Investigation of interest rate derivatives by Quantum Finance

A thesis submitted

by

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Contents

Introduction

Quantum Finance, which refer to applying the mathematical formalism of quantum mechanics and quantum field theory to finance, shows real advantage in the study of interest rate. In debt market, there is an entire curve of forward interest rates which are imperfect correlated that evolves randomly. Baaquie has pioneered the work of modelling forward interest rates using the formalism of quantum field theory. In the framework of 'Quantum Finance', I present in this dissertation, the investigation of interest rate derivatives from empirical, numerical and theoretical aspects.

In the first chapter, I present a very brief introduction on interest rate and interest rate derivatives. The introduction is very elementary but should be sufficient for the purpose of this dissertation. I explain the concepts and notation needed for detailed investigation in later chapters.

In the second chapter, I provide the review of interest rate models, especially market standard HJM model. The quantum field theory model of interest rate is then presented as a generalization of these models. Market measure in quantum finance is given in this chapter. I carry out the key steps of the derivation of cap and swaption pricing formula in quantum finance.

In the third chapter, I empirically study cap and floor and demonstrate that the field theory model generates the prices fairly accurately based on three different ways of obtaining information from data. Comparison of field theory model with Black's model is also given.

In chapter four, I study the hedging of Libor derivatives. Two different approach, stochastic hedging and minimizing residual variance, are used. Both approaches utilize field theory models to instill imperfect correlation between LIBOR of different maturities as a parsimonious alternative to the existing theory. I then demonstrate the ease with which our formulation is implemented and the implications of correlation on the hedge parameters.

Pricing formula of coupon bond option given in chapter two is empirically studied in chapter five. Besides the price of swaption, volatility and correlation of swaption are computed. An efficient algorithm for calculating forward bond correlators from historical data is given.

Pricing formula for a new instrument, the option on two correlated coupon bonds, will be derived in chapter six. Since this is not a traded instrument yet, both market drift and martingale drift is used.

In chapter seven, I study the American style interest rate derivatives. An efficient algorithm based on 'Quantum Finance' is introduced. New inequalities satisfied by American coupon bond option are verified by the numerical solution. Cap, Floor, Swaption and Coupon bond option with early exercise opportunities are studied in this chapter. Thus the dissertation shows an integrated picture on the subject of applying Quantum Finance to the study of interest rate derivatives.

CHAPTER 1

Interest Rate and Interest Rate Derivatives

§ 1.1 Simple Fixed Income Instruments

The **zero-coupon bond**, denoted as $B(t, T)$ at present time t, is a contract paying a known fixed amount say L, the principal, at some given date in the future, the maturity date T. This promise of future wealth is worth something now: it cannot have zero or negative value. Furthermore, except in extreme circumstances, the amount you pay initially will be smaller than the amount you receive at maturity.

A **coupon-bearing bond** noted as $B_c(t, T)$ at present time t, is similar to the zero-coupon bond except that as well as paying the principal L at maturity, it pays smaller fixed quantities c_i, the coupons, at intervals $T_i, \quad i = 1, 2, \ldots N$ up to and including the maturity date where $T \equiv T_N$. We can think of the coupon bond as a portfolio of zero coupon bonds; one zero-coupon bond for each coupon date with a principal being the same as the original bond's coupon, and a final zero-coupon bond with the same maturity as the original. Then the value of the coupon bond at time $t < T_1$ is given by

$$\sum_{i=1}^{N} c_i B(t, T_i) + LB(t, T) = \sum_{i=1}^{N} a_i B(t, T_i) \tag{1.1}$$

where for simplicity of notation the final payment is included in the sum by setting $a_N = c_N + L$.

Everyone who has a bank account has a **money market account**. This is an account that accumulates interest compounded at a rate that varies from time to time. The rate at which interest accumulates is usually a short-term and unpredictable rate. Suppose at some time t, the account has an amount of money as M. Interest rate for the small interval $t \to t + \Delta t$ is r, then the increase of money in this interval is given by

$$dM = rMdt \tag{1.2}$$

The money market account is very important since the rate is used to discount future cash flow to get time value of money.

In its simplest form a **floating interest rate** is the amount that you get on your bank account. This amount varies from time to time, reflecting the state of the economy and in response to pressure from other banks for your business.

§ 1.2 Interest Rate

§ 1.2.1 Convention of Interest Compounding

To be able to compare fixed-income products we must decide on a convention for the measurement of interest rate. From the money market account equation 1.2, we have a continuously compounded rate, meaning that the present value of 1$ paid at time T in the future is

$$e^{-rT} \times \$1 \tag{1.3}$$

for some constant r. This rate is also the *discounting* rate. [1] Note the rate in real world is always a function of time or even a unpredictable rate. The above convention is used in the options world.

Another common convention is to use the formula

$$\frac{1}{(1 + \epsilon r')^{T/\epsilon}} \times \$1 \tag{1.4}$$

for present value, where r' is some interest rate per year. This represents discretely compounded interest (ϵ=1 year for simplest case) and assumes that interest is accumulated for T years. The formula is derived from calculating the present value from a single-period payment, and then compounding this for each year. This formula is commonly used for the simpler type of instruments such as zero-coupon bond. The two formula are identical, of course, when

$$\epsilon r = \log(1 + \epsilon r') \tag{1.5}$$

[1] The term discounting is fundamental to finance. Consider the interest on a fixed deposit that is rolled over; this leads to an exponential compounding of the initial fixed deposit. Discounting, the inverse of the process of compounding, is the procedure that yields the present day value of a future pre-fixed sum of money.

§ 1.2.2 Yield to Maturity

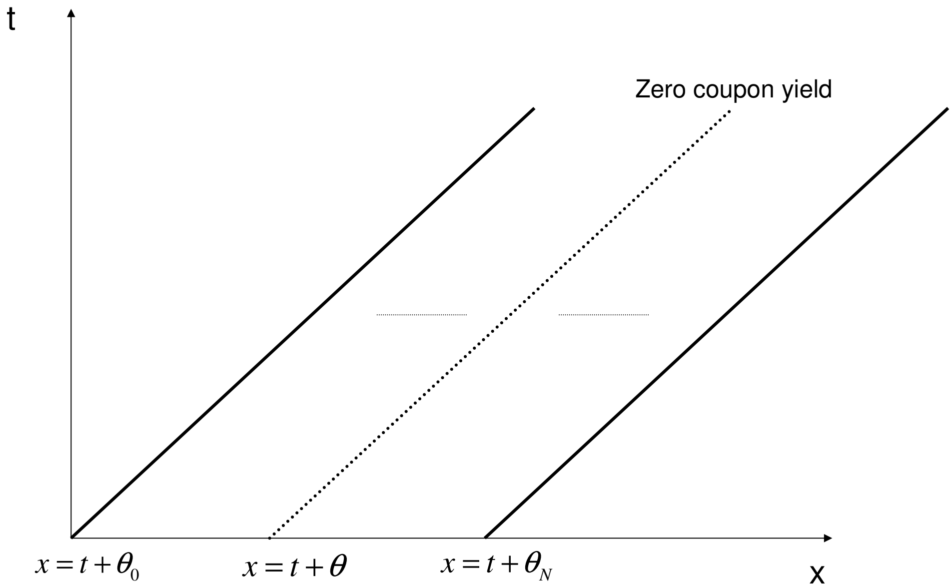

Figure 1.1: Zero coupon yield curve data on lines of constant θ; the θ interval is not a constant. $\theta_N = 30$ years

There is such a variety of fixed-income products, with different coupon structure, fixed and/or floating rates, that it is necessary to be able to compare different products consistently. One way to do this is through measure of how much each contract earns. Suppose that we have a zero-coupon bond maturing at time T when it pays one dollar. At time t is has a value $B(t,T)$. Applying a constant rate of return of y between t and T, then one dollar received at time T has a present value of $B(t,T)$ at time t, where using continuously compounding

$$B(t,T) = e^{-y(T-t)} \tag{1.6}$$

It follows that

$$y = -\frac{\log B(t,T)}{T-t} \tag{1.7}$$

If the bond is a traded security then we know the price at which the bond can be bought. If this is the case then we can calculate the **yield to maturity** or **internal rate of return** as the value y computed from Eq. 1.7. This can be generalized to coupon bond by discounting all coupons and the principal to the present by using some rate y, which is yield to maturity when the present value of the bond is equal to the traded price.

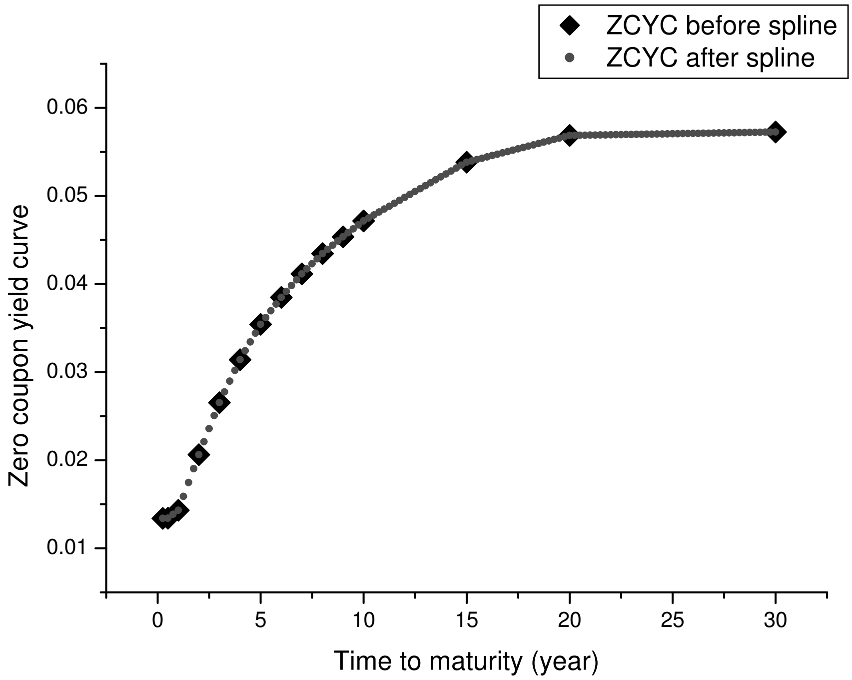

Figure 1.2: Zero coupon yield curve at 2003.1.29 with maturity up to 30 year. Original data and data after interpolation

The plot of yield to maturity against time to maturity is called the **yield curve**. For the moment assume that this has been calculated from zero-coupon bonds and that these bonds have been issued by a perfectly creditworthy source.

The **zero coupon yield curve** (called ZCYC later) provided by Bloomberg is given in $\theta = x - t$ =constant direction, where x is future time, as shown in Fig.1.1 with the interval of θ between two data points as 3m, 6m, 1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y, 15y, 20y, 30y.

Of course, the yield need not be a constant through the interval between two data points. Cubic spline is used to interpolate points every three month, we choose three month as minimum interval since it is the basis of Libor time. The zero coupon yield curve is plotted at time 2003.1.29 for both original data and data after interpolation in Fig.1.2.

Unlike the definition of yield to maturity in 1.6 and 1.7, in this real case discrete compounding convention has to be used. As discussed in § 1.2.1, for zero coupon bond, the

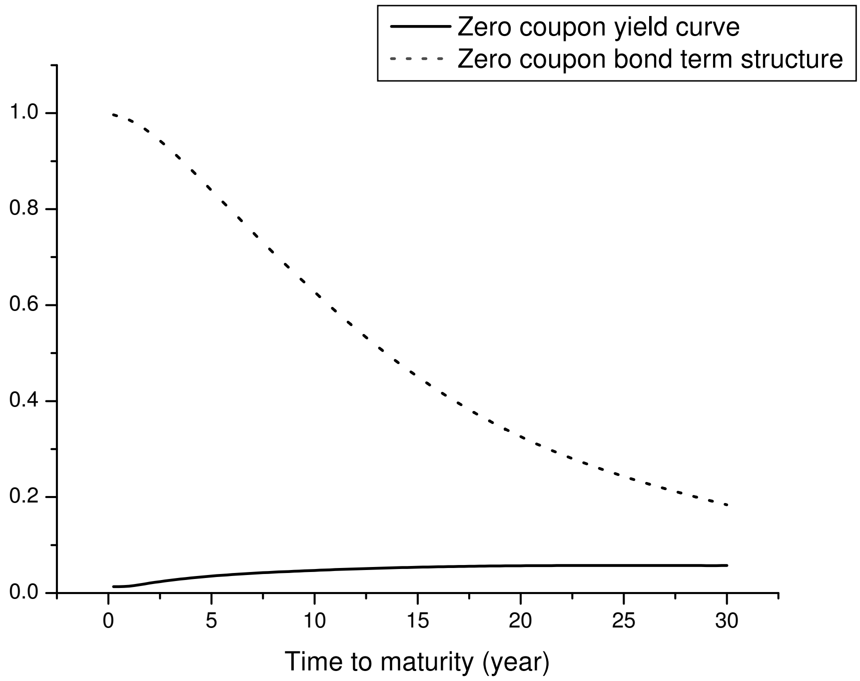

Figure 1.3: Zero coupon bond price and zero coupon yield curve at 2003.1.29 with maturity up to 30 year.

compounding convention is discrete. Also the interest is discretely compounded every three month, thus the zero coupon bond prices for different maturities (denoted as zero coupon bond term structure)are given by

$$B(t, T) = \frac{1}{(1 + y(t, T)/4)^{4(T-t)}} \tag{1.8}$$

and are plotted together with zero coupon yield curve at time 2003.1.29 in Fig. 1.3.

§ 1.2.3 Forward Rates

The main problem with the use of yield to maturity as a measure of interest rates is that it is not consistent across instruments. One five year bond may have a different yield from another five year bond if they have different coupon structures. It is therefore difficult to say that there is a single interest rate associated with a maturity.

One way of overcoming this problem is to use **forward rates**.

Forward interest rates $f(t, x)$ are the interest rates, fixed at time t, for an instantaneous loan at future times $x > t$ that are assumed to apply for all instruments. This contrasts with yields which are assumed to apply up to maturity, with a different yield for each bond. $f(t,x)$ has the dimensions of 1/time.

Now, the price of a zero coupon bond can be given by discounting the payoff of \$1, paid at time T, to present time t by using the prevailing forward interest rates.

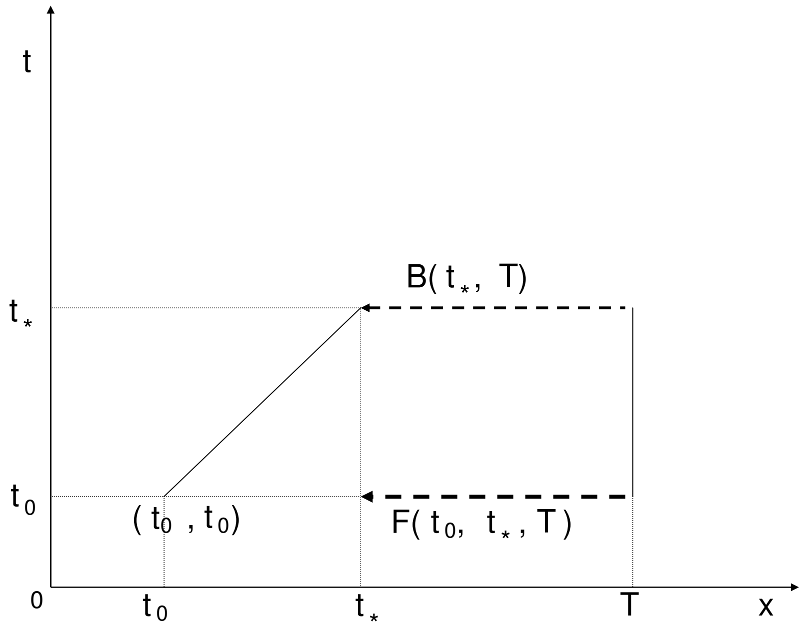

Figure 1.4: The forward interest rates, indicated by the dashed lines, that define a Treasury Bond $B(t_*, T)$ and it's forward price $F(t_0, t_*, T)$.

Discounting the \$1 payoff, paid at maturity time T, is obtained by taking infinitesimal backward time steps ϵ from T to present time t, and yields [2]

$$B(t,T) = e^{-\epsilon f(t,t+\epsilon)}e^{-\epsilon f(t,t+2\epsilon)}..e^{-\epsilon f(t,x)}...e^{-\epsilon f(t,T)}\$1 \quad (1.9)$$

$$\Rightarrow B(t,T) = \exp\{-\int_t^T dx f(t,x)\} \quad (1.10)$$

Suppose a Treasury Bond $B(t_*, T)$ is going to be issued at some future time $t_* > t_0$, and expires at time T; the **forward price** of the Treasury Bond is the price that one pays at time t to lock-in the delivery of the bond when it is issued at time t_*, and is given by

$$F(t_0, t_*, T) = \exp\{-\int_{t_*}^T dx f(t,x)\} = \frac{B(t_0,T)}{B(t_0,t_*)} \quad : \text{ Forward Bond Price} \quad (1.11)$$

[2]The fixed payoff \$ 1 is assumed and is not written out explicitly.

Treasury Bond $B(t_*, T)$, to be issued at time t_* in the future, is graphically represented in Figure 1.4, together with its (present day) forward price $F(t_0, t_*, T)$ at $t_0 < t_*$.

From Eqn. 1.10, the forward rate is given by

$$f(t, x) = -\frac{\partial}{\partial T}(\log B(t, T)) \tag{1.12}$$

Writing this in terms of yields $y(t, T)$ we have

$$B(t, T) = e^{-y(t,T)(T-t)} \tag{1.13}$$

and also

$$f(t, T) = y(t, T) + \frac{\partial y}{\partial T} \tag{1.14}$$

This is the relationship between yields and forward rates when everything is nicely differentiable.

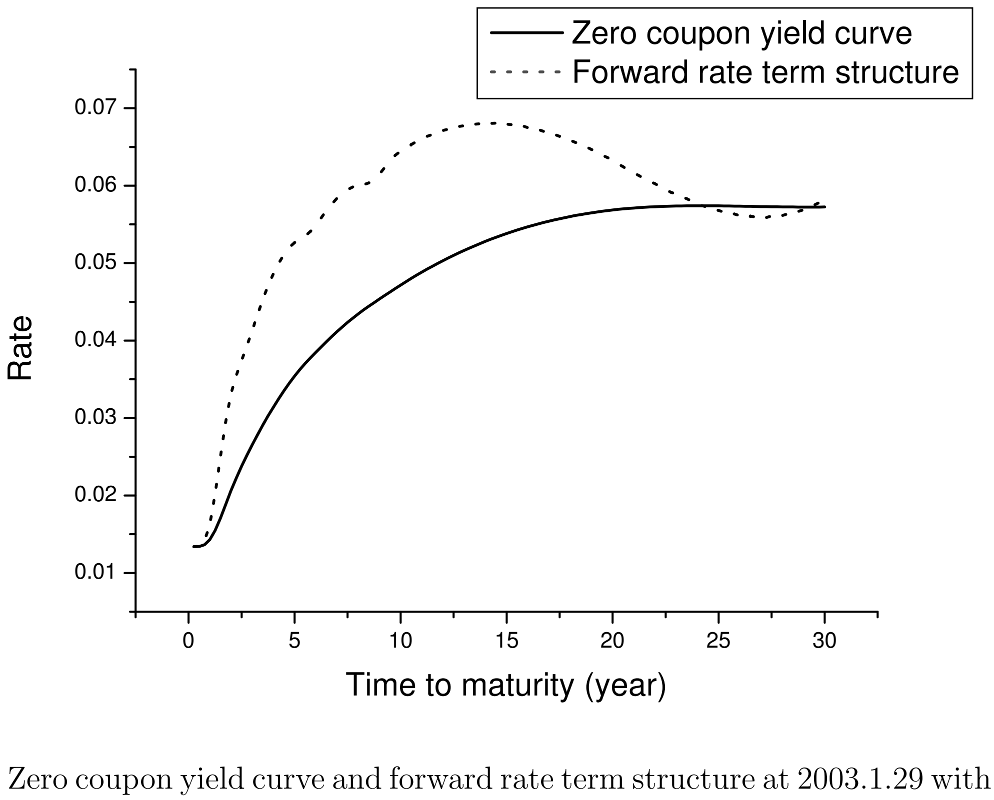

Figure 1.5: Zero coupon yield curve and forward rate term structure at 2003.1.29 with maturity up to 30 year.

However, in the less-than-perfect world we have to deal with only discrete set of data points. The discrete compounding convention has to be used. Thus for $f(t, t+\epsilon)$ we have

$$\begin{aligned} B(t, t+\epsilon) &= \frac{1}{1+\epsilon f(t, t+\epsilon)} \\ \rightarrow f(t, t+\epsilon) &= \frac{B(t, t+\epsilon)^{-1}-1}{\epsilon} \end{aligned} \tag{1.15}$$

This rate will be applied to all instruments whenever we want to discount over this period. For the next period we have

$$\begin{aligned} B(t, t+2\epsilon) &= \frac{1}{(1+\epsilon f(t, t+\epsilon))(1+\epsilon f(t, t+2\epsilon))} \\ \rightarrow f(t, t+2\epsilon) &= \frac{1}{\epsilon}\left(\frac{B(t, t+2\epsilon)}{B(t, t+\epsilon)}-1\right) \end{aligned} \tag{1.16}$$

By this method of bootstrapping we can build up the forward rate curve. The forward rate curve is plotted with zero coupon yield curve at 2003.1.29 in Fig. 1.5.

§ 1.2.4 Libor

We briefly review the main features of the Libor market for the readers who are unfamiliar with this financial instrument. The discussion follows [6]

Eurodollar refer to US$ bank deposits in commercial banks outside the US. These commercial banks are either non-US banks or US banks outside the US. The deposits are made for a fixed time, the most common being 90- or 180-day time deposits, and are exempt from certain US government regulations that apply to time deposits inside the US.

The Eurodollar deposit market constitutes one of the largest financial markets. The Eurodollar market is dominated by London, and the interest rates offered for these US$ time deposits are often based on Libor, the **London Interbank Offer Rate**. The Libor is a simple interest rate derived from a Eurodollar time deposit of 90 days. The minimum deposit for Libor is a par value of $1000000. Libor are interest rates for which commercial banks are willing to lend funds in the interbank market.

Eurodollar futures contracts are amongst the most important instrument for short term contracts and have come to dominate this market. The Eurodollar futures contract, like other futures contracts, is an undertaking by participating parties to loan or borrow a fixed amount of principal at an interest rate fixed by Libor and executed at a specified future date.

Eurodollar futures as expressed by Libor extends to up to ten years into the futures, and hence there are underlying forward interest rates driving all Libor with different maturities.

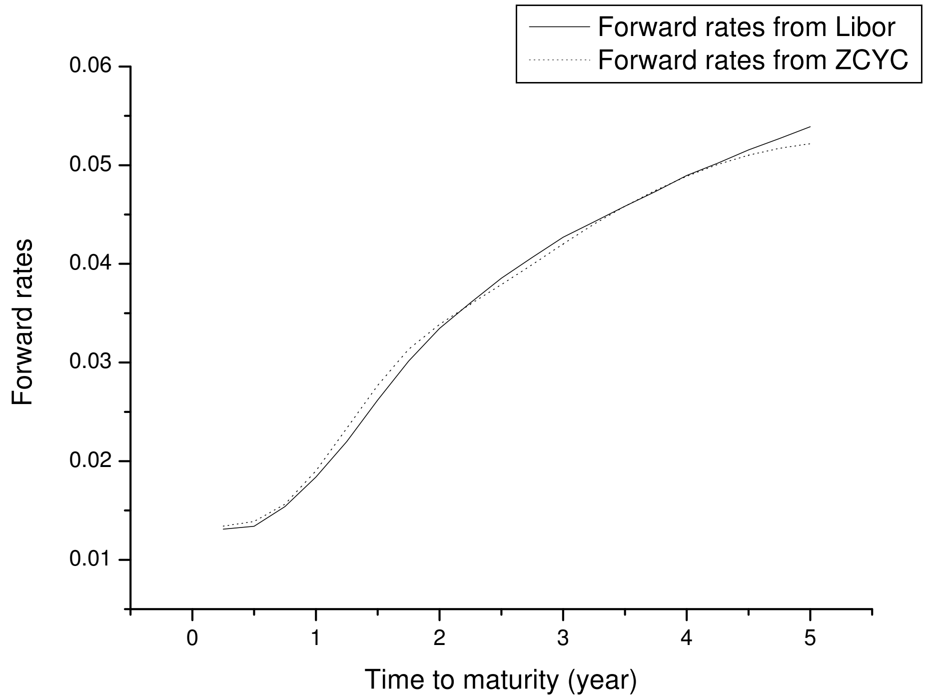

Figure 1.6: Forward rate term structure at 2003.1.29 both from zero coupon yield curve and from Libor with maturity up to 5 year.

For a futures contract entered into at time t for a 90-day deposit of \$1 million from future time T to $T + \ell$ (ℓ=90/360year), the principal plus simple interest that will accrue- on maturity- to an investor long on the contract is given by

$$P + I = 1 + \ell L(t, T)$$

where $L(t, T)$ is the (annualized) three-month (90-day) Libor. Let the forward interest rates for the three-month Libor be denoted by $f(t, x)$. One can express the principal plus interest based on the compounded forward interest rates and obtain

$$P + I = e^{\int_T^{T+\ell} dx f(t,x)}$$

hence the relationship between Libor and its forward rates is given by

$$L(t, T) = \frac{e^{\int_T^{T+\ell} dx f(t,x)} - 1}{\ell} \tag{1.17}$$

Some time one may need to assume that the Eurodollar futures Libor prices are equal to the forward rates. More precisely, from eq1.17

$$L(t, T) \simeq f(t, T) + O(\ell) \tag{1.18}$$

Forward interest rates derived from Libor carry a small element of credit risk that is not present in the forward interest rates derived from zero risk US Treasury Bonds; in this paper the difference is considered neglible and ignored. Fig. 1.6 shows the forward rate term structure at 2003.1.29 from both zero coupon yield curve and Libor.

§ 1.3 Review of Derivative and Rational Pricing

§ 1.3.1 Derivatives

A derivative is an instrument whose value is dependent on other securities (called the underlying securities). The derivative value is therefore a function of the value of the underlying securities. Derivatives can be based on different types of assets such as commodities, equities or bonds, interest rates, exchange rates, or indices (such as a stock market index, consumer price index (CPI) or even an index of weather conditions). Their performance can determine both the amount and the timing of the payoffs. The main use of derivatives is to either remove risk or take on risk depending if one is a hedger or a speculator. The diverse range of potential underlying assets and payoff alternatives leads to a huge range of derivatives contracts traded in the market. The main types of derivatives are futures, forwards, options and swaps. In today's uncertain world, derivatives are increasingly being used to protect assets from drastic fluctuations and at the same time they are being re-engineered to cover all kinds of risk and with this the growth of the derivatives market continues.

Broadly speaking there are two distinct groups of derivative contracts, which are distinguished by the way that they are traded in market:

Over-the-counter (OTC) derivatives are contracts that are traded (and privately negotiated) directly between two parties, without going through an exchange or other intermediary. Products such as swaps, forward rate agreements, and exotic options are almost always traded in this way. The OTC derivatives market is huge. According to the Bank for International Settlements, the total outstanding notional amount is USD 298 trillion (as of 2005)[3].

Exchange-traded derivatives are those derivatives products that are traded via Derivatives exchanges. A derivatives exchange acts as an intermediary to all transactions, and takes initial

[3]BIS survey: The Bank for International Settlements (BIS), in their semi-annual OTC derivatives market activity report from May 2005 that, at the end of December 2004, the total notional amounts outstanding of OTC derivatives was 248 trillion with a gross market value of 9.1 trillion.

margin from both sides of the trade to act as a guarantor. The world's largest[4] derivatives exchanges (by number of transactions) are the Korea Exchange (which lists KOSPI Index Futures & Options), Eurex (which lists a wide range of European products such as interest rate & index products), Chicago Mercantile Exchange and the Chicago Board of Trade. According to BIS, the combined turnover in the world's derivatives exchanges totalled USD 344 trillion during Q4 2005.

There are three major classes of derivatives: Futures/Forwards, which are contracts to buy or sell an asset at a specified future date. Options, which are contracts that give the buyer the right (but not the obligation) to buy or sell an asset at a specified future date. Swaps, where the two parties agree to exchange cash flows.

§ 1.3.2 Option

Since this thesis focuses on interest rate derivatives, further details of these derivatives are reviewed in§ 1.4. Only the general idea of the option which is the most crucial form of derivative is given here. And if one values all options, one can value any derivative whatsoever.

There are two basic types of options that are traded in the market. A call option gives the holder the right to buy the underlying asset by a certain date for a certain price. A put option gives the holder the right to sell the underlying asset by a certain date for a certain price. This price is called the strike price and the date is called the exercise date or maturity of the contract.

There is a further classification of options according to when they can be exercised. An European option can only be exercised at maturity while an American option can be exercised at any time up to maturity. The Bermudan option can only be exercised on certain fixed days between the present time and the maturity of the contract.

From the definition of a call option, we can see that the value of an European call option at maturity is given by the payoff

$$C = (S-K)_+ \equiv \begin{cases} S-K, & S > K \\ 0, & S < K \end{cases}$$

(if $S < K$ then the option will not be exercised and if $S > K$, the profit on the option will be $S - K$). Note

$$(a-b)_+ \equiv (a-b)\Theta(a-b) \tag{1.19}$$

[4]Futures and Options Week: According to figures published in F&O Week 10 October 2005.

and the Heaviside step function $\Theta(x)$ is defined by

$$\Theta(x) \equiv \begin{cases} 1 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases} \tag{1.20}$$

where C is the value of the call option at maturity, S is the value of the underlying security at maturity and K is the strike price of the option. Define

$$C(t, S, K) = E[e^{-r(T-t)}(S(T) - K)_+] \tag{1.21}$$

Similarly, the payoff of a put option at maturity is given by

$$P = (K - S)_+$$

(if $K < S$ then the option will not be exercised and if $K > S$, the profit on the option will be $K - S$) where P is the value of the put option at maturity.

From eq. 1.19 the payoff for the call and a put option are generically given by

$$(a - b)_+ = (a - b)\Theta(a - b)$$

The derivation of **put-call parity** hinges on the identity, which follows from eq. 1.20, that

$$\Theta(x) + \Theta(-x) = 1 \tag{1.22}$$

since it yields

$$(a - b)_+ - (b - a)_+ = (a - b)\Theta(a - b) - (b - a)\Theta(b - a) = a - b \tag{1.23}$$

Thus the difference in the call and put payoff function satisfies

$$(S - K)_+ - (S - K)_+ = S - K \tag{1.24}$$

Hence

$$C(t, S, K) - P(t, S, K) = S - e^{-r(T-t)}K \qquad \text{Put-call parity} \tag{1.25}$$

§ 1.3.3 Rational Pricing

Arbitrage is the practice of taking advantage of a state of imbalance between two (or possibly more) markets. Where this mismatch can be exploited (i.e. after transaction costs, storage costs, transport costs etc.) the arbitrageur "locks in" a risk free profit above the prevailing risk free return say from the money market.

In general, arbitrage ensures that "the law of one price" will hold; arbitrage also equalises the prices of assets with identical cash flows, and sets the price of assets with known future cash flows.

The **principle of no arbitrage** effectively states that there is no such thing as a free lunch in the financial markets. It is one of the most important and central principles of finance. The logic behind the existence of this principle is that if a free lunch exists it will be used by everyone so that is ceases to be free or that the lunch is exhausted.

More concretely, the principle of no arbitrage states that there exists no trading strategy which guarantees a riskless profit above the money market with no initial investment. This statement is equivalent to the statement that one cannot get a riskless return above the risk free interest rate in the market provided that there are no transaction costs (in the presence of transactions, one can only say that one can not get a riskless return more than the risk free interest rate plus the transaction costs). The main assumption behind this principle is that people prefer more money to less money.

Rational pricing is the assumption in financial economics that asset prices (and hence asset pricing models) reflect the arbitrage-free price of the asset, as any deviation from this price will be "arbitraged away". This assumption is useful in pricing fixed income securities, particularly bonds, and is fundamental to the pricing of derivative instruments. The fundamental theorem of asset pricing given by Harrison and Pliska[61] has two parts to it. The first is that the absence of arbitrage in the market implies the existence of a measure under which all the discounted asset prices are martingales. The second part of the theorem basically states that in a complete market without transaction costs or arbitrage opportunities, the price of all options are the expectation value of the future payoff of the option under a unique measure in which all discounted asset prices are martingales.

The concept of martingale in probability theory is the mathematical formulation of the concept of a fair game, and is equivalent, in finance, to the principle of an efficient market.

Suppose a gambler is playing a game of tossing a fair coin, represented by a discrete random variable Y with two equally likely possible outcomes ± 1; that is, $P(Y = 1) = P(Y = -1) = \frac{1}{2}$. Let X_n represent the amount of cash that the gambler has after n identical throws. That is, $X_n = \sum_{i=1}^{n} Y_i$, where Y_i's are independent random variables all identical to Y; let x_n denote some specific outcome of random variable X_n. The martingale condition states that the expected value of the cash that the gambler has on the $(n+1)$th throw must be equal to the cash that he is holding at the nth throw. Or in equations

$$E[X_{n+1}|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n] = x_n \tag{1.26}$$

In other words, in a fair game, the gambler - on the average - simply leaves the casino with the cash that he came in with.

The martingale framework was proposed by Harrison and Kreps[39] and extended by Artzner and Delbaen[5] and Heath, Jarrow, Morton[21] for term structure modelling. An essential point is the choice of the numeraire, that is, the common unit on the basis of which asset prices are expressed. Any asset price can be selected as a numeraire, as long as it has strictly positive value in any state of the world.

In the different projects of this thesis, different measure for martingale evolution[63] is chosen for convenience. I briefly review all of them below with detail calculation discussed in later chapters after Quantum Finance has been introduced in chapter 2.

In Heath, Jarrow and Morton [21], a martingale was defined by discounting Treasury Bonds using the money market account, with money market numeraire $M(t, t_*)$ defined by

$$M(t, t_*) = e^{\int_t^{t_*} r(t')dt'} \tag{1.27}$$

The quantity $B(t, T)/M(t, T)$ is defined to be a martingale

$$\begin{aligned} \frac{B(t,T)}{M(t,t)} &= E_M\left[\frac{B(t_*,T)}{M(t,t_*)}\right] \\ \Rightarrow B(t,T) &= E_M[e^{\int_t^{t_*} r(t')dt'} B(t_*,T)] \end{aligned} \tag{1.28}$$

where $E_M[\ldots]$ denotes expectation values taken with respect to the **money market measure**.

It is often convenient to have a discounting factor that renders the futures price of (Libor or Treasury) bonds into a martingale. Consider the forward value of bond given by

$$F(t_0, T_n + \ell) = e^{-\int_{Tn}^{Tn+\ell} dx f(t_0,x)} = \frac{B(t, T_n + \ell)}{B(t, T_n)} \tag{1.29}$$

The forward numeraire is given by $B(t, T_n)$

$$e^{-\int_{Tn}^{Tn+\ell} dx f(t_0,x)} = E_F[e^{-\int_{Tn}^{Tn+\ell} dx f(t_*,x)}] \tag{1.30}$$

In effect, as expressed in the equation above, the **forward measure** makes the forward bond price a martingale.

In Baaquie [9], a common measure that yields a martingale evolution for all Libor is presented. To understand the discounting that yields a martingale evolution of Libor rate $L(t, T_n)$, rewrite Libor in 1.17 as follows

$$\begin{aligned} L(t, T_n) &= \frac{1}{\ell}(e^{\int_{T_n}^{T_n+\ell} dx f(t,x)} - 1) \\ &= \frac{1}{\ell}\left[\frac{B(t, T_n) - B(t, T_n + \ell)}{B(t, T_n + \ell)}\right] \end{aligned} \tag{1.31}$$

The Libor is interpreted as being equal to the bond portfolio $(B(t, T_n) - B(t, T_n + \ell))/\ell$ with discounting factor for the **Libor market measure** being equal to $B(t, T_n + \ell)$. Hence, the martingale condition for the Libor market measure, denote by $E_L[\ldots]$, is given by

$$\frac{B(t_0, T_n) - B(t_0, T_n + \ell)}{B(t_0, T_n + \ell)} = E_L\left[\frac{B(t_*, T_n) - B(t_*, T_n + \ell)}{B(t_*, T_n + \ell)}\right] \tag{1.32}$$

In other words, the Libor market measure is defined such that the Libor $L(t, T_n)$ for each T_n is a martingale; that is, for $t_* > t_0$

$$L(t_0, T_n) = E_L[L(t_*, T_n)] \tag{1.33}$$

§ 1.4 Interest Rate Derivatives

An interest rate derivative is a derivative where the underlying asset is the right to pay or receive a (usually notional) amount of money at a given interest rate.

Interest rate derivatives are the largest derivatives market in the world. Market observers estimate that \$60 trillion dollars by notional value of interest rate derivatives contract had been exchanged by May 2004.

According to the International Swaps and Derivatives Association, 80% of the world's top 500 companies at April 2003 used interest rate derivatives to control their cashflow. This compares with 75% for foreign exchange options, 25% for commodity options and 10% for stock options.

§ 1.4.1 Swap

An interest rate swap is contracted between two parties. Payments are made at fixed times T_n and are separated by time intervals ℓ, which is usually 90 or 180 days. The swap contract has a notional principal V, with a pre-fixed period of total duration and with the last payment being made at time T_N. One party pays, on the notional principal V, a fixed interest rate denoted by R_S and the other party pays a floating interest rate based on the prevailing market rate, or vise versa. The floating interest rate is usually determined by the prevailing value of Libor at the time of the floating payment.

In the market, the usual practice is that floating payments are made every 90 days whereas fixed payments are made every 180 days; for simplicity of notation we will only analyze the case when both fixed and floating payments are made on the same day.

A swap of the first kind, namely swap$_I$, is one in which a party pays at fixed rate R_S and receives payments at the floating rate [82]. Hence, at time T_n the value of the swap is the difference between the floating payment received at the rate of $L(t, T_n)$, and the fixed payments paid out at the rate of R_S. All payments are made at time $T_n + \ell$, and hence need to be discounted by the bond $B(T_0, T_n+\ell)$ for obtaining its value at time T_0. Similarly, swap$_{II}$ – a swap of the second kind – is one in which the party holding the swap pays at the floating rate and receives payments at fixed rate R_S.

Consider a swap that starts at time T_0 and ends at time $T_N = T_0 + N\ell$, with payments being made at times $T_0 + n\ell$, with $n = 1, 2, ..., N$. The value of the swaps are given by [9], [82]

$$\text{swap}_I(T_0, R_S) = V\left[1 - B(T_0, T_0 + N\ell) - \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell)\right] \tag{1.34}$$

$$\text{swap}_{II}(T_0, R_S) = V\left[\ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell) + B(t, T_0 + N\ell) - 1\right]$$

Note that, since swap$_I$+swap$_{II}$ $= 0$, an interest swap is a zero sum game, with the gain of one party being equal to the loss of the other party.

The par value of the swap when it is initiated at time T_0 is zero; hence the par fixed rate R_P, from eq. 1.43, is given by

$$\begin{aligned} \text{swap}_I(T_0, R_P) &= 0 = \text{swap}_{II}(T_0, R_P) \\ \Rightarrow \ell R_P &= \frac{1 - B(T_0, T_0 + N\ell)}{\sum_{n=1}^{N} B(T_0, T_0 + n\ell)} \end{aligned}$$

The **forward swap** or a **deferred swap**, similar to the forward price of a Treasury Bond, is a swap entered into at time $t_0 < T_0$, and it's price is given by [9]

$$\text{swap}_I(t_0; T_0, R_S) = V\big[B(t_0, T_0) - B(t_0, T_0 + N\ell) - \ell R_S \sum_{n=1}^{N} B(t_0, T_0 + n\ell)\big] \tag{1.35}$$

A deferred swap matures at time T_0.

At time t_0 the par value for the fixed rate of the deferred swap, namely $R_P(t_0)$, is given by [9]

$$\begin{aligned} \text{swap}_I(t_0; T_0, R_P(t_0)) &= 0 = \text{swap}_{II}(t_0; T_0, R_P(t_0)) \\ \Rightarrow \ell R_P(t_0) &= \frac{B(t_0, T_0) - B(t_0, T_0 + N\ell)}{\sum_{n=1}^{N} B(t_0, T_0 + n\ell)} \end{aligned} \tag{1.36}$$

§ 1.4.2 Cap and Floor

Financial market's participants sometimes have to enter into financial contracts in which they pay or receive cash flows tied to some floating rate such as Libor. In order to hedge the risk caused by the Libor's variability, participants often enter into derivative contracts with a fixed upper limit or lower limit of Libor at cap rate. These types of derivatives are known as interest-rate caps and floors.

A cap gives its holder a series of European call options or caplets on the Libor rate, where all caplet has the same strike price, but a different expiration dates. Typically, the expiration dates for the caplets are on the same cycle as the frequency of the underlying Libor rate.

A midcurve caplet[5] is defined as a caplet that is exercised at time t_* that is before the time at which the caplet is operational. Suppose the midcurve caplet is for the Libor rate for time interval T_n to $T_n + \ell$, where ℓ is 90 days, and matures at time t_*. Let the caplet price, at time $t_0 < t_*$, be given by $Caplet(t_0, t_*, T_n)$. The payoff for the caplet is given by [9]

$$Caplet(t_*, t_*, T_n) = \ell V B(t_*, T_n + \ell)\big[L(t_*, T_n) - K\big]_+$$

where $B(t_*, T_n + \ell)$ is the Treasury Bond and V is the principal for which the interest rate caplet is defined. $L(t_*, T_n)$ is the value at time t_* of the Libor rate applicable from time T_n

[5]Midcurve options, analyzed in this thesis, are options that mature before the instrument becomes operational. For example a caplet may cap interest rates for a duration of three months say one year in the future, and a midcurve option on such a caplet can have a maturity time only six months, hence expiring six months before the instrument becomes operational. Similarly a midcurve option on a coupon bond may mature in say six months time with the bond starting to pay coupons only a year from now. Midcurve options are widely traded in the market and hence need to be studied.

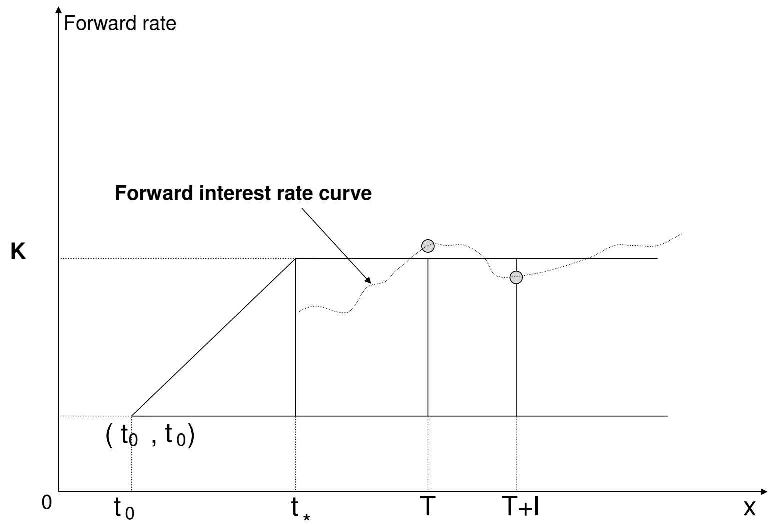

Figure 1.7: Diagram reprsenting a caplet $\ell V B(t_*, T + \ell)[L(t_*, T) - K]_+$. During the time interval $T \le t \le T + \ell$, the borrower holding a **caplet** needs to pay only K interest rate, regardless of the values of forward interest rate curve during this period.

to $T_{n+\ell}$, and K is the cap rate(the strike price). Note that while the cash flow on this caplet is received at time $T_n + \ell$, the Libor rate is determined at time t_*, which means that there is no uncertainty about the case flow from the caplet after Libor is set at time t_*. Figure 1.7 shows how a caplet provides a cutoff to the maximum interest rate that a borrower holding a caplet will need to pay.

From the fundamental theorem of finance the price of the $Caplet(t_0, t_*, T_n)$ is given by the expectation value of the pay-off function discounting – using the spot interest rate $r(t) = f(t, t)$ – from future time t_* to present time t_0, and yields [6]

$$Caplet(t_0, t_*, T_n) = \ell V E\Big[e^{-\int_{t_0}^{t_*} r(t)} B(t_*, T_n + \ell)\big[L(t_*, T_n) - K\big]_+\Big]$$

with the price of a floorlet defined by

$$Floorlet(t_0, t_*, T_n) = \ell V E\Big[e^{-\int_{t_0}^{t_*} r(t)} B(t_*, T_n + \ell)\big[K - L(t_*, T_n)\big]_+\Big]$$

Figure 1.8 shows the domain over which the midcurve caplet is defined.

Put-call parity relation is given by [9]

$$Caplet(t_0, t_*, T_n) - Floorlet(t_0, t_*, T_n) = \ell V B(t_0, T_n + \ell)[L(t_0, T_n) - K] \qquad (1.37)$$

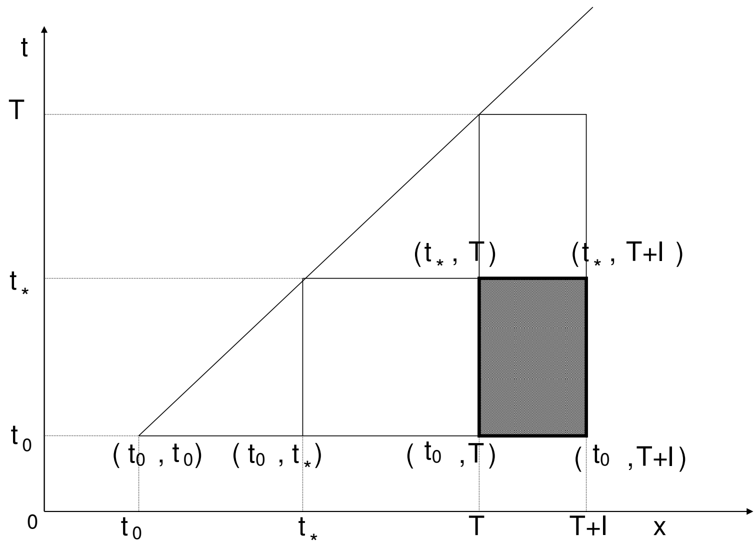

Figure 1.8: The domain of the midcurve caplet in the xt plane; the payoff $\ell VB(t_*, T+\ell)[L(t_*, T) - K]_+$ is defined at time t_*. The shaded portion shows the domain of the forward interest rates that define the price $Caplet(t_0, t_*, T)$ for a midcurve caplet.

Thus, we can get floorlet price from this put-call parity and independent derivation is not necessary.

An interest rate cap with a duration over a longer period is made from the sum over caplets spanning the requisite time interval. Consider a midcurve cap, to be exercised at time t_*, with cap starting from time $T_m = m\ell$ and ending at time $T_{n+1} = (n+1)\ell$; its price is given by

$$Cap(t_0, t_*) = \sum_{j=m}^{n} Caplet(t_0, t_*, T_j; K_j) \tag{1.38}$$

Figure 3.9 shows the structure of the an interest cap in terms of it's constituent caplets.

It follows from above that the price of an interest cap only requires the prices of interest rate caplets. Hence, in effect, one needs to obtain the price of a single caplet for pricing interest rate caps.

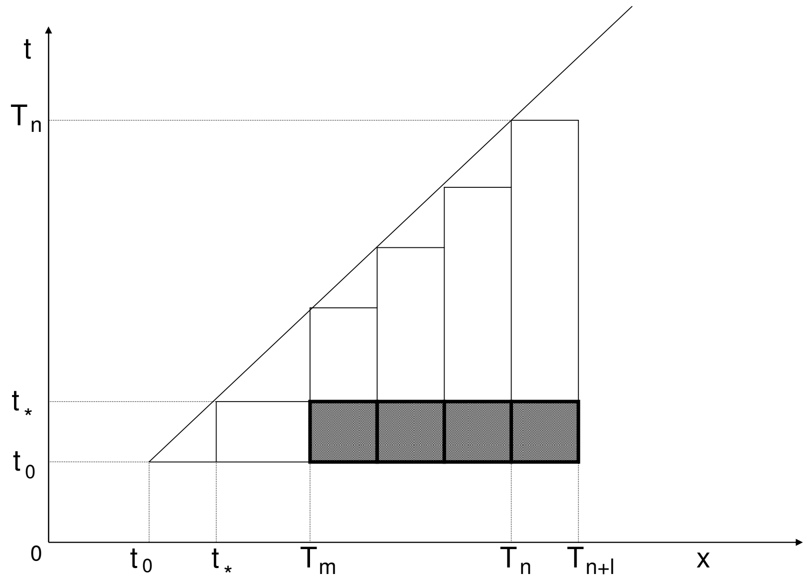

Figure 1.9: The domain of the midcurve interest rate cap $Cap(t_0, t_*) = \sum_{j=m}^{n} Caplet(t_0, t_*, T_j; K_j)$, defined from future time T_m to time T_n in terms of the portfolio of midcurve caplets. The shaded portion indicates the domain of the forward interest rated required for the pricing of the midcurve $Cap(t_0, t_*)$.

§ 1.4.3 Coupon Bond Option

The payoff function $S(t_*)$ of a European call option maturing at time t_*, for strike price K, is given by

$$S(t_*) \;=\; \Big(\sum_{i=1}^{N} c_i B(t_*, T_i) - K\Big)_+ \tag{1.39}$$

The price of a European call option at time $t_0 < t_*$ is given by discounting the payoff $S(t_*)$ from time t_* to time t. Any measure that satisfies the martingale property can be used for this discounting [6]; in particular the money market numeraire is given by $\exp(\int r(t)dt)$ where $r(t) = f(t, t)$ is the spot interest rate. In terms of the money market measure, discounting the payoff function by the money market numeraire yields the following price of a European call and put options

$$C(t_0, t_*, K) = E\big[e^{-\int_{t_0}^{t_*} dtr(t)} S(t_*)\big] = E\big[e^{-\int_{t_0}^{t_*} dtr(t)} \big(\sum_{i=1}^{N} c_i B(t_*, T_i) - K\big)_+\big] \quad (1.40)$$

$$P(t_0, t_*, K) = E\big[e^{-\int_{t_0}^{t_*} dtr(t)} \big(K - \sum_{i=1}^{N} c_i B(t_*, T_i)\big)_+\big]$$

In particular, Treasury Bonds are martingales for the money market numeraire; hence

$$E[e^{-\int_{t_0}^{t_*} dtr(t)} B(t_*, T)] = B(t_0, T) \quad (1.41)$$

§ 1.4.4 Swaption

A swaption, denoted by CS_I and CS_{II}, is an option on swap_I and swap_{II} respectively; suppose the swaption matures at time T_0; it will be exercised only if the value of the swap at time T_0 is greater than its par value of zero; hence, the payoff function is given by

$$CS_I(T_0; R_S) = V\big[1 - B(T_0, T_N) - \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell)\big]_+$$

and a similar expression for CS_{II}. The value of the swaption at an earlier time $t < T_0$ is given for the money market numeraire by

$$\begin{aligned} CS_I(t, R_S) &= V\big\langle e^{-\int_t^{T_0} r(t')dt'} CS_I(T_0; R_S)\big\rangle \\ &= V\big\langle e^{-\int_t^{T_0} r(t')dt'} \big[1 - B(T_0, T_N) - \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell)\big]_+\big\rangle \end{aligned} \quad (1.42)$$

and similarly for $CS_{II}(t, R_S)$.

One can see that a swap is equivalent to a specific portfolio of coupon bonds, and all techniques that are used for coupon bonds can be used for analyzing swaptions.

Eq. 1.23, together with the martingale property of zero coupon bonds under the money market measure given in eq. 1.41 that $\langle e^{-\int_t^{T_0} r(t')dt'} B(T_0, T_n)\rangle = B(t, T_n)$, yields the put-call parity for the swaptions as [9]

$$\begin{aligned} CS_I(t, R_S) - CS_{II}(t, R_S) &= V\big\langle e^{-\int_t^{T_0} r(t')dt'} \big[1 - B(T_0, T_0 + N\ell) - \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell)\big]\big\rangle \\ &= V\big[B(t, T_0) - B(t, T_0 + N\ell) - \ell R_S \sum_{n=1}^{N} B(t, T_0 + n\ell)\big] \\ &= \text{swap}_I(t; T_0, R_S) \end{aligned} \quad (1.43)$$

where recall $\mathrm{swap}_I(t; T_0, R_S; t)$ is the price at time t of a deferred swap that matures at time $T_0 > t$.

The price of swaption CS_{II}, in which the holder has the option to enter a swap in which he receives at a fixed rate R_S and pays at a floating rate, is given by the formula for the call option for a coupon bond. Suppose the swaption CS_{II} matures at time T_0; the payoff function on a principal amount V is given by

$$CS_{II}(T_0, R_S) = V\big[B(T_0, T_0 + N\ell) + \ell R_S \sum_{n=1}^{N} B(T_0, T_0 + n\ell) - 1\big]_+ \qquad (1.44)$$

Comparing the payoff for CS_{II} given above with the payoff for the coupon bond call option given in eq. 1.39, one obtains the following for the swaption coefficients

$$\begin{aligned} c_n &= \ell R_S \;\; ; \;\; n = 1, 2, ..., (N-1) \;\; ; \;\; \text{Payment at time } T_0 + n\ell \qquad (1.45)\\ c_N &= 1 + \ell R_S \;\; ; \;\; \text{Payment at time } T_0 + N\ell \\ K &= 1 \end{aligned}$$

The price of CS_I is given from CS_{II} by using the put-call relation given in eq. 1.43.

There are swaptions traded in the market in which the floating rate is paid at $\ell = 90$ days intervals, and with the fixed rate payments being paid at intervals of $2\ell = 180$ days. For a swaption with fixed rate payments at 90 days intervals – at times $T_0 + n\ell$, with $n = 1, 2.., N$ – there are N payments. For payments made at 180 days intervals, there are only $N/2$ payments [6] made at times $T_0 + 2n\ell$, $n = 1, 2, ..., N/2$, and of amount $2R_S$. Hence the payoff function for the swaption is given by

$$\begin{aligned} CS_I(T_0; R_S) &= V\big[1 - B(T_0, T_0 + N\ell) - 2\ell R_S \sum_{n=1}^{N/2} B(T_0, T_0 + 2n\ell)\big]_+ \\ &= V\big[1 - \sum_{n=1}^{N/2} \tilde{c}_n B(T_0, T_0 + 2n\ell)\big]_+ \qquad (1.46) \end{aligned}$$

The par value at time t_0 is fixed by the forward swap contract, and from eq. 1.36 is given by

$$2\ell R_P(t_0) = \frac{B(t_0, T_0) - B(t_0, T_0 + N\ell)}{\sum_{n=0}^{N/2} B(t_0, T_0 + 2n\ell)} \qquad (1.47)$$

and reduces at $t_0 = T_0$ to the par value of the fixed interest rate payments being given by

$$2\ell R_P = \frac{1 - B(T_0, T_0 + N\ell)}{\sum_{n=1}^{N/2} B(T_0, T_0 + 2n\ell)}$$

[6] Suppose the swaption has a duration such that N is even. Note that $N = 4$ for a year long swaption.

The equivalent coupon bond put option payoff function is given by

$$S_{\text{Put}}(t_*) = \big(K - \sum_{n=1}^{N/2} \tilde{c}_n B(t_*, T_0 + 2n\ell)\big)_+ \tag{1.48}$$

and from eq. 5.1, has the coefficients and strike price given by

$$\tilde{c}_n = 2\ell R_S \ \ ; \ \ n = 1, 2, ..., (N-1)/2 \ \ ; \ \ \text{Payment at time } T_0 + 2n\ell$$
$$\tilde{c}_{N/2} = 1 + 2\ell R_S \ \ ; \ \ \text{Payment at time } T_0 + N\ell$$
$$K = 1$$

The price of CS_I for the 180 days fixed interest payment case is given from CS_{II} by using the put-call relation similar to the given in eq. 1.43.

