NUMERICAL SIMULATION OF SEDIMENT TRANSPORT AND MORPHOLOGICAL EVOLUTION

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NATIONAL UNIVERSITY OF SINGAPORE

2009
NUMERICAL SIMULATION OF SEDIMENT TRANSPORT
AND MORPHOLOGICAL EVOLUTION

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A THESIS SUBMITTED
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF CIVIL ENGINEERING
NATIONAL UNIVERSITY OF SINGAPORE
2009
To My Parents
Acknowledgements

First and foremost, I would like to express my gratitude to my supervisors, Professor Cheong Hin Fatt and Professor Lin Pengzhi, for their guidance, support and encouragement throughout my study at National University of Singapore. Numerous meetings and discussions are the origins of the research ideas and the directions of the way going forward. Their attitude for the research will lead me further in the future career. The time spent with me and the patience allowing me to improve myself should be appreciated. Without them, this thesis would not have been possible.

I also like to thank my previous supervisor, Professor Zhang Qinghe at Tianjin University during my study for the Master of Engineering from 2001 to 2004. His knowledge and virtue are always worthy of my respect.

I have also benefited from the generosity of many others and special thanks go to the following persons. The numerical model developed in this study is partially based on the PhD thesis of Dr. Yong-Sik Cho at Cornell University. And the program for the turbulence spectrum analysis was generously provided by Dr. Ren-Chieh Lien at the University of Washington, who also gave me valuable guidance in this research field. In addition, analytical solutions of the shock wave for the numerical testing of the morphological evolution equation were kindly provided by Dr. Wen Long at University of Maryland. Their generosity is appreciated.

I would like to acknowledge the Research Scholarship provided by National University of Singapore from 2004 to 2008. I am grateful for the financial support from the Research Engineer position provided by Professor Cheong Hin Fatt from 2008 to 2009.
I am happy to thank Mr. Zhang Dan, Mr. Zhang Wenyu, Dr. Liu Dongming, Mr. Chen Haoliang, Mr. Sun Yabin, Mr. Xu Haihua, Dr. Ma Peifeng, Dr. Anuja Karunaratna, Dr. Pradeep Fernando, Dr. Cheng Yonggang, Mr. Shen Wei, Mr. Chen Zhuo, Mr. Lim Kian Yew, Mr. Satria Negara, Dr. Gu Hanbin, and Dr. Zhang Jinfeng, for their friendship and valuable discussion during the study. Special thanks go to Dr. Wang Zengrong, for his helpful discussion about the signal analysis with me.

Thanks are extended to Mr. Krishna and Ms. Norela for their help between office and laboratory and to Mr. Semawi and Mr. Roger for their assistance my experiments at Hydraulics Laboratory.

Last but not least, I would like to express the gratitude from my heart to my parents and my sister, who have been giving me the unconditional love in my life. I also like to thank my wife for her care, patience and love. I could not finish my study without the support from all of them.
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Summary

A two-dimensional depth-averaged numerical model has been developed to simulate long-term sediment transport and morphological evolution. Furthermore, considering the fact that the detailed experimental studies on the turbulent flows involving sediment transport and morphological evolution are few, a series of experiments have been conducted in the laboratory flume to provide valuable measured data for purposes of model validation.

The numerical model consists of three modules: the hydrodynamic module, the sediment transport module and the morphological evolution module. Firstly, the hydrodynamic conditions are computed by solving the shallow-water equations with the depth-averaged $\hat{k}-\hat{\varepsilon}$ turbulence closure. Based on the flow conditions, the suspended sediment concentration is evaluated by solving the convection-diffusion equation while the bed load transport is predicted from an empirical equation. Finally, the bed evolution is calculated using fifth-order accurate WENO (Weighted Essentially Non-Oscillatory) scheme. In order to improve the prediction, the bed shear stress obtained from the traditional Manning’s formula is corrected according to the secondary flow effect with the assumption of a “triangular model” for the main flow and the cross flow components. To simulate the sediment transport on the sloping bed more realistically, the effect of the bed slope, i.e., the effect of gravity on the sediment particle, is incorporated into the model. Both the critical shear stress for the sediment incipient motion and the sediment transport direction are corrected according to the local bed slope. In addition, utilizing the difference of the stability criteria between flow and sediment transport calculations, an
approximate method is proposed for the gradually varying sediment bed to improve the computational efficiency.

After careful numerical testing, the model is first applied to study the sediment transport in a trench with different slopes and over a dune respectively under the open channel flow conditions. In the long-term simulations, the numerical model gives good predictions for the whole process of bed evolution.

Moreover, the studies are extended to the two-dimensional situations covering the turbulent flow and sediment transport and the morphological evolution in the channels with abrupt cross-section changes. The experiments are conducted in a channel with an abrupt expansion and in a channel with an abrupt contraction. Three-dimensional velocity components are measured from which both the mean flow and turbulent flow fields are obtained. The dissipation rate of the turbulent kinetic energy is estimated from the inertial subrange in Kolmogorov spectrum. Under the same flow conditions, the morphological evolution resulted from the bed load transport is investigated and the evolution of the bed profiles are recorded. Using the present model, the numerical simulation is carried out and good predictions for the trend of the bed evolution are obtained. Lastly, the hydrodynamic conditions and the morphological evolution in a channel consisting of a contraction and an expansion are studied numerically. Compared to the available experimental data and the numerical results from a 3D model, the present model gives reasonably good predictions with high computational efficiency.
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List of Symbols

\( a \)  
reference level

\( a_{i+1/2} \)  
Roe speed of bed-form propagation

\( A \)  
parameter

\( B \)  
channel width

\( c \)  
sediment concentration

\( C \)  
depth-averaged sediment concentration

\( C_a \)  
reference concentration at reference level \( a \)

\( C_{a,e} \)  
equilibrium near-bed concentration

\( C_b \)  
depth-averaged bed-load concentration

\( C_D \)  
drag force coefficient

\( C_{\text{wave}} \)  
wave celerity

\( c_f \)  
friction coefficient

\( c_fx \)  
friction coefficient in x-direction

\( c_k, c_e \)  
empirical constants

\( C_\mu, C_{1e}, C_{2e} \)  
empirical constants in turbulence model

\( C(z_b) \)  
bed-form propagation phase speed

\( Cr_{\text{Flow}}, Cr_{\text{Sedi}} \)  
Courant numbers for flow and suspended load computations

\( d \)  
sediment particle diameter

\( d_{50} \)  
median diameter of the sediment particle
$d_{10}, d_{90}$  sediment particle diameter such that 10% and 90% of all the grain sizes are smaller than $d_{10}$ and $d_{90}$, respectively

d_{\text{sphere}}  
diameter of the sphere

$D_s$  
particle size parameter

$F_D$  
drag force

$F_{cr,0}$  
drag force for sediment particle on a flat bed

$F_{cr,\beta}$  
drag force for sediment particle on a slope

$Fr$  
Froude number

$F\hat{k}X, F\hat{k}Y$  
convection terms in $\hat{k}$–equation

$F\hat{\epsilon}X, F\hat{\epsilon}Y$  
convection terms in $\hat{\epsilon}$–equations

$F_{cx}, F_{cy}$  
sediment fluxes in $x$- and $y$-directions

$g_i$  
i-th component of the gravitational acceleration

$H$  
water depth

$H_f$  
flooding depth

$i, j$  
spatial nodes when subscript

$k$  
turbulent kinetic energy

$\hat{k}$  
depth-averaged turbulent kinetic energy

$\tilde{k}$  
wave number

$k_s$  
Nikuradse roughness

$k_1, k_2$  
correction factors for streamwise and transverse sloping beds

$n$  
Manning’s roughness coefficient; time level when superscript
$nx, ny$ grid numbers in $x$- and $y$-directions

$p$ pressure

$P, Q$ volume flux components in $x$- and $y$-directions

$P$ production of turbulent kinetic energy

$P_n, P_r$ flow flux components normal and tangential to the solid boundary

$P_h$ horizontal production term of turbulent kinetic energy

$P_{kV}, P_{eV}$ vertical production terms of turbulent kinetic energy and its dissipation rate

$poro$ porosity factor of sediment

$q_b$ bed load transport rate

$q_{bx}, q_{by}$ bed load transport rates in $x$- and $y$-directions

$R$ resultant force of the drag force and gravitational force component along the steepest slope

$Re$ Reynolds number

$Re_*$ grain Reynolds number

$S$ energy slope

$s$ specific gravity of the particle

$S_D, S_E$ sediment deposition and entrainment fluxes

$t$ time

$T$ excess bed shear stress parameter

$T_{xx}, T_{yx}, T_{xy}, T_{yy}$ depth-averaged effective stresses

$u, v, w$ velocity components in $x$-, $y$- and $z$-directions
\( \bar{u}, \bar{v}, \bar{w} \) mean velocities in \( x-, y- \) and \( z \)-directions

\( u', v', w' \) velocity fluctuations in \( x-, y- \) and \( z \)-directions

\( U, V \) depth-averaged velocity components in \( x- \) and \( y \)-directions

\( U_{\text{max}}, V_{\text{max}} \) maximum depth-averaged velocities in \( x- \) and \( y \)-directions

\( U_{\text{main}} \) main stream velocity

\( U_R \) resultant velocity

\( U_t \) tangential velocity

\( u_* \) friction velocity

\( u_f \) velocity in a hypothetical two-dimensional boundary layer

\( VIS\hat{k}X, VIS\hat{k}Y \) diffusion terms in \( \hat{k} \) – equation

\( VIS\hat{e}X, VIS\hat{e}Y \) diffusion terms in \( \hat{e} \) – equations

\( w_f \) settling velocity of sediment particle

\( x, y, z \) coordinates in Cartesian coordinate system

\( W \) submerged weight of sediment particle

\( y_0 \) zero-velocity level in the logarithmic law-of-the-wall

\( z_b \) bed elevation which is reckoned negative when measured vertically upwards with respect to the datum

\( \alpha \) angle between flow and \( x \)-axis

\( \beta \) slope angle

\( \beta_x, \beta_y \) angles that the slope makes with \( x \) and \( y \)-axes
γ  
angle between the drag force and the gravitational force component along the steepest slope

γ_w  
angle of near wall streamline with respect to mainstream velocity

γ_1  
angle between the flow direction and the resultant force

γ_2  
angle of γ_1 projected on the horizontal plane

Δ  
bed form height

Δt  
time step

Δt_{Flow, ∆t_{Sed}}  
time steps used in the flow and suspended load computations

Δx , Δy  
spatial steps

δ_{ij}  
Kronecker delta

ε  
dissipation rate of turbulent kinetic energy

\hat{ε}  
depth-averaged dissipation rate of turbulent kinetic energy

η  
free surface elevation measured vertically with respect to the datum

θ  
Shields parameter

θ_{cr}  
critical Shields parameter

θ  
angle that the base line of the slope makes with x-axis

θ'  
angle that the base line of the slope makes with flow direction

κ  
von Karman constant

μ  
molecular viscosity of the fluid

μ  
frictional coefficient

ν_m  
fluid kinematic viscosity

ν_s  
sediment diffusivity
\( \dot{v}_s \) depth-averaged sediment diffusivity

\( \nu_t \) turbulent or eddy viscosity

\( \dot{v}_t \) depth-averaged turbulent viscosity

\( \rho \) density of fluid

\( \rho_s \) density of sediment

\( \sigma_{ij} \) rate of the strain tensor

\( \sigma_k, \sigma_\epsilon \) Prandtl numbers

\( \tau_b \) bed shear stress

\( \tau_{bx}, \tau_{by} \) bottom friction stresses in \( x \)- and \( y \)-directions

\( \tau_{b,cr} \) critical bed shear stress

\( \tau_{cr,0} \) critical bed shear stress on a flat bed

\( \tau_{cr,\beta} \) critical bed shear stress on a slope

\( \tau_{ij} \) molecular viscous stress tensor

\( \tau_w \) shear stress on solid wall

\( \phi \) angle of repose

\( \langle \cdot \rangle \) mean quantities
Chapter 1

Introduction

1.1 Background of Sediment Transport Study

Sediment transport under hydrodynamic conditions plays an essential role in the morphological evolution of rivers, estuaries and coastal areas (Guo and Jin, 1999). From the long-term point of view, it determines, for example, the local scour or deposition in the vicinity of the river structures (e.g., groins and bridge piers) and consequently their instability. In addition, sediment transport is the crucial factor in the morphological migration including the formation of estuarine delta, the beach erosion and shoreline retreat. On the other hand, strong flows such as those induced by dam-break may generate intense erosion and transport and cause the drastic topographic deformation in a very short time (Soares-Frazao et al., 2007). All these phenomena will have great influence on the earth and the human being. Therefore, it is very important to study the sediment transport phenomena and the resulting morphological change.

Due to the complexity of the sediment transport mechanism, both numerical modeling and experimental investigation are very important methods for studying the sediment transport.

Compared with the experimental study, numerical simulation is probably a very convenient and effective method in sediment transport study. Generally speaking, there are two kinds of numerical models which are used in the sediment transport simulations, i.e., depth-resolved models and depth-averaged models. For the depth-resolved models,
the Reynolds-averaged Navier-Stokes (RANS) equations with some kind of turbulence closure are solved for the flow field. Based on the information on the flow field, the sediment transport and thereafter the bed changes can be calculated. For example, Nagata et al. (2005) developed a full three-dimensional model to simulate the flow and the bed deformation around river hydraulic structures. The model solved the RANS equations with $k-\varepsilon$ turbulence closure for the flow field. The change of the bed topography was calculated by coupling a stochastic model for sediment pickup and deposition. The model was validated in the situations of rivers with the spur dike and the bridge pier, respectively. The results predicted by the model were compared with the laboratory observations and sufficient accuracy could be found in terms of the flow and the scour geometry around the structures. Similarly, Minh-Duc and Rodi (2008) used a full three-dimensional model to calculate the flow and the sediment transport in a contracted channel with movable bed. As the emphasis of their study, the nonequilibrium adaptation length used in the calculation of the bed load transport was investigated systematically.

In the depth-resolved models, it is of essential importance to understand physically and describe mathematically the sediment particle exchanges between on the bed and in the water when dealing with the suspended load transport. van Rijn (1984c) performed a series of tests in the laboratory flume in order to determine the pick-up rate experimentally. The experimental results yielded a simple pick-up function which was evaluated by comparison with other existing functions. In addition, van Rijn (1985) used mathematical models to study the concentration profiles for the net entrainment situation, the net deposition situation and the situation containing both entrainment and deposition, respectively. Similar cases have also been studied by Celik and Rodi (1985).
Because of the lack of sufficient theoretical description of the sediment entrainment, a variety of empirical expressions have been proposed, such as those given by van Rijn (1984c), Garcia and Parker (1991), Cao (1997) and Pizzuto (1987). Among them, the one based on the concept of equilibrium near-bed concentration is adopted extensively by many researchers. For example, in the mathematical model developed by Celik and Rodi (1985, 1988) for calculating suspended sediment transport in open channel flow under steady, non-equilibrium situations, a net flux boundary condition is applied near the bed for the concentration equation and is prescribed as the difference between deposition of sediment to and entrainment from the bed. The deposition rate is known from the local concentration and settling velocity, while the entrainment is assumed to occur at the same rate as it does under equilibrium conditions, provided sufficient sediment material is available on the bed. The empirical expression of equilibrium near-bed concentration proposed by van Rijn (1984b) has shown some good performance (e.g., van Rijn, 1984c; van Rijn, 1985; van Rijn, 1986; van Rijn, 1989a; Toro et al., 1989).

Although the depth-resolved models can provide better accuracy of the computation, they are more computationally time-consuming and less efficient than the depth-averaged ones. For long-term and large-scale simulations, depth-averaged models are necessary and show their advantages under the condition of the current computational power (Jia and Wang, 1999). Therefore, the depth-averaged models are applied extensively in the areas of rivers, estuaries and coastal regions.

In the depth-averaged models, the flow field is usually calculated based on the depth-averaged equations, which include the shallow-water equations, the Saint-Venant equations and the Boussinesq equations. The horizontal two-dimensional convection-diffusion equation including the terms describing the sediment movement in the vertical
direction is solved for suspended load transport. Similar to the depth-resolved model, empirical formulas and the sediment conservation equation are solved in the depth-averaged model for the bed load transport and the bed change, respectively.

The depth-averaged models can be used to study the sediment transport and morphological evolution in many circumstances. Firstly, they can be used for calculating the sediment transport and morphological changes in the channels or rivers. For example, Minh-Duc et al. (2004) developed a depth-averaged model using a finite-volume method with boundary-fitted grids to simulate the bed deformation in alluvial channels. In the model system, the hydrodynamic module was based on the shallow-water equations. The sediment transport module comprised of the semi-empirical suspended load formula and the nonequilibrium bed load formula. The bed deformation module was based on the sediment mass balance. In addition, both secondary flow effect and bed slope effect were taken into account in the model. The applications of the model included the bed scour and deposition in a shallow pool with a jet discharge and the bed deformation in curved channels under steady and unsteady flow conditions. Generally good agreement was shown when comparing the numerical predictions with the laboratory measurements.

Guo and Jin (2002) used a two-dimensional model for nonuniform suspended sediment transport to simulate riverbed deformation. The sediment mixture was divided into several size groups and each group was considered to be composed of uniform particles. After the verification with laboratory data, the model was applied to an alluvial river and encouraging results were obtained in terms of water level, sediment concentration, suspended sediment size distribution and riverbed variation.

Secondly, the depth-averaged models can predict the bed evolution in curved or meandering channels. Although the flow and the sediment transport processes in the
meandering channels are three-dimensional in nature, the depth-averaged models can still predict the flow field and bed evolution with reasonable accuracy. For example, Kassem and Chaudhry (2002) developed a two-dimensional model to predict the bed deformation in alluvial channel bends. In their model, the depth-averaged water flow equations were solved along with the constant eddy viscosity assumption. Only bed load transport was considered to contribute to the bed evolution. This model was applied to model the bed evolution in flumes with 140° and 180° bends. The numerical predictions agreed quite well with the laboratory data. The morphological modeling in the same 180° curved flumes was also conducted by Abad et al. (2008) using their 2D depth-averaged model named STREMR HySeD.

Vasquez et al. (2008) reported their numerical investigation on the bed changes in the meandering Waal River using a two-dimensional depth-averaged model. Comparisons between numerical results and observed data showed good agreement.

Duan and Julien (2005) employed a depth-averaged two-dimensional numerical model to study the inception and development of channel meandering processes. Both bed load and suspended load were calculated assuming equilibrium sediment transport and the bank erosion consisted of the basal erosion and the bank failure. The numerical results showed the potential of the depth-averaged models in the simulation of the channel meandering process.

Thirdly, the depth-averaged models can be used to predict the strong bed erosions due to the dam-break flows. Over the recent decades, continuing efforts have been made to investigate the dam-break hydraulics and the depth-averaged models have become an important means for the study. For example, Cao et al. (2004) presented a one-
dimensional model for studies on the mobile bed hydraulics of dam-break flow and the induced sediment transport and bed evolution. The model was based upon the conservative laws of the shallow water hydrodynamics.

Wu and Wang (2007) also established a one-dimensional depth-averaged model to simulate the dam-break flow over movable beds. Being tested in two experimental cases, the model showed reliable performance with fairly good agreement between numerical results and measurements.

Zech et al. (2008) adopted a two-layer depth-averaged model to study the dam-break induced sediment movement. The model was applied to both a flat bed channel and a trapezoidal channel and provided quite good results compared with the laboratory observations.

In addition, the depth-averaged model can also be applied to study the formation processes and configuration of channel-flow dominated alluvial deltas (Tseng et al., 2006), the behavior of the alternate bars in a channel (Jang and Shimizu, 2005), the formation of channel and shoal patterns in well-mixed elongated estuaries (Hibma et al., 2003), and the longshore sediment transport by nonlinear waves and currents (Karambas and Karathanassi, 2004), etc.

In the depth-averaged modeling, since the information across the water depth is not provided for the suspended load transport, it is crucial to search for reasonably accurate ways to represent the vertical information. Promisingly, the way to describe sediment deposition on or entrainment from beds in depth-resolved models can be extended to the depth-averaged ones. For example, the calculations of the suspended sediment deposition and entrainment by Guo and Jin (1999) are similar to the methods used in depth-resolved models. In their calculations, the entrainment rate was equal to the sediment carrying
capacity under the equilibrium condition multiplied by a factor and the deposition rate was equal to the settling velocity times the near-bed concentration which was equal to the depth-averaged concentration multiplied by a factor. The same idea but different expression for transport capacity was also used by Zhou and Lin (1998) in their 1D model.

In addition to the numerical simulation, experimental study is another important method to study the sediment transport. Many researchers have conducted various experiments for better understanding the mechanism of sediment transport. For example, Shields (1936) was the pioneer in the experimental investigation of the sediment incipient motion and his result, i.e., Shields diagram, is still enjoying extensive popularity nowadays.

Sumer and Fredsoe (2001) conducted the experimental study on the scour around a pile subject to the combined waves and current. In their study, two kinds of experiments were carried out: one was the waves and current were in the same direction while the other was the direction of the wave propagation was perpendicular to the current. Some hydraulic parameters were measured and some conclusions were drawn for the scour depth.

In addition, experiments provide important measured data for the validation of the numerical models. For example, Gonzalez et al. (2008) presented the experimental validation of a two-dimensional depth-averaged numerical model of sediment transport using laser technologies such as particle image velocimetry (PIV) and three-dimensional scanner. The study was carried out by through a series of tests with bed load transport. The comparisons between the numerical and experimental results showed that the depth-averaged model could accurately reproduce the bed profile evolution as well as the velocity fields.
Due to the importance of the experimental study and efficiency of numerical simulation, researchers usually combine both of them to study the sediment transport and morphological evolution. For example, when investigating the vegetation effects on the morphological behavior of alluvial channels, Jang and Shimizu (2007) carried out both laboratory experiments and numerical simulations. Zhang et al. (2007) also investigated the flow and bed deformation around groins under flood conditions in a river restoration project with both experimental and numerical methods.

1.2 Background of Shallow-Water Equations Models

The success of modeling sediment transport relies on the computational accuracy of hydrodynamics which drive sediment movement. As long as the water depth $H$ is small relative to the wave length $L$, i.e., $H < L/20$, the flow field can be described by using the shallow-water equations (SWE). Open-channel flows, tidal waves and tsunamis are all included in the range of shallow-water waves or long waves (Dean and Dalrymple, 1991).

Without question, SWE models can be applied to the circumstances mentioned above with satisfactory accuracy. At the same time, one can also enjoy the good computational efficiency due to their depth-averaged nature. Therefore, with these merits, SWE models have extensive applications in the simulation of hydrodynamics.

For example, Chapman and Kuo (1985) applied a SWE model to study the recirculating flow in a rectangular channel with symmetrically abrupt expansion in width. McGuirk and Rodi (1978) used their SWE model to calculate a side discharge into open channel flow. The recirculation zone developing downstream of the discharge was well predicted with the comparison with the experiments. Molls et al. (1995) numerically
simulated the flow near a groin by solving the SWE. The numerical results compared favorably with the experimental measurements.

To study the complex hydrodynamic phenomena in well-mixed estuaries, Loose et al. (2005) solved the SWE using the finite volume method in combination with an advection-diffusion equation for the salt. The saline intrusion in the Rio Maipo estuary was studied by using their model and the numerical results showed that due to the presence of a littoral bar at the river mouth, the salt was precluded from transporting upstream and the salinity intrusion was almost negligible. These results were validated with the field measurements.

In addition, SWE models have been widely applied in other circumstances, such as the flow around bridge abutments in a compound channel (Biglari and Sturm, 1998), the flow fields in navigation installations induced by hydropower releases (Bravo and Holly Jr., 1996), the flow in a strongly curved channel containing a 180° bend (Puri and Kuo, 1985 and Molls and Chaudhry, 1995), the flow in a meandering channel containing two 90° bends in alternating directions (Ye and McCorquodale, 1997) and the transverse mixing layer in shallow open-channel flows (Babarutsi and Chu, 1998).

In addition to modeling the subcritical flows, SWE models have the ability to simulate the rapidly varied flows. In these cases, the water depth or flow velocity changes abruptly over a short distance. Since the hydrostatic pressure assumption held in the depth-averaged models is usually broken, it is a great challenge for all depth-averaged models. However, with careful numerical treatments, SWE models can still simulate the rapidly varied flows with satisfactory accuracy. For example, Zhou and Stansby (1999) simulated several hydraulic jumps occurring in different situations using a 2D SWE model. The equations of the model were discretised using the finite volume method in a strong
conservation form. Their results were compared with the available experimental data and numerical results and showed that their SWE model could provide good results.

Using a numerical technique essentially based upon the staggered grids and the conservative numerical schemes, Stelling and Duinmeijer (2003) could simulate the flows over bathymetry with strongly large gradient. Moreover, as reported by Ye and McCorquodale (1997), the SWE model can be applied supercritical flow occurring in a Parshall flume. Younus and Chaudhry (1994) also adopted a SWE model in the numerical simulations of some rapidly varied open channel flows including a hydraulic jump in a diverging channel, a supercritical flow in a diverging channel and a circular hydraulic jump. Furthermore, the formation, evolution and dissipation of the tidal bore could even be reproduced by using SWE models (Pan et al., 2007).

SWE models can be applied to some extreme cases such as the flooding caused by the dam failures. For example, Zhou et al. (2004) simulated numerically the dam-break flows in general geometries with complex bed topography using a model based on the SWE. The tests included a channel with a 90° bend, a channel with a 45° bend and a straight channel with a triangular bump on the bed. The numerical results were compared with the experimental data and good agreement between them was shown.

Using 2D SWE model, Wang et al. (2000) successfully simulated the reflection and interactions for 1D dam-break bores, 2D partial dam-break and the dam-break bore diffraction around a rectangular barrier. Similar 2D partial dam-break flows were simulated by Aureli et al. (2008) using a finite volume numerical model based on the classical SWE. The comparisons between the numerical and experimental results showed good agreement in terms of the water depth.
Without turbulence closure, the applications of the SWE models are limited in some ideal cases, e.g., Wang et al. (2000) and Stelling and Duinmeijer (2003). To simulate the realistic flows, the turbulence models are usually needed.

