the formula in [Mac72, p. 136]. □

Formula 3.3.20 (D_n: Representations of $(q)_\infty^{2n^2-n}$, $n > 1$)**.**

$$(q)_\infty^{2n^2-n} = \frac{2^{n-2}}{\prod_{k=1}^{n-1}(-1)^k(2k)!} \times \det_{1\le j,k\le n}\left(\sum_{\ell=-\infty}^{\infty}(-1)^{k-1}\left((2n-2)\ell+j-1\right)^{2k-2}q^{(n-1)\ell^2+(j-1)\ell}\right). \tag{3.48}$$

Proof. Apply $\left(\frac{\partial}{\partial z_k}\right)^{2(k-1)}\Big|_{z_k=0}$ to identity (3.22). □

Appendix A

A.1 Theta functions

There is a more general notion of a theta function [FK01, Pg. 72] defined in the following way.

Definition A.1.1. *The theta function with characteristic* $\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} \in \mathbb{R}^2$ *is defined by*

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,t) = \sum_{n=-\infty}^{\infty} \exp 2\pi i\left(\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2 t + \left(n+\frac{\epsilon}{2}\right)\left(z+\frac{\epsilon'}{2}\right)\right).$$

Comparing with Definition 1.2.1, our m-th order Jacobi theta function

$$T^l_{m,j}(\pi z) = e^{-\pi i \frac{jl}{2m}} q^{-(\frac{j}{2m})^2} \theta\begin{bmatrix}\frac{j}{m}\\ l\end{bmatrix}(mz, mt).$$

We list below, some useful results mentioned in Chapters 1 and 3, starting with a proof of Theorem 1.1.8.

Theorem A.1.2 (Jacobi's Triple Product Identity)**.** *For $x \neq 0$ and $|q| < 1$, we have*

$$\sum_{n=-\infty}^{\infty} x^n q^{n^2} = (-xq;q^2)_\infty(-x^{-1}q;q^2)_\infty(q^2;q^2)_\infty.$$

Proof. There are many proofs of this theorem. In the following, we reproduce the proof in [KL03]. (As explained in [Coo98], this proof was actually first given by

Macdonald [Mac72].) Let

$$F(x) = (-xq; q^2)_\infty(-x^{-1}q; q^2)_\infty(q^2; q^2)_\infty$$

and consider the Laurent series expansion

$$F(x) = \sum_{n=-\infty}^{\infty} c_n(q)x^n.$$

The quotient

$$\frac{F(q^2x)}{F(x)} = \frac{(-xq^3; q^2)_\infty(-(xq)^{-1}; q^2)_\infty}{(-xq; q^2)_\infty(-x^{-1}q; q^2)_\infty} = \frac{1}{xq}.$$

Hence

$$xqF(xq^2) = F(x).$$

Equating coefficients, we have the following recurrence

$$c_n(q) = c_{n-1}(q)q^{2n-1} = \ldots = c_0(q)q^{n^2},$$

which gives us

$$F(x) = (-xq; q^2)_\infty(-x^{-1}q; q^2)_\infty(q^2; q^2)_\infty = c_0(q) \sum_{n=-\infty}^{\infty} x^n q^{n^2}. \tag{A.1}$$

To evaluate $c_0(q)$, let ω denote the primitive cube root of unity and substitute $x = -q$, $-\omega q$ and $-\omega^2 q$ respectively, into (A.1). Summing the three resulting equations, we get

$$\begin{aligned} 3(q^6; q^6)_\infty &= c_0(q) \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n}(1 + w^n + w^{2n}) \\ &= c_0(q) \sum_{\substack{n=-\infty \\ 3|n}}^{\infty} 3(-1)^n q^{n^2+n}. \end{aligned}$$

Finally, set q as q^9 and x as $-q^3$ in (A.1) to get

$$(q^6; q^6)_\infty = c_0(q^9) \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+3n}.$$

Comparing these last two equations, we can calculate that $c_0(q)$ is actually independent of q and equals 1. $\square$

Proposition A.1.3. *Let $F^l_{m,j}(z) \in V^l_{m,k}$, $j \equiv k \pmod 2$. Then $F^l_{m,j}(z)$ has exactly m zeroes in Π, the fundamental parallelogram, whose sum is*

$$\left(\frac{m-l}{2} + K\right)\pi + \left(\frac{m-j}{2} + K'\right)\pi t,$$

for some integers K and K'.

Proof. The fact that each $F^l_{m,j}(z)$ has exactly m zeroes is given in Proposition 1.2.4. For the second part, we use the fact that

$$\frac{1}{2\pi i}\int_C \frac{z\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)}dz = \sum \text{zeroes} - \sum \text{poles},$$

where C is the positive contour about $a + \Pi$, for some a.