Note that it is only due to asymmetric nature of the last coefficient, namely c_N and $\tilde{c}_{N/2}$ for the two cases discussed above, that the swap interest rate R_S does not completely factor out (upto a re-scaling of the strike price) from the swaption price.

Options on swap$_I$ and swap$_{II}$, namely CS_I and CS_{II}, are both **call options** since it gives the holder the option to either receive fixed or receive floating payments, respectively. When expressed in terms of coupon bond options, it can be seen from eqs. 1.42 and 1.44 that the swaption for receiving fixed payments is equivalent to a coupon bond put option, whereas the option to receive floating payments is equivalent to a coupon bond call option.

§ 1.5 Appendix: De-noising time series financial data

Time series financial data like zero coupon yield , Libor or price of instruments can be studied directly to get hidden mechanisms that make any forecasts work. The point, in other words, is to find the causal, dynamical structure intrinsic to the process we are investigating, ideally to extract all the patterns in it that have any predictive power. Also, we need to get the drift velocity of infinitesimal change of daily forward rates. This requires **smooth** time series data without high frequency white noise. Wavelet analysis[56, 24, 25] can often compress or de-noise a signal without appreciable degradation.

We use the graphical interface tools in wavelet toolbox in matlab to do the one-dimensional stationary wavelet analysis. Select DB8 to decompose the signal, where DB8 stands for the Daubechies[19] family wavelets and 8 is the order.[7] After decomposed the signal and got

[7]Ingrid Daubechies invented what are called compactly supported orthonormal wavelets – thus making discrete wavelet analysis practicable.

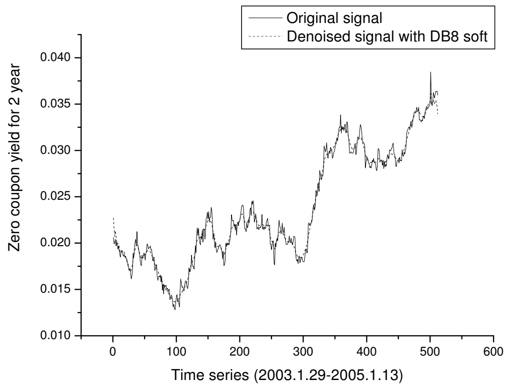

Figure 1.10: The original and de-noised two year zero coupon yield data versus time (2003.1.29-2005.1.13)

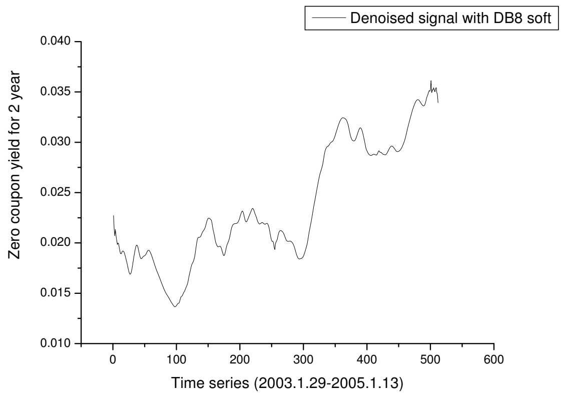

Figure 1.11: The smooth two year zero coupon yield data versus time (2003.1.29-2005.1.13) after de-noising

detail coefficients of the decomposition, a number of options are available for fine-tuning the de-noising algorithm, we'll accept the defaults of fixed form soft thresholding[24, 25] and unscaled white noise. An example of de-noising time series zero coupon yield data is given in Fig. 1.10, 1.11 and 1.12. Another example of de-noising time series Libor rate is given in Fig. 1.13, 1.14 and 1.15.

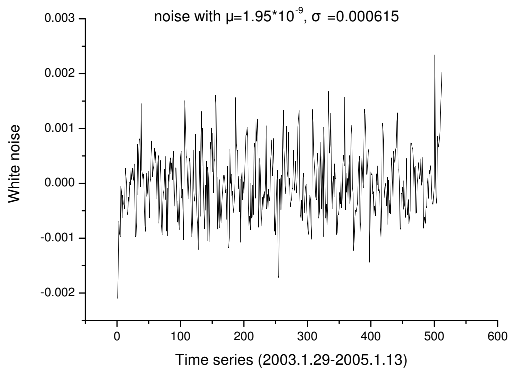

Figure 1.12: The white noise de-noised from original two year zero coupon yield data versus time (2003.1.29-2005.1.13), with $\mu = 1.95 \times 10^{(} - 9)$ and $\sigma = 6.15 \times 10^{-4}$

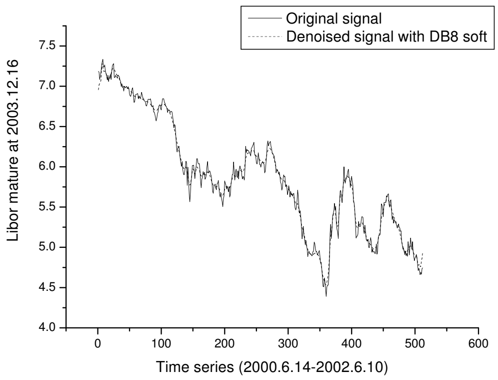

Figure 1.13: The original and de-noised Libor forward rates which mature at 2003.12.16 versus time (2000.6.14-2002.6.10)

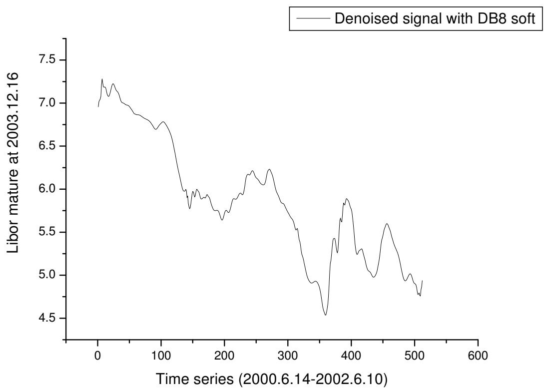

Figure 1.14: The smooth Libor forward rates which mature at 2003.12.16 versus time (2000.6.14-2002.6.10) after de-noising

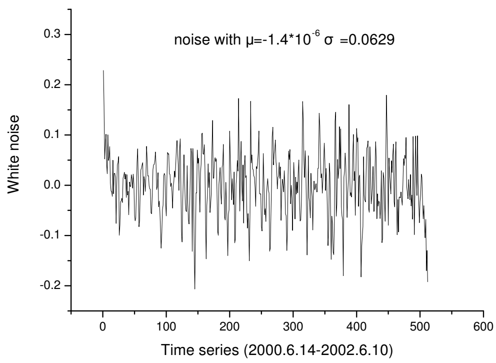

Figure 1.15: The white noise de-noised from original Libor forward rates which mature at 2003.12.16 versus time (2000.6.14-2002.6.10), with $\mu = -1.4 \times 10^(-6)$ and $\sigma = 6.29 \times 10^{-2}$

Chapter 2

Quantum Finance of Interest Rate

[1]Under the fundamental theorem of asset pricing, in order to price interest rate derivatives, one need to get the expectation of future payoff under a martingale measure. This lead us to study the dynamics of interest rate term structure.

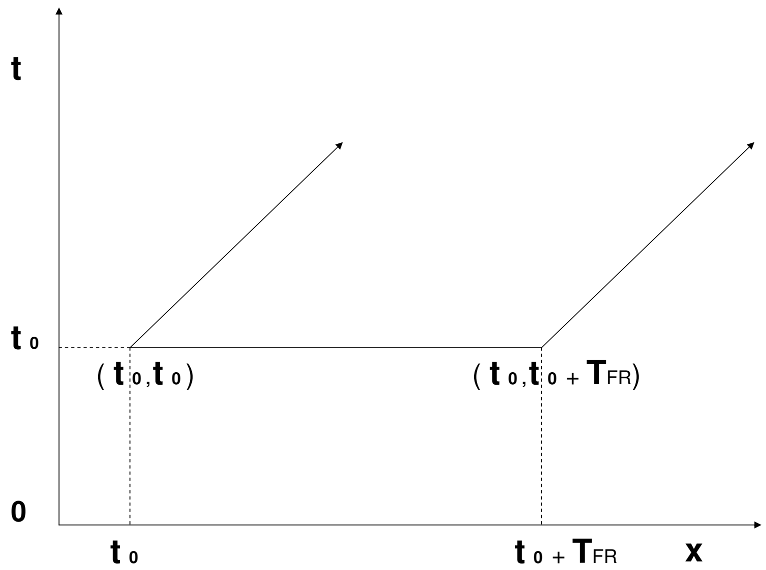

Figure 2.1: The domain for the forward rates.

The shape of the domain for the forward rates is shown in Fig. 2.1. In the figure, it

[1]Quantum finance [6] refers to the application of the formalism of quantum mechanics and quantum field theory to finance.

has been assumed that the forward rates are defined only up to a time T_{FR} into the future. Theoretically, forward rates can exist for all future time, so in most cases we will take the limit $T_{FR} \to \infty$. The forward rate for the current time f(t,t) is usually denoted by $r(t)$ and is called the spot rate. For a long time, it was thought that the spot rate alone determined the dynamics of all the bond prices but modern models tend to introduce dynamics to the entire forward rate curve.

§ 2.1 Review of interest rate models

Early models of the term structure attempted to model the bond price dynamics. Their results did not allow for a better understanding of the term structure, which is hidden in the bond prices. However, many interest rate models are simply models of the stochastic evolution[87, 88] of a given interest rate (often chosen to be the short term rate). An alternative is to specify the stochastic dynamics of the entire term structure of interest rates, either by using all yields or all forward rates.

Merton was the first to propose a general stochastic process as a model for the short rate. Then Vasicek [65] in his seminal paper showed how to price bonds and derive the market price of risk based on diffusion models of the spot rate. He also introduced his famous Vasciek model in that paper. Cox, Ingersoll and Ross [41] have developed an equilibrium model in which interest rates are determined by the supply and demand of individuals. Jamshidian [32, 33, 34] derives analytic solutions for the prices of European call and put option on both zero coupon bond and coupon bearing bond based on these models. However, these models are all time-invariant models and suffer from the shortcomings that the short term rate dynamics implies an endogenous term structure, which is not necessarily consistent with the observed one. This is why Hull and White[44] introduced a class of one factor time varying models which is consistent with a whole class of existing models. Although models have undergone improvements that more terms have been added in to simulate the complexity of spot rate dynamics, these models are still classified into a wide class of spot rate model- called affine model-all of which has a positive probability of negative values. This has led some authors to propose models with lognormal rates, thus avoiding negative rates. Later non-affine models have been developed such as Black, Derman and Toy [28] who proposed a one factor binomial model. Later, Black and Karasinski [29] has proposed the Black-Karasinski model which is an extension of the Black, Derman and Toy model with a time varying reversion speed. However, as noted in Heath, Jarrow and Morton [81], they all have one serious problem, since all of them only model the spot rate, they make very specific predictions for the forward rate structure. These predictions are usually not stratified in reality and this leads to model specification problems. The specification of arbitrary market prices of risk in these models tends to alleviate this problem but introduces the even more severe problem of introducing arbitrage opportunities as noted in Cox, Ingersoll and Ross [41]. Also, the debt market directly trades in the forward rates and provides an enormous amount of data on these. It is sensible to create models that take the forward rates as the primary instrument so as to match the behavior of the market.

This led Heath , Jarrow and Morton to develop their famous model where all the forward rates are modelled together. This model, usually called the HJM model is, together with its variants, now the industry standard interest rate model.

§ 2.1.1 Heath-Jarrow-Morton (HJM) model

In K-factor HJM model[21], the time evolution of the forward rates is modelled to behave in a stochastic manner driven by K-independent white noises $W_i(t)$, and is given by

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sum_{i=1}^{K} \sigma_i(t,x) W_i(t) \tag{2.1}$$

where $\alpha(t,x)$ is the drift velocity term and $\sigma_i(t,x)$ are the deterministic volatilities of the forward rates.

Note that although the HJM model evolves an entire curve $f(t,x)$, at each instant of time t it is driven by K random variables given by $W_i(t)$, and hence has only K degrees of freedom.

From Eq.2.1

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} dt' \alpha(t',x) + \int_{t_0}^{t} dt' \sum_{i=1}^{K} \sigma_i(t',x) W_i(t') \tag{2.2}$$

The initial forward rate curve $f(t_0,x)$ is determined from the market, and so are the volatility functions $\sigma_i(t,x)$. Note the drift term $\alpha(t,x)$ is fixed to ensure that the forward rates have a martingale time evolution, which makes it a function of the volatilities $\sigma(t,x)$.

For every value of time t, the stochastic variables $W_i(t), i = 1, 2, \ldots, K$ are independent Gaussian random variables given by

$$E(W_i(t) W_j(t')) = \delta_{ij} \delta(t - t') \tag{2.3}$$

The forward rates $f(t,x)$ are driven by random variables $W_i(t)$ which gave the same random 'shock' at time t to all the future forward rates $f(t,x), x > t$. To bring in the maturity dependence of the random shocks on the forward rate, the volatility function $\sigma_i(t,x)$, at given time t, weights this 'shocks' differently for each x.

The action functional is

$$S[W] = -\frac{1}{2}\sum_{i=1}^{K}\int_{t_1}^{t_2} dtW_i(t)^2 \tag{2.4}$$

We can use this action to calculate the generating functional which is

$$\begin{aligned} Z[j,t_1,t_2] &= \int DW e^{\sum_{i=1}^{K}\int_{t_1}^{t_2} dtj_i(t)W_i(t)} e^{S[W]} \\ &= e^{\frac{1}{2}\sum_{i=1}^{K}\int_{t_1}^{t_2} dtj_i(t)^2} \end{aligned} \tag{2.5}$$

However, this model is still restricted by the fact that it has only a finite number of factors which each influence the entire forward rate curve. This restricts the possible correlation structure of the forward rates. This restriction can be removed by taking the number of factors to infinity as pointed out in Cohen and Jarrow [43]. This is however unrealistic from a specification point of view as an infinite number of parameters cannot, of course, be estimated. Hence, models where a rich correlation structure could be imposed with a small number of parameters were developed. The earliest such model was proposed by Kenendy [26] and was followed by Goldstein [80], Santa-Clara and Sornette [75] and Baaquie [7]. Besides Baaquie's field theory generalisation of the HJM model, all the other models is written with a stochastic partial differential equation in infinitely many variables. The approach based on quantum field theory proposed by Baaquie[7] is in some sense complimentary to the approach based on stochastic partial differential equations since the expressions for all financial instruments are formally given as functional integral. One advantage of the approach based on quantum field theory is that it offers a different perspective on financial processes, offers a variety of computational algorithms, and nonlinearities in the forward rates as well as its stochastic volatility can be incorporated in a fairly straightforward manner. On the other hand, the field theory generalisation of the HJM model has been theoretically proved adequate for modelling the infinite degree of freedom with correlation since quantum field theory in physics has been developed exactly for cases including imperfect correlated infinite parameters.

§ 2.2 Quantum Field Theory Model for Interest Rate

The quantum field theory of forward interest rates is a general framework for modelling the interest rates that provides a particularly transparent and computationally tractable formulation of interest rate instruments.

Forward interest rates $f(t,x)$ are related to the two dimensional stochastic (random) field $\mathcal{A}(t,x)$ that drives the time evolution of the forward interest rates, and is given by

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sigma(t,x)\mathcal{A}(t,x) \tag{2.6}$$

The drift of the forward interest rates $\alpha(t,x)$ is fixed by a choice of numeraire [6], [9], and $\sigma(t,x)$ is the volatility function that is fixed from the market [6].

The value of all financial instruments are given by averaging the stochastic field $\mathcal{A}(t,x)$ over all it's possible values. This averaging procedure is formally equivalent to a quantum field theory in imaginary (Euclidean) time and hence, in effect, $\mathcal{A}(t,x)$ is equivalent to a two dimensional quantum field.

Integrating eq. 2.6 yields

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} dt'\alpha(t',x) + \int_{t_0}^{t} dt'\sigma(t',x)\mathcal{A}(t',x) \tag{2.7}$$

where $f(t_0,x)$ is the initial forward interest rates that is specified by the market.

One is free to choose the dynamics of the quantum field $\mathcal{A}(t,x)$. Following Baaquie and Bouchaud [16, 10], the Lagrangian that describes the evolution of instantaneous forward rates is defined by three parameters μ, λ, η and is given by[2]

$$\mathcal{L}(A) = -\frac{1}{2}\left\{\mathcal{A}^2(t,z) + \frac{1}{\mu^2}\left(\frac{\partial \mathcal{A}(t,z)}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 \mathcal{A}(t,z)}{\partial^2 z}\right)^2\right\} \tag{2.8}$$

where market (psychological) future time is defined by $z = (x-t)^\nu$.

A more general Gaussian Lagrangian is nonlocal in future time z and has the form

$$\mathcal{L}(A) = -\frac{1}{2}\mathcal{A}(t,z)D^{-1}(t,z,z')\mathcal{A}(t,z') \tag{2.9}$$

The action $S[\mathcal{A}]$ of the Lagrangian is defined as

$$S[\mathcal{A}] = \int_{t_0}^{\infty} dt \int_{0}^{\infty} dz dz' \mathcal{L}(\mathcal{A}) \tag{2.10}$$

In order to compare with empirical data, the normalized correlation function is given as [16]

$$\mathcal{C}(\theta,\theta') = \frac{D(\theta,\theta')}{\sqrt{D(\theta,\theta')D(\theta,\theta')}} \tag{2.11}$$

[2]More complicated nonlinear Lagrangians have been discussed in [6].

where $\theta = x - t, \theta' = x' - t$ and can be expressed explicitly as

$$\begin{aligned} \mathcal{C}(\theta_+;\theta_-) &= \frac{g_+(z_+) + g_-(z_-)}{\sqrt{[g_+(z_+ + z_-) + g_-(0)][g_+(z_+ - z_-) + g_-(0)]}} \\ z_\pm(\theta_+;\theta_-) &\equiv z(\theta) \pm z(\theta') \end{aligned} \tag{2.12}$$

with, in the real case that will be of relevance for fitting the empirical data

$$\begin{aligned} g_+(z) &= e^{-\lambda z \cosh(b)} \sinh\{b + \lambda z \sinh(b)\} \\ g_-(z) &= e^{-\lambda|z| \cosh(b)} \sinh\{b + \lambda|z| \sinh(b)\} \\ e^{\pm b} &= \frac{\lambda^2}{2\mu^2}\left[1 \pm \sqrt{1 - 4(\frac{\mu^4}{\lambda})}\right] \end{aligned} \tag{2.13}$$

Baaquie and Bouchaud [16] have determined the empirical values of the three constants μ, λ, ν, and have demonstrated that this formulation is able to accurately account for the phenomenology of interest rate dynamics. Ultimately, all the pricing formulae for interest rate instruments stems from the volatility function $\sigma(t, x)$ and correlation parameters μ, λ, ν contained in the Lagrangian, as well as the initial term structure $f(t_0, x)$.

The market value of all financial instruments based on the forward interest rates are obtained by performing a path integral over the (fluctuating) two dimensional quantum field $\mathcal{A}(t, z)$. The expectation value for an instrument, say $F[\mathcal{A}]$, is denoted by $\langle F[\mathcal{A}]\rangle \equiv E[F[\mathcal{A}]]$ and is defined by the functional average over all values of $\mathcal{A}(t, z)$, weighted by the probability measure e^S/Z. Hence

$$\langle F[\mathcal{A}]\rangle \equiv E\big(F[\mathcal{A}]\big) \equiv \frac{1}{Z}\int D\mathcal{A}\; F[\mathcal{A}]\; e^{S[\mathcal{A}]} \;\;;\;\; Z = \int D\mathcal{A} e^{S[\mathcal{A}]} \tag{2.14}$$

The quantum theory of the forward interest rates is defined by the generating (partition) function [6] given by

$$\begin{aligned} Z[h] &= E\big[e^{\int_{t_0}^\infty dt \int_0^\infty dz h(t,z)\mathcal{A}(t,z)}\big] \equiv \big\langle e^{\int_{t_0}^\infty dt \int_0^\infty dz h(t,z)\mathcal{A}(t,z)}\big\rangle \\ &\equiv \frac{1}{Z}\int D\mathcal{A}\; e^{S[\mathcal{A}] + \int_{t_0}^\infty dt \int_0^\infty dz h(t,z)\mathcal{A}(t,z)} \\ &= \exp\Big(\frac{1}{2}\int_{t_0}^\infty dt \int_0^\infty dz dz' h(t,z) D(z,z';t) h(t,z')\Big) \end{aligned} \tag{2.15}$$

which follows from the correlator of the $\mathcal{A}(t, x)$ quantum field given by

$$\langle \mathcal{A}(t,z)\mathcal{A}(t',z')\rangle = E\big[\mathcal{A}(t,z)\mathcal{A}(t',z')\big] = \delta(t - t')D(z,z';t) \tag{2.16}$$

For simplicity of notation $\langle F[\mathcal{A}]\rangle$ will be used for denoting expectation values and only the case of $\nu = 1$ will be considered; all integrations over z are replaced with those over future time x. For $\nu = 1$ from eq. 2.10 the dimension of the quantum field $\mathcal{A}(t,x)$ is 1/time and from eq. 2.7 the volatility $\sigma(t,x)$ of the forward interest rates also has dimension of 1/time.

§ 2.3 Market Measures in Quantum Finance

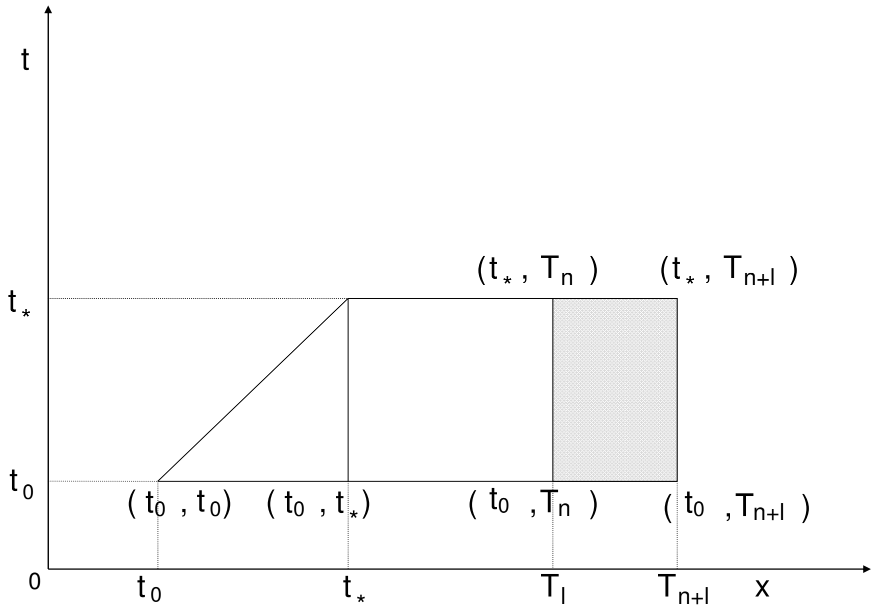

Figure 2.2: The domain of integration $\mathcal{M}$ for evaluating the drift of the Libor market numeraire.

For the purpose of modeling Libor term structure, it is convenient to choose an evolution such that all the Libor rates have a martingale evolution. For **Libor market measure**, recall § 1.3.3, in terms of the underlying forward interest rates, one has from eq. 1.33

$$\begin{aligned} e^{\int_{T_n}^{T_n+l} dx f(t_0,x)} &= E_L[e^{\int_{T_n}^{T_n+\ell} dx f(t_*,x)}] \\ \Rightarrow e^{F_0} &= E_L[e^{F_*}] \end{aligned} \tag{2.17}$$

$$F_0 \equiv \int_{T_n}^{T_n+l} dx f(t_0,x) \quad ; \quad F_* \equiv \int_{T_n}^{T_n+l} dx f(t_*,x) \tag{2.18}$$

Denote the drift for the market measure by $\alpha_L(t,x)$, and let $T_n < x \le T_n + \ell$; the evolution equation for the Libor forward interest rates is given, similar to eq. 2.7, by

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} dt'\alpha_L(t',x) + \int_{t_0}^{t} dt'\sigma(t',x)A(t',x) \tag{2.19}$$

Hence

$$E_L\big[e^{F_*}\big] = e^{F_0+\int_{\mathcal{M}}\alpha_L(t',x)} \int DAe^{\int_{\mathcal{M}}\sigma(t',x)A(t',x)}e^{S[A]} \tag{2.20}$$

where the integration domain $\mathcal{M}$ is given in Fig. 2.2. From eqs. 2.15, 2.17 and 2.20

$$\begin{aligned} e^{-\int_{\mathcal{M}}\alpha_L(t,x)} &= \int DAe^{\int_{\mathcal{M}}\sigma(t,x)A(t,x)}e^{S[A]} \\ &= \exp\{\frac{1}{2}\int_{t_0}^{t_*} dt \int_{T_n}^{T_n+\ell} dxdx'\sigma(t,x)D(x,x';t)\sigma(t,x')\} \end{aligned} \tag{2.21}$$

Hence the Libor drift velocity is given by

$$\alpha_L(t,x) = -\sigma(t,x)\int_{T_n}^{x} dx' D(x,x';t)\sigma(t,x') \quad ; \quad T_n \le x < T_n + \ell \tag{2.22}$$

The Libor drift velocity $\alpha_L(t,x)$ is **negative**, as is required for compensating growing payments due to the compounding of interest. Fig. 2.3 shows the behavior of the drift velocity $-\alpha_L(t,x)$, with the value of $\sigma(t,x)$ taken from the market.

For the **Forward measure**, recall § 1.3.3, to determine the corresponding drift velocity $\alpha_F(t,x)$, the right hand side of Eq.1.30 is explicitly evaluated. Note from Eq. 2.7

$$E_F\big[e^{-\int_{Tn}^{T} dxf(t_*,x)}\big] \quad = \quad e^{-\int_{Tn}^{T} dxf(t_0,x)-\int_{\mathcal{T}}\alpha_F(t',x)} \int DAe^{-\int_{\mathcal{T}}\sigma(t',x)A(t',x)}e^{S[A]} \tag{2.23}$$

where the integration domain $\mathcal{T}$ is given in Fig. 2.2.

Hence, from eqs. 2.15 and 2.23

$$\begin{aligned} e^{\int_{\mathcal{T}}\alpha_F(t,x)} &= \int DAe^{-\int_{\mathcal{T}}\sigma(t,x)A(t,x)}e^{S[A]} \\ &= \exp\{\frac{1}{2}\int_{t_0}^{t_*} dt \int_{T_n}^{T} dxdx'\sigma(t,x)D(x,x';t)\sigma(t,x')\} \end{aligned} \tag{2.24}$$

Hence the drift velocity for the forward measure is given by

$$\alpha_F(t,x) = \sigma(t,x)\int_{T_n}^{x} dx' D(x,x';t)\sigma(t,x') \quad ; \quad T_n \le x < T_n + \ell \tag{2.25}$$

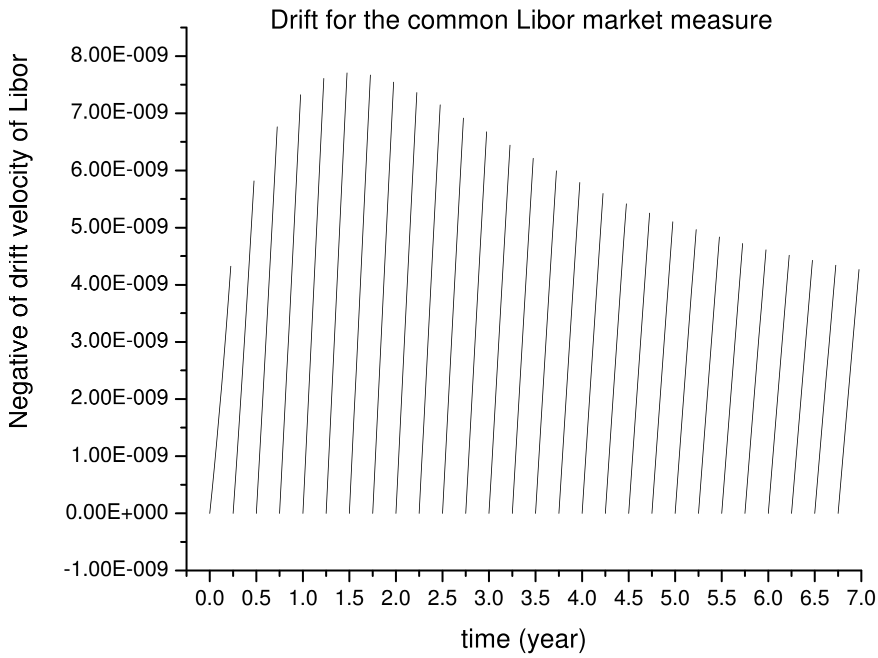

Figure 2.3: Negative of the drift velocity, namely $-\alpha_L(t,x)$, for the common Libor market measure, which is equal to the drift velocity $\alpha_F(t,x)$ for the forward Libor measure.

The Libor drift $\alpha_L(t,x)$ is the negative of the drift for the forward measure, that is

$$\alpha_L(t,x) = -\alpha_F(t,x)$$

For the **money market measure**, from eq. 1.28 the drift velocity is given by [6] as

$$\alpha_M(t,x) = \sigma(t,x)\int_t^x dx' D(x,x';t)\sigma(t,x') \tag{2.26}$$

§ 2.4 Pricing a caplet in quantum finance

Recall the discussion in § 1.4.2, the price of a midcurve caplet, issued at time t_0 and maturing at time $t_* \in [t_0, T]$, is denoted by $Caplet(t_0, t_*, T)$.[3]

Let the principal amount be equal to ℓV, and the caplet rate be K.The **payoff function**

[3] A European midcurve caplet can be exercised only at maturity time t_*.

of the caplet from eq.1.37 is given as

$$Caplet(t_*, t_*, T) = \ell V B(t_*, T+\ell)[L(t_*, T) - K]_+ \tag{2.27}$$
$$= \tilde{V} B(t_*, T)(X - F_*)_+ \tag{2.28}$$

where

$$L(t_*, T) = \frac{e^{\int_T^{T+\ell} dx f(t_*, x)} - 1}{\ell} \quad ; \quad F_* = F(t_*, T, T+\ell) = \exp\{-\int_T^{T+\ell} dx f(t_*, x)\}$$
$$X = \frac{1}{1+\ell K} \quad ; \quad \tilde{V} = (1+\ell K)V$$

The payoff function for a floorlet is given by

$$Floorlet(t_*, t_*, T) = \tilde{V} B(t_*, T)(F_* - X)_+$$

and ensures the lender holding the floorlet option receives a minimum rate of K for the interest payments.

The European caplet at time t_0 is computed using the forward measure with numeraire $B(t, T)$[4] yields

$$\frac{Caplet(t_0, t_*, T)}{B(t_0, T)} = E_F\left[\frac{Caplet(t_*, t_*, T)}{B(t_*, T)}\right] \tag{2.30}$$
$$\Rightarrow Caplet(t_0, t_*, T) = \tilde{V} B(t_0, T) E_F (X - F_*)_+ \tag{2.31}$$

Baaquie [6, 9] has derived the price by evaluating the expectation value using field theory, the evaluation procedure is reviewed for a midcurve caplet below. Similar technic will be used when pricing or hedging other interest rate derivatives in the frame of quantum finance.

The payoff function is re-written in a form that is more suited to path integral using the following identity

$$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp e^{ipz} \tag{2.32}$$

Hence, from eq.2.28 and 2.32, one has the following

$$(X - F_*)_+ = \int_{-\infty}^{+\infty} dG \delta[G + \int_T^{T+\ell} dx f(t_*, x)](X - e^G)_+$$
$$= \int_{-\infty}^{+\infty} dG \frac{dp}{2\pi} e^{ip(G + \int_T^{T+\ell} dx f(t_*, x))}(X - e^G)_+ \tag{2.33}$$

[4]For any traded financial instrument $\mathcal{I}$, the forward martingale property in eq.1.30 yields

$$\frac{\mathcal{I}(t_0, \tau)}{B(t_0, \tau)} = E_F\left[\frac{\mathcal{I}(t_*, \tau)}{B(t_*, \tau)}\right] \tag{2.29}$$

where $\mathcal{I}(t_*, \tau)$ is the payoff function at maturity time t_*.

Re-write eq.2.31 as

$$Caplet(t_0, t_*, T) \;=\; \tilde{V}B(t_0, T)\int_{-\infty}^{+\infty} dG\Psi(G, t_*, T)(X - e^G)_+ \tag{2.34}$$

where

$$\Psi(G, t_*, T) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} E_{F[t_0,t_*]}\left[e^{ip(G+\int_T^{T+\ell} dx f(t_*,x))}\right] \tag{2.35}$$

Following eq.2.15, one obtains

$$\Psi(G, t_*, T) = \frac{1}{\sqrt{2\pi q^2}} e^{-\frac{1}{2q^2}(G+\int_T^{T+\ell} dx f(t_*,x)+\frac{q^2}{2})^2} \tag{2.36}$$

where

$$q^2 = q^2(t_0, t_*, T) = \int_{t_0}^{t_*} dt \int_T^{T+\ell} dx dx' \sigma(t, x) D(x, x'; t)\sigma(t, x') \tag{2.37}$$

Thus, by solving the path integral in eq.2.34, one obtains a closed form of the European caplet price. At time $t_0 < t_*$ the caplet price is given by the following Black-Scholes type formula

$$Caplet(t_0, t_*, T) = \tilde{V}B(t_0, T)\left[XN(d_+) - FN(d_-)\right] \tag{2.38}$$

where $N(d_\pm)$ is the cumulative distribution for the normal random variable with the following definitions[5]

$$\begin{aligned} F &= \exp\{-\int_T^{T+\ell} dx f(t_0, x)\} \\ d_\pm &= \frac{1}{q}\left[\ln\left(\frac{X}{F}\right) \pm \frac{q^2}{2}\right] \end{aligned} \tag{2.39}$$

§ 2.5 Feynman Perturbation Expansion for the Price of Coupon Bond Options and Swaptions

Recall that the price of interest rate Cap is a summation of single Caplet which has duration of only three month, correlation between Libor forward rates with interval of three month is close to one. However, for coupon bond option and swaption, the underlying may span for longer duration up to twenty years where the imperfect correlation plays crucial role. This led

[5] Note one recovers the normal caplet result by setting $t_* = T$.

us to study the European coupon bond option[6] in the quantum finance frame work. The field theory for the forward interest rates is Gaussian, but when the payoff function for the coupon bond option is included it makes the field theory nonlocal and nonlinear. A perturbation expansion using Feynman diagrams gives a closed form approximation for the price of coupon bond option based on the fact that the volatility of the forward interest rates is a small quantity. I will review the results given in Baaquie [11] in this section in order to carry out the empirical study in chapter 5.

Recall any numeraire can be used for discounting the payoff function for options for a financial instrument as long as the numeraire yields a martingale evolution for the financial instrument. The choice of the numeraire that yields a martingale measure also fixes the drift $\alpha(t,x)$ [9].

Recall from § 2.3, the forward price $F_i \equiv F(t_0,t_*,T_i)$ can be chosen as a martingale [6], and is called the forward measure. The forward measure is more convenient for the option pricing problem since one can dispense discounting with the stochastic (money market) numeraire, namely by $\exp\{\int_{t_0}^{t_*} r(t)dt\}$, and instead discount using the non-stochastic (present value of a) zero coupon bond $B(t_0,t_*)$.

Call and put options for the coupon bonds using the forward measure are given by

$$
\begin{aligned}
C(t_0,t_*,K) &= B(t_0,t_*)E_F\big[\big(\sum_{i=1}^{N} c_i B(t_*,T_i) - K\big)_+\big] = B(t_0,t_*)\langle S(t_*)\rangle_F \qquad (2.40)\\
P(t_0,t_*,K) &= B(t_0,t_*)E_F\big[\big(K - \sum_{i=1}^{N} c_i B(t_*,T_i)\big)_+\big]
\end{aligned}
$$

The price of the coupon bond can be re-written as

$$
\begin{aligned}
\sum_{i=1}^{N} c_i B(t_*,T_i) &= \sum_{i=1}^{N} c_i e^{-\alpha_i - Q_i} F(t_0,t_*,T_i)\\
&= \sum_{i=1}^{N} c_i F_i + \sum_{i=1}^{N} c_i[B(t_*,T_i) - F_i]\\
&\equiv F + V \qquad (2.41)
\end{aligned}
$$

[6]As discussed in § 1.4.4, swaption can be written in the same form of coupon bond option, thus all the investigation done on coupon bond option can be applied on swaption automatically.

with definitions

$$J_i \equiv c_i F_i \quad ; \quad F_i = \exp\{-\int_{t_*}^{T_i} dx f(t_0, x)\} \tag{2.42}$$

$$F \equiv \sum_{i=1}^{N} c_i F_i = \sum_{i=1}^{N} J_i \tag{2.43}$$

$$V \equiv \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i] = \sum_{i=1}^{N} J_i \big[e^{-\alpha_i - Q_i} - 1\big] \tag{2.44}$$

$$\alpha_i = \int_{R_i} \alpha(t, x) \tag{2.45}$$

$$Q_i = \int_{R_i} \sigma(t, x)\mathcal{A}(t, x) \equiv \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \sigma(t, x)\mathcal{A}(t, x) \tag{2.46}$$

The payoff function is re-written using the properties of the Dirac delta function. It follows from eq. 2.41 that

$$\begin{aligned}
\Big(\sum_{i=1}^{N} c_i B(t_*, T_i) - K\Big)_+ &= \big(F + V - K\big)_+ = \int_{-\infty}^{+\infty} dW \delta(V - W)\big(F + W - K\big)_+ \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta e^{i\eta(V-W)}\big(F + W - K\big)_+
\end{aligned}$$

Hence the price of the call option, from eq. 2.40, can be written as

$$C(t_0, t_*, K) = B(t_0, t_*) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta \big(F + W - K\big)_+ e^{-i\eta W} Z(\eta) \tag{2.47}$$

with the partition function given by

$$\begin{aligned}
Z(\eta) &= \langle e^{i\eta V} \rangle_F \\
&= \frac{1}{Z} \int D\mathcal{A} e^S e^{i\eta V} \quad ; \quad Z = \int D\mathcal{A} e^S
\end{aligned} \tag{2.48}$$

A perturbation expansion is developed that evaluates the partition function $Z(\eta)$ as a series in the volatility function $\sigma(t, x)$. A cumulant expansion of the partition function in a power series in η yields

$$Z(\eta) = e^{i\eta D - \frac{1}{2}\eta^2 A - i\frac{1}{3!}\eta^3 B + \frac{1}{4!}\eta^4 C + \ldots} \tag{2.49}$$

The coefficients $A, B, C, \ldots$ are evaluated using Feynman diagrams.

Expanding the right hand side of eq. 2.48 in power series to fourth order in η yields

$$\begin{aligned} Z(\eta) &= \frac{1}{Z}\int D\mathcal{A} e^{i\eta V} e^{S[\mathcal{A}]} \\ &= \frac{1}{Z}\int D\mathcal{A} e^{S[\mathcal{A}]}\big[1 + i\eta V + \frac{1}{2!}(i\eta)^2 V^2 \\ &+ \frac{1}{3!}(i\eta)^3 V^3 + \frac{1}{4!}(i\eta)^4 V^4 + ...\big] \end{aligned} \tag{2.50}$$

Comparing eqs. 2.49 and 2.50 and carrying out a field theory, we have

$$A = \sum_{ij=1}^{N} J_i J_j [G_{ij} + \frac{1}{2} G_{ij}^2] + O(G_{ij}^3) \tag{2.51}$$

$$\begin{aligned} B &= 3\sum_{ijk=1}^{N} J_i J_j J_k G_{ij} G_{jk} + O(G_{ij}^3) \\ C &= 16\sum_{ijkl=1}^{N} J_i J_j J_k J_l G_{ij} G_{jk} G_{kl} + O(G_{ij}^4) \end{aligned} \tag{2.52}$$

Where the dimensionless forward bond price correlator is given by

$$\begin{aligned} G_{ij} &\equiv G_{ij}(t_0, t_*, T_i, T_j; \sigma) \\ &= \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x, x'; t) \\ &= G_{ji} \quad : \quad \text{real and symmetric} \end{aligned} \tag{2.53}$$

The evaluation of G_{ij} is illustrated in Figure 2.4, and Figure 2.5 shows it's dependence on T_i and T_j. G_{ij} is the forward bond propagator that expresses the correlation in the fluctuations of the forward bond prices $F_i = F(t_0, t_*, T_i)$ and $F_j = F(t_0, t_*, T_j)$.

From eqn. 2.47, one can do an expansion for the partition function of the cubic and quartic terms in η, and then perform the Gaussian integrations over η; this yields

$$\begin{aligned} C(t_0, t_*, K) &= B(t_0, t_*)\frac{1}{2\pi}\int_{-\infty}^{+\infty} dW d\eta \big(F + W - K\big)_+ e^{-i\eta W} e^{-\frac{1}{2}\eta^2 A - i\frac{1}{3!}\eta^3 B + \frac{1}{4!}\eta^4 C + ...} \\ &= B(t_0, t_*)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW (F + W\sqrt{A} - K)_+ f(\partial_w) e^{-\frac{1}{2}W^2} + O(\sigma^4) \end{aligned} \tag{2.54}$$

where, for $\partial_w \equiv \partial/\partial W$, one has the following

$$f(\partial_w) \equiv 1 - (\frac{B}{6A^{3/2}})\partial_w^3 + (\frac{C}{24A^2})\partial_w^4 + \frac{1}{2}(\frac{B}{6A^{3/2}})^2 \partial_w^6 + O(\sigma^4) \tag{2.55}$$

$$A \simeq O(\sigma^2) \ ; \ \frac{B}{A^{3/2}} \simeq O(\sigma) \ ; \ \frac{C}{A^2} \simeq O(\sigma^2)$$

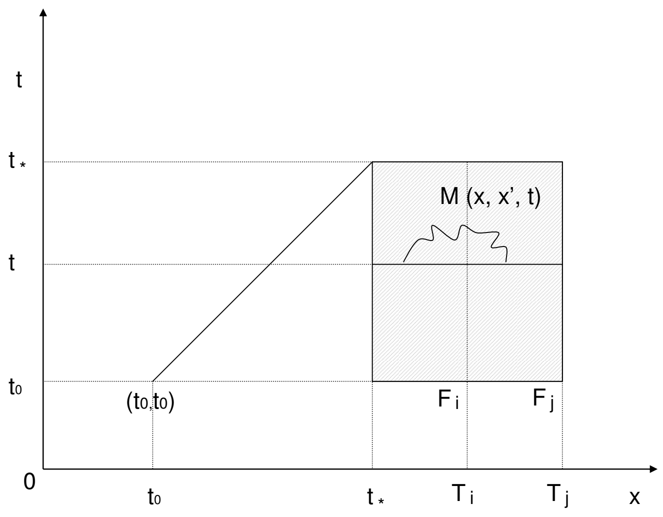

Figure 2.4: The shaded domain of the forward interest rates contribute to the correlator $G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x, x'; t)$. For a typical point t in the time integration the figure shows the correlation function $M(x, x'; t)$ connecting two different values of the forward interest rates at future time x and x'.

Due to the properties of $\Theta(x)$, the Heaviside theta function, the second derivative of the payoff is equal to the Dirac delta function, namely

$$\partial_w^2 (F + W\sqrt{A} - K)_+ = \sqrt{A}\delta(W - X) \tag{2.56}$$

$$X = \frac{K - F}{\sqrt{A}} \quad ; \text{ Dimensionless} \tag{2.57}$$

Using equation above and eqs. 2.54, 2.55 yields, after integration by parts, the following

$$C(t_0, t_*, K) = B(t_0, t_*) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dW \Big[(F + W\sqrt{A} - K)_+ + \sqrt{A}\delta(W - X) \{ -\frac{B}{6A^{3/2}} \partial_w + \frac{C}{24A^2} \partial_w^2 + \frac{1}{2} (\frac{B}{6A^{3/2}})^2 \partial_w^4 \} \Big] e^{-\frac{1}{2}W^2} + O(\sigma^4) \tag{2.58}$$

Note

$$I(X) = \int_{-\infty}^{+\infty} dW (W - X)_+ e^{-\frac{1}{2}W^2} = e^{-\frac{1}{2}X^2} - \sqrt{\frac{\pi}{2}} X \big[1 - \Phi(\frac{X}{\sqrt{2}}) \big] \tag{2.59}$$

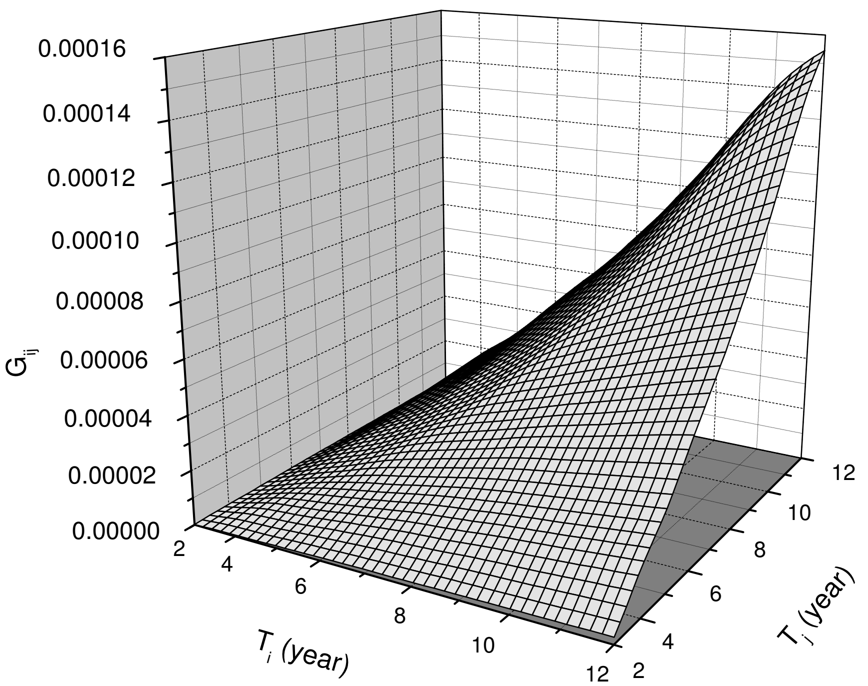

Figure 2.5: The forward bond price correlator $G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x, x'; t) = G_{ij}(t_0, t_*, T_i, T_j)$, is plotted against T_i and T_j with $t_* - t_0 = 2$ years, where $M(x, x'; t)$ is taken from swaption data.

where the error function is given by

$$\Phi(u) = \frac{2}{\sqrt{\pi}} \int_0^u dW e^{-W^2}$$

Hence the price of the coupon bond is given by

$$C(t_0, t_*, K) = B(t_0, t_*)\sqrt{\frac{A}{2\pi}}\Big[\frac{B}{6A^{3/2}}X + \frac{C}{24A^2}(X^2 - 1) + \frac{1}{72}\frac{B^2}{A^3}(X^4 - 6X^2 + 3)\Big]e^{-\frac{1}{2}X^2}$$
$$+ B(t_0, t_*)\sqrt{\frac{A}{2\pi}}I(X) + O(\sigma^4) \tag{2.60}$$

and is graphed in Figure 2.6; the reason the surface is smooth is because the variables X and A are varied continuously.

The asymptotic behaviour of the error function $\Phi(u)$ yields the following limits

$$I(X) = \begin{cases} 1 - \sqrt{\frac{\pi}{2}}X + O(X^2) & X \approx 0 \\ \frac{e^{-\frac{1}{2}X^2}}{X^2}[1 + O(\frac{1}{X^2})] & X >> 0 \end{cases}$$

For the coupon bond and swaption at the money $F = K$; hence the option's price close to at the money has $X \approx 0$ and to leading order yields the price to be

$$C(t_0, t_*, K) \approx B(t_0, t_*)\sqrt{\frac{A}{2\pi}} - \frac{1}{2}B(t_0, t_*)(K - F) + O(X^2) \tag{2.61}$$

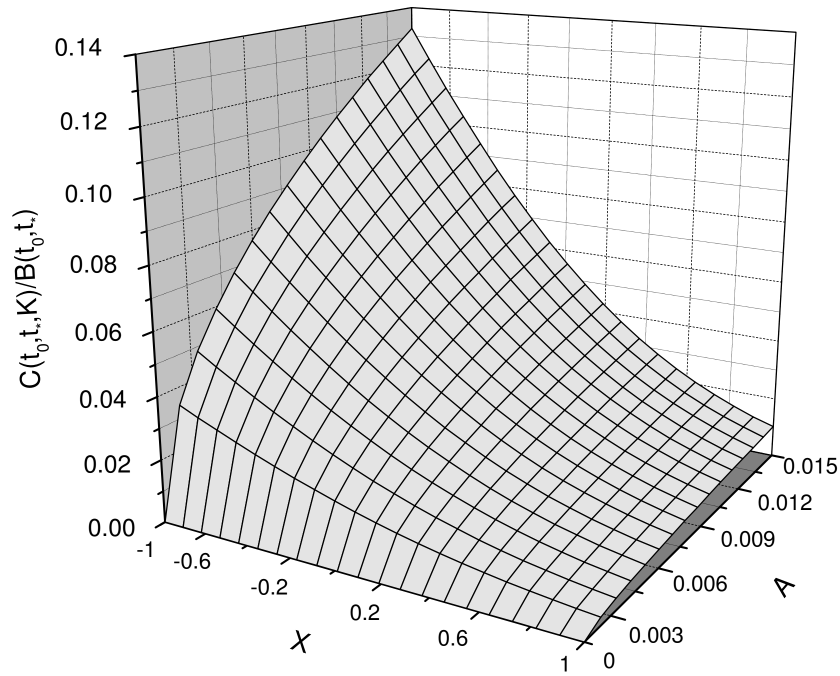

Figure 2.6: The value of the swaption $C(t_0, t_*, K)/B(t_0, t_*) = \sqrt{\frac{A}{2\pi}} I(X) + O(\sigma^2)$, plotted as a function of A and X.

Chapter 3

Empirical Study of Interest Rate Caplet

The industry standard for pricing an interest rate caplet is Black's formula. Another distinct price of the same caplet is derived using a quantum field theory model of the forward interest rates in § 2.4. In this chapter, an empirical study is carried out to compare the two caplet pricing formulae. Historical volatility and correlation of forward interest rates are used to generate the field theory caplet price; another approach is to fit a parametric formula for the effective volatility using market caplet price. The result in this chapter shows that the field theory model generates the price of a caplet and cap fairly accurately. Black's formula for a caplet is compared with field theory pricing formula. It is seen that the field theory formula for caplet price has many advantages over Black's formula.

§ 3.1 Introduction

Liquid interest rate option like Cap and Floor have been embedded with all available information in the price, thus the main challenge for market participants is to extract information from these option and use these information and a no-arbitrage model to price other exotic options. The underlying Libor rates are common for these option, consequently one may need to extract information purely depend on the Libor rates.

In Black's formula for caplets the implied volatility σ_B for a caplet is quoted in the market and the market prices are computed from this by Black's formula, which is similar to the Black-Scholes option formula [82]. Caplets with different maturities and different strike prices are quoted with different σ_B which means σ_B is not purely depend on Libor and can not be used to price other Libor options. Also, the Black's formula is in effect has no predictive power but instead is used as a non-linear transformation from caplet volatilities to prices: implied volatility is simply another way of quoting the price of the caplet itself. The main utility of Black's formula is that implied σ_B is more stable than the price itself and can give a quick explanation of the market.

In contrast to Black's formula, the field theory pricing formula is a new formula of pricing which is derived from an arbitrage-free model of the term structure of interest rates. Recall from 2, the main advantage of modelling the forward interest rates using field theory is that there are infinitely many random variables at each instant driving the forward interest rates. Hence we need not consider exactly correlated forward interest rates; for the field theory model the correlation of forward rates for different maturities is accurately explained by propagator of field theory in [16].

An empirical study of the field theory pricing and Black's caplet formulae are conducted in this chapter. The main result in the field theory pricing formula is that the effective volatility **q** of a caplet is computed by a three dimensional integration on the correlation between forward interest rates in future time. The following are three different approaches which will be discussed in this chapter for fixing the effective field theory volatility q for pricing caplets.

- The volatility function σ_H and parameters in the correlation of the field theory model, say μ λ and η are all fitted from historical Libor data.
- The market correlator is computed directly from Libor market data.
- A parametric formula for the effective volatility q, and consequently the implied volatility σ_I for the field theory model is determined from historical caplet price. [1] The value of σ_I is quite distinct from σ_B since σ_I is a function of future time and can be used for a for extrapolating to the futre. In contrast, σ_B is a value that has to be computed every single day from the caplet price.

[1]Note in contrast σ_H is obtained from the historical data of the forward interest rates.

§ 3.2 Comparison with Black's formula for interest rate caps

Recall the field theory caplet pricing formula evaluated in § 2.4, note that q is the effective volatility for the pricing formula. Observe that the propagator for forward rates is required for pricing the caplet. Ultimately, the pricing formulae for caplets and floorlets stem from the volatility function $\sigma(t,x)$ and correlation parameters μ, λ, η contained in the Lagrangian for the forward interest rates, as well as the initial interest rates term structure. The pricing formula in eq. 2.38 for at the money case for a normal caplet ($t_* = T_n$) has $X = e^{\mathcal{F}}$, and is given by

$$Cap(t_0, t_*) = VB(t_{0,t_*})\big[N(d_+) - N(d_-)\big] \tag{3.1}$$

where

$$d_\pm = \pm\frac{q}{2} \tag{3.2}$$

We will use this formula for a comparison with the Black's model.

Black's model for interest rate cap is briefly reviewed in order to compare it with the field theory model. Black's formula is based on the assumption that the spot interest rate is a log normal random variable. The payoff function for Black's formula at time t_* is [42]

$$g_B(t_*) = \frac{\ell V}{1 + \ell R(t_*)}(R(t_*) - R)_+ \tag{3.3}$$

$R(t_*)$ is the 3-month LIBOR rate at the beginning of the quarter at time t_*, and R is the strike price on the interest rate.

Black's formula for the value of the cap at time $t_0 < t_*$ is [82]

$$Cap_B(t_0, t_*, R) = \frac{V\ell}{1 + \ell f(t_0, t_*)}B(t_0, t_*)[f(t_0, t_*)N(d_+^B) - RN(d_-^B)] \tag{3.4}$$

where

$$d_\pm^B = \frac{1}{\sigma_B\sqrt{t_* - t_0}}\left[\ln\frac{f(t_0, t_*)}{R} \pm \frac{\sigma_B^2(t_* - t_0)}{2}\right]$$

and $f(t_0, t_*)$ denote the ℓ period forward rate from t_* to $t_* + \ell$. At time $t_* = n\ell$, $f(t_0, t_*) = L(t_0, t_*)$.

When at the money, $f(t_0, t_*) = R$, the pricing is given by

$$Cap_B(t_0, t_*, R) = \frac{f(t_0, t_*)}{1 + \ell f(t_0, t_*)}V\ell B(t_0, t_*)[N(d_+^B) - N(d_-^B)] \tag{3.5}$$

with

$$d_{\pm}^{B} = \frac{\sigma_B\sqrt{(t_* - t_0)}}{2}$$

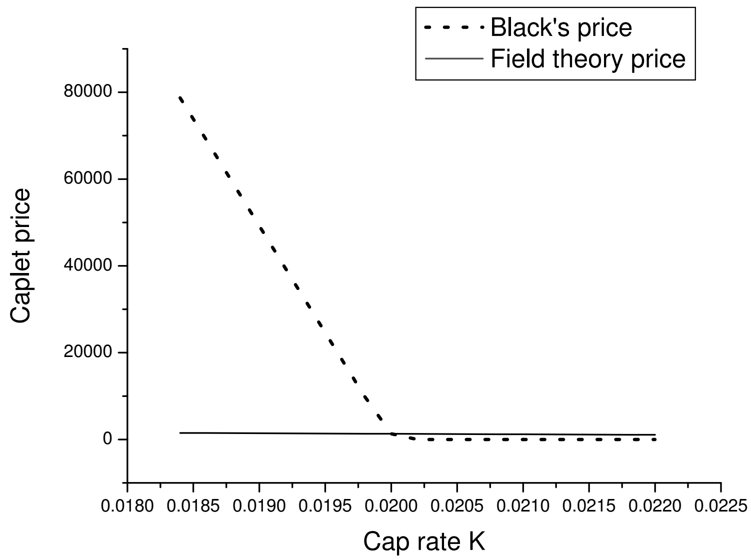

Figure 3.1: Caplet price from normalized Black's formula and field theory formula versus Cap rate K. Libor is given at 0.02. The caplet is at the money when K=0.02

Clearly the price of a cap derived from an arbitrage free model, as in eq. 2.38, and Black's formula do not agree. Even at the money, since the prefactor of two pricing formula are different, the effective volatility q in field theory formula and σ_B are not equal. We can normalize the prefactor by multiplying Black's formula by $\frac{1+\ell f(t_0,t_*)}{\ell f(t_0,t_*)}$, and then the two pricing formula are exactly the same and with q equal to σ_B. We change the cap rate K to compare the normalized pricing formula away from the money. The comparison is shown in Fig. 3.1, and the field theory model price is shown more clearly in Fig. 3.2. We can see that only at the money when cap rate K=0.02, the two pricing formula have the same result. The caplet pricing for the two formulae in general are not equal and deviate quite rapidly when K is no longer at the money.