For simplicity, zero-equation models can be used as the turbulence closure in the SWE models. One of these models is the constant eddy viscosity model and assumes that the eddy viscosity is constant throughout the flow field. The value of the constant eddy viscosity is found from empirical information or from trial and error calculations (Molls et al., 1995).

Another zero-equation model relates the depth-averaged eddy viscosity to the friction velocity and the water depth. This method assumes a linear distribution of the shear stress and a logarithmic distribution of the velocity leading to a parabolic distribution of the eddy viscosity. After depth-averaging the eddy viscosity, the depth-averaged eddy viscosity is obtained. This model implies that the turbulence is generated and dissipated locally and there is no transport of turbulence in the flow field. The examples of using this model include Zhou (1995) and Zhou and Stansby (1999).

The advanced turbulence closures used in the SWE models are the two-equation models among which the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence model has been adopted by many researchers. By solving $\hat{k}$ - and $\hat{\varepsilon}$ -equations, the generation, transport and dissipation of the turbulence quantities in the flow field can be determined.

Since the introduction of the $\hat{k} - \hat{\varepsilon}$ turbulence model into the depth-averaged models by Rastogi and Rodi (1978), the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence model has been tested extensively. For example, with the help of the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence model, McGuirk and Rodi (1978) successfully predicted the recirculation zone, jet trajectories,
dilution and isotherms in the problem of the side discharges into open-channel flow. The empirical constants simply adopted from three-dimensional $k-\varepsilon$ turbulence model were also proved to be satisfactory. Similar $\hat{k}-\hat{\varepsilon}$ turbulence model with slight difference on the expressions of the production terms was adopted by Chapman and Kuo (1985) to study the separated flow in a rectangular channel with abrupt expansion in width. As reported by Younus and Chaudhry (1994), it was observed that the numerical simulations of the supercritical flow in a diverging channel and the radial hydraulic jump were improved with the adoption of the depth-averaged $\hat{k}-\hat{\varepsilon}$ turbulence model. In the simulation of the flow in a strongly curved channel, the depth-averaged two-equation $\hat{k}-\hat{\varepsilon}$ turbulence model yielded excellent agreement with the experimental data (Puri and Kuo, 1985). The depth-averaged $\hat{k}-\hat{\varepsilon}$ turbulence model also has extensive applications in other circumstances and has shown a good behavior (e.g., Bravo and Holly Jr., 1996; Ye and McCorquodale, 1997; Babarutsi and Chu, 1998; Biglari and Sturm, 1998; Minh-Duc et al. 2004; Wu, 2004).

1.3 Review on Considerations of Slope Effect on Sediment Transport

Most of the bed load transport equations are derived and calibrated based on the flat bed data. On the other hand, the direction of the sediment transport is normally considered to be coincident with the direction of the bed shear stress. When sediment transport happens on flat or relatively gentle beds, the normal way of determining sediment transport and its direction would be acceptable. However, when the sediment transport on steep beds is considered, the effect of the bed slope on the sediment transport rate as well
as its direction should be included in the calculations to avoid the large deviations in the computed results.

Many researchers have made strenuous efforts in the calculations of sediment transport on sloping beds. Some of them have proposed new transport equations to replace those equations derived from flat beds. For example, Smart (1984) investigated the sediment transport capacity under the flow condition in a flume with steep downsloping bed up to $11.3^\circ$ and reported that the Meyer-Peter and Muller equation seriously underestimated sediment transport for slopes steeper than $1.7^\circ$. Therefore, a new equation was proposed for the slopes from $2.3^\circ$ to $11.3^\circ$ on the basis of his experimental data. Similarly, Dey and Debnath (2001) also conducted an experimental investigation on sediment pickup under unidirectional flow condition in a closed duct. The angle of the bed slope was varied from upsloping $15^\circ$ to downsloping $25^\circ$. A sediment pickup equation determined from the experimental data was suggested to be used on horizontal, downsloping and upsloping beds. Although these equations are proposed based on the data on sloping bed, their applications are still limited due to the limited verification.

In addition to the effect of the streamwise bed slope, the effect of the transverse slope has been studied too. For example, Sekine and Parker (1992) determined a relation for the ratio of transverse to streamwise bed load transport by using a stochastic model of saltating grains.

Instead of adding new equations to the numerous existing equations, an alternative way is to extend the transport equations to the situations of sloping beds through some modifications. Most of the transport equations contain two important variables: bed shear stress and critical shear stress, of which the former is normally from the drag force of the
flow and the latter is normally determined from the Shields diagram based on the flat bed experiments. Therefore, these two variables are usually modified accordingly to include the bed slope effect into the transport equations.

A conventional treatment to modify the critical shear stress, as suggested by van Rijn (1989b), is to resolve the bed slope into a streamwise slope and a transverse slope according to the flow direction. For the streamwise slope, the critical shear stress has a correction factor which is a function of the streamwise slope angle. Similarly, for the transverse slope, the critical shear stress has another correction factor which is the function of the transverse slope angle. These two factors can be derived through analyzing the force balances on a sediment particle resting on a longitudinal sloping bed and a transverse sloping bed respectively. To consider the effect of a generalized slope, the critical shear stress will be corrected by both streamwise and transverse correction factors.

More rigorous derivation of the correction factor for an arbitrarily sloping bed has been introduced by Brook (1963), Zhang (2007) and Apsley and Stansby (2008). In their analyses, a sediment particle resting on a sloping bed with arbitrary orientation was considered and the forces acting on it included the drag force from the flow, the gravitational force and the frictional force. After analyzing the relationships among these forces, a correction factor for the critical shear stress was obtained. Note that only the magnitude of the critical shear stress has been corrected by them.

In some occasions, the modification for the critical shear stress is not enough and additional modifications for the transport equation would be needed. For example, on the basis of the experiments on bed load transport on steep longitudinal slopes, Damgaard et al. (1997) found that for upsloping bed and small downsloping bed, the slope effect on bed load transport was described adequately by correcting the critical Shields parameter in the
Meyer-Peter and Muller equation for the slope. For large downsloping bed, they suggested that an additional correction factor was required.

Rather than modifying the critical shear stress, another way to consider the bed slope effect is to modify the bed shear stress. For example, Wu (2004) added the streamwise component of the gravitational force to the bed shear stress. Therefore, the bed shear stress in the sediment transport formulas was replaced by the effective shear stress in which the gravitational contribution was derived by considering the extreme case with bed angle equal to the angle of repose and then was modified further for the downsloping bed. It should be noted that when considering the gravitational contribution to the effective shear stress, only the streamwise component of the gravitational force has been included.

The bed slopes will affect not only the magnitude of the sediment transport, but also its direction. When the bed slope is present, the trajectory of the sediment movement will be affected by the component of the gravitational force. To consider the influence of the bed slope on the direction of the sediment transport, Struiksma (1985), Mosselman (2005), Abad et al (2008) and Vasquez et al. (2008) corrected the direction of the bed load transport, which originally coincides with the direction of the bed shear stress, by taking into account the gravitational effect along the bed slopes in x- and y-directions, respectively. This mainly considers the transverse effect of the bed slope on the sediment transport direction. A parameter contained in the function of calculating the transport direction is derived theoretically from a force balance or even empirically from flume experiments (Mosselman, 2005).
1.4 Objective and Scope of Present Study

The present study consists of two objectives. One is to develop a two-dimensional depth-averaged numerical model to simulate long-term sediment transport and morphological evolution. Considering the fact that the detailed experimental studies on the turbulent flows, sediment transport and morphological evolution are rare, the other objective is to conduct a series of well-designed experiments in the laboratory flume to provide valuable experimental measurements.

In the numerical model development, a previous shallow-water equation model (Cho, 1995; Liu and Cho, 1995) has been extended to simulate the turbulence, sediment transport and the morphological evolution. The present model consists of three modules: the hydrodynamic module, the sediment transport module and the morphological module. Firstly, the hydrodynamic conditions are computed by solving the shallow-water equations with a depth-averaged $\hat{k} - \hat{e}$ turbulence closure model. Based on the flow conditions, the suspended sediment concentration can be evaluated by solving the convection-diffusion equation while the bed load transport can be predicted from an empirical equation. Finally, the bed evolution can be calculated. In order to improve the prediction for the bed shear stress, the bed shear stress obtained from the traditional Manning’s formula is corrected according to the secondary flow effect. When the sloping beds are present in the calculation of the sediment transport, the bed slope effect is incorporated into the model. The equation giving the local bed slope correction to both the critical shear stress and the transport direction of the sediment particles is derived in this study. Thereafter, the equation is validated against the measurements reported in the literature based on the experiments of the sediment transport on sloping beds. In addition, an approximate
method which is suitable for gradually varying beds is proposed in order to improve the computational efficiency. After the careful testing and validations, the developed model is then used to study the sediment transport in one- and two-dimensional situations.

In order to provide detailed experimental data for the numerical model, three experiments are conducted in this study. For the one-dimensional situation, a sand dune evolution under the open channel flow condition is studied experimentally. For the two-dimensional situation, on the other hand, the channels with abrupt expansion and abrupt contraction in cross-sections are considered. The turbulent flow fields and the morphological evolutions are measured extensively for both cases. These will also provide useful experimental data bases for other numerical models, both two- and three-dimensional.

In this study, we first present the mathematical basis of the model in Chapter 2, including the shallow-water equations, the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence closure, both the bed load and suspended load transport equations, the morphological change equation along with the initial conditions and boundary conditions. Next, the correction for the bed shear stress under the secondary flow effect is given in detail. The consideration of the bed slope effect on the sediment transport and its validation are also presented in this chapter. In Chapter 3, details of the numerical implementation of the model are given and this is followed by the stability analysis and some special numerical treatments. An approximate calculation method for the sediment transport and morphological evolution is also proposed in this chapter.

In Chapter 4, the numerical model, including shallow-water equations, depth-averaged $\hat{k} - \hat{\varepsilon}$ equations, suspended load transport equation and morphological evolution equation,
is tested for both the one- and two-dimensional cases. The results obtained from the model are compared to the analytical solutions and the experimental measurements.

The model is then employed to study the sediment transport and morphological evolution in the one-dimensional situations in Chapter 5. Firstly, the bed changes in trenches under open channel flow conditions are investigated. Three tests of the trenches with different side slopes are considered and both suspended load and bed load transport have contributions to the bed evolution. The numerical predictions for the bed shapes are compared with the experimental measurements. Secondly, the evolution of a sand dune under the open channel flow conditions is studied both experimentally in the laboratory flume and numerically using the present model. Only bed load transport occurs in this case. The comparisons between numerical results and the experimental measurements are carried out in terms of the bed elevations at different times. In addition, the consideration of the bed slope effect and the approximate calculation method proposed in Chapter 2 and Chapter 3 respectively are also verified in this chapter.

In Chapter 6, the turbulent flow, the sediment transport and the morphological evolution in the channels with abrupt cross-section change are investigated both experimentally and numerically. Firstly, the flow field in a channel with an abrupt expansion is studied in the laboratory flume. Based on the experimental measurements, the flow parameters, including the velocity, the turbulent kinematic energy and its dissipation rate and the turbulent viscosity can be calculated or estimated for the comparison with the numerical results obtained using the present model. Secondly, the experiment to study the sediment transport and morphological evolution in the same channel are conducted. The model is used to study this case and the numerical results are compared with the experimental measurements in terms of the bed elevations. Thirdly, the
flow field in a channel with an abrupt contraction is investigated in the laboratory flume. The model is employed to study this case and the obtained numerical results are compared with the experimental data. Fourthly, the sediment transport and morphological evolution in the contracted channel are studied experimentally in the laboratory and numerically with the present model. The comparisons between the experimental measurements and the numerical results are conducted. Fifthly, the present model is employed to study the flow and bed deformation in a channel consisting of a contraction and an expansion. The numerical results including the water surface, velocities and the bed elevations are compared with the experimental measurements and the numerical results obtained using another three-dimensional model.

In the last Chapter, i.e., Chapter 7, the summaries of the study are given. The model performance is evaluated and summarized. In addition, suggestions for possible future research in improving the numerical model and the sediment transport study are discussed.
Chapter 2

Mathematical Formulation of the Numerical Model

2.1 Shallow-Water Equations

For completeness, the derivation of shallow-water equations (SWE) is presented here and the starting point is the incompressible Navier-Stokes Equations (NSE) which are given by:

\[
\frac{\partial u_i}{\partial x_i} = 0 \tag{2.1}
\]

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + g_i + \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} \tag{2.2}
\]

where \( i, j = 1, 2, 3 \) for three-dimensional flows, \( u_i \) is the velocity component in the \( x_i \)-direction, \( \rho \) is fluid density, \( p \) is the pressure, \( g_i \) is the \( i \)-th component of the gravitational acceleration, and \( \tau_{ij} \) represents the molecular viscous stress tensor. For a Newtonian fluid, \( \tau_{ij} = 2\mu\sigma_{ij} \) with \( \mu \) being the molecular viscosity of the fluid and

\[
\sigma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \text{ the rate of the strain tensor.}
\]

2.1.1 Continuity equation

The depth averaged continuity equation is obtained by integrating the continuity equation (2.1) from the bottom to the water surface. Here, the \( x \) coordinate is chosen to represent the general direction of flow, \( y \) is the lateral coordinate and \( z \) is the vertical
coordinate so that \( u, v, \) and \( w \) represent the \( x, y \) and \( z \) components of the velocity respectively.

\[
\int_{z_b}^{\eta} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = \int_{-\eta}^{\eta} \frac{\partial u}{\partial x} dz + \int_{z_b}^{\eta} \frac{\partial v}{\partial y} dz + w(x, y, \eta) - w(x, y, -z_b) = 0
\]  

(2.3)

where \( \eta \) is the free surface elevation measured vertically with respect to the datum, \( z_b \) is the bed elevation which is reckoned negative when measured vertically upwards with respect to the datum (as depicted in Figure 2.1). The total water depth is given by \( H = \eta + z_b \).

By using the Leibnitz rule of integration which has the following general form

\[
\frac{\partial}{\partial x} \int_{a(x)}^{b(x)} Q(x, y) dy = \int_{a(x)}^{b(x)} \frac{\partial Q(x, y)}{\partial x} dy + Q(x, \beta(x)) \frac{\partial \beta(x)}{\partial x} - Q(x, \alpha(x)) \frac{\partial \alpha(x)}{\partial x}
\]

(2.4)

the integrated continuity equation is rewritten as

\[
\frac{\partial}{\partial x} \int_{-z_b}^{\eta} u dz - u(x, y, \eta) \frac{\partial \eta}{\partial x} - u(x, y, -z_b) \frac{\partial z_b}{\partial x} + w(x, y, \eta) - w(x, y, -z_b) + \frac{\partial}{\partial y} \int_{-z_b}^{\eta} v dz - v(x, y, \eta) \frac{\partial \eta}{\partial y} - v(x, y, -z_b) \frac{\partial z_b}{\partial y} = 0
\]

(2.5)

Defining the depth-averaged velocity components by

\[
U = \frac{1}{H} \int_{-z_b}^{\eta} u dz \quad \text{and} \quad V = \frac{1}{H} \int_{-z_b}^{\eta} v dz
\]

(2.6)

and introducing the boundary conditions at the free surface and the bottom, i.e., the kinematic free surface boundary condition (KFSBC):

\[
\frac{\partial \eta}{\partial t} + u(x, y, \eta) \frac{\partial \eta}{\partial x} + v(x, y, \eta) \frac{\partial \eta}{\partial y} = w(x, y, \eta)
\]

(2.7)

and the bottom boundary condition (BBC) for a changing (with time) surface when dealing with sediment bed forms:
the final form of the depth-integrated continuity equation is obtained as
\[ \frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \] (2.9)
where \( P(=UH) \) and \( Q(=VH) \) denote the \( x \)- and \( y \)- components of the volume flux respectively. This equation is also called the height function transport equation which can be used to track the water surface movement.

2.1.2 Momentum equations

The momentum equation in the \( x \)-direction is taken as an example and the equation in \( y \)-direction can be derived similarly. By rewriting the momentum equation of the NSE in \( x \)-direction to its conservative form and with the application of continuity equation
\[ \frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \] (2.10)
If the assumption is made that the vertical accelerations are negligible compared to gravity so that the pressure distribution with depth is hydrostatic, the pressure term in (2.10) can be expressed as follows

\[-\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial \eta}{\partial x}\]  

(2.11)

where \(g\) is the gravitational acceleration. After the vertical integration from the bottom to the water surface and with the help of Leibnitz rule together with both the KFSBC and the BBC, the left hand side of (2.10) becomes

\[\frac{\partial}{\partial t} \int_{-z_b}^{\eta} u dz + \frac{\partial}{\partial x} \int_{-z_b}^{\eta} u^2 dz + \frac{\partial}{\partial y} \int_{-z_b}^{\eta} uv dz\]  

(2.12)

Take the second term in the above equation as an example again:

\[\frac{\partial}{\partial x} \int_{-z_b}^{\eta} u^2 dz = \frac{\partial}{\partial x} \left[ \int_{-z_b}^{\eta} \left( U + u - U \right)^2 dz \right] \]

\[= \frac{\partial}{\partial x} \left[ \int_{-z_b}^{\eta} U^2 dz + \int_{-z_b}^{\eta} (u - U)^2 dz + \int_{-z_b}^{\eta} 2U(u - U) dz \right]\]

(2.13)

\[= \frac{\partial}{\partial x} \int_{-z_b}^{\eta} u^2 dz + \frac{\partial}{\partial x} \int_{-z_b}^{\eta} (u - U)^2 dz\]

Therefore, the left hand side of (2.10) finally becomes

\[\frac{\partial}{\partial t} (UH) + \frac{\partial}{\partial x} (U^2 H) + \frac{\partial}{\partial y} (U VH) + \frac{\partial}{\partial x} \int_{-z_b}^{\eta} (u - U)^2 dz + \frac{\partial}{\partial y} \int_{-z_b}^{\eta} (u - U)(v - V) dz\]  

(2.14)

On the right hand side of (2.10), the integration of the stress terms yields in a similar way

\[\int_{-z_b}^{\eta} \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) dz = \frac{\partial}{\partial x} \int_{-z_b}^{\eta} \tau_{xx} dz + \frac{\partial}{\partial y} \int_{-z_b}^{\eta} \tau_{xy} dz\]

\[+ \left[ - \tau_{xx}(\eta) \frac{\partial \eta}{\partial x} - \tau_{xy}(\eta) \frac{\partial \eta}{\partial y} + \tau_{xz}(\eta) \right] - \left[ \tau_{xx}(-z_b) \frac{\partial z_b}{\partial x} + \tau_{xy}(-z_b) \frac{\partial z_b}{\partial y} + \tau_{xz}(-z_b) \right]\]  

(2.15)

The terms within square brackets can be interpreted as the stress tensor components in the surface and bottom planes. Ways for further simplification have been introduced by
Kuipers and Vreugdenhil (1973), Dean and Dalrymple (1991) and Lin (2008) to name a few. Here, only the one used by Kuipers and Vreugdenhil (1973) will be introduced. By considering a sloping free surface plane with a new orthogonal coordinate system $(x', y', z')$ where the plane $(x', y')$ is aligned with the tangential plane of the sloping surface and $z'$ is normal to the plane and pointing upwards, one is able to establish the coordinate transformation between the original coordinate system and the new coordinate system in the following general form

$$(2.16)$$

$$
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
= \begin{pmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
  n_1 & n_2 & n_3
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
$$

where $l$, $m$ and $n$ are determined by the rotational angles in the new coordinate. This will lead to the relationship between the stress tensors in the two coordinates as follows

$$(2.17)$$

$$
\begin{pmatrix}
  \tau_{xx}' & \tau_{xy}' & \tau_{xz}' \\
  \tau_{yx}' & \tau_{yy}' & \tau_{yz}' \\
  \tau_{zx}' & \tau_{zy}' & \tau_{zz}'
\end{pmatrix}
= \begin{pmatrix}
  l_1 & l_2 & l_3 \\
  m_1 & m_2 & m_3 \\
  n_1 & n_2 & n_3
\end{pmatrix}
\begin{pmatrix}
  \tau_{xx} & \tau_{xy} & \tau_{xz} \\
  \tau_{yx} & \tau_{yy} & \tau_{yz} \\
  \tau_{zx} & \tau_{zy} & \tau_{zz}
\end{pmatrix}
$$

To make an easier example, a 2D problem in $(x, z)$ plane will be solved. Since the new coordinate $(x', z')$ rotates in the counter-clockwise direction at an angle of $\theta$ to follow the surface, we have the following relationship to be satisfied

$$(2.18)$$

$$
\begin{pmatrix}
  \tau_{xx}' & \tau_{xz}' \\
  \tau_{zx}' & \tau_{zz}'
\end{pmatrix}
= \begin{pmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  \tau_{xx} & \tau_{xz} \\
  \tau_{zx} & \tau_{zz}
\end{pmatrix}
$$

from which we have
\[
\tau_{xz} = \tau_{sx} = -\sin \theta \tau_{sx} + \cos \theta \tau_{sx} = -\sin \theta \tau_{sx} + \cos \theta \tau_{sx} \\
(2.19)
\]

With the further assumption that the water surface slope is small, \( \sin \theta \approx \tan \theta = \frac{\partial \eta}{\partial x} \) and \( \cos \theta \approx 1 \), which lead to

\[
\tau_{xz}(\eta) = -\frac{\partial \eta}{\partial x} \tau_{sx}(\eta) + \tau_{xz}(\eta) \\
(2.20)
\]

For a 3D case, we have

\[
\tau_{xz}(\eta) = -\frac{\partial \eta}{\partial x} \tau_{sx}(\eta) - \frac{\partial \eta}{\partial y} \tau_{xy}(\eta) + \tau_{xz}(\eta) \\
(2.21)
\]

Similarly, for the bottom surface plane, we have

\[
\tau_{xz}(-h) = -\frac{\partial h}{\partial x} \tau_{sx}(-h) - \frac{\partial h}{\partial y} \tau_{sy}(-h) + \tau_{xz}(-h) \\
(2.22)
\]

Here \( \tau_{xz}(\eta) \) and \( \tau_{xz}(-h) \) can be regarded as the stresses along the free surface and bottom surface respectively, such as wind stress \( \tau_{sx} \) and bottom friction \( \tau_{bx} \). Substituting (2.21) and (2.22) into (2.15) and taking no consideration of free surface stresses in this study, one finally obtains

\[
\frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) + gH \frac{\partial \eta}{\partial x} = -\frac{1}{\rho} \tau_{bx} + \frac{1}{\rho} \frac{\partial (HT_{sx})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{sy})}{\partial y} \\
(2.23)
\]

Similarly, the momentum equation in y-direction is

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{PQ}{H} \right) + \frac{\partial}{\partial y} \left( \frac{Q^2}{H} \right) + gH \frac{\partial \eta}{\partial y} = -\frac{1}{\rho} \tau_{by} + \frac{1}{\rho} \frac{\partial (HT_{sy})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{yy})}{\partial y} \\
(2.24)
\]

The bottom friction stresses \( \tau_{bx} \) and \( \tau_{by} \) (in \( \text{N/m}^2 \)) are written in terms of Manning’s formula which is derived from momentum equation under the assumption of steady and uniform flow.
\[ \tau_{hx} = \frac{\rho g n^2}{H^{7/3}} P \sqrt{P^2 + Q^2}, \quad \tau_{hy} = \frac{\rho g n^2}{H^{7/3}} Q \sqrt{P^2 + Q^2} \] (2.25)

where \( n \) is the Manning’s roughness coefficient (in \( \text{s/m}^{1/3} \)). The depth-averaged effective stresses \( T_{xx}, T_{yx}, T_{xy} \) and \( T_{yy} \) (also in \( \text{N/m}^2 \)) are given by (Kuipers and Vreugdenhil, 1973) as follows

\[ T_{xx} = \frac{1}{H} \left[ 2 \mu \frac{\partial u}{\partial x} - \rho \langle u' u' \rangle - \rho (u - U)^2 \right] dz \]

\[ T_{yx} = T_{xy} = \frac{1}{H} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \rho \langle u' v' \rangle - \rho (u - U) (v - V) \right] dz \] (2.26)

\[ T_{yy} = \frac{1}{H} \left[ 2 \mu \frac{\partial v}{\partial y} - \rho \langle v' v' \rangle - \rho (v - V)^2 \right] dz \]

where \( \langle \ \rangle \) and the prime “‘” denote the mean quantities and the turbulent fluctuations respectively. For the effective stresses mentioned above, the contribution of the viscous stresses can be taken as negligible; the dispersion terms which do not represent turbulent transport (Rodi, 1984; Bravo and Holly, 1996) will not be considered in this study and the attention will be focused on the closure of the depth-averaged Reynolds stresses which will be interpreted in detail in the next section.