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{z\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)}dz \\
=\ & \frac{1}{2\pi i}\int_a^{a+\pi}\left(\frac{z\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)} - \frac{(z+\pi t)\left(F^l_{m,j}(z+\pi t)\right)'}{F^l_{m,j}(z+\pi t)}\right)dz \\
& \quad -\frac{1}{2\pi i}\int_a^{a+\pi t}\left(\frac{z\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)} - \frac{(z+\pi)\left(F^l_{m,j}(z+\pi)\right)'}{F^l_{m,j}(z+\pi)}\right)dz \\
=\ & \frac{1}{2\pi i}\int_a^{a+\pi}\left(-\pi t\frac{\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)} + 2mi(z+\pi t)\right)dz \\
& \quad +\frac{1}{2\pi i}\int_a^{a+\pi t}\left(\pi\frac{\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)}\right)dz \\
=\ & \frac{m}{2}(2a+\pi+2\pi t) - \frac{t}{2i}\int_a^{a+\pi}\frac{\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)}dz + \frac{1}{2i}\int_a^{a+\pi t}\frac{\left(F^l_{m,j}(z)\right)'}{F^l_{m,j}(z)}dz.
\end{aligned}$$

To evaluate the last integral, we consider an open domain containing the segment a to $a + \pi t$, where $F^l_{m,j}(z)$ has neither poles nor zeroes. In this domain, the function has an analytic logarithm and we can write

$$F^l_{m,j}(z) = \exp\left(h^l_{m,j}(z)\right).$$

When $z = a + \pi t$, by (1.13),

$$\begin{aligned} F_{m,j}^l(a+\pi t) &= (-1)^l q^{-m} e^{-2mia} F_{m,j}^l(a) \\ &= \exp\left(\pi i l - m\pi i t - 2mia + h_{m,j}^l(a) + 2K\pi i\right), \end{aligned}$$

where K is some integer.

Hence

$$\begin{aligned} \frac{1}{2i}\int_a^{a+\pi t} \frac{\left(F_{m,j}^l(z)\right)'}{F_{m,j}^l(z)} dz &= \frac{1}{2i}\left(h_{m,j}^l(a+\pi t) - h_{m,j}^l(a)\right) \\ &= \frac{\pi l}{2} - \frac{m\pi t}{2} - ma + K\pi. \end{aligned}$$

The other integral can be calculated in a similar fashion to obtain

$$-\frac{t}{2i}\int_a^{a+\pi} \frac{\left(F_{m,j}^l(z)\right)'}{F_{m,j}^l(z)} dz = \left(-\frac{j}{2} + K'\right)\pi t.$$

$\square$

Lemma A.1.4. *$V_{m,k}^l$ has dimension at most m.*

Proof. Suppose on the contrary that the dimension of $V_{m,k}^l$ is strictly greater than m. Let $x_1, x_2, \ldots, x_{m-1}$ be distinct points. Consider the following evaluation map

$$\begin{aligned} \phi : V_{m,k}^l &\to \mathbb{C}^{m-1} \\ f &\mapsto \left(f(x_1), f(x_2), \ldots, f(x_{m-1})\right). \end{aligned}$$

Since $\ker(\phi)$ has dimension at least two, we can find independent $f, g \in \ker(\phi)$. Choose a point α such that

$$\alpha \neq \left(\frac{m-l}{2} + K\right)\pi + \left(\frac{m-j}{2} + K'\right)\pi t - \sum_{k=1}^{m-1} x_k.$$

Then the function $f(\alpha)g(z) - f(z)g(\alpha)$ vanishes at α and each x_i contradicting Proposition A.1.3.

$\square$

A.2 Modular forms

We list below, some standard facts about modular forms. See [Ser73, Chpt. VII] or [Kob93, Chpt. III] for a more detailed account.

Let $\mathrm{SL}_2(\mathbb{Z})$ denote

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z} \, , ad - bc = 1 \right\}.$$

Definition A.2.1. *A modular form $f(\tau)$ of weight k is a holomorphic function on the complex upper half plane, i.e.* $\mathrm{Im}(\tau) > 0$*, satisfying*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{A.2}$$

Definition A.2.2. *A cusp form $f(\tau)$ is a modular form that vanishes at infinity.*

We use $M_k(\mathrm{SL}_2(\mathbb{Z}))$ (and $S_k(\mathrm{SL}_2(\mathbb{Z}))$) to denote the set of modular (resp. cusp) forms of weight k.

Theorem A.2.3. *A holomorphic function $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ if and only if*

$$f(\tau + 1) = f(\tau), \qquad f(-1/\tau) = \tau^k f(\tau). \tag{A.3}$$

Since modular forms satisfy the first relation in the previous equation, we can express f as a function of q where $q = e^{2\pi i\tau}$, i.e.

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

Definition A.2.4. *Ramanujan's Eisenstein series are defined as*

$$P = P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k},$$

$$Q = Q(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k},$$

$$R = R(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1 - q^k}.$$

$Q(q)$ and $R(q)$ are modular forms of weight 4 and 6 respectively, while $P(q)$ is not a modular form on $\mathrm{SL}_2(\mathbb{Z})$. We usually omit the dependence on q.

Theorem A.2.5. *Ramanujan's Eisenstein series satisfies the following differential equations [Ram16, Eq. 30]:*

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \qquad q\frac{dR}{dq} = \frac{PR - Q^2}{2}.$$

Theorem A.2.6. *[Kob93, Pg. 118] Let $k > 2$, then any $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ can be written in the form,*

$$f(\tau) = \sum_{4i+6j=k} c_{i,j} Q^i R^j.$$

Definition A.2.7. *Dedekind's eta-function is defined as*

$$\eta = \eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k) = q^{\frac{1}{24}} (q)_\infty.$$

$\eta(\tau)$ satisfy the following transformation formula [Kob93, Pg. 121]:

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau), \qquad \eta(-1/\tau) = \sqrt{-i\tau}\, \eta(\tau). \tag{A.4}$$

Theorem A.2.8. *[Kob93, Pg. 117] $S_{12}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}\eta^{24}$. Moreover for $k > 14$, we have*

$$S_k(\mathrm{SL}_2(\mathbb{Z})) = \eta^{24} M_{k-12}(\mathrm{SL}_2(\mathbb{Z})).$$

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