We will see later that q has many advantages over σ_B. Most importantly, the effective volatility q is obtained in the field theory model, as computed in eqn. 2.37, from the underlying forward interest rates that are common to all Libor based options.

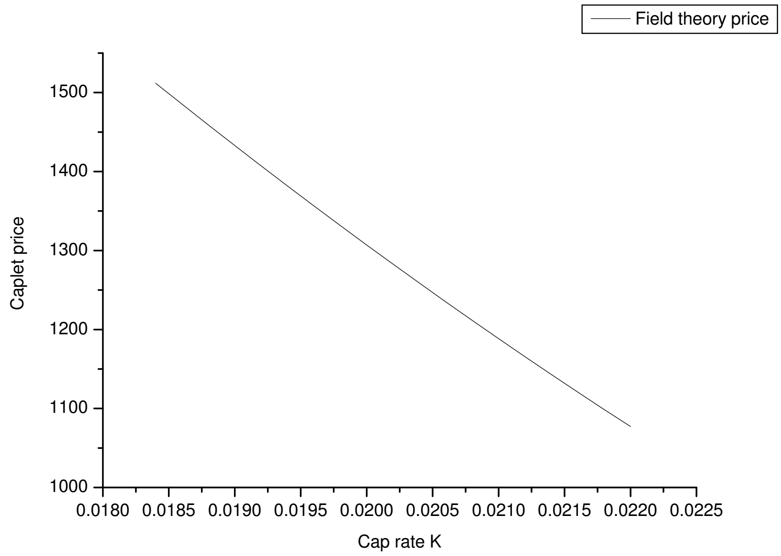

Figure 3.2: Caplet price from field theory formula versus Cap rate K. Libor is given at 0.02. The caplet is at the money when K=0.02

§ 3.3 Empirical Pricing of Field Theory Caplet Price

In this section, the Cap data and underlying forward interest rates data are discussed, put-call parity is empirically tested. An empirical study of three different approaches for implementing the field theory caplet pricing is carried out and the results are discussed.

§ 3.3.1 Data

We analyze the price of the option on Eurodollar futures contracts expiring 13 Dec 2004 with a strike price of 98. Daily price is from 7th Mar 2003 to 28th May 2004. All the prices are presented with interest rate in basis points (100 basis points=1% annual interest rate) and has to multiplied by the notional value of one million Dollars.

The put option of this data is equivalent to the caplet price with fixed maturity date say 13 Dec 2004. The importance of put call parity for pricing and choosing numeraire has been emphasized in Baaquie [9]; we examine market prices for the caplet and floorlet to see how the put-call parity is obeyed by the market. From eq. 1.37 for put-call partiy, for the case $t_* = T_n$ =13 Dec 2004, we form a portfolio

$$\Pi(t_0) = Caplet(t_0, T_n) - Floorlet(t_0, T_n) - \ell V B(t_0, T_n + \ell)(L(t_0, T_n) - K) \tag{3.6}$$

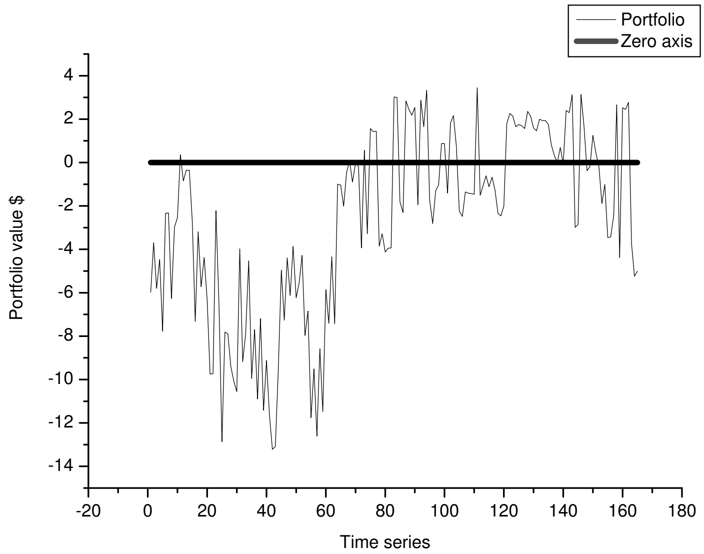

Figure 3.3: Value of portfolio $\Pi(t_0)$ as in Eqn 3.6 with notional value 1 million versus time t_0 (12.9.2003–7.5.2004)

The market value of the portfolio directly from data is plotted in Fig. 3.3, and shows that the market automatically obey put-call parity very well. The deviation of $\Pi(t_0)$ is negligible compared to the price of a caplet. Hence there are no arbitrage opportunities in the pricing of caplets and floorlets. We price the Caplets from the pricing formula, the floorlet's price being given by the put-call parity relation.

The following are three distinct approaches to fitting the field theory caplet pricing formula.

§ 3.3.2 Parameters for the Field Theory Caplet Price using Historical Libor

For the field theory pricing formula for the daily prices of the caplets, we need as input the daily initial term structure, the input volatility function and parameters μ λ and η for propagator. Daily fit of the volatility function and propagator parameters can be derived by a daily moving average on 60 days Libor rates history[2]. For the sake of simplicity, from [6, 48] we take volatility function to depend only on future time, namely

$$\sigma(t, x) = \sigma(x - t)$$

[2]The moving average can be any length of history days, and 60 days is chosen as most convenient.

Although this assumption cannot be indefinitely extended, it can be valid for up to 3 years [48] which is enough for our empirical study. Thus we only need to do the parametric fit once, then use these parameter for the whole data set projected to 1.5 years in the future. It should be noted that one can always do the parametric fit more frequently to get more accurate results.

Since the field theory is defined on a domain $x \geq t$, the propagator satisfies

$$D(t, x, x^{'}) = D(x - t, x' - t) \tag{3.7}$$

We fit a parametric curve for the effective volatility [47] using historical data from the prices of Libor for before 2003 May.

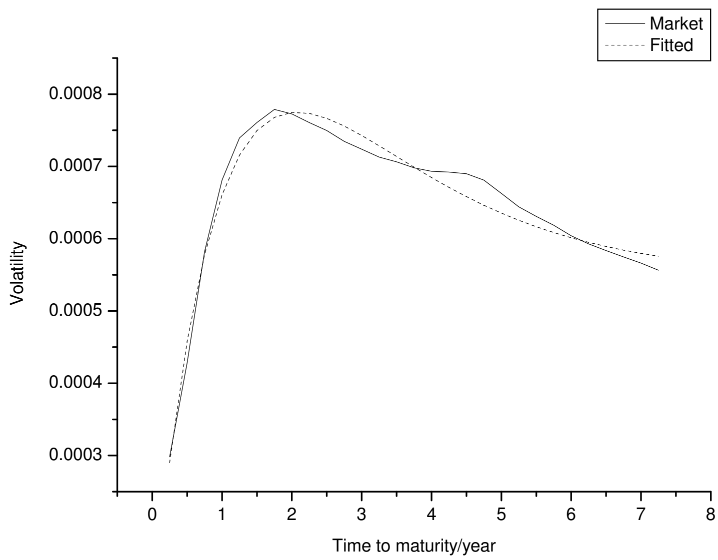

Figure 3.4: Volatility of Libor forward rates $\sigma(\text{year}^{-3/2})$ versus time to maturity, both from data and from formula with fitted parameters. Data select from 29.1.2003 to 29.4.2003. Normalized root mean square error is 2.76%

More precisely, the forward interest rates that we use to fix input volatility and propagator are data from the Eurodollar futures covers from 1998 May 4th to 2003 Apr 29th; the length of the dataset is 1256 trading days for daily prices of Libor 7 years into the future. Since, for $\ell = 3$ months, we have

$$L(t, T) = \frac{e^{\int_T^{T+\ell} f(t,x)dx} - 1}{\ell} \tag{3.8}$$

$$\simeq f(t, T) \tag{3.9}$$

Hence we use the Libor as being exactly equal to the forward rates, and then select a moving average over the last 63 days, from 29th Jan 2003 to 29th April 2003 since these carry the most relevant information.

Following Bouchaud and Matacz [47], we give a parametric formula for volatility and fix the parameters from the data which we select and the function is given as follows[3]

$$\begin{aligned}\sigma_H(\theta) &= 0.00055 - 0.00026\exp(-0.71826(\theta-\theta_{min})) \\ &\quad +0.0006(\theta-\theta_{min})\exp(-0.71826(\theta-\theta_{min}))\end{aligned} \tag{3.10}$$

where $\theta_{min} = 3$month

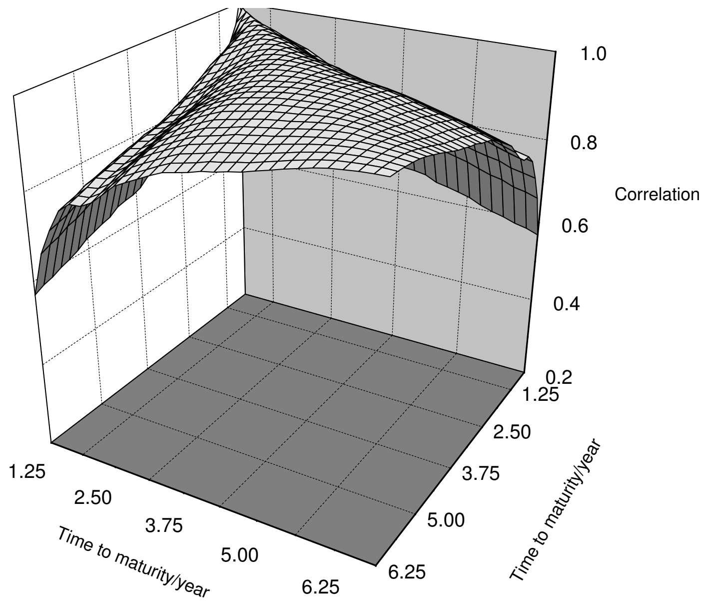

Figure 3.5: Correlation of Libor forward rates versus time to maturity, data selected from 29.1.2003 to 29.4.2003.

Following Baaquie and Bouchaud in [16] we can determine the empirical values of the three parameters μ, λ, η for the stiff Lagrangian by fitting the propagator to market correlation by using Levenberg Marquadrt method, the three parameters are fixed by Libor forward rates data from 29th Jan 2003 to 29th Apr 2003 as follow. The fits for volatility of Libor is given in Fig. 3.4. And the correlation is given in Fig. 3.5, the graph shows that the underlying forward rates are not perfectly correlated, the parameters are given as follow

λ	μ	b	η	root mean square error for the entire fit
16.578657/year	8.0761/year	1.376644	0.044127	1.09%

[3]We use σ_H to present volatility from historical Libor rates

Before pricing the caplet with all the information we have, one more thing need to be noted. Denoting by $\langle ... \rangle$ the expectation value of a stochastic quantity, we have from Baaquie[6] for the connected correlator

$$\langle \big(\delta f(t,\theta)\big)^2 \rangle_c \equiv \langle \big(\delta f(t,\theta)\big)^2 \rangle - \langle \delta f(t,\theta) \rangle \langle \delta f(t,\theta) \rangle = \epsilon \sigma_H^2(\theta) D(\theta,\theta) \tag{3.11}$$

setting $\epsilon = 1/260$, where 260 is the trading days in one year.

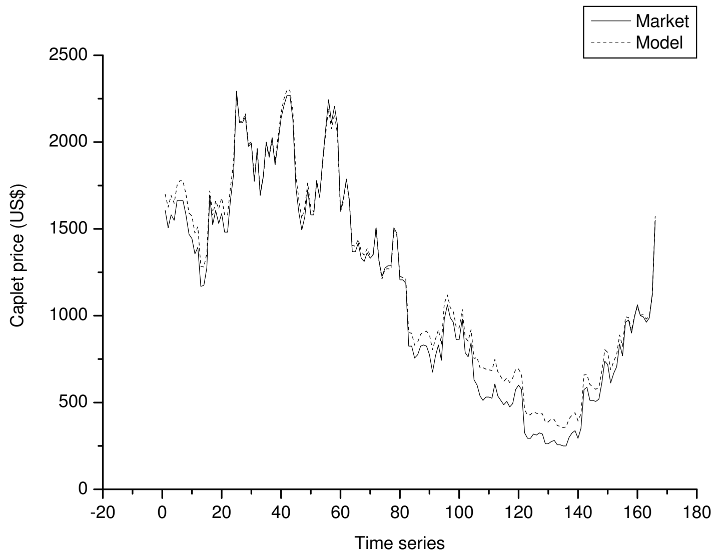

Figure 3.6: Caplet price which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004), from Market and Model Computed based on Historical Volatility and Correlation which is fitted from historical Libor rates, normalized root mean square error=17.39%

To be able to compare the volatilities of different Gaussian models, we can re-scale the field $A(t,\theta)$ so that $D(\theta,\theta) = \frac{1}{\epsilon}$. The re-scaled frame yields the usual definition of volatility of the forward rates, given as follow

$$\langle \big(\delta f(t,\theta)\big)^2 \rangle_c = \sigma_H^2(t,\theta) \tag{3.12}$$

where note the ϵ has been absorbed in the correlator. Thus, when we use the correlator as an input in real calculation, D without normalization has to be used; hence for the effective volatility q we have

$$\begin{aligned} q^2 &= q^2(t_0, t_*, T_n) \\ &= \frac{1}{\epsilon} \int_{t_0}^{t_*} dt \int_{T_n}^{T_n+\ell} dx dx' \sigma_H(t,x) \tilde{D}(x,x';t) \sigma_H(t,x') \end{aligned} \tag{3.13}$$

Using as input the initial forward rates curve and volatility as well as correlation and from the pricing formula from eq. 2.38, we can obtain the empirical field theory caplet price. We see from Fig. 3.6 that the computed caplet price matches the market value very well, the normalized root means square error is 17.39%.

§ 3.3.3 Market Correlator for Field Theory Caplet Price

Note the parametric fit for σ and propagator finally yields the market correlator given by

$$M(t,x,x') = \sigma(t,x)D(x,x';t)\sigma(t,x')$$

Although these parameters give us insights on the field theory model itself we can also obtain M directly from data without fitting any of the parameters. Note we have

$$M(t,x,x^{'}) = \frac{1}{\epsilon}\langle\delta f(t,x)\delta f(t,x^{'})\rangle_c = M(x-t,x^{'}-t) \tag{3.14}$$

and this in turn is sufficient to determine the effective volatility q.

Libor data can be interpolated since it only depend on $\theta = x - t$. Furthermore caplets are instrument that have only a short duration, being based on the 3 month Libor. We can re-express the formula for q^2 in the following manner

$$q^2 = \int_{t_0}^{t_*} dt \int_{T_n-t}^{T_n+\ell-t} d\theta d\theta^{'} M(\theta,\theta^{'}) \tag{3.15}$$

The integration on time requires us to have future M; since M is a function of future time θ and θ', we can shift the average block of $M(\theta,\theta')$ back to it's historical values. For calculating q^2 we need to do one integration on t, which is reduced to a summation of the average value on different block of history Libor data; the difference among the parallelogram blocks is only a horizontal shift and all of them end at time t_0.

Furthermore, since the Libor data we have are expressed in eqn 3.8 as integration of forward interest rates , we can save two integrations and directly use the Libor data without approximating by the forward interest rates as in eqn 3.9. We can hence price the caplet by directly obtaining q^2 from the data;, this is more efficient and more accurate. Market data yields q^2 by the following correlator

$$q^2 = \int_{t_0}^{t^*} dt\langle\delta Y(t,T_n)\delta Y(t,T_n)\rangle_c \tag{3.16}$$

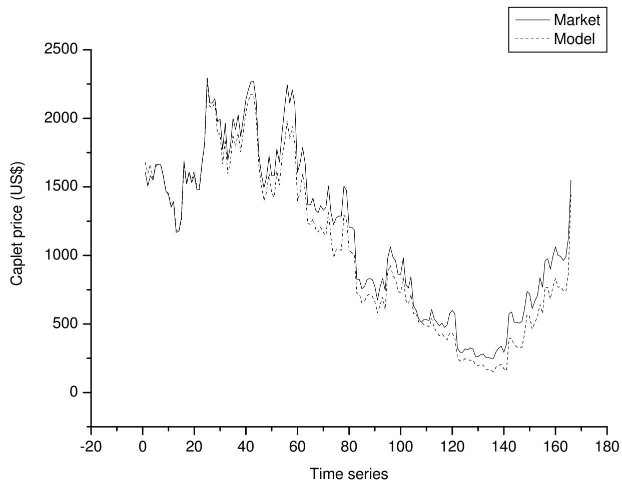

Figure 3.7: Caplet price which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004), both market and model price with effective volatility q computed directly from Libor rates. normalized root mean square error=17.89%

where

$$\begin{aligned} Y(t, T_n) &= \int_{T_n}^{T_n+\ell} dx f(t,x) \\ &= \ln(1 + \ell L(t, T_n)) \end{aligned} \tag{3.17}$$

The computational cap price is given by Fig.3.7. where we can see it again matches the market price very well, the normalized root mean square error is 17.89%.

§ 3.3.4 Market fit for Effective Volatility from Caplet Price

Recall that we have computed q both by fitting the parameters of the field theory model and directly by using the market correlator – both of which use historical Libor data. Another alternative of is that of directly fitting q from the market caplet prices, thus yielding the implied volatility σ_I. In contrast σ_H is obtained from $\langle(\delta f(t,\theta))^2\rangle_c$.

The first approximate fit (both accurate and simple) for the effective volatility is to fit q as a linear function $q = b\theta$ and then implied volatility is a square root function of future time[4]. The linear fit in future time $\theta = x - t$ obviously cannot explain the phenomena of

[4]Fitting effective volatility is much easier than fitting correlation from option price data. Furthermore, the impact of changing correlation is insignificant compared with changing effective volatility since a caplet only

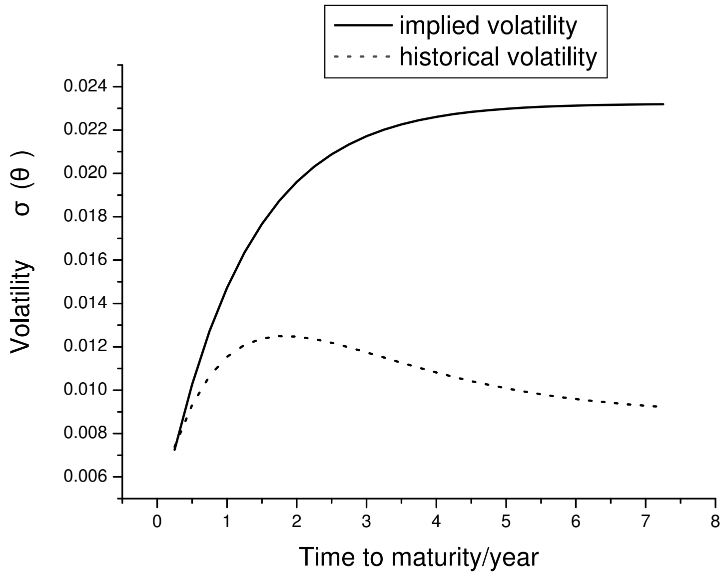

Figure 3.8: Implied volatility σ_I (year$^{-3/2}$) fitted from Caplet data (12.9.2003-4.2.2004) versus time to maturity

implied volatility since it blows up as time going into future. However for the market price of caplet over only a short duration, and square root volatility provides a very good fit. For time far into the future we directly fit the implied volatility with an exponential formula, as in eq. 3.10. The fitting is for the first 100 days in the same data set, say from 12.9.2003 to 4.2.2004. The best fit for σ_I is given in Fig.3.8, and is the following

$$\begin{aligned}\sigma_I(\theta) &= 0.00144 - 0.00122\exp(-0.71826(\theta-\theta_{min})) \\ &\quad +0.00014(\theta-\theta_{min})\exp(-0.71826(\theta-\theta_{min}))\end{aligned} \tag{3.18}$$

We use the effective volatility fitted using the first 100 days to price the whole 168 days cap using the field theory caplter pricing formula 2.38. Results are shown in Fig.3.9 with normalized root mean square error 6.67% and also floorlet price is shown in Fig. 3.10 with normalized root mean square error 7.9%.

One can always do a daily moving fit for q to improve the accuracy of the calculation, and the technique used is the same as here. We only fit q once and use the fitted volatility to price the whole time series that we select. The empirical result shows that the fit is already good enough to show some important features of the field theory caplet pricing formula.

involves the correlation between two neighboring forward interest rates within the range of a single caplet, and hence over a maximum future time difference of 90 days; to a good approximation within a single ℓ=90 days, $D(x,x') \simeq 1$.

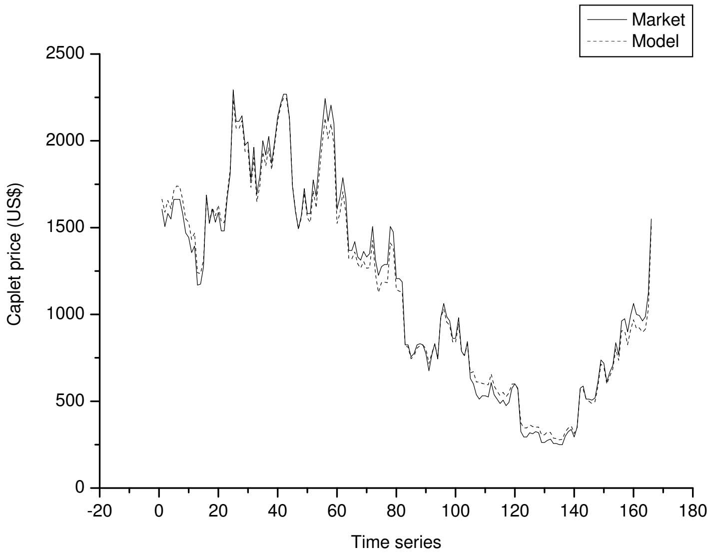

Figure 3.9: Caplet price which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004), both market and model with implied volatility fitted directly from first 100 days caplet price. normalized root mean square error=6.67%

Given below is the root means square error of the above three approaches for fitting the field theory caplet price.

	σ_H from Libor	Market correlator	σ_I from Caplet
normalized root mean square error in caplet price	17.39%	17.89%	6.67%

§ 3.3.5 Comparison of Field Theory caplet price with Black's formula

Recall that the price of a caplet is equivalent to an effective value for Black's implied volatility σ_B, and one obtains an implied volatility everyday from the price; σ_B is shown in 3.11. The shape of Black's implied volatility is very irregular and cannot be fitted well by any formula. No prediction can be made for the future value of Black's volatility and hence we cannot extrapolate it to the future and make a prediction for the price of a caplet for future time.

By comparison with the Black's formula, we see that the field theory model yields a nontrivial result. The effective volatility q and thus the implied volatility σ_I is fitted from caplet prices by the field theory formula as shown in Fig.3.8 and Fig.3.12, and can be used

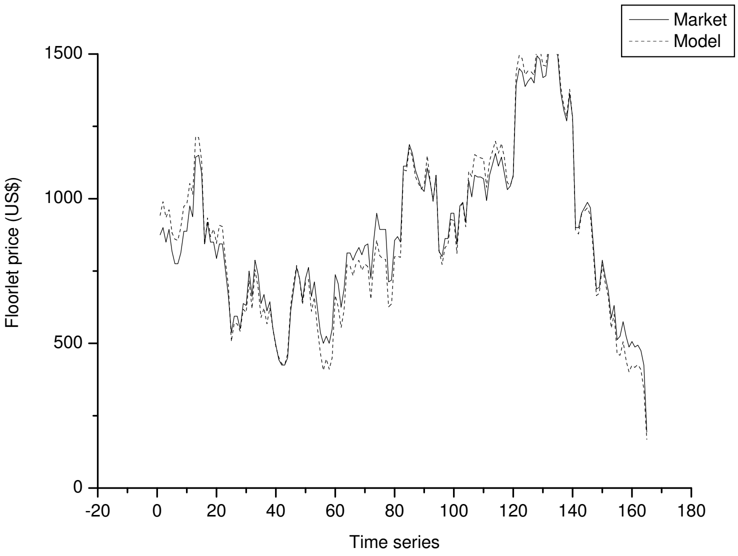

Figure 3.10: Floorlet price which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004), both market and model with implied volatility fitted directly from first 100 days floorlet price. normalized root mean square error=7.9%

for pricing caplet prices in the future since σ_I is only a function of $x - t$.

§ 3.4 Pricing an Interest Rate Cap

We apply the field theory caplet pricing formula to the pricing of interest rate cap and study fixed maturity cap data. We analyze 494 trading days market cap price which mature 1, 2 and 3 years in the future. We try to price the same cap using the field theory formula and compare it to the market price.

Fixed maturity cap is a sum of caplets shows as below

$$cap(t_0, T_N) = \sum_{n=1}^{N-1} caplet(t_0, T_n) \tag{3.19}$$

The caplet price is based on 3 month Libor, and the first caplet matures at 3 month. A one year cap can be expressed as a sum of 3 caplets, a two year cap is a sum of 7 caplets; the domain of the cap can be seen in fig3.13.

We first use the field theory implied volatility σ_I, fitted by the fixed maturity date cap on the same period, to price the two year and three year cap. The computed price are shown

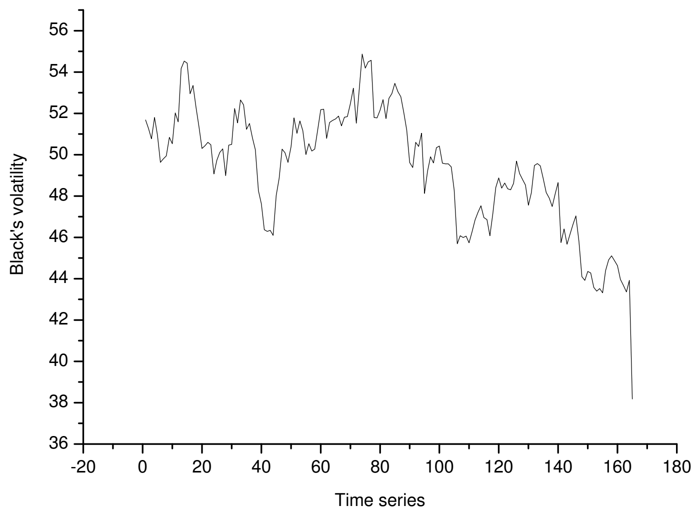

Figure 3.11: Market Black volatility σ_B for caplet which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004).

in Fig. 3.14 and Fig. 3.15. Since now there is a new instrument everyday, one can always improve accuracy by fitting moving lattice effective volatiltiy[5] directly from the 1,2,3 year cap.

We can also price the cap by the historical Libor data using eqn3.16 . Results are shown in Fig.3.16. The normalized root mean square error in cap price are [6]

	cap(t_0,2) from σ_I	cap(t_0,3) from σ_I	cap(t_0,3) from market correlator
normalized root mean square error in Cap price	6.7%	5.54%	5.59%

§ 3.5 Conclusion

Libor based Caps and Floors are important financial instruments for managing interest rate risk. However, the multiple payoffs underlying these contracts complicates their pricing as

[5] only Libor time q need to be fitted.

[6] the value of normalized root means square here are smaller than those for caplet in Fig. 3.6, 3.7, 3.9 since RMS is normalized by the price and the cap price here are much bigger than those for caplets. But the original point of the graph here are not zero and the differences between fit and observation are stretched and seems inconsistent with the normalized root means square.

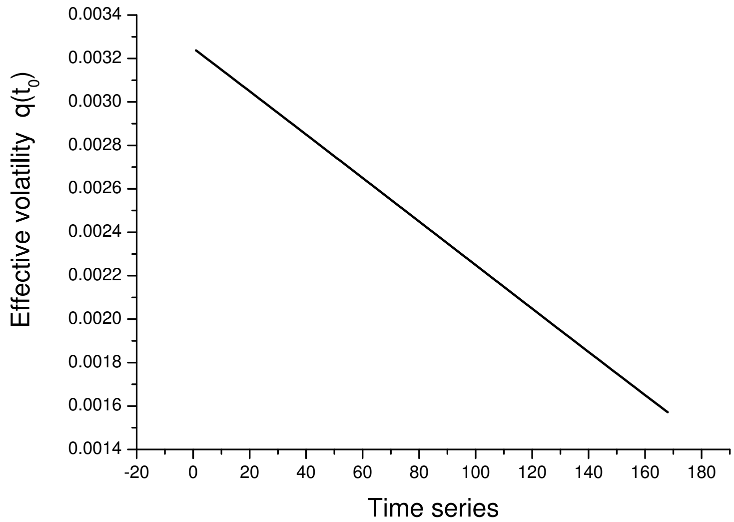

Figure 3.12: Effective volatility q for caplet which mature at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004).

the Libor term structure dynamics are not perfectly correlated. A field theory model which allows for imperfect correlation between every Libor maturity overcomes this difficulty while maintaining model parsimony.

We did an empirical study of the field theory pricing formula of interest rates caplets, and used three alternative approach – with all of the three approaches show satisfactory results. Unlike Black's model, the effective volatility q for the field theory caplet pricing formula can be derived from the underlying historical Libor rate, and hence the field theory caplet pricing formula yields a **prediction** for the caplet price: given the input Libor data, the field theory model generates the daily caplet price. More importantly, it can be used to price other Libor based options; in contrast Black's formula is just a (non-linear) representation of the market price, with a one to one relation between market price and implied Black volatility σ_B.

Historical caplet prices also served to obtain a best fit for the effective volatility q needed for the field theory caplet pricing; the best q obtained from hisotorical data can be used to generate long future caplet prices.

We also empirically studied different interest rate Cap data, and demonstrated the accuracy of the field theory model for pricing interest rate Caps.

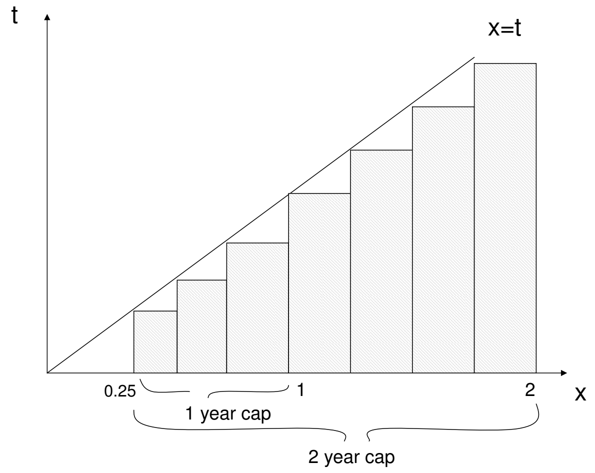

Figure 3.13: Domain for 1 year and 2 year cap. For 1 year Cap, 3 caplet involved. For 2 year Cap, 7 caplet evolved

§ 3.6 Appendix: Example of Black's formula

We illustrate Black's formula for pricing cap by working out a real life example.

Consider a contract that caps the interest rate on a \$1 million loan for three month with Libor rate. The contract is written on t=2003.9.13 which mature on t_*=2004.12.12 with a cap rate R given by 2 percent. The Libor $L(t, t_*)$ at 2003.9.13 for 3 month Eurodollar deposit from 2004.12.12 to 2005.3.12 is given by 2.95 percent per annum. The bond price $B(t, t_*)$ is 0.984. Referred to section § 3.2 eqn 3.4 and eqn 3.5, we have

$$f(t, t_*) = L(t, t_*) = 0.0295$$

then

$$d_+^B = \frac{1}{0.5168 * \sqrt{1.25}}[\ln \frac{0.0295}{0.02} + \frac{0.5168^2 * 1.25}{2}] = 0.527$$
$$d_-^B = d_+^B - 0.5168 * \sqrt{1.25} = -0.0508$$

Thus

$$\begin{aligned} Caplet(t, t_*, 0.02) &= \$\frac{1000000 * 0.25}{1 + 0.25 * 0.0295} * 0.984[0.0295 * N(0.527) - 0.02 * N(-0.0508)] \\ &= \$1587.655 \end{aligned}$$

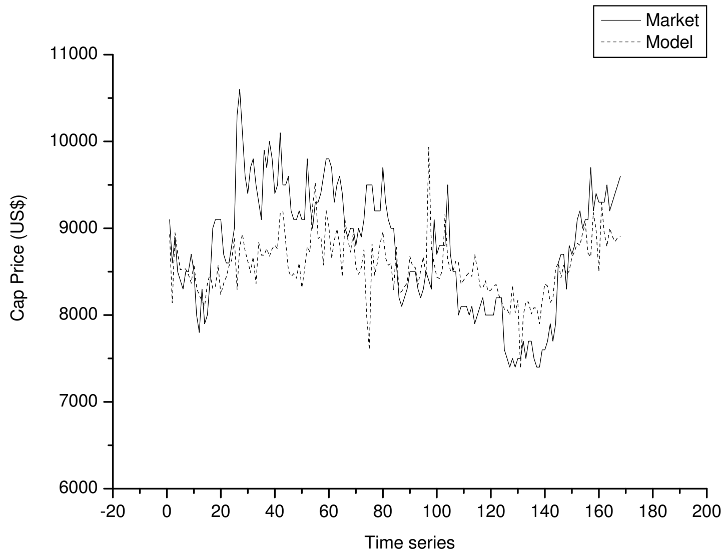

Figure 3.14: Cap price which mature 2 year in the future versus time t_0 (12.9.2003-7.5.2004), both market and model with effective volatility fitted by caplet price. normalized root mean square error=6.7%

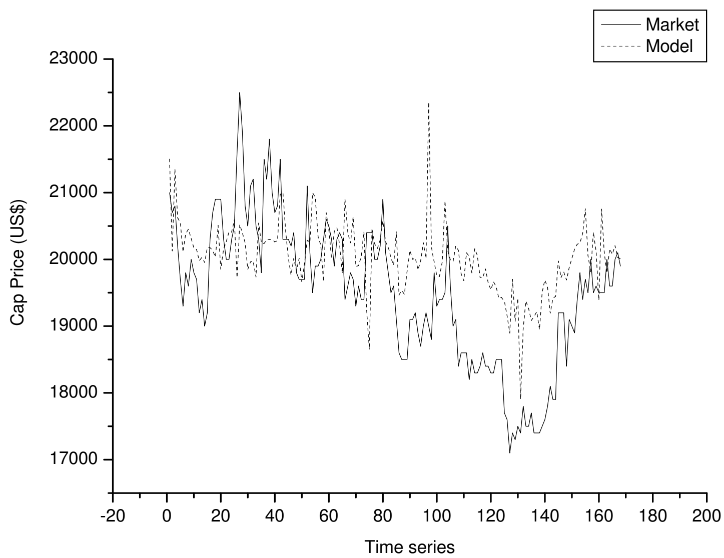

Figure 3.15: Cap price which mature 3 year in the future versus time t_0 (12.9.2003-7.5.2004), both market and model with effective volatility fitted by caplet price. normalized root mean square error=5.54%

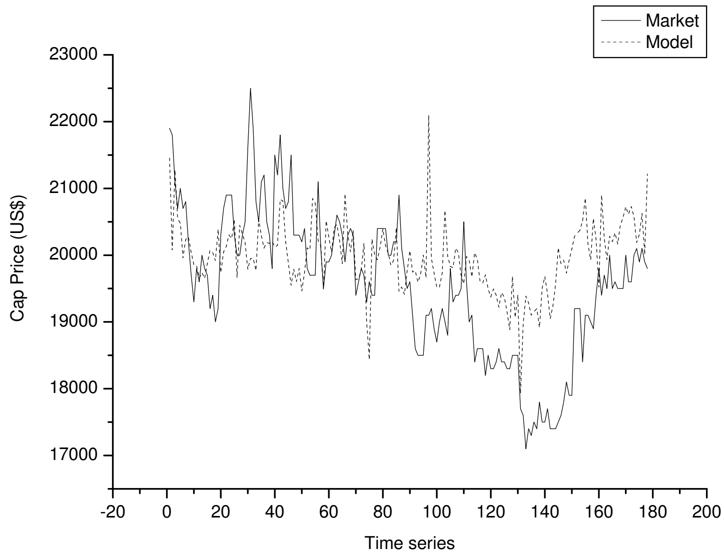

Figure 3.16: Cap price which mature 3 year in the future versus time t_0 (12.9.2003-21.5.2004), both market and model with effective volatility computed directly from historical Libor rates. normalized root mean square error=5.59%

Chapter 4
Hedging Libor Derivatives

All financial instruments are subject to the random behavior of underlying variables such as stock prices, interest rates, exchange rates, etc. There are many ways of defining risk as discussed in Bouchaud and Potters [48]. Hedging is the general concept for the procedure of reducing, if not completely eliminating, an investor's exposure to randomness by constructing a portfolio of correlated instruments. For the interest rate derivatives discussed in this paper, the underlying source of risk is defined by interest rate fluctuations.

The seminal paper of Black Scholes (1973) [30] was the first to recognize that perfectly hedging a derivative enabled one to price the security by no-arbitrage. Specifically, in the absence of market frictions such as short-selling constraints, the ability to hedge a derivative security coincides with one's ability to replicate its payoffs. The seller of an option assumes the risk of a potential liability at its maturity. In particular, the buyer of a Call option is entitled to receive a non-negative payoff from the seller if the stock price is above a certain threshold. Thus, an increase (decrease) in the stock price increases (decreases) the value of a Call option on this underlying stock. However, the terminal value of a Call option can be replicated by buying stock and borrowing from the money market account (temporary cash loan). In particular, there exists a trading strategy involving the stock and money market account which forms a replicating portfolio that mimics the Call option's value across time. Intuitively, by purchasing a portion of the underlying stock, fluctuations in the replicating portfolio are identical to those of the Call option. Therefore, if one sells a Call option, they can hedge this possible liability by replicating the option's payoffs to ensure they have the required funds available to pay the buyer. Hence, selling a Call option while replicating its payoffs creates a riskless portfolio containing the Call option, a certain amount of stock and the money market (cash) account. The critical amount of stock that needs to be purchased and included in the replicating portfolio is referred to as the option's Delta hedge parameter. Similarly, transactions in the bond and futures market can replicate as well as hedge interest rate options. Overall, ascertaining the trading strategy or Delta hedge parameter which replicates a derivative enables one to price this security by no-arbitrage as risk preferences become irrelevant once a riskless portfolio is created by hedging. Moreover, the initial cost involved in forming a replicating portfolio that provides identical payoffs as the derivative must equal the price of the derivative by no-arbitrage (law of one price).

Hedge parameters that minimize the risk associated with a finite number of random fluctuations in the forward interest rates is provided in Baaquie, Srikant, and Warachka [17]. Previously, field theory research has focused on applications involving traditional Heath, Jarrow, and Morton [21] forward interest rates, and on the pricing of LIBOR-based derivatives as is Baaquie [9]. This chapter extends the concept of stochastic Delta hedging developed in Baaquie [6] to the hedging of LIBOR derivatives.

§ 4.1 Hedging

This section details the implications of our field theory model on hedging LIBOR derivatives. The impact of correlation is examined in the context of the residual variance and the Delta hedge parameter for a portfolio. In particular, a stochastic Delta hedging technique is given in Subsection § 4.1.1.

A portfolio $\Pi(t)$ composed of a $Cap(t, t_*, T)$[1] [2] and N LIBOR futures contracts, with the futures chosen to ensure fluctuations in the value of the portfolio are minimized, is studied. This portfolio equals

$$\Pi(t) = Cap(t, t_*, T) + \sum_{i=1}^{N} \eta_i(t)\mathcal{F}(t, T_i)\,, \tag{4.1}$$

where $\eta_i(t)$ represents the hedge parameter for the i^{th} futures contract included in the portfolio. These $\eta_i(t)$ terms ensure movements in the Cap and futures contracts "offset" one another to minimize the fluctuations in $\Pi(t)$. The LIBOR futures and Cap prices are denoted by

$$\mathcal{F}(t, T_i) \;=\; V[1 - \ell L(t, T_i)] \tag{4.2}$$

$$Cap(t, t_*, T) \;=\; \tilde{V}B(t, T)\int_{-\infty}^{+\infty} \frac{dG}{\sqrt{2\pi {q_*}^2}} e^{-\frac{1}{2q_*^2}\left(G - \int_T^{T+\ell} dx f(t,x) - \frac{q_*^2}{2}\right)^2} (X - e^{-G})_+ \tag{4.3}$$

[1]This is a more general expression for a Cap referred to as the midcurve Cap.

[2]We will only consider the simple case that the $Cap(t, t_*, T)$ is composed by only one $Caplet(t, t_*, T)$

where for the midcurve Cap, we have

$$q_*^2 = \frac{1}{t_* - t}\int_t^{t_*} d\tilde{t}\int_{T_n}^{T_n+\ell} dx dx^{'} \sigma(\tilde{t},x)D(x,x^{'};\tilde{t})\sigma(\tilde{t},x^{'}) \,. \tag{4.4}$$

From equation (4.1), we have

$$\Pi(t) = Cap(t,t_*,T) + V\sum_{i=1}^{N}\eta_i(t)(1-\ell L(t,T_i)) \,.$$

For the sake of brevity, we suppress the constant term $V\sum_{i=1}^{N}\eta_i$ in the above equation, which is irrelevant for hedging, and change the negative sign before the LIBOR futures to positive as follows

$$\begin{aligned}\Pi(t) &= Cap(t,t_*,T) + V\sum_{i=1}^{N}\eta_i(t)\ell L(t,T_i)) \\ &= Cap(t,t_*,T) + V\sum_{i=1}^{N}\eta_i(t)\left(e^{\int_{T_i}^{T_i+\ell} f(t,x)dx} - 1\right) \,. \end{aligned} \tag{4.5}$$

In the next two subsections, we discuss two methods for obtaining the $\eta_i(t)$ parameters. We first study stochastic hedging which requires us to specify which forward rate movements are to be hedged against. We then investigate a portfolio's residual variance, a technique which enables us to control the effectiveness of the hedging procedure. Specifically, instead of only hedging a subset of forward rate movements, residual variance applies to all forward rate fluctuations, including those that increase the portfolio's value. The material on stochastic hedging constitutes the main results of this chapter.

§ 4.1.1 Stochastic Hedging

Stochastic hedging of interest rate derivatives has been introduced by Baaquie [6], where the specific case of hedging Treasury Bonds is considered in detail. We focus on applying this technique to the hedging of a LIBOR Cap. Consider the hedging of a Cap against fluctuations in the forward rate $f(t,x)$ which also influence the futures price.

A portfolio $\Pi(t)$ composed of a $Cap(t,t_*,T)$ and one LIBOR futures contract is studied. As in equation (4.5), we set $N = 1$ to obtain

$$\Pi(t) = Cap(t,t_*,T) + V\eta_1(t)\left(e^{\int_{T_1}^{T_1+\ell} f(t,x)dx} - 1\right) \,.$$

where the hedging of this portfolio at instant time t is given by

$$\begin{aligned}\delta\Pi(t) &= \frac{\partial\Pi}{\partial t}\Delta t + \int dx \frac{\delta\Pi}{\delta f(t,x)}\Delta f(t,x) + \frac{1}{2}\int dx \frac{\delta^2\Pi}{\delta f(t,x)^2}(\Delta f(t,x))^2 \\ &+\frac{1}{2}\int dx \int dx' \frac{\delta^2\Pi}{\delta f(t,x)\delta f(t,x')}\Delta f(t,x)\Delta f(t,x') + O(\epsilon^2) \qquad (4.6)\end{aligned}$$

where $\Delta f(t,x) = f(t+\epsilon,x) - f(t,x)$. And $\Delta t \equiv \epsilon = 1/360$ year, while higher orders of ϵ are negligible. Furthermore, the dynamics $\dot{\Pi} = \delta\Pi/\delta t$ equal

$$\begin{aligned}\frac{d\Pi}{dt} &= \dot{\Pi}(t) = \frac{\partial\Pi}{\partial t} + \int dx \frac{\delta\Pi}{\delta f(t,x)}\dot{f}(t,x) + \frac{1}{2}\int dx \frac{\delta^2\Pi}{\delta f(t,x)^2}\dot{f}^2(t,x) \\ &+\frac{1}{2}\int dx \int dx' \frac{\delta^2\Pi}{\delta f(t,x)\delta f(t,x')}\dot{f}(t,x)\dot{f}(t,x') + O(\epsilon)\,. \qquad (4.7)\end{aligned}$$

Since $\langle \dot{f}\dot{f}\rangle \sim \frac{1}{\epsilon}$ as in Baaquie [6], $\epsilon\dot{f}(t,x)^2 \sim 0(1)$, the second order terms are as important as the first order terms. Normal calculus retains the first order terms since ϵ is infinitesimally small.

We study the delta hedging first. The portfolio is required to be invariant to *small* changes in the forward rate. Thus, Delta hedging this portfolio requires

$$\frac{\delta}{\delta f(t,x)}\Pi(t) = 0\,. \qquad (4.8)$$

This Delta hedge involves a first-order approximation for the change in a portfolio's value as a result of forward rate fluctuations.

In field theory, for each time t, there are infinitely many random variables driving the forward rate term-structure indexed by x. Therefore, for any N, one can never perfectly Delta hedge by satisfying equation (4.8). The best alternative is to Delta hedge on average, and this scheme is referred to as stochastic Delta hedging as detailed in Baaquie [6]. To implement stochastic Delta hedging, one considers the conditional expectation value of the portfolio $\Pi(t)$, conditioned on the occurrence of some specific value of the forward rate $f_h \equiv f(t,x_h)$, namely $E[\Pi(t)|f(t,x_h)]$. Define the conditional probability of a Cap and a LIBOR futures by

$$\begin{aligned}\tilde{Cap}(t,t_*,T;f_h) &= E[Cap(t,t_*,T)|f_h] \qquad (4.9)\\ \tilde{L}(t,T_1;f_h) &= E[L(t,T_1)|f_h]\,.\end{aligned}$$

From Baaquie [6], we have the conditional probability of a Cap given by

$$\begin{aligned}\tilde{Cap}(t,t_*,T;f_h) &= \tilde{V}\int_{-\infty}^{\infty} dG\left\{(x-e^G)_+\Psi(G|f_h)\right\} \qquad (4.10)\\ \Psi(G|f_h) &= \frac{\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-\frac{q_h^2}{2}p^2}e^{ip(G-\frac{q_h^2}{2})}\int Df e^{-\int_t^T f(t,x)}e^{ip\int_T^{T+l}dxf(t,x)}\delta(f(t,x_h)-f)e^S}{\int Df\delta(f(t,x_h)-f)e^S}\,,\end{aligned}$$

while the conditional probability of a LIBOR futures is

$$\begin{aligned} \tilde{L}(t,T_1;f_h) &= \int_{-\infty}^{\infty} dG e^G \Phi(G|f;t,T_1) \\ \Phi(G|f;t,T_1) &= \frac{\int Df\delta(G-\int_{T_1}^{T_1+\ell} f(t,x)dx)\delta(f(t,x_h)-f)e^S}{\int Df\delta(f(t,x_h)-f)e^S}\,. \end{aligned} \tag{4.11}$$

Stochastic Delta hedging is defined by approximating equation (4.8) as

$$\frac{\partial}{\partial f_h} E[\Pi(t)|f_h] = 0\,, \tag{4.12}$$

for a given forward rate f_h among an infinite number of possible forward rates. Hence, from equation (4.12), stochastic Delta hedging yields

$$\eta_1 = -\frac{\partial \tilde{Cap}(t,t_*,T;f_h)}{\partial f_h} / \frac{\partial \tilde{L}(t,T_1;f_h)}{\partial f_h}\,. \tag{4.13}$$

Thus, changes in the hedged portfolio $\Pi(t)$ are, on average, sensitive to fluctuations in the forward rate $f(t,x_h)$.

The conditional probability in equation (4.10) and equation (4.11) along with the hedge parameter η_1 is evaluated explicitly for the field theory description of forward rates in the appendix which also contains the relevant notation. One should notice that nontrivial correlations appear in all the terms. The final result, from equation (4.42), is given by

$$\eta_1 = \frac{C\cdot \tilde{Cap}(t,t_*,T;f_h) - B\cdot\chi\cdot\tilde{V}\cdot\left[XN'(d_+)/Q + e^{-G_0+\frac{Q^2}{2}}N(d_-) - e^{-G_0+\frac{Q^2}{2}}N'(d_-)/Q\right]}{e^{G_1+\frac{Q_1^2}{2}}\cdot B_1} \tag{4.14}$$

As a comparison, the HJM limit is also analyzed in the appendix.

Furthermore, one can Gamma hedge the same forward rate. The second-order Gamma hedge recognizes that *large* movements in the forward rate may cause the first-order Delta approximation to be inaccurate. In particular, if hedging is not performed frequently, the Delta hedge parameter can become outdated. However, Gamma evaluates changes in the Delta hedge parameter as the forward rate term structure evolves over time.

To hedge against the $\partial^2\Pi(t)/\partial f^2$ fluctuations, one needs to form a portfolio with two LIBOR futures contracts that minimizes the change in the value of $E[\Pi(t)|f_h]$ by both Delta and Gamma hedging. The hedge parameters are solved analytically, with empirical results presented in Section 4.

Suppose a Cap needs to be hedged against the fluctuations of two forward rates, namely f_h for $h = 1, 2$. The conditional probabilities for the Cap and LIBOR futures, with two forward rates fixed at f_h, are

$$\begin{aligned} \tilde{Cap}(t, t_*, T; f_1, f_2) &= E[Cap(t, t_*, T)|f_1, f_2] \\ \tilde{L}(t, T_1; f_1, f_2) &= E[L(t, T_1)|f_1, f_2]\,. \end{aligned}$$

A portfolio of two LIBOR futures contracts with different maturities $T_i \neq T$ is defined as

$$\Pi(t) = Cap(t, t_*, T) + \sum_{i=1}^{N} \eta_i(t) L(t, T_i)\,, \tag{4.15}$$

where the hedging of this portfolio at instant time t is given by

$$\delta\Pi(t, f_1, f_2) = \frac{\partial \Pi}{\partial t}\delta t + \sum_{i=1}^{2} \frac{\partial \Pi}{\partial f_i}\delta f_i + \frac{1}{2}\sum_{i=1}^{2} \frac{\partial^2 \Pi}{\partial f_i^2}\delta^2 f_i + \frac{1}{2}\frac{\partial^2 \Pi}{\partial f_1 \partial f_2}\delta f_1 \delta f_2 + O(\epsilon^2) \tag{4.16}$$

The dynamics $\dot{\Pi} = \delta\Pi/\delta t$ equal

$$\dot{\Pi}(t, f_1, f_2) = \frac{\partial \Pi}{\partial t} + \sum_{i=1}^{2} \frac{\partial \Pi}{\partial f_i}\dot{f}_i + \frac{\epsilon}{2}\sum_{i=1}^{2} \frac{\partial^2 \Pi}{\partial f_i^2}\dot{f}_i^2 + \frac{\epsilon}{2}\frac{\partial^2 \Pi}{\partial f_1 \partial f_2}\dot{f}_1 \dot{f}_2 + O(\epsilon)\,. \tag{4.17}$$

The stochastic Delta hedging conditions are given by

$$\frac{\partial}{\partial f_h} E[\Pi(t)|f_1, f_2] = 0 \ \text{ for } \ h = 1, 2$$

while stochastic Gamma hedging involves

$$\frac{\partial^2}{\partial f_h^2} E[\Pi(t)|f_1, f_2] = 0 \ \text{ for } \ h = 1, 2$$

with **Cross Gamma** hedging

$$\frac{\partial^2}{\partial f_1 \partial f_2} E[\Pi(t)|f_1, f_2] = 0$$

being unique to this paper. This Cross Gamma hedging only make sense in field theory models where movements in any specific forward rate can be hedged.

One can solve the above system of N simultaneous equations to determine the N hedge parameters. The volatility of the hedged portfolio is reduced by increasing the number of forward interest rates being hedged.

For this portfolio, we can analytically prove that Delta hedge parameters for the two forward rates differ by a prefactor $\frac{A_2}{A_{12}}$

$$\frac{\partial}{\partial f_1} E[\Pi(t)|f_1, f_2] = -\frac{A_2}{A_{12}} \frac{\partial}{\partial f_2} E[\Pi(t)|f_1, f_2] \tag{4.18}$$

where A_2 and A_{12} are defined in Appendix D. Therefore, Delta hedging against two forward rates only determines the portfolio's hedge parameters for one LIBOR futures. Gamma hedging two forward rates is also the same except for a prefactor $\frac{A_2}{A_{12}}$.

Overall, for hedging against two forward rates we are left with three independent constraints from the above six constraints. In order to study the effect of each set of constraints separately, we form portfolios which include two LIBOR futures, and adopt hedging strategies that involve more than Delta hedging to solve for the two hedge parameters. The first strategy implements one Delta and one Gamma hedge against a single forward rate. Two hedge parameters can also be solved in the context of a one Delta hedge and an additional Cross Gamma hedge.

All of these hedge strategies are evaluated explicitly in the appendix. Intuitively, we expect the portfolio to be hedged more effectively with the inclusion of the Cross Gamma parameter. Generally speaking, the field theory framework allows us to form portfolios that hedge against any number of forward rates by including more LIBOR futures contracts.

Until now, we obtained the parameter for each choice of the LIBOR futures and forward rates being hedged. Furthermore, we can minimize the following

$$\sum_{i=1}^{N} |\eta_i| \tag{4.19}$$

to find the *minimum* portfolio. This additional constraint finds the most effective futures contracts, where effectiveness is measured by requiring the smallest number of contracts, hence transactions.

In general, stochastic Delta hedging against N forward rates for large N is complicated, and closed-form solutions are difficult to obtain.

§ 4.1.2 Residual Variance

Hedging a Cap using LIBOR futures contracts can also be accomplished by minimizing the residual variance of the hedged portfolio. It is the instantaneous *change* in the portfolio value that is stochastic. Therefore, the volatility of this change is computed to ascertain the efficacy of the hedge portfolio.