2.2 Depth-Averaged \( \hat{k} - \hat{\epsilon} \) Turbulence Closure

2.2.1 Three-dimensional \( k - \epsilon \) model

According to the turbulent viscosity hypothesis, the turbulent stresses are proportional to the mean rate-of-strain

\[ -\langle u_i' u_j \rangle = \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = -\frac{2}{3} k \delta_{ij} \] (2.27)

where $\nu_t$ is the turbulent or eddy viscosity, $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$) and $k = \frac{1}{2}\langle u'_i u'_j \rangle$ is the turbulent kinetic energy. By defining the rate of dissipation of turbulent kinetic energy $\varepsilon = \nu_m \left( \frac{\partial u'_i}{\partial x_j} \right)^2$ where $\nu_m$ is the fluid kinematic viscosity, the state of turbulence can be characterized by two parameters: $k$ and $\varepsilon$. Dimensional analysis yields the following relation for turbulent viscosity with $k$ and $\varepsilon$:

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}$$

in which $C_\mu$ is empirical constant. At high Reynolds numbers, the distribution of $k$ and $\varepsilon$ over the flow field can be determined from the following semi-empirical transport equations for $k$ and $\varepsilon$ (Rodi, 1984)

$$\frac{\partial k}{\partial t} + u_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \nu_t \frac{\partial k}{\partial x_j} \right) + P - \varepsilon$$

(2.29)

$$\frac{\partial \varepsilon}{\partial t} + u_j \frac{\partial \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \nu_t \frac{\partial \varepsilon}{\partial x_j} \right) + C_{1\varepsilon} \frac{\varepsilon}{k} P - C_{2\varepsilon} \frac{\varepsilon^2}{k}$$

(2.30)

where $P = \nu_t \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_i}{\partial x_j}$ is called the production of turbulent kinetic energy due to the mean motion and represents the transfer of kinetic energy from the mean to the turbulent motion, $\sigma_k$ and $\sigma_\varepsilon$ are Prandtl numbers which relate the diffusion of $k$ and $\varepsilon$ to $\nu_t$, $C_{1\varepsilon}$ and $C_{2\varepsilon}$ are further empirical constants governing the rate of production and dissipation. The empirical constants appearing in (2.28) ~ (2.30) can take the following values as recommended by Launder and Spalding (1974):
\[ C_\mu = 0.09, \sigma_k = 1.0, \sigma_\varepsilon = 1.3, C_{1\varepsilon} = 1.44 \text{ and } C_{2\varepsilon} = 1.92 \quad (2.31) \]

### 2.2.2 Depth-averaged \( \hat{k} - \hat{\varepsilon} \) model

Originally, Rastogi and Rodi (1978) adapted the above three-dimensional \( k - \varepsilon \) model into the depth-averaged version. They assumed that the depth-averaged state of turbulence can be characterized by two depth-averaged parameters, i.e., turbulent kinetic energy \( \hat{k} \) (in \( \text{m}^2/\text{s}^2 \)) and its dissipation rate \( \hat{\varepsilon} \) (in \( \text{m}^2/\text{s}^3 \)), and that the depth-averaged turbulent stresses can be related to these parameters via

\[
-\int_{-z_0}^{z} \langle u_i u_j \rangle \, dz = \hat{v}_t H \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \hat{k} H \delta_{ij} \quad (2.32)
\]

where \( i, j = 1, 2 \) and \( \hat{v}_t \) is the depth-averaged turbulent viscosity (in \( \text{m}^2/\text{s} \)) and can be calculated from the following relation

\[
\hat{v}_t = C_\mu \frac{\hat{k}^2}{\hat{\varepsilon}} \quad (2.33)
\]

The transport equations of \( \hat{k} \) and \( \hat{\varepsilon} \) are accordingly incorporated into the depth averaged equations:

\[
\frac{\partial (H \hat{k})}{\partial t} + \frac{\partial (H U_i \hat{k})}{\partial x_i} + \frac{\partial (H V_i \hat{k})}{\partial y_i} = \frac{\partial}{\partial x} \left[ \hat{v}_t \frac{\partial (H \hat{k})}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \hat{v}_t \frac{\partial (H \hat{k})}{\partial y} \right] + P_h + P_{k\varepsilon} - \hat{\varepsilon} H \quad (2.34)
\]

\[
\frac{\partial (H \hat{\varepsilon})}{\partial t} + \frac{\partial (H U_i \hat{\varepsilon})}{\partial x_i} + \frac{\partial (H V_i \hat{\varepsilon})}{\partial y_i} = \frac{\partial}{\partial x} \left[ \hat{v}_t \frac{\partial (H \hat{\varepsilon})}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \hat{v}_t \frac{\partial (H \hat{\varepsilon})}{\partial y} \right] + C_{1\varepsilon} \frac{\hat{\varepsilon}^2}{\hat{k}} P_h + P_{\varepsilon\varepsilon} - C_{2\varepsilon} \frac{\hat{\varepsilon}^2}{\hat{k}} H \quad (2.35)
\]

where
is the production term due to interactions of turbulent stresses with horizontal velocity gradients while the main contribution to $P_{kV}$ and $P_{eV}$ stems from significant vertical velocity gradients near the bottom of the water body. Because it depends strongly on the bottom roughness, Rastogi and Rodi (1978) related them to the friction velocity $u_*$ with the help of dimensional analysis by writing

$$P_{kV} = c_k u_*^3 \text{ and } P_{eV} = c_e \frac{u_*^4}{H}$$

where $u_* = \sqrt{c_f \left( U^2 + V^2 \right)}$ with dimensionless friction coefficient $c_f$; $c_k$ and $c_e$ are empirical constants and can be determined in the ensuing analysis. For a uniform flow, the gradient terms in x- and y-directions in (2.34) and (2.35) can be cancelled and the $\hat{k}$- and $\hat{\varepsilon}$-equations reduce to

$$c_k u_*^3 - \hat{\varepsilon} H = 0$$

$$c_e \frac{u_*^4}{H} - C_{2e} \frac{\hat{\varepsilon}^2}{k} H = 0$$

In this case, $\hat{\varepsilon}$ is related to the energy slope, $S$, via $\hat{\varepsilon} = SgU$, and $u_*$ to $S$, via $u_* = \sqrt{SgH}$. With these relations and with the relation between $u_*$ and $U$, i.e.,

$$\frac{\tau_b}{\rho} = u_*^2 = c_f U^2,$$

the following results can be obtained from (2.38):

$$c_k = \frac{1}{\sqrt{c_f}}$$
Determination of $c_\varepsilon$ from (2.39) needs further empirical input on $\hat{v}$, taken from the measurements in developed channel flow. Rastogi and Rodi (1978) adopted the data from Laufer (1951):

$$\frac{\hat{v}}{u_* H} = 0.0765$$  \hspace{1cm} (2.41)

With (2.33), $c_\varepsilon$ is obtained from (2.39)

$$c_\varepsilon = 3.6 \frac{C_{2\varepsilon}}{C_f^{3/4}} \sqrt{C_\mu}$$  \hspace{1cm} (2.42)

Therefore,

$$P_{kv} = \frac{1}{\sqrt{C_f}} u_*^3$$ and $$P_{ev} = 3.6 \frac{C_{2\varepsilon}}{C_f^{3/4}} \sqrt{C_\mu} \frac{u_*^4}{H}$$  \hspace{1cm} (2.43)

where $u_* = \sqrt{C_f (U^2 + V^2)}$ with dimensionless friction coefficient $C_f = \frac{n^2 g}{H^{1/3}}$ for rough beds.

It is noted that these equations developed from the uniform flow condition will be used in the generalized flow conditions, similar to the Manning’s formula. In addition, the empirical constants (2.31) are simply adopted by the depth-averaged $\hat{k} - \hat{\varepsilon}$ model.

### 2.3 Sediment Transport Model

The transport of bed material under the general hydrodynamic conditions can be divided into two modes: bed load and suspended load. Before introducing the sediment transport models for bed load and suspended load, some parameters which will be used in the next stage will be introduced.
2.3.1 Some parameters for sediment transport

The particle size parameter $D_s$ reads as

$$D_s = \left[ \frac{(s-1)g}{\nu^2} \right]^{\frac{1}{3}} d_{50}$$  \hspace{1cm} (2.44)

where $d_{50}$ is the median diameter of the sediment particles, $s = \rho_s / \rho$ is the specific gravity of the particle, $\rho_s$ is the sediment density.

The settling velocity $w_f$ can be obtained from experiments or calculated from some formula. The formula for the terminal settling velocity $w_f$ of a sphere in a still fluid can be obtained from the equilibrium between the net particle weight and the fluid drag force on the particle, giving

$$w_f = \left[ \frac{4(s-1)gd_{\text{sphere}}}{3C_D} \right]^{\frac{1}{2}}$$  \hspace{1cm} (2.45)

where $d_{\text{sphere}}$ is the diameter of the sphere, $C_D$ is the drag coefficient which is a function of the Reynolds number $Re = w_fd_{\text{sphere}}/\nu$. In the Stokes regime, i.e., laminar flow ($Re < 1$) in which the viscous force predominates, $C_D = \frac{24}{Re}$; in the regime of transition flow ($1 < Re < 1000$) in which both the viscous force and inertial force are in the same importance, $C_D = \frac{24}{Re} + \frac{3}{\sqrt{Re}} + 0.34$ and the iteration is needed; in the regime of turbulent flow ($Re > 10000$) in which the inertial force predominates, $C_D = 0.4$. However, the expressions valid for a sphere will be difficult to be applied for a natural sediment particle because of the differences in shape. van Rijn (1989) suggested the following formulae for non-spherical sediment particles:
\[ w_f = \frac{(s-1)gd^2}{18 \nu_m} \quad \text{for } 1 < d < 100\mu m \]

\[ w_f = \frac{10 \nu_u}{d} \left[ \left( 1 + \frac{0.01(s-1)gd^3}{\nu_m^2} \right)^{1/2} - 1 \right] \quad \text{for } 100 < d < 1000\mu m \]  
\[ w_f = 1.1[(s-1)gd]^{1/2} \quad \text{for } d > 1000\mu m \]  

(2.46)

where \( d \) is the sieve diameter.

The Shields parameter \( \theta \) is the ratio of the hydrodynamic drag to lift force of fluid acting on the particle and the submerged particle weight. As the former is in general proportional to \( \rho d^2 u_s^2 \) and the latter is proportional to \( (\rho_s - \rho)gd_{50}^2 \), this parameter has the following form

\[ \theta = \frac{u_s^2}{(s-1)gd_{50}} = \frac{\tau_b}{(\rho_s - \rho)gd_{50}} \]  

(2.47)

where \( \tau_b \) is the bed shear stress acting on the bed material particle. The critical Shields parameter \( \theta_{cr} \) is the effective parameter at which critical hydraulic forces on a particle for the initiation of sediment movement occurs and reads as

\[ \theta_{cr} = \frac{\tau_{b,cr}}{(\rho_s - \rho)gd_{50}} \]  

(2.48)

where \( \tau_{b,cr} \) is the critical bed shear. It has been found from many experiments that \( \theta_{cr} \) is a function of the grain Reynolds number \( Re_s = u_s d / \nu_m \). The pioneering experiments by Shields in 1936 on a flat bed are most widely used as the so-called Shields diagram. Yalin (1972) showed that the Shields curve can be expressed in terms of \( \theta_{cr} \) and \( D_s \) as follows
The excess bed shear stress parameter is defined as follows:

\[
T = \frac{\tau_b - \tau_{b,cr}}{\tau_{b,cr}}
\]  

(2.50)

### 2.3.2 Bed load transport equations

Usually, the sediment particles transporting in the form of rolling, sliding and saltation near the bed are called the bed load. There are many formulae to predict the bed load transport rate in the literature and some of them will be presented here.

**Meyer-Peter and Muller (1948):**

Meyer-Peter and Muller conducted extensive experimental work in a laboratory flume and proposed a relatively simple empirical formula which is still frequently used:

\[
\frac{q_b}{\sqrt{(s-1)gd_{s0}^{1.5}}} = 8(\theta - \theta_{cr})^{3/2}
\]  

(2.51)

where \(q_b\) is the bed load transport rate (in m³/s/m). This equation is based on particle diameters in the range of 3.17-28.6 mm.

**van Rijn (1984a):**

van Rijn (1984a) defined the bed load transport rate as the product of the particle velocity, the saltation height and the bed load concentration and proposed the following equation:
where $T$ and $D_* \text{ are defined by (2.44) and (2.50). This equation can provide a reliable estimate of the bed load transport in the particle range of 200-2000 \mu m, which is based on a verification study using 580 flume and field data.}

### 2.3.3 Suspended load transport equation

Based on the general concept of mass conservation, the transport process of suspended sediment can be described by the three-dimensional convection-diffusion equation as follows

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + (w - w_f) \frac{\partial c}{\partial z} = \nu_s \left[ \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right]$$

(2.53)

where $c$ is the volumetric sediment concentration, i.e., volume replaced by sediments per unit fluid volume as a normalized concentration, $u$, $v$ and $w$ are the velocities in $x$, $y$- and $z$-directions which are often taken as the fluid velocities at the same location (inertial effects of sediment particles, assumed to be very small, being neglected), $\nu_s$ is the sediment diffusivity which is assumed to be isotropic without any difference in all directions.

Again in analogy to the derivation of the SWE, one can obtain the depth-averaged sediment transport equation by taking the depth-integration of (2.53) and applying Liebniz rule

$$\frac{\partial (CH)}{\partial t} + \frac{\partial (UCH)}{\partial x} + \frac{\partial (VCH)}{\partial y} = \nu_s \left[ \frac{\partial}{\partial x} \frac{\partial (CH)}{\partial x} + \frac{\partial}{\partial y} \frac{\partial (CH)}{\partial y} \right] - (S_D - S_k)$$

(2.54)
where $C$ is the depth-averaged sediment concentration, $\hat{v}_z$ is the depth-averaged sediment diffusivity which is strongly related to the turbulent diffusion and will be assumed to be equal to $\hat{v}_z$ in this study (Guo and Jin, 1999; Zhou and Li, 2005), $S_D$ and $S_E$ are source and sink terms representing sediment deposition and entrainment fluxes respectively (m/s).

It is assumed in this study that deposition and entrainment of sediment particles are separate mechanisms and may happen simultaneously (Celik and Rodi, 1985, 1988). While $S_E$ is computed from some entrainment function, $S_D$ only depends on the near-bed concentration as well as particle property itself (i.e. settling velocity). Thus, the sediment concentration in vertical space will reach the equilibrium condition (no net deposition or erosion) when the net sediment flux between entrainment and deposition approaches zero.

### 2.3.4 Sediment deposition function

The description of sediment deposition is represented as the product of sediment settling velocity, $w_f$, and the near-bed concentration, i.e., the reference concentration being set at reference level $z = a$, $C_a$ (Celik and Rodi, 1985 and 1988): \[ S_D = w_f C_a \] (2.55)

The next step involves the determination of the near-bed concentration $C_a$ its relationship with the averaged concentration $C$. Under the equilibrium condition across the water depth, the upward and downward sediment fluxes are balanced \[ w_f c + v_s \frac{dc}{dz} = 0 \] (2.56)
Assuming $v_s$ is constant across the water depth, one can integrate (2.56) over the water depth from the reference level $a$ to obtain the vertical distribution of sediment concentration with respect to $C_a$ (Zhou and Li, 2005):

$$c(z) = C_a \cdot \exp\left[-w_f \frac{(z-a)/v_s}{\hat{v}_s}\right]$$  \hspace{1cm} (2.57)

Taking the depth-averaging of the above exponential profile, one can set up the relationship between $C_a$ and $C$ as follows (Zhou and Li, 2005):

$$C_a = \frac{w_f (H-a)}{\hat{v}_s \left[1-\exp\left(-w_f \frac{(H-a)}{\hat{v}_s}\right)\right]} C$$  \hspace{1cm} (2.58)

### 2.3.5 Sediment entrainment function

Since a complete theoretical description of sediment entrainment is not yet available, empirical expressions have to be employed (van Rijn, 1984c). Some of them are expressed as functions of bed shear stress. For example, van Rijn (1984) performed a series of tests in the laboratory flume in order to determine the pick-up rate experimentally. In all, five types of almost uniform sand material were used with diameters in the range of 130-1500 μm. For each type of sand, the mean flow velocities were in the range of 0.5-1.0 m/s and the water depth was kept constant at a value of 0.25m. The experimental results yielded a simple function for pick-up rate as follows:

$$S_E = 0.00033D^{0.3}T^{1.5}\sqrt{(s-1)gd_{50}}$$  \hspace{1cm} (2.59)

Some entrainment functions were constructed based on the concept of equilibrium near-bed concentration, $C_{a,e}$, and have shown some good performance reported in the past
literature (van Rijn, 1984b, 1985, 1986, 1989; Celik and Rodi, 1985, 1988; Garcia and Parker, 1991). According to this concept, the flow is always postulated to have the ability to entrain as much sediment from the bed as it can as long as sufficient sediment is available on the bed. As the entrainment rate $S_E$ is equal to the deposition rate $S_D$ under the equilibrium condition, the value of $S_E$ should be the product of sediment settling velocity and the near-bed concentration under this condition:

$$S_E = w_j C_{a,e}$$  

Therefore, the determination of pick-up function becomes the determination of a suitable expression for $C_{a,e}$ (Garcia and Parker, 1991). From the relation between the reference concentration $C_a$ and the bed load transport rate $q_b$, for example, van Rijn (1984b) derived an expression for $C_{a,e}$ with a factor which was determined later based on the 20 flume and field data. The flow depths varied from 0.1-25 m, the flow velocities from 0.4-1.6 m/s and the sediment sizes from 180-700 $\mu$m. The final expression for $C_{a,e}$ was proposed by van Rijn (1984b) as follows

$$C_{a,e} = 0.015 \frac{d_{so}}{a} T^{1.5} \frac{T^{0.3}}{D_s^{0.3}}$$  

The reference level $a$ is assumed to be equal to half the bed form height ($\Delta$), or the equivalent Nikuradse roughness height ($k_s$) if the bed form dimensions are not known, while a minimum value $a = 0.01H$ is used for reasons of accuracy in the concentration profile, i.e., $a = 0.5\Delta$ or $a = k_s$ with $a_{min} = 0.01H$. van Rijn (1984b) also suggested that this function yields good results for particles in the range of 100-500 $\mu$m.
2.4 Morphological Change Model

By considering the sediment volumetric conservation in a control volume, the rate of morphological change can be expressed as follows (van Rijn, 1985):

\[
(1 - \text{poro}) \frac{\partial z_b}{\partial t} = \frac{\partial (aC_b)}{\partial t} + \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} + S_e - S_D
\]

(2.62)

where \( \text{poro} \) is porosity factor, i.e., the void ratio of sediment in deposition and \( C_b \) is depth-averaged bed-load concentration. In most transport situations, \( C_b \ll 1 \) therefore the storage term \( \frac{\partial (aC_b)}{\partial t} \) will be neglected in this study (Cui et al., 2005 and Lanzoni, 2008).

2.5 Correction for Bed Shear Stress

Johnston (1960) has presented a study of the secondary flow type of three-dimensional turbulent boundary layer where he has presented a simple “triangular model” for the polar plot of the main flow component \( (u) \) and the cross flow component \( (v) \) and formulated the relationships between the momentum integral quantities and the skin friction coefficient.

Johnston introduced the streamline coordinate system shown in Figure 2.2 where \( (x,y,z) \) represent the streamline coordinate system and \( \alpha \) is the main flow streamline turning angle relative to a fixed reference streamline and which is measured in anti-clockwise direction. Figure 2.3 shows the velocity component profiles and the wall shear stress components. \( U_{\text{main}} \) is the main stream velocity magnitude and \( (u,v,w) \) are the mean components of the boundary layer velocity vector related to the main flow streamline. We note that at any elevation \( z \) along the vertical \( Oz \), there will be a streamline with velocity...
components \((u,v,w)\). Hence, at the limiting wall streamline very near \(O\), the all the components \((u,v,w)\) are equal to zero on account of the no-slip condition at the wall.
Figure 2.2: Streamline coordinate system; z-axis points out of paper.

Figure 2.3: Sketch of velocity component profiles and wall shear stress components.
If we assume that along Oz, the vertical component of the velocity is everywhere small such that \( w \approx 0 \) so that the significant velocity components are \( u \) and \( v \). Johnston (1960) proposed the polar plot of \( u/U_{\text{main}} \) against \( v/U_{\text{main}} \) (as shown in Figure 2.4) for his experimental results for air flow in a rectangular duct around a 90° bend transition to another rectangular duct of the same height but different width, for the experimental results of Grushwitz (1935) for air flow in a duct of constant cross section around a 90° bend and for the experimental results of Kuethe et al (1949) where measurements were taken on the turbulent boundary layer on a yawed wing of elliptical plan form follow a triangular form. Two distinct regions were identified: a frictional sublayer near the wall arising from the cross flow (Region I) and the outer part of the boundary layer away from the wall (Region II).
Johnston (1960) assumed that in the very small Region I, the skewed boundary layer is represented by a hypothetical boundary layer with collateral velocity profiles \( u_f \) whose direction is coincident with the direction of the wall shear stress of the skewed boundary layer such that

\[
\begin{align*}
    u &\approx u_f \cos \gamma_w, \quad v \approx u_f \sin \gamma_w \\
    \frac{v}{U_{main}} &= \frac{u}{U_{main}} \tan \gamma_w
\end{align*}
\]  

(2.63)

By definition, the wall shear stress whose direction is shown in Figure 2.3 is given by

\[
\tau_b = c_f \frac{1}{2} \rho U_{main}^2
\]  

(2.64)

Its component along the \( x \)-direction of the main flow streamline is

\[
\tau_{bx} = c_f \frac{1}{2} \rho U_{main}^2 \\
\frac{\tau_{bx}}{\tau_b} = \frac{c_f}{c_f} = \frac{\tau_{bx}}{\sqrt{\tau_{bx}^2 + \tau_{by}^2}} = \frac{1}{\sqrt{1 + \left(\frac{\tau_{by}}{\tau_{bx}}\right)^2}} = \frac{1}{\sqrt{1 + \tan^2 \gamma_w}}
\]  

(2.65)

\[
\tau_b = \tau_{bx} \sqrt{1 + \tan^2 \gamma_w}
\]

Assuming further that Region I lies within the laminar sublayer, the junction point \( z_p \) between Regions I and II in Johnston’s triangular model can be derived. At \( P \), since the velocity profile in the laminar sublayer is linear,

\[
\tau_b = \rho u^2_s = \rho V \left(\frac{u_f}{z_p}\right)_p \rightarrow \left(\frac{u_f}{v}\right)_p = \frac{z_p u^2_s}{V}
\]  

(2.66)
Further,

\[
(u_f^2)_p = (u^2 + v^2)_p = u_p^2 \left(1 + \frac{v_p^2}{u_p^2}\right) = u_p^2 \left(1 + \tan^2 \gamma_w\right) = u_p^2 \frac{c_f^2}{c_{fx}^2}
\]

\[
z_p \frac{u_p^2}{v} = u_p \frac{c_f}{c_{fx}}
\]

\[
\frac{u_p}{U_{main}} = \frac{1}{U_{main}} \frac{z_p}{v} \frac{u_p^2 c_{fx}}{c_f} = \frac{z_p U_{main} c_{fx}}{v \frac{2}{v}}
\]

Johnston (1960) also showed that under certain conditions, the outer part of the boundary layer (Region II) can be represented by

\[
\frac{v}{U_{main}} = A \left(1 - \frac{u}{U_{main}}\right) = 2\alpha \left(1 - \frac{u}{U_{main}}\right)
\]

From (2.63),

\[
v_p = u_p \tan \gamma_w
\]

Substitution into (2.68)

\[
\frac{u_p \tan \gamma_w}{U_{main}} = A \left(1 - \frac{u_p}{U_{main}}\right) \rightarrow \frac{u_p}{U_{main}} = \frac{1}{1 + \frac{\tan \gamma_w}{A}}
\]

Johnston further introduced the dimensionless form for \( z_p \) such that

\[
z_p = \frac{z_p u_s}{v}
\]

Introducing this into (2.67) and (2.70), the following equation is obtained.

\[
z_p \frac{U_{main} c_{fx}}{v \frac{2}{v}} = \frac{1}{1 + \frac{\tan \gamma_w}{A}}
\]

\[
1 + \frac{\tan \gamma_w}{A} = \frac{2v}{z_p U_{main} c_{fx}} = \frac{2}{z_p U_{main} c_{fx}} \frac{u_s}{c_f} = \frac{2}{z_p U_{main} c_{fx}} \sqrt{c_f} = \frac{\sqrt{2}}{z_p U_{main} c_{fx}} \sqrt{c_f}
\]

\[
= \frac{\sqrt{2}}{z_p U_{main} c_{fx}} \left(1 + \tan^2 \gamma_w\right)^{\frac{1}{4}}
\]
From experimental evidence, Johnston showed that \( z_p \approx 14 \) and

\[
\frac{\tan \gamma_w}{A} = 0.1 \left[ \frac{\left(1 + \tan^2 \gamma_w\right)^{1/4}}{\sqrt{C_{f_x}}} \right] - 1
\] (2.73)

This equation allows us to determine \( \gamma_w \) at any point in the channel when the angle \( \alpha \) at that point is known. From the depth averaged hydrodynamics, the angle \( \alpha \) can be determined. The shear stress obtained from the Manning equation gives \( c_{f_x} \) since \( x \) is along the direction of the main flow streamline. From (2.73), the angle \( \gamma_w \) is found by some iteration methods such as the Newton-Raphson method. Once determined, one can proceed to find the actual direction of the bed shear stress as well as its magnitude through \( c_f \). This is the correction for both magnitude and direction of the bed shear stress in the presence of cross flows.

### 2.6 Effect of Bed Slope on Sediment Transport

In the calculation of sediment transport on a flat bed, the critical bed shear stress is normally obtained from Shields diagram which is also based on the flat bed experiments. However, when the local bed is no longer flat, the critical condition for the sediment incipient motion should be determined according to the local bed situation rather than just adopting the Shields diagram. Due to the effect of gravity, the bed slope will affect the critical shear stress as well as the direction of the sediment transport. In this section, a derivation will be made for the general situation of the sediment incipient motion on a sloping bed and a new method to consider the effect of bed slope on the sediment transport will be proposed herein.
2.6.1 Effect of bed slope on critical shear stress

Let us consider a sediment particle resting on a sloping plane \( ABP \) (Figure 2.5). The original coordinate system for the computational domain is \( xoy \) and one can define a new coordinate system \( x'Oy' \) according to the flow direction aligning with \( Ox' \). The steepest slope of this plane is along line \( PC \) with the slope angle \( \beta \). In addition, this plane intersects planes \( xOz, yOz, x'Oz \) and \( y'Oz \) with the angles of \( \beta_x, \beta_y, \beta'_x \) and \( \beta'_y \) respectively. The base line of the slope makes an angle \( \theta \) with \( ox \) and \( \theta' \) with \( Ox' \).