In terms of notation, $C(t,T)$ equal $C(t,t_*,T)$ in the previous subsection where $t_* = T$ and $T = t_* + \delta$ for δ being 3 months. The variance of the portfolio fluctuations, $Var\left[\frac{d\Pi(t)}{dt}\right]$, equals

$$Var\left[\frac{dCap(t,T)}{dt}\right] + Var\left[\sum_{i=1}^{N}\Delta_i \frac{dL(t,T_i)}{dt}\right]$$
$$+\sum_{i=1}^{N}\Delta_i Var\left[\left\langle\frac{dCap(t,T)}{dt},\frac{dL(t,T_i)}{dt}\right\rangle - \left\langle\frac{dCap(t,T)}{dt}\right\rangle\left\langle\frac{dL(t,T_i)}{dt}\right\rangle\right]. \quad (4.20)$$

The detailed calculation for determining the hedge parameters and portfolio variance is carried out in the appendix. As in Baaquie, Srikant, and Warachka [17], the following notation is introduced for simplicity

$$\begin{aligned} K_i &= \chi\hat{L}(t,T_i)\int_T^{T+\ell}dx\int_{T_i}^{T_i+\ell}dx'\sigma(t,x)\sigma(t,x')D(x,x';t)\,, \\ M_{ij} &= \hat{L}(t,T_i)\hat{L}(t,T_j)\int_{T_i}^{T_i+\ell}dx\int_{T_j}^{T_j+\ell}dx'\sigma(t,x)\sigma(t,x')D(x,x';t)\,. \end{aligned} \quad (4.21)$$

Equation (4.21) allows the residual variance in equation (4.37) to be succinctly expressed as

$$\chi^2\int_T^{T+\ell}dx\int_T^{T+\ell}dx'\sigma(t,x)\sigma(t,x')D(x,x';t) + 2\sum_{i=1}^{N}\Delta_i K_i + \sum_{i=1}^{N}\sum_{j=1}^{N}\Delta_i\Delta_j M_{ij} \quad (4.22)$$

which contains covariance terms. When at-the-money, the value of χ below facilitates our empirical estimation of the model in Section 4

$$\begin{aligned} \chi &= -VB(t,T)\int_{-\infty}^{+\infty}\frac{dG}{\sqrt{2\pi q^2}}\frac{1}{q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right) \\ &\quad\times\left\{e^{-\frac{1}{2q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right)^2}(X-e^{-G})_+\right\} \\ &= VB(t,T)\left\{\frac{1}{\sqrt{2\pi q^2}}e^{-1/2d_+^2}+\left(\frac{1+\ell K}{1+\ell L}\right)\left[-\frac{1}{\sqrt{2\pi q^2}}e^{-1/2d_-^2}+N(d_-)\right]\right\} \end{aligned} \quad (4.23)$$

where $d_\pm = \left(\ln\frac{X}{F}\pm q^2/2\right)/q$. The value of χ for an at-the-money options yields $d_\pm = \pm q/2$ which implies

$$\chi(t,T)|_{\text{at-the-money}} = VB(t,T)N(d_-)\,. \quad (4.24)$$

Observe that the residual variance depends on the correlation between forward rates described by the propagator. Ultimately, the effectiveness of the hedge portfolio is an empirical question since perfect hedging is not possible. This empirical question is addressed in Section § 4.2 when the propagator is calibrated to market data.

Hedge parameters Δ_i that minimize the residual variance in equation (4.22) are

$$\Delta_i \;=\; -\sum_{j=1}^{N} K_j M_{ij}^{-1} \,. \tag{4.25}$$

These parameters represent the optimal amounts of the futures contracts to include in the hedge portfolio.

Equation (4.25) is proved by differentiating equation (4.22) with respect to Δ_i and subsequently solving for its value. The variance of the hedged portfolio in equation (4.26) is proved by substituting the result of equation (4.25) into equation (4.22)

$$V_R \;=\; \chi^2 \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t,x)\sigma(t,x')D(x,x';t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K_i M_{ij}^{-1} K_j \tag{4.26}$$

which declines monotonically as N increases.

The residual variance in equation (4.26) enables the effectiveness of the hedge portfolio to be evaluated. Therefore, equation (4.26) is the basis for studying the impact of including different LIBOR futures contracts in the hedge portfolio. For $N = 1$, a single maturity T_i is evaluated, and the residual variance in equation (4.26) reduces to

$$\begin{aligned} &\chi^2 \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t,x)\sigma(t,x')D(x,x';t) \\ -\; &\left(\frac{\left(\int_T^{T+\ell} dx \int_{T_1}^{T_1+\ell} dx' \sigma(t,x)\sigma(t,x')D(x,x';t)\right)^2}{\int_{T_1}^{T_1+\ell} dx \int_{T_1}^{T_1+\ell} dx' \sigma(t,x)\sigma(t,x')D(x,x';t)} \right) . \end{aligned} \tag{4.27}$$

The second term in equation (4.27) represents the reduction in variance attributable to the hedge portfolio. To obtain the HJM limit, the propagator is constrained to equal one, reducing the residual variance V_R in equation (4.27)

$$\chi^2 \left[\left(\int_T^{T+\ell} dx \sigma(t,x) \right)^2 - \left(\frac{(\int_T^{T+\ell} dx \int_{T_1}^{T_1+\ell} dx' \sigma(t,x)\sigma(t,x'))^2}{\int_{T_1}^{T_1+\ell} dx \sigma(t,x)\sigma(t,x')} \right) \right] \tag{4.28}$$

to zero. This HJM limit is consistent with our intuition that the residual variance is identical zero for any LIBOR maturity since all forward rates are perfectly correlated. This result is also shown empirically in Section § 4.2. However, results from hedging with two LIBOR futures contracts in HJM model are not presented since one degree of freedom cannot be hedged with two instruments. Indeed, in this circumstance, M_{ij}^{-1} is singular.

§ 4.2 Empirical Implementation

This section illustrates the implementation of our field theory model and provides preliminary results for the impact of correlation on the hedge parameters. The correlation parameter for the propagator of LIBOR rates is estimated from historical data on LIBOR futures and at-the-money options. We calibrate the term structure of the volatility, $\sigma(\theta)$, (see [49], [47]) and the propagator with the parameters λ and μ as in Baaquie and Bouchaud [16]. All the empirical results showed below are calculated from the derivation expressed in this paper.

§ 4.2.1 Empirical Results on Stochastic Hedging

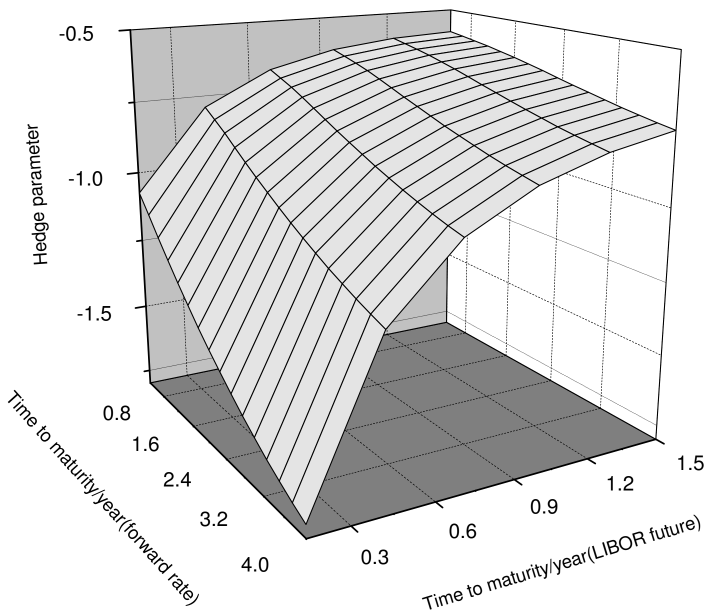

Figure 4.1: *Hedge parameter* η_1 *for stochastic Delta hedging of* $Cap(t, 1, 4)$ *using LIBOR futures maturity* T_1 *and forward rate maturity* x_h *involving* $\Pi(t) = Cap(t, 1, 4) + \eta_1(t)\mathcal{F}(t, T_1)$.

Stochastic hedging mitigates the risk of fluctuations in specified forward rates. The focus of this section is on the stochastic hedge parameters η_i, with the best strategy chosen to

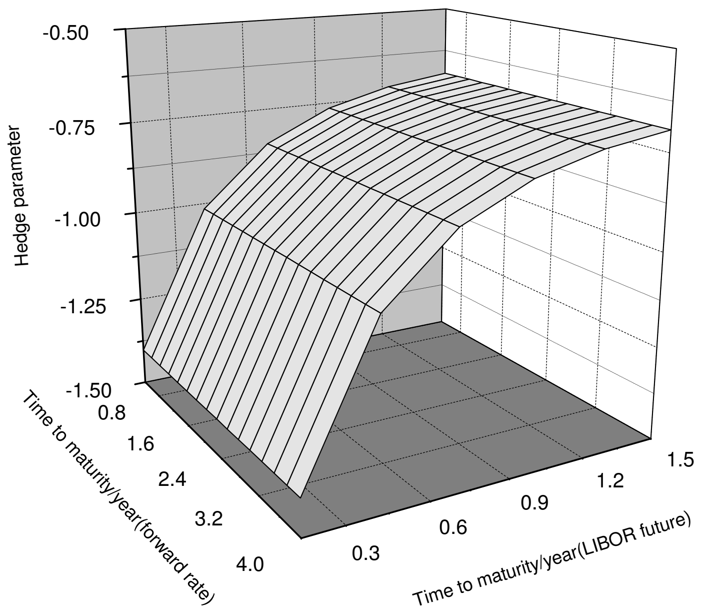

Figure 4.2: *Hedge parameter* η_1 *for stochastic hedging of* $Cap(t, 1, 4)$ *using LIBOR futures maturity* T_1 *and forward rate maturity* x_h *in the HJM limit of* $D = 1$ *(forward rates perfectly correlated) involving* $\Pi(t) = Cap(t, 1, 4) + \eta_1(t)\mathcal{F}(t, T_1)$.

ensure the LIBOR futures portfolio involves the smallest possible long and short positions since $\sum_{i=1}^{N} |\eta_i|$ is minimized.

Hedging in Field Theory Models Compared to HJM

The comparison is carried out in the simplest portfolio where one forward rate is hedged by one LIBOR futures, with a detailed empirical study in Subsection § 4.2.1. As an illustration, Fig. 4.1 plots the hedge parameter η_1 in our field theory model against the LIBOR futures maturity T_1, and the forward rate maturity x_h being hedged. One advantage of the field theory model is that, in principle, a hedge strategy against the movements of infinitely many correlated forward rates is available. To illustrate the contrast between our field theory model and a single-factor HJM model, we plot the identical hedge portfolio as above when $D = 1$, which has been shown to be the HJM limit of field theory models. From Fig. 4.2, for the HJM limit, the hedge parameter η_1 is invariant to the forward rate maturity x_h, which is expected since all forward rates $f(t, x_h)$ are perfectly correlated in a single-factor HJM model. Therefore, it makes no difference which of the forward rates is being hedged.

Hedging Against One Forward Rate with One LIBOR Futures

We first study a portfolio with one LIBOR futures and one Cap to hedge against a single term structure movement. The portfolio is given by

$$\Pi(t) = Cap(t, t_*, T) + \eta_1(t)\mathcal{F}(t, T_1)$$

where the hedging is done by stochastic Delta hedging $\frac{\partial}{\partial f_h} E[\Pi(t)|f_h] = 0$ on forward rate $f(t, x_h)$.

Hedge parameters η_1 for different LIBOR futures maturities T_1, and the forward rate maturity x_h, are shown in Fig. 4.1. This figure describes the selection of the LIBOR futures in the minimum portfolio that requires the fewest number of long and short positions.

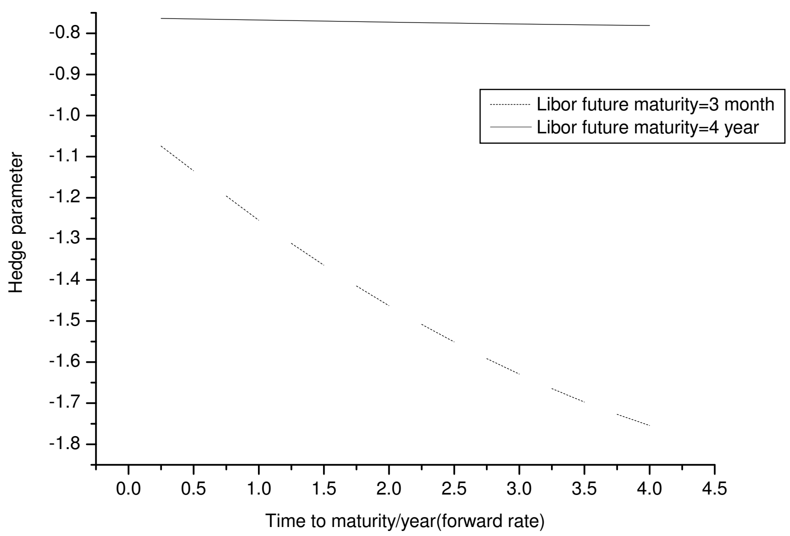

Figure 4.3: *Hedge parameter* η_1 *for stochastic hedging of* $Cap(t, 1, 4)$ *for forward maturity* x_h *of forward rate* $f(t, x_h)$, *with fixed LIBOR futures contract maturity* T_1, *involving* $\Pi(t) = Cap(t, 1, 4) + \eta_1(t)\mathcal{F}(t, T_1)$.

Fig. 4.3 shows how the hedge parameters depend on x_h for a fixed T. Two limits $T_1 = \delta = \frac{1}{4}$ (3 months) and $T_1 = 16\delta$ are chosen. We find that $x_h = \delta$ is always the most important forward rate to hedge against. Another graph describing the parameter dependence on T_1 is given in Fig. 4.4 with $x_h = \delta$. The minimum of hedge parameter η_1 at $x_h \simeq 1.5$years reflects the maximum of $\sigma(t, x)$ around the same future time. For greater generality, we also hedge $Cap(t, t_*, T)$ for different t_* and T values, and find that although the value of the parameter changes slightly, the shape of the parameter surface is almost identical.

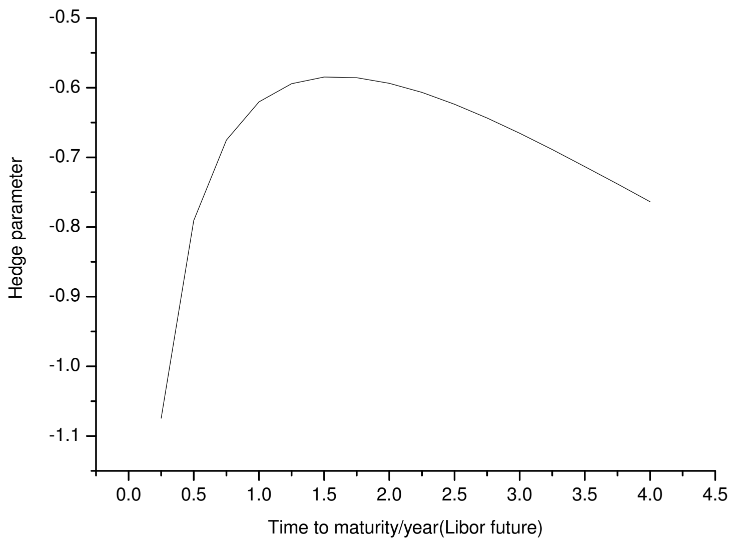

Figure 4.4: *Hedge parameter η_1 for stochastic hedging of $Cap(t,1,4)$ for LIBOR futures maturity T_1 when hedging against $f(t,t+\delta)$ with $\delta = 3/12$, involving $\Pi(t) = Cap(t,1,4) + \eta_1(t)\mathcal{F}(t,T_1)$.*

Hedging Against One Forward Rate with Two LIBOR Futures

In Fig. 4.5, we investigate hedging one forward rate with two LIBOR futures by employing both Delta and Gamma hedging. The portfolio is given by

$$\Pi(t) = Cap(t,t_*,T) + \sum_{i=1}^{2} \eta_i(t)\mathcal{F}(t,T_i)$$

where stochastic Delta hedging $\frac{\partial}{\partial f_1}E[\Pi(t)|f_1] = 0$ and stochastic Gamma hedging $\frac{\partial^2}{\partial f_1^2}E[\Pi(t)|f_1] = 0$ are employed.

From the previous case, we can hedge against $f(t,\delta)$ in order to obtain a minimum portfolio involving the least amount of short and long positions. The diagonal reports that two LIBOR futures with the same maturity reduces to Delta hedging with one LIBOR futures. The data from which Fig. 4.5 is plotted illustrates that selling 38 contracts of $L(t,t+6\delta)$ and buying 71 $L(t,t+\delta)$ contracts identifies the minimum portfolio. More explicitly, the variables in the portfolio are given as

T_1	T_2	x_{h1}	η_1	η_2
1.5 year	0.25 year	0.25 year	-38	71

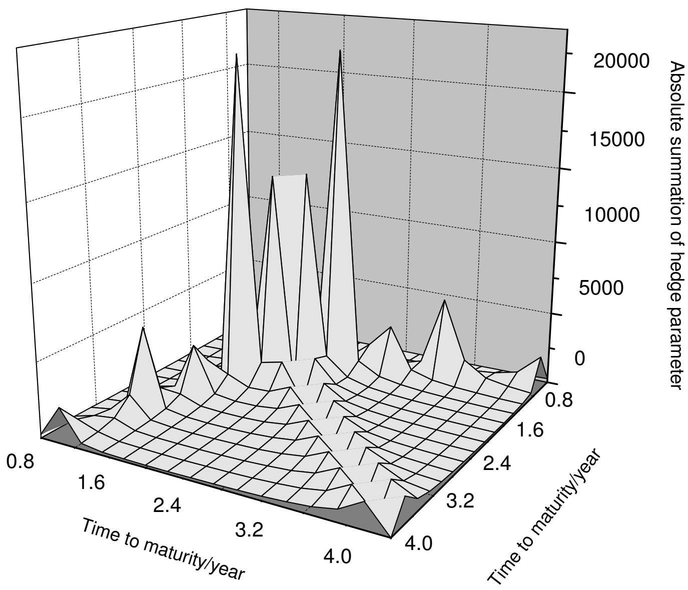

Figure 4.5: *Summation of absolute hedge parameters* $|\eta_1| + |\eta_2|$ *for two LIBOR futures,* T_1 *and* T_2*. The portfolio* $\Pi(t) = Cap(t, 1, 4) + \sum_{i=1}^{2} \eta_i(t)\mathcal{F}(t, T_i)$ *involves a stochastic hedge against one forward rate with both Delta and Gamma hedging.*

Hedging Against Two Forward Rates with Two LIBOR Futures

In addition, we consider hedging fluctuations in two forward rates. Specifically, we study a portfolio comprised of two LIBOR futures and one Caplet $\Pi(t) = Cap(t, t_*, T) + \sum_{i=1}^{2} \eta_i(t)\mathcal{F}(t, T_i)$ where the parameters η_i are fixed by Delta hedging $\frac{\partial}{\partial f_1} E[\Pi(t)|f_1, f_2] = 0$ and and Cross Gamma hedging $\frac{\partial^2}{\partial f_1 \partial f_2} E[\Pi(t)|f_1, f_2] = 0$.

The result is displayed in Fig. 4.6 where we hedge against two short maturity forward rates, such as $f(t, \delta)$ and $f(t, 2\delta)$. Again the data from which Fig. 4.6 is plotted illustrates that buying 45 contracts of $L(t, t+15\delta)$ and selling 25 $L(t, t+3\delta)$ contracts forms the minimum portfolio. More explicitly, the variables in the portfolio are given as

T_1	T_2	x_{h1}	x_{h2}	η_1	η_2
3.75 year	0.75 year	0.25 year	0.5 year	45	-25

Fig. 4.5 and Fig. 4.6 result from summing the absolute values of the hedge parameters (as in equation (4.19)) which depend on the maturities of the LIBOR futures T_i. The corresponding empirical results are consistent with our earlier discussion.[3]

[3]If we choose the hedge portfolio by minimizing $\sum_{i=1}^{N} \eta_i$, we find that the minimum portfolio requires 1500 contracts (long the short maturity and short their long maturity counterparts).

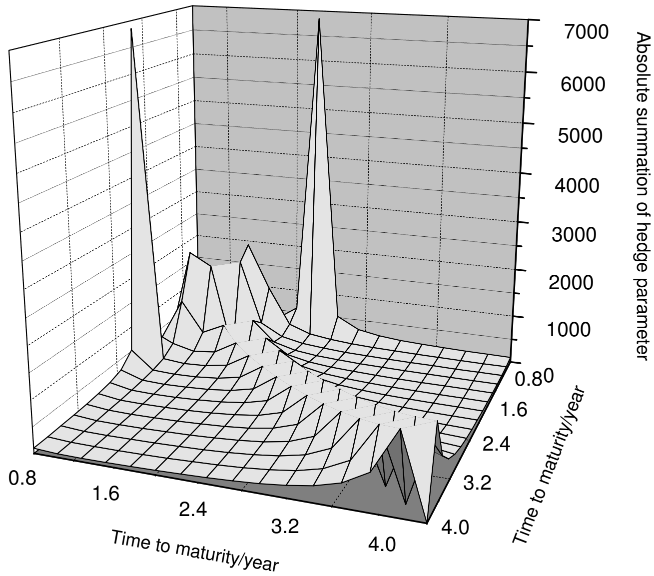

Figure 4.6: *Summation of absolute hedge parameters* $|\eta_1| + |\eta_2|$ *for two LIBOR futures maturities,* T_1 *and* T_2*. The portfolio* $\Pi(t) = Cap(t, 1, 4) + \sum_{i=1}^{2} \eta_i(t)\mathcal{F}(t, T_i)$ *involves stochastic hedging against two forward rates, with both Delta and Cross Gamma hedging.*

§ 4.2.2 Empirical Results on Residual Variance

The reduction in variance achievable by hedging a Cap with LIBOR futures is the focus of this section. The portfolio

$$\Pi(t) = Cap(t, T) + \sum_{i=1}^{N} \Delta_i(t)\mathcal{F}(t, T_i)$$

is considered with $Var\left[\frac{d\Pi(t)}{dt}\right]$ being minimized. The residual variance for hedging a 1 and 4 year Cap with a LIBOR futures is shown in Fig. 4.7, along with its HJM counterpart. Observe that the residual variance drops to exactly zero when the same maturity LIBOR futures is used to hedge the Cap.

By considering the changes of residual variance with respect of parameters λ and μ, we find the neighboring points create no disparities, at least one cannot tell which offers the better hedge. An explanation of this is effect is that forward rates with similar maturities are strongly correlated. Furthermore, the HJM residual variance for both hedging a 1 year and 4 year Cap are identical to the residual variance=0 axis. This is consistent with our analytical result in equation (4.28).

The residual variance for hedging a 4 year Cap with two LIBOR futures is provided in Fig.

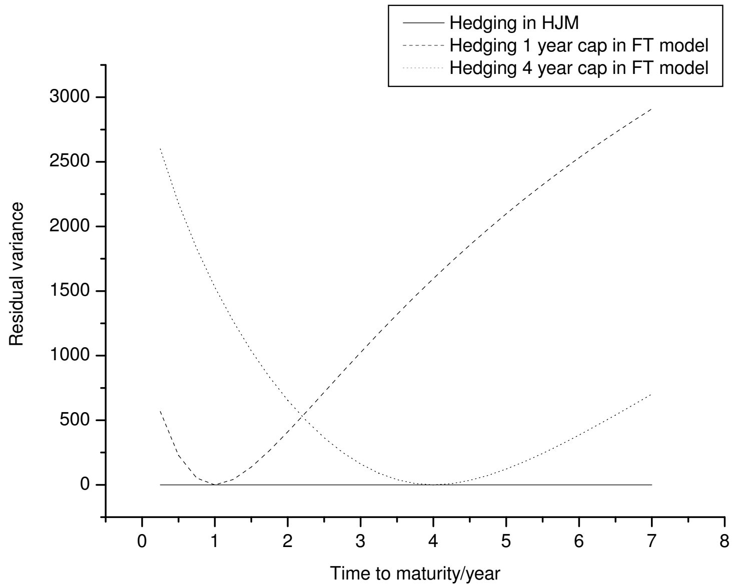

Figure 4.7: *Residual variance* $Var\left[\frac{d\Pi(t)}{dt}\right]$ *for a one and four year Cap versus LIBOR futures maturity* T_1 *used to hedge portfolio* $\Pi(t) = Cap(t,T) + \Delta_1(t)\mathcal{F}(t,T_1)$.

4.8. It is interesting to note that hedging with two instruments, even with similar maturities, entails a significant decrease in residual variance compared to hedging with one futures. This is illustrated in Fig. 4.8 where $\theta = \theta'$ represents hedging with one LIBOR futures. The residual variance in this situation is higher than the nearby points, and increases in a discontinuous manner.

§ 4.3 Appendix1: Residual Variance

First, consider the variance of a Cap in the field theory model. Define the Delta of the Cap, $\frac{\partial Cap(t,T)}{\partial \int_T^{T+\ell} f(t,x)dx}$, as χ

$$\begin{aligned}\chi \;\equiv\; & -VB(t,T)\int_{-\infty}^{+\infty}\frac{dG}{\sqrt{2\pi q^2}}\frac{1}{q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right)\\ & \times\left\{e^{-\frac{1}{2q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right)^2}(X-e^{-G})_+\right\}. \end{aligned} \quad (4.29)$$

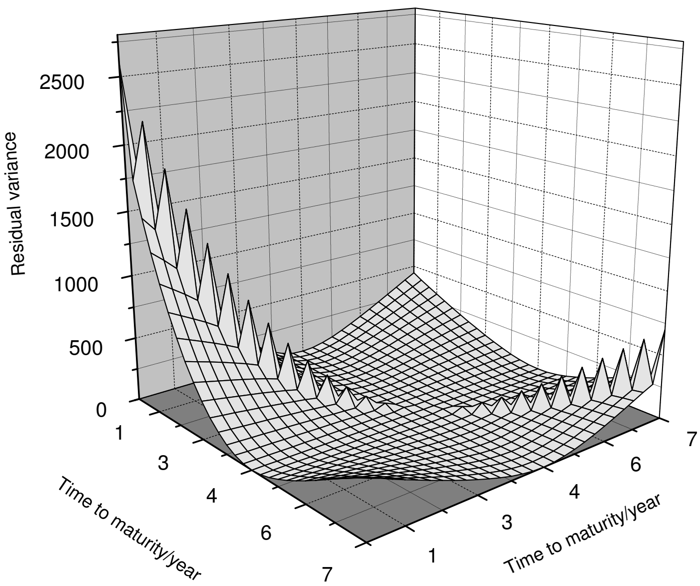

Figure 4.8: *Residual variance* $Var\left[\frac{d\Pi(t)}{dt}\right]$ *for a four year Cap versus two LIBOR futures maturities* T_i *used to hedge* $\Pi(t) = Cap(t,4) + \sum_{i=1}^{2}\Delta_i(t)\mathcal{F}(t,T_i)$.

The result in equation (4.3) for the Cap price implies that

$$\begin{aligned}\frac{dCap(t,T)}{dt} &= -VB(t,T)\int_{-\infty}^{+\infty}\frac{dG}{\sqrt{2\pi q^2}}\frac{1}{q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right)\int_T^{T+\ell}\frac{\partial f(t,x)}{\partial t}dx \\ &\quad\times\left\{e^{-\frac{1}{2q^2}\left(G-\int_T^{T+\ell}dxf(t,x)-\frac{q^2}{2}\right)^2}\left(X-e^{-G}\right)_+\right\} &(4.30)\\ &= \chi\left(\int_T^{T+\ell}\frac{\partial f(t,x)}{\partial t}dx\right) &(4.31)\\ &= \chi\left(\int_T^{T+\ell}dx\alpha(t,x)+\int_T^{T+\ell}dx\sigma(t,x)A(t,x)\right) &(4.32)\end{aligned}$$

with $E\left[\frac{dCap(t,T)}{dt}\right] = \left(\int_T^{T+\ell}dx\alpha(t,x)\right)dt$ since $E[A(t,x)] = 0$. Therefore, the resulting variance equals

$$\frac{dCap(t,T)}{\epsilon} - E\left[\frac{dCap(t,T)}{\epsilon}\right] = \chi\int_T^{T+\ell}dx\sigma(t,x)A(t,x)\,. \tag{4.33}$$

With $\delta(\cdot) = \frac{1}{\epsilon}$ representing a delta function, squaring this expression and invoking the property that $E[A(t,x)A(t,x')] = \delta(0)D(x,x';t) = \frac{D(x,x';t)}{dt}$ results in the instantaneous Cap price variance being

$$Var\left[\frac{dCap(t,T)}{\epsilon}\right] = \frac{1}{\epsilon}\chi^2\int_T^{T+\ell}dx\int_T^{T+\ell}dx'\sigma(t,x)D(x,x';t)\sigma(t,x')\,. \tag{4.34}$$

The quantity ϵ signifies a small step forward in time. The underlying intuition is that we are converting a portfolio of futures contracts to one involving another function of LIBOR rates. Then, the instantaneous variance of a LIBOR portfolio is considered. For a LIBOR portfolio, $\hat{\Pi}(t) = V\ell \sum_{i=1}^{N} \Delta_i L(t,T_i)$, the following result holds,

$$\frac{d\hat{\Pi}(t)}{dt} - E\left[\frac{d\hat{\Pi}(t)}{dt}\right] = \sum_{i=1}^{N} \Delta_i \hat{L}(t,T_i) \int_{T_i}^{T_i+\ell} dx \sigma(t,x) A(t,x) \tag{4.35}$$

where $\hat{L}(t,T_i) = V e^{\int_{T_i}^{T_i+\ell} f(t,x)dx} = \frac{V}{f(t,T_i,T_i+\ell)}$ and

$$Var\left[\frac{d\hat{\Pi}(t)}{dt}\right] = \frac{1}{\epsilon}\sum_{i=1}^{N}\sum_{j=1}^{N} \Delta_i \Delta_j \hat{L}(t,T_i)\hat{L}(t,T_j) \int_{T_i}^{T_i+\ell} dx \int_{T_j}^{T_j+\ell} dx \sigma(t,x) D(x,x';t)\sigma(t,x')\,. \tag{4.36}$$

The (residual) variance of the hedged portfolio

$$\Pi(t) = Cap(t,T) + \sum_{i=1}^{N} \Delta_i \mathcal{F}(t,T_i)$$

is then computed in a straightforward manner. Equation (4.36) implies the hedged portfolio's variance equals

$$\begin{aligned} &\chi^2 \int_T^{T+\ell} dx \int_T^{T+\ell} dx' \sigma(t,x)\sigma(t,x') D(x,x';t) \\ &+2\chi \sum_{i=1}^{N} \Delta_i \hat{L}(t,T_i) \int_T^{T+\ell} dx \int_{T_i}^{T_i+\ell} dx' \sigma(t,x)\sigma(t,x') D(x,x';t) \\ &+\sum_{i=1}^{N}\sum_{j=1}^{N} \Delta_i\Delta_j \hat{L}(t,T_i)\hat{L}(t,T_j) \int_{T_i}^{T_i+\ell} dx \int_{T_j}^{T_j+\ell} dx' \sigma(t,x)\sigma(t,x') D(x,x';t)\,. \end{aligned} \tag{4.37}$$

§ 4.4 Appendix2: Conditional Probability of Hedging One Forward Rate

Using the results of the Gaussian models in Baaquie [?], after a straightforward but tedious calculation, the following is derived from equations (4.10) and (4.11)

$$\Psi(G|f_h) = \frac{\chi}{\sqrt{2\pi Q^2}} \exp\left[-\frac{1}{2Q^2}(G-G_0)^2\right] \tag{4.38}$$

$$\Phi(G|f;t_h,T_{n1}) = \frac{1}{\sqrt{2\pi Q_1^2}} \exp\left[-\frac{1}{2Q_1^2}(G-G_1)^2\right]\,. \tag{4.39}$$

The notations are shown as follow

$$
\begin{aligned}
X &= \frac{1}{1+\ell k} \;\; ; \;\; \tilde{V} = (1+\ell k)V \\
\chi &= \exp\left\{ -\int_{t_h}^{T_n} dx f(t_0,x) - \int_{M_1} \alpha(t,x) + \frac{1}{2}E + \frac{C}{A}(f(t_0,x_h) + \int_{t_0}^{t_h} dt\alpha(t,x_h) - f - \frac{C}{2}) \right\} \\
d_+ &= (\ln x + G_0)/Q \qquad ; \qquad d_- = (\ln x + G_0 - Q^2)/Q \\
G_0 &= \int_{T_n}^{T_n+\ell} dx f(t_0,x) - F - \frac{B}{A}(f(t_0,x_h) - C - f + \int_{t_0}^{t_h} dt\alpha(t,x_h)) + \frac{q^2}{2} \\
Q^2 &= q^2 - \frac{B^2}{A} \\
G_1 &= \int_{T_{n1}}^{T_{n1}+\ell} dx f(t_0,x) + \int_{M_3} \alpha(t,x) - \frac{B_1}{A}(f(t_0,x_h) - \int_{t_0}^{t_h} dt\alpha(t,x_h) - f) \\
Q_1^2 &= D - \frac{B_1^2}{A} \\
A &= \int_{t_0}^{t_h} dt\sigma(t,x_h)^2 D(t,x_h,x_h;T_{FR}) \\
B &= \int_{M_2} \sigma(t,x_h)D(t,x_h,x;T_{FR})\sigma(t,x) \\
B_1 &= \int_{\tilde{M}_1} \sigma(t,x_h)D(t,x_h,x;T_{FR})\sigma(t,x) \\
C &= \int_{M_1} \sigma(t,x_h)D(t,x_h,x;T_{FR})\sigma(t,x) \\
D &= \int_{\tilde{\mathcal{Q}}_1} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x') \\
q^2 &= \int_{\mathcal{Q}_2+\mathcal{Q}_4} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x')
\end{aligned}
$$

$$
\begin{aligned}
E &= \int_{\mathcal{Q}_1} \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x') \\
F &= \int_{t_0}^{t_h} dt \int_{t_h}^{T_n} dx \int_{T_n}^{T_n+\ell} dx' \sigma(t,x)D(t,x,x';T_{FR})\sigma(t,x')\,.
\end{aligned}
$$

The domain of integration is given in Figs. 4.9 and 4.10. It can be seen that the unconditional probability distribution for the Cap and LIBOR futures yields volatilities q^2 and D respectively. Hence the conditional expectation reduces the volatility of Cap by $\frac{B^2}{A}$, and by $\frac{B_1^2}{A}$ for the LIBOR futures. This result is expected since the constraint imposed by the requirement of a conditional probability reduces the allowed fluctuations of the instruments.

It could be the case that there is a special maturity time x_h which causes the largest reduction in conditional variance. The answer is found by minimizing the conditional variance

$$\tilde{Cap}(t_h, t_*, T_n; f_h) = \chi\tilde{V}(xN(d_+) - e^{-G_0+\frac{Q^2}{2}}N(d_-)) \tag{4.40}$$

$$\tilde{L}(t_h, T_{n1}; f_h) = e^{G_1+\frac{Q_1^2}{2}}\,. \tag{4.41}$$

Recall the hedging parameter is given by equation (4.13). Using equation (4.41) and setting $t_0 = t$, $t_h = t + \epsilon$, we get an (instantaneous) stochastic Delta hedge parameter $\eta_1(t)$ equal to

$$\frac{C \cdot \tilde{Cap}(t, t_*, T_n; f_h) - B \cdot \chi \cdot \tilde{V} \cdot \left[xN'(d_+)/Q + e^{-G_0+\frac{Q^2}{2}}N(d_-) - e^{-G_0+\frac{Q^2}{2}}N'(d_-)/Q\right]}{e^{G_1+\frac{Q_1^2}{2}} \cdot B_1} \tag{4.42}$$

§ 4.5 Appendix3: HJM Limit of Hedging Function

The HJM-limit of the hedging functions is analyzed for the specific exponential function considered by Jarrow and Turnbull [82]

$$\sigma_{hjm}(t, x) = \sigma_0 e^{\beta(x-t)}\,, \tag{4.43}$$

which sets the propagator $D(t, x, x'; T_{FR})$ equal to one. It can be shown that

$$\begin{aligned}
A &= \frac{\sigma_0^2}{2\beta}e^{-2\beta x_h}(e^{2\beta t_h} - e^{2\beta t_0}) \\
B &= \frac{\sigma_0^2}{2\beta^2}e^{-\beta x_h}(e^{-\beta T_n} - e^{-\beta T_n+\ell})(e^{2\beta t_h} - e^{2\beta t_0}) \\
B_1 &= \frac{\sigma_0^2}{2\beta^2}e^{-\beta x_h}(e^{-\beta T_{n1}} - e^{-\beta T_{n1}+\ell})(e^{2\beta t_h} - e^{2\beta t_0}) \\
C &= \frac{\sigma_0^2}{2\beta^2}e^{-\beta x_h}(e^{-\beta t_h} - e^{-\beta T_n})(e^{2\beta t_h} - e^{2\beta t_0}) \\
D &= \frac{\sigma_0^2}{2\beta^3}(e^{-\beta T_{n1}+\ell} - e^{-\beta T_{n1}})^2(e^{2\beta t_h} - e^{2\beta t_0}) \\
E &= \frac{\sigma_0^2}{2\beta^3}(e^{-\beta T_n} - e^{-\beta t_h})^2(e^{2\beta t_h} - e^{2\beta t_0}) \\
F &= \frac{\sigma_0^2}{2\beta^3}(e^{-\beta T_n+\ell} - e^{-\beta T_n})(e^{-\beta T_n} - e^{-\beta t_h})(e^{2\beta t_h} - e^{2\beta t_0})\,.
\end{aligned}$$

The exponential volatility function given in equation (4.43) has the remarkable property, similar to the case found for the hedging of Treasury Bonds in Baaquie [**?**], that

$$Q_1^2(hjm) = D_{hjm} - \frac{B_{1hjm}^2}{A_{hjm}} \equiv 0\,. \tag{4.44}$$

Hence, the conditional probability for the LIBOR futures is deterministic. Indeed, once the forward rate f_h is fixed, the following identity is valid

$$\tilde{L}_{hjm}(t_h, T_{n1}; f_h) \equiv L(t_h, T_{n1}) . \tag{4.45}$$

In other words, for the volatility function in equation (4.43), the LIBOR futures for the HJM model is exactly determined by one of the forward rates.

However, the conditional probability for the Cap is not deterministic since the volatility from t_h to t_*, before the Cap's expiration, is not compensated for by fixing the forward rate.

§ 4.6 Appendix4: Conditional Probability of Hedging Two Forward Rates

When hedging against two forward rates, equations (4.10) and (4.11) imply we have the conditional probability of a Cap given by

$$\Psi(G|f_1, f_2) = \frac{\int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{-\frac{q_h^2}{2}p^2} e^{ip(G-\frac{q_h^2}{2})} \int Df e^{-\int_{t_h}^{Tn} f(t_h,x)} e^{ip\int_{Tn}^{Tn+l} dx f(t_h,x)} \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^S}{\int Df \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^S} \tag{4.46}$$

and the conditional probability of LIBOR being

$$\Phi(G|f_1, f_2, T_{nj}) = \frac{\int Df \delta(G - \int_{T_{nj}}^{T_{nj}+\ell} f(t_h, x) dx) \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^S}{\int Df \prod_{i=1}^{2} \delta(f(t_h, x_i) - f_i) e^S} \quad j = 1, 2 \tag{4.47}$$

which yields

$$\Psi(G|f_1, f_2) = \frac{\chi}{\sqrt{2\pi Q^2}} \exp\left[-\frac{1}{2Q^2}(G - G_0)^2\right] \tag{4.48}$$

$$\Phi(G|f_1, f_2, T_{nj}) = \frac{1}{\sqrt{2\pi \tilde{Q}_j^2}} \exp\left[-\frac{1}{2\tilde{Q}_j^2}(G - \tilde{G}_j)^2\right] \quad j = 1, 2 \tag{4.49}$$

under the following notation

$$X = \frac{1}{1 + \ell k} \;;\; \tilde{V} = (1 + \ell k)V$$

$$\chi = \exp\left\{-\int_{t_h}^{T_n} dx f(t_0, x) - \int_{M_1} \alpha(t, x) + \frac{1}{2}E + \frac{C_{12}}{\tilde{A}_{12}}(R_{12} - \frac{C_{12}}{2})\right\}$$

$$
\begin{aligned}
d_+ &= (\ln x + G_0)/Q \qquad ; \qquad d_- = (\ln x + G_0 - Q^2)/Q \\
G_0 &= \int_{T_n}^{T_n+\ell} dx f(t_0,x) - F - \frac{B_{12}}{\tilde{A}_{12}}(R_{12} - C_{12}) + \frac{q^2}{2} \\
Q^2 &= q^2 - \frac{B_{12}^2}{\tilde{A}_{12}} \\
\tilde{G}_j &= \int_{T_{nj}}^{T_{nj}+\ell} dx f(t_0,x) + \int_{\tilde{M}_j} \alpha(t,x) - \frac{\tilde{B}_{12j}}{\tilde{A}_{12}} R_{12} \quad j=1,2 \\
\tilde{Q}_j^2 &= D_j - \frac{\tilde{B}_{12j}^2}{\tilde{A}_{12}} \quad j=1,2 \\
R_i &= f(t_0,x_i) + \int_{t_0}^{t_h} dt\alpha(t,x_i) - f_i \quad i=1,2 \\
R_{12} &= R_1 - \frac{A_{12}}{A_2} R_2 \\
A_i &= \int_{t_0}^{t_h} dt\sigma(t,x_i)^2 D(t,x_i,x_i;T_{FR}) \quad i=1,2 \\
A_{12} &= \int_{t_0}^{t_h} dt\sigma(t,x_1) D(t,x_1,x_2;T_{FR}) \sigma(t,x_2) \\
\tilde{A}_{12} &= A_1 - \frac{A_{12}}{A_2} \\
B_i &= \int_{M_2} \sigma(t,x_i) D(t,x_i,x;T_{FR}) \sigma(t,x) \quad i=1,2 \\
B_{12} &= B_1 - \frac{A_{12}}{A_2} B_2 \\
\tilde{B}_{ij} &= \int_{\tilde{M}_j} \sigma(t,x_i) D(t,x_i,x;T_{FR}) \sigma(t,x) \quad i=1,2; \quad j=1,2 \\
\tilde{B}_{12j} &= \tilde{B}_{1j} - \frac{A_{12}}{A_2} \tilde{B}_{2j} \quad j=1,2\ldots,5 \\
C_i &= \int_{M_1} \sigma(t,x_i) D(t,x_i,x;T_{FR}) \sigma(t,x) \quad i=1,2 \\
C_{12} &= C_1 - \frac{A_{12}}{A_2} C_2 \\
D_j &= \int_{\tilde{\mathcal{Q}}_j} \sigma(t,x) D(t,x,x';T_{FR}) \sigma(t,x') \quad j=1,2 \\
q^2 &= \int_{\mathcal{Q}_2+\mathcal{Q}_4} \sigma(t,x) D(t,x,x';T_{FR}) \sigma(t,x') \\
E &= \int_{\mathcal{Q}_1} \sigma(t,x) D(t,x,x';T_{FR}) \sigma(t,x') \\
F &= \int_{t_0}^{t_h} dt \int_{t_h}^{T_n} dx \int_{T_n}^{T_n+\ell} dx' \sigma(t,x) D(t,x,x';T_{FR}) \sigma(t,x') \, .
\end{aligned} \tag{4.50}
$$

The domain of integration is given in Figs 4.9 and 4.10.

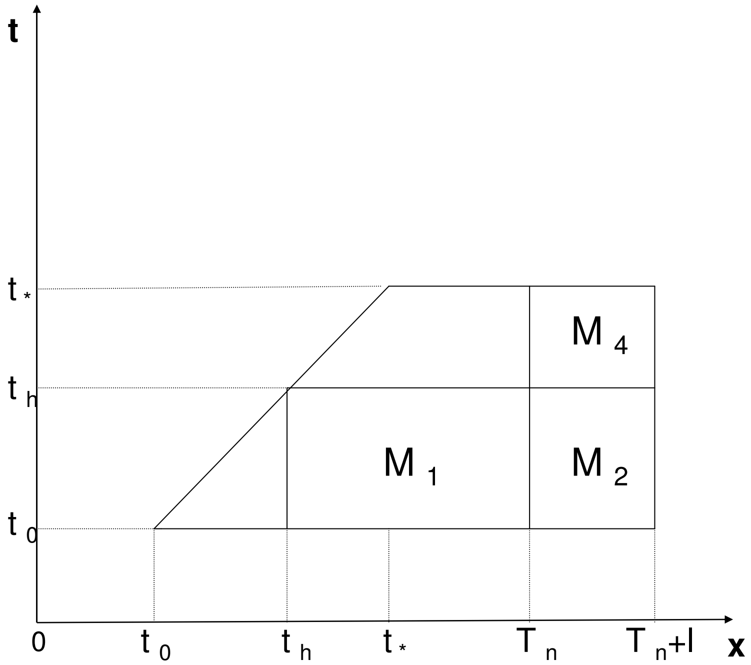

Figure 4.9: *Domain of integration M_1, M_2 and integration cube $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_4$ where the x' axis has the same limit as its corresponding x axis.*

Furthermore, an N-fold constraint on the instruments would further reduce the variance of the instruments

$$\tilde{Cap}(t_h, t_*, T_n; f_1, f_2) = \chi\tilde{V}(xN(d_+) - e^{-G_0 + \frac{Q^2}{2}} N(d_-)) \quad (4.51)$$

$$\tilde{L}(t_h, T_{nj}; f_1, f_2) = e^{\tilde{G}_j + \frac{\tilde{Q}_j^2}{2}} . \quad (4.52)$$

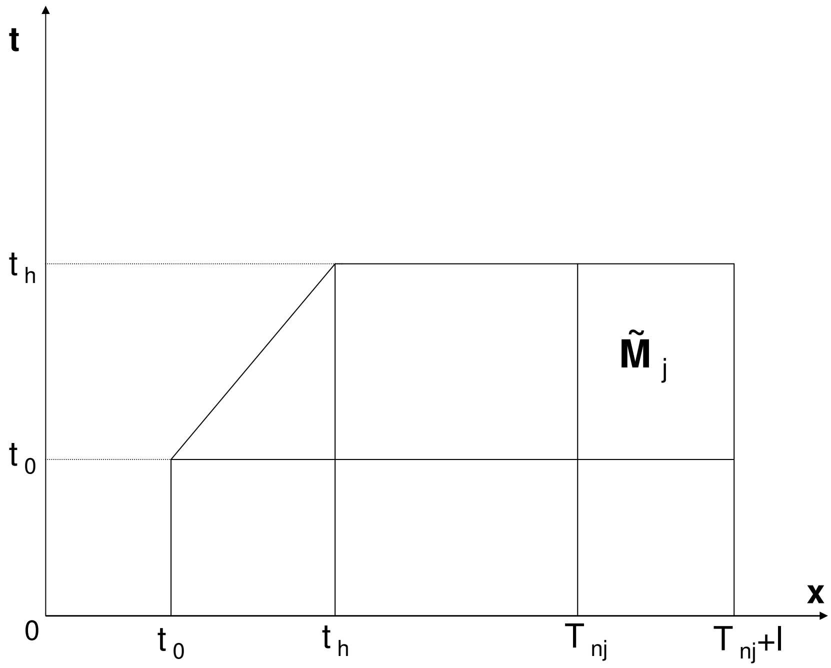

Figure 4.10: *Domain of integration* $\tilde{M}_j$ *and integration cube* $\tilde{\mathcal{Q}}_j$ *where the* $x^{'}$ *axis has the same limit as its corresponding* x *axis.*

Chapter 5

Empirical Study of Coupon Bond option

Followed § 2.5, we give the form for correlation and volatility. We empirically study the swaption market and propose an efficient computational procedure for analyzing the data. Empirical results of the swaption price, volatility, swaption correlation are compared with the predictions of quantum finance. The quantum finance model generates the market swaption price to over 90% accuracy.

§ 5.1 Swaption at the money and Correlation of Swaptions

Recall from § 1.4.4, the swaptions that we are studying have floating interest rate payments that are paid at $\ell = 3$ month intervals and fixed rate payments that are paid at intervals of $2\ell = 6$ months. The 3 monthly floating rate payments are paid at times $T_0 + n\ell$, with $n = 1, 2.., N$; there are N payments. For 6 monthly fixed rate payments there are only $N/2$ payments [1] of amount $2R_S$, made at times $T_0 + 2n\ell$, $n = 1, 2, ..., N/2$.

The payoff function for the interest rate swaption, in which the holder of the option receives

[1] Suppose the swaption has a duration such that N is even. Note $N = 4$ for a year long swaption.

at the fixed rate and pays at the floating rate, is given by [82]

$$\begin{aligned} CS(T_0; R_S) &= V\big[B(T_0, T_0 + N\ell) + 2\ell R_S \sum_{n=1}^{N/2} B(T_0, T_0 + 2n\ell) - 1\big]_+ \\ &= V\big[\sum_{n=1}^{N/2} c_n B(T_0, T_0 + 2n\ell) - 1\big]_+ \end{aligned} \quad (5.1)$$

where $B(t, T)$ is the price of a zero coupon Treasury Bond at time t that matures at time $T > t$. The coefficients and strike price for a swaption are hence given by

$$\begin{aligned} &c_n = 2\ell R_S \ \ ; \ \ n = 1, 2, ..., (N-1)/2 \ \ ; \ \ \text{Payment at time } T_0 + 2n\ell \\ &c_{N/2} = 1 + 2\ell R_S \ \ ; \ \ \text{Payment at time } T_0 + N\ell \\ &K = 1 \end{aligned} \quad (5.2)$$

The fixed interest rate par value R_P, at time t_0, is such that the value of the interest rate swap has zero value. Hence

$$\begin{aligned} &\sum_{n=1}^{N/2} c_n B(T_0, T_0 + 2n\ell) - 1 = 0 \\ \Rightarrow \ \ &2\ell R_P(t_0) = \frac{B(t_0, T_0) - B(t_0, T_0 + N\ell)}{\sum_{n=0}^{N/2} B(t_0, T_0 + 2n\ell)} \end{aligned} \quad (5.3)$$

The price of a swaption $C(t_0, t_*, R_S)$ at time $t_0 < t_*$, using the money market measure and discounting the value of the payoff function using the spot interest rate $r(t) = f(t, t)$, is given from eq. 5.1 by

$$C(t_0, t_*, R_S) = VE\Big[e^{-\int_{t_0}^{t_*} dt r(t)}\big(\sum_{n=1}^{N/2} c_n B(t_*, t_* + 2n\ell) - 1\big)_+\Big] \quad (5.4)$$

where V is the notional deposit on which the interest is calculated; we set $V = 1$.

The option price has been derived in chapter § 2.5 and the market correlator G_{ij} of the forward bond prices in eq.2.53 for different quantities is defined over different domains of the forward interest rates and this results in the integration of the forward interet rates correlation function over different integration limits. The exact form of the various integrations will be discussed later with the other correlators that are required for the computation of swaption volatility.

The input data that we need for computing the swaption price can be derived from the underlying forward interest rates' data and yields the coupon bond price, the forward bond price and the fixed rate par value R_p.

§ 5.1.1 Swaption At The Money

Recall the par value of the fixed interest rate $R_p(t_0)$ is the value for the fixed interest payments for which the swap at time t_0 is zero. From eqn 5.3 fixed interest rate equal to par value, namely $R_S = R_P$, implies the following

$$\begin{aligned} F &\equiv F(t_0) = \sum_{i=1}^{N/2} c_i F(t_0, T_0, T_0 + 2i\ell) = \sum_{i=1}^{N/2} 2\ell R_p F(t_0, T_0, T_0 + 2i\ell) + F(t_0, T_0, T_0 + N\ell) \\ &= \frac{B(t_0, T_0) - B(t_0, T_0 + N\ell)}{\sum_{n=0}^{N/2} B(t_0, T_0 + 2n\ell)} \sum_{i=1}^{N/2} F(t_0, T_0, T_0 + 2i\ell) + F(t_0, T_0, T_0 + N\ell) = 1 \end{aligned} \tag{5.5}$$

In the coupon bond option pricing formula $X = (F - K)/\sqrt{A}$ and for swaptions $K = 1$. Hence when the fixed interest rate R_S for the swaption is at the money $F = 1$ and this leads to $X = (F - K)/\sqrt{A} = 0$. As discussed in chapter § 2.5, the asymptotic behavior of the error function yields the following

$$I(x) = 1 - \sqrt{\frac{\pi}{2}} X + 0(x^2) \qquad X \approx 0 \tag{5.6}$$

and hence the swaption close to at the money, to leading order, has the form

$$C(t_0, t_*, R_p) \simeq B(t_0, t_*) \sqrt{\frac{A}{2\pi}} - \frac{1}{2} B(t_0, t_*)(K - F) + 0(X^2) \tag{5.7}$$

§ 5.1.2 Volatility and Correlation of Swaptions

The volatility and correlation of swaption prices are important quantities since they are indicators of the market's direction and also give us insights on portfolio study.

Consider the volatility and correlation of the change of swaption price for infinitesimal time steps. Let $C_1 \equiv C(t_0, t_1, R_1)$ and $C_2 \equiv C(t_0, t_2, R_2)$ denote two swaptions. Denote time derivative by an upper dot; for infinitesimal time step ϵ we have

$$\begin{aligned} \langle \dot{C}_1 \dot{C}_2 \rangle_c &= \frac{1}{\epsilon^2} \langle (C_1(t_0 + \epsilon) - C_1(t_0))(C_2(t_0 + \epsilon) - C_2(t_0)) \rangle_c \\ &= \frac{1}{\epsilon^2} \langle \delta C_1(t_0) \delta C_2(t_0) \rangle_c \end{aligned} \tag{5.8}$$

where the connected correlator is defined by $\langle AB \rangle_c \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$.

Note the swaption prices C_1, C_2 depend on the forward bond prices F_i, which take random values every day. The random changes in the price of the forward bond prices lead to changes in the price of the swaption. The correlation function $\langle \delta C_1(t_0)\delta C_2(t_0)\rangle_c$ can be evaluated by a historical average over the daily swaption prices, considered as the random outcomes of the swaption price due to the random changes in the forward bond price. Hence a historical average of the correlator of changes in the swaption price can be equated to the ensemble average of the correlator taken over the random fluctuations of the forward bond prices.

The field theory of forward interest rates yields

$$\langle \dot{f}(t,x)\dot{f}(t,x')\rangle_c = \frac{1}{\epsilon}M(t,x,x') \tag{5.9}$$

From the pricing formula given in eq. 5.7, the swaption's rate of change at the money, namely $X = 0$, is given by the following.

$$\begin{aligned}
\sqrt{2\pi}\frac{dC(t_0,t_I,R_p)}{dt_0} &= \frac{dB(t_0,t_I)}{dt_0}\sqrt{A_I} + \frac{1}{2\sqrt{A_I}}\frac{dA_I}{dt_0} + \sqrt{\frac{\pi}{2}}B(t_0,t_I)\frac{dF}{dt_0} \\
&= D_I - C(t_0,t_I,R_p)\int_{t_0}^{t_I} dx\dot{f}(t_0,x) - \frac{B(t_0,t_I)}{\sqrt{A_I}}\sum_{ij=1}^{I} J_iJ_jG_{ij}\int_{t_I}^{T_j} dx\dot{f}(t_0,x) \\
&\quad -\sqrt{\frac{\pi}{2}}B(t_0,t_I)\sum_{i=1}^{N} J_i\int_{t_I}^{T_i} dx\dot{f}(t_0,x)
\end{aligned} \tag{5.10}$$

where t_I denotes t_1 or t_2 and D_I contains all the deterministic (non-stochastic) factors that are subtracted out in forming the connected correlation functions.

To determine $\dot{C}$, as seen from equation above, one needs $\dot{f}((t_0,x)$, namely the evolution equation of the quantum field $f(t,x)$. The evolution equation of $f(t,x)$, together with eqs.

5.9 and 5.10, yields the following

$$
\begin{aligned}
2\pi\epsilon\langle\delta C_1(t_0)\delta C_2(t_0)\rangle_c \quad &= \\
C_1C_2\int_{t_0}^{t_1}dx\int_{t_0}^{t_2}dx^{'}M(t_0,x,x') &+ \frac{B(t_0,t_2)}{\sqrt{A_2}}C_1\sum_{jj'=1}^{N2}G_{jj'}J_jJ_{j'}\int_{t_0}^{t_1}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \\
+\ & \frac{B(t_0,t_1)}{\sqrt{A_1}}C_2\sum_{ii'=1}^{N1}G_{ii'}J_iJ_{i'}\int_{t_0}^{t_2}dx\int_{t_1}^{T_i}dx'M(t_0,x,x') \\
+\ & \frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_1A_2}}C_1C_2\sum_{ii'=1}^{N1}\sum_{jj'=1}^{N2}G_{ii'}J_iJ_{i'}G_{jj'}J_jJ_{j'}\int_{t_1}^{T_i}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \\
+\ & \sqrt{\frac{\pi}{2}}B(t_0,t_2)C_1\sum_{j=1}^{N2}J_j\int_{t_0}^{t_1}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \\
+\ & \sqrt{\frac{\pi}{2}}B(t_0,t_1)C_2\sum_{i=1}^{N1}J_i\int_{t_0}^{t_2}dx\int_{t_1}^{T_i}dx'M(t_0,x,x') \\
+\ & \sqrt{\frac{\pi}{2}}\frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_1}}\sum_{ii'=1}^{N1}\sum_{j=1}^{N2}G_{ii'}J_iJ_{i'}J_j\int_{t_1}^{T_i}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \\
+\ & \sqrt{\frac{\pi}{2}}\frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_2}}\sum_{i=1}^{N1}\sum_{jj'=1}^{N2}J_iG_{jj'}J_jJ_{j'}\int_{t_1}^{T_i}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \\
+\ & \frac{\pi}{2}B(t_0,t_1)B(t_0,t_2)\sum_{i=1}^{N1}\sum_{j=1}^{N2}J_iJ_j\int_{t_1}^{T_i}dx\int_{t_2}^{T_j}dx'M(t_0,x,x') \qquad (5.11)
\end{aligned}
$$

where A_1 and A_2 denote A for the two swaptions respectively.