For the forces acting on the sediment particle are the drag force from the flow is \( F_D \) and is along line \( PA' \) while the component of sediment submerged weight along line \( PC \) is \( W \sin \beta \). And the angle between them, i.e., \( \angle A'PC \), is \( \gamma \).
Figure 2.5: Diagram of the drag force and gravitational force component acting on a sediment particle resting on a sloping bed.
According to the geometry shown in the Figure 2.5, one can obtain the relationships among these angles as follows:

\[
\tan \theta = \frac{OB}{OA} = \frac{OP / \tan \beta_y}{OP / \tan \beta_x} = \frac{\tan \beta_x}{\tan \beta_y}
\]

(2.74)

\[
\theta = \alpha + \theta
\]

(2.75)

\[
OA = OP / \tan \beta_x, \ OB = OP / \tan \beta_y, \ OC = OP / \tan \beta
\]

\[
\Rightarrow AC = OP \cdot \sqrt{\frac{1}{\tan^2 \beta_x} - \frac{1}{\tan^2 \beta}} \quad CB = OP \cdot \sqrt{\frac{1}{\tan^2 \beta_y} - \frac{1}{\tan^2 \beta}}
\]

\[
AB = OP \cdot \sqrt{\frac{1}{\tan^2 \beta_x} + \frac{1}{\tan^2 \beta_y}}, \ AC + CB = AB
\]

\[
\Rightarrow \tan^2 \beta = \tan^2 \beta_x + \tan^2 \beta_y
\]

\[
\tan \gamma = \frac{A'C}{PC} = \frac{OA' \cos \theta'}{OP / \sin \beta} = \frac{(OC / \sin \theta') \cos \theta'}{OP / \sin \beta} = \frac{OC \sin \beta}{OP \tan \theta'} = \frac{1}{\tan \beta} \tan \theta'
\]

(2.76)

\[
= \frac{\cos \beta}{\tan \theta'}
\]

(2.77)

For the case of a sediment particle resting on a slope with the bed angle equal to the angle of repose \( \phi \), the gravity force component along the slope is equal to the frictional force

\[
W \sin \phi = \mu \cdot W \cos \phi
\]

(2.78)

where \( \mu \) is the frictional coefficient. Therefore, one can obtain \( \mu = \tan \phi \). We have, for the case of a flat bed, the drag force being resisted by frictional force which becomes fully mobilized at the point of incipient motion.

\[
F_{cr,\theta} = \mu W = W \tan \phi
\]

(2.79)
For the case of a general sloping bed, the forces $F_D$ and $W \sin \beta$ give rise to a resultant force which will be resisted by the frictional force. From the triangle of forces, the resultant force $R$ is

$$R^2 = (\mu W \cos \beta)^2 = F_D^2 + (W \sin \beta)^2 - 2(F_D)(W \sin \beta)\cos(\pi - \gamma)$$

$$\rightarrow \left( F_{cr,0} \cos \beta \right)^2 = F_D^2 + 2F_D \left( \frac{F_{cr,0}}{\mu} \sin \beta \right) \cos \gamma + \left( \frac{F_{cr,0}}{\mu} \sin \beta \right)^2$$

$$\rightarrow F_D^2 + 2F_D \left( \frac{F_{cr,0}}{\mu} \sin \beta \cos \gamma \right) + \left( \frac{F_{cr,0}}{\mu} \sin \beta \right)^2 - (F_{cr,0} \cos \beta)^2 = 0$$  \hspace{1cm} (2.80)

$$\rightarrow F_D = -\left( \frac{F_{cr,0}}{\mu} \sin \beta \cos \gamma \right) + \sqrt{\left( \frac{F_{cr,0}}{\mu} \sin \beta \cos \gamma \right)^2 - F_{cr,0}^2 \left( \frac{\sin^2 \beta}{\mu^2} - \cos^2 \beta \right)}$$

$$= \frac{F_{cr,0}}{\mu} \left\{ \sqrt{\mu^2 \cos^2 \beta - \sin^2 \beta \sin^2 \gamma - \sin \beta \cos \gamma} \right\}$$

i.e.,

$$F_{cr,0} = F_{cr,0} \cos \beta \left\{ \sqrt{1 - \frac{\sin^2 \gamma \tan^2 \beta}{\tan^2 \phi} - \frac{\cos \gamma \tan \beta}{\tan \phi}} \right\}$$  \hspace{1cm} (2.81)

Note that when $\mu$ is fully mobilized, $F_D$ takes on the critical value on the slope and the friction angle kicks in. This result is for the flow over a downslope which makes an angle of $\theta$ with the fluid stream.

For the case of a fluid stream on the upslope, we see that $F_D$ should now point upwards and the angle it makes with $W \sin \beta$ is $(\pi - \gamma)$. The force triangle becomes

$$R^2 = (\mu W \cos \beta)^2 = F_D^2 + (W \sin \beta)^2 - 2(F_D)(W \sin \beta)\cos \gamma$$  \hspace{1cm} (2.82)

And we have the result for upslope

$$F_{cr,0} = F_{cr,0} \cos \beta \left\{ \sqrt{1 - \frac{\sin^2 \gamma \tan^2 \beta}{\tan^2 \phi} + \frac{\cos \gamma \tan \beta}{\tan \phi}} \right\}$$  \hspace{1cm} (2.83)
Note that the above relationships for the force are also valid for the shear stress. Therefore, we have

\[
\tau_{cr,\beta} = \tau_{cr,0} \cos \beta \left\{ \sqrt{1 - \frac{\sin^2 \gamma \tan^2 \beta}{\tan^2 \phi}} \pm \frac{\cos \gamma \cdot \tan \beta}{\tan \phi} \right\}
\] (2.84)

in which the positive sign is for the upslope face and negative sign is for the downslope face.

### 2.6.2 van Rijn (1989)’s method

For the purpose of the comparison, it is necessary to mention the method used by van Rijn (1989) to consider the influence of the bed slope. In case of a streamwise sloping bed (in flow direction), the critical shear stress will be modified as

\[
\tau_{cr,\beta} = k_1 \tau_{cr,0}
\] (2.85)

in which \( k_1 = \sin (\phi - \beta) / \sin \phi \) for a downsloping bed and \( k_1 = \sin (\phi + \beta) / \sin \phi \) for a upsloping bed.

In case of a transverse sloping bed (normal to flow direction):

\[
\tau_{cr,\beta} = k_2 \tau_{cr,0}
\] (2.86)

in which \( k_2 = \cos \beta \sqrt{1 - \frac{\tan^2 \beta}{\tan^2 \phi}} \).

For a combination of a streamwise and transverse sloping bed, it follows that

\[
\tau_{cr,\beta} = k_1 k_2 \tau_{cr,0}
\] (2.87)
2.6.3 Application to some cases

In this section, (2.84) will be applied to some particular cases to obtain the modifications for the critical shear stress of sediment on the slope.

(a) When $\theta' = 0$, the flow direction is perpendicular to the steepest slope or we have a transverse slope

$$\theta' = 0 \rightarrow \tan \theta' = 0$$

$$\tan \gamma = \infty \rightarrow \gamma = \pi / 2$$

This yields

$$\tau_{cr,\beta} = \tau_{cr,0} \cos \beta \left\{ \frac{1}{\sqrt{1 - \frac{\tan^2 \beta}{\tan^2 \phi}}} \right\}$$

(b) When $\theta' = \pi / 2$ and the flow is going downslope, the gravitational and drag forces are in the same direction

$$\theta' = \pi / 2 \rightarrow \tan \theta' = \infty$$

$$\tan \gamma = 0 \rightarrow \gamma = 0$$

This gives

$$\tau_{cr,\beta} = \tau_{cr,0} \cos \beta \left\{ \frac{1}{1 - \frac{\tan \beta}{\tan \phi}} \right\}$$

(c) When $\theta' = \pi / 2$ and the flow is going upslope, the gravitational and drag forces are opposite in directions

$$\theta' = \pi / 2 \rightarrow \tan \theta' = \infty$$

$$\tan \gamma = 0 \rightarrow \gamma = 0$$

This leads to

$$\tau_{cr,\beta} = \tau_{cr,0} \cos \beta \left\{ 1 + \frac{\tan \beta}{\tan \phi} \right\}$$
From the results obtained in (a), (b) and (c), we can see that the modification factors calculated from the equations we propose are same as those factors suggested by van Rijn (1989) when the flow direction is in line with or perpendicular to the direction of the steepest slope.

In the case of a general sloping bed, a simple multiplication of the two factors obtained individually for streamwise and transverse slopes would be physically questionable (Zhang et al., 2007). However, using equation (2.84), the modification factor can still be calculated for a general sloping bed without losing simplicity.

### 2.6.4 Verification of the slope effect equation

In this section, the equation for modifying critical shear stress will be applied in the calculations of bed load transport rate. For the purpose of verification, the calculated bed load transport rate will be compared with the experimental measurements provided by Smart (1984) and Damgaard et al. (1997) for the upslope and downslope cases. Smart (1984) performed experiments in a tilting flume to measure the sediment transport capacity on steep slopes. The bed angle of the flume was downslope with a maximum angle of 11.3°. The median size of the sediment used in the experiments ranged from 2 to 10.5mm. In the experiments of Damgaard et al. (1997), the bed angles covered from upslope 30° to downslope 29°. The median size of the sand was 0.208mm.

Based on the information (e.g., bed slope, bed shear stress, etc.) provided from the experiments, we can calculate the sediment transport rate. Both the van Rijn (1984a) equation and Meyer-Peter and Muller (1948) equation are used in the calculations. The effect of the bed slope is included to modify the critical shear stress.
Figure 2.6 shows the comparisons between measured and calculated bed load transport rates. It can be seen that Meyer-Peter and Muller (1948) equation gives better prediction for the sediment transport rate than the van Rijn (1984a) equation in these cases. Therefore, the Meyer-Peter and Muller (1948) equation will be adopted in our study if no special statement is given.

Figure 2.6: Comparisons between measured and calculated bed load transport rates, $q_b$ ($\text{m}^2/\text{s}$): Left: calculated using van Rijn (1984a) equation; Right: calculated using Meyer-Peter and Muller (1948) equation.
2.6.5 Modification of sediment transport direction

In addition to the effect on the critical shear stress, bed slope will also affect the direction of sediment transport. On a sloping bed, sediment movement will not follow the direction of bed shear stress, but the direction of the resultant force.

**Downslope:**

Consider a downslope case first. Recall that the angle between flow direction and the steepest slope is $\angle A'PC = \gamma$. Define the angle between the flow direction and the resultant force is $\angle A'PD = \gamma_1$, thus the angle between the steepest slope and the resultant force is $\angle CPD = \gamma - \gamma_1$ (see Figure 2.7(a)). Recall again the component of gravitational force along the steepest slope is $W \sin \beta$. The resultant force will be $R = \mu \cdot W \cos \beta = W \cos \beta \tan \phi$. Considering the triangle formed by $R$ and $W \sin \beta$ shown in Figure 2.7(b), we can use the law of sines as follows

$$\frac{\sin \gamma_1}{W \sin \beta} = \frac{\sin (180^\circ - \gamma)}{W \cos \beta \tan \phi}$$

from which we can obtain

$$\gamma_1 = \arcsin \left(\frac{\tan \beta}{\tan \phi} \sin \gamma\right)$$

Next is to map $\gamma_1$ onto the plane $xOy$ to obtain the angle $\gamma_2$, i.e., $\angle A'OD$. According to the geometric relationships, we can obtain the following expressions:
\[ OC = \frac{OP}{\tan \beta}, \quad PC = \frac{OP}{\sin \beta} \]

\[ PD = \frac{OP}{\sin \beta \cos(\gamma - \gamma_1)}, \quad OD = OP \cdot \sqrt{\frac{1}{\sin^2 \beta \cdot \cos^2 (\gamma - \gamma_1)} - 1} \]  

(2.96)

\[ \angle COD = \arcsin \left[ \frac{1}{\tan \beta \cdot \sin \beta \cdot \cos^2 (\gamma - \gamma_1) - 1} \right] \]

Thus, we have

\[ \gamma_2 = \angle COD - \theta' = \arcsin \left[ \frac{1}{\tan \beta \cdot \sin \beta \cdot \cos^2 (\gamma - \gamma_1) - 1} \right] - \theta' \]  

(2.97)

**Upslope:**

In case of the upslope, the angle between flow direction and the steepest slope is still

\[ \angle A'PC = \gamma. \]

The angle between the flow direction and the resultant force \( \gamma_1 \), thus

\[ \angle A'PD = \gamma_1 \] (see Figure 2.8(a)). Considering the triangle shown in Figure 2.8(b) and using the law of sines, we will obtain

\[ \gamma_1 = \arcsin \left( \frac{\tan \beta}{\tan \phi} \sin \gamma \right) \]  

(2.98)

which has the same form as in the case of the downslope. Similarly, when mapping \( \gamma_1 \) onto the plane \( xOy \) to obtain the angle \( \gamma_2 \), i.e., \( \angle A'OD \), we have the following expressions:
\[ OC = \frac{OP}{\tan \beta}, \quad PC = \frac{OP}{\sin \beta} \]

\[ PD = \frac{OP}{\sin \beta \cos(\gamma + \gamma_1)}, \quad OD = OP \cdot \sqrt{\frac{1}{\sin^2 \beta \cos^2(\gamma + \gamma_1)} - 1} \quad (2.99) \]

\[ \angle COD = \arcsin \left[ \frac{1}{\tan \beta \cdot \sqrt{\frac{1}{\sin^2 \beta \cos^2(\gamma + \gamma_1)} - 1}} \right] \]

Thus, we have

\[ \gamma_2 = \theta' - \angle COD = \theta' - \arcsin \left[ \frac{1}{\tan \beta \cdot \sqrt{\frac{1}{\sin^2 \beta \cos^2(\gamma + \gamma_1)} - 1}} \right] \quad (2.100) \]
Figure 2.7: Diagram of the angle relationships among the forces acting on a sediment particle resting on a sloping bed in case of downslope flow: (a) 3D view; (b) Force triangle.
Figure 2.8: Diagram of the angle relationships among the forces acting on a sediment particle resting on a sloping bed in case of upslope flow: (a) 3D view; (b) Force triangle.
2.6.6 Procedure of considering the effect of bed slope

It is necessary to give the detailed procedure of considering the effect of bed slope as a summary.

(1) Calculate the sloping bed angles with respect to the axes Ox and Oy, i.e., $\beta_x$ and $\beta_y$, according to the bed elevations.

(2) Calculate the bed angle $\beta$ with respect to the steepest slope

$$\tan^2 \beta = \tan^2 \beta_x + \tan^2 \beta_y$$

(2.76)

(3) Calculate the flow direction $\alpha$ according to the flow velocities $U$ and $V$

$$\tan \alpha = \frac{V}{U}$$

(2.101)

(4) Calculate the angle between the base line of the slope and the $x$-axis, $\theta$, and the angle between the base line of the slope and the flow direction, $\theta'$

$$\tan \theta = \frac{\tan \beta_x}{\tan \beta_y}$$

(2.74)

$$\theta = \alpha + \theta$$

(2.75)

(5) Calculate the angle between the flow direction and the steepest line on the sloping bed

$$\tan \gamma = \frac{\cos \beta}{\tan \theta'}$$

(2.77)

(6) Calculate the modification factor of the critical shear stress

$$\tau_{cr,\beta} = \tau_{cr,0} \cos \beta \left\{ \sqrt{1 - \frac{\sin^2 \gamma \tan^2 \beta}{\tan^2 \phi} + \left| \frac{\cos \gamma \tan \beta}{\tan \phi} \right|^2} \right\}$$

(2.84)

in which the positive sign is for upslope and negative sign is for downslope.

(7) Calculate the modification angle for the sediment transport direction, $\gamma_2$
\[ \gamma_1 = \arcsin \left( \frac{\tan \beta \sin \gamma}{\tan \phi} \right) \]  

\[ \gamma_2 = \begin{cases} \arcsin \left( \frac{1}{\tan \beta \cdot \sqrt{\frac{1}{\sin^2 \beta \cdot \cos^2 (\gamma - \gamma_1)} - 1}} \right) - \theta' & \text{for downslope} \\ \theta' - \arcsin \left( \frac{1}{\tan \beta \cdot \sqrt{\frac{1}{\sin^2 \beta \cdot \cos^2 (\gamma + \gamma_1)} - 1}} \right) & \text{for upslope} \end{cases} \]  

(2.102)

2.7 Initial and Boundary Conditions

In this section, the initial and boundary conditions will be discussed. Especially in practical computations, the resolution of the mesh near the physical boundaries is generally too coarse to resolve the boundary layers. Therefore, some modifications of boundary conditions need to be made so that the numerical model can predict reasonable results in the vicinity area of boundaries.

2.7.1 Initial conditions

In most cases, the mean flow can be assumed to be zero flux with certain water depth, i.e., \( P = Q = 0 \) and \( H = H_0 \), at the initial moment; that is, the computation is given a cold start. For the turbulence field, small finite values are necessary to “seed” the initial disturbance. According to Lin and Liu (1998), \( \hat{k} = \frac{1}{2} u_i^2 \) with \( u_i = \delta U_0 \), where \( U_0 \) is the
inflow velocity and \( \delta \) is chosen as \( 2.5 \times 10^{-3} \). \( \hat{\epsilon} \) is estimated through equation
\[ \hat{\epsilon} = C_T \frac{\hat{k}^2}{\hat{\nu}_t}, \]
with \( \hat{\nu}_t = \xi \nu_m \), where \( \xi \) is chosen to be 5.0 at present.

The sediment concentration can be initially assumed to be zero, i.e., \( C = 0 \), since the it is negligibly low during a cold start.

### 2.7.2 Boundary conditions

#### Inflow and outflow boundary conditions

The specification of flow boundary condition depends on the flow state. For subcritical flow, the flow rate is specified at the upstream boundary, i.e., \( P = P_0 \), \( Q = 0 \), and the water depth is specified at the downstream boundary, i.e., \( H = H_0 \). For other quantities, the gradient is set to be zero, i.e., \( \partial/\partial x = 0 \).

For supercritical flow, both flow rate and water depth are specified at the upstream boundary, i.e., \( P = P_0 \), \( Q = 0 \), \( H = H_0 \). Similarly, other quantities are set to be zero gradient, i.e., \( \partial/\partial x = 0 \).

Where a hydraulic jump occurs, the inclusion of a transition from supercritical to subcritical flow would require the specification of the water depths at the upstream end for supercritical flow as well as at the downstream end for subcritical flow. The inflow rate again is defined at the upstream boundary.

For the suspended sediment transport, the concentration is usually defined at the upstream boundary, i.e., \( C = C_0 \).
Boundary conditions at solid boundary

If the solid boundary is fixed, the no-slip boundary condition requires the velocity flux on the solid boundary to be zero, i.e., \( P_n = P_r = 0 \) where \( P_n \) and \( P_r \) represent the velocity flux components normal and tangential to the solid boundary respectively. For the \( \hat{k} \)– and \( \hat{\varepsilon} \)– equations, the boundary conditions near the solid boundary are also needed. In principle, \( \hat{k} \) is zero on the solid boundary. However, the grid size normally cannot adequately resolve the turbulent boundary layer in practical computations. Thus, the boundary conditions require \( \hat{k} \) and \( \hat{\varepsilon} \) to be specified just inside the turbulent boundary layer instead of right on the wall.

Consider a turbulent boundary layer close to a flat wall with the y-axis pointing away from it. Under the thin shear-layer approximation, the time variation, convection and pressure gradient are small compared with the cross-stream shear stress, thus one can simplify the streamwise momentum equation and \( \hat{k} \)-equation as follows (Lemos, 1992)

\[
-\frac{\partial \left\langle u' v' \right\rangle}{\partial y} + \nu_m \frac{\partial^2 U_r}{\partial y^2} = 0 \tag{2.103}
\]

\[
-\left\langle u' v' \right\rangle \frac{\partial U_r}{\partial y} - \hat{\varepsilon} = 0 \tag{2.104}
\]

Integrating (2.103) from the wall to the place out of the viscous sublayer where the viscous effect can be neglected, one obtains

\[
-\left\langle u' v' \right\rangle \bigg|_{y = y_w} = \nu_m \frac{\partial U_r}{\partial y} \bigg|_{y = 0} = \frac{1}{\rho} \tau_w = u_*^2 \tag{2.105}
\]

Based on the dimensional analysis, the mean velocity gradient can be expressed as follows

\[
\frac{\partial U_r}{\partial y} = \frac{u_*}{\kappa y} \tag{2.106}
\]
where $\kappa = 0.41$ is the von Karman constant.

After integrating the above equation, the logarithmic law-of-the-wall is obtained

$$\frac{U_s}{u_*} = \frac{1}{\kappa} \ln \left( \frac{y}{y_0} \right) \quad \text{with} \quad y_0 = \begin{cases} \frac{v_m}{Eu_*} & \text{for hydraulically smooth regime} \\ \frac{k_s}{30} & \text{for hydraulically rough regime} \end{cases}$$

(2.107)

where $y_0$ is the zero-velocity level and $E = 9.0$ for hydraulically smooth walls.

In the region of $y^+ = u_* y / \nu_m > 40$, the viscous contribution to the shear stress is negligible compared with Reynolds stress and turbulence is in a state of local equilibrium

$$\hat{\varepsilon} = -\langle u'v' \rangle \frac{\partial U_s}{\partial y} = \frac{u_*^2}{\kappa y}$$

(2.108)

From eddy viscosity concept,

$$\hat{\nu}_t = \frac{\langle u'v' \rangle}{\partial U_s} = \kappa u_* y$$

(2.109)

i.e., the eddy viscosity is proportional to the distance from the wall. Substituting (2.108) and (2.109) into (2.33), one obtains

$$\hat{k} = \frac{u_*^2}{\sqrt{C_\mu}}$$

(2.110)

Equations (2.110) and (2.108) constitute the boundary conditions for $\hat{k}$ and $\hat{\varepsilon}$ after $u_*$ has been found out from (2.107).

### 2.8 Summary of Governing Equations

Before we leave this chapter, it is useful to have a brief summary of governing equations.
The shallow-water equations read:

\[ \frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \]  \hspace{1cm} (2.9) \\

\[ \frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) + gH \frac{\partial \eta}{\partial x} = -\frac{1}{\rho} \tau_{bx} + \frac{1}{\rho} \frac{\partial (HT_{x})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{yy})}{\partial y} \]  \hspace{1cm} (2.23) \\

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{PQ}{H} \right) + \frac{\partial}{\partial y} \left( \frac{Q^2}{H} \right) + gH \frac{\partial \eta}{\partial y} = -\frac{1}{\rho} \tau_{by} + \frac{1}{\rho} \frac{\partial (HT_{y})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{yy})}{\partial y} \]  \hspace{1cm} (2.24) \\

The depth-averaged \( \hat{k} - \hat{\varepsilon} \) turbulence closure reads:

\[ \frac{\partial (H\hat{k})}{\partial t} + \frac{\partial (H\hat{u}\hat{k})}{\partial x} + \frac{\partial (H\hat{v}\hat{k})}{\partial y} = \frac{\partial}{\partial x} \left[ \hat{v}_t \frac{\partial (H\hat{k})}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \hat{v}_t \frac{\partial (H\hat{k})}{\partial y} \right] \]  \hspace{1cm} (2.34) \\

\[ \frac{\partial (H\hat{e})}{\partial t} + \frac{\partial (H\hat{u}\hat{e})}{\partial x} + \frac{\partial (H\hat{v}\hat{e})}{\partial y} = \frac{\partial}{\partial x} \left[ \hat{v}_t \frac{\partial (H\hat{e})}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \hat{v}_t \frac{\partial (H\hat{e})}{\partial y} \right] \]  \hspace{1cm} (2.35) \\

\[ + C_{1e} \frac{\hat{\varepsilon}}{k} \hat{P}_h + P_{ve} - C_{2e} \frac{\hat{\varepsilon}^2}{k} H \]

Bed load transport equations read:

Meyer-Peter and Muller (1948): \[ \frac{q_b}{\sqrt{(s-1)g d_{50}^{1.5}}} = 8(\theta - \theta_{cr})^3 \]  \hspace{1cm} (2.51) \\

van Rijn (1984a): \[ \frac{q_b}{\sqrt{(s-1)g d_{50}^{1.5}}} = 0.053 \frac{T^{2.1}}{D^{0.5}} \]  \hspace{1cm} (2.52) \\

The suspended load transport equation reads:

\[ \frac{\partial (CH)}{\partial t} + \frac{\partial (UCH)}{\partial x} + \frac{\partial (VCH)}{\partial y} = \hat{v}_s \left[ \frac{\partial}{\partial x} \frac{\partial (CH)}{\partial x} + \frac{\partial}{\partial y} \frac{\partial (CH)}{\partial y} \right] \]  \hspace{1cm} (2.54) \\

The morphological change model reads:

\[ (1 - \text{poro}) \frac{\partial z_h}{\partial t} = \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} + S_e - S_D \]  \hspace{1cm} (2.62)
These equations will serve as the governing equations for the model. In the next chapter, the numerical solutions to these equations will be discussed in detail.
Chapter 3

Numerical Implementation

In this chapter, the implementation of the numerical model, including the SWE, the depth-averaged \( \hat{k} - \hat{\epsilon} \) equations, the suspended load transport equation and the morphological evolution equation, will be discussed in detail. In addition, a numerical stability analysis is performed for the model followed by some special numerical treatments.

3.1 Model Implementation

3.1.1 Sketch of computational domain

In this study, the finite difference method constructed on a staggered grid system will be used throughout the computation. As shown in Figure 3.1, all the scalar quantities, i.e., \( H, C, \hat{k} \) and \( \hat{\epsilon} \), are defined at the center of the cell and all the vectors, i.e., \( U, V, P \) and \( Q \), are defined at the faces of the cell. For the stress terms, \( T_{xx} \) and \( T_{yy} \) are evaluated at the center of the cell and \( T_{yx} \) and \( T_{xy} \) are evaluated at the top right corner grid (Lin, 1998).
Figure 3.1: A single cell of the staggered grid and the locations of variables.
3.1.2 Shallow-water equations

Firstly, let us consider the linear shallow-water equations, i.e.,

\[
\frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \quad (2.9)
\]

\[
\frac{\partial P}{\partial t} + gH \frac{\partial \eta}{\partial x} = 0 \quad (3.1)
\]

\[
\frac{\partial Q}{\partial t} + gH \frac{\partial \eta}{\partial y} = 0 \quad (3.2)
\]

These equations are discretized by using the explicit leap-frog finite difference scheme which has second-order accuracy in both time and space:

\[
\frac{H_{i,j}^{n+1/2} - H_{i,j}^{n-1/2}}{\Delta t} + \frac{P_{i+1/2,j}^{n} - P_{i-1/2,j}^{n}}{\Delta x_{i}} + \frac{Q_{i,j+1/2}^{n} - Q_{i,j-1/2}^{n}}{\Delta y_{j}} = 0 \quad (3.3)
\]

\[
\frac{P_{i+1/2,j}^{n+1} - P_{i+1/2,j}^{n}}{\Delta t} + gH_{i+1/2,j}^{n+1/2} \frac{\eta_{i+1/2,j}^{n+1/2} - \eta_{i+1/2,j}^{n}}{\Delta x_{i+1/2}} = 0 \quad (3.4)
\]

\[
\frac{Q_{i,j+1/2}^{n+1} - Q_{i,j+1/2}^{n}}{\Delta t} + gH_{i,j+1/2}^{n+1/2} \frac{\eta_{i,j+1/2}^{n+1/2} - \eta_{i,j+1/2}^{n}}{\Delta y_{j+1/2}} = 0 \quad (3.5)
\]

where the subscript \((i,j)\) denotes spatial nodes, the superscript \(n\) denotes the time level, \(\Delta x\) and \(\Delta y\) represent the spatial step sizes in the \(x\)- and \(y\)-directions respectively, \(\Delta t\) represents the time step size.

It is noted that in the finite difference form, the information needed but not at the originally defined locations can be linearly interpolated from neighboring values which are known at the defined locations. Some commonly used interpolated variables are given as follows:

\[
\Delta x_{i+1/2} = \frac{1}{2} (\Delta x_{i} + \Delta x_{i+1}) \quad (3.6)
\]
\[ \Delta y_{j+1/2} = \frac{1}{2}(\Delta y_j + \Delta y_{j+1}) \]  \hspace{1cm} (3.7)

\[ H_{i+1/2,j}^{n+1/2} = \frac{H_{i+1/2,j}^{n+1/2} \Delta x_i + H_{i,j}^{n+1/2} \Delta x_{i+1}}{\Delta x_i + \Delta x_{i+1}} \]  \hspace{1cm} (3.8)

\[ H_{i,j+1/2}^{n+1/2} = \frac{H_{i+1/2,j}^{n+1/2} \Delta y_j + H_{i,j}^{n+1/2} \Delta y_{j+1}}{\Delta y_j + \Delta y_{j+1}} \]  \hspace{1cm} (3.9)

\[ H_{i+1/2,j}^n = \frac{\Delta x_i (H_{i+1/2,j}^{n+1/2} + H_{i,j}^{n-1/2}) + \Delta x_{i+1} (H_{i+1/2,j}^{n+1/2} + H_{i,j}^{n-1/2})}{2(\Delta x_i + \Delta x_{i+1})} \]  \hspace{1cm} (3.10)

\[ H_{i,j+1/2}^n = \frac{\Delta y_j (H_{i+1/2,j}^{n+1/2} + H_{i,j}^{n-1/2}) + \Delta y_{j+1} (H_{i+1/2,j}^{n+1/2} + H_{i,j}^{n-1/2})}{2(\Delta y_j + \Delta y_{j+1})} \]  \hspace{1cm} (3.11)

**Bottom frictional terms**

The bottom frictional terms in (2.23) and (2.24) are discretized as (Cho, 1995; Liu and Cho, 1995)

\[ \frac{1}{\rho} \tau_{hx} = v_x \left( P_{i+1/2,j}^{n+1} - P_{i+1/2,j}^{n} \right), \quad \frac{1}{\rho} \tau_{hy} = v_y \left( Q_{i,j+1/2}^{n+1} - Q_{i,j+1/2}^{n} \right) \]  \hspace{1cm} (3.12)

in which \( v_x \) and \( v_y \) are given in terms of the Manning’s formula

\[ v_x = \frac{1}{2} \frac{gn^2}{(H_{i+1/2,j}^n)^{7/3}} \left[ \left( P_{i+1/2,j}^n \right)^2 + \left( Q_{i+1/2,j}^n \right)^2 \right]^{1/2} \]  \hspace{1cm} (3.13)

\[ v_y = \frac{1}{2} \frac{gn^2}{(H_{i+1/2,j}^n)^{7/3}} \left[ \left( P_{i,j+1/2}^n \right)^2 + \left( Q_{i,j+1/2}^n \right)^2 \right]^{1/2} \]  \hspace{1cm} (3.14)

Therefore, (2.9), (2.23) and (2.24) become

\[ H_{i,j}^{n+1/2} = H_{i,j}^{n-1/2} - \frac{\Delta t}{\Delta x_i} \left( P_{i+1/2,j}^n - P_{i-1/2,j}^n \right) - \frac{\Delta t}{\Delta y_j} \left( Q_{i,j+1/2}^n - Q_{i,j-1/2}^n \right) \]  \hspace{1cm} (3.14)
\[ P_{i+1/2,j} = \frac{1}{1 + v_x \Delta t} \left[ (1 - v_x \Delta t) P_{i+1/2,j}^n + \frac{\Delta t}{\Delta x_{i+1/2}} g H_{i+1/2,j}^{n+1/2} \left( \eta_{i+1,j}^{n+1/2} - \eta_{i,j}^n \right) \right] \]

\[ - \frac{\Delta t}{1 + v_x \Delta t} \left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i+1/2,j}^n + \frac{\Delta t}{1 + v_x \Delta t} \frac{1}{\rho} \left[ \frac{\partial}{\partial x} \left( HT_{x} \right) + \frac{\partial}{\partial y} \left( HT_{y} \right) \right]_{i+1/2,j}^n \] (3.15)

\[ Q_{i,j+1/2} = \frac{1}{1 + v_y \Delta t} \left[ (1 - v_y \Delta t) Q_{i,j+1/2}^n + \frac{\Delta t}{\Delta y_{i,j+1/2}} g H_{i,j+1/2}^{n+1/2} \left( \eta_{i,j+1}^{n+1/2} - \eta_{i,j}^n \right) \right] \]

\[ - \frac{\Delta t}{1 + v_y \Delta t} \left[ \frac{\partial}{\partial x} \left( \frac{PQ}{H} \right) + \frac{\partial}{\partial y} \left( \frac{Q^2}{H} \right) \right]_{i,j+1/2}^n + \frac{\Delta t}{1 + v_y \Delta t} \frac{1}{\rho} \left[ \frac{\partial}{\partial x} \left( HT_{x} \right) + \frac{\partial}{\partial y} \left( HT_{y} \right) \right]_{i,j+1/2}^n \] (3.16)

**Convection terms**

As appearing in the \( x \)- and \( y \)-momentum equations, all the convection terms and diffusion terms will be evaluated at the \( n \)-th time step. The nonlinear convection terms in the \( x \)- and \( y \)-momentum equations are evaluated at the right face and the top face of the cell respectively, i.e.

\[ \left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i+1/2,j}^n \quad \text{and} \quad \left[ \frac{\partial}{\partial x} \left( \frac{PQ}{H} \right) + \frac{\partial}{\partial y} \left( \frac{Q^2}{H} \right) \right]_{i,j+1/2}^n \] (3.17)

To calculate the spatial derivatives in the convection terms, the upwind scheme is used.

As an example, only the details for the \( x \)-momentum equation will be given here. The upwind scheme is represented by

\[ \left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) \right]_{i+1/2,j}^n = \begin{cases} \left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) \right]_{i,j}^n & \text{if } P_{i+1/2,j}^n \geq 0 \\ \left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) \right]_{i+1,j}^n & \text{if } P_{i+1/2,j}^n < 0 \end{cases} \] (3.18)
\[
\left[ \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i+1/2,j}^n = \begin{cases} 
\left[ \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i,j}^n & \text{if } Q_{i+1/2,j}^n \geq 0 \\
\left[ \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i+1,j}^n & \text{if } Q_{i+1/2,j}^n < 0
\end{cases}
\]

\[ (3.19) \]

where the following finite difference forms are defined

\[
\left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) \right]_{i,j}^n = \frac{1}{\Delta x_i} \left[ \left( \frac{P_{i+1/2,j}^n}{H_{i+1/2,j}^n} \right)^2 - \left( \frac{P_{i-1/2,j}^n}{H_{i-1/2,j}^n} \right)^2 \right]
\]

\[ (3.20) \]

\[
\left[ \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) \right]_{i+1,j}^n = \frac{1}{\Delta x_{i+1}} \left[ \left( \frac{P_{i+3/2,j}^n}{H_{i+3/2,j}^n} \right)^2 - \left( \frac{P_{i+1/2,j}^n}{H_{i+1/2,j}^n} \right)^2 \right]
\]

\[ (3.21) \]

\[
\left[ \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i,j}^n = \frac{1}{\Delta y_{j-l/2}} \left[ \frac{(PQ)_{i+1/2,j-l/2}^n}{H_{i+1/2,j-l/2}^n} - \frac{(PQ)_{i+1/2,j-l-1/2}^n}{H_{i+1/2,j-l-1/2}^n} \right]
\]

\[ (3.22) \]

\[
\left[ \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) \right]_{i+1,j}^n = \frac{1}{\Delta y_{j+1/2}} \left[ \frac{(PQ)_{i+1/2,j+1/2}^n}{H_{i+1/2,j+1/2}^n} - \frac{(PQ)_{i+1/2,j+1}^n}{H_{i+1/2,j+1}^n} \right]
\]

\[ (3.23) \]

Since the upwind scheme is employed, the discretized momentum equations are only first order in accuracy in terms of spatial grid sizes.

**Stress terms**

Once again, the gradients of the total stresses in the \(x\)- and \(y\)-directions are computed at the right face and the top face of the cell respectively, i.e.

\[
\left[ \frac{\partial (HT_{xx})}{\partial x} + \frac{\partial (HT_{yx})}{\partial y} \right]_{i+1/2,j}^n \quad \text{and} \quad \left[ \frac{\partial (HT_{xy})}{\partial x} + \frac{\partial (HT_{yy})}{\partial y} \right]_{i+1/2,j}^n
\]

\[ (3.24) \]

In this study, the central difference scheme is employed to discretize the stress terms. An example will be given in the \(x\)-direction as follows
\[
\left[ \frac{\partial (HT_{xx})}{\partial x} + \frac{\partial (HT_{yx})}{\partial y} \right]_{i+1/2,j}^{n} = \frac{(HT_{xx})_{i+1,j}^{n} - (HT_{xx})_{i,j}^{n}}{\Delta x_{i+1/2}} + \frac{(HT_{yx})_{i+1/2,j+1/2}^{n} - (HT_{yx})_{i+1/2,j-1/2}^{n}}{\Delta y_{j}}
\]

(3.25)

As mentioned before, the total stresses (both the normal and shear stresses) above are the summation of molecular stresses and turbulent or Reynolds stresses. The former are the products of the molecular viscosity and the rates of strain of the mean flow, while the latter can be obtained based on the eddy viscosity concept in (2.32). Both of them involve the evaluation of the rates of the strain of the mean flow. The normal stress in the x-direction evaluated at the center of the cell, for example, involves the computation of the normal strain rate of the mean flow which has the following finite difference form

\[
\left( \frac{\partial P}{\partial x} \right)_{i,j}^{n} = \frac{P_{i+1/2,j}^{n} - P_{i-1/2,j}^{n}}{\Delta x_{i}}
\]

(3.26)

The shear stress evaluated at the top right corner of the grid involves the following computations

\[
\left( \frac{\partial P}{\partial y} \right)_{i+1/2,j+1/2}^{n} = \frac{P_{i+1/2,j+1}^{n} - P_{i+1/2,j}^{n}}{\Delta y_{j+1/2}}, \quad \left( \frac{\partial Q}{\partial x} \right)_{i+1/2,j+1/2}^{n} = \frac{P_{i+1,j+1/2}^{n} - P_{i,j+1/2}^{n}}{\Delta x_{i+1/2}}
\]

(3.27)

### 3.1.3 Depth-averaged \( \hat{k} \) – \( \hat{e} \) equations

Both the \( \hat{k} \) – and \( \hat{e} \) – equations (2.34) and (2.35) are treated in a semi-implicit way and can be symbolically written as (Lemos, 1992)

\[
\frac{(H\hat{e})_{i,j}^{n+1/2} - (H\hat{e})_{i,j}^{n-1/2}}{\Delta t} + F\hat{e}X + F\hat{e}Y = VIS\hat{e}X + VIS\hat{e}Y + C_{i}^{e} \hat{e}_{i,j}^{n-1/2} P_{hi,j}^{n+1/2} + P_{e}^{e} - C_{2e} \hat{e}_{i,j}^{n-1/2} H_{i,j}^{n-1/2}
\]

(3.28)
\[
\frac{(H\hat{k})_{i,j}^{n+1/2} - (H\hat{k})_{i,j}^{n-1/2}}{\Delta t} + F\hat{k}X + F\hat{k}Y = VIS\hat{k}X + VIS\hat{k}Y + P_{hi,j}^{n+1/2} + P_{hv}^{n+1/2} + \frac{P_{kv} - C_{\mu} \hat{k}_{i,j}^{n-1/2} \hat{V}_{k,j}^{n-1/2}}{\hat{V}_{k,j}^{n-1/2}} H_{i,j}^{n-1/2}
\]

(3.29)

Therefore, the final finite difference forms for the \( \hat{k} - \hat{\epsilon} \) equations are written as follows

\[
\hat{\epsilon}_{i,j}^{n+1/2} = \frac{(H\hat{\epsilon})_{i,j}^{n-1/2}}{\Delta t} - F\hat{\epsilon}X - F\hat{\epsilon}Y + VIS\hat{\epsilon}X + VIS\hat{\epsilon}Y + C_{1e} \hat{\epsilon}_{i,j}^{n-1/2} P_{hi,j}^{n+1/2} + P_{dv}^{n+1/2} + P_{hv}^{n+1/2} + \frac{H_{i,j}^{n+1/2}}{\Delta t} + C_{2e} \hat{\epsilon}_{i,j}^{n-1/2} H_{i,j}^{n-1/2}
\]

(3.30)

\[
\hat{k}_{i,j}^{n+1/2} = \frac{(H\hat{k})_{i,j}^{n-1/2}}{\Delta t} - F\hat{k}X - F\hat{k}Y + VIS\hat{k}X + VIS\hat{k}Y + P_{hi,j}^{n+1/2} + P_{hv}^{n+1/2} + P_{kv}^{n+1/2} + \frac{H_{i,j}^{n+1/2}}{\Delta t} + C_{\mu} \hat{k}_{i,j}^{n-1/2} H_{i,j}^{n-1/2}
\]

(3.31)

In (3.30) and (3.31), the convection terms (\( F\hat{k}X, F\hat{k}Y, F\hat{\epsilon}X \) and \( F\hat{\epsilon}Y \)), the diffusion terms (\( VIS\hat{k}X, VIS\hat{k}Y, VIS\hat{\epsilon}X \) and \( VIS\hat{\epsilon}Y \)) and the horizontal production terms \( P_{hi,j}^{n+1/2} \) need to be evaluated in space. Here only the details of finite difference forms for the \( \hat{\epsilon} \) equation are given as examples.

**Convection terms**

Similar to the shallow-water equations, the convection terms here are discretized with an upwind scheme as follows

\[
F\hat{\epsilon}X = \begin{cases} 
\frac{\partial P\hat{\epsilon}^{n-1/2}}{\partial x} & \text{if } P_{i,j}^{n+1/2} \geq 0 \\
\frac{\partial P\hat{\epsilon}^{n-1/2}}{\partial x} & \text{if } P_{i,j}^{n+1/2} < 0
\end{cases}
\]

(3.32)
\[ F \hat{e} Y = \left[ \frac{\partial Q \hat{e}}{\partial y} \right]_{i,j}^{n^{-1/2}} = \begin{cases} \left[ \frac{\partial Q \hat{e}}{\partial y} \right]_{i,j}^{n^{-1/2}} & \text{if } Q_{i,j}^{n^{-1/2}} \geq 0 \\ \left[ \frac{\partial Q \hat{e}}{\partial y} \right]_{i,j}^{n^{-1/2}} & \text{if } Q_{i,j}^{n^{-1/2}} < 0 \end{cases} \quad (3.33) \]

where the following finite difference forms are defined

\[ \left[ \frac{\partial P \hat{e}}{\partial x} \right]_{i-1/2,j}^{n^{-1/2}} = \frac{(P \hat{e})_{i,j}^{n^{-1/2}} - (P \hat{e})_{i-1,j}^{n^{-1/2}}}{\Delta x_{i-1/2}} \quad (3.34) \]

\[ \left[ \frac{\partial P \hat{e}}{\partial x} \right]_{i+1/2,j}^{n^{-1/2}} = \frac{(P \hat{e})_{i+1,j}^{n^{-1/2}} - (P \hat{e})_{i,j}^{n^{-1/2}}}{\Delta x_{i+1/2}} \quad (3.35) \]

\[ \left[ \frac{\partial Q \hat{e}}{\partial y} \right]_{i,j-1/2}^{n^{-1/2}} = \frac{(Q \hat{e})_{i,j}^{n^{-1/2}} - (Q \hat{e})_{i,j-1}^{n^{-1/2}}}{\Delta y_{j-1/2}} \quad (3.36) \]

\[ \left[ \frac{\partial Q \hat{e}}{\partial y} \right]_{i,j+1/2}^{n^{-1/2}} = \frac{(Q \hat{e})_{i,j+1}^{n^{-1/2}} - (Q \hat{e})_{i,j}^{n^{-1/2}}}{\Delta y_{j+1/2}} \quad (3.37) \]

**Diffusion terms**

The diffusion terms in (3.30) are defined as:

\[ VIS \hat{e} X = \left[ h \frac{\partial (H \hat{e})}{\partial x} \right]_{i,j}^{n^{-1/2}} \quad \text{and} \quad VIS \hat{e} Y = \left[ h \frac{\partial (H \hat{e})}{\partial y} \right]_{i,j}^{n^{-1/2}} \quad (3.38) \]

The first term can be discretized with the central difference scheme

\[ \left[ h \frac{\partial (H \hat{e})}{\partial x} \right]_{i,j}^{n^{-1/2}} = \frac{1}{\Delta x_i} \left[ \left[ \frac{\partial (H \hat{e})}{\partial x} \right]_{i+1/2,j}^{n^{-1/2}} - \left[ \frac{\partial (H \hat{e})}{\partial x} \right]_{i-1/2,j}^{n^{-1/2}} \right] \]

\[ = \frac{1}{\Delta x_i} \left[ \frac{\partial (H \hat{e})_{i+1/2,j}^{n-1/2}}{\Delta x_{i+1/2}} \frac{\partial (H \hat{e})_{i,j}^{n-1/2}}{\Delta x_{i-1/2,j}} - \frac{\partial (H \hat{e})_{i+1/2,j}^{n-1/2}}{\Delta x_{i+1/2}} \frac{\partial (H \hat{e})_{i,j}^{n-1/2}}{\Delta x_{i-1/2,j}} \right] \quad (3.39) \]

and the second term can be similarly obtained
Horizontal production term

The horizontal production term $P_{hi,j}^{n+1/2}$ involves the evaluation of the rates of the strain of the mean flow and has the following form

\[
P_{hi,j}^{n+1/2} = \frac{\hat{v}_{hi,j}^{n-1/2}}{H_{i,j}^{n+1/2}} \left\{ 2 \left[ \frac{\partial P}{\partial x} \right]^2 + 2 \left[ \frac{\partial Q}{\partial y} \right]^2 + \left[ \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right]^2 \right\}_{i,j}^{n+1/2}
\]

\[
= \frac{\hat{v}_{hi,j}^{n-1/2}}{H_{i,j}^{n+1/2}} \left\{ 2 \left[ \frac{P_{i,j+1/2}^{n+1/2} - P_{i,j-1/2}^{n+1/2}}{\Delta x_i} \right]^2 + 2 \left[ \frac{Q_{i,j+1/2}^{n+1/2} - Q_{i,j-1/2}^{n+1/2}}{\Delta y_j} \right]^2 \right\}_{i,j}^{n+1/2}
\]

\[
+ \left[ \frac{P_{i,j}^{n+1/2} - P_{i,j-1/2}^{n+1/2}}{\Delta y_j} + \frac{Q_{i,j}^{n+1/2} - Q_{i,j-1/2}^{n+1/2}}{\Delta x_i} \right]^2 \right\}
\]

3.1.4 Suspended load transport equation

The horizontal 2D convection-diffusion equation for the suspended load transport is discretized based on the same grid system as that for the SWE.

Convection terms

Firstly, the two-step Lax-Wendroff scheme with second-order accuracy is used to discretize the convection terms which, for easy interpretation, are represented by the following expression

\[
\frac{\partial (CH)}{\partial t} + \frac{\partial F_{CX}}{\partial x} + \frac{\partial F_{CY}}{\partial y} = 0
\]

(3.42)
in which $F_{CX} = P \cdot C$ and $F_{CY} = Q \cdot C$ represent the sediment fluxes in $x$- and $y$-directions respectively. In the first step, $CH$ is computed at the intermediate nodes $(i+1/2, j)$ and $(i, j+1/2)$ and at half time step $n$

$$
(CH)_{i+1/2,j}^n = \frac{\Delta x_i \cdot (CH)_{i+1,j}^{n-1/2} + \Delta x_{i+1} \cdot (CH)_{i,j}^{n-1/2}}{\Delta x_i + \Delta x_{i+1}}
$$

$$- \frac{\Delta t/2}{\Delta x_{i+1/2}} (F_{CX,i+1,j}^{n-1/2} - F_{CX,i,j}^{n-1/2}) - \frac{\Delta t/2}{\Delta y_j} (F_{CY,i+1/2,j+1/2}^{n-1/2} - F_{CY,i+1/2,j-1/2}^{n-1/2})$$

In the second step, $CH$ is updated at defined grid $(i, j)$ and at time level $n + 1/2$

$$
(CH)_{i,j}^{n+1/2} = (CH)_{i,j}^{n-1/2} - \frac{\Delta t}{\Delta x_i} (F_{CX,i+1/2,j}^n - F_{CX,i-1/2,j}^n) - \frac{\Delta t}{\Delta y_j} (F_{CY,i,j+1/2}^n - F_{CY,i,j-1/2}^n)
$$

**Diffusion terms**

The diffusion terms are evaluated at the center of the cell, i.e.,

$$
\nu_s \left[ \frac{\partial}{\partial x} \frac{\partial (CH)}{\partial x} + \frac{\partial}{\partial y} \frac{\partial (CH)}{\partial y} \right]_{i,j}^{n-1/2}
$$

To calculate the spatial derivatives in the above terms, central difference scheme is used in discretization. Thus, the first term in the square brackets can be written in the following finite difference form
\[
\begin{bmatrix}
\frac{\partial}{\partial x} (CH)_{i,j}^{n-1/2} \\
\frac{\partial}{\partial y} (CH)_{i,j}^{n-1/2}
\end{bmatrix} = \frac{1}{\Delta x_i} \left\{ \left[ \frac{\partial (CH)}{\partial x} \right]_{i+1/2,j}^{n-1/2} - \left[ \frac{\partial (CH)}{\partial x} \right]_{i-1/2,j}^{n-1/2} \right\}
\]
\[= \frac{1}{\Delta x_i} \left[ \frac{(CH)_{i+1,j}^{n-1/2} - (CH)_{i,j}^{n-1/2}}{\Delta x_{i+1/2}} - \frac{(CH)_{i,j}^{n-1/2} - (CH)_{i-1,j}^{n-1/2}}{\Delta x_{i-1/2}} \right] \quad (3.47)
\]

and the second term can be written as
\[
\begin{bmatrix}
\frac{\partial}{\partial y} (CH)_{i,j}^{n-1/2} \\
\frac{\partial}{\partial x} (CH)_{i,j}^{n-1/2}
\end{bmatrix} = \frac{1}{\Delta y_j} \left\{ \left[ \frac{\partial (CH)}{\partial y} \right]_{i,j+1/2}^{n-1/2} - \left[ \frac{\partial (CH)}{\partial y} \right]_{i,j-1/2}^{n-1/2} \right\}
\]
\[= \frac{1}{\Delta y_j} \left[ \frac{(CH)_{i,j+1}^{n-1/2} - (CH)_{i,j}^{n-1/2}}{\Delta y_{j+1/2}} - \frac{(CH)_{i,j}^{n-1/2} - (CH)_{i,j-1}^{n-1/2}}{\Delta y_{j-1/2}} \right] \quad (3.48)
\]

### 3.1.5 Morphological evolution equation

A simple forward difference scheme can be used for the temporal discretization of the morphological evolution equation

\[
(1 - \text{poro}) \frac{z_{bi,j}^{n+1/2} - z_{bi,j}^{n-1/2}}{\Delta t} = \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} + S_E - S_D \quad (3.49)
\]

For the spatial discretization, there are numerous schemes for the convection-dominated equations. To avoid the numerical dispersion (oscillation) and numerical dissipation and to ensure shock capturing, the Weighted Essentially Non-Oscillatory (WENO) scheme introduced by Long et al. (2008) is adopted in the study. We will present the 2D WENO for approximating the term \( \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} \) in the following.