§ 5.1.3 Market correlator

The forward bond price correlator G_{ij}, the swaption correlator and volatility are all computed from a set of three dimensional integrations on $M(t,x,x')$ with various integration limits. A general form of all the integration is given as follows

$$
\mathcal{I}=\int_{t_0}^{m1}dt\int_{m2}^{d1}dx\int_{m3}^{d2}dx'M(t,x,x') \qquad (5.12)
$$

and the limits of integrations are listed in the Table 5.1 below.

Note that for quantities appearing in swaption price and volatility function, the swaption maturity is at t_* and the two indices i and j run from 1 to N, with the last payment being

	m1	m2	m3	d1	d2
G_{ij}	t_*	t_*	t_*	T_i	T_j
$G_{ii'}$	t_1	t_1	t_1	T_i	$T_{i'}$
$G_{jj'}$	t_2	t_2	t_2	T_j	$T_{j'}$

Table 5.1: The various domains of integration for evaluating the integral $\mathcal{I} = \int_{t_0}^{m1} dt \int_{m2}^{d1} dx \int_{m3}^{d2} dx' M(t,x,x')$ that are required for computing the coefficients in the swaption price and correlators.

made at T_N. For the swaption correlation, options mature at two different times $t_2 \geq t_1$, and hence two indices i, j have the range $i = 1, 2, ..N1$ and $j = a, a+1, ..N2$ where the last payments are made at T_{N1} and T_{N2} respectively. In the next section, we examine the data in detail in order to compute $\mathcal{I}$.

§ 5.2 Data from Swaption Market

The swaption market provides daily data for X by Y swaptions. These swaptions mature X years from today, with the underlying swap starting at time X and the last payment being paid $X + Y$ years in the future. The domain for the swaption instrument is given in the time and future time $tx-$plane in Figure 5.1.

All the prices are presented with interest rates in basis points (100 basis points=1% annual interest rate) and has to be multiplied by the notional value of one million Dollars. Daily swaption prices 'at the money' are quoted from 29.1.2003 to 28.1.2005, a total of 523 daily data. In order to get accurate results, actual days in the real 6 months are divided by 360, since the convention for total number of days in a year is 360.

§ 5.2.1 ZCYC data

In order to generate swaption prices and swaption correlation from the model, both the historical and current underlying forward interest rates are required. The value of the coupon bond and forward bond price and the par fixed rate R_p are computed from the current forward interest rates. The integrand of the forward bond correlator G_{ij}, namely $M(t,x,x')$, is derived from historical forward interest rates' data.

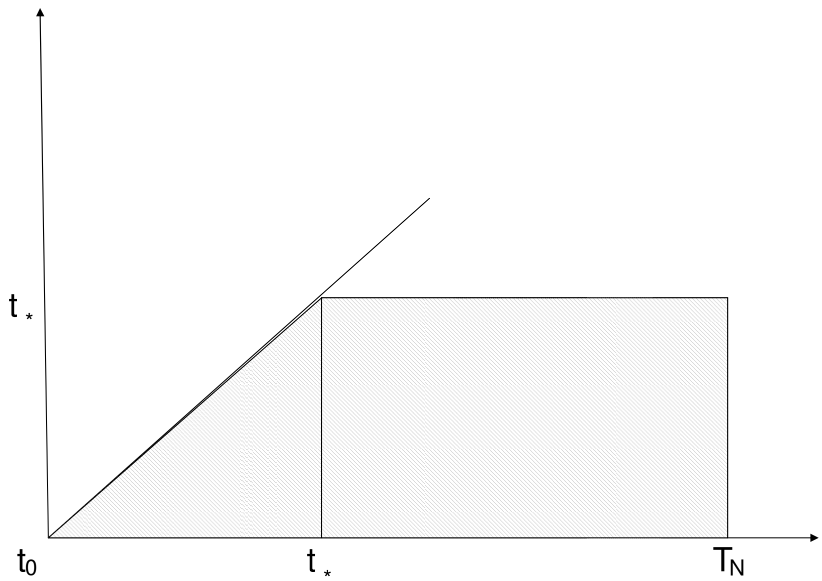

Figure 5.1: The shaded area is the domain for evaluating the price of a swaption. For 2by10 swaption $t_* = t_0 + 2$ year and $T_N = t_* + 10$ year.

Our analysis uses Bloomberg data for the **Zero Coupon Yield Curve** (ZCYC), denoted by $Z(t_0, T)$, from 29.1.2003 to 28.1.2005, and which yields, in total, 523 daily ZCYC data. The ZCYC is necessary for evaluating long duration swaptions since Libor data exists for maturity of only upto a maximum of 7 years in the future, whereas ZCYC has data with maturity of up to 30 years.

The ZCYC is given in the $\theta = x - t$ =constant direction, with the interval of θ between two data points not being a constant. Cubic spline is used for interpolating the data to a 3 month interval.

From [82] we have that the zero coupon bond is given by

$$B(t_0, T) \quad = \quad \frac{1}{(1 + Z(t_0, T)/c)^{(T-t_0)*c}} \tag{5.13}$$

where c represents how many times the bond is compounded per year. For ZCYC c is given as half yearly, and hence we have $c = 2/\text{year}$. As expected the forward bond price is given by

$$F(t_0, t_*, T) = \frac{B(t_0, T)}{B(t_0, t_*)} \tag{5.14}$$

From the definition of the zero coupon bond

$$B(t_0, T) = \exp\{-\int_{t_0}^{T} dx f(t_0, x)\}$$

we obtain, from eq. 5.13, the following

$$\int_{t_0}^{T} f(t_0, x) dx = \log[(1 + Z(t_0, T)/c)^{(T-t_0)*c}] \tag{5.15}$$

Note the important fact that the bond market directly provides the ZCYC, which is the *integral* of the forward interest rates over future time x. One can numerically differentiate the ZCYC to extract $f(t, x)$; this procedure does yield an estimate of $f(t, x)$, but with such large errors that it makes the estimate quite useless for empirically analyzing swaption pricing. Hence we develop numerical procedures direclty based on the ZCYC.

All data required for calculating a swaption's price can be obtained directly from the ZCYC data. The interpolation of ZCYC data and the convention used by Bloomberg has been empirically tested by comparing the computed R_p (from eqn 5.3) with the one given by market, and the result confirms the correctness of our computation.

§ 5.3 Numerical Algorithm for the Forward Bond Correlator

The market value of the forward bond price correlator $\mathcal{I}$ given in eq. 5.9 can be derived from ZCYC data. From eq. 5.9 and for discrete time $\dot{f} \simeq \delta f/\epsilon$ the correlation for changes in the forward interest rates is given by [6]

$$M(t, x, x') = \frac{1}{\epsilon}\langle \delta f(t, x) \delta f(t, x')\rangle_c \quad ; \quad \delta f(t, x) = f(t + \epsilon, x) - f(t, x) \tag{5.16}$$

Thus, we have for the forward bond correlators the following

$$\mathcal{I} = \frac{1}{\epsilon} \int_{t_0}^{m1} dt \int_{m2}^{d1} dx \int_{m3}^{d2} dx' \langle \delta f(t, x) \delta f(t, x')\rangle_c \tag{5.17}$$

From Table 5.1 we see that none of the limits on the integrations over x, x' depend on the time variable t; hence the finite time difference operator δ can be moved out of the x, x' integrations and yields

$$\mathcal{I} = \frac{1}{\epsilon} \int_{t_0}^{m1} dt \langle \big[\delta \int_{m2}^{d1} dx f(t, x)\big] \big[\delta \int_{m3}^{d2} dx' f(t, x')\big]\rangle_c \tag{5.18}$$

We keep to the x and x' integration variables instead of changing them to θ and θ' since, as discussed earlier, ZCYC data directly yields the integrals of forward interest rates on future time x. The numerical values of $\int_{m2}^{d1} dx f(t,x)$ and $\int_{m3}^{d2} dx' f(t,x')$ are obtained from the market values of the ZCYC.

To evaluate the market correlator $\mathcal{I}$ one needs to know the value of the correlator $M(t,x,x')$ in the *future*; the reason being that the time integration t in $\mathcal{I}$ runs from present time t_0 to time $m1 > t_0$ in the future. The problem of obtaining the future values of $M(t,x,x')$ can be solved by assuming that the correlation function for changes in the forward interest rates is invariant under time translations; that is

$$M(t,x,x') = M(t-a, x-a, x'-a) \tag{5.19}$$

The assumption of time translation invariance of the forward rates correlation function has been empirically tested in [48]; although this assumption cannot be indefinitely extended, a two years shift is considered to be reasonable [48].

The integration on the t axis can be converted to a summation by discretizing time into a lattice with spacing ϵ'; one then obtains

$$\mathcal{I} = \epsilon' \sum_{t_k=0}^{m1-t_0} \int_{m2}^{d1} dx \int_{m3}^{d2} dx' M(t_0+t_k, x, x') \tag{5.20}$$

From eq. 5.20 and a change of variables yields

$$x = y + t_k; \qquad x' = y' + t_k$$

We hence have from eq. 5.20

$$\begin{aligned} \mathcal{I} &= \epsilon' \sum_{t_k} \int_{m2-t_k}^{d1-t_k} dy \int_{m3-t_k}^{d2-t_k} dy' M(t_0+t_k, y+t_k, y'+t_k) \\ &= \epsilon' \sum_{t_k} \int_{m2-t_k}^{d1-t_k} dy \int_{m3-t_k}^{d2-t_k} dy' M(t_0, y, y') \end{aligned} \tag{5.21}$$

where condition given in eq. 5.19 has been used to obtain eq. 5.21. The integration on future data has been replaced by a summation on the *current* value of $M(t_0,x,x')$, with x,x' taking values on various intervals. The current value of $M(t_0,x,x')$ in turn is evaluated by taking averages of the correlator over it's *past* values.

From above and eqs. 5.16, 5.18 and 5.21 we have

$$\begin{aligned}\mathcal{I} &= \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle \int_{m2-t_k}^{d1-t_k}\delta f(t_0,y)dy\int_{m3-t_k}^{d2-t_k}\delta f(t_0,y')dy'\Big\rangle_c \\ &= \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle\Big[\delta\int_{m2-t_k}^{d1-t_k} f(t_0,y)dy\Big]\Big[\delta\int_{m3-t_k}^{d2-t_k} f(t_0,y')dy'\Big]\Big\rangle_c\end{aligned}$$

As discussed earlier, in order to directly use the ZCYC data the finite time difference operator δ is taken outside the future time integrations. Note ϵ' is the time integration interval and is equal to ϵ; for the time summation with daily intervals $\epsilon = \epsilon' = \frac{1}{260}$ (260 is the actual number of trading days in one year).

Re-expressing $\mathcal{I}$ in terms of the ZCYC data we obtain

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle\delta Y(t_0, m2-t_k, d1-t_k)\delta Y(t_0, m3-t_k, d2-t_k)\Big\rangle_c$$

where, from eq. 5.15, we have

$$\begin{aligned}Y(t_0,t_*,T) &= \int_{t_*}^{T} f(t_0,x)dx = \int_{t_0}^{T} f(t_0,x)dx - \int_{t_0}^{t_*} f(t_0,x)dx \\ &= \log((1+Z(t_0,T)/c)^{(T-t_0)*c}) - \log((1+Z(t_0,t_*)/c)^{(t_*-t_0)*c}) \qquad (5.22)\end{aligned}$$

The forward bond price correlator's present value (at time t_0) is obtained by averaging the value of the correlator $\big\langle\delta Y(t_0, m2-t_k, d1-t_k)\delta Y(t_0, m3-t_k, d2-t_k)\big\rangle$ over the last $t_0 - t_A$ days with $t_A = 180$ days.[2] Since the computation requires the value of δY for different future time intervals x, x' we have to again use cubic splines to interpolate ZCYC for obtaining daily values of the ZCYC. The shift of the future time integration to the present and the domain used for doing the averages for the correlator is illustrated in Figure 5.2.

§ 5.4 Empirical results

The 2by10 and 5by10 swaptions are priced for time series 6.4.2004- 28.1.2005 using the pricing formula from Section 2. When computing the forward interest rates' correlator $M(t;x,x')$ we found that daily swaption prices are stable when more than 270 days of historical data for ZCYC are used; but a 270-day average does not give the best fit of the predictions of model

[2]We ran the program by adding 30 days to the time averaging for evaluating the expectation values of the correlators; the best fit is given when the averaging is done over the past 180 days; see Section 5.

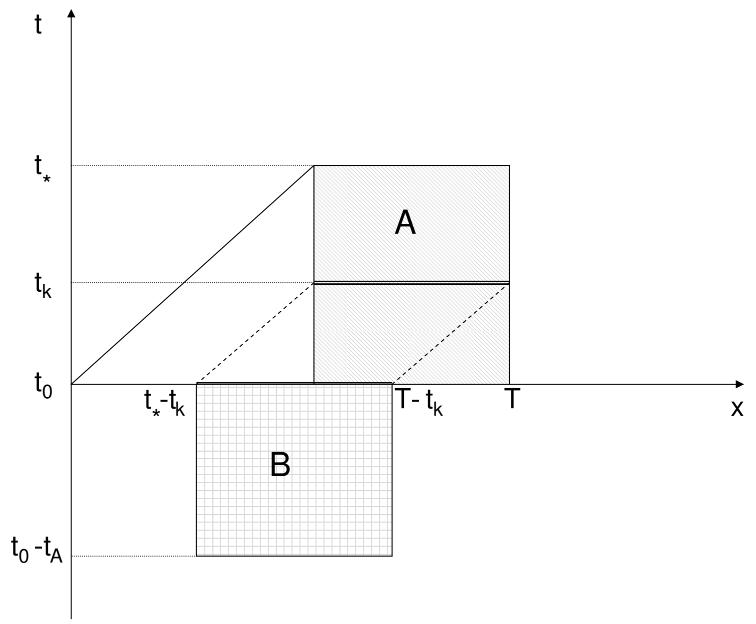

Figure 5.2: Shaded area A is the integration domain of $\mathcal{I}$. For the case when $t = t_0 + t_k$, the integration of x and x' for evaluating the expression for $Y(t_0 + t_k, t_*, T)$ inside $\langle \delta Y(t_0 + t_k, t_*, T)\delta Y(t_0 + t_k, t_*, T)\rangle$ is shifted back to t_0. Invariance in time yields this to be equal to $\langle \delta Y(t_0, t_* - t_k, T - t_k)\delta Y(t_0, t_* - t_k, T - t_k)\rangle$. A historical average is done over the shaded area B, which is in the past of t_0. $t_A = 180$ days is the optimum number of past data for evaluating the historical averages.

swaption price with the swaption's market value. This may be due to too much old information creating large errors in the predictions for the present day swaption prices. However, averaging on less historical data causes the swaption price curve to fluctuate strongly since it is likely that new information dominates swaption pricing and makes the price too sensitive to small changes.

Our empirical studies and results show that a moving averaging of 180 days historical data gives the best result for this period. One can most likely improve the accuracy by higher frequency sampling of 180 days of historical data.

The results obtained from the field theory model is compared with daily market data and is shown in Fig. 5.3 and Fig.5.4, with normalized root mean square of error being 3.31% and

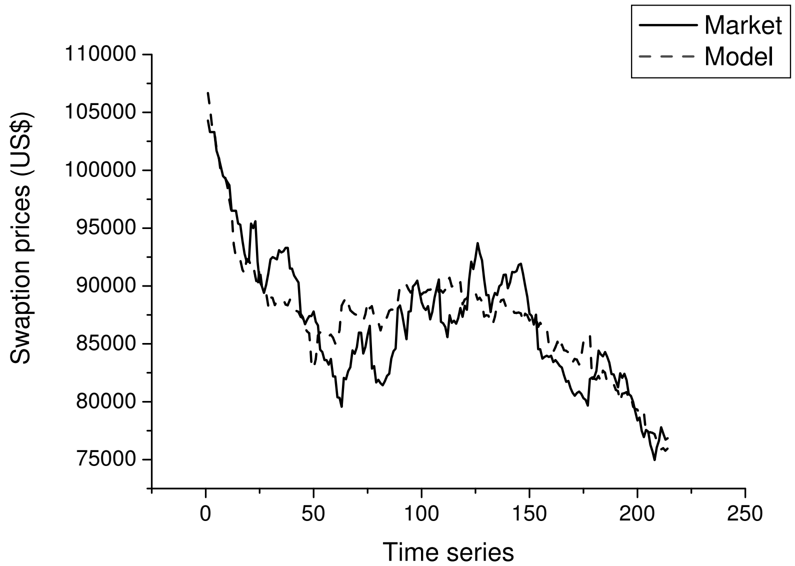

Figure 5.3: 2by10 Swaption price versus time t_0 (6.4.2004-28.1.2005), for both market and and model. Normalized root mean square error=3.31%

6.31% respectively.

The results for the swaption volatility and correlation discussed in Section § 5.1.2 are derived for the change on the same instruments; from eq. § 5.1.2

$$\delta C_1 \equiv C_1(t_0+\epsilon) - C_1(t_0) \equiv C_1(t_0+\epsilon, R_s) - C_1(t_0, R_s) \tag{5.23}$$

where $C_1(t_0+\epsilon)$ and $C_1(t_0)$ are the same contract being traded on successive days. Par fixed rate R_p is determined when the contract is initiated at time t_0, and the swaption $C_1(t_0)$ is at the money. However, in general $C_1(t_0+\epsilon)$ is away from the money; the reason being that the swaption depends on the forward bond prices F_i, and these change every day and hence there is a daily change in the par fixed rate R_p. From the market we only have the price of the swaption at the money. Historical data for the daily prices of swaptions in the money and out of the money are not quoted by Bloomberg. Hence, only the swaption volatility and correlation computed from the model is shown in Figure 5.5, without any comparison made with the market value for these quantities.

§ 5.4.1 Comparison of Field Theory Pricing with HJM-model

In order to see how the field theory model compares with the industry standard one factor HJM model, we empirically study swaption pricing in the HJM model. By considering the

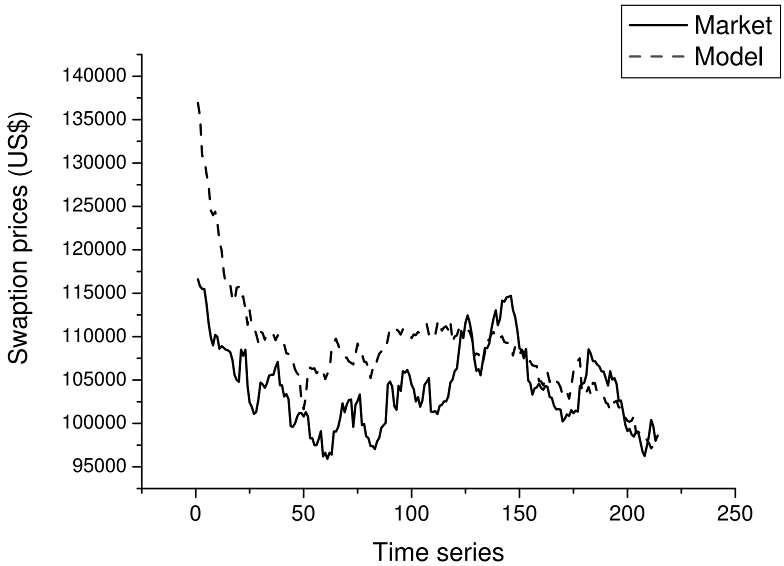

Figure 5.4: 5by10 Swaption price versus time t_0 (6.4.2004-28.1.2005), both market and model. The normalized root mean square error=6.31%

volatility function to have the special form of $\sigma(t,x) = \sigma_0 e^{-\lambda(x-t)}$ Jarrow and Turnbul [82] obtained, for the one-factor HJM model, the following explicit expression for the coupon bond option

$$\begin{aligned} C_{\text{HJM}}(t_0,t_*,K) &= \sum_{i=1}^{N} c_i B(t_0,T_i)N(d_i) - KB(t_0,t_*)N(d) \qquad (5.24)\\ d_i &\equiv \frac{r'}{\sigma_R} + W(t_*,T_i)\sigma_R \;\;;\;\; d = \frac{r'}{\sigma_R}\\ W(t_*,T_i) &\equiv \frac{1}{\lambda}[1-e^{-\lambda(T_i-t_*)}] \;\;;\;\; \sigma_R^2 = \frac{\sigma_0^2}{2\lambda}[1-e^{-2\lambda(t_*-t_0)}] \end{aligned}$$

The quantity r' is related to the strike price K by a nonlinear transformation that depends on the initial coupon bond price [82]. As shown in Paper I, to leading order in σ_0 the HJM limit of the field theory pricing formula with exponential volatility given by $\sigma(t,x) = \sigma_0 \exp(-\lambda(x-t))$ yields the HJM pricing formula.

We estimate σ_0 for the exponential volatility function in the HJM model from historical ZCYC data. By using exponential volatility and daily forward bond prices obtained from ZCYC, we price the swaption with HJM pricing formula and in Fig. 5.6 compare it with the market price and the field theory price.

The results show that HJM model is inadequate for pricing swaptions. Both because it systematically overprices the swaption by a large amount, and also because the instability

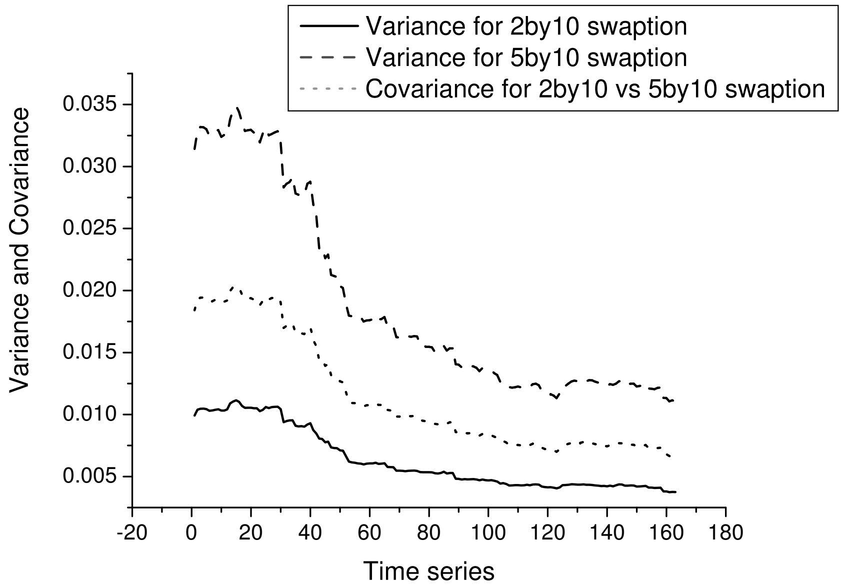

Figure 5.5: Swaption Variance $\langle \dot{C}_1^2 \rangle_c$, $\langle \dot{C}_2^2 \rangle_c$ and Covariance $\langle \dot{C}_1 \dot{C}_2 \rangle_c$ versus time t_0 (15.6.2004-27.1.2005) computed from the quantum finance model, with the value of the forward bond prices taken from market data.

of the price itself would give incorrect results if one tries to hedge the swaption using the one-factor HJM-pricing formula.

Instead of using the HJM-formula for pricing the coupon bond options partitioners may consider representing the price of the swaption by an implied volatility using the HJM pricing formula. However, unlike the case for the price of caps where this procedure is possible [14], the entire swaption curve cannot be fitted by adjusting only one quantity σ_0. Furthermore, the implied volatility $\sigma(t, x)$ in the first place may not be able to fit the price of all swaptions, and secondly it will depend on time; it is quite impractical to numerically evaluate daily implied volatility from daily swaption prices.

§ 5.5 Conclusion

The quantum finance swaption pricing formula was empirically tested, for various durations, by comparing it's predictions with the market values. There is over 90% agreement of the theoretical predictions for the swaption's price with it's market value, with errors around 6% for most swaptions and with an accuracy of about 3% for the shorter maturity swaptions.

A comparison of the field theory model and the industry standard HJM-model shows that

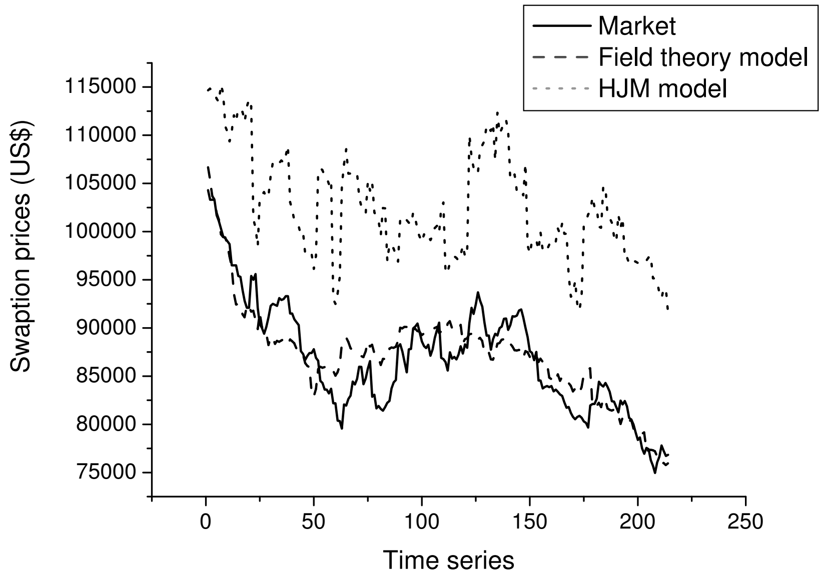

Figure 5.6: 2by10 swaption price, at the money, from the market, from the quantum finance model and from the HJM model. Time t_0 is in the range (6.4.2004-28.1.2005). The normalized root mean square error for HJM =18.87% compared with the far more accurate quantum finance swaption formula with error =3.31%.

the field theory model gives a more accurate and stable result than the HJM-model.

The HJM-model is not suited for pricing swaptions because the volatility parameter that goes into the pricing formula cannot be extracted from the swaption data. In contrast since the field theory model directly uses the market correlator $M(t, x, x')$ all the market information is fully accounted for in the swaption price.

The correlation of different swaptions and their volatilities are central ingredients in forming swaption portfolio's and hedging these portfolios. The quantum finance swaption pricing formula provides an approximate analytic result that can in turn be used to compute the correlation and volatility of swaptions; based on these analytic reaults one can form and hedge interest rate portfolios.

§ 5.6 Appendix: Test of algorithm for computing $\mathcal{I}$

The computation of $\mathcal{I}$ is the key step in calculating swaption prices. We test the programm used for numerically computing $\mathcal{I}$ by an analytically solvable formula of the forward interest rates.

Consider an analytical formula for forward interest rates as given below

$$f(t,x) = 1 - e^{-\lambda(x-t)} \tag{5.25}$$

The forward interest rates have an exponential form and increase from $f(t,t) = 0$ to the maximum value 1. Furthermore, $f(t,x)$ depends only on $x-t$, which is what we need for carrying out the the shift of time as explained in eq. 5.19.

Since we can analytically perform the integration of forward interest rates, one can directly determine Y. The analytic expression for the Y is given by

$$\begin{aligned} Y &\equiv Y(t,t_*,T_i) \\ &= \int_{t_*}^{T_i} dx f(t,x) \\ &= T_i - t_* + \frac{1}{\lambda}e^{\lambda t}(e^{-\lambda T_i} - e^{-\lambda t_*}) \end{aligned} \tag{5.26}$$

The input data is generated from our test forward interest rates and processed using the same algorithm as employed in Section § 6.6. Using the forward interest rates itself as input data will cause new errors since does not directly appear in the program being checked. Note that Y depends on three variables and is not suitable as input data. In analogy with eq. 5.22 we form a new variable $z(t,x)$, similar to ZCYC, such that

$$\begin{aligned} Y(t,t_*,T_i) &= z(t,T_i) - z(t,t_*) \\ z(t,x) &= x - t + \frac{1}{\lambda}(e^{-(x-t)} - 1) \end{aligned} \tag{5.27}$$

The function $z(t,x)$ from above formula is used as input data since this is starting point for the analysis of market data.

Since we are checking $\mathcal{I}$, more concretely G_{ij}, we have the exact analytical result

$$G_{ij} = \int_{t_0}^{t_*} dt \langle \dot{Y}(t,t_*,T_i)\dot{Y}(t,t_*,T_j)\rangle_c \tag{5.28}$$

and from eq. 5.26 we have

$$\dot{Y}(t,t_*,T_i) = e^{\lambda t}(e^{-\lambda T_i} - e^{-\lambda t_*}) \equiv b_i e^{\lambda t} \tag{5.29}$$

Changing the integration on t to a summation, we have

$$\begin{aligned}
G_{ij} &= \epsilon' \sum_{k=0}^{N} \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x-(t_0+k\epsilon'), x'-(t_0+k\epsilon')); \qquad N = \frac{t_*-t_0}{\epsilon'} \\
&= \epsilon' \sum_{k=0}^{N} \int_{t_*-k\epsilon'}^{T_i-k\epsilon'} dy \int_{t_*-k\epsilon'}^{T_j-k\epsilon'} dy' M(y-t_0, y'-t_0) \\
&= \epsilon' \sum_{k=0}^{N} \langle \dot{Y}(t_0, t_*-k\epsilon', T_i-k\epsilon') \dot{Y}(t_0, t_*-k\epsilon', T_j-k\epsilon') \rangle_c \qquad (5.30)
\end{aligned}$$

where

$$\begin{aligned}
&\langle \dot{Y}(t_0, t_*-k\epsilon', T_i-k\epsilon') \dot{Y}(t_0, t_*-k\epsilon', T_j-k\epsilon') \rangle_c \\
= \;& \frac{1}{N'} \sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon, t_*-k\epsilon', T_i-k\epsilon') \dot{Y}(t_0-n\epsilon, t_*-k\epsilon', T_j-k\epsilon') \\
&- \frac{1}{N'^2} \sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon, t_*-k\epsilon', T_i-k\epsilon') \sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon, t_*-k\epsilon', T_j-k\epsilon') \\
= \;& \frac{1}{N'} b_i b_j \frac{(1-e^{-2\lambda\epsilon N'})}{1-e^{-2\lambda\epsilon}} - \frac{1}{N'^2} b_i b_j \frac{(1-e^{-\lambda\epsilon N'})^2}{(1-e^{-\lambda\epsilon})^2} \qquad (5.31)
\end{aligned}$$

with

$$b_i = e^{-\lambda(T_i-k\epsilon')} - e^{-\lambda(t_*-k\epsilon')}$$

We ran the program for $\mathcal{I}$ with our artificial test input and compared the result from the program with the known analytical results. The numerical test for the algorithm shows that it exactly reproduces the analytical results, verifying the correctness of the algorithm used for our empirical analysis of swaptions.

Chapter 6

Price of Correlated and Self-correlated Coupon Bond Option

We now can move on to a more complicated case to show the powerfulness of the perturbation expansion which appears naturally in the context of field theory model. In order to do so, we can price an exotic multi-assets option. Since coupon bond with different maturities are driven by the same underlying forward interest rates, discounted payoff functions for coupon bonds are correlated. New instruments, option rely on two correlated coupon bond (correlated and self correlated coupon bond option), are proposed and the pricing formula using both martingale and market drift is derived in this chapter

§ 6.1 Correlated Coupon Bond Options

The Let the two underlying coupon bond start at times $t_2 \geq t_1$; in a notation that generalizes the payoff function given in eq. 2.41 let the two payoff functions for the options be given as below

$$\begin{aligned} S_1(t_1) &= \Big(\sum_{i=0}^{N1} c_i B(t_1, T_i) - K_1\Big)_+ \\ S_2(t_2) &= \Big(\sum_{j=a}^{N2} c_i B(t_2, T_j) - K_2\Big)_+ \end{aligned} \tag{6.1}$$

where $T_i = T_0 + i\ell$ are the fixed times, and can be taken to be the times at which Libor is exercised. Note the first payoff function starts with bonds maturing at time $T_0 > t_1$ whereas the second payoff function starts with bonds maturing at $T_a > t_2$

The price at today of the correlated coupon bond option can be given by

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}(t_0, t_1, t_2, K_1, K_2) \qquad (6.2)\\
&= \big\langle e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1) e^{-\int_{t_0}^{t_2} dtr(t)} S_2(t_2)\big\rangle - \big\langle e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1)\big\rangle\big\langle e^{-\int_{t_0}^{t_2} dtr(t)} S_2(t_2)\big\rangle\\
&\equiv \mathcal{M}_{12} - \mathcal{M}_1\mathcal{M}_2 \qquad (6.3)
\end{aligned}$$

We also can define correlated coupon bond option as $\mathcal{M}_{12}$, the difference between this definition and the connected piece($\mathcal{M}$) is simply multiplication of the prices of the two coupon bond options which has been derived in § 2.5.

Since the correlated coupon bond option is not a traded financial instrument and hence one does not expect its value to be the price of a financial instrument. The expectation value of the payoff consequently need not be evaluated using the martingale measure, and could equally consistently be evaluated using the market evolution of the underlying forward interest rates with a market drift of the forward interest rates not equal to the martingale drift. The precise probability measure used for performing the averaging $\langle..\rangle$ and hence evaluate the instrument need not, for now, be completely specified.

Thus quantities $\mathcal{M}_1$, $\mathcal{M}_2$ are only similar to the price of a coupon bond option, except that, unlike the coupon bond option, they can be evaluated using the market drift. To evaluate new piece $\mathcal{M}_{12}$, the following natural generalization of the notation of eq. 2.47 yields

$$\begin{aligned}
\mathcal{M}_{12} &= B(t_0, t_2)B(t_0, t_1)(\frac{1}{2\pi})^2 \int_{-\infty}^{+\infty} dW_1 d\eta_1 dW_2 d\eta_2 \big(\mathcal{F}_1 + W_1 - K_1\big)_+\\
&\qquad \times \big(\mathcal{F}_2 + W_2 - K_2\big)_+ e^{-i(\eta_1 W_1 + \eta_2 W_2)} Z(\eta_1, \eta_2) \qquad (6.4)
\end{aligned}$$

and where the partition function for the new instrument, namely $Z(\eta_1, \eta_2)$, is given by the appropriate generalization of eq. 2.48, as follows

$$Z(\eta_1, \eta_2) = \langle M_1 e^{i\eta_1 V_1} M_2 e^{i\eta_2 V_2}\rangle \qquad (6.5)$$

with the following definitions

$$\begin{aligned}
M_1 &= \frac{e^{-\int_{t_0}^{t_1} dtr(t)}}{B(t_0, t_1)} = e^{-\int_{\Delta_1}\alpha - \int_{\Delta_1}\sigma A} \quad ; \quad M_2 = \frac{e^{-\int_{t_0}^{t_2} dtr(t)}}{B(t_0, t_2)} = e^{-\int_{\Delta_2}\alpha - \int_{\Delta_2}\sigma A}\\
\mathcal{F}_1 &= \sum_{i=1}^{N1} J_{1i} \quad ; \quad \mathcal{F}_2 = \sum_{j=a}^{N2} J_{2j} \quad ; \quad J_{1i} = c_{1i}F_{1i} \quad ; \quad J_{2j} = c_{2j}F_{2j}\\
V_1 &= \sum_{i=1}^{N1} J_{1i}\big[e^{-\alpha_{1i} - Q_{1i}} - 1\big] \quad ; \quad V_2 = \sum_{j=a}^{N2} J_{2j}\big[e^{-\alpha_{2j} - Q_{2j}} - 1\big]
\end{aligned}$$

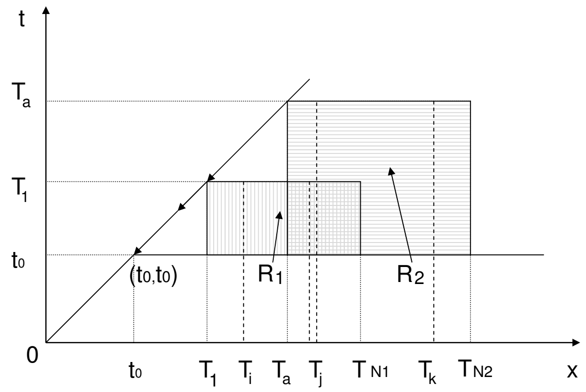

Figure 6.1: The rectangular domains R_1 and R_2 for the path integration of the correlated coupon bond option $\mathcal{M}$. Domains R_1 and R_2 overlap, and the double dashed line at time T_j indicates that both the options have a bond maturing at T_j.

To leading order the perturbation expansion for the partition function, from eq. 6.5, yields

$$Z(\eta_1, \eta_2) = \langle M_1 M_2 \rangle \; e^{ia_1\eta_1 + ia_2\eta_2 - \frac{1}{2}\sum_{ij=1}^{2} \eta_i A_{ij} \eta_j} + O(\eta_1^3, \eta_2^3) \tag{6.6}$$

where the explicit expressions for the coefficients a_1 , a_2 and matrix A_{ij} are computed in next sections by evaluating their respective path integrals for both martingale drift and market drift.

From eqs. 6.4, 6.5 and 6.6 the correlation function, after performing the η_1, η_2 integrations and some simplifications, is given by

$$\mathcal{M}_{12} = \mathcal{M}_0 \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} dW_1 dW_2 \big(W_1 - X_1\big)_+ \big(W_2 - X_2\big)_+ e^{-\frac{1}{2(1-\rho^2)}\left(W_1^2 + W_2^2 - 2\rho W_1 W_2\right)} \tag{6.7}$$

where

$$\begin{aligned}
\mathcal{M}_0 &= \frac{1}{2\pi} B(t_0,t_1)B(t_0,t_2)\mathcal{A} \\
\mathcal{A} &= \langle M_1 M_2 \rangle \sqrt{A_{11}A_{22}} \\
\rho &= \frac{A_{12}}{\sqrt{A_{11}A_{22}}} \\
X_1 &= \frac{K_1 - \mathcal{F}_1 - a_1}{\sqrt{A_{11}}} \quad ; \quad X_2 = \frac{K_2 - \mathcal{F}_2 - a_2}{\sqrt{A_{22}}}
\end{aligned} \tag{6.8}$$

The correllator $\mathcal{M}_{12}$ has two possible expansions, namely the case where

- $X_1,\ X_2$ are small and ρ is arbitrary
- ρ is small and $X_1,\ X_2$ are arbitrary

Expansion in $X_1,\ X_2$
Expanding the payoff function about $X_1,\ X_2 \simeq 0$ yields

$$(W - X)_+ \simeq (W - X)\theta(W) + \frac{X^2}{2}\delta(W) + O(X^3) \tag{6.9}$$

Hence, from eq. 6.7 and performing the integrations using the properties of the error function yields

$$\mathcal{M}_{12} = \mathcal{M}_0\big[m_0 + m_1(X_1 + X_2) + m_2 X_1 X_2 + m_3(X_1^2 + X_2^2)\big] + O(X_1^3, X_2^3) \tag{6.10}$$

with the coefficients being given by

$$\begin{aligned}
m_0 &= \begin{cases} \rho\big[\pi + \sqrt{\frac{1-\rho^2}{\rho^2}} - \tan^{-1}(\sqrt{\frac{1-\rho^2}{\rho^2}})\big] & ; \quad \rho \geq 0 \\ |\rho|\big[\sqrt{\frac{1-\rho^2}{\rho^2}} - \tan^{-1}(\sqrt{\frac{1-\rho^2}{\rho^2}})\big] & ; \quad \rho \leq 0 \end{cases} \\
m_1 &= \sqrt{\frac{\pi}{2}}(1+\rho) \\
m_2 &= \begin{cases} \pi - \tan^{-1}(\sqrt{\frac{1-\rho^2}{\rho^2}}) & ; \quad \rho \geq 0 \\ \tan^{-1}(\sqrt{\frac{1-\rho^2}{\rho^2}}) & ; \quad \rho \leq 0 \end{cases} \\
m_3 &= \frac{1}{2}\sqrt{1-\rho^2}
\end{aligned}$$

Note that $\mathcal{M}_{12}$ is a continuous function of ρ, with the graph of m_0 given in Figure 6.2.

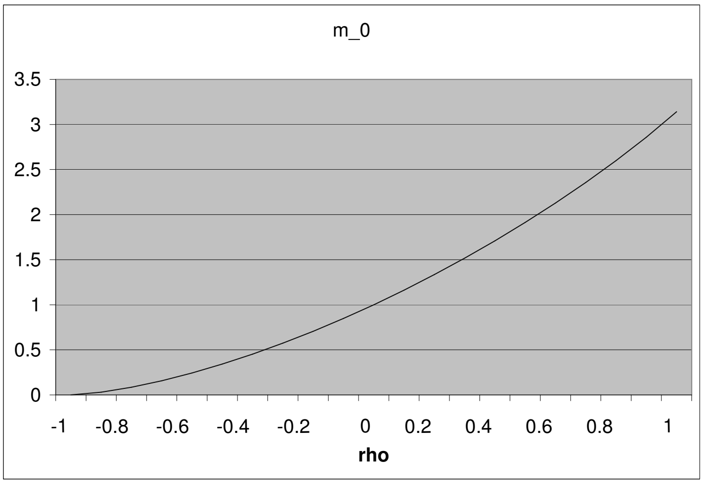

Figure 6.2: Graph of m_0, with ρ plotted along the x-axis and value of m_0 along the y-axis.

Expansion in ρ

To $O(\rho^2)$ the W_1, W_2 integrations completely factorize. Expanding $\mathcal{M}_{12}$ in a power series yields

$$\begin{aligned} \mathcal{M}_{12} &= \mathcal{M}_0 \int_{-\infty}^{+\infty} dW_1 dW_2 \big[1 + \rho W_1 W_2 + O(\rho^2)\big] \big(W_1 - X_1\big)_+ \big(W_2 - X_2\big)_+ e^{-\frac{1}{2}(W_1^2 + W_2^2)} \\ &= \mathcal{M}_0 \big[I(X_1) I(X_2) + \rho J(X_1) J(X_2)\big] + O(\rho^2) \end{aligned} \tag{6.11}$$

where $I(X)$ is given in eq. 2.59 and

$$\begin{aligned} J(X) &= \int_{-\infty}^{+\infty} dW \big(W - X\big)_+ W e^{-\frac{1}{2}W^2} \\ &= \sqrt{\frac{\pi}{2}}[1 - \Phi(\frac{X}{\sqrt{2}})] - 2Xe^{-\frac{X^2}{2}} \end{aligned} \tag{6.12}$$

§ 6.2 Self-Correlated Coupon Bond Option

The self-correlated coupon bond option is the limit of taking the two coupon bonds in the correlated case to be identical, that for $S(t_1) = S(t_2) = S(t)$. Since the computation of self-correlated coupon bond option is a bit simpler than the case for the correlated coupon bond option, the computation is carried out for it's own interest and for providing a check on the correlated case.

The self-correlated coupon bond option price $\mathcal{N}$, not to be confused with the volatility of the forward rates $\sigma(t,x)$, is given by

$$\begin{aligned}\mathcal{N}(S_1) &= \left\langle [e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1)]^2 \right\rangle - \left[\langle e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1) \rangle \right]^2 \\ &= \left\langle \left[e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1) \right]^2 \right\rangle - C^2(t_0, t_1; K) \end{aligned} \quad (6.13)$$

since the option price of the option is given by $C(t_0, t_1; K) = \langle e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1) \rangle$. Again, this will not the true when we consider the market drift case.

Since the Θ function has the property that $\Theta^2(x) = \Theta(x)$; hence the payoff function, given in eq. 1.39, yields the following

$$\begin{aligned} S_1(t_1) &= \Big(\sum_{i=0}^{N1} c_i B(t_1, T_i) - K_1 \Big)_+ = (F + V - K)\Theta(F + V - K) \\ \Rightarrow S_1^2(t_1) &= (F + V - K)^2 \Theta(F + V - K) \end{aligned}$$

Since there is only one Θ function in the expectation value needed to evaluate the self-correlated coupon bond option, it can be evaluated in a manner similar to the price of coupon bond option. Similar to eqs. 2.47 and **??**, one has

$$\left\langle [e^{-\int_{t_0}^{t_1} dtr(t)} S_1(t_1)]^2 \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta (F + W - K)^2 \Theta(F + W - K) e^{-i\eta W} Z_{\text{Vol}}(\eta) \quad (6.14)$$

with the partition function, from eq. 7.4, given by

$$Z(\eta) = \langle e^{-2\int_{t_0}^{t_1} dtr(t)} e^{i\eta V} \rangle \quad (6.15)$$

Expanding the partition function a power series in η and performing the average over the forward interest rates, and factoring out a prefactor, yields to second order

$$Z(\eta) = \left\langle e^{-2\int_{t_0}^{t_1} dtr(t)} \right\rangle \; e^{i\eta D_V - \frac{1}{2}\eta^2 A_V + \dots} \quad (6.16)$$

where

$$\begin{aligned} \left\langle e^{-2\int_{t_0}^{t_1} dtr(t)} \right\rangle &= B^2(t_0, t_1) e^{\Omega} \\ \Omega &= \int_{\Delta_1} M \end{aligned} \quad (6.17)$$

The coefficients are given by

$$D_V = \frac{1}{B^2(t_0, t_1) e^{\Omega}} \langle e^{-2\int_{t_0}^{t_1} dtr(t)} V \rangle \quad (6.18)$$

$$A_V = \frac{1}{B^2(t_0, t_1) e^{\Omega}} \langle e^{-2\int_{t_0}^{t_1} dtr(t)} V^2 \rangle - D_V^2 \quad (6.19)$$

Eq. 2.44 yields $V = \sum_{i=1}^{N} J_i(e^{-\alpha_i - Q_i} - 1)$; the coefficient D_V is consequently given by

$$D_V = \frac{1}{B^2(t_0, t_1)e^{\Omega}} \sum_{i=1}^{N} J_i \langle e^{-2\int_{t_0}^{t_1} dt r(t)} (e^{-\alpha_i - Q_i} - 1)\rangle$$

$$= \sum_{i=1}^{N} J_i [e^{2\int_{\Delta_1 R_i} M} - 1] \qquad (6.20)$$

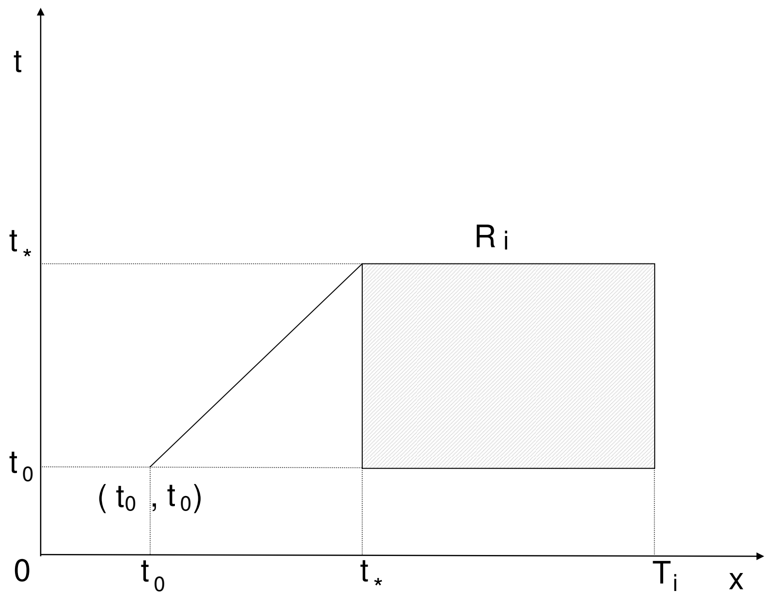

Figure 6.3: The shaded area is the domain of integration R_i.

since $\left\langle e^{-2\int_{t_0}^{t_1} dt r(t)} \right\rangle = B^2(t_0, t_1)e^{\Omega}$. The domain of integration Δ_1 is the unshaded portion of Figure 6.3 (for $t_1 = t_*$), and the domain of $\Delta_1 R_i$ is Δ_1 together with the shaded portion of Figure 6.3.

Similarly, the coefficient A_V, from eqs. 6.19 and 6.20, is given by

$$A_V = \frac{1}{B^2(t_0, t_1)e^{\Omega}} \sum_{i,j=1}^{N} J_i J_j \langle e^{-2\int_{t_0}^{t_1} dt r(t)} (e^{-\alpha_i - Q_i} - 1)(e^{-\alpha_j - Q_j} - 1)\rangle - D_V^2$$

$$= \sum_{i,j=1}^{N} \tilde{J}_i \tilde{J}_j [e^{G_{ij}} - 1] \qquad (6.21)$$

$$\tilde{J}_i = J_i e^{2\int_{\Delta_1 R_i} M}$$

Collecting the results from eqs. 6.14 and 6.15 yields the following result for the self-correlated coupon bond option price

$$\begin{aligned}\Big\langle[e^{-\int_{t_0}^{t_1} dtr(t)}S_1(t_1)]^2\Big\rangle &= \frac{B^2(t_0,t_1)e^{\Omega}A_V}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW(W-X_V)^2\Theta(W-X_V)e^{-\frac{1}{2}W^2} \\ &= \frac{B^2(t_0,t_1)e^{\Omega}A_V}{2}H(X_V) \qquad (6.22)\end{aligned}$$

where

$$H(X_V) = (1+X_V^2)\big[1-\Phi(\frac{X_V}{\sqrt{2}})\big] - \sqrt{\frac{2}{\pi}}X_V e^{-\frac{1}{2}X_V^2} \qquad (6.23)$$

$$X_V = \frac{K-F-D_V}{\sqrt{A_V}} \qquad (6.24)$$

§ 6.3 Coefficients for martingale drift

The explicit expressions for the coefficients a_1 , a_2 and matrix A_{ij} that are required for obtaining $\mathcal{M}_{12}$ are now explicitly computed.

For the purpose of illustrating the computation required, the calculation for the correlatted coupon bond option is analytically carried out using the money market numeraire [9] such that the numeraire given by $\exp(\int_{t_0}^{t} dt'r(t'))$ yields a martingale measure for all Treasury Bonds. All expectation values in this Section are defined using the money market numeraire, and the martingale condition states that

$$\begin{aligned}\langle e^{-\int_{t_0}^{t} dt'r(t')}B(t,T)\rangle &\equiv E_{\text{Money market}}[e^{-\int_{t_0}^{t} dt'r(t')}B(t,T)] \\ &= B(t_0,T) \qquad (6.25)\end{aligned}$$

with the associated drift velocity given by [6]

$$\alpha(t,x) = \int_t^x dx'M(x,x';t) \qquad (6.26)$$

The money market numeraire is the most suitable numeraire for finding the correlated coupon bond option as it treats both the discount factors on an equal basis.

The partition function from eq. 6.5 is given by

$$Z(\eta_1,\eta_2) = \langle M_1e^{i\eta_1V_1}M_2e^{i\eta_2V_2}\rangle \qquad (6.27)$$

where recall the definitions

$$
\begin{aligned}
M_1 &= e^{-\int_{t_0}^{t_1} dt r(t)}/B(t_0,t_1) \quad ; \quad M_2 = e^{-\int_{t_0}^{t_2} dt r(t)}/B(t_0,t_2) \\
\mathcal{F}_1 &= \sum_{i=1}^{N1} c_i F_{1i} \quad ; \quad \mathcal{F}_2 = \sum_{j=a}^{N2} c_i F_{2i} \\
V_1 &= \sum_{i=1}^{N1} J_{1i}\big[e^{-\alpha_{1i}-Q_{1i}} - 1\big] \quad ; \quad V_2 = \sum_{j=a}^{N2} J_{2j}\big[e^{-\alpha_{2j}-Q_{2j}} - 1\big]
\end{aligned}
$$

Using the money market drift velocity given in eq. 6.26 yields

$$
\begin{aligned}
\alpha_{1i} &= \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \alpha(t,x) \\
&= \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \int_t^x dx' M(x,x';t) \\
Q_{1i} &= \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \sigma(t,x) A(t,x) \\
\alpha_{2j} &= \int_{t_0}^{t_2} dt \int_{t_2}^{T_j} dx \int_t^x dx' M(x,x';t) \\
Q_{2j} &= \int_{t_0}^{t_2} dt \int_{t_2}^{T_j} dx \sigma(t,x) A(t,x)
\end{aligned}
\tag{6.28}
$$

and the discount factors yield, from eq. 2.7, and with $i = 1, 2$, the following

$$
\begin{aligned}
M_i &= e^{-\int_{\Delta_i}\alpha - \int_{\Delta_i}\sigma A} \\
\int_{\Delta_i} \alpha &= \int_{t_0}^{t_i} dt \int_t^{t_i} dx \alpha(t,x) \\
\int_{\Delta_i} \sigma A &= \int_{t_0}^{t_i} dt \int_t^{t_i} dx \sigma(t,x) A(t,x)
\end{aligned}
$$

The partition function for the correlated coupon bond option, from eq. 6.5, is given by

$$
Z(\eta_1,\eta_2) = \langle M_1 M_2 \rangle \; e^{ia_1\eta_1 + ia_2\eta_2 - \frac{1}{2}\sum_{ij=1}^{2} \eta_i A_{ij} \eta_j} + O(\eta_1^3, \eta_2^3) \tag{6.29}
$$

Using the fact that for the money market numeraire $\langle M_1 \rangle = \langle M_2 \rangle = 1$ the coefficients upto terms of $O(\eta_1^3, \eta_2^3)$ are given, from eqs. 6.5 and 6.29, by the following.

$$
a_1 = \frac{1}{\langle M_1 M_2 \rangle} \langle M_1 M_2 V_1 \rangle \tag{6.30}
$$

$$
a_2 = \frac{1}{\langle M_1 M_2 \rangle} \langle M_1 M_2 V_2 \rangle \tag{6.31}
$$

$$
A_{ii} = \frac{1}{\langle M_1 M_2 \rangle} \langle M_1 M_2 V_i^2 \rangle - a_i^2 \quad ; \quad i = 1, 2 \tag{6.32}
$$

$$
A_{12} = \frac{1}{\langle M_1 M_2 \rangle} \langle M_1 M_2 V_1 V_2 \rangle - a_1 a_2 = A_{21} \tag{6.33}
$$

All the calculations for the coefficients of η_1, η_2 are given by Gaussian path integrations, as was the case for evaluating the price of the coupon bond option. The path integrals for evaluating all the five coefficients a_1, a_2 and A_{ij} are carried out on the various sub-domains of $R_1 \mathrm{U} R_2$ shown in Figure 6.1.

The definition of M_1 and M_2 yields the following

$$\begin{aligned} \langle M_1 M_2 \rangle &= e^{\Omega_{12}} \\ \Omega_{12} &= \int_{\mathcal{T}_{12}} M(x, x'; t) \\ &\equiv \int_{t_0}^{t_1} dt \int_t^{t_1} dx \int_t^{t_2} dx' M(x, x'; t) \end{aligned} \tag{6.34}$$

The domain $\mathcal{T}_{12}$ is given in Figure 6.4.