WENO is based on the ENO (Essentially Non-Oscillatory) scheme whose key idea is to use the smoothed stencil among several candidates to approximate the fluxes \( q_{bx} \) and \( q_{by} \) at the cell interfaces \((i \pm 1/2, j)\) and \((i, j \pm 1/2)\) respectively to high order and at the
same time to avoid spurious oscillations near shocks or discontinuities. WENO takes the process one step further by taking a weighed average of the candidate stencils. Weights are adjusted to obtain local smoothness.

The sediment transport rates $q_{bx}$ and $q_{by}$ can be split into two parts which associated with bedform propagation in the positive and negative directions, namely $q_{bx}^+ \cdot q_{bx}^-$ and $q_{by}^+ \cdot q_{by}^-$. For $\frac{\partial q_{bx}}{\partial x}$ and $\frac{\partial q_{by}}{\partial y}$, the WENO gives

$$
\frac{\partial q_{bx}}{\partial x} = \frac{\hat{q}_{\text{bx},i+1/2,j} - \hat{q}_{\text{bx},i-1/2,j}}{\Delta x_j}
$$

$$
\frac{\partial q_{by}}{\partial y} = \frac{\hat{q}_{\text{by},i,j+1/2} - \hat{q}_{\text{by},i,j-1/2}}{\Delta y_j}
$$

(3.50)

where the key is to estimate $\hat{q}_{\text{bx},i+1/2,j}$, $\hat{q}_{\text{bx},i-1/2,j}$, $\hat{q}_{\text{by},i,j+1/2}$ and $\hat{q}_{\text{by},i,j-1/2}$. Again, $\hat{q}_{\text{bx},i+1/2,j}$ and $\hat{q}_{\text{by},i,j+1/2}$ can be split into left-biased-fluxes $\hat{q}_{\text{bx},i+1/2,j}^-$ and $\hat{q}_{\text{by},i,j+1/2}^-$ and right-biased-fluxes $\hat{q}_{\text{bx},i+1/2,j}^+$ and $\hat{q}_{\text{by},i,j+1/2}^+$ respectively

$$
\hat{q}_{\text{bx},i+1/2,j} = \hat{q}_{\text{bx},i+1/2,j}^- + \hat{q}_{\text{bx},i+1/2,j}^+ \quad \hat{q}_{\text{by},i,j+1/2} = \hat{q}_{\text{by},i,j+1/2}^- + \hat{q}_{\text{by},i,j+1/2}^+
$$

(3.51)

The left-biased-fluxes are calculated using

$$
\hat{q}_{\text{bx},i+1/2,j}^- = \begin{cases} 
\omega_x q_{\text{bx},i+1/2,j}^{(1)} + \omega_x q_{\text{bx},i+1/2,j}^{(2)} + \omega_x q_{\text{bx},i+1/2,j}^{(3)} & \text{if } C_x (z_b)_{i+1/2,j} \geq 0 \\
0 & \text{if } C_x (z_b)_{i+1/2,j} < 0
\end{cases}
$$

$$
\hat{q}_{\text{by},i,j+1/2}^- = \begin{cases} 
\omega_y q_{\text{by},i,j+1/2}^{(1)} + \omega_y q_{\text{by},i,j+1/2}^{(2)} + \omega_y q_{\text{by},i,j+1/2}^{(3)} & \text{if } C_y (z_b)_{i,j+1/2} \geq 0 \\
0 & \text{if } C_y (z_b)_{i,j+1/2} < 0
\end{cases}
$$

(3.52)

where $C_x (z_b)$ and $C_y (z_b)$ are the bed-form propagation phase speeds in $x$- and $y$-directions respectively and $\omega_x, \omega_x, \omega_x, \omega_y, \omega_y, \omega_y$ are the weights used.
\begin{align*}
q_{bxi,j+1/2}^{(1)} &= \frac{1}{3} q_{bxi-1,j} + \frac{5}{6} q_{bxi,j} + \frac{1}{3} q_{bxi+1,j} - \frac{7}{6} q_{bxi,j-1} + \frac{11}{6} q_{bxi,j}, \\
q_{byi,j+1/2}^{(1)} &= \frac{1}{3} q_{byi,j-2} + \frac{5}{6} q_{byi,j-1} + \frac{1}{3} q_{byi,j+1} - \frac{7}{6} q_{byi,j} + \frac{11}{6} q_{byi,j}.
\end{align*}

(3.53)

\begin{align*}
q_{bxi,j+1/2}^{(2)} &= \frac{1}{6} q_{bxi-1,j} + \frac{5}{6} q_{bxi,j} + \frac{1}{3} q_{bxi+1,j} - \frac{7}{6} q_{bxi,j-1} + \frac{11}{6} q_{bxi,j}, \\
q_{byi,j+1/2}^{(2)} &= \frac{1}{6} q_{byi,j-1} + \frac{5}{6} q_{byi,j} + \frac{1}{3} q_{byi,j+1} - \frac{7}{6} q_{byi,j} + \frac{11}{6} q_{byi,j}.
\end{align*}

(3.53)

\begin{align*}
q_{bxi,j+1/2}^{(3)} &= \frac{1}{3} q_{bxi,j} + \frac{5}{6} q_{bxi+1,j} - \frac{1}{3} q_{bxi,j-1} - \frac{5}{6} q_{bxi,j} + \frac{1}{3} q_{bxi+1,j} - \frac{1}{6} q_{bxi,j+2} \\
q_{byi,j+1/2}^{(3)} &= \frac{1}{3} q_{byi,j} + \frac{5}{6} q_{byi,j+1} - \frac{1}{3} q_{byi,j-1} - \frac{5}{6} q_{byi,j} + \frac{1}{3} q_{byi,j+1} - \frac{1}{6} q_{byi,j+2}.
\end{align*}

are six candidate stencils for estimating $q_{bx}$ and $q_{by}$ at the grid interfaces $(i + 2, j)$ and $(i, j + 2)$ with third order accuracy (left-biased in the sense of the x-direction that 3 grid points $(i - 2, j)$ to $(i, j)$ to the left of location $(i + 2, j)$ are used but only 2 grid points $(i + 1, j)$ and $(i + 2, j)$ to the right of location $(i + 1/2, j)$ are used.). $\omega_{x1}$, $\omega_{x2}$, $\omega_{x3}$, $\omega_{y1}$, $\omega_{y2}$ and $\omega_{y3}$ are carefully chosen weights such that $\hat{q}_{bxi,j+1/2}$ and $\hat{q}_{byi,j+1/2}$ given by (3.52) are the fifth order accurate approximations of $q_{bx}$ and $q_{by}$ at the grid interfaces $(i + 1/2, j)$ and $(i, j + 1/2)$ and near the discontinuity no Gibbs phenomena occur. The weights are calculated as follows

\begin{align*}
\omega_{x1} &= \frac{\alpha_{x1}}{\alpha_{x1} + \alpha_{x2} + \alpha_{x3}}, \quad \omega_{x2} = \frac{\alpha_{x2}}{\alpha_{x1} + \alpha_{x2} + \alpha_{x3}}, \quad \omega_{x3} = \frac{\alpha_{x3}}{\alpha_{x1} + \alpha_{x2} + \alpha_{x3}}, \\
\omega_{y1} &= \frac{\alpha_{y1}}{\alpha_{y1} + \alpha_{y2} + \alpha_{y3}}, \quad \omega_{y2} = \frac{\alpha_{y2}}{\alpha_{y1} + \alpha_{y2} + \alpha_{y3}}, \quad \omega_{y3} = \frac{\alpha_{y3}}{\alpha_{y1} + \alpha_{y2} + \alpha_{y3}}.
\end{align*}

(3.54)

where

\begin{align*}
\alpha_{x1} &= \frac{0.1}{(S_{x1} + \epsilon)^2}, \quad \alpha_{x2} = \frac{0.6}{(S_{x2} + \epsilon)^2}, \quad \alpha_{x3} = \frac{0.3}{(S_{x3} + \epsilon)^2}, \\
\alpha_{y1} &= \frac{0.1}{(S_{y1} + \epsilon)^2}, \quad \alpha_{y2} = \frac{0.6}{(S_{y2} + \epsilon)^2}, \quad \alpha_{y3} = \frac{0.3}{(S_{y3} + \epsilon)^2}.
\end{align*}

(3.55)
with $\epsilon \approx 10^{-20}$ as a small number to avoid division by zero and the smoothness measurements $S_{x1}$, $S_{x2}$, $S_{x3}$, $S_{y1}$, $S_{y2}$ and $S_{y3}$ are calculated as follows

\[
S_{x1} = \frac{13}{12} (v_{x1} - 2v_{x2} + v_{x3})^2 + \frac{1}{4} (v_{x1} - 4v_{x2} + 3v_{x3})^2
\]
\[
S_{x2} = \frac{13}{12} (v_{x2} - 2v_{x3} + v_{x4})^2 + \frac{1}{4} (v_{x2} - 4v_{x4} - v_{x4})^2
\]
\[
S_{x3} = \frac{13}{12} (v_{x3} - 2v_{x4} + v_{x5})^2 + \frac{1}{4} (3v_{x3} - 4v_{x4} + v_{x5})^2
\]
\[
S_{y1} = \frac{13}{12} (v_{y1} - 2v_{y2} + v_{y3})^2 + \frac{1}{4} (v_{y1} - 4v_{y2} + 3v_{y3})^2
\]
\[
S_{y2} = \frac{13}{12} (v_{y2} - 2v_{y3} + v_{y4})^2 + \frac{1}{4} (v_{y3} - 4v_{y4} - v_{y4})^2
\]
\[
S_{y3} = \frac{13}{12} (v_{y3} - 2v_{y4} + v_{y5})^2 + \frac{1}{4} (3v_{y3} - 4v_{y4} + v_{y5})^2
\]

where

\[
v_{x1} = q_{bx1-2,j}, v_{x2} = q_{bx1-1,j}, v_{x3} = q_{bx1,j}, v_{x4} = q_{bx1+1,j}, v_{x5} = q_{bx1+2,j}
\]
\[
v_{y1} = q_{by1,j-2}, v_{y2} = q_{by1,j-1}, v_{y3} = q_{by1,j}, v_{y4} = q_{by1,j+1}, v_{y5} = q_{by1,j+2}
\]

Similarly, one can calculate the right-biased-fluxes $\hat{q}_{bxi+1/2,j}^+$ and $\hat{q}_{byi,j+1/2}^+$ (3 grid points to the right of the grid interface $(i+1/2, j)$ and only 2 grid points to the left are used in the sense of the x-direction)

\[
\hat{q}_{bxi+1/2,j}^+ = \begin{cases} 
\hat{\omega}_{x1} \tilde{q}_{bxi+1/2,j} + \hat{\omega}_{x2} \tilde{q}_{bxi+1/2,j}^2 + \hat{\omega}_{x3} \tilde{q}_{bxi+1/2,j}^3 & \text{if } C_x(z_h)_{i+1/2,j} < 0 \\
0 & \text{if } C_x(z_h)_{i+1/2,j} \geq 0
\end{cases}
\]
\[
\hat{q}_{byi,j+1/2}^+ = \begin{cases} 
\hat{\omega}_{y1} \tilde{q}_{byi,j+1/2} + \hat{\omega}_{y2} \tilde{q}_{byi,j+1/2}^2 + \hat{\omega}_{y3} \tilde{q}_{byi,j+1/2}^3 & \text{if } C_y(z_h)_{i,j+1/2} < 0 \\
0 & \text{if } C_y(z_h)_{i,j+1/2} \geq 0
\end{cases}
\]

where
\[
\begin{align*}
\hat{q}^{1}_{bi+1/2,j} &= -\frac{1}{6} q_{bi-1,j} + \frac{5}{6} q_{bi,j} + \frac{1}{3} q_{bi+1,j}, \\
\hat{q}^{2}_{bi+1/2,j} &= \frac{1}{3} q_{bi,j} + \frac{5}{6} q_{bi-1,j} - \frac{1}{6} q_{bi+2,j}, \\
\hat{q}^{3}_{bi+1/2,j} &= \frac{11}{6} q_{bi+1,j} - \frac{7}{6} q_{bi+2,j} + \frac{1}{3} q_{bi+3,j}, \\
\hat{q}^{4}_{bi+1/2,j} &= \frac{1}{3} q_{bi,j} + \frac{5}{6} q_{bi+1,j} - \frac{1}{6} q_{bi+2,j}, \quad (3.59)
\end{align*}
\]

\[
\begin{align*}
\tilde{\omega}_x &= \frac{\tilde{\alpha}_x}{\tilde{\alpha}_x + \tilde{\alpha}_y + \tilde{\alpha}_z}, \\
\tilde{\omega}_y &= \frac{\tilde{\alpha}_y}{\tilde{\alpha}_x + \tilde{\alpha}_y + \tilde{\alpha}_z}, \quad \tilde{\omega}_z = \frac{\tilde{\alpha}_z}{\tilde{\alpha}_x + \tilde{\alpha}_y + \tilde{\alpha}_z}
\end{align*}
\]

\[
\begin{align*}
\tilde{\alpha}_x &= \frac{0.3}{(\tilde{S}_x + \varepsilon)^2}, \quad \tilde{\alpha}_y = \frac{0.6}{(\tilde{S}_y + \varepsilon)^2}, \quad \tilde{\alpha}_z = \frac{0.1}{(\tilde{S}_z + \varepsilon)^2}, \\
(3.60)
\end{align*}
\]

\[
\begin{align*}
\tilde{S}_x &= \frac{13}{12} (v_{x,2} - 2v_{x,3} + v_{x,4})^2 + \frac{1}{4} (v_{x,2} - 4v_{x,3} + 3v_{x,4})^2, \\
\tilde{S}_y &= \frac{13}{12} (v_{y,3} - 2v_{y,4} + v_{y,5})^2 + \frac{1}{4} (v_{y,3} - v_{y,5})^2, \\
\tilde{S}_z &= \frac{13}{12} (v_{z,4} - 2v_{z,5} + v_{z,6})^2 + \frac{1}{4} (3v_{z,4} - 4v_{z,5} + v_{z,6})^2, \\
\tilde{S}_x &= \frac{13}{12} (v_{y,1} - 2v_{y,3} + v_{y,4})^2 + \frac{1}{4} (v_{y,2} - 4v_{y,3} + 3v_{y,4})^2, \\
\tilde{S}_y &= \frac{13}{12} (v_{y,3} - 2v_{y,4} + v_{y,5})^2 + \frac{1}{4} (v_{y,3} - v_{y,5})^2, \\
\tilde{S}_z &= \frac{13}{12} (v_{y,4} - 2v_{y,5} + v_{y,6})^2 + \frac{1}{4} (3v_{y,4} - 4v_{y,5} + v_{y,6})^2, \\
\end{align*}
\]

and \(v_{x,6} = q_{bi+3,j}\) and \(v_{y,6} = q_{bi,j+3}\) in addition to (3.57). The bed-form propagation phase speed \(C(z_b)\) can be estimated numerically. For the 2D case, the bed variation equation due to the bed load transport reads

\[
\frac{\partial z_b}{\partial t} = \frac{1}{1 - \text{poro}} \left( \frac{\partial q_{bx}}{\partial x} + \frac{\partial q_{by}}{\partial y} \right) = 0
\]

(3.63)
If the propagations of $z_h$ can be described by speeds $C_x(z_h)$ and $C_y(z_h)$ in $x$- and $y$-directions respectively, then (3.63) can be re-written as

$$\frac{\partial z_h}{\partial t} + C_x(z_h)\frac{\partial z_h}{\partial x} + C_y(z_h)\frac{\partial z_h}{\partial y} = 0 \quad (3.64)$$

Therefore, (3.63) and (3.64) give

$$C_x(z_h) = -\frac{\partial q_{bx}}{\partial x} \frac{\partial z_h}{\partial x}, \quad C_y(z_h) = -\frac{\partial q_{by}}{\partial y} \frac{\partial z_h}{\partial y} \quad (3.65)$$

which can be estimated numerically. In WENO, it is important that upwinding is used, and the easiest and the most inexpensive way to achieve upwinding is to calculate the Roe speeds

$$a_{i+1/2,j} = \frac{q_{bi+1,j} - q_{bi-1,j}}{(1 - \text{poro})(z_{bi+1,j} - z_{bi,j})}, \quad a_{i,j+1/2} = \frac{q_{bi,j+1} - q_{bi,j-1}}{(1 - \text{poro})(z_{bi,j+1} - z_{bi,j})} \quad (3.66)$$

If $a_{i+1/2,j} \geq 0$, the flow is from left to right, and so is the corresponding bed form phase speed; vice versa. Similar conclusion can be drawn for $a_{i,j+1/2}$. Since only the signs of $a_{i+1/2,j}$ and $a_{i,j+1/2}$ are needed, we can only compute the signs of

$$\left( q_{bi+1,j} - q_{bi-1,j} \right) \left( z_{bi+1,j} - z_{bi,j} \right) \quad \text{and} \quad \left( q_{bi,j+1} - q_{bi,j-1} \right) \left( z_{bi,j+1} - z_{bi,j} \right)$$

thus avoid the problem when the denominator is zero. Therefore, only requiring the sign of the phase speed to judge the “upwind” direction, WENO is less demanding and more stable than the schemes (e.g., Lax-Wendroff scheme and Warming-Beam scheme) that require the estimation of phase speed in both magnitude and sign.
3.1.6 Computational cycle

Finally, the complete cycle for updating the field variables within one time step is summarized as follows:

1. Compute the continuity equation (3.14) of SWE to get the new water depths.
2. Apply the boundary conditions of water depth.
3. Compute the momentum equations (3.15) and (3.16) of SWE to get the new flow fluxes.
4. Apply the boundary conditions of flux.
5. Update $\hat{k}$ and $\hat{\varepsilon}$ using new velocities.
6. Compute suspended load transport equation to get the new $C$.
7. Compute bed load transport equation to get $q_b$.
8. Compute morphological evolution equation to update bed elevation.

3.2 Stability Analysis

In this section, the stability condition is determined for the numerical algorithms used in the study. To achieve it, one can use the von Neumann stability analysis after linearizing the PDE. The solutions for these equations can be written as a series of Fourier modes which will then be substituted into the finite difference equations. A stable scheme requires that the amplification factor for each Fourier mode should not exceed unity. In this way, it is possible to find out whether a finite difference scheme is stable and the stability condition.

Here one presents the following stability conditions for SWE as an example:
\[ \Delta t \leq \min \left\{ \frac{\Delta x}{U_{\text{max}} + C_{\text{wave}}}, \frac{\Delta y}{V_{\text{max}} + C_{\text{wave}}} \right\} \quad \text{and} \quad \Delta t \leq \min \left\{ \frac{(\Delta x)^2}{2(v_r + v_m)}, \frac{(\Delta y)^2}{2(v_r + v_m)} \right\} \]  

(3.67)

where \( U_{\text{max}} \) and \( V_{\text{max}} \) are the maximum velocities in \( x \)- and \( y \)-directions respectively; \( C_{\text{wave}} = \sqrt{gh} \) is the wave celerity. The first constraint above is set by the convection terms and the second the diffusion terms. Similar stability conditions can be found for the convection-diffusion equation for the suspended load transport.

For the morphological change equation (which is Exner equation), since the WENO scheme is a nonlinear scheme in the sense that the coefficients used in the scheme depend upon the transport rate adaptively rather than being constants, no theoretical stability criterion is available (Long, et al., 2008). Time step \( \Delta t_{\text{sedi}} \) used in the morphological computation must satisfy the condition that Courant number is less than one.

However, under some assumptions or using numerical estimation, the Exner equation could be linearized and the stability analysis could be started based on the propagation phase speeds \( C_x(z_b) \) and \( C_y(z_b) \) in \( x \)- and \( y \)-directions. Therefore, the stability condition for the morphological equation is as follows

\[ \Delta t \leq \min \left\{ \frac{\Delta x}{C_x(z_b)}, \frac{\Delta y}{C_y(z_b)} \right\} \]  

(3.68)

In the following section, two methods to obtain the propagation phase speed will be introduced. For simplicity, 1D situation will be considered and the propagation phase speed \( C(z_b) \) will be calculated based on the bedload transport rate \( q_b \).

One method is presented by Jansen (1979). For the bedload transport, the general form is
\( q_b = q_b(U, \text{other parameters}) \)  

(3.69)

Assuming that all parameters but \( U \) remain constant during the propagation of a small disturbance, we have

\[
\frac{\partial q_b}{\partial x} = \frac{dq_b(U)}{dU} \frac{\partial U}{\partial x}
\]

(3.70)

To further simplify the above equation, it is necessary to simplify both \( \frac{dq_b(U)}{dU} \) and \( \frac{\partial U}{\partial x} \) as follows. If it is assumed that the transport rate varies linearly with the flow velocity, then \( \frac{dq_b(U)}{dU} = m_1 = \text{constant} \). Since \( U = \frac{P}{H} = \frac{P}{\eta + z_b} \), the following is obtained,

\[
\frac{\partial U}{\partial x} = \frac{-P}{(\eta + z_b)^2} \frac{\partial z_b}{\partial x} \text{ if the gradient of water surface can be neglected, i.e., } \frac{\partial \eta}{\partial x} = 0.
\]

Therefore, one obtains

\[
\frac{\partial q_b}{\partial x} = \frac{-m_1 P}{(\eta + z_b)^2} \frac{\partial z_b}{\partial x}
\]

(3.71)

Substituting (3.71) into the Exner equation, the linearization is achieved and along with the propagation speed \( C(z_b) \)

\[
\frac{\partial z_b}{\partial t} = \frac{-m_1 P}{(1 - \text{poro})(\eta + z_b)^2} \frac{\partial z_b}{\partial x} = C(z_b) \frac{\partial z_b}{\partial x}
\]

(3.72)

The other method to obtain \( C(z_b) \) is using numerical estimation. As described in section 3.1.5, for the 1D case, the propagation speed can be defined as follows

\[
C(z_b) = \frac{\frac{\partial q_b}{\partial x}}{(1 - \text{poro}) \frac{\partial z_b}{\partial x}}
\]

(3.73)

Hudson et al. (2005) employed a central difference scheme to solve (3.73) numerically
\[ C(z_h) = \frac{q_{bh+1} - q_{bh-1}}{(1 - \text{poro})(z_{bh+1} - z_{bh-1})}, \text{ for } z_{bh+1} \neq z_{bh-1} \]  \hspace{1cm} (3.74)

Note that the above equation can be used only when \( z_{bh+1} \neq z_{bh-1} \) and it will produce quite inaccurate approximation for \( C(z_b) \) when the gradient of bed level approaches zero or changes sign.

### 3.3 Special Numerical Treatments

#### 3.3.1 Boundary condition for \( \hat{k} - \hat{c} \) equations on solid boundary

As discussed before, the boundary conditions for \( \hat{k} \) and \( \hat{c} \) are applied at the location within the log-law region of turbulent boundary layer and (2.107) is used to connect the outside tangential velocity \( U_x \) to the frictional velocity \( u_* \). Since the velocities are defined at the center of the interfaces of the cells, the normal distance to the walls \( y_n = \Delta x/2 \) when \( U_x = V \) and \( y_n = \Delta y/2 \) when \( U_x = U \) depending upon the location of the wall.

The equation of the logarithmic law-of-the-wall, i.e., (2.107) is implicit when considering the hydraulically smooth regime. It can be solved iteratively by using the Newton-Raphson method to get the \( u_* \) (Lemos, 1992; Lin, 1998)

\[ X^{(n+1)} = X^{(n)} + \frac{\kappa - X^{(n)}(A + \ln X^{(n)})}{1 + \frac{\kappa}{X^{(n)}}} \]  \hspace{1cm} (3.75)

where the superscript \( n \) is the iteration number and \( X = u_*/U_x \) and \( A = \ln \frac{EU_xy_n}{V_m} \).
3.3.2 Approximate calculation method for gradually varied beds

The numerical model developed in the study is to satisfy the long term and large scale computations thus its efficiency is an important issue to be considered. In this section, an approximate calculation method will be presented. For the computation of gradually varied beds, this method can accelerate the computational speed greatly while keeping accuracy.

To ensure the stability of the finite difference scheme of SWE, the following CFL (Courant-Friedrichs-Lewy) condition should be satisfied for convection terms

\[
Cr_{\text{Flow}} = \left\{ \left( \frac{|U_{\text{max}}| + \sqrt{gH}}{\Delta x} \right), \left( \frac{|V_{\text{max}}| + \sqrt{gH}}{\Delta y} \right) \right\} \leq 1
\]  
(3.76)

where \( Cr_{\text{Flow}} \) is the Courant number for the flow computation, \( \Delta t_{\text{Flow}} \) is the time step used in the flow computation. On the other hand, for the suspended load transport equation, the CFL condition for convection terms is as follows

\[
Cr_{\text{Sedi}} = \left\{ \left( \frac{\Delta t_{\text{Sedi}}}{\Delta x} \right), \left( \frac{\Delta t_{\text{Sedi}}}{\Delta y} \right) \right\} \leq 1
\]  
(3.77)

where \( Cr_{\text{Sedi}} \) is the Courant number for the suspended load calculation, \( \Delta t_{\text{Sedi}} \) is the time step used in the suspended load calculation.

In the regular computational method, the same time step would be used for flow and sediment computations, i.e., \( \Delta t = \min \{ \Delta t_{\text{Flow}}, \Delta t_{\text{Sedi}} \} \). However, comparing the difference of these two Courant number criteria, we can find that \( \Delta t_{\text{Sedi}} \) could be greater than \( \Delta t_{\text{Flow}} \) for any subcritical flow. Hence, an approximate method is proposed in this study and the details are given as follows.
Firstly, run the hydrodynamic module until steady state corresponding to the initial topography. After that, the sediment transport module and the morphological evolution module will be launched with a larger time step $\Delta t_{\text{Sedi}}$. The flow calculation will be carried out once after each sediment calculation with $\Delta t_{\text{Flow}}$ assuming the flow remains the same from $\Delta t_{\text{Flow}}$ to $\Delta t_{\text{Sedi}}$. Accordingly the flow computation and the sediment computation will be carried out alternately. For small Froude number, $\Delta t_{\text{Sedi}}$ could be many times of $\Delta t_{\text{Flow}}$. Therefore, the whole calculation procedure would be accelerated with satisfactory accuracy if the sediment bed varies gradually.
Chapter 4

Numerical Testing

In this chapter, the numerical model, including SWE, depth-averaged $\hat{k} - \hat{e}$ equations, suspended load transport equation and morphological evolution equation, will be tested individually. The available analytical solutions as well as the experimental measurements will be used for the comparisons.

4.1 1D Hydrodynamic Module

4.1.1 Solitary wave propagation

The solitary wave propagation in a constant water depth is a classical benchmark test for a hydrodynamic model. The solitary wave is a shallow water wave with finite amplitude and theoretically infinite wave length. During its propagation, the solitary wave keeps its shape unchanged, with the nonlinearity and frequency dispersion perfectly balanced. A definition sketch of a solitary wave is shown in Figure 4.1.
Based on the potential flow approximation, the Boussinesq analytical solutions for the solitary wave are as follows (Lee et al., 1982):

\[
\eta(x,t) = a \text{sech}^2 \left[ \frac{3a}{4h_0^2} (x - C_{\text{wave}} t) \right] \tag{4.1}
\]

\[
C_{\text{wave}} = \sqrt{g (h_0 + a)} \tag{4.2}
\]

\[
\frac{u}{\sqrt{gh_0}} = \frac{\eta}{h_0} \left( 1 - \frac{1}{4} \frac{\eta}{h_0} \right) \tag{4.3}
\]

where \( \eta \) is the free surface displacement measured from still water level; \( a \) is the maximum wave amplitude; \( h_0 \) is the still water depth; \( C_{\text{wave}} \) is the wave celerity; \( u \) is the horizontal component of the water particle velocity.

The solitary wave with the ration of the wave amplitude to the still water depth of \( a / h_0 = 0.002 \) is investigated. In the numerical computation, \( h_0 = 1 \text{ m} \) and \( a = 0.002 \text{ m} \) are
used. It should be noted that due to the choice of very small wave amplitude, the solitary wave has degenerated to a non-dispersive linear long wave and can be simulated by using SWE. In addition, the turbulence model is turned off in the simulation and the bottom frictional term is set to zero. Therefore, the governing equations of the model used in this test are as follows:

\[
\frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} = 0 \tag{4.4}
\]

\[
\frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + gH \frac{\partial \eta}{\partial x} = 0 \tag{4.5}
\]

The computational domain of 800 m long is discretized with the uniform grid of \( \Delta x = 0.5 \) m. A time step is chosen as \( \Delta t = 0.1 \) s to satisfy the numerical stability constraint. Initially, the water within the computational domain is still. As a boundary condition, the flow flux \( P_0 = u(h_0 + \eta) \) is specified at the left boundary based on the analytical solutions (4.1) ~ (4.3) thus a solitary wave is sent from left boundary and starts propagating within the computational domain.

Figure 4.2(a) shows the comparisons of the solitary wave profiles at different time \( t\sqrt{g/h_0} = 125.26, 250.53, 375.79, 501.06 \) and 626.32 between the analytical solutions and the numerical results. It is seen that the numerical results agree with the analytical solutions nearly perfectly even after the propagation of about 700 \( d \).

Next, the numerical model will be checked for the conservation of both the mass and the energy. Since the wave is inside the computational domain within the whole computation, the total mass should be conserved if the scheme is accurate. In addition, since both the diffusion term and the bottom frictional term are excluded in this test, the
energy should be conserved, if the model accurately predicts both the velocity, which is related to the kinetic energy, and the free surface displacement, which is related to the potential energy. Figure 4.2(b) shows the time histories of the mass, total energy, kinetic energy and potential energy. It can be seen that the model conserves both mass and energy very well. Based on the linear wave theory, the kinetic energy should be exactly equal to the potential energy. This phenomenon can also be found in Figure 4.2(b).
Figure 4.2: (a) Comparisons of the solitary wave profiles at different time $t\sqrt{g/h_0} = (A): 125.26$, (B): 250.53, (C): 375.79, (D): 501.06 and (E): 626.32 between the analytical solutions (dashed line) and the numerical results (solid line). (b) Time histories of the mass (dash-dot line), total energy (solid line), kinetic energy (dashed line) and potential energy (dotted line); the mass has been normalized by the calculated mass at $t\sqrt{g/h_0} = 250.53$ and the energy has been normalized by the calculated total energy at $t\sqrt{g/h_0} = 250.53$. 
**Numerical convergence**

To investigate the numerical convergence of the model, the following test of simulating a solitary wave propagation in constant water depth using different grid discretizations is conducted. The same wave as used in the previous test is used here. Four grid systems from coarse to fine as well as their corresponding time steps are as follows:

(A) $\Delta x = 4.0 \text{ m}$ and $\Delta t = 0.8 \text{ s}$

(B) $\Delta x = 2.0 \text{ m}$ and $\Delta t = 0.4 \text{ s}$

(C) $\Delta x = 1.0 \text{ m}$ and $\Delta t = 0.2 \text{ s}$

(D) $\Delta x = 0.5 \text{ m}$ and $\Delta t = 0.1 \text{ s}$

The comparisons of the solitary wave profiles at time $t \sqrt{g/h_0} = 501.06$ are shown in Figure 4.3(a). It is observed that when the grid becomes finer, the numerical results agree better with the analytical solutions in terms of both the wave height and the wave phase. This indicates that the model solutions converge to the true solution when the grid size approaches zero.

A more direct view of the numerical convergence can be seen in Figure 4.3(b). In this figure, the calculated wave height at time $t \sqrt{g/h_0} = 501.06$ is plotted against $h_0/\Delta x$. As $h_0/\Delta x$ increase, the numerical results are approaching the true solution asymptotically. For this particular test, the finest grid system, which corresponds to resolving one wave length using 200 meshes, seems sufficient enough to give an accurate numerical solution.

It should be noted that the numerical convergence characteristics may vary case by case. In the following study, unless otherwise mentioned, all problems have been tested by using different grid systems and the final results are presented in the range of acceptable
numerical convergence, i.e., the numerical results do not change significantly by reducing
the grid size by half.

Figure 4.3: (a) Comparisons of the solitary wave profiles at time $t \sqrt{g/h_0} = 501.06$ among
the analytical solutions (dashed line), the numerical results using $\Delta x = 4.0$ m (circles),
$\Delta x = 2.0$ m (dotted line), $\Delta x = 1.0$ m (dash-dot line) and $\Delta x = 0.5$ m (solid line). (b)
Numerical convergence in terms of the wave height at time $t \sqrt{g/h_0} = 501.06$; analytical
solution (dashed line) and the numerical solutions (circles).
4.1.2 Idealized dam-break

In this section, an idealized dam-break problem in a rectangular channel with a horizontal and frictionless bed is considered. As shown in Figure 4.4 (a), a dam is located at \( x = 0 \) of an infinite channel. For \( x > 0 \), the water depth is \( h_0 \) and for \( x < 0 \), the water depth is \( h_1 \) with \( h_0 < h_1 \). Initially, the water is at rest on both sides of the dam. At the time \( t = 0 \), the dam is suddenly removed. At any time \( t = t_0 \), four zones can be identified (see Figure 4.4 (b)): zone 0 is the quiescent undisturbed downstream water; zone 2 is the shock wave with constant state; zone 3 is a centred simple wave connecting zone 2 and zone 1; and zone 1 is the undisturbed upstream water. Analytical solutions are available for this problem (e.g., Stoker, 1957 and Wu et al., 1999).

The theoretical solution of the shock wave, i.e., zone 2, is as follows:

\[
\frac{u_2}{C_{\text{wave}0}} = \frac{\xi}{C_{\text{wave}0}} - \frac{1}{4} \frac{C_{\text{wave}0}}{\xi} \left[ 1 + \sqrt{1 + 8 \left( \frac{\xi}{C_{\text{wave}0}} \right)^2} \right] \tag{4.6}
\]

\[
\frac{C_{\text{wave}2}}{C_{\text{wave}0}} = \left[ \frac{1}{2} \left( \sqrt{1 + 8 \left( \frac{\xi}{C_{\text{wave}0}} \right)^2} - 1 \right) \right]^{1/2} \tag{4.7}
\]

\[
u_2 + 2C_{\text{wave}2} = 2C_{\text{wave}1} \tag{4.8}
\]

where \( C_{\text{wave}} = \sqrt{gh} \) with subscripts 0, 1 and 2 denoting zone 0, 1 and 2 respectively; \( u \) is mean velocity of the flow, \( \xi \) is the velocity of the shock.

The solution to zone 3 is Ritter’s solution:

\[
\frac{h}{h_1} = \frac{1}{9} \left( 2 - \frac{x}{C_{\text{wave}1}t} \right)^2 \tag{4.9}
\]
To simulate this dam-break problem, the turbulence model is turned off and the bottom frictional term is set to zero again and the governing equations are same as (4.4) and (4.5). Consider a channel with 1000m length and a dam located at the middle of it. Initially, the upstream and downstream water depths are 10m and 1m respectively. In the numerical computation, $\Delta x = 1\text{ m}$ and $\Delta t = 0.03\text{ s}$.

Figure 4.5 shows the comparisons of the numerical and analytical solutions of water depth, $H$ and velocity $U$ at $t=30$s after the dam failure. Overall, the numerical results have a very good agreement with the analytical solutions. However, they show some numerically diffusive effect at the discontinuities. This is caused by the upwind scheme of the convection terms and will not be expected to be serious in the real cases with bottom friction.
Figure 4.4: Breaking of a dam: (a) at $t = 0$; (b) at $t = t_0$. 
Figure 4.5: Comparisons of both water depth and velocity between the analytical solutions (solid line) and the numerical results (dashed line). Initial water depth before dam-break is also plotted (dotted line).
4.1.3 Partial dam-break

In this test, a dam-break conducted experimentally will be repeated numerically using the present model. Available measurements are used for the comparisons. This case is also used by Ying, et al. (2004) and Tseng and Yen (2004) for testing their numerical models. Waterways Experiment Station (WES), U.S. Army Corps of Engineers (WES, 1960) studied a dam-break problem experimentally. The flume was 122m long and 1.22m wide with rectangular cross-section. The slope of the bed of the flume was 0.005 and the Manning coefficient was 0.009. As shown in Figure 4.6, a dam with 0.305m height was located at the midpoint of the channel. Initially, the water surface at the upstream side of the dam was at the same height as the top of the dam, while the channel downstream the dam was dry. At time t=0, a part of the dam with 0.183m height was breached instantaneously and the water started flooding the downstream channel. There were four measurement stations recording the time histories of the water depth: STA100, STA150, STA225 and STA350. The velocities were also recorded at STA225 and STA350 which were downstream of the dam.

To simulate this problem, the above experimental values are adopted in the computation. The turbulence model is turned on and the bottom friction term is included with the provided Manning $n = 0.009$. Therefore, the governing equations for this test are as follows:

\[
\frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} = 0 \tag{4.5}
\]

\[
\frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + gH \frac{\partial \eta}{\partial x} = -\frac{1}{\rho} \tau_{bx} + \frac{1}{\rho} \frac{\partial (HT_{xx})}{\partial x} \tag{4.10}
\]
\[
\frac{\partial (Hk)}{\partial t} + \frac{\partial (HUK)}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{\hat{\nu}_i}{\sigma_k} \frac{\partial (Hk)}{\partial x} \right] + P_h + P_{k\nu} - \hat{\nu} H \quad (4.11)
\]

\[
\frac{\partial (H\hat{\epsilon})}{\partial t} + \frac{\partial (HUK\hat{\epsilon})}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{\hat{\nu}_i}{\sigma_{\epsilon}} \frac{\partial (H\hat{\epsilon})}{\partial x} \right] + C_{1e} \frac{\hat{\epsilon}}{k} P_h + P_{\epsilon\nu} - C_{2e} \frac{\hat{\epsilon}^2}{k} H \quad (4.12)
\]

The simulation lasts 200s. In addition, \(\Delta x = 0.5\) m and \(\Delta t = 0.1\) s are used for this test. Figure 4.7 shows the numerical results of the time histories of the water depth and velocity corresponding to the same locations of the four measurement stations. Compared to the available experimental data, the general agreement between them is reasonably good. Note that the numerical experiments show that even with the turbulence model being turned off, the numerical results are almost the same as those with the turbulence model. This suggests that the turbulence effect is negligible for this particular problem which happened within a very short time.
Figure 4.6: Definition sketch of the initial condition of the partial dam-break problem and the positions of four measurement stations, i.e., STA100, STA150, STA225 and STA350.
Figure 4.7: Comparisons of both water depth and velocity between the experimental data (circle) and the numerical results (solid line) at stations STA100, STA150, STA225 and STA350.
4.1.4 Hydraulic jump

Generally, the hydraulic jump is used to verify the shock-capturing capability of the model. In this section, the laboratory data of a hydraulic jump from Gharangik and Chaudhry (1991) which have also been adopted to test models in many papers are used (e.g., Gharangik and Chaudhry, 1991; Molls and Chaudhry, 1995; Zhou and Stansby, 1999). The experiments were conducted in a rectangular metal flume with a length of $L = 14.0\,\text{m}$ and a width of $B = 0.46\,\text{m}$. The water was supplied from a large tank through a sharp-edged sluice gate and the jump was formed by controlling the tailwater depth with an adjustable downstream gate. A weighing tank was used to measure the discharge. The jump profile and location as well as the water surface upstream and downstream of the jump were measured in the experiments. The flow conditions were as follows: upstream depth $H_u = 0.064\,\text{m}$, upstream velocity $U_u = 1.826\,\text{m/s}$, upstream Froude number $Fr = 2.30$, downstream depth $H_d = 0.168\,\text{m}$, bottom slope $S_b = 0$, Manning coefficient $n = 0.008 \sim 0.011$.

This hydraulic jump generated experimentally is simulated using the present model. The governing equations used in this test are the same as those used in the partial dam-break problem, i.e., (4.5), (4.10), (4.11) and (4.12). The model is run with $\Delta x = 0.2\,\text{m}$ and $\Delta t = 0.02\,\text{s}$ and Manning $n$ is selected as 0.009 after trial and error. Since the hydraulic jump includes a transition from super- to sub-critical flow, both the velocity and water depth are specified at the upstream boundary, while only the water depth is specified at the downstream boundary (Gharangik and Chaudhry, 1991; Zhou and Stansby, 1999). For the initial condition, the flow is assumed to be supersonic in the entire channel. After the
increase in the water depth at downstream boundary, jump will form around downstream end and then travel towards the upstream end until it is stabilized at one location.

Figure 4.8 shows the simulated water surface profiles at different times after the water depth is raised at the downstream boundary. It is clearly seen that the hydraulic jump is generated downstream and then travels upstream before its stabilization. Figure 4.9 shows the comparison between measured jump profile and numerical results. It is found that the numerical simulation reasonably predicts the location and the profile of the jump.
Figure 4.8: Numerical results of the water surface profile at different time $t=0, 15, 30, 45$ and 60 seconds and at final steady state.

Figure 4.9: Comparisons of water surface profile between the experimental data (cross) and the numerical results (solid line).
4.2 2D Hydrodynamic Module

4.2.1 Sloshing in a tank

In this section, the water sloshing in a confined tank will be studied. The tank has the dimensions of \( L_x \times L_y \), in which the coordinate origin is defined at the left-bottom corner of the basin. The initial free surface displacement is a Gaussian distribution about the center of the basin, i.e.,

\[
\eta_0(x, y) = H_0 \exp\left\{ -\beta \left[ \left( x - L_x / 2 \right)^2 + \left( y - L_y / 2 \right)^2 \right] \right\}
\]

(4.13)

where \( H_0 \) is the initial height of the hump and \( \beta \) is the peak enhancement factor. Wei and Kirby (1995) proposed the linear analytical solution for the free surface evolution as follows:

\[
\eta(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \overline{\eta}_{nm} e^{-\nu_{nm} t} \cos(n\lambda x) \cos(m\lambda y)
\]

(4.14)

in which:

\[
\overline{\eta}_{nm} = \frac{4}{(1 + \delta_{n0})(1 + \delta_{m0})L_x L_y} \int_0^{L_x} \int_0^{L_y} \eta_0(x, y) \cos(n\lambda x) \cos(m\lambda y) \, dx \, dy
\]

(4.15)

where \( \delta_{nm} \) is the Kronecker delta function and

\[
\lambda = \frac{\pi}{L_x} = \frac{\pi}{L_y}
\]

(4.16)

The \((n, m)\) wave modes have the corresponding natural frequency which is determined from the linear dispersion equation

\[
\omega_{nm}^2 = gk_{nm} \tanh(k_{nm} h_0)
\]

(4.17)
where $h_0$ is the still water depth and

$$k_{nm}^2 = (n\lambda)^2 + (m\lambda)^2 = \left(\frac{\pi}{L_x}\right)^2 (n^2 + m^2)$$  \hspace{1cm} (4.18)

In the simulation, the following parameters are taken: $L_x = L_y = 10$ m, $h_0 = 0.1$ m, $H_0 = 0.001$ m and $\beta = 0.4$. Note that the water depth is chosen to satisfy the shallow water assumption, i.e., $h_0 = 0.1$ m $< \frac{L_x}{20} = 0.5$ m. In addition, a relatively small ratio of the initial hump height to the water depth, i.e., $H_0/h_0 = 0.01$, makes only a little nonlinearity present during the wave transformation. Without the turbulence effect or the bottom friction involved, the governing equations of the model for this test are as follows:

$$\frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$$  \hspace{1cm} (2.9)

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(\frac{P^2}{H}\right) + \frac{\partial}{\partial y}\left(\frac{PQ}{H}\right) + gH \frac{\partial \eta}{\partial x} = 0$$  \hspace{1cm} (4.19)

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{PQ}{H}\right) + \frac{\partial}{\partial y}\left(\frac{Q^2}{H}\right) + gH \frac{\partial \eta}{\partial y} = 0$$  \hspace{1cm} (4.20)

The uniform grid systems are used to discretize the computational domain. In order to investigate the numerical convergence of the model in two-dimensional situation, we will use three different grid systems presented with their corresponding time steps as follows:

(A) $nx = ny = 50$ and $\Delta t = 0.1$ s

(B) $nx = ny = 100$ and $\Delta t = 0.05$ s

(C) $nx = ny = 200$ and $\Delta t = 0.025$ s

After $t = 0$ s, the gravitational force plays the role as the restoring force and the water inside the tank will start to slosh forever due to no energy loss. Figure 4.10 shows the
comparisons of the time histories of the water surface elevation at the center and the corner of the tank among the linear analytical solution and the numerical results using different grid systems. It is found that generally the numerical results match the analytical solutions quite well. There are some small phase discrepancies observed at some time between the numerical results and analytical solutions due to the wave dispersions which may not be caught by the SWE. In addition, when the grid becomes finer, the numerical results show convergence to the true solutions.

Figure 4.11 shows the time histories of the mass and total energy during the water sloshing. It can be seen that the model conserves both mass and energy perfectly during the whole computation. Figure 4.12 shows the free surface profiles during the water sloshing. It can be observed that the symmetry of the water surface has been preserved very well throughout the whole simulation, regardless of the complex wave transformation happening.
Figure 4.10: Comparisons of the time histories of the normalized water surface elevation $\eta / H_0$ (a) at the center (5m, 5m) and (b) at the corner (0, 0) of the tank among the linear analytical solution (solid line), the numerical results using $nx = ny = 50$ (dashed line), $nx = ny = 100$ (dash-dot line) and $nx = ny = 200$ (dotted line)
Figure 4.11: Time histories of the mass (dashed line) and total energy (solid line); the mass has been normalized by the calculated mass at $t = 0$ and the energy has been normalized by the calculated total energy at $t = 0$. 
Figure 4.12: Snap shots of the free surface profiles during the water sloshing at $t = (a) 0$, (b) 5 s, (c) 10 s, (d) 15 s, (e) 20 s and (f) 25 s.
4.2.2 Uniform flow in a straight channel

Both Rodi (1980) and Younus and Chaudhry (1994) reported the measured depth-averaged longitudinal velocities across the channel cross-sections for a developed uniform flow in a straight channel. These experimental data can be used to test the performance of the depth-averaged $k-\varepsilon$ model and boundary conditions near solid walls.

The experiment was conducted in an open rectangular channel with the following channel parameters: width to depth ratio $B/H = 30$, water depth $H = 0.305$ m, channel width $B = 9.15$ m, longitudinal velocity $U = 0.152$ m/s, Manning roughness factor $n = 0.029$. Depth-averaged horizontal velocity distributions across different channel cross-sections were measured (Figure 4.13).

In the present simulation, the two-dimensional SWE including both the turbulence closure and the bottom friction terms are used, i.e.,

$$ \frac{\partial H}{\partial t} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0 \tag{2.9} $$

$$ \frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( \frac{P^2}{H} \right) + \frac{\partial}{\partial y} \left( \frac{PQ}{H} \right) + gH \frac{\partial \eta}{\partial x} = -\frac{1}{\rho} \tau_{bx} + \frac{1}{\rho} \frac{\partial (HT_{xx})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{sy})}{\partial y} \tag{2.23} $$

$$ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{PQ}{H} \right) + \frac{\partial}{\partial y} \left( \frac{Q^2}{H} \right) + gH \frac{\partial \eta}{\partial y} = -\frac{1}{\rho} \tau_{by} + \frac{1}{\rho} \frac{\partial (HT_{sy})}{\partial x} + \frac{1}{\rho} \frac{\partial (HT_{yy})}{\partial y} \tag{2.24} $$

The computational domain is $30 \text{ m} \times 9.15 \text{ m}$ and the above experimental conditions are used in the numerical simulation. Since the flow is subcritical, uniform-distributed flow flux of $0.0464 \text{ m}^3/\text{s}$ is specified at the upstream boundary, while a water depth of $0.305$ m is fixed at the downstream end. At the walls, no-slip conditions are imposed. In addition, $n_x = 50$, $n_y = 200$ and $\Delta t = 0.02 \text{ s}$ are used in the computation.
It can be seen from Figure 4.13 that as the model with the help of the turbulence closure is able to resolve the near-wall velocities very well. Note that the difference in the velocity distributions at the different cross-sections found in the experiment is not readily discerned in the numerical simulation since the flow is fully developed after some distance away from the upstream end.

![Figure 4.13: Comparisons of depth-averaged longitudinal velocities between the experimental data (x/h=60: square; x/h=100: triangle; x/h=150: circle) and numerical results.](image-url)
4.2.3 Recirculating flow near a groyne

In this section, the present model will be tested in the situation of the recirculating flow near a groyne. Available measured data of the velocities and bed shear stresses will be used for the comparisons with numerical results. In addition, the method proposed for correcting the bed shear stress in Chapter 2 will be verified.

Experimental setup

Rajaratnam and Nwachukwu (1983) conducted the experiments to study the characteristics of the turbulent flow near thin plate groynes projecting perpendicularly into a fully developed turbulent flow in a long rectangular channel. The flume used in the experiments was 37m long, 0.9m wide and 0.76m deep with smooth bed and sides. The groyne was an aluminum plate with 3mm thickness and 152mm length and projected well above the water surface. The flow velocity and water depth in Test A1 were $U_0 = 0.253 \text{ m/s}$ and $H = 0.189 \text{ m}$ respectively. The recirculating length measured in the experiment is about 1.9m. As shown in Figures 4.16 and 4.17, the resultant velocity and bed shear stress profiles measured at $y/b = 1.0, 1.5, 2.0, 3.0$ and 4.0 are plotted after normalized by $U_0 = 0.253 \text{ m/s}$ and $\tau_0 = 0.1293 \text{ N/m}^2$ measured in upstream region respectively.

Numerical setup and results

To simulate the recirculating flow generated in the laboratory, the governing equations used in the model are (2.9), (2.23) and (2.24). The computational domain is 6m in length and 0.9m in width. The upstream and downstream boundaries are located at 2m and 4m away from the groyne respectively. In order to catch the high gradient near the groyne, the non-uniform grid ($200 \times 60$) is packed near the groyne with the minimum grid size 1.5mm
in both $x$- and $y$-directions (see Figure 4.14). Flow flux of $0.047817 \text{ m}^2/\text{s}$ and water depth of $0.189\text{ m}$ are specified at the upstream and downstream boundaries respectively. At the channel walls, no-slip boundary condition is applied. Manning coefficient $n$ is assigned as a typical value for smooth bottom, i.e., $0.01 \text{s}/\text{m}^{1/3}$. In addition, the time step $\Delta t = 0.0003 \text{s}$.

Figure 4.15 shows the streamline pattern of the recirculating flow. From the figure, it is seen that the recirculating length obtained from the model is $1.433\text{ m}$, which is smaller than the measured value, i.e., $1.9\text{ m}$. Figure 4.16 shows the comparison of the resultant velocity profiles among the experimental data, the numerical results from the present model, from Molls et al. (1995) and from Tingsanchali and Maheswaran (1990). It is seen that due to the presence of the groyne, the velocity of the flow coming upstream will increase greatly in the vicinity of the tip of the groyne. The velocity has a large drop in the region behind the groyne. The numerical results show good agreement with the experimental data and capture the high velocities near the tip of the groyne. The deviation is observed at $y/b = 2.0$, where the calculated velocities underpredict the experimental data in the downstream region of the groyne. Similar phenomena of the underprediction are also reported by Molls et al. (1995) and Tingsanchali and Maheswaran (1990) (see Figure 4.16). The possible reason may come from the measurement errors in the experiment.