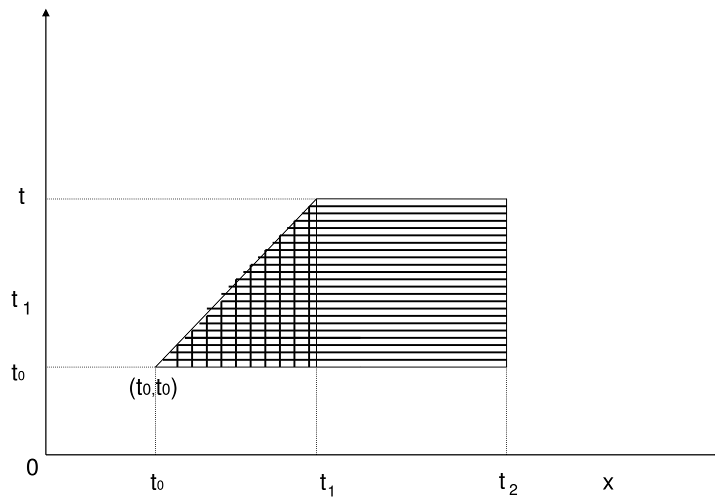

Figure 6.4: Domain $\mathcal{T}_{12}$ for evaluating $\langle M_1 M_2 \rangle$.

Furthermore Gaussian integrations yields

$$\begin{aligned} a_1 &= \frac{1}{\langle M_1 M_2 \rangle} \langle M_1 M_2 V_1 \rangle \\ &= e^{-\Omega_{12}} \langle M_1 M_2 V_1 \rangle \\ &= \sum_{i=1}^{N1} J_i \big(e^{\Gamma_{2i}} - 1 \big) \end{aligned} \tag{6.35}$$

$$\text{where} \quad \Gamma_{2i} = \int_{\Delta_2 R_i} M = \int_{t_0}^{t_1} dt \int_t^{t_2} dx \int_{t_1}^{T_i} dx' M(x, x'; t)$$

Similarly, since $\langle M_2 V_2 \rangle = 0$, the coefficient a_2 is given by

$$a_2 = \sum_{j=a}^{N2} J_j \big(e^{\Gamma_{1j}} - 1 \big) \tag{6.36}$$

$$\text{where} \quad \Gamma_{1j} = \int_{\Delta_1 R_j} M = \int_{t_0}^{t_1} dt \int_t^{t_1} dx \int_{t_2}^{T_j} dx' M(x, x'; t)$$

Since eq. 6.26 yields $\langle M_1 V_1 \rangle = 0$, one obtains the following

$$\langle M_1 V_1^2 \rangle = \sum_{ii'=1}^{N1} J_i J_{i'} \big(e^{G_{ii'}} - 1 \big) \tag{6.37}$$

$$\langle M_2 V_2^2 \rangle = \sum_{jj'=a}^{N2} J_j J_{j'} \big(e^{G_{jj'}} - 1 \big) \tag{6.38}$$

$$\begin{aligned} \text{with} \quad G_{ii'} &= \int_{R_i R_{i'}} M = \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_1}^{T_{i'}} dx' M(x, x'; t) \\ G_{jj'} &= \int_{R_j R_{j'}} M = \int_{t_0}^{t_2} dt \int_{t_2}^{T_j} dx \int_{t_2}^{T_{j'}} dx' M(x, x'; t) \end{aligned}$$

Consider the expectation values

$$\langle M_1 M_2 V_1^2 \rangle = e^{\Omega_{12}} \sum_{ii'=1}^{N1} J_i J_{i'} \big(e^{\Gamma_{2i} + \Gamma_{2i'} + G_{ii'}} - e^{\Gamma_{2i}} - e^{\Gamma_{2i'}} + 1 \big) \tag{6.39}$$

$$\langle M_1 M_2 V_2^2 \rangle = e^{\Omega_{12}} \sum_{jj'=a}^{N2} J_j J_{j'} \big(e^{\Gamma_{1j} + \Gamma_{1j'} + G_{jj'}} - e^{\Gamma_{1j}} - e^{\Gamma_{1j'}} + 1 \big) \tag{6.40}$$

$$\langle M_1 M_2 V_1 V_2 \rangle = e^{\Omega_{12}} \sum_{i=1}^{N1} \sum_{j=a}^{N2} J_i J_j \big(e^{\Gamma_{1j} + \Gamma_{2i} + \gamma_{ij}} - e^{\Gamma_{2i}} - e^{\Gamma_{1j}} + 1 \big) \tag{6.41}$$

$$\text{with} \quad \gamma_{ij} = \int_{R_i R_j} M = \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_2}^{T_j} dx' M(x, x'; t)$$

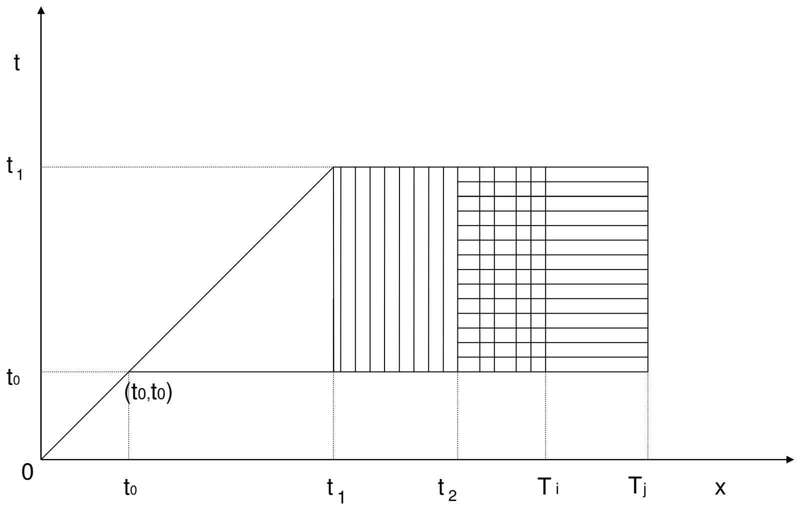

Figure 6.5: Domain R_iR_j for evaluating γ_{ij} .

The domain R_iR_j for evaluating the coefficient γ_{ij} is shown in Figure 6.5.

From equations above and from eqs. 6.34, 6.30, 6.45, 6.46 and 6.37 and 6.38 the coefficients A_{ij} are explicitly given by

$$
\begin{aligned}
A_{11} &= \sum_{ii'=1}^{N1} \mathcal{J}_i \mathcal{J}_{i'} \left(e^{G_{ii'}} - 1\right) \\
A_{22} &= \sum_{jj'=a}^{N2} \mathcal{J}_j \mathcal{J}_{j'} \left(e^{G_{jj'}} - 1\right) \\
A_{12} &= A_{21} = \sum_{i=1}^{N1} \sum_{j=a}^{N2} \mathcal{J}_i \mathcal{J}_j \left(e^{\gamma_{ij}} - 1\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J}_i &= J_i e^{\Gamma_{1i}} \quad ; \quad i = 1, 2, ..N1 \\
\mathcal{J}_j &= J_j e^{\Gamma_{2j}} \quad ; \quad j = a, a+1, ..N2
\end{aligned}
$$

To determine $\mathcal{M} = \mathcal{M}_{12} - \mathcal{M}_1 \mathcal{M}_2$ one needs to also determine $\mathcal{M}_1$ and $\mathcal{M}_2$. For the martingale measure $\mathcal{M}_1$ and $\mathcal{M}_2$ are simply the coupon bond option price, and hence

$$\begin{aligned}\mathcal{M}_1 &= C_1(t_0, t_1; X_1)\\ \mathcal{M}_2 &= C_2(t_0, t_2; X_2)\end{aligned}$$

The result obtained for the correlated coupon bond option $\mathcal{M}$ in eq. 6.2 contains the special case of the self-correlated coupon bond option.

The self-correlated coupon bond option price derived in Section § 6.2 provide a check for the correlated coupon bond option $\mathcal{M}$.

§ 6.4 Coefficients for market drift

Recall as discussed earlier, the explicit expressions for the coefficients a_1 , a_2 and matrix A_{ij} that are required for obtaining $\mathcal{M}_{12}$ can also be evaluated completely from the market evolution, in particular using the market drift, of the underlying forward interest rates.

Following section § 6.3, some terms need to be redefined in order to account for the market drift.

$$\begin{aligned}\alpha_{1i} &= \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx\alpha(t,x)\\ \alpha_{2j} &= \int_{t_0}^{t_2} dt \int_{t_2}^{T_j} dx\alpha(t,x)\end{aligned} \tag{6.42}$$

where $\alpha(t,x)$ is the market drift which has to be computed from the underlying forward interest rates.

The coefficients in eqn.6.29 are now given by the following

$$a_1 = \frac{1}{\langle M_1 M_2 \rangle_c} (\langle M_1 M_2 V_1 \rangle - \langle M_1 V_1 \rangle \langle M_2 \rangle) \tag{6.43}$$

$$a_2 = \frac{1}{\langle M_1 M_2 \rangle_c} (\langle M_1 M_2 V_2 \rangle - \langle M_2 V_2 \rangle \langle M_1 \rangle) \tag{6.44}$$

$$A_{ii} = \frac{1}{\langle M_1 M_2 \rangle_c} (\langle M_1 M_2 V_i^2 \rangle - \langle M_i V_i^2 \rangle \langle M_j \rangle) - a_i^2 \quad ; \quad i = 1,2 \quad j \neq i \tag{6.45}$$

$$A_{12} = \frac{2}{\langle M_1 M_2 \rangle_c} (\langle M_1 M_2 V_1 V_2 \rangle - \langle M_1 V_1 \rangle \langle M_2 V_2 \rangle) - 2a_1 a_2 = A_{21} \tag{6.46}$$

The definition of M_i for nonmartingale drift yields the following

$$\langle M_i \rangle = e^{-\int_{\Delta_i}\alpha + \frac{1}{2}\int_{\Delta_i} M(x,x';t)} \qquad i = 1,2 \tag{6.47}$$

$$\langle M_1 M_2 \rangle = e^{\Omega_{12} - \int_{\Delta_1}\alpha - \int_{\Delta_2}\alpha + \frac{1}{2}\int_{\Delta_1} M(x,x';t) + \frac{1}{2}\int_{\Delta_2} M(x,x';t)} \tag{6.48}$$

$$\langle M_1 M_2 \rangle_c = \langle M_1 M_2 \rangle - \langle M_1 \rangle \langle M_2 \rangle \tag{6.49}$$

where

$$\begin{aligned}
\int_{\Delta_i} \alpha &= \int_{t_0}^{t_i} dt \int_t^{t_i} dx \alpha(t,x) \\
\int_{\Delta_i} M(x,x';t) &= \int_{t_0}^{t_i} dt \int_t^{t_i} dx \int_t^{t_i} dx' M(x,x';t) \\
\Omega_{12} &= \int_{\mathcal{T}_{12}} M(x,x';t) \\
&\equiv \int_{t_0}^{t_1} dt \int_t^{t_1} dx \int_t^{t_2} dx' M(x,x';t)
\end{aligned}$$

Furthermore Gaussian integrations yields

$$\langle M_1 M_2 V_1 \rangle = \langle M_1 M_2 \rangle \sum_{i=1}^{N1} J_i \left(e^{-\alpha_{1i} + \frac{1}{2}G_{ii'} + \Gamma_{1i} + \Gamma_{2i}} - 1 \right) \tag{6.50}$$

$$\begin{aligned}
\text{where} \quad \Gamma_{1i} &= \int_{\Delta_1 R_i} M = \int_{t_0}^{t_1} dt \int_t^{t_1} dx \int_{t_1}^{T_i} dx' M(x,x';t) \\
\Gamma_{2i} &= \int_{\Delta_2 R_i} M = \int_{t_0}^{t_1} dt \int_t^{t_2} dx \int_{t_1}^{T_i} dx' M(x,x';t) \\
G_{ii'} &= \int_{R_i R_{i'}} M = \int_{t_0}^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_1}^{T_{i'}} dx' M(x,x';t)
\end{aligned}$$

Similarly,

$$\langle M_1 M_2 V_2 \rangle = \langle M_1 M_2 \rangle \sum_{j=a}^{N2} J_j \left(e^{-\alpha_{2j} + \frac{1}{2}G_{jj'} + \Gamma_{1j} + \Gamma_{2j}} - 1 \right) \tag{6.51}$$

$$\begin{aligned}
\text{where} \quad \Gamma_{1j} &= \int_{\Delta_1 R_j} M = \int_{t_0}^{t_1} dt \int_t^{t_1} dx \int_{t_2}^{T_j} dx' M(x,x';t) \\
\Gamma_{2j} &= \int_{\Delta_2 R_j} M = \int_{t_0}^{t_1} dt \int_t^{t_2} dx \int_{t_2}^{T_j} dx' M(x,x';t) \\
G_{jj'} &= \int_{R_j R_{j'}} M = \int_{t_0}^{t_2} dt \int_{t_2}^{T_j} dx \int_{t_2}^{T_{j'}} dx' M(x,x';t)
\end{aligned}$$

and

$$\langle M_1 V_1 \rangle = \langle M_1 \rangle \sum_{i=1}^{N1} J_i \big(e^{-\alpha_{1i} + \frac{1}{2} G_{ii'} + \Gamma_{1i}} - 1 \big) \qquad (6.52)$$

$$\langle M_2 V_2 \rangle = \langle M_2 \rangle \sum_{j=a}^{N2} J_j \big(e^{-\alpha_{2j} + \frac{1}{2} G_{jj'} + \Gamma_{2j}} - 1 \big) \qquad (6.53)$$

Now, a_1 and a_2 can be evaluated by eqns 6.43,6.44 with the above explicit expressions.

The coefficients A_{ij} are now evaluated in a similar manner and yields

$$\langle M_1 V_1^2 \rangle = \langle M_1 \rangle \sum_{i,i'=1}^{N1} J_i J_{i'} \big(e^{-\alpha_{1i} - \alpha_{1i'} + \frac{1}{2} G_{ii} + \frac{1}{2} G_{i'i'} + G_{ii'} + \Gamma_{1i} + \Gamma_{1i'}} - e^{-\alpha_{1i} + \frac{1}{2} G_{ii} + \Gamma_{1i}} - e^{-\alpha_{1i'} + \frac{1}{2} G_{i'i'} + \Gamma_{1i'}} + 1 \big) \qquad (6.54)$$

$$\langle M_2 V_2^2 \rangle = \langle M_2 \rangle \sum_{j,j'=a}^{N2} J_j J_{j'} \big(e^{-\alpha_{2j} - \alpha_{2j'} + \frac{1}{2} G_{jj} + \frac{1}{2} G_{j'j'} + G_{jj'} + \Gamma_{2j} + \Gamma_{2j'}} - e^{-\alpha_{2j} + \frac{1}{2} G_{jj} + \Gamma_{2j}} - e^{-\alpha_{2j'} + \frac{1}{2} G_{j'j'} + \Gamma_{2j'}} + 1 \big) \qquad (6.55)$$

and

$$\langle M_1 M_2 V_1^2 \rangle = \langle M_1 M_2 \rangle \sum_{i,i'=1}^{N1} J_i J_{i'} \Big(e^{-\alpha_{1i} - \alpha_{1i'} + \Omega + \frac{1}{2} G_{ii} + \frac{1}{2} G_{i'i'} + G_{ii'} + \Gamma_{1i} + \Gamma_{1i'} + \Gamma_{2i} + \Gamma_{2i'}} - e^{-\alpha_{1i} + \Omega + \frac{1}{2} G_{ii} + \Gamma_{1i} + \Gamma_{2i}} - e^{-\alpha_{1i'} + \Omega + \frac{1}{2} G_{i'i'} + \Gamma_{1i'} + \Gamma_{2i'}} + 1 \Big) \qquad (6.56)$$

$$\langle M_1 M_2 V_2^2 \rangle = \langle M_1 M_2 \rangle \sum_{j,j'=a}^{N2} J_j J_{j'} \Big(e^{-\alpha_{2j} - \alpha_{2j'} + \Omega + \frac{1}{2} G_{jj} + \frac{1}{2} G_{j'j'} + G_{jj'} + \Gamma_{1j} + \Gamma_{1j'} + \Gamma_{2j} + \Gamma_{2j'}} - e^{-\alpha_{2j} + \Omega + \frac{1}{2} G_{jj} + \Gamma_{1j} + \Gamma_{2j}} - e^{-\alpha_{2j'} + \Omega + \frac{1}{2} G_{j'j'} + \Gamma_{1j'} + \Gamma_{2j'}} + 1 \Big) \qquad (6.57)$$

$$\langle M_1 M_2 V_1 V_2 \rangle = \langle M_1 M_2 \rangle \sum_{i=1}^{N1} \sum_{j=a}^{N2} J_i J_j \Big(e^{-\alpha_{1i} - \alpha_{2j} + \Omega + \frac{1}{2} G_{ii} + \frac{1}{2} G_{jj} + G_{ij} + \Gamma_{1i} + \Gamma_{1j} + \Gamma_{2i} + \Gamma_{2j}} - e^{-\alpha_{1i} + \Omega + \frac{1}{2} G_{ii} + \Gamma_{1i} + \Gamma_{2i}} - e^{-\alpha_{2j} + \Omega + \frac{1}{2} G_{jj} + \Gamma_{1j} + \Gamma_{2j}} + 1 \Big) \qquad (6.58)$$

Furthermore, $\mathcal{M}_1$ and $\mathcal{M}_2$ are not the coupon bond option price if one consistently use the market drift. Different from the one derived in Baaquie [11], one has

$$D = \frac{1}{\langle M_i \rangle} \langle M_i V_i \rangle \qquad (6.59)$$

$$A = \frac{1}{\langle M_i \rangle} \langle M_i V_i^2 \rangle - D^2 \qquad i = 1, 2 \qquad (6.60)$$

§ 6.5 Market correlator and drift

Since the new instruments are also computed from a set of three dimensional integrations on $M(t,x,x')$ with various integration limits. A general form of all the integration is then given as follows

$$\mathcal{I} = \int_{t_0}^{m1} dt \int_{m2}^{d1} dx \int_{m3}^{d2} dx' M(t,x,x') \tag{6.61}$$

and the limits of integrations are listed in the Table 5.1 below.

	Γ_{*i}	G_{ij}	Γ_{1i}	Γ_{2i}	Γ_{1j}	Γ_{1j}	$G_{ii'}$	$G_{jj'}$	γ_{ij}
m1	t_*	t_*	t_1	t_1	t_1	t_1	t_1	t_2	t_1
m2	t	t_*	t	t	t	t	t_1	t_2	t_1
m3	t_*	t_*	t_1	t_1	t_2	t_2	t_1	t_2	t_2
d1	t_*	T_i	t_1	t_2	t_1	t_2	T_i	T_j	T_i
d2	T_i	T_j	T_i	T_i	T_j	T_j	$T_{i'}$	$T_{j'}$	T_j

Table 6.1: The various domains of integration for evaluating the integral $\mathcal{I} = \int_{t_0}^{m1} dt \int_{m2}^{d1} dx \int_{m3}^{d2} dx' M(t,x,x')$ that are required for computing the coefficients in the swaption price, correlators and new instruments.

Besides, we also have a set of two dimensional integration on market drift $\alpha(t,x)$ which have the general form

$$\mathcal{D} = \int_{t_0}^{m1} dt \int_{m_2}^{d1} dx \alpha(t,x) \tag{6.62}$$

and the limits of integration are listed the Table 6.2

For the new instruments, underlying coupon bond mature at two different times $t_2 \geq t_1$,

	α_{Δ_1}	α_{Δ_2}	α_{1i}	α_{2j}
m1	t_1	t_2	t_1	t_2
m2	t	t	t_1	t_2
d1	t_1	t_2	T_i	T_j

Table 6.2: The various domains of integration for evaluating the integral $\mathcal{D} = \int_{t_0}^{m1} dt \int_{m_2}^{d1} dx \alpha(t,x)$ that are required for computing the coefficients in the new instruments.

and hence two indices i,j have the range $i = 1,2,..N1$ and $j = a, a+1,..N2$ where the last payments are made at T_{N1} and T_{N2} respectively. Also, for new instruments, m2 has the value of t, it means it is not a fixed value and depend on the t integration In the next section, we present the algorithm of computing $\mathcal{I}$ and $\mathcal{D}$ which is similar as those in § 5.3.

§ 6.6 Numerical Algorithm for the Forward Bond Correlator and drift

Following the algorithm presented in § 5.3, most of the procedures are exactly the same. However, as in the table 6.1, one of the lower limit of integration can be variable t, then two different results are shown respectively.

Re-expressing $\mathcal{I}$ in terms of the ZCYC data we obtain

- When both the lower limit of y and y' are fixed

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\big\langle \delta Y(t_0, m2-t_k, d1-t_k)\delta Y(t_0, m3-t_k, d2-t_k)\big\rangle$$

where, from eq. 5.15, we have

$$\begin{aligned} Y(t_0, t_*, T) &= \int_{t_*}^{T} f(t,x)dx = \int_{t_0}^{T} f(t_0,x)dx - \int_{t_0}^{t_*} f(t_0,x)dx \\ &= \log((1+Z(t_0,T)/c)^{(T-t_0)*c}) \\ &\quad -\log((1+Z(t_0,t_*)/c)^{(t_*-t_0)*c}) \end{aligned} \tag{6.63}$$

- One of the lower limit is not fixed, say m2=t

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\big\langle \int_{t_0}^{d1-t_k}\delta f(t_0,y)dy\int_{m3-t_k}^{d2-t_k}\delta f(t_0,y')dy'\big\rangle \tag{6.64}$$

since

$$\int_{t_0}^{T}\delta_{t_0} f(t,x)dx = \delta_{t_0}\int_{t_0}^{T} f(t_0,x) + \epsilon f(t_0,t_0) \tag{6.65}$$

we have

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\big\langle(\delta Y(t_0,t_0,d1-t_k) + \epsilon f(t_0,t_0))\delta Y(t_0,m3-t_k,d2-t_k)\big\rangle$$

Similarly, the market drift can be derived from ZCYC data as

$$\alpha(t,x) = \frac{1}{\epsilon}\langle \delta f(t,x)\rangle \tag{6.66}$$

From eqn 6.62 by using the same shift technic, instead of integrating on the future data, we have integration on current and past data as follow

$$\mathcal{D} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle \int_{m2-t_k}^{d1-t_k} \delta f(t_0,y)dy\Big\rangle \tag{6.67}$$

Again, two different results are shown blow

- When the lower limit of y is fixed

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle \delta Y(t_0, m2-t_k, d1-t_k)\Big\rangle$$

- The lower limit is not fixed, say m2=t

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle \int_{t_0}^{d1-t_k} \delta f(t_0,y)dy\Big\rangle \tag{6.68}$$

we have

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k}\Big\langle (\delta Y(t_0, t_0, d1-t_k) + \epsilon f(t_0,t_0))\Big\rangle$$

Thus we can price the new instruments using the algorithm presented above, we first computed the self-correlated coupon option and the correlated coupon bond option with $t_1 = t_2$ in section § 6.1 and the numerical results show identity. We then plot the results for correlation of coupon bond options in Fig. 6.6 for martingale drift and in Fig. 6.7 for market drift. No market data is compared since it is not a traded instrument yet.

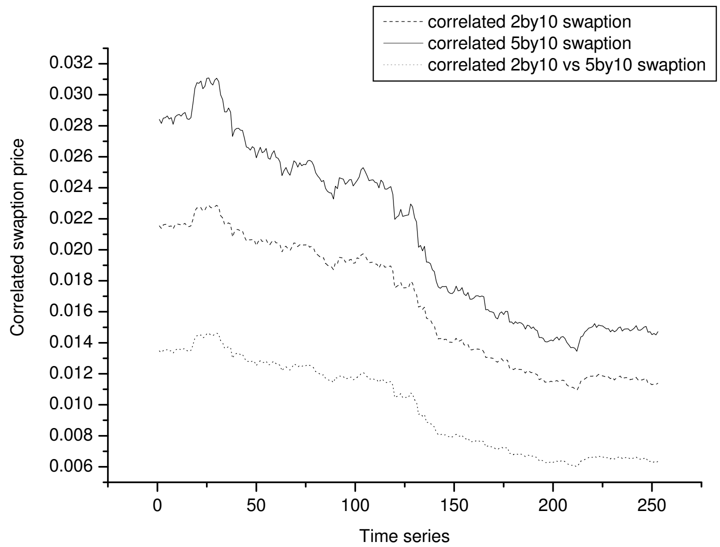

Figure 6.6: Correlated coupon bond options with martingale drift plotted versus time t_0 (15.6.2004-27.1.2005), computed from model.

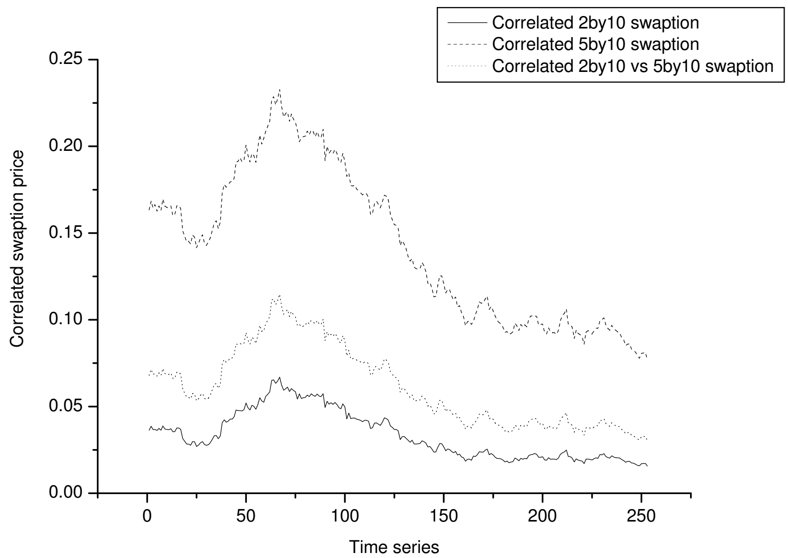

Figure 6.7: Correlated coupon bond options with market drift plotted versus time t_0 (15.6.2004-27.1.2005), computed from model.

Chapter 7

American Option Pricing for Interest Rate Caps and Coupon Bonds in Quantum Finance

American option for interest rate caps and coupon bonds are analyzed in the formalism of quantum finance. Calendar time and future time are discretized to yield a lattice field theory of interest rates that provides an efficient numerical algorithm for evaluating the price of American options. The algorithm is shown to hold over a wide range of strike prices and coupon rates. All the theoretical constraints that American options have to obey are shown to hold for the numerical prices of American interest rate caps and coupon bond options. Non-trivial correlation between the different interest rates are efficiently incorporated in the numerical algorithm. New inequalities are conjectured, bsed on the results of the numerical study, for American options on interest rate instruments.

§ 7.1 Introduction

American options for interest rate caps and for fixed income securities are amongst the most widely traded financial instruments. An accurate and arbitrage free pricing of American interest rate options has far reaching applications. American options for interest rates are rather complex instruments since at any moment in time, there are a large number of future interest rates that exist in the market. All of the interest rates evolve randomly and have strong correlations with the other interest rates. In principle, interest rate instruments, are described **at every instant** by infinitely many degrees of freedom (random variables).

In the simple case of a European option on equity, the Black-Scholes equation can be explicitly solved to obtain an analytical formula for the price of the option [42]. When one considers other financial derivatives that allow anticipated early exercise or depend on the history of the underlying assets, numerical approaches need to be used. Appropriate numerical procedures have been developed in the literature to price exotic financial derivatives on equity with path-dependent features, as discussed in detail in [42]. These procedures involve the use of Monte Carlo simulations, binomial tree (and their improvements) and finite difference methods.

The pricing of European and American options for the interest rates is far more complicated than for equity options. In order to price interest derivatives, one needs to model the underlying interest rates dynamics. The leading model at present for modelling interest rates and its derivatives is the HJM (Heath, Jarrow and Morton)-model; for the N-factor model, the interst rates at every instant are driven by N random variables [42], [83]. Numerical techniques for pricing American interest rates options [83] are all based on the generalization of the binomial tree approach.

Baaquie [6] has developed the formalism of quantum finance to model non-trivial correlations between forward interest rates with different maturities as a parsimonious alternative to the existing interest rate theories in finance, in particular to the HJM-model [83], [82]. Interest rate derivatives are shown in this paper to have a natural numerical algorithm for their pricing that directly follows from the quantum finance formulation of the forward rates.

In quantum finance, the stochastic forward interest rates are averaged over all their possible values to evaluate interest rate options and other derivatives; the averaging over the stochastic field is mathematically identical to the averaging in quantum field theory. In effect, from a mathematical point of view, the forward interest rate is a two dimensional (stochastic) quantum field. Hence in quantum finance one uses the techniques of quantum field theory for modelling the interest rates. An efficient algorithm is developed in this paper for obtaining the price of the American option using the formalism of quantum finance.

To price American options for equity an efficient computational algorithm, using path integrals, has been developed by Montagna and Nicrosini [38] and is reviewed in Appendix § 7.10. The quantum field theory describing the forward interest rates is discretized and yields a lattice field theory model; an algorithm that generalizes the path integral approach of [38] to the case of interest rate options is obtained using the lattice field theory.

§ 7.2 Field Theory Model of Forward Interest Rates

The field theory of forward rates is discussed in 2, however, since the value of American option is more convenient to be given by averaging the quantum field $f(t,x)$ over all it's possible values instead of the quantum field $\mathcal{A}(t,x) = [\dot{f}(t,x) - \alpha(t,x)]/\sigma(t,x)$. Thus the Lagrangian that describes the evolution of instantaneous forward interest rates is re-written as, and for $\partial f(t,x)/\partial t \equiv \dot{f}(t,x)$,

$$
\begin{aligned}
\mathcal{L}[f] &= -\frac{1}{2}\Big[\left\{\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\right\}^2 + \frac{1}{\mu^2}\left\{\frac{\partial}{\partial x}\left(\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\right)\right\}^2 \\
&+ \frac{1}{\lambda^4}\left\{\frac{\partial^2}{\partial x^2}\left(\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\right)\right\}^2\Big]
\end{aligned}
\tag{7.1}
$$

Then the action $S[f]$ of the Lagrangian is given as

$$
S = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_0^{\infty} dx \left(\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\right) D^{-1}(x,x',t)\left(\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\right) \tag{7.2}
$$

All expectation values, denoted by $E[..]$, are evaluated by integrating over all possible values of the quantum field $f(t,x)$. The quantum theory of the forward interest rates is defined by the generating (partition) function [6] for the following combination of the field $[\dot{f}(t,x) - \alpha(t,x)]/\sigma(t,x)$ since it is this combination that will appear in the American option. Hence the generating function is given by

$$
\begin{aligned}
Z[h] &= E\left[\exp\Big\{\int_{t_0}^{\infty} dt \int_0^{\infty} dx h(t,x)\big[\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\big]\Big\}\right] \\
&\equiv \frac{1}{Z}\int Df \; \exp\Big\{S[f] + \int_{t_0}^{\infty} dt \int_0^{\infty} dx h(t,x)\big[\frac{\dot{f}(t,x) - \alpha(t,x)}{\sigma(t,x)}\big]\Big\} \\
&= \exp\Big(\frac{1}{2}\int_{t_0}^{\infty} dt \int_0^{\infty} dx dx' h(t,x) D(x,x';t) h(t,x')\Big)
\end{aligned}
\tag{7.3}
$$

The prices of interest rates instruments are obtained by performing a path integral over all possible values of the (fluctuating) two dimensional quantum field $f(t,x)$, weighted by the probability measure e^S/Z. The expectation value for an instrument, say $F[f]$, is defined by the following

$$
E\big(F[f]\big) \equiv \frac{1}{Z}\int Df \; F[f] \; e^{S[f]} \;\; ; \;\; Z = \int Df e^{S[f]} \tag{7.4}
$$

§ 7.2.1 American Caplet and Coupon Bond Options

The American option has the same payoff function as the European option, but with the additional feature that it can be exercised at any time before the expiration day. To define the numerical algorithm the time interval $[t_0, t_*]$ is discretized; let $t_0 = 0$. It is assumed that an option can be exercised at any time before maturity but only at the discrete times. Since one has to evolve the payoff function backwards in time, for the numerical algorithm it is more convenient to label time **backwards**, with the origin of the time lattice being placed at t_*, the maturity of the payoff function. Hence define lattice time by $t_i = t_* - (i-1)\epsilon,\ i = 1, 2, ...M+1$, where $t_1 = t_*$; present time $t_0 = 0$ yields for lattice time $t_{M+1} = 0$ and hence fixes $M = t_*/\epsilon$. In other words the option can only be exercised at time $t_1 = t_*, t_2 = t_* - \epsilon, t_3 = t_* - 2\epsilon, \dots, t_M = t_* - (M-1)\epsilon$.

Let $C(t_i, t_*)$ denote the price of both caplet and coupon bond option, the third argument T in $Caplet(t_i, t_*, T)$ being suppressed. In the forward measure the ratio $C(t_i, t_*)/B(t_i, t_*)$ is a martingale. The initial trial value of the American option at earlier time t_i is given from the option price at time t_{i+1} by the martingale property as follows

$$\begin{aligned} g_I(t_{i+1}) = \frac{C(t_{i+1}, t_*)}{B(t_{i+1}, t_*)} &= E_F\left[\frac{C(t_i, t_*)}{B(t_i, t_*)}\right] \\ \Rightarrow g_I(t_{i+1}) &= E_F[g(t_i)] \end{aligned} \tag{7.5}$$

The subscript I in $g_I(t_{i+1})$ denotes the initial trial value of the American option at t_{i+1}. The trial option price is compared with the payoff function (divided by the appropriate numeraire) and yields the actual value of the option at time t_{i+1} being equal the maximum of the two [42].

Caplet

The scaled caplet payoff is given by

$$V\frac{F(t_i, t_*, T)\,(X - F(t_i, T, T+\ell))_+}{B(t_i, t_*)} \quad ; \quad F(t_i, T, T+\ell) = \exp\{-\int_T^{T+\ell} dx f(t_i, x)\} \tag{7.6}$$

The important point to note that the form of the payoff does not change with time; the discounting factor at t_* that appears in the payoff at maturity, namely the bond $B(t_*, T)$ is changed into the forward bond $F(t_i, t_*, T)$ as one moves to an intermediate time t_i; there is no additional discounting factor. The American option price at time t_{i+1} is equal to the maximum of the initial trial option value $g_I(t_{i+1})$ and the payoff function at time t_{i+1}; hence

$$\frac{C(t_{i+1},t_*)}{B(t_{i+1},t_*)} = g(t_{i+1}) = Max\Big[g_I(t_{i+1}), V\frac{F(t_{i+1},t_*,T)\,(X-F(t_{i+1},T,T+\ell))_+}{B(t_{i+1},t_*)}\Big] \tag{7.7}$$

Figure 7.1 shows the forward interest rates that define the caplet payoff function at different times t_*, t_i, t_0.

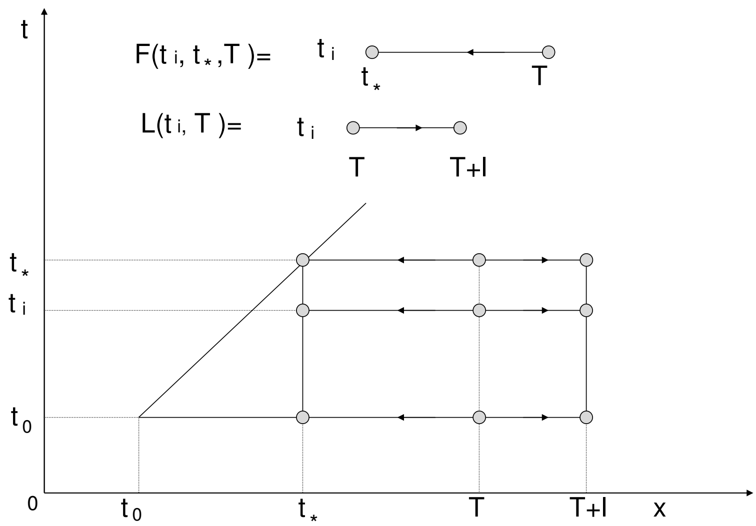

Figure 7.1: The (non-discounted) payoff function $VF(t_i,t_*,T)(X-F(t_i,T,T+\ell))_+/B(t_i,t_*)$ for the caplet at intermediate time $t_i \in [t_0,t_*]$.

Coupon Bond

The scaled coupon bond payoff is given by

$$\frac{\big(\sum_{j=1}^{N} c_j F(t_i,t_*,T_j) - K\big)_+}{B(t_i,t_*)} \quad ; \quad F(t_i,t_*,T_j) = \exp\{-\int_{t_*}^{T_j} dx f(t_i,x)\} \tag{7.8}$$

where, as is the case for the interest rate caplet, in the payoff function at intermediate time $t_i \in [t_0,t_*]$ the bond price at time t_*, namely $B(t_*,T_j)$ has been replaced, at time t_i, by the forward bond price $F(t_i,t_*,T_j)$. The American option price at time t_{i+1} is given by

$$\frac{C(t_{i+1},t_*)}{B(t_{i+1},t_*)} = g(t_{i+1}) = Max\Big[g_I(t_{i+1}), \frac{\big(\sum_{j=1}^{N} c_j F(t_{i+1},t_*,T_j) - K\big)_+}{B(t_{i+1},t_*)}\Big] \tag{7.9}$$

Note the important fact that for both the caplet and coupon bond, the payoff function at each time t_i is identical to the form of the payoff function at maturity time t_*; in particular, the payoff function is not discounted when it is compared to the trial value of the option $g_I(t_i)$.

§ 7.3 Lattice Field Theory of Interest Rates

The field theory of forward interest rates is defined on the semi-infinite continuous xt plane. To obtain a numerical algorithm the xt plane needs to be discretized into a lattice consisting of a finite number of points. The time direction, as mentioned earlier, is discretized into a lattice with spacing ϵ and future time direction x is also discretized into a lattice with spacing a.

Recall from eqs.(7.1) and (7.2) that the action for continuous time and future time is given

$$S = -\frac{1}{2}\int dt \int dx \Big[(\frac{\dot{f}-\alpha}{\sigma})^2 + \frac{1}{\mu^2}(\frac{\partial}{\partial x}(\frac{\dot{f}-\alpha}{\sigma}))^2 + \frac{1}{\lambda^4}(\frac{\partial^2}{\partial x^2}(\frac{\dot{f}-\alpha}{\sigma}))^2\Big] \tag{7.10}$$

The time lattice is defined as discussed in Section § 7.2.1, and for simplicity of notation $t_0 = 0$. Future time, similar to calendar time, is labelled running **backwards**, with the origin of future time being placed at the payoff function. At maturity t_*, future time is taken to have N lattice points corresponding to N forward rates; given the trapezoidal shape of the forward rates domain defined by $x \geq t$, the range of discretized x_j depends on discretized time t_i. Hence, for $t \in [0, t_*]$, the discretized calendar and future time are given by

$$t \to t_i = t_* - (i-1)\epsilon \;\; ; \;\; i = 1, 2, ..., M+1 \;\; ; \;\; M = \frac{t_*}{\epsilon} \tag{7.11}$$

$$t_i \;:\; x \to x_j = T - (j-1)a \;\; ; \;\; j = 1, 2, ..., N+i \;\; ; \;\; N = \frac{T - t_*}{a}$$

The total number of lattice sites is equal to $N(M+1)+M(M+1)/2$. Note for most numerical calculations one usually takes $\epsilon = a$.

Figure 7.2 shows the lattice on which the forward interest rates are defined.

To define the lattice theory, one needs to rescale the field $f(t, x)$ and all the parameters so that all the quantities that appear in the theory are dimensionless. For this reason, define the following dimensionless lattice quantities

$$\begin{aligned} f_{mn} &= af(t, x) = af(t_* - (m-1)\epsilon, T - (n-1)a) \\ \tilde{\alpha}_{mn} &= a\epsilon\alpha(t, x) \;\; ; \;\; s_{mn} = \sqrt{\epsilon a}\sigma(t, x) \\ \tilde{\mu} &= a\mu \;\; ; \;\; \tilde{\lambda} = a\lambda \end{aligned}$$

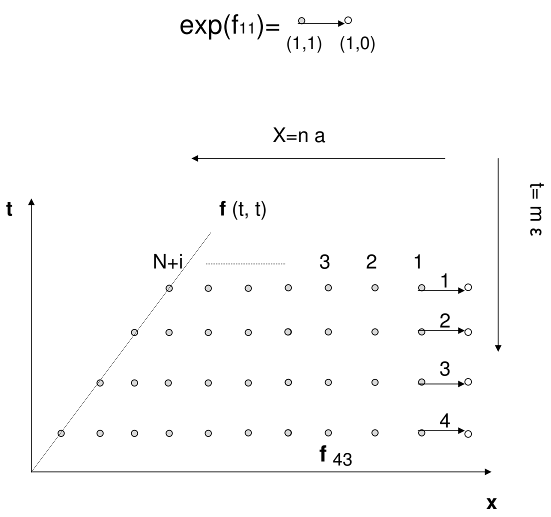

Figure 7.2: Forward interest rates on the lattice, with each dot representing a forward rate $f(t,x) \to f_{ij}$; an arrow indicates that a forward rate $e^{f_{ij}}$ connects lattice (ij) to $(i, j+1)$. The lattice points take values in the range $i = 1, 2, ..., M+1$ and $j = 1, 2, ..., N+i$.

The dimensionless field variables f_{mn} yield the following discretizations

$$\dot{f} = \frac{1}{a\epsilon}(f_{m,n} - f_{m+1,n}) \equiv \frac{1}{a\epsilon}\delta_t f_{mn} \tag{7.12}$$

$$\partial_x f = \frac{1}{a^2}(f_{m,n} - f_{m,n+1}) \equiv \frac{1}{a^2}\delta_x f_{mn} \tag{7.13}$$

Thus, from eq. 7.10, one obtains the lattice action S, expressed completely in terms of dimensionless field variables and parameters and is given by

$$S_L = -\frac{1}{2}\sum_{m,n}\left[(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}})^2 + \frac{1}{\tilde{\mu}^2}(\delta_x(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}}))^2 + \frac{1}{\tilde{\lambda}^4}(\delta_x^2(\frac{\delta_t f - \tilde{\alpha}}{s_{mn}}))^2\right] \tag{7.14}$$

Doing an integration by parts, the action in eq. 7.14 yields

$$S_L = -\frac{1}{2}\sum_{m=1}^{M+1}\sum_{n=1}^{N+m}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn}\tilde{D}^{-1}_{m,nn'}\left(\frac{\delta_t f - \tilde{\alpha}}{s}\right)_{mn'} \tag{7.15}$$

where $\tilde{D}^{-1}_{m,nn'}$ is the dimensionless inverse of the propagagtor with dimensionless parameters

$\tilde{\mu}, \tilde{\lambda}$. The dimensionless lattice Lagrangian is given from S by the following [6]

$$\begin{aligned} S_L &= \sum_{m=1}^{M} \mathcal{L}[\mathbf{f}_{m+1}; \mathbf{f}_m] \\ \mathcal{L}[\mathbf{f}_{m+1}; \mathbf{f}_m] &= -\frac{1}{2} \sum_{n,n'=1}^{N+m} \left(\frac{\delta_t f - \tilde{\alpha}}{s} \right)_{mn} \tilde{D}^{-1}_{m,nn'} \left(\frac{\delta_t f - \tilde{\alpha}}{s} \right)_{mn'} \end{aligned} \tag{7.16}$$

where $\mathbf{f}_m = (f_{m1}, f_{m2}, ..., f_{m,N+m})$. Note the length of the forward interest rate vector $\mathbf{f}_m$ depends on the time lattice m and has $N+m$-components; the reason being that forward interest rates are defined for all $x \geq t$, which on the lattice implies that the number of forward rates for a given t_i depends on t_i.

The functional integral is discretized and yields the lattice field theory of forward interest rates given by

$$Z = \int Dfe^S \to Z_L = \tilde{\mathcal{N}} \prod_{m=1}^{M+1} \prod_{n=1}^{N+m} \int_{-\infty}^{+\infty} df_{mn} e^{S_L} \tag{7.17}$$

with normalization $\tilde{\mathcal{N}}$.

The American option for the caplet is formulated on the lattice. For starters discretize the payoff function of caplet in eqn7.6; at maturity time t_* the discretized caplet is denoted by $C_1 \equiv C(t_*, t_*, T)$.

On the lattice, Libor is given by

$$1 + \ell L(t,T) = \exp\{\int_T^{T+\ell} dx f(t,x)\} \simeq \exp\{\frac{\ell}{a} f_{m1}\} = \exp\{f_{m1}\} \tag{7.18}$$

where, for simplicity and because Libor data is given only on a future time lattice with spacing ℓ, one takes $\ell = a$. Hence

$$\begin{aligned} Caplet(t_*, t_*, T) &= \tilde{V} B(t_*, T)(X - F_*)_+ \equiv C_1 \\ \Rightarrow C_1 &= \tilde{V} \exp\{-\sum_{j=1}^{N+1} f_{1j}\}(X - e^{-f_{11}})_+ \\ &= C_1(f_{11}, f_{12}, \ldots f_{1,N+1}) = C_1(\mathbf{f}_1) \end{aligned} \tag{7.19}$$

The payoff function is evolved backwards in time to obtain the price of the option from the payoff function. To illustrate the general procedure, consider the first step backwards; one starts from the payoff function at time $t_* = t_1$ and find the value of the option at time t_2, since recall increasing the index of lattice time ones goes backwards in time. In taking one step backwards in time, the number of independent forward interest rates on the lattice increases by one rate, starting from forward rates $\mathbf{f}_i = (f_{i,1}, f_{i,2}, \dots f_{i,N+i})$ and ending up with $\mathbf{f}_{i+1} = (f_{i+1,1}, f_{i+1,2}, \dots f_{i+1,i}, f_{i+1,N+i+1})$. Hence the option price given by $C(\mathbf{f}_i) = C_i(f_{i,1}, f_{i,2}, \dots f_{i,N+i})$ evolves to $C(\mathbf{f}_{i+1}) = C_{i+1}(f_{i+1,1}, f_{i+1,2}, \dots f_{i+1,N+i+1})$, where sometimes the notation $C_i \equiv C_i(f_{i,1}, f_{i,2}, \dots f_{i,N+i})$ is used.

The expression $\mathcal{N}\exp\{\mathcal{L}[\mathbf{f}_{i+1}, \mathbf{f}_i]\}$ is the **pricing kernel** for interest rate options, analogous to the pricing kernel for the (simpler) case of equity given in eq. 7.44. Similar to eq. 7.47 for equity, the pricing kernel propagates the option price $C_i(\mathbf{f}_i)$ backwards in time and yields the option price $C_{i+1}(\mathbf{f}_{i+1})$ at earlier time t_{i+1}.

From eq. 2.30, the convention being used for future lattice time is that for all t_i, $T \to x_1$, that is, the minimum value of the future lattice index x_n; hence the zero coupon bond in the payoff function is given by $B(t_i, T) \to \exp\{-\sum_{j=1}^{N+i} f_{ij}\}$ and similarly for $B(t_{i+1}, 1)$.

The pricing kernel yields, similar to eq. 7.47 for the case of equity options, the option price C_{i+1} at earlier time t_{i+1} from option price C_i by taking one step backward in time, and generates the initial trial value for the option C_{i+1}. Hence, eq. 2.30 yields the following result [6][1]

$$\frac{C_{i+1}}{B(t_{i+1},1)} \equiv \frac{C_{i+1}(\mathbf{f}_{i+1})}{B(t_{i+1},1)} = \mathcal{N}\int d\mathbf{f}_i \; e^{\mathcal{L}[\mathbf{f}_{i+1},\mathbf{f}_i]} \; \frac{C_i(\mathbf{f}_i)}{B(t_i,1)} \tag{7.20}$$

$$= \mathcal{N}\prod_{p=1}^{N+i} \int df_{ip} \; \exp\left(-\frac{1}{2}\sum_{j,k=1}^{N+i}\left(\frac{f_{i,j}-\bar{f}_{i+1,j}}{s_{ij}}\right) D^{-1}_{i;jk}\left(\frac{f_{i,k}-\bar{f}_{i+1,k}}{s_{ik}}\right)\right)\frac{C_i}{B(t_i,1)} \tag{7.21}$$

$$\text{where} \quad \bar{f}_{i+1,j} \doteq f_{i+1,j} + \tilde{\alpha}_{i+1,j} \tag{7.22}$$

Similar to the case of American option for an equity discussed in Appendix § 7.10, the interest rate dimensionless volatility s_{mn} is quite small, that is $s_{mn} \simeq 0$. Hence in the f_{ij} integrations given in eq. 7.21, the path integral will be dominated by values $f_{i,j}$ that are close to $\bar{f}_{i+1,j} = f_{i+1,j} + \tilde{\alpha}_{i+1,j}$. The most accurate way for evaluating the functional integral in eq. 7.21 is to Taylor expand the function $g_i = C_i/B_{i,1}$ about $\bar{f}_{i+1,j}$. A Taylor's expansion of the kernel

[1] In terms of the Hamiltonian H of the forward interest rates the option price C_{i+1} is given by

$$\frac{C_{i+1}}{B_{i+1,1}} = \prod_{j=1}^{N+i}\int df_{ij}\langle f_{i+1,1}, f_{i+1,2}, \dots f_{i+1,N+i}|e^{-\epsilon H}|f_{i,1}, f_{i,2}\dots f_{i,N+i}\rangle\frac{C_i}{B_{i,1}}$$

See [6] for more details.

function g_i around $\bar{f}_{i+1,1}, \ldots, \bar{f}_{i+1,i}$ yields, for $\bar{g}_i \equiv g_i(\bar{f}_{i+1,j})$, the following

$$g_i = \bar{g}_i + \sum_{j=1}^{N+i} \frac{\partial \bar{g}_i}{\partial f_{ij}}(f_{ij} - \bar{f}_{i+1,j}) + \frac{1}{2} \sum_{j,k=1}^{N+i} \frac{\partial^2 \bar{g}_i}{\partial f_j \partial f_k}(f_{ij} - \bar{f}_{i+1,j})(f_{ik} - \bar{f}_{i+1,k}) + \ldots \tag{7.23}$$

The Gaussian integrations over all the forward rates f_{ij} is carried out using the following results

$$\mathcal{N} \prod_{p=1}^{N+i} \int df_{ip} \;\; e^{\mathcal{L}[\mathbf{f}_{i+1}, \mathbf{f}_i]} = 1 \;\; ; \;\; \mathcal{N} \prod_{p=1}^{N+i} \int df_{ip} \;\; e^{\mathcal{L}[\mathbf{f}_{i+1}, \mathbf{f}_i]}(f_{ij} - \bar{f}_{i+1,j}) = 0$$

$$\mathcal{N} \prod_{p=1}^{N+i} \int df_{ip} \;\; e^{\mathcal{L}[\mathbf{f}_{i+1}, \mathbf{f}_i]}(f_{ij} - \bar{f}_{i+1,j})(f_{ik} - \bar{f}_{i+1,k}) = s_{ij} \tilde{D}_{i;jk} s_{ik} \tag{7.24}$$

Hence, eqs. 7.21 and 7.24 yield the result that

$$C_{i+1} = B_{i+1,1} \left[\bar{g}_i + \frac{1}{2} \sum_{j,k=1}^{N+i} \frac{\partial^2 \bar{g}_i}{\partial f_{i,j} \partial f_{ik}} s_{i,j} \tilde{D}_{i;jk} s_{ik} \right] + \ldots \tag{7.25}$$

Recall the function $g_i = g_i[\mathbf{f}_i]$ depends on the vector $\mathbf{f}_i = (f_{i,1}, f_{i,2}, \ldots f_{i,N+i})$. The value of $\bar{g}_i = \bar{g}_i[\bar{f}_{i+1,j}]$, that is, on the entire forward rate tree at time t_{i+1}. Since $g_i[\mathbf{f}_i]$ is being differentiated with respect to only two of the components, namely $f_{i,j}, f_{i,k}$, only these (two) components will be explicitly indicated, with the rest of the components in $g_i[\mathbf{f}_i]$ being suppressed.

The second derivative of g_i is numerically estimated using the symmetric second order difference. The spacing with δ is taken to be $O(s)$, as dictated by the Lagrangian in eq. 7.21. The symmetric second order derivative, using $\delta^i_{\pm} g(f_i) = g(f_i \pm \delta)/\delta$, yields the following discretization

$$\frac{\partial^2 \bar{g}_i}{\partial f_{i,j} \partial f_{i,k}} \equiv \frac{\partial^2 g_i}{\partial f_{i,j} \partial f_{i,k}}|_{f_{i,j} = \bar{f}_{i+1,j}} = \frac{1}{2}(\delta^i_- \delta^j_+ + \delta^i_+ \delta^j_-) g_i|_{f_{i,j} = \bar{f}_{i+1,j}} \tag{7.26}$$

$$= \frac{1}{2\delta^2} \left[g_i(\bar{f}_{i+1,j} + \delta, \bar{f}_{i+1,k}) - 2g_i(\bar{f}_{i+1,j}, \bar{f}_{i+1,k}) - g_i(\bar{f}_{i+1,j} + \delta, \bar{f}_{i+1,k} - \delta) \right.$$

$$\left. + g(\bar{f}_{i+1,j}, \bar{f}_{i+1,k} - \delta) + g(\bar{f}_{i+1,j}, \bar{f}_{i+1,k} + \delta) - g_i(\bar{f}_{i+1,j} - \delta, \bar{f}_{i+1,k} + \delta) + g_i(\bar{f}_{i+1,j} - \delta, \bar{f}_{i+1,k}) \right]$$

The result in eq. 7.26 above has a very significant feature. To evaluate $\partial^2 \bar{g}_i / \partial f_{ij} \partial f_{ik}$ one needs to know the values of g_i at the points $f_{ij} = \bar{f}_{i+1,j}$, for all f_{ij} , $j = 1, 2, .., N + i$. Moreover, as required by eq. 7.26, the forward rate tree at time t_{i+1} must also contain the following three points, namely $\bar{f}_{i+1,j}, \bar{f}_{i+1,j} \pm \delta$. This feature of the recursion equation is a

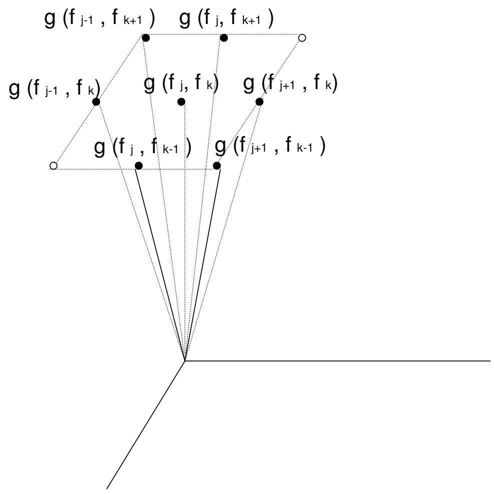

Figure 7.3: The approximation for $\partial^2 \bar{g}_i / \partial f_{ij} \partial f_{ik}$ is computed from seven values of the function g_i indicated in the diagram; the time index i has suppressed in figure for greater clarity.

reflection of a similar property for the case of the American option for equity as in eq. 7.52 and shown in Figure 7.20.

Figure 7.3 is a graphical representation of the seven terms required for the computation of $\partial^2 \bar{g}_i / \partial f_{i,j} \partial f_{i,k}$.

The price of the European option for caplets and coupon bonds can be obtained by repeating the backward recursion up to the present time. In Section § 7.6 and § 7.7 the numerical price[2] of the European option will be compared with those computed from the closed form result in § 2.4 and chapter § 2.5 to assess the accuracy of the algorithm. For American option, one needs to perform, for **each** step up to present time, a comparison of the trial option price with the payoff as given in eq. 7.7 and 7.9.

[2]In the numerical studies of both the caplets and coupon bond options the midcurve option will not be priced; instead, for simplicity, only options that mature when the instrument becomes operational are studied. It should be noted that all the numerical procedures used in this paper can be generalized in a straight forward manner to the midcurve case.

§ 7.4 Tree Structure of Forward Interest Rates

To minimize the computational complexity and the time of execution, it is mandatory to limit as far as possible the number of possible forward rates for which the option price is computed.

The forward bond measure for the forward interest rates is chosen for Libor instruments and is defined only for future Libor time lattice. The numeraire used for discounting all financial instruments is equal to $B(t, T_n)$ for $T_n \leq x < T_{n+\ell}$. The numerair makes all forward bond prices on the Libor future time lattice martingales[9]; the definiton of the martingale for this numeraire, together with the forward drift, yields the following result

$$e^{-\int_{T_n}^{T_n+\ell} f(t_0,x)} \equiv F(t_0, T_n, T_{n+\ell}) = E_F[F(t_*, T_n, T_{n+\ell})]$$
$$\Rightarrow \alpha(t, x) = \sigma(t, x) \int_{T_n}^{x} dx' D(x, x'; t)\sigma(t, x') \quad ; \quad T_n \leq x < T_n + \ell$$

For the American option, future time is discretized as $x = na$ and Libor interval $\ell = a$; hence $T_n = n\ell = na$ lies on the future time lattice with $x_n = T_n$. This in turn, from above equation, yields

$$\alpha(t, x_n) = 0 \qquad ; \quad x_n = T_n \tag{7.27}$$

and there is no drift for the lattice on which the American option is being computed. In fact, this simplification is the main reason for taking $\ell = a$.

The option price is fixed by, among other parameters, the initial forward interest rate curve $f(t_0, x)$. The option price for only those intermediate (virtual) values of the forward rates need to be considered that contribute to the final option price. For a given initial forward interest rates curve, what this means is that the option price needs to evaluated only for those values of the forward rates that lie on a **tree** or **grid**.

In order to ascertain the forward interest rates grid, it is necessary to start from the initial forward interest rate curve, which from eq. 7.11 is given by $f(t_0, x) \rightarrow f_{M+1,n}$, $n = 1, 2, ..N+M+1$. Similar to the case for the American equity option discussed in the Appendix, **each** initial value of the forward rates generates an independent tree. Recall that, starting from the initial forward rates $f_{M+1,n}$, to reach the forward interest rates at calendar time m, with forward rates f_{mn}, one needs to take $M+1-m$ steps. Hence the structure of the forward rate tree is given by

$$f_{mn}^{k} \doteq f_{M+1,n} + k\delta, \qquad -(M + 1 - m) \leq k \leq +(M + 1 - m) \tag{7.28}$$

At lattice time m the forward rate tree has $2(M + 1 - m) - 1 = 2(M - m) + 1$ number of values for f_{mn}, centered on the initial forward rate $f_{M+1,n}$. The spacing of the tree is taken to

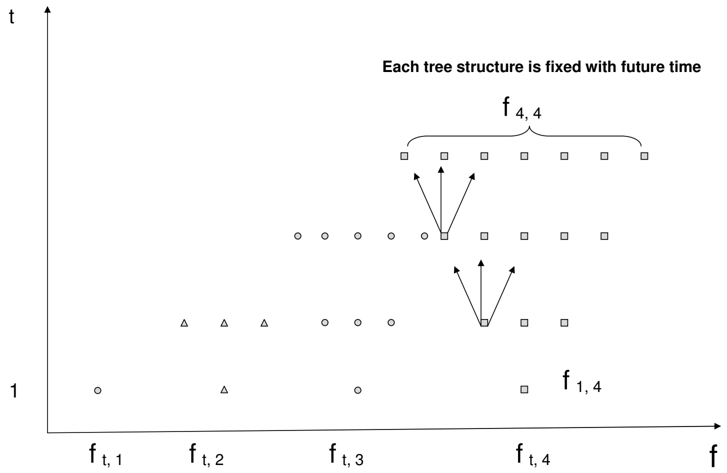

Figure 7.4: Forward rates tree structures. Each symbol at bottom (a hollow circle, square and triangle) represents a forward rate f_{mn} at present time. Each initial value of the forward rates evolves as independent tree, with rates evolving from the same initial rate shown with the same symbol. Note for the tree of forward rates, those f_{mn} that emanate from different initial forward rates are independent of each other, and this is indicated by different symbols.

have a **fixed** value δ, which is of $O(s)$. A fixed value of δ is required to obtain a re-combining tree; conversely, if δ is taken to vary with time, one gets a dense tree with exponentially more points than the recombining tree [82].

The same result as given in eq. 7.28 is obtained if one recurses backwards from the payoff at calendar time t_* to the initial forward curve at t_0. The reason being that for zero drift, that is $\alpha(t, x) = 0$, one has from eq. 7.26 that the values of the function g_i at with forward rates on the grid $f_{i,j} = f_{i+1,j}, f_{i+1,j} \pm \delta$ are required to obtain the value of g_{i+1}; if one recurses backwards $M + 1 - m$ times, one hits the initial forward rate curve and in effect obtains the result given in eq. 7.28.

The forward interest rate tree is illustrated in Figure 7.4. Compared with Figure 7.2, the forward rate at a lattice point (with fixed time t and future time x) has been expanded into a tree structure in a direction orthogonal to the xt lattice. The full forward interest rates simultaneously have infinitely many tree structures and all these tree structures are correlated by the action S given in eq. 7.14.

Thus, as one recurses backwards from the payoff function, at any intermediate time the price of the option needs to determined for the forward interest rates only on the grid points. From 7.7, 7.9 and 7.26, the initial trial values $g_I(i)$ that one needs are the option values from the previous step, with the values of the forward rates taken only from the tree structure for lattice time t_i.

§ 7.5 Numerical Algorithm

Given the initial forward interest rates curve, the tree structure from Section § 7.4 can be formed and yields the grid of the forward interest rates up to the expiration date of the option. To get the option price today, one starts from expiration time t_* and evolves backwards in calendar time. All computations are carried out using the lattice theory given in eq. 7.17.[3]

Lattice points are labelled by i, j with i labelling calendar time and j labelling future time. Maturity time t_* is labelled by t_1, that is $i = 1$, as shown in Figure 7.2. The scaled payoff function g_1 is, in general, a function of all forward interest rates f_{1j} with future time taking $N + 1$ values, as shown in Figure 7.2, being labelled as $j = 1, 2, ..N + 1$. Thus, g_1, which depends on N forward rates, can be represented as $g(f_{11}, \dots f_{1i}, \dots f_{1N+1})$.

Since the step size ϵ is fixed, the total number of steps $M = t_*/\epsilon$ for the evolutions backwards in time is consequenlty also fixed. At time t_i the number of tree points for each forward interest rate is given by $2(M - (i - 1)) + 1 = 2(M - i) + 3$; there are $N + i$ number of independent forward rates. Since each forward rate tree has $2(M - i) + 3$ points, this leads to the total number of points of the tree at time i being given by $(2(M - i) + 3)^{N+i}$. The tree is organized into a multidimensional array,[4] namely $g[2(M - i) + 3]_{N+i}$, which is a $N + i$ dimensional array with each index running from $1, 2, .., 2(M - i) + 3$.

The size of the multidimensional array for realistic cases can be very big, requiring a large amount of computer memory and leading to codes for the American option that are inefficient. In order to develop an efficient algorithm the multidimensional array is mapped into a **vector array** with a length of $(2(M - i) + 3)^{N+i}$. The multidimensional matrix representation of the tree has some advantages since the index of each forward interest rates is presented explicitly. Hence in the recursive steps required for evaluating eqs. 7.7, 7.9 and 7.26, the matrix representation is the most transparent way of keeping track of the grid points from the previous step that are required for deriving the trial option price for the present step.

[3] Note in this Section $t_0 = 0$

[4] $g[M]_N \equiv g\underbrace{[M][M]\dots[M]}_{N}$.

To go from the matrix representation to the vector array one needs an algorithm for mapping the indices of the matrix to the index of the array; in particular the multidimensional matrix array $g[j_{N+i}]\dots[j_p]\dots[j_2][j_1]$ needs to be mapped into a vector array $g[j]$. For time i let the matrix indices $[j_{N+i}]\dots[j_p]\dots[j_2][j_1]$ be assigned specific numerical values; the corresponding vector index j is given by the following mapping.

$$\begin{aligned} & g[j_{N+i}]\dots[j_p]\dots[j_2][j_1] = g[j] \\ & [j_{N+i}]\dots[j_p]\dots[j_2][j_1] \rightarrow j = \sum_{p=1}^{N+i}(j_p-1)[2(M-(i-1))+1]^{p-1} \end{aligned} \tag{7.29}$$

One should note that the matrix representation is never used in writing the codes for this algorithm. The vector array is used for avoiding the use of the multidimensional matrix; only the indices of the matrix are needed as intermediate step to address grid points in the recursion process.

To find the grid points, in particular those that are nearest neighbor and next nearest neighbor as required in evaluating eq. 7.26 one needs the mapping in the reverse direction. The mapping from the vector array index to matrix indices is given by the following. The notation used is that the vector index j is recursively updated to $j^{(1)}, j^{(2)}, ..., j^{(p)}, ...j^{(N+i)}$; recall the notation $j_1, j_2, ...j_p, ...j_{N+i}$ are the indices labelling the multidimensional matrix. The inverse mapping is a composed of a two-step algorithm, with the matrix index j_p being determined and the vector index j being updated to $j^{(p)}$. More precisely, the following is the mapping.

$$\begin{aligned} & g[j] \rightarrow g[j_{N+i}]\dots[j_p]\dots[j_2][j_1] \\ & \text{where, using modular arithmetic yields} \\ & \begin{cases} j_{N+i} = \text{Integer}[(j-1)/(2(M-i)+3)^{N+i-1}]+1 \\ j^{(1)} = j - \text{Integer}[j/(2(M-i)+3)^{N+i-1}] \end{cases} \\ & \vdots \\ & \begin{cases} j_p = \text{Integer}[(j^{(p-1)}-1)/(2(M-i)+3)^{p-1}]+1 \\ j^{(p)} = j^{(p-1)} - \text{Integer}[j^{(p-1)}/(2(M-i)+3)^{p-1}] \end{cases} \\ & \vdots \\ & \begin{cases} j_1 = j^{(N+i-1)} \\ j^{(N+i)} = j^{(N+i-1)} - j^{(N+i-1)} = 0 \end{cases} \end{aligned} \tag{7.30}$$

Note the inverse mapping stops after $N+i$ steps, as indeed it must as this is the total number of forward interest rates at calendar time i. The inverse map returns all the $N+i$ indices j_p of the matrix representation $g[j_{N+i}]\dots[j_2][j_1]$ from the vector index j of the vector array $g[j]$.

In summary, at step i, the final values of option prices are evaluated and stored in a vector array $g_i[(2(M-i)+3)^{N+i}]$. Evolving one step back from t_i to t_{i+1}, in order to evaluate the trial value of $g_{i+1}[j]$, one needs to first map the vector index j to matrix indices $j_p,\ p = 1, 2 \ldots N+i$ using eq. 7.30. The matrix indices are needed for tracking those option prices at step i that are required for deriving $g_{i+1}[j]$ on the grid points. Then, one reverts back to the vector array index from these matrix indices by eq. 7.29, and furthermore obtain the corresponding option values at step i. The recursion process in 7.7, 7.9 and 7.26 is then performed to obtain the trial value of $g_{i+1}[j]$.

Completing one recursion step results in trial values $g_I[(2(M-i)+3]_{N+i}$ for the $(i+1)$th step. The grid points are **dynamic** in nature since one more forward interest rate, namely $f_{i+1,N+i+1}$ has to be **added** at $(i+1)$th step, as shown in Fig.7.2. The dimension of the matrix of trial values has to be increased from $N+i$ to $N+i+1$ dimensions, that is $g[j_{N+i}]\ldots[j_1] \rightarrow g[j_{N+i+1}][j_{N+i}]\ldots[j_1]$. The new forward interest rate does not directly influence the option values, but only through the scaling function $B(t_{i+1}, T)$.

The expanded matrix has to be assigned numerical values for the new index j_{N+i+1} in the range of 1 to $2(M-((i+1)-1))+1 = 2(M-i)+1$. The way this is done is to make the values of the expanded matrix **independent** of the new forward rate; in other words, the following assignment is made for the initial trial option price

$$g[j_{N+i+1}][j_{N+i}]\ldots[j_1] \equiv g[j_{N+i}]\ldots[j_1]\ ;\ 1 \leq j_{N+i+1} \leq (2(M-i)+1) \tag{7.31}$$

For vector array representation, j now takes additional values from $[2(M-i)+1]^{N+i}+1$ to $[2(M-i)+1]^{N+i+1}$. The option values of the vector array for the new values of j, similar to eq. 7.31, are made independent of the new value of j; hence

$$g[j] \equiv g[j-[2(M-i)+1]^{N+i)}]\quad ;\quad 2(M-i)+1]^{N+i}+1 \leq j \leq [2(M-i)+1]^{N+i+1} \tag{7.32}$$

The mapping in eq. 7.32, in going from the left hand to the right hand side of the equation **shifts**, by a constant, the argument of the new index j and thus bringing it back into the old range; as j runs through the (additional) new values, the expanded vector takes values from the old array that are a constant shift from the new values of j; it can be seen by inspection that eq. 7.32 assigns values to the expanded vector array consistent with the labelling of the new matrix elements given in eq. 7.31.

The trial option value is compared with the payoff value at $(i+1)$th step. The higher value is retained at the $(i+1)$th step as the final value of $g[j_{N+i+1}][j_{N+i}]\ldots[j_1]$, which is the American option at the grid points.

The price of the American option at present is obtained by repeating the recursion until $i = M + 1$.The European option price can also be derived following the same algorithm, but without making the comparison with payoff value at each step of the recursion.[5]

To initiate the numerical algorithm the initial forward rates curve and all the parameters in the lattice Lagrangian have to be specified. The numerical algorithm is given as follows.

- Input initial forward rates curve and parameters.
- Generate the payoff for maturity time $i = 1$ and store in (both) the vector arrays $Geuro_{old}$ and $Gamerican_{old}$ for European and American option.
- For $i = 2$ to $M + 1$
  1. Recurse one step back from $Geuro_{old}$ and $Gamerican_{old}$ to get trial values $Geuro_{new}$ and $Gamerican_{new}$ using eqs. 7.7, 7.9 and 7.26.
  2. Expand $Geuro_{new}$ and $Gamerican_{new}$ from $N + i - 1$ to $N + i$ dimension so as to address the dynamics of grid using eq. 7.32.
  3. Compute the payoff value at step i without discounting and store result in $Gamerican_{old}$.
  4. Compare $Gamerican_{old}$ with $Gamerican_{new}$ and store the larger one in $Gamerican_{old}$. Replace values in $Geuro_{old}$ with values in $Geuro_{new}$.
  5. End for.

The vector array has to be used to assign option values for each point in the forward rate grid. The length of the vector array can be vary large, and frequently addressing the elements of the array may cause problem of over the stack or even giving wrong values. However, programming languages have a feature of dynamicaly addressing the the location of the vector array, which helps to avoid these problems.

Note that the first order of the recursion contributes significantly to the final vaule. Furthermore the drift in forward bond measure is zero at 3 monthly lattice of points. The tree structure has to be enough wide to include information about the changes in the value of the forward rates. One has the freedom of increasing the width of the tree by setting the prefactor of $\delta = O(s)$. Since σ that is being used is the volatility for a one day change of forward interest rate, to obtain s the real days in each step must be multiplied into it.

[5]The European option price is always evaluated (at the same time as the Americna option) for carrying out consistency checks on the numerical results.

The above numerical algorithm yields only one value for both American option and European option; in order to get values for a time series or values depending on different values of the various parameters, the entire algorithm needs to be repeated.

§ 7.6 Numerical Results for Caplets

A caplet on Libor and maturing when the caplet becomes operational was analyzed. The initial forward interest rate curve as well as the volatility function was taken from the Libor market; the propagator $\tilde{D}_{i,jk}$ is assigned numerical values taken from caplet data [14].[6] For simplicity, take the time lattice $\epsilon = 3$ months. The present time for the caplet is taken to be from 12th Sept 2003 to 7th may 2004, with maturity at fixed time in the 12th Dec 2004. For early exercise the American option on the caplet can expire at **five** fixed times.

For M time steps and $N+1$ forward rates in first step, at step i there are $Q = (2(M-i)+3)^{N+i}$ option prices that need to be determined. The total number of option prices for the whole algorithm is $\mathcal{O} = \sum_{i=1}^{M+1}(2(M-i)+3)^{N+i}$. Thus for a caplet at 12th Sept 2003, $N = 0$ and $M = 5$ (since $M = t_*/\epsilon$ and $\epsilon = 3$ month), the number of option prices that need to be determined is $\mathcal{O} = 1,304$. The total number of option prices $\mathcal{O}$ increases rapidly with increasing M, with $\mathcal{O} = 14,758,719$ for $M = 10$.

The caplet tree of the (relevant) forward rates is built with $\delta = 2s$; all computations are carried out only for the values of the forward rates taking values in the tree. Caplet volatility is taken from the market by moving average on the historical data, and at 12th Sept 2003 is given in Figure 7.21 [14].

In [14], the daily price from 12th Sept 2003 to 7th May 2004 of the option on Eurodollar futures contracts expiring 13 Dec 2004 with a strike price 98 were computed. It was shown the field theory pricing formula is fairly accurately; the same instrument is studied numerically using the lattice field theory of interest rates.

Both European and American options are computed and the European results are used to check the accuracy of the algorithm by comparing the numerical results with results from closed form pricing formula for European options.

The American option can be exercised at any time before it's expiry day, which means one should set ϵ to be very small and N to be very big; doing so would require a huge memory and very long time to run the program since the possible option values Q for each step would

[6]Market data was used for pricing caplets to demonstrate the flexibility of the numerical algorithm.

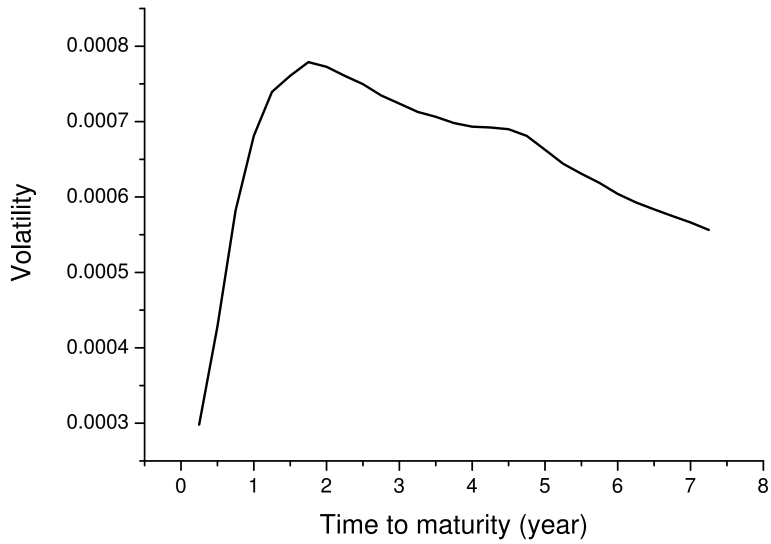

Figure 7.5: Forward interest rates volatility $\sigma(\theta)$, with dimension year^{-1}, on 12th Sept 2003 versus maturity of time. $\sigma(\theta)^2 = \sqrt{\langle \delta f^2(t,\theta) \rangle_c}$; the average < ... > is obtained by an averaging over 60 days of historical Libor data.

then be a large number.

In Figure 7.6 the numerical results of caplet are shown, and it is seen the numerical results are quite accurate even for a large value of $\epsilon =$ 3months; for this value of ϵ this program can generate 167 daily prices by running for less than two seconds on a desktop.

Floorlet prices result is shown in Fig.7.7

Besides accuracy, the numerical results need to be consistent with the general properties of the various options. In particular [82], the American caplet (put option) must always be more expensive than European caplet since the American option includes the European option as a special case; however, American floorlet (call option), in the absence of a dividend, is always equal to European floorlet. The normalized difference between American and European options is shown in Figure 7.8[7].

The results are seen to be consistent with the general properties of the American and European options; the normalized difference between American and European caplet is strictly

[7]One expects that by decreasing the time lattice size ϵ, the difference between American caplet and European caplet should be enlarged and become more significant compared with the error (deviating from zero) of the difference between American floorlet and European floorlet.

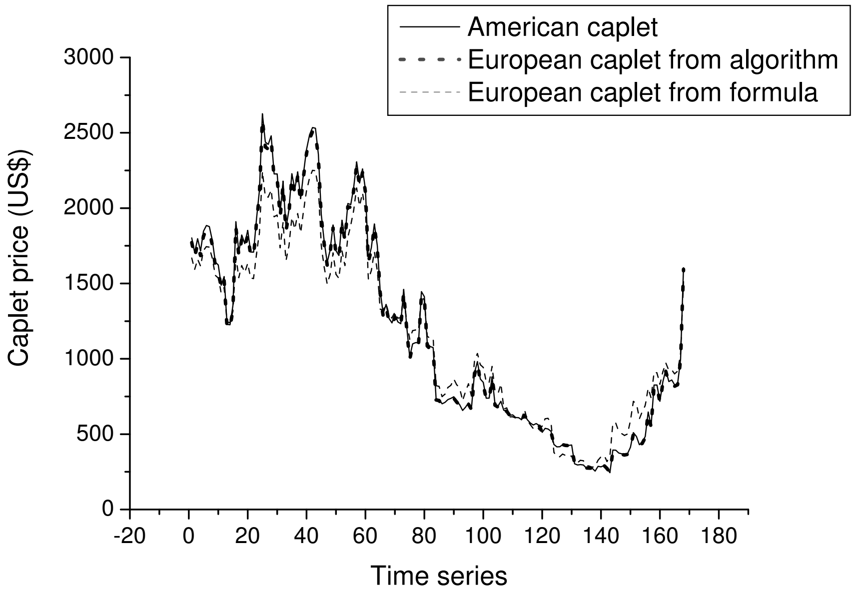

Figure 7.6: American and European caplet prices for fixed maturity at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004). (European capletfrom formula and algorithm). The normalized root mean square error for the European caplet price between the numerical value and formula is 7.3%.

positive showing that the American caplet is always more expensive than the European caplet; on the other hand, the gap between American and European floorlet can have negative values, showing that, within the accuracy of the numerical algorithm, their difference is zero.

Although the interval between evolving steps ϵ is set equal to 3 months, one can always decrease this interval to get more accurate results. The American option is more expensive on decreasing the interval ϵ since one needs to pay more to have an option that can be exercised on more occasions before the expiry date. One can consider the American option being exercised at fewer instants of time as Bermudan options. A **Bermudan option** can be exercised at a number of pre-fixed times and is equal to a basket of European options, with the difference that once the Bermudan option is exercised all the remaining European options become invalid. A Bermudan option is always cheaper than an American option but more expensive than European option. Some numerical results for the European, Bermudan and American caplets are shown in Figure 7.9, and are seen to be consistent with the general requirement for these options.

Put call parity for the European caplet and floorlet is given by [9], [14]

$$Caplet(t_0, t_*) - Floorlet(t_0, t_*) = \ell VB(t_0, t_* + \ell)[L(t_0, t_*) - K] \tag{7.33}$$

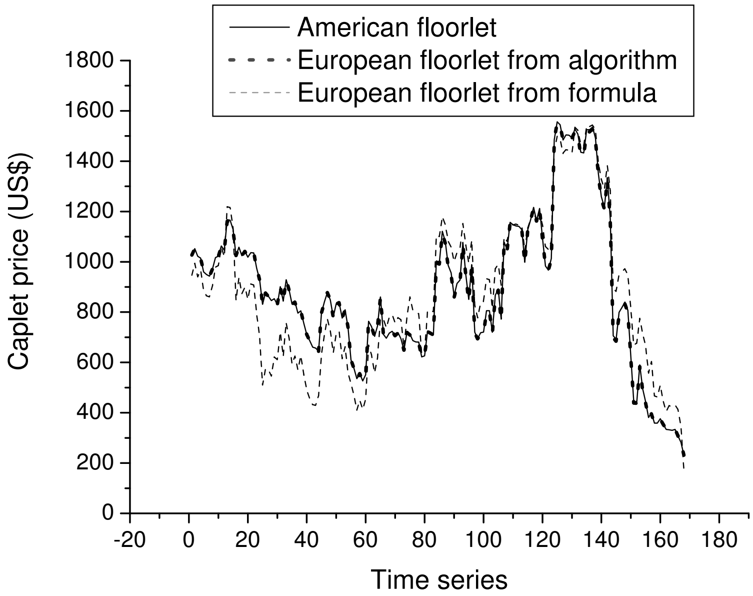

Figure 7.7: American and European floorlet prices for fixed maturity at 12.12.2004 versus time t_0 (12.9.2003-7.5.2004), (European floorlet from formula and algorithm). The normalized root mean square error of the European floorlet price from numerical algorithm and formula for floorlet is 8.8%

The third argument T, indicating when the caplet becomes operational, is suppressed since the numerical algorithm only studies the price of a caplet and floorlet for $t_* = T$. The result in Figure 7.10 verifies that put call parity is valid for the European option prices generated by the numerical algorithm.

§ 7.7 Numerical Results for Coupon Bond Options

For the coupon bond option case, the forward rates tree is built with the value of $\delta = 6s$. The main focus of the study of the American coupon bond option is not empirical, but instead is to develop an efficient and accurate algorithm. Given the complexity of the instrument, a model is assumed for the volatility and the initial value of the forward rates and the numerical study analyzes the accuracy of the algorithm for the model. No market data is used for studying the American coupon bond option price.

The inital lattice forward interest rates is taken as below

$$f_{mn} = f_0(1 - e^{-\lambda(n-m)}) \tag{7.34}$$

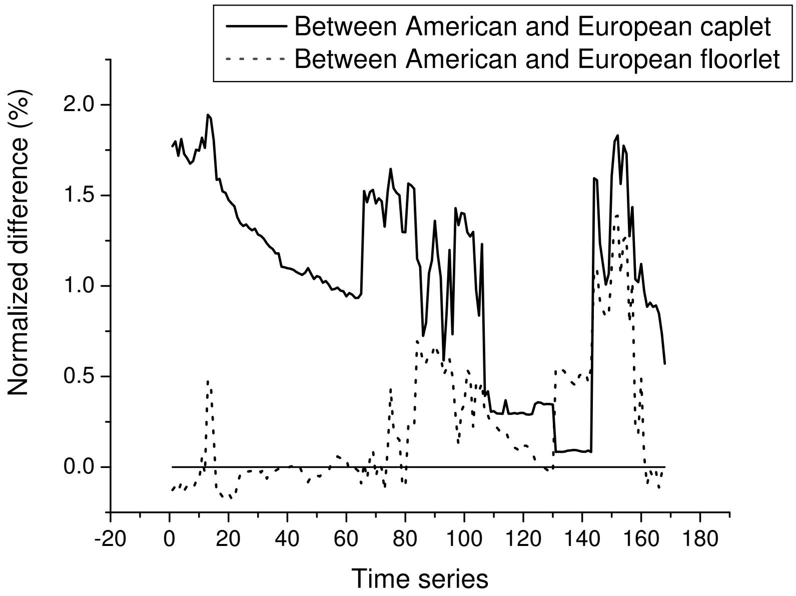

Figure 7.8: The (%) normalized difference between the American and European options for both caplet and floorlet versus time t_0 (12.9.2003-7.5.2004). Within the numerical accuracy of the computation, the European and American floorlet prices are equal, whereas the caplet prices for the European case is always less than the American case, as expected.

where f_0 is a prefactor used to get the same magnitude as the real market forward rates; let $f_0 = 0.1$ and choose $\lambda = 1$ so that f_{mn} is of the order of 10^{-2}.

Following Bouchaud and Matacz [47], volatility is taken to have the following form (the parameters are fixed by historical forward interest rates data)

$$\begin{aligned} \sigma(\theta) &= 0.00055 - 0.00026 \exp(-0.71826(\theta - \theta_{min})) \\ &\quad +0.0006(\theta - \theta_{min}) \exp(-0.71826(\theta - \theta_{min})) \ \ ; \ \ \theta = x - t \end{aligned} \tag{7.35}$$

where $\theta_{min} = 3$ month.

The volatility, computed as volatility of daily change in the forward rates, is given from historical data by $\sigma^2(t,\theta) = \langle \delta f^2(t,\theta) \rangle_c$, $\delta f(t,\theta) = f(t+1,\theta) - f(t,\theta)$. Thus, in building the tree where each step ϵ is 3 months, we have to multiply the actual days of 3 month into $\sigma(\theta)$ to obtain the dimensionless volatility s_{mn}; from the definitions of the lattice variables $\theta = x - t \rightarrow (N - n + m)\epsilon$. Since the number of trading days in 3 months is 65 and $\epsilon = a = 3/12 = 0.25$ years, the dimensionless volatility for the case of the coupon bond option

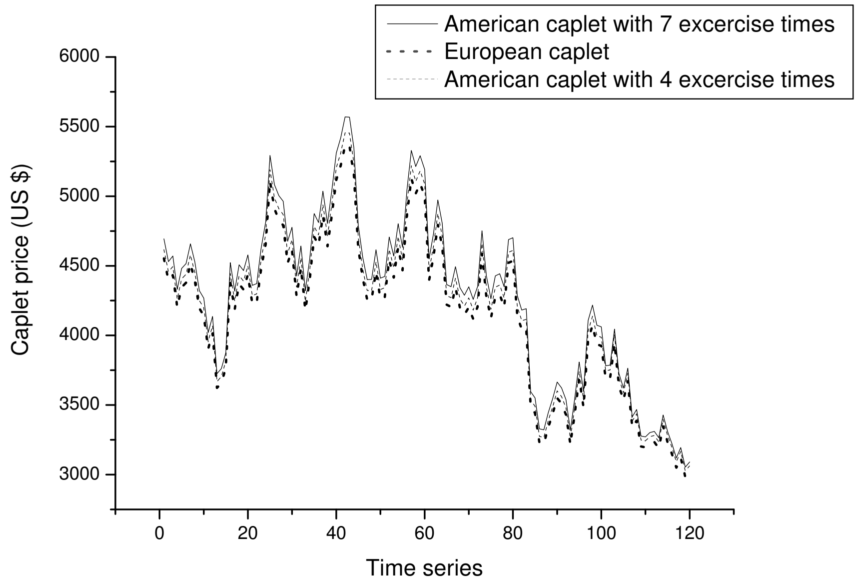

Figure 7.9: Caplet prices versus time t_0 for the European and American options, with four and seven possible exercise times

is given by ($\theta_{min} = \epsilon$)

$$\begin{aligned} s_{mn} &= 65\sqrt{0.25}\sqrt{0.25}\ \ \sigma(\sqrt{0.25}(N-n+m)) \\ &= 16.25\big[0.00055 - 0.00026\exp(-0.35913(N-n+m-1) \\ &\quad +0.0003(N-n+m-1)\exp(-0.35913(N-n+m-1))\big] \end{aligned} \tag{7.36}$$

The stiff propagator is given by Baaquie and Bouchaud [16], with parameters taken to have the following values $\tilde{\lambda} = 1.790$/year; $\tilde{\mu} = 0.403$/year; $\eta = 0.34$.

The numerical study considers the coupon bond option $c_1 B(t_*, 1/4) + c_2 B(t_*, 1/2)$ that matures in one years time, that is $t_* - t_0 = t_* = 1$ year and has a duration of six months, with two coupon payments and each is paid every three months; the fixed coupon rate is taken to be equal to c and the principal amount equal to 1. Thus the payoff function at time $t_* = 1$ year for the put option is given by

$$S(t_*) = (K - \sum_{i=1}^{2} c_i B(t_*, T_i))_+ \tag{7.37}$$

where $T_1 = 1.25$ year, $T_2 = 1.5$ year, $c_1 = c$ and $c_2 = c + 1$. Note taking the $c = 0$ limit converts the coupon bond into a zero coupon bond.

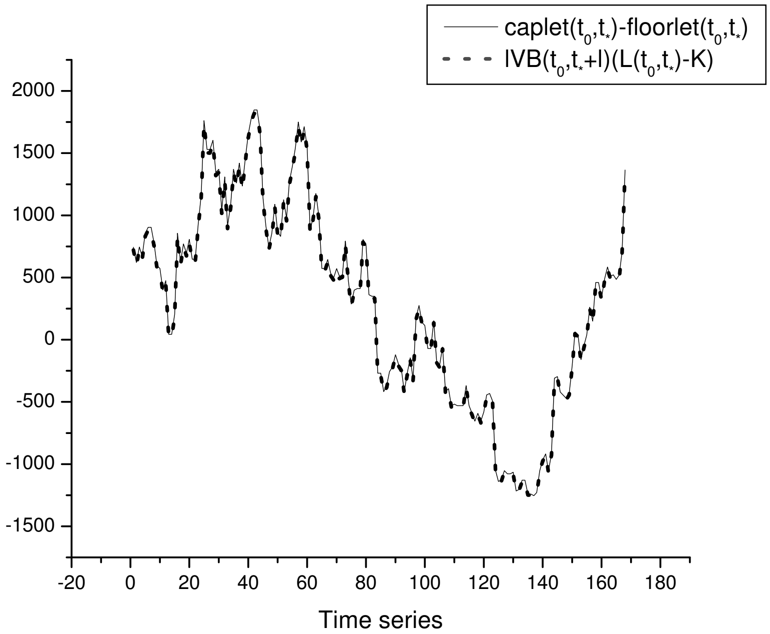

Figure 7.10: Put-call parity for the European caplet and floorlet versus time t_0 (12.9.2003-7.5.2004). The normalized root mean square error is 3.2%

For this coupon bond option, $N = 1$ and set $M = 4$; this yields the total number of option prices to be evaluated equal to $\mathcal{O} = 1,293$. This number increases more rapidly than for the caplet case, and for $M = 10$ it reaches $\mathcal{O} = 118,507,277$.

Comparing the numerical value of the European coupon bond option with the approximate formula given in eq. 2.60 provides a check on the numerical result. The approximate formula is an expansion in the volatility of the forward rates s_{mn}, and as long as this volatility is small, the numerical and approximate results should agree.

For the specific case that is being studied numerically the coefficient A that appears in European coupon bond option given in eq. 2.60 has the closed form expression given by

$$A = \sum_{ij=1}^{2} J_i J_j \left[G_{ij} + \frac{1}{2} G_{ij}^2 \right] + O(G_{ij}^3) \tag{7.38}$$

Note and $J_i = c_i F_i$, with $F_1 = 0.982321$ and $F_2 = 0.963426$.

The numerical values for G_{ij}, the correlator of the forward bond prices, are given in Table 7.1.

Numerical results for coupon bond option prices with changing strike price K and coupon

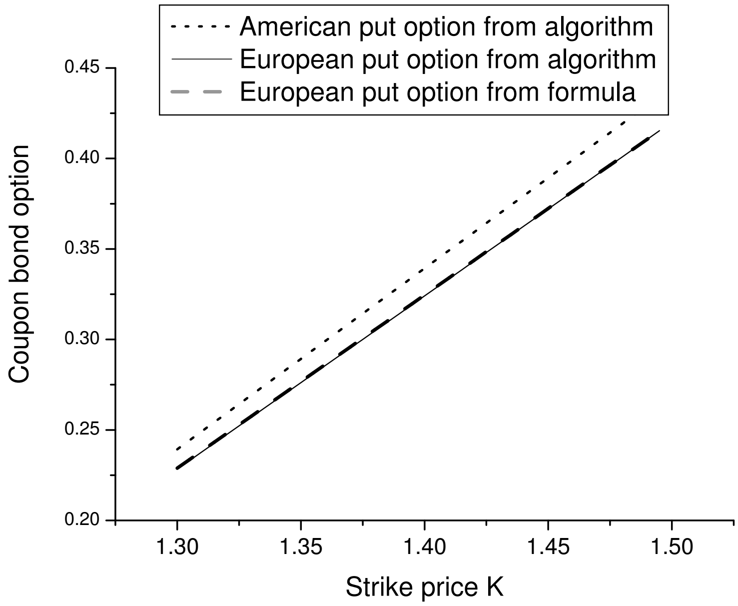

Figure 7.11: Prices of coupon bond American and European put option that matures in one year and with a duration of six month, during which two coupons at the rate of $c = 0.05$ are paid every three months versus strike price K (European option from formula and algorithm). The normalized root mean square error between the numerical value and formula for the European option is 0.17%.

G_{ij}	i=1	i=2
j=1	1.669×10^{-8}	3.624×10^{-8}
j=2	3.624×10^{-8}	7.924×10^{-8}

Table 7.1: The correlators G_{ij} between different forward bond prices.

rate c are given in Figures 7.11 and 7.12. The numerical value of the European coupon bond option is seen to be approximately equal to the closed form approximate formula in eq. 2.60; as required by consistency, the American put option always has a higher price than European put option.

For completeness, the special case of the American option on a zero coupon bond, obtained by setting $c = 0$ in eq. 7.37, is given in Figure 7.13 and shows all the features required by the consistentcy of the option prices.

The algorithm has been checked for internal consistency by plotting the prices of the coupon bond American put options, European put options and the payoff function against the value of the coupon bond $B(t_0)$, which were generated by varying the coupon rate c.

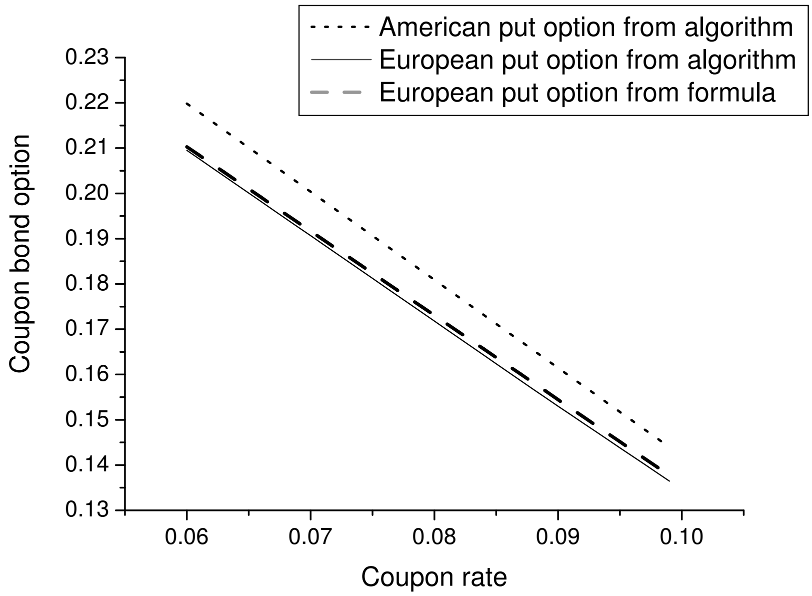

Figure 7.12: Prices of coupon bond American and European put option that matures in one year and with a duration of six month, during which two coupons at the rate of $c =$ are paid every three months with strike price $K = 1.3$ versus coupon rate c (European option from formula and algorithm) are shown. The normalized root mean square error between the numerical result and formula for European option is 0.73%.

Figure 7.14 shows that the results are consistent with the general properties of these options, with the price of the American option always being higher than the European option, and the American option joins the payoff with the same slope as the payoff function for small coupon bond value $B(t_0)$, where for large values of $B(t_0)$ the American options joints the European option, as discussed in [42].

Another check of the algorithm is the put-call parity for coupon bond option. For European option the numerical value have to obey the equation [11]

$$C_E(t_0, t_*, K) - P_E(t_0, t_*, K) = \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*) \tag{7.39}$$

The numerical results in Fig.7.15 shows the accuracy of the algorithm and the normalized root mean square error is only 1.53%.

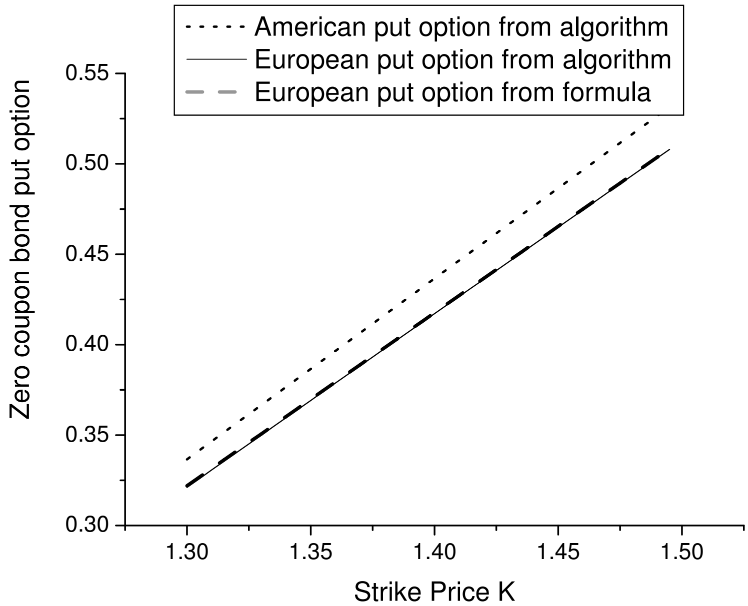

Figure 7.13: Prices of American and European put options for zero coupon bond with half year duration and one year maturity versus strike price K (European options from formula and algorithm). The normalized root mean square error between the numerical value and formula for the European option is 0.16%.

§ 7.8 Put Call Inequalities for American Coupon Bond Option

In analogy with the inequalities for the put and call American options for equity given in eq. 7.54, one can consider the following inequalities for the case of American options on coupon bonds

$$B(t_0) - K \leq C(t_0, t_*, K) - P(t_0, t_*, K) \leq B(t_0) - B(t_0, t_*)K \quad : \quad \text{Incorrect}$$

where $B(t_0)$ is value of the coupon bond at t_0

$$B(t_0) = \sum_{i=1}^{N} c_i B(t_0, T_i) \tag{7.40}$$

On graphing the three expressions in the above equation, as shown in Figure 7.16, it is seen that the put-call inequalities are **incorrect**.

Instead of the above incorrect inequality, a **conjecture** is made that the American coupon bond options satisfy the following modified inequalities in which the coupon bond value $B(t_0)$

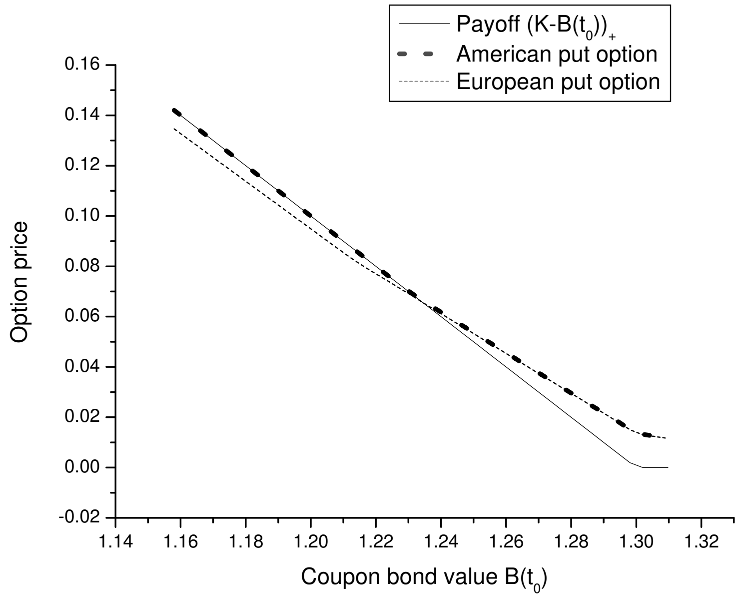

Figure 7.14: The coupon bond payoff function and the American and European coupon bond put option prices versus the underlying coupon bond value $B(t_0) = \sum_{i=1}^{2} c_i B(t_0, T_i)$.

at time t_0 has been replaced by the present value of the payoff function, called $F(t_0)$. Hence we have postulate the following inequalities

$$F(t_0) - K \leq C(t_0, t_*, K) - P(t_0, t_*, K) \leq F(t_0) - KB(t_0, t_*)$$

where $F(t_0)$ is value of the forward coupon bond price at t_0, namely

$$F(t_0) \equiv \sum_{i=1}^{N} c_i F(t_0, t_*, T_i)$$

On numerically checking this inequality, as shown in Fig.7.17, it is seen that the American coupon bond option in fact does satisfy the conjectured inequalities!

In analogy with the conjecture for the inequalities obeyed by American coupon bond options, the following inequalities are conjectured for the American caplet and floorlet

$$\begin{aligned} F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - K] &\leq Caplet(t_0, t_*) - Floorlet(t_0, t_*) \\ &\leq F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] \end{aligned} \tag{7.41}$$

which can also be expressed as follows

$$\begin{aligned} Caplet(t_0, t_*) - Floorlet(t_0, t_*) - F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - K] &\geq 0 \\ F(t_0, t_*, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] - Caplet(t_0, t_*) - Floorlet(t_0, t_*) &\geq 0 \end{aligned}$$

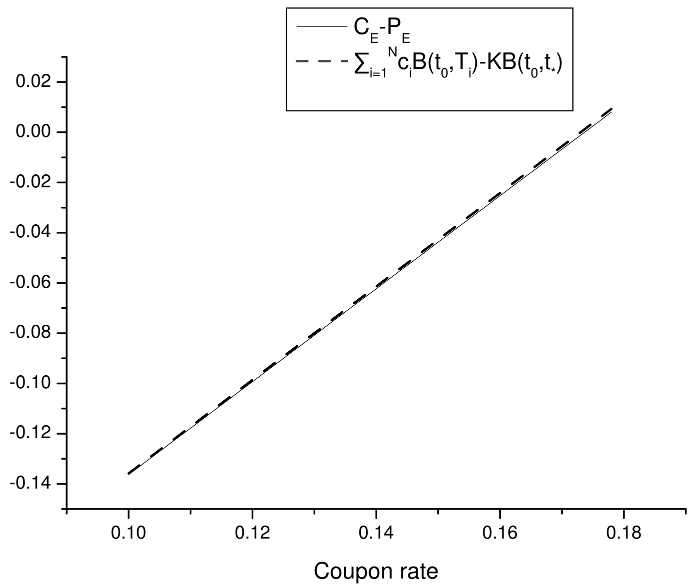

Figure 7.15: Graph of put-call parity $C_E(t_0, t_*, K) - P_E(t_0, t_*, K) = \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*)$ for numerical prices of the European coupon bond options versus coupon rate. The normalized root mean square error for put-call parity is 1.53%.

Figure 7.18 shows that the conjectured inequalities indeed do hold for the numerical prices of the American caplet options.

The conjecture for the American caplet and floorlet is not as significant as the one for the American coupon bond option since the numerical prices also satisfy the inequalities that are similar to the equity inequalities in eq. 7.54, namely

$$\begin{aligned} B(t_0, t_* + \ell)[L(t_0, t_*) - K] &\leq Caplet(t_0, t_*) - Floorlet(t_0, t_*) \\ &\leq B(t_0, t_* + \ell)[L(t_0, t_*) - B(t_0, t_*)K] \end{aligned} \tag{7.42}$$

The numerical results for American caplet and floorlet show that they obey the inequalities given in 7.19.

§ 7.9 Conclusions

An efficient and accurate numerical algorithm has been developed and implemented for pricing American options for interest rate instruments. The procedures that are presently being used [83] are all based on the HJM-model and use variants of the binomial tree to build the tree

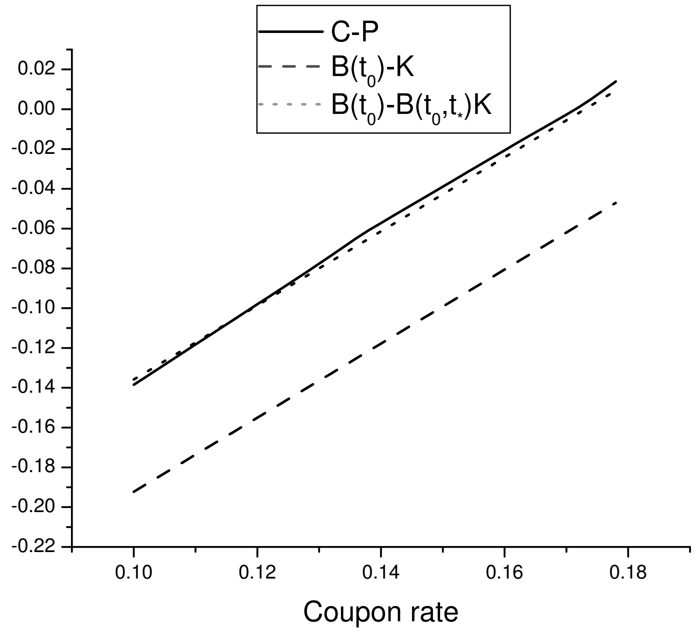

Figure 7.16: Numerical result showing the **incorrectness** of put-call inequalities $B(t_0) - K \leq C(t_0, t_*, K) - P(t_0, t_*, K) \leq B(t_0) - B(t_0, t_*)K$ of American coupon bond option that are analogous to the equity case versus coupon rate.

for the interest rates and coupon bonds; the complexity of the tree in the HJM-model is determined by how many factors are driving the interest rates.

The approach of quantum finance to American options is radically different. One starts with the pricing kernel, for which a model is written from first principles. The payoff function for the American option is propagated backwards on a time lattice using the pricing kernel, and entails performing a path integral numerically. The numerical path integral can be interpreted as generating the values of the American option on a tree of forward interest rates. At each step the trial American option value obtained is compared with the payoff function.

The (lattice) forward interest rates $f(t, x) \rightarrow f_{mn}$ are directly involved in the recursion equation. Any attempt to replace f_{mn} by any collection of white noise, as is the case for the HJM-model, makes the whole numerical computation intractable. Furthermore, the non-trivial correlation between the changes in the option price due to the propagator $\tilde{D}_{m,jk}$ is easily incorporated in the recursion equation, as can be seen from eq. 7.25; this correlation cannot be incorporated in the numerical procedures based on the HJM-model. The prices of the American option for caplets and coupon bonds have been shown to be consistent, obeying all the contraints that necessarily follow from the principles of finance.

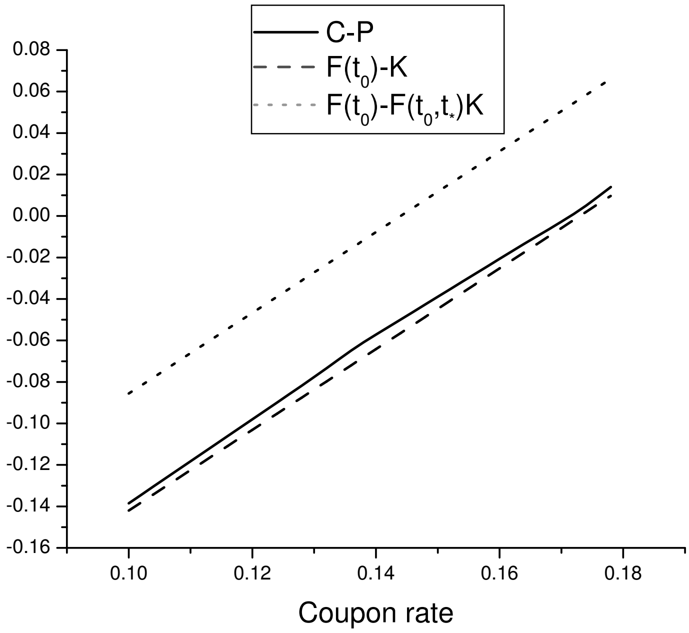

Figure 7.17: Numerical result confirming the **conjectured** put-call inequalities of American coupon bond option $F(t_0)-K \leq C(t_0,t_*,K)-P(t_0,t_*,K) \leq F(t_0)-KB(t_0,t_*)$ verus coupon rate; $F(t_0) \equiv \sum_{i=1}^{N} c_i F(t_0,t_*,T_i)$.

The comparison of the numerical values of the European coupon with the caplet formula and the approximate coupon bond option formula showed that the numerical algorithm in general is over 95% accurate, reaching an accuracy of 99% for the coupon bond option. An additional benefit of the comparison is that it provides a proof of the accuracy of the approximate coupon bond optin price for low volatility.

The entire computation has been carried out on a desktop computer and with a small lattice of about 10 to 20 lattice points, with the option price for each set of parameters requiring only a few seconds of computation time. Even such a crude approximation gives excellent results, showing the possibilty of using such algorithms for practical applications.

A conjecture for the put-call inequalities was made for the American coupon bond option and caplet was made based on the analysis of the numerical results. The fact that these inequalities seem to hold quite robustly for the numerical results obtained adds confidence to the correctness of this conjecture. A derivation of the conjecture from the principles of finance would confirm the correctness of the conjecture.

The numerical algorithm developed in this paper opens the way to the study of all varieties

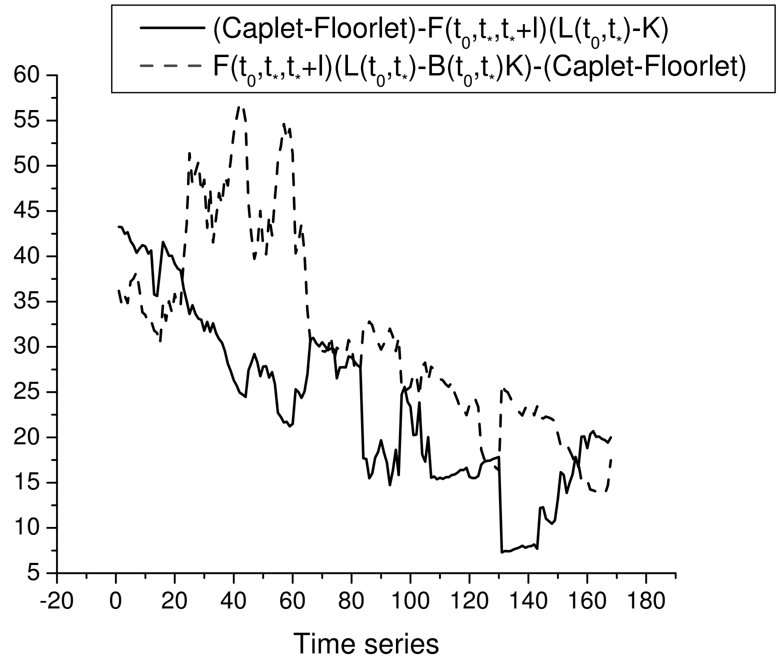

Figure 7.18: Put call inequality obeyed by the price of a caplet American put and call options. $F(t_0,t_*,t_*+\ell)[L(t_0,t_*)-K] \leq Caplet(t_0,t_*) - Floorlet(t_0,t_*) \leq F(t_0,t_*,t_*+\ell)[L(t_0,t_*) - B(t_0,t_*)K]$

of American options, both for interest rate instruments and for correlated equity instruments as well.

§ 7.10 Appendix: American option on equity

The Black-Scholes case provides a simple example for understanding the more complicated case. Montagna and Nicrosini [38] have given a path integral numerical evaluation of the American option for Black-Scholes equity. In order to understand the American option for the more complex case of interest rates, the derivation of [38] is briefly reviewed. The notation followed in this Appendix is the same as the one used for analyzing lattice interest rates in Section 4.

The Black-Scholes models the time evolution of asset prices. Consider a time lattice with time running **backwards**, that is, $t_i = (N-i)\epsilon,\ i = 0,1,..N$, with $t_* = N\epsilon$ usually being the expiration time for an option. In terms of the logarithm of the asset price $S_i = e^{z_i}$, with $z_i \equiv z(t_i)$ and discretized velocity $dz/dt = (z_i - z_{i+1})/\epsilon$, the Black-Scholes Lagrangian [6] is

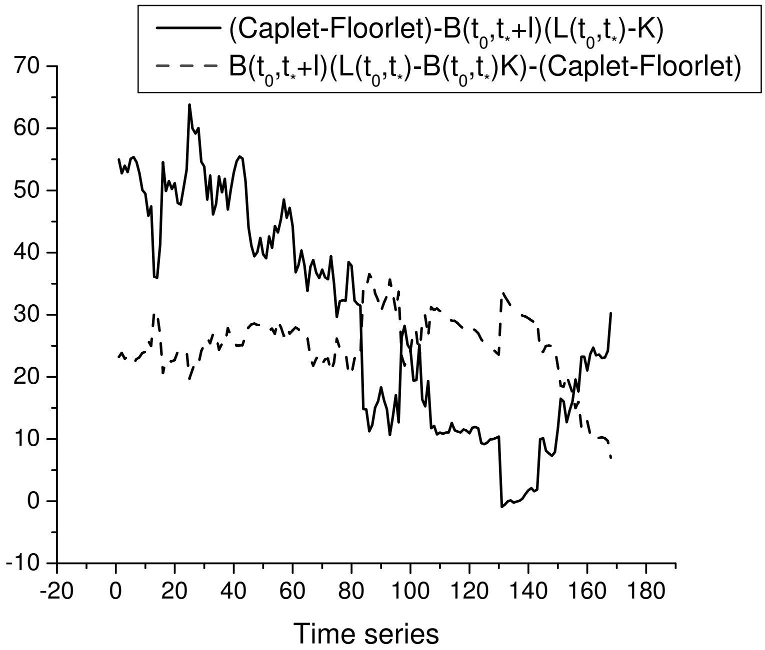

Figure 7.19: Put call inequality obeyed by the price of a caplet American put and call options. $B(t_0, t_*+\ell)[L(t_0, t_*)-K] \leq Caplet(t_0, t_*)-Floorlet(t_0, t_*) \leq B(t_0, t_*+\ell)[L(t_0, t_*)-B(t_0, t_*)K]$

given by

$$\mathcal{L}_{BS}(i) = -\frac{1}{2\sigma^2}\left(\frac{z_i - z_{i+1}}{\epsilon} - \alpha\right)^2 - r \tag{7.43}$$

where $\alpha = r - \sigma^2/2$. Let the boundary conditions be given by $z_0 = z$; $z_N = z'$; the action and the pricing kernel are then given by [6]

$$\begin{aligned}
S_{BS} &= \epsilon \sum_{i=0}^{N-1} \mathcal{L}_{BS}(i) = \sum_{i=0}^{N-1} \mathcal{L}(i) \\
\mathcal{L}(i) &= \epsilon \mathcal{L}_{BS}(i) = -\frac{1}{2s^2}\left(z_i - z_{i+1} - \tilde{\alpha}\right)^2 - \tilde{r} \\
p(z', z; N) &= \tilde{\mathcal{N}} \prod_{i=0}^{N-1} \int dz_i e^{S_{BS}} \\
p(z', z; 1) &= \mathcal{N} \exp\{\mathcal{L}\} = \sqrt{\frac{1}{2\pi s^2}} \exp\{-\frac{1}{2s^2}\left(z - z' - \tilde{\alpha}\right)^2 - \tilde{r}\}
\end{aligned} \tag{7.44}$$

with dimensionless parameters $s^2 = \epsilon\sigma^2$, $\tilde{\alpha} = \epsilon\alpha$ and $\tilde{r} = \epsilon r$.

Consider a European put option P_i maturing at time $N\epsilon = t_*$ in the future, with maturity time labelled by $i = 0$; the payoff function is given by $(K - e^z)_+$, where K is the strike price.[8] Since time is running backwards, the pricing kernel gives the price of the European put option at time $i = N$ by the following equation

$$P_E(z', N) = \int_{-\infty}^{+\infty} dzp(z', z; N)(K - e^z)_+ \quad (7.45)$$

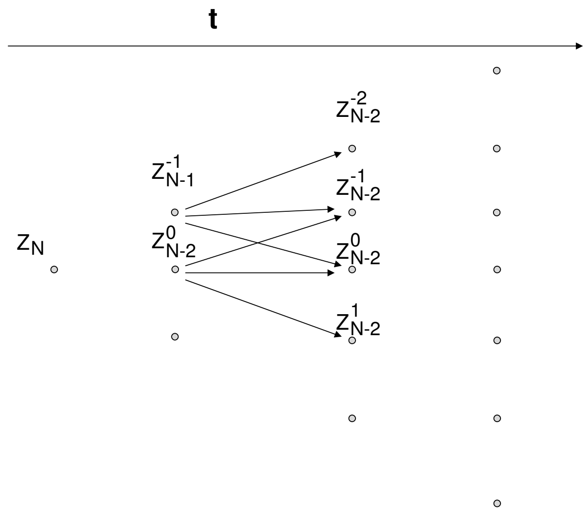

Figure 7.20: Tree of stock values, with number of points growing linearly with maturity time.

Consider the case of the American put option with possibility of an early exercise. The payoff of the American option is the same as the European option, with the additional freedom that the holder of the option can exercise the option anytime from the present to its theoretical maturity date t_*. Since time is divided into short intervals of spacing ϵ, early exercise of the option can only take place at the discrete time instants of $t_i = i\epsilon$. To find the price of the American option $P(t)$ one propagates the payoff function backwards in time. At time slice t_i the American option has a price given by $P(t_i)$. To determine the option price at next (earlier) instant t_{i+1} one propagates $P(t_i)$ backwards in time to obtain an initial trial value of the American option at t_{i+1}, called $P_I(t_{i+1})$. The actual value of the American option at t_{i+1} is given by the maximum of the (non-discounted) payoff function and $P_I(t_{i+1})$; that is

$$P(t_{i+1}) = Max\{P_I(t_{i+1}), (K - e^{z_{i+1}})_+)\} \quad (7.46)$$

[8]The price of an American call option, for a non-dividend paying stock, can be shown to be equal to the European call option [42].

In the path integral the pricing kernel is used for computing the initial trial option price $P_I(t_{i+1})$; eqs. 7.45 and 7.44 yield

$$P_I(t_{i+1}, z') = \int_{-\infty}^{+\infty} dz p(z', z; 1) P(t_i, z) = \mathcal{N} \int_{-\infty}^{+\infty} dz e^{\mathcal{L}(z',z)} P(t_i, z) \tag{7.47}$$

$$= e^{-\tilde{r}} \sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} dz \exp\{-\frac{1}{2s^2}(z - z' - \tilde{\alpha})^2\} P(t_i, z) \tag{7.48}$$

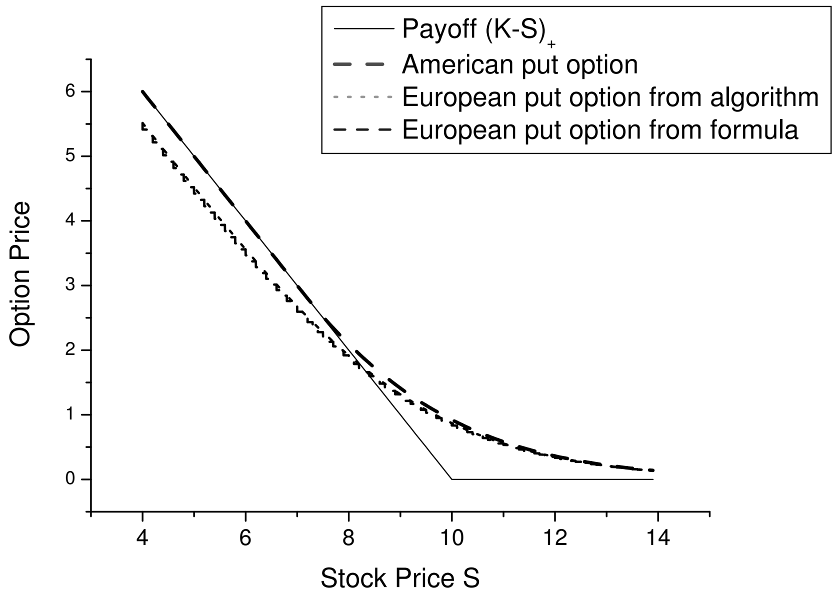

Figure 7.21: Price of the American stock option versus stock price S.

Almost all cases of interest have fairly small volatility, that is, $s \simeq 0$; for small s the most efficient procedure for evaluating the integral in eq. 7.48 is to Taylor expand the function $P(t_i, z)$ about the very sharp maximum of the Gaussian part of the integrand located at the point $z' + \tilde{\alpha} \equiv \bar{z}$. Denoting differentiation with respect to z by prime yields the Taylors expansion

$$P(t_i, z) = P(t_i, \bar{z}) + (z - \bar{z}) P'(t_i, \bar{z}) + \frac{1}{2}(z - \bar{z})^2 P''(t_i, \bar{z}) + .. \tag{7.49}$$

Using the fact that

$$\sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} dz e^{-\frac{1}{2s^2}(z-\bar{z})^2} = 1 \;\; ; \;\; \sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} dz e^{-\frac{1}{2s^2}(z-\bar{z})^2} (z - \bar{z}) = 0$$

$$\sqrt{\frac{1}{2\pi s^2}} \int_{-\infty}^{+\infty} dz e^{-\frac{1}{2s^2}(z-\bar{z})^2} (z - \bar{z})^2 = s^2 \tag{7.50}$$

S	American Put	Numerical European Put	European from Black-Scholes
6.0	4.00	3.558	3.558
8.0	2.095	1.918	1.918
10.0	0.922	0.870	0.870
12.0	0.362	0.348	0.348
14.0	0.132	0.128	0.128

Table 7.2: Numerical prices of American and European put options for the parameters $T = 0.5$ year, $r = 0.1$/year, $\sigma = 0.4$, $K = 10$, $\epsilon = T/100$, as a function of the possible present time stock prices S.

yields, from eqs. 7.48 and 7.49, the following recursion equation

$$P_I(t_{i+1}, z') = e^{-\tilde{r}}\big[P(t_i, \bar{z}) + \frac{1}{2}s^2 P''(t_i, \bar{z})\big] + O(s^4) \tag{7.51}$$

Discretizing the values of $\bar{z}$ into a grid of spacing δ of O(s) yields

$$P_I(t_{i+1}, z') \simeq e^{-\tilde{r}}\big[P(t_i, \bar{z}) + \frac{1}{\delta^2}[P(t_i, \bar{z}+\delta) - 2P(t_i, \bar{z}) + P(t_i, \bar{z}-\delta)]\big] \tag{7.52}$$

Note that to obtain the value of $P_I(t_{i+1}, z')$ in eq. 7.52 one needs the values of option prices at the earlier time at three distinct points, namely $P(t_i, \bar{z}), P(t_i, \bar{z} \pm \delta)$. By induction, it follows that as one recurses back in time, the number of points at which the option price can be obtained collapses into a single point. Hence, in order to find the option price at a some particular value at present, which in the notation being used is at $t_N = 0$, one needs to create a **tree** of values for z_i, at which points the recursion equation will evaluate the value of the option; the tree is illustrated in Figure 7.20.

As shown in fig 7.20, the points on the tree grow linearly with each step in time. The values of z_i on the tree are taken to have a spacing of $\delta = s$ so that the spread of the z_i values on the tree can span the interval required for obtaining an accurate result from the integration. The tree at time t_i has the following values for z_i, namely

$$z_i^{(k)} \doteq z_N + \tilde{\alpha} + ks, \qquad k = -(N-i), \ldots, +(N-i) \tag{7.53}$$

At a given time t_i, the tree consists of $2(N-i)+1$ values of z_i, centered on the $S = e^{z_N}$, namely the value of the stock at initial time for which the price of the American option is being computed.

The algorithm expressed in eqs. 7.46 and 7.52 were numerically tested and yield results that are fairly accurate as well as consistent with those obtained in [38].

The American and European put option prices, together with the payoff function for the put option, are shown in Fig.7.21 and are seen to be consistent with the discussions in [42]; in particular note that the American put option is always more expensive than the European put option, as indeed it must be since it has more choice; furthermore, the American put option, for small values of the stock price S, has the same slope as the payoff function, hence smoothly joining it.

From [42], the inequalities obeyed by the price of American call and put options C and P respectively, on a stock with stock price S, strike price K and maturing at future time T is given by the following

$$S - K \leq C - P \leq S - e^{-rT}K \tag{7.54}$$

where r is the spot interest rate. The put call inequality for American option of a stock is seen in Figure reffig:sputcall to hold for the numerical option prices.

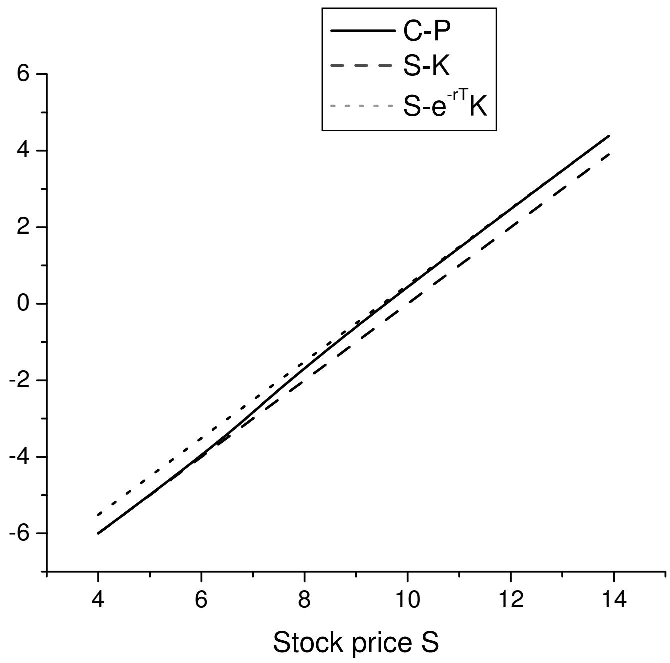

Figure 7.22: Put call inequality for the American stock option.

All the basic features of the algorithm for pricing an American put option for equity are simpified expressions that appear in the more complex algorithm needed to price American options for interest rate instruments.

Conclusion

This dissertation shows the advantages of Quantum Finance in modelling the term structure of forward interest rates. As a generalization of HJM model, the field theory model has been proved adequate for modelling interest rates using infinite degree of freedom. The model offers a different perspective on financial processes, offers a variety of computational algorithm and many of the pricing which can be derived analytically in a much easier manner.

In order to show all the advantages, three different point of views were given in the dissertation. I empirically studied the pricing for European cap, floor, swaption and coupon bond option, showing that the field theory model is consistent with all instruments unlike Black's formula which is only internally consistent. The empirical results show the field theory model generates the prices of interest rated derivatives for the market to an accuracy of about 95% and matches all the trend of the market. In contrast, the HJM model systemically over-prices swaption by $7-9\%$.

I also theoretically studied the hedging of Libor derivatives using two different approaches. In particular, the stochastic delta hedging can perform the hedging on any specific forward interest rates, which can not be done by any of the prevalent model. New instrument, the option on two correlated coupon bond, was priced by using the field theory model and perturbation expansion with both market drift and martingale drift.

I also show that an efficient algorithm yields from Quantum Finance for numerically studied the American and Bermudan style interest rate derivatives. The field theory model of forward rates was discretized to yield the lattice theory. And it starts from the pricing kernel, for which a model is written from first principles. Multidimensional tree structure with correlation was generated to numerically integrate the path integral. New inequalities of American coupon bond option were verified by the numerical solution.

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Program for swaption pricing

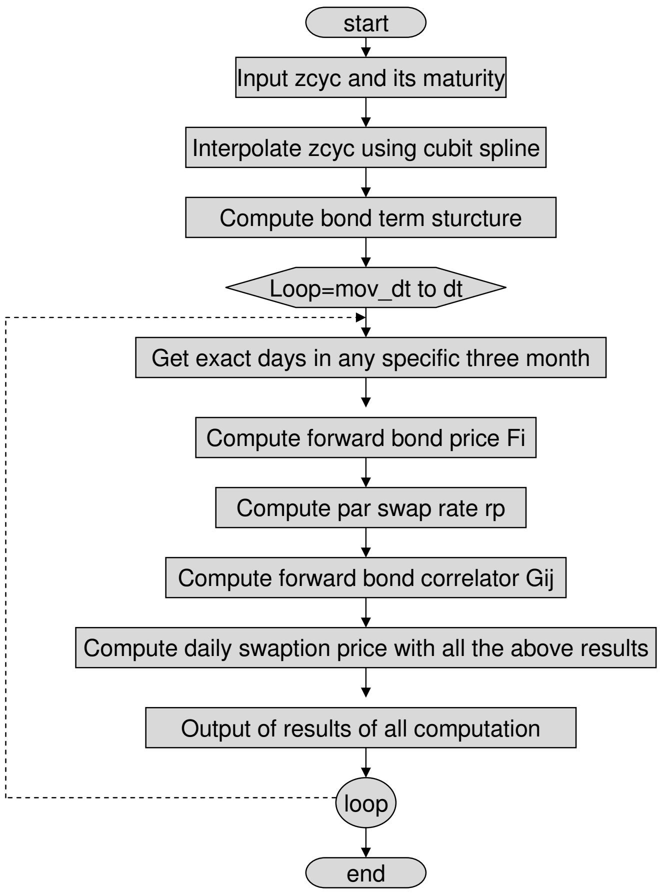

Figure 7.23: Flow chart of program for swaption pricing.

```
/*
*File: Swaption II 2by10.cpp
*
*Description: This program reads zero coupon yield
*curve data up to 30 years from 2003.1.29 to 2005.1.28,
*generate 2by10 swaption price from 2004.2.11 to 2005.1.28.
*using 270 days moving average to compute G_ij
*
*
*/

#include <math.h>
#include <stdlib.h>
#include <stdio.h>
#include <malloc.h>
#include "clresource.h"

/* parameter for calculate the cumulative distribution function*/
#define a1 (0.31938153)
#define a2 (-0.356563782)
#define a3(1.781477937)
#define a4 (-1.82125597)
#define a5 (1.330274429)
#define gamma (0.2316419)

#define Pi (3.1415926)
#define V (1000000)  /* notional value */
#define dt (523)   /* range from 271~523days to compute 1~253
                      daily swaption prices */
#define mov_dt (270)    /* moving days history for M */
#define interval (0.25)
#define theta (120)  /*number of lattice points in theta direction */
#define Y1 (2)  /*time to maturity */
#define Y2 (10)  /* tenor of swap */
#defineday_quarter (65) /* trading days in three month */
```

```
#define pre (1)

int maturity=Y1*4;  /* number of lattice points in maturity of
                  swaption */
int duration=Y2*2; /* number of lattice points in
                    tenor of swaption */
double dayinterval=double(1)/260; /* one trading day used in year unit */
double compounding=2; /* compounding frequency */
double year=360; /* number of days in calendar year */

/* Code from Numerical Receipies for cubic spline */ void
spline(double x[], double y[], int n, double yp1, double ypn,
double y2[])
{
    int i,k;
    double p,qn,sig,un,*u;
    u=vector(1,n-1);
    if (yp1 > 0.99e30)
        y2[1]=u[1]=0.0;
    else
    {
        y2[1] = -0.5;
        u[1]=(3.0/(x[2]-x[1]))*((y[2]-y[1])/(x[2]-x[1])-yp1);
    }
    for (i=2;i<=n-1;i++)
    {
        sig=(x[i]-x[i-1])/(x[i+1]-x[i-1]);
        p=sig*y2[i-1]+2.0;
        y2[i]=(sig-1.0)/p;
        u[i]=(y[i+1]-y[i])/(x[i+1]-x[i]) - (y[i]-y[i-1])/(x[i]-x[i-1]);
        u[i]=(6.0*u[i]/(x[i+1]-x[i-1])-sig*u[i-1])/p;
    }
    if (ypn > 0.99e30)
        qn=un=0.0;
    else {
        qn=0.5;
```

```
        un=(3.0/(x[n]-x[n-1]))*(ypn-(y[n]-y[n-1])/(x[n]-x[n-1]));
    }
    y2[n]=(un-qn*u[n-1])/(qn*y2[n-1]+1.0);
    for (k=n-1;k>=1;k--)
        y2[k]=y2[k]*y2[k+1]+u[k];
    free_vector(u,1,n-1);
}

/* Code from Numerical Receipies for cubic spline */
void splint(double xa[], double ya[], double y2a[], int n, double
x, double *y)
{
    void nrerror(char error_text[]);
    int klo,khi,k;
    double h,b,a;
    klo=1;
    khi=n;
    while (khi-klo > 1)
    {
        k=(khi+klo) >> 1;
        if (xa[k] > x) khi=k;
        else klo=k;
    }
    h=xa[khi]-xa[klo];
    if (h == 0.0) nrerror("Bad xa input to routine splint");
    a=(xa[khi]-x)/h;
    b=(x-xa[klo])/h;
    *y=a*ya[klo]+b*ya[khi]+((a*a*a-a)*y2a[klo]+(b*b*b-b)*y2a[khi])*(h*h)/6.0;
}

/* This subroutine give exact calendar days in any specific three
month */
void lmaker (double month[],double l[])
{
    int i;
    for (i=0;i<theta;i++)
```

```
    {
        if (month[i]==1) l[i]=90;
        if (month[i]==2) l[i]=89;
        if (month[i]==3) l[i]=92;
        if (month[i]==4) l[i]=91;
        if (month[i]==5) l[i]=92;
        if (month[i]==6) l[i]=92;
        if (month[i]==7) l[i]=92;
        if (month[i]==8) l[i]=92;
        if (month[i]==9) l[i]=91;
        if (month[i]==10) l[i]=92;
        if (month[i]==11) l[i]=92;
        if (month[i]==12) l[i]=90;
    }
}

/* This subroutine calculate bond term structure by using zero
coupon yield curve */
void bondmaker(double f[][theta],double bond[][theta])
{
    int i,j;
    double time;
    for (j=0;j<dt;j++)
    {
    for (i=0;i<theta;i++)
    {
        time=double(i+1)*0.25;
        bond[j][i]=1/pow((1+f[j][i]/compounding),time*compounding);
    }
    }
}

/* This subroutine calculate par swap rate by using bond term
structure */
void Rpmaker(double bond[dt][theta],int j,double l[],double *rp)
{
```

```
    double a,b;
    int i,x,y;
    x=Y1;
    y=Y2;
    a=0;
    b=0;
    b=bond[j][4*(x+y)-1];
    for (i=1;i<=2*y;i++)
    {
        a=a+l[i+2*x-1]/year*bond[j][2*(i+2*x)-1];
    }
    *rp=(bond[j][4*x-1]-b)/(a);
}

/* This subroutine calculate forward bond prices from bond term
structure */
void Fimaker(double bond[dt][theta],int i,double F[duration])
{
    int j;
    for (j=0;j<duration;j++)
    {
        F[j]=bond[i][j*2+Y1*4+1]/bond[i][Y1*4-1];
    }
}

/* This subrouitne calculate forward bond correlator Gij from
historical zcyc data */
void Gijmaker (double **ff,int Ti, int Tj,double *G,int day)
{
    int i,j,k,s,d,ii,jj,a;
    double temp,temp2,temp3,gap[mov_dt-1][2],c,b;
    temp3=0;
    for (k=0;k<=maturity;k++)
    {
        temp2=0;
        i=(maturity-k)*day_quarter;
```

```
        ii=(Ti-k)*day_quarter;
        jj=(Tj-k)*day_quarter;
        a=0;
        for (j=1;j<mov_dt;j++)
        {
            s=day-j;
            gap[j-1][0]=(log(pow(1+ff[s][ii+a]/compounding,
             compounding*(ii+a)*dayinterval))-log(pow(1+ff[s][i+a]/compounding,
             compounding*(i+a)*dayinterval)))-(log(pow(1+ff[s-1][ii+a+1]/compounding,
             compounding*(ii+a+1)*dayinterval))-log(pow(1+ff[s-1][i+a+1]/compounding,
             compounding*(i+a+1)*dayinterval)));
            gap[j-1][1]=(log(pow(1+ff[s][jj+a]/compounding,
             compounding*(jj+a)*dayinterval))-log(pow(1+ff[s][i+a]/compounding,
             compounding*(i+a)*dayinterval)))-(log(pow(1+ff[s-1][jj+a+1]/compounding,
             compounding*(jj+a+1)*dayinterval))-log(pow(1+ff[s-1][i+a+1]/compounding,
             compounding*(i+a+1)*dayinterval)));
            a++;
        }
        temp=0;
        c=0;
        b=0;
        for (d=0;d<mov_dt-1;d++)
        {
            temp=temp+gap[d][0]*gap[d][1];
            c=c+gap[d][0];
            b=b+gap[d][1];
        }
        temp2=fabs(temp/(mov_dt-1)-c*b/((mov_dt-1)*(mov_dt-1)));
        if (k==0 || k==maturity)
            temp3=temp3+0.5*interval*temp2;
        else
            temp3=temp3+interval*temp2;
    }
    *G=260*temp3;
}
```

```
/* This subroutine compute F from forward bond prices and par
swapt rate */
double Fmake(double F[duration],double l[],double rp)
{
    int i,x;
    double FF,temp,c;
    temp=0;
    x=Y1;
    for (i=0;i<duration;i++)
    {
        if (i!=duration-1)
        {
            c=l[i+2*x]/year*rp;
            temp=temp+c*F[i];
        }
        else
        {
            c=1+l[i+2*x]/year*rp;
            temp=temp+c*F[i];
        }
    }
    FF=temp;
    return FF;
}


/* This subroutine compute A from F, Gij and par swap rate */
double Amake(double F[duration],double
G[duration][duration],double l[],double rp)
{
    int i,j,x;
    double A,c1,c2,temp;
    temp=0;
    x=Y1;
    //s=2*(duration-1)-1-1;
```

```
for (i=0;i<duration;i++)
{
    for (j=0;j<duration;j++)
    {
        if (i!=duration-1 && j!=duration-1)
        {
            c1=l[i+2*x]/year*rp;
            c2=l[j+2*x]/year*rp;
            temp=temp+c1*c2*F[i]*F[j]*(G[i][j]+G[i][j]*G[i][j]/2);
        }
        else
        {
            if (i!=duration-1)
            {
                c1=l[i+2*x]/year*rp;
                c2=(1+l[j+2*x]/year*rp);
                temp=temp+c1*c2*F[i]*F[j]*(G[i][j]+G[i][j]*G[i][j]/2);
            }
            else
            {
                if (j!=duration-1)
                {
                    c1=(1+l[i+2*x]/year*rp);
                    c2=l[j+2*x]/year*rp;
                    temp=temp+c1*c2*F[i]*F[j]*(G[i][j]+G[i][j]*G[i][j]/2);
                }
                else
                {
                    c1=1+l[i+2*x]/year*rp;
                    c2=1+l[j+2*x]/year*rp;
                    temp=temp+c1*c2*F[i]*F[j]*(G[i][j]+G[i][j]*G[i][j]/2);
                }
            }
        }
    }
}
```

```
    A=temp;
    return A;

}

/* This subroutine compute daily swaption price */

void swaptionmake (double Gij[duration][duration],double
Fi[duration],double l[],double rp, double Bond,double *swaption)
{
    double d1,k1,k3,temp1,temp3,n1,phi,A,chi,k,F;
    k=1;
    A=Amake(Fi,Gij,l,rp);
    F=Fmake(Fi,l,rp);
    chi=-(k-Fmake(Fi,l,rp))/sqrt(A);
    d1 = chi ;
    k1 = 1 / (1 + gamma * d1);
    k3 = 1 / (1 + gamma * (-d1));
    temp1 = 1 - exp(-d1*d1 / 2) / sqrt(2 * Pi) * (a1 * k1 + a2 * k1*k1 +
          a3 * pow(k1,3)+ a4 * pow(k1,4) + a5 * pow(k1,5));
    temp3 = 1 - exp(-d1*d1 / 2) / sqrt(2 * Pi) * (a1 * k3 + a2 * k3*k3 +
          a3 * pow(k3,3)+ a4 * pow(k3,4) + a5 * pow(k3,5));
    if(d1 >= 0.0)
        n1 = temp1;
    else if(d1<0.0)
        n1 = 1 - temp3;
    phi = 2*n1 - 1;
    *swaption =  sqrt(pre)*V* Bond *sqrt(A)*(sqrt(1/ (2*Pi))*exp(-chi*chi/2)
              - chi/2* (1 - phi));
}



void main()
{
    /*declare input out files*/
```

```
FILE *forwardf;
FILE *monthf;
FILE *result;
FILE *testrp;
FILE *testFi;
FILE *testB;
FILE *testGij;
/*declare variables for main program*/
int i,j,ii,jj,kk;
double monthquarter[theta],lquarter[theta],lhalf[theta/2]
    ,Gij[duration][duration],Fi[duration],f[dt][theta],B[dt][theta]
    ,Rs,Bond,swapmodel[dt],x[theta+2],y2[theta+2],yp1,ypn,y[theta+2];
/*declare and define size of pointer for 2 dimensional ZCYC*/
double **ff;
ff=matrix(0,dt-1,0,theta*day_quarter);
/*open files which have declared, txt for input, dat for output*/
forwardf=fopen("ZCYC.txt","r+");
monthf=fopen("month.txt","r+");
result=fopen("Swaption 2by10 price.dat","w");
testrp=fopen("testrp.dat","w");
testFi=fopen("testFi_price.dat","w");
testB=fopen("testB_price.dat","w");
testGij=fopen("testGij.dat","w");
/*declare and define size of pointer for  variables*/
double *swaption;
swaption=(double *)calloc(1,sizeof(double));
double *rp;
rp=(double *)calloc(1,sizeof(double));
double *covariance;
covariance=(double *)calloc(1,sizeof(double));
double *G;
G=(double *)calloc(1,sizeof(double));
double *z;
z=(double *)calloc(1,sizeof(double));
/*initial points of x axis at which the ZCYC is given for using in the spline*/
for (i=1;i<=theta+1;i++)
```

```
{
    x[i]=double((i-1)*0.25);
}
/*read ZCYC from file into matrix f[][]*/
for (i=0;i<dt;i++)
{
    for (j=0;j<theta;j++)
    {
        fscanf(forwardf,"%lf",&f[i][j]);
    }
}
/* call subroutine for computing the bond from ZCYC,feed f[][]
and get back B[][] */
bondmaker(f,B);
/* recursion for cubic spline of all ZCYC, i for trading time */
for (i=0;i<dt;i++)
{
        for (kk=2;kk<=theta+1;kk++)
        {
            y[kk]=f[i][kk-2];
        }
        y[1]=0;
        yp1=(y[2]-y[1])/(x[2]-x[1]);
        ypn=(y[theta+1]-y[theta])/(x[theta+1]-x[theta]);
        /* only y2(second derivative at x axis point) is output,
        the rest is input */
        spline(x, y, theta, yp1, ypn, y2);
        /* j for future time of 30 years (daily)
        smooth function of any point you need on x axis,
        "double(j)*dayinterval" give the x value you input,
        z is the value of ZCYC at the input x point */
        for (j=0;j<=theta*day_quarter;j++)
        {
                splint(x, y, y2, theta, double(j)*dayinterval, z);
                ff[i][j]=*z;
        }
```

```
        printf("%d\n",i);
}
/* main recursion for swaption price, it started at mov_dt since the days
before is set as hitory to compute Gij */
for (i=mov_dt;i<dt;i++)
{

    printf("%d\n",i);
    /* read which month day i belongs to */
    fscanf(monthf,"%lf\n",&monthquarter[0]);
    for (ii=1;ii<theta;ii++)
    {
        monthquarter[ii]=monthquarter[ii-1]+3;
        if (monthquarter[ii]>12) monthquarter[ii]=monthquarter[ii]-12;
    }
    lmaker(monthquarter,lquarter);
    for (ii=0;ii<theta/2;ii++)
    {
        lhalf[ii]=lquarter[ii*2]+lquarter[ii*2+1];
    }
    /* input B and i, output Fi[] */
    Fimaker(B,i,Fi);
    for (j=0;j<duration;j++)
    {
        fprintf(testFi,"%f  ",Fi[j]);
    }
    fprintf(testFi,"\n");
    /* call subroutine for par value of strike price */
    Rpmaker(B,i,lhalf,rp);
    Rs=*rp
    fprintf(testrp,"%f\n",Rs);
    /* discounting bond for 2by10 swaption */
    Bond=B[i][Y1*4-1];
    fprintf(testB,"%f\n",Bond);
    /* recursion for Gij[][] */
    for (ii=0;ii<duration;ii++)
```

```
        {
            for (jj=0;jj<duration;jj++)
            {
                Gijmaker(ff,(ii+1)*2+maturity, (jj+1)*2+maturity, G,i);
                Gij[ii][jj]=*G;
                fprintf(testGij,"%18.16f  ",Gij[ii][jj]);
            }
            fprintf(testGij,"\n");
        }
        /* call subroutine to compute daily swaption price */
        swaptionmake(Gij,Fi,lhalf,Rs,Bond,swaption);
        swapmodel[i]=*swaption;
        /* output daily value for swaption */
        fprintf(result,"%f\n",swapmodel[i]);
    }
    free_matrix(ff,0,dt-1,0,theta*day_quarter-1);
    fclose(forwardf);
    fclose(result);
    fclose(monthf);
    fclose(testrp);
    fclose(testFi);
    fclose(testB);
    fclose(testGij);
    exit(0);
}
```

The simulation program for American option of interest rate derivative

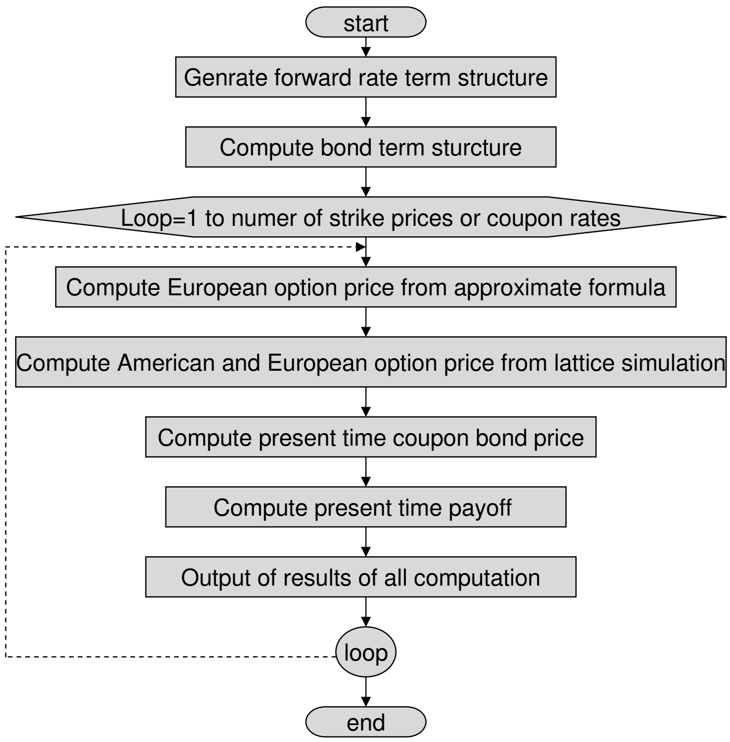

Figure 7.24: Flow chart of main program for American coupon bond option pricing.

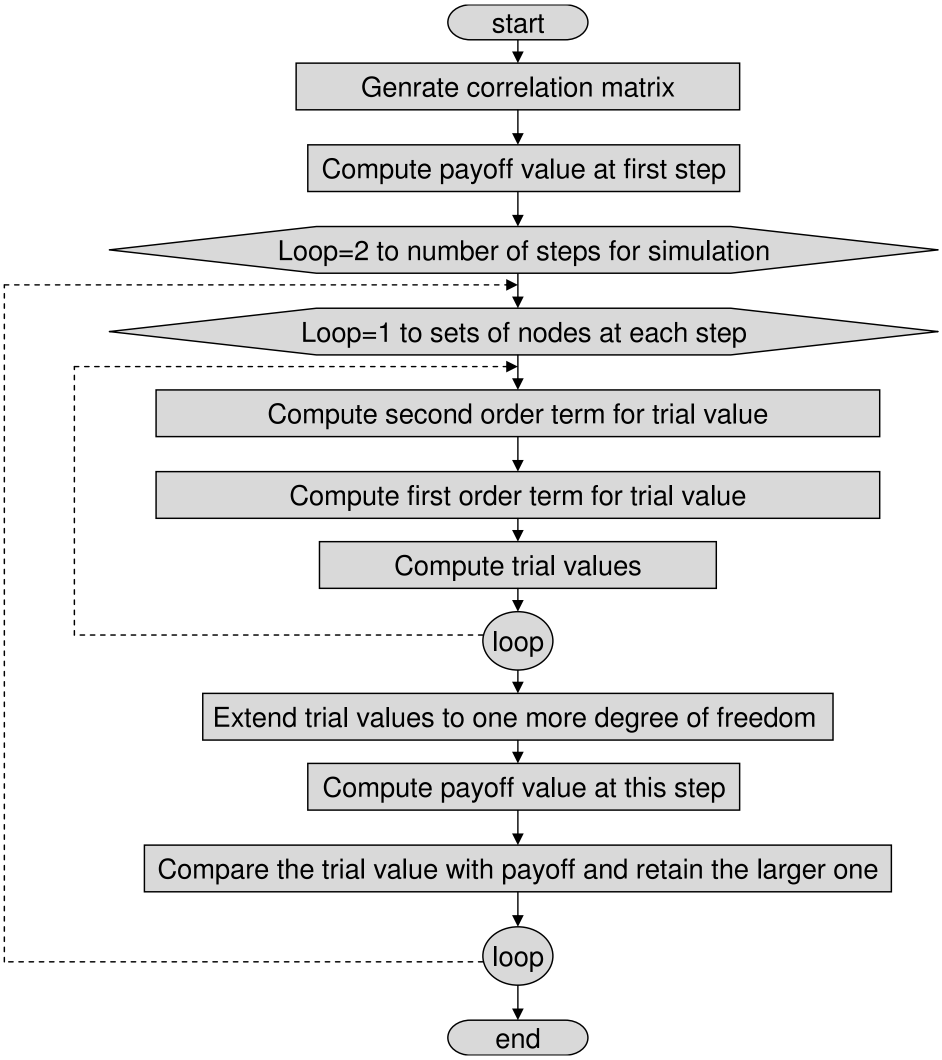

Figure 7.25: Flow chart of subroutine for American coupon bond option simulation.

```
/*
*File: American swaption.cpp
*
*
*Description: This program used forward rate term structure to
*generate American and European coupon bond option
*from lattice field theory model simulation and compare with
*European coupond option price from approximate formula
*
*/
#include <iostream.h>
#include <math.h>
#include <stdlib.h>
#include <stdio.h>
#include <malloc.h>
#include "clresource.h"

#define NR_END 1
#define FUNC(x) ((*func)(x))
#define JMAX 20
#define JMAXP (JMAX+1)
#define K 5
#define EPS 1.0e-4

/* parameter for calculate the cumulative distribution function*/
#define a1 (0.31938153)
#define a2 (-0.356563782)
#define a3(1.781477937)
#define a4 (-1.82125597)
#define a5 (1.330274429)
#define gamma (0.2316419)

#define Pi (3.1415926)
#define V (1000000) /* notional value */
#define N (1000000) /* maximum legth of vector array */
```

```
#define Y1 (1) /* time to maturity */
#define Y2 (0.5) /* tenor of swap */
#define compounding (4) /* compounding frequency */
#define day_quarter (65) /* trading days in three month */
/* number of lattice points in tenor of swaption=compounding*Y2 */
#define duration (2)
/* number of lattice points of forward rates in future
time=compounding*(Y1+Y2); */
#define number_points (6)
/* number of lattice points in calendar time=compounding*Y1+1; */
#define step (5)

/* parameters for integration */
static double xsav,ysav;
static double (*nrfunc)(double,double,double);
double ti,tj;
double tstar=Y1;

double cp=-1; /* cp=1 means call option; cp=-1 means put option */
double strike;
double coupon=0.05;
double epsilon=0.25;
int maturity=compounding*Y1;
double pre=6;
/* prefactor for generating of forward rate term strucutre */
double f0=0.1;


/* function of integration limits */ double yy1(double x) {
    double yl;
    yl=tstar;
    return yl;
}

double yy2(double x) {
    double yh;
```

```
    yh=tstar+ti;
    return yh;
}

double z1(double x,double y) {
    double zl;
    zl=tstar;
    return zl;
}

double z2(double x,double y) {
    double zh;
    zh=tstar+tj;
    return zh;
}

/* function of volatility */ double funcsigma(double x,double y) {
    double fx;
    double yita=0.34;
    fx=0.00055-0.00026*exp(-0.71821*(y-x-0.25))+
        0.00061*(y-x-0.25)*exp(-0.71821*(y-x-0.25));
    return fx;
}

/* function of stiff correlation with psycholgical time */

double funcpropgator(double x,double y,double z)
{
   double fx,miutwidle,yita,lemdatwidle,lembda,ch,b,zp,zm,
       gp,gm,gpm,gmp,gz;
    miutwidle=0.4;
   /*since cosh(b) has lower limit, (lemda/miu)^(2*yita) has can not
   smaller than 2,miu<0.6 lemda>1.69; miu<0.5, lemda>1.39 */
   yita=0.34;
   lemdatwidle=1.79;
   lemda=pow(lemdatwidle,yita);
```

```
   ch=0.5*pow(lemdatwidle/miutwidle,2*yita);
   /* printf("ch=%f\n",ch); */
   b=log(ch+sqrt(pow(ch,2)-1));  /* ch must bigger than 1 */
   zp = pow((y-x), yita) + pow((z-x), yita);
   zm = fabs(pow((y-x), yita) - pow((z-x),  yita));
   gp = exp(-lemda * zp * cosh(b)) * sinh(b + lemda * zp * sinh(b));
   gm = exp(-lemda * zm * cosh(b)) * sinh(b + lemda * zm * sinh(b));
   gpm = exp(-lemda * (zp + zm) * cosh(b)) * sinh(b + lemda * (zp + zm) * sinh(b));
   gmp = exp(-lemda * (zp - zm) * cosh(b)) * sinh(b + lemda * (zp - zm) * sinh(b));
   gz = sinh(b);
   fx = (gp + gm) / pow(((gpm + gz) * (gmp + gz)) ,0.5);
   /* printf("b=%f cosh(b)=%f fx=%f, y=%f, z=%f\n",b, cosh(b),fx, y, z); */
   /* printf("Check-1: b=%f cosh(b)=%f fx=%f\n",b, cosh(b),fx); */
   return fx;
}

/*The 3-dimensional function to be integrated.*/

double func(double x,double y, double z)
{
   double fx;
   fx=funcsigma(x,y)*funcpropgator(x,y,z)*funcsigma(x,z);
   /* printf("Check-1: b=%f cosh(b)=%f fx=%f\n",b, cosh(b),fx); */
   return fx;
}

/* Code for gaussian integration from numerical receipies */
double qgaus(double (*func)(double), double a, double b)
{
int j;
double xr,xm,dx,s; static double
x[]={0.0,0.1488743389,0.4333953941,
0.6794095682,0.8650633666,0.9739065285}; static double
w[]={0.0,0.2955242247,0.2692667193,0.2190863625,0.1494513491,0.0666713443};
xm=0.5*(b+a); xr=0.5*(b-a); s=0; for (j=1;j<=5;j++) { dx=xr*x[j];
s += w[j]*((*func)(xm+dx)+(*func)(xm-dx)); } return s *= xr; }
```

```
/* Code for three dimensional integration from numerical
receipies*/
double quad3d(double (*func)(double, double, double),
double x1, double x2)
{
        double qgaus(double (*func)(double), double a, double b);
        double f1(double x);
        nrfunc=func;
        return qgaus(f1,x1,x2);
}

double f1(double x)
{
        double qgaus(double (*func)(double), double a, double b);
        double f2(double y);
        double yy1(double),yy2(double);
        xsav=x;
        return qgaus(f2,yy1(x),yy2(x));
}

double f2(double y)
{
        double qgaus(double (*func)(double), double a, double b);
        double f3(double z);
        double z1(double,double),z2(double,double);
        ysav=y;
        return qgaus(f3,z1(xsav,y),z2(xsav,y));
}

/*The integrand f(x, y, z) evaluated at fixed x and y.*/

double f3(double z)
{
        return (*nrfunc)(xsav,ysav,z);
}
```

```
/* This subroutine calculate bond term structure by using forward
rate term structure */
void bondmaker(double *f,double *bond)
{
    int i;
    double temp;
    temp=0;
    for (i=0;i<number_points;i++)
    {
        temp=temp+f[i];
        bond[i]=exp(-l*temp);
    }
}

/* This subroutine calculate European swaption price from
approximate formula */
void europeanmake (double *f, double *bond,double *eof)
{
    double (*fun)(double,double,double);
    int i,j;
    double temp,c1,c2,Gij[duration][duration],Fi[duration];
    double d1,k1,k3,temp1,temp3,n1,phi,A,chi,F;

    temp=0;
    for (i=0;i<duration;i++)
    {
        Fi[i]=bond[maturity+i]/bond[maturity-1];
        temp=temp+Fi[i];
        ti=(i+1)*l;
        for (j=0;j<duration;j++)
        {
            tj=(j+1)*l;
            fun=&func;
            Gij[i][j]=quad3d(fun, 0, tstar);
            /* printf("%18.16f\n",Gij[i][j]); */
```

```
    }
}
F=coupon*temp+Fi[duration-1];
temp=0;
for (i=0;i<duration;i++)
{
    for (j=0;j<duration;j++)
    {
        if (i==duration-1)
        {
            c1=coupon+1;
        }
        else
        {
            c1=coupon;
        }
        if (j==duration-1)
        {
            c2=coupon+1;
        }
        else
        {
            c2=coupon;
        }
        temp=temp+c1*c2*Fi[i]*Fi[j]*(Gij[i][j]+0.5*Gij[i][j]*Gij[i][j]);
    }
}
A=temp;
chi=cp*(strike-F)/sqrt(A);
d1 = chi ;
k1 = 1 / (1 + gamma * d1);
k3 = 1 / (1 + gamma * (-d1));
temp1 = 1 - exp(-d1*d1 / 2) / sqrt(2 * Pi) * (a1 * k1 + a2 * k1*k1
      + a3 * pow(k1,3) + a4 * pow(k1,4) + a5 * pow(k1,5));
temp3 = 1 - exp(-d1*d1 / 2) / sqrt(2 * Pi) * (a1 * k3 + a2 * k3*k3
      + a3 * pow(k3,3) + a4 * pow(k3,4) + a5 * pow(k3,5));
```

```
    if(d1 >= 0.0)
        n1 = temp1;
    else if(d1<0.0)
        n1 = 1 - temp3;
    phi = 2*n1 - 1;
    *eof =  bond[maturity-1] *sqrt(A)*(sqrt(1/ (2*Pi))*exp(-chi*chi/2)
          - chi/2* (1 - phi));
}

/* subroutine mapping matrix indicies from vector array index */
void index( int i,int kk,int jj,int indexf[])
{
    int m,j;
    j=jj;
    for (m=i;m>=2;m--)
    {
        if (j%(int(pow(2*(step-kk)+1,m-1)))!=0)
        {
            indexf[m]=j/int(pow(2*(step-kk)+1,m-1))+1;
        }
        else
        {
            indexf[m]=j/int(pow(2*(step-kk)+1,m-1));
        }
        if (indexf[m]==0)
        {
            indexf[m]=2*(step-kk)+1;
        }
        j=j%(int(pow(2*(step-kk)+1,m-1)));
    }
    if (j==0)
    {
        indexf[1]=2*(step-kk)+1;
    }
    else
        indexf[1]=j;
```

```
}

/* this subroutine calculate the first order term of the trail
value of swaption at each step */

double first_order(double *cold,int i,int j)
{
    double first;
    int kk,index_f[number_points+1];
    int TEMP;
    index(i-1,(i-(duration-1)),j,index_f);
    for (kk=1;kk<=i-1;kk++)
    {
        index_f[kk]++;
    }
    TEMP=0;
    for (kk=i-1;kk>=1;kk--)
    {
        if (kk!=1)
        {
            TEMP=TEMP+(index_f[kk]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,kk-1));
        }
        else
        {
            TEMP=TEMP+index_f[kk];
        }
    }
    first=cold[TEMP];
    return first;
}

/* this subroutine calculate the second order term of the trail
value of swaption at each step */
double laplace(double *cold,int i,int j,int kk, int m)
{
    double lapl;
```

```
int ii,index_f[number_points+1];
double term_laplace[8];
int TEMP;
index(i-1,(i-(duration-1)),j,index_f);
for (ii=1;ii<=i-1;ii++)
{
    index_f[ii]++;
}
TEMP=0;
for (ii=i-1;ii>1;ii--)
{
     TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,ii-1));
}
TEMP=TEMP+index_f[1];
term_laplace[1]=cold[TEMP];
TEMP=0;
index_f[kk]++;
for (ii=i-1;ii>1;ii--)
{
     TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,ii-1));
}
TEMP=TEMP+index_f[1];
term_laplace[2]=cold[TEMP];
TEMP=0;
index_f[m]--;
for (ii=i-1;ii>1;ii--)
{
     TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-1))+1,ii-1));
}
TEMP=TEMP+index_f[1];
term_laplace[3]=cold[TEMP];
TEMP=0;
index_f[kk]--;
for (ii=i-1;ii>1;ii--)
{
     TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-1))+1,ii-1));
```

```
    }
    TEMP=TEMP+index_f[1];
    term_laplace[4]=cold[TEMP];
    index_f[m]++;
    TEMP=0;
    index_f[m]++;
    for (ii=i-1;ii>1;ii--)
    {
         TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,ii-1));
    }
    TEMP=TEMP+index_f[1];
    term_laplace[5]=cold[TEMP];
    TEMP=0;
    index_f[kk]--;
    for (ii=i-1;ii>1;ii--)
    {
         TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,ii-1));
    }
    TEMP=TEMP+index_f[1];
    term_laplace[6]=cold[TEMP];
    TEMP=0;
    index_f[m]--;
    for (ii=i-1;ii>1;ii--)
    {
         TEMP=TEMP+(index_f[ii]-1)*int(pow(2*(step-(i-(duration-1)-1))+1,ii-1));
    }
    TEMP=TEMP+index_f[1];
    term_laplace[7]=cold[TEMP];

    lapl=term_laplace[2]-2*term_laplace[1]-term_laplace[3]+term_laplace[4]+
        term_laplace[5]-term_laplace[6]+term_laplace[7];

    return lapl;
}

/* this subroutine calculate the payoff for every set of forward
```

```
rates at one step */
void payoff(double *f,int i,double *g)
{
    double c,dsum,driftsum,forward[number_points+1],
    forfinal[number_points+1][2*(step-1)+1+1]
        ,d[number_points+1],TEMP,bondtemp,drift[number_points+1];
    int j,kk,m,index_f[number_points+1];
    for (j=1;j<=number_points;j++)
    {
        forward[j]=f[number_points-j];
        d[j]=pre*sqrt(epsilon)*sqrt(l)*funcsigma(0,l*(number_points-j+1));
        drift[j]=0;
    }
    for (kk=1;kk<=i;kk++)
    {
        for (m=1;m<=2*(step-(i-(duration-1)))+1;m++)
        {
            if (m-(step-(i-(duration-1)))-1<0)
            {
                dsum=0;
                for (j=1;j<=abs(m-(step-(i-(duration-1)))-1);j++)
                {
                    dsum=dsum+quyear*d[kk+j-1];
                }
                driftsum=0;
            forfinal[kk][m]=l*forward[kk]+epsilon*l*driftsum
                -dsum;
            }
            else if (m-(step-(i-(duration-1)))-1>0)
            {
                dsum=0;
                for (j=1;j<=abs(m-(step-(i-(duration-1)))-1);j++)
                {
                    dsum=dsum+quyear*d[kk+j-1];
                }
                driftsum=0;
```

```
        forfinal[kk][m]=l*forward[kk]+epsilon*l*driftsum
            +dsum;
        }
        else
        {
            driftsum=0;
        forfinal[kk][m]=l*forward[kk]+epsilon*l*driftsum
            +(m-(step-(i-(duration-1)))-1)*d[kk];
        }
    }
}
for (j=1;j<=pow(2*(step-(i-(duration-1)))+1,i);j++)
{
    index(i,i-(duration-1),j,index_f);
    TEMP=0;
    bondtemp=0;
    for (kk=1;kk<=duration;kk++)
    {
        if (kk==duration)
        {
            c=coupon+1;
        }
        else
        {
            c=coupon;
        }
        bondtemp=bondtemp+forfinal[duration-kk+1][index_f[duration-kk+1]];
        TEMP=TEMP+c*exp(-bondtemp);
    }
    g[j]=cp*(TEMP-strike); /* payoff for cap */
    if (g[j]<=0)
    {
        g[j]=0;
    }
    TEMP=0;
    bondtemp=0;
```

```
        for (m=i;m>=3;m--)
        {
            bondtemp=bondtemp+forfinal[m][index_f[m]];
        }
        g[j]=exp(bondtemp)*g[j];
    }
}

/* this subroutine generate american swaption price by recursing
comparison of trial value and payoff at each step up to present
time */
void americanmake(double *f,double *b,double *apnew,double *epnew)
{
    int i,j,kk,m;
    double TEMP1,TEMP2;
    double **prop;
    prop=matrix(1,number_points,1,number_points);
    double *poptold;
    poptold=vector(0,N);
    double *peuroold;
    peuroold=vector(0,N);
    double *poptnew;
    poptnew=vector(0,N);
    double *peuronew;
    peuronew=vector(0,N);
    for (i=1;i<=number_points;i++)
    {
        for (j=1;j<=number_points;j++)
        {
            prop[i][j]=funcpropgator(0,l*i,l*j);
        }
    }
    payoff(f,duration,poptold);
    for (i=1;i<=pow(2*step-1,duration);i++)
    {
        peuroold[i]=poptold[i];
```

```
}
for (i=2;i<=step;i++)
{
    for (j=1;j<=int(pow(2*(step-i)+1,duration+i-2));j++)
    {
        TEMP1=0;
        TEMP2=0;
        for (kk=1;kk<=duration+i-2;kk++)
        {
            for (m=1;m<=duration+i-2;m++)
            {
                TEMP1=TEMP1+1/(2*pre*pre)*laplace(poptold,i+duration-1,j,kk,m)
                     *(prop[duration+i-kk][duration+i-m]);
                TEMP2=TEMP2+1/(2*pre*pre)*laplace(peuroold,i+duration-1,j,kk,m)
                     *(prop[duration+i-kk][duration+i-m]);
            }
        }
        poptnew[j]=first_order(poptold,i+duration-1,j)+TEMP1;
        peuronew[j]=first_order(peuroold,i+duration-1,j)+TEMP2;
    }
    for (j=int(pow(2*(step-i)+1,duration+i-2))+1;
         j<=int(pow(2*(step-i)+1,duration+i-1));j++)
    {
        poptnew[j]=poptnew[j-int(pow(2*(step-i)+1,duration+i-2))];
        peuronew[j]=peuronew[j-int(pow(2*(step-i)+1,duration+i-2))];
    }
    payoff(f,i+duration-1,poptold);
    for (j=1;j<=int(pow(2*(step-i)+1,duration+i-1));j++)
    {
        if (poptnew[j]>poptold[j])
        {
            poptold[j]=poptnew[j];
        }
        peuroold[j]=peuronew[j];
    }
}
```

```
    *apnew=b[maturity-1]*poptold[1];
    *epnew=b[maturity-1]*peuroold[1];
    free_vector(poptold,0,N);
    free_vector(peuroold,0,N);
    free_vector(poptnew,0,N);
    free_vector(peuronew,0,N);
    free_matrix(prop,1,number_points,1,number_points);
}


void main()
{
    /* declare input out files */
    FILE *result;
    /* declare variables for main program */
    int i,j;
    double atmcoupon,S,pay,c,temp;
    double *f;
    f=vector(0,number_points-1);
    double *B;
    B=vector(0,number_points-1);
    double *eof;
    eof=(double *)calloc(1,sizeof(double));
    double *eoa;
    eoa=(double *)calloc(1,sizeof(double));
    double *aoa;
    aoa=(double *)calloc(1,sizeof(double));
    result=fopen("american swaption.dat","w");
    /* generate forward rate term structure from formula */
    for (i=0;i<number_points;i++)
    {
        f[i]=f0*(1-exp(-lambda*l*(i+1)));
        printf("%f, ",f[i]);
    }
    printf("\n");
    /* call subroutine for computing the bond from forward rates,
```

```
feed f[] and get back B[] */
bondmaker(f,B);
/* atmcoupon=(strike-B[number_points-1]/B[maturity-1])
/(B[number_points-2]/B[maturity-1]
+B[number_points-1]/B[maturity-1]);
printf("%f\n ",atmcoupon); */
/* recursing for different coupon rate or strike price,
output daily value for swaption */
for (i=0;i<40;i++)
{
    /* coupon=0.1+i*0.002; */
    strike=1.3+i*0.005;
    cout<<"i="<<i<<endl;
europeanmake(f,B,eof);
americanmake(f,B,aoa,eoa);
temp=0;
for (j=1;j<=duration;j++)
{
    if (j==duration)
    {
        c=coupon+1;
    }
    else
    {
        c=coupon;
    }
    temp=temp+c*B[maturity-1+j]/B[maturity-1];
}
S=temp;
if (strike>S) /* for put option */
/* if (strike<S) for call */
{
    pay=cp*(S-strike);
}
else
{
```

```
        pay=0;
    }
    fprintf(result,"%f,%f,%f,%f,%f\n",S,pay,(*aoa),(*eoa),*eof);
    }
    free_vector(f,0,number_points-1);
    free_vector(B,0,number_points-1);
    fclose(result);
    exit(0);
}
```