Figure 4.17 shows the comparisons of the bed shear stress among the experimental data, the numerical results from the present model and from Tingsanchali and Maheswaran (1990). It is seen that the bed shear stress has an extremely high value at the tip of groyne. Without correcting the bed shear stress obtained from the Manning’s
formula, the calculated value of the bed shear stress will largely underestimate the measured value. However, with the help of the correction for the bed shear stress according to the secondary flow structure, much improvement has been made in the prediction of this high bed shear stress. At $y/b = 1.5$ and $2.0$, some overestimations for the high values of the bed shear stress are found. Note that the correction for the bed shear stress occurs around the tip of the groyne where the flow may show strong secondary structure. Outside the recirculating region, since the secondary flow effect becomes weak, this correction becomes negligible.

Overall, the present model can simulate the velocity field of the recirculating flow near a groyne quite well. Considering the secondary flow effect and conducting the correction for the bed shear stress obtained from the original Manning’s formula, the model can predict the bed shear stress reasonably and capture the maximum bed shear stress occurring at the tip of the groyne.
Figure 4.14: Grid arrangement in the computational domain; groyne is located at $x=2\text{m}$; lines are plotted every two grid nodes for easier visibility.

Figure 4.15: Computed streamline pattern and the recirculating length; $x/b=0$ is the groyne position along flume direction.
Figure 4.16: Comparisons of the normalized depth-averaged resultant velocity profiles among the experimental data (circle), the numerical results from the present model (solid line), from Molls et al. (1995) (dash-dot line) and from Tingsanchali and Maheswaran (1990) (dotted line); all the velocities are normalized by $U_0 = 0.253 \text{ m/s}$; $x/b = 0$ is the groyne position along flume direction.
Figure 4.17: Comparisons of the normalized bed shear stress profiles among the experimental data (circle), the numerical results from the present model (solid line), from the present model with the correction of the bed shear stress (dashed line) and from Tingsanchali and Maheswaran (1990) (dotted line); all the shear stresses are normalized by measured $\tau_0 = 0.1293$ N/m$^2$ in upstream region; $x/b = 0$ is the groyne position along flume direction.
4.3 Convection-Diffusion Equation

After testing the hydrodynamic model in both 1D and 2D situations, the numerical testing for the suspended load transport equation which essentially is a convection-diffusion equation is carried out next.

4.3.1 1D Gaussian hump

The analytical solution for 1D convection-diffusion of a Gaussian hump with unit height is given by Noye and Tan (1988):

\[
C(x,t) = \frac{1}{\sqrt{4\pi t + 1}} \exp\left(-\frac{(x-x_0-Ut)^2}{\hat{\nu}_s (4t + 1)}\right) \tag{4.21}
\]

where \(x_0\) is the centre of the initial Gaussian hump. The initial condition can also be obtained from above formula by setting \(t = 0\).

The governing equation corresponding to this problem is as follows:

\[
\frac{\partial (CH)}{\partial t} + \frac{\partial (UCH)}{\partial x} = \hat{\nu}_s \frac{\partial}{\partial x} \frac{\partial (CH)}{\partial x} \tag{4.22}
\]

In the test, the computational domain is 9 m long and the simulation will be made up to \(t = 8\) s. In addition, the following parameters are taken: \(x_0 = 1.0\) m, \(U = 0.8\) m/s, \(\hat{\nu}_s = 0.005\) m²/s, \(\Delta x = 0.025\) m, \(\Delta t = 0.0125\) s.

The convection and diffusion process of the Gaussian hump is shown in Figure 4.18. The figure also gives the comparisons between numerical results and analytical solutions at \(t = 0, 2, 4, 6\) and 8s. Almost perfect agreement can be found.

Since no concentration goes into or out of the computational domain, the total volume of the concentration should remain invariant with the time. Figure 4.19 shows the time
history of the total volume of concentration normalized by its initial value. It can be seen that in this 1D situation, the volume conservation has been maintained perfectly within the range of computer runoff error.

Figure 4.18: Comparisons of the concentration distributions between the analytical solution (solid lines) and the numerical results (dashed lines) at \( t=0, 2, 4, 6 \) and 8s (from left to right).
Figure 4.19: Time history of the total volume of the concentration; the total volume of concentration is normalized by its initial value.
4.3.2 2D Gaussian hump

In this section, the convection-diffusion test of a Gaussian hump is extended into two-dimension. Similar to (4.21), the analytical solution for 2D convection-diffusion of a Gaussian hump is given by Noye and Tan (1989):

\[
C(x, y, t) = \frac{1}{4t+1} \exp \left\{ \frac{(-x-x_0-Ut)^2 + (y-y_0-Vt)^2}{\hat{v}_s (4t+1)} \right\}
\]

(4.23)

where \((x_0, y_0)\) is the centre of the initial Gaussian hump.

The governing equation corresponding to this 2D problem is as follows:

\[
\frac{\partial (CH)}{\partial t} + \frac{\partial (UCH)}{\partial x} + \frac{\partial (VCH)}{\partial y} = \hat{v}_s \left[ \frac{\partial}{\partial x} \frac{\partial (CH)}{\partial x} + \frac{\partial}{\partial y} \frac{\partial (CH)}{\partial y} \right]
\]

(4.24)

Similar to the 1D case, the initial condition can be obtained from above formula by setting \(t = 0\). In the test, a square domain of \(0 \leq x, y \leq 2\) m will be considered and the following parameters are chosen in the computation: \(x_0 = y_0 = 0.5\) m, \(U = V = 0.8\) m/s, \(\hat{v}_s = 0.01\) m\(^3\)/s, \(\Delta x = \Delta y = 0.0125\) m, \(\Delta t = 0.0025\) s.

Figure 4.20 shows the three-dimensional perspective view of the initial hump and the hump at \(t = 1.25\) s, for the numerical results which are very similar to their counterparts of analytical solution. More direct comparisons between numerical and analytical results are made from the concentric circular contour lines shown in Figure 4.21. Remarkably good agreement has been found.

The volume conservation of concentration has been monitored during the whole computation. Figure 4.22 shows the time history of total volume of concentration normalized by its initial value. It can be seen that in the 2D situation, the volume conservation has been maintained perfectly within the range of computer runoff error.
Figure 4.20: Three-dimensional perspective view of the initial hump (left) and the hump at $t=1.25$ s (right), for the numerical results.
Figure 4.21: The contours of (a): initial hump and (b): hump at $t=1.25s$. Dashed lines: numerical results; solid lines: analytical solution.
Figure 4.22: Time history of the total volume of the concentration; the total volume of concentration is normalized by its initial value.
4.3.3 2D point source

In this section, the study is focused on a point source which is convected and diffused horizontally in an infinite region. According to Lardner and Song (1991), the initial condition for a point source $C(x, y, 0) = K\delta(x-x_0)\delta(y-y_0)$ can be discretized as

$$C_y(0) = \begin{cases} \frac{K}{\Delta x_i \Delta y_{j_0}} & \text{if } (i, j) = (i_0, j_0) \\ 0 & \text{otherwise} \end{cases}$$ (4.25)

where $K$ is a scaling factor and $(i_0, j_0)$ is the grid corresponding to the point source location $(x_0, y_0)$. The exact solution for this problem is as follows

$$C(x, y, t) = \frac{K}{4\pi\hat{V}_s t} \exp\left(-\frac{(x-x_0-Ut)^2 + (y-y_0-Vt)^2}{4\hat{V}_s t}\right)$$ (4.26)

The governing equation used to simulate this problem is still (4.24). A square region of $0 \leq x, y \leq 2\times10^4$ m is chosen as the computational domain so that the solution remains zero at the boundaries when the computation is conducted up to $t=36000$s. In addition, the following parameters are used: $x_0 = y_0 = 8\times10^4$ m, $U = V = 0.5$ m/s, $K = 10^{12}$, $\hat{V}_s = 10^4$ m$^2$/s, $\Delta x = \Delta y = 1250$ m, $\Delta t = 25$ s.

Figure 4.23 shows the three-dimensional perspective view for numerical results of the concentration distribution after the initial point source has been convected and diffused for 36000s. Moreover, the contour plots of the concentration distribution in Figure 4.24 show that the agreement between numerical and analytical solutions is almost perfect.

The volume conservation of concentration has also been monitored during the whole computation. Figure 4.25 shows that the reduction of the volume at $t=36000$s is around
0.01%. Therefore, it can be concluded that the volume conservation can be maintained very well even after a very long time of computation.

It should be noted that Lardner and Song (1991) attributed the errors of numerical results to the setting of the initial condition as a point source and pointed out that a considerable improvement in accuracy could be obtained if smooth initial values were used, i.e., use the solution from (4.26) at some time-step as initial condition. Surprisingly, this phenomenon was not observed in our simulation and very good numerical results have been obtained using a discretized initial condition for a point source, i.e., (4.25).

Figure 4.23: Three-dimensional perspective view of the concentration distribution at $t=36000s$, for the numerical results.
Figure 4.24: The contour of the concentration distribution at $t=36000$ s. Dashed line: numerical results; solid line: analytical solution.
Figure 4.25: Time history of the total volume of the concentration; the total volume of concentration is normalized by its initial value.
4.4 1D Morphological Equation

In this section, the propagation of an initial hump in a unidirectional flow is used to verify the WENO scheme of morphological equation. 1D case with only bedload transport will be considered and the governing equation is as follows

\[
(1 - \text{poro}) \frac{\partial z_b}{\partial t} = \frac{\partial q_b}{\partial x}
\]  

(4.27)

Assuming that \( q_b \) is a power function of flow velocity and the flow rate is steady in a channel with a rigid lid, we have

\[
q_b = a U^b, \quad U = P / H, \quad H = -z_b
\]  

(4.28)

where \( a \) and \( b \) are constants.

From the following two equations

\[
\frac{\partial z_b}{\partial t} - \frac{1}{1 - \text{poro}} \frac{\partial q_b}{\partial x} = 0
\]  

(4.29)

\[
\frac{\partial z_b}{\partial t} + C(z_b) \frac{\partial z_b}{\partial x} = 0
\]  

(4.30)

the propagation speed can be obtained for this particular case

\[
C(z_b) = - \frac{1}{1 - \text{poro}} \frac{\partial q_b}{\partial z_b} = \frac{1}{(1 - \text{poro}) z_b} abu^b
\]  

(4.31)

The initial bed elevation is given as a Gaussian hump

\[
z_b(x, 0) = -h_0 + 2 \exp \left( -\beta (x - x_c)^2 \right)
\]  

(4.32)

where \( x_c \) is the center of the Gaussian hump.

In this test, the following quantities are used

\[ a = 0.001 \text{ s}^2/\text{m}, \quad b = 3.0, \quad P = 10 \text{ m}^2/\text{s}, \quad \text{poro} = 0.4 \]
Long et al. (2008) solved (4.30) by the method of characteristics and provided the weak solutions which employ shock-fitting to develop a solution satisfying jump conditions (Whitham, 1974, Chapter 2). This will serve as analytical solution which will be used to verify the numerical results.

Figure 4.26 shows the simulated evolution of the initial Gaussian hump up to 10,000 s. It can be shown that the WENO scheme simulates the propagation of bed elevation stably without generating any numerical dispersion. Moreover, the shock fronts of the bed can be captured very well with very small numerical dissipation by this scheme during the whole simulation, shown clearly in Figure 4.27 which compares the numerical results to the analytical solutions at $t=600$ s, $2000$ s and $6000$ s.

The volume conservation of the WENO scheme has been checked according to the following definition of total volume of sand bed

$$Vol(t) = \int_{-\infty}^{\infty} (z_b(x,t) - z_{bx})dx$$

(4.33)

where $z_{bx}$ is the bed level away from the hump.

Figure 4.28 shows the time history of total volume of sand bed normalized by the initial total volume $Vol(0)$. It can be seen that the volume conservation has been maintained perfectly by WENO within the range of computer runoff error.
Figure 4.26: Numerical simulation of Gaussian hump evolution up to 10,000 s.
Figure 4.27: Comparisons of the bed elevation between the analytical solution (solid line) and the numerical result (circle) at $t=600$ s (left), 2000 s (middle) and 6000 s (right).
Figure 4.28: Time history of the total volume of the sand bed; the total volume of the sand bed is normalized by its initial total volume.
Chapter 5

Sediment Transport in 1D Situations

5.1 Introduction

This chapter describes the application of the depth-averaged hydrodynamic and sediment model to two experimental situations in laboratory flumes – one which records the evolution of a sediment laden flow over a sediment trench (Tests 1, 2 and 3) and another which records the evolution of a sediment dune under flowing water. As a good starting point, one-dimensional modeling can provide clear views on the numerical methods as well as the physical mechanisms. In addition, extensive sensitivity analyses can be conducted to show the influences of the model parameters.

As a benchmark test, the trench experiment has been used by many model developers to validate their numerical models. For example, van Rijn (1980, 1985 and 1986) used the experimental data of the trenches to validate his depth-resolved model developing for predicting the siltation in dredged trenches. To verify their depth-averaged model, Guo and Jin (2002) also adopted those trench experimental data. Therefore, the evolution of the trenches due to the sediment transport is modeled in this study. Available experimental data is used for the comparison with the numerical results.

The profile of the dune can be seen as the opposite counterpart of the trench. To study the dune evolution under the open-channel flow condition, the experiment was conducted in the laboratory flume and the bed profiles at the different time were recorded to form a
complete data set describing the whole evolution process. This process of the dune evolution is simulated using the present model.

To simulate the morphological evolution, the time steps for the hydrodynamic and sediment models are set to be the same in the usual computation scheme. In order to speed up the computations, the time step for the sediment model is set to be 10 times the time step for the hydrodynamic model. What is implicit here is that the hydrodynamics show little or no change during the sediment model computations. This scheme was verified against the results of Test 1 in the trench experiments and adopted for validation against the remaining results for the trench and dune.

5.2 Sediment Transport in a Trench

5.2.1 Experimental setup

A series of experiments were conducted in a flume (30m long, 0.5m wide and 0.7m deep) at the Delft Hydraulics Laboratory (van Rijn, 1980) to investigate the morphological evolution of different initial trench profiles under open channel flow conditions. The initial configurations of the trench profiles are shown in Figure 5.1. In all, three tests with different side slopes 1:3, 1:7 and 1:10 have been performed. The mean flow velocity and the water depth were kept constant as 0.51 m/s and 0.39 m respectively for all three tests. The sand bed consisted of fine particles with $d_{10} = 115\mu m$, $d_{50} = 160\mu m$ and $d_{90} = 200\mu m$ and the settling velocity was 0.013 m/s with a relative error of 25%. To maintain the equilibrium bed conditions (no scour or deposition) upstream of the channel, sand of the same size and composition was fed at a constant rate of 0.04 kg/s/m (relative error of about 10%). The flow velocity and sediment concentration profiles across the
water depth were measured at selected locations (see Figure 5.1) along the channel when the flow and sediment fields reached the quasi-steady state with trivial change of the initial topography. There were 8 measurement locations in Test 1 and 5 measurement locations in both Test 2 and Test 3.

Based on the measured velocities and sediment concentrations upstream of the trench, the suspended load transport rate was estimated to be $0.030 \pm 0.006$ kg/s/m giving a bed load transport rate of about 0.01 kg/s/m. Consequently, the contribution of the suspended load transport to the total load transport was in the range of 60% to 90%. During each test, the flow was maintained across the trench for up to 15 hours to allow for the evolution of the bed profiles which were measured after 7.5hrs and 15hrs.
Figure 5.1: Sketches of the initial trench profiles and locations of measurements for flow velocity and sediment concentration profiles: (a) Test 1 with measurement locations 1 ~ 8; (b) Test 2 with measurement positions 1 ~ 5; (c) Test 3 with measurement locations 1 ~ 5. All dimensions are in meter.
Figure 5.2: Cont’d.
5.2.2 Velocity and concentration fields

In the experiment, the flow velocity profiles and sediment concentration profiles were measured at some positions along the channel (see Figure 5.1) at the state when the flow and sediment fields reached the quasi-steady state and the beds had little changes. In this section, we will calculate the flow conditions and suspended sediment concentrations under the fixed bed condition and compare them with the experimental data.

For all three tests, the computational domain is 12m long with a flat bed of 1m length in front of the upstream slope. The initial trench morphologies are set as shown in Figure 5.1. Flow flux of $20.1989\text{m}^2/\text{s}$ and water depth of 0.39m are specified at the upstream and downstream boundaries respectively. For the bottom friction, $k_s = 3d_{90}$ leads to Manning’s coefficient $n = 0.0122$ according to the relation $n = \frac{H^{1/6}}{18 \cdot \log_{10} \frac{12H}{k_s}}$. For the suspended load transport, the sediment concentration of $8.1 \times 10^{-5}$ is specified at the upstream boundary as input data. This is the depth-averaged value of the measured concentration profile at the place 1m before the upstream slope reach of the trench in the Test 1. To keep the upstream part of channel in equilibrium condition, $a = 0.01\text{m}$ is employed in the computation. The spatial step is chosen as $\Delta x = 0.2\text{m}$ after testing the grid independence and the time step is $\Delta t = 0.06\text{s}$.

Starting from the static flow, the numerical calculations are carried out for 6 minutes to reach the steady solution. Throughout the calculations, the initial trench morphologies are kept unchanged.

Figure 5.2, 5.4 and 5.6 show the computed depth-averaged velocities at the positions where the measurements were made in all three tests. For the comparisons, measured
velocity profiles and their depth-averaged values are also plotted in the figures. It is seen that the numerical results of the velocity capture the main trend of the flow variations along the channel, i.e., the flow deceleration in the trench due to the increase of the water depth (water surface difference <2%) and the acceleration in the downstream channel due to the decrease of the water depth. However, the discrepancies between the depth-averaged values of the measured velocity profiles and the numerical predictions are quite noticeable. The reason is because the model has to conserve the flow flux all along and the velocity is computed as the quotient of the flow rate and the water depth. This is also proved by the good agreement between the numerical results and the depth-averaged velocities calculated based on the flow fluxes.

Similar to the velocities, Figure 5.3, 5.5 and 5.7 show the comparisons between the numerical results of the depth-averaged concentration and the depth-averaged values of the measured concentration profiles. It can be seen from the figures that the main phenomenon of the sediment transport is the deposition of the suspended sediment into the trench due to the slowdown of the flow velocity. Some underestimation of the sediment concentration is observed which means the amount of the sediment deposition calculated by the model is larger than that measured in the experiment. This discrepancy is attributed mainly to the sediment deposition function adopted in the model. In this function, the near-bed concentration $C_a$ is determined from the relationship with the depth-averaged concentration $C$ and the vertical distribution of the concentration is assumed to follow the exponential profile under the equilibrium condition. This assumption may overestimate the near-bed concentration $C_a$ and in turn overestimates the sediment deposition rate when the concentration profile is not under the equilibrium condition.
In the downstream flat channel, the flow speeds up since the water depth becomes shallower. Sediment is entrained from the bed and transported downstream. As seen in the figures, the depth-averaged concentrations predicted by the numerical model clearly show this trend with some underestimation compared to the measurements.
Figure 5.2 (a): Flow velocities at positions 1~8 in Test 1. Circle: measured velocity profiles across water depth; Solid line: depth-averaged values of measured velocity profiles; Dashed line: numerical results of depth-averaged velocities; Dotted line: depth-averaged velocities calculated based on flow fluxes; (b): Measurement positions 1~8 in Test 1.
Figure 5.3 (a): Sediment concentrations at positions 1~8 in Test 1. Circle: measured concentration profiles across water depth; Solid line: depth-averaged values of measured concentration profiles; Dashed line: numerical results of depth-averaged concentrations; (b): Measurement positions 1~8 in Test 1.
Figure 5.4 (a): Flow velocities at positions 1~5 in Test 2. Circle: measured velocity profiles across water depth; Solid line: depth-averaged values of measured velocity profiles; Dashed line: numerical results of depth-averaged velocities; Dotted line: depth-averaged velocities calculated based on flow fluxes; (b): Measurement positions 1~5 in Test 2.
Figure 5.5 (a): Sediment concentrations at positions 1~5 in Test 2. Circle: measured concentration profiles across water depth; Solid line: depth-averaged values of measured concentration profiles; Dashed line: numerical results of depth-averaged concentrations; (b): Measurement positions 1~5 in Test 2.
Figure 5.6 (a): Flow velocities at positions 1~5 in Test 3. Circle: measured velocity profiles across water depth; Solid line: depth-averaged values of measured velocity profiles; Dashed line: numerical results of depth-averaged velocities; Dotted line: depth-averaged velocities calculated based on flow fluxes; (b): Measurement positions 1~5 in Test 3.
Figure 5.7 (a): Sediment concentrations at positions 1~5 in Test 3. Circle: measured concentration profiles across water depth; Solid line: depth-averaged values of measured concentration profiles; Dashed line: numerical results of depth-averaged concentrations; (b): Measurement positions 1~5 in Test 3.
5.2.3 Verification of approximate calculation method

In this section, we will verify the approximate calculation method for sediment transport and morphological evolution which has been proposed in Chapter 3. Test 1 among these three tests is chosen for this verification. The numerical results calculated using the approximate method will be compared with those calculated using the regular method.

The numerical settings are same as those in the last section except that the bed change is being simulated. For better comparison, the bed slope effect on the sediment transport is not included in this verification.

For the regular method, we choose the same time steps, i.e., $\Delta t_{\text{Flow}} = \Delta t_{\text{Sedi}} = 0.06 \text{s}$, in hydrodynamic and sediment computations. Nevertheless, for the approximate method we propose in the study, we can choose $\Delta t_{\text{Sedi}} = 0.6 \text{s}$ which is 10 times of $\Delta t_{\text{Flow}} = 0.06 \text{s}$. Before the sediment computation starts, the flow model has been run for 6 minutes to ensure the flow field reaches the “initial steady state” after which the computations of both flow field and sediment field are carried out alternately up to 15 hours.

The numerical results at $t=1, 3, 5, \ldots, 13$ and 15hr using these two methods are plotted in Figure 5.8 for comparison between each other. It can be seen that the differences between them are very small. A more direct comparison is showed in Figure 5.9 in which the numerical results calculated using the regular method are plotted versus those calculated using the approximate method. The same conclusion can be drawn that the approximate method can produce almost the same results as the regular method. However, only 1/10 of computational time is needed in this case.
Therefore, by using the approximate calculation method, the great computational efficiency can be achieved with almost same accuracy. After the verification, the approximate method will be applied for all computations of sediment transport and morphological evolution in this study.

Figure 5.8: Comparisons of numerical results of bed elevations at $t=1, 3, 5, \ldots, 13$ and 15hr in Test 1 calculated from regular method (dashed line) and from approximate method (dotted line). Initial trench profile (solid line) and water surface (dash-dot line) are also shown.
Figure 5.9: Bed elevations at $t=1, 3, 5, \ldots, 13$ and 15hr in Test 1 calculated from regular method versus from approximate method (dots). Solid line: perfect agreement.
5.2.4 Calculations of morphological evolution

In this section, all three trench tests of the morphological evolution will be simulated by using the present numerical model. The approximate calculation method will be used to improve the computational efficiency. The bed slope effect on the sediment transport will be included in the computations as well.

Similar to the experiment settings, the hydrodynamic and sediment conditions in the simulations are kept identical for all the three tests. The only difference is the side slope of the trench. In addition, the angle of repose is chosen as $\phi = 31^\circ$. For each test, the computation is conducted up to 15 hours.

Figures 5.10~12 show the comparisons of the trench profiles after 7.5 hours and 15 hours among the numerical results from the present model and from a depth-resolving model (van Rijn, 1980) and the experimental measurements for all three tests. It is observed from these figures that the present model captures the main phenomena of the sediment transport happening in the channel with a trench. On the one hand, the sediment coming from the upstream channel deposits into the trench due to the flow deceleration. On the other hand, the sediment bed downstream the trench is eroded due to the flow acceleration. Regarding the numerical simulation of these two processes, the present model can simulate the upstream siltation process quite well. However, the present model systematically underestimates the erosion levels downstream the trench for all three tests. This is probably caused by an underestimation for the suspended sediment concentration in the channel downstream the trench. It can be seen from the last concentration profiles in Figure 5.3, 5.5 and 5.7 that the measured near-bed concentrations have very high values while the depth-averaged concentrations predicted by the present model are relatively low.
The underestimation for the concentration results in the underestimation for the amount of the sediment eroded from the bed and transported downstream. Similar phenomena can be found from the numerical results of the depth-resolving model (van Rijn, 1980), i.e., good prediction for the siltation and underestimation for the erosion. Especially, the present depth-averaged model gives very similar numerical results of the erosion levels as the depth-resolving model.

Another phenomenon seen from Figures 5.10~12 is that the present model gives better prediction for the morphological evolution when the side slope of the trench becomes milder. For the mildest side slope in Test 3, very good agreement is obtained between the numerical predictions and the experimental measurements. This illustrates that the depth-averaged model has better performance in the mild topography than in the topography with big changes.

In summary, the morphological evolution of the trench under the open-channel flow condition can be numerically modeled using the present model. Overall, the numerical predictions show good agreement with the experimental measurements.
Figure 5.10: Bed elevation comparisons after 7.5 and 15 hours between numerical results and experimental data in Test 1. Solid line: initial bed; Circles and triangles: bed measured after 7.5 and 15 hours respectively; Dash-dot and dashed lines: present numerical results after 7.5 and 15 hours respectively; Lines with plus and with cross: van Rijn’s numerical results after 7.5 and 15 hours respectively; Dotted line: numerical result of water surface after 15 hours.
Figure 5.11: Bed elevation comparisons after 7.5 and 15 hours between numerical results and experimental data in Test 2. Solid line: initial bed; Circles and triangles: bed measured after 7.5 and 15 hours respectively; Dash-dot and dashed lines: present numerical results after 7.5 and 15 hours respectively; Lines with plus and with cross: van Rijn’s numerical results after 7.5 and 15 hours respectively; Dotted line: numerical result of water surface after 15 hours.
Figure 5.12: Bed elevation comparisons after 7.5 and 15 hours between numerical results and experimental data in Test 3. Solid line: initial bed; Circles and triangles: bed measured after 7.5 and 15 hours respectively; Dash-dot and dashed lines: present numerical results after 7.5 and 15 hours respectively; Lines with plus and with cross: van Rijn’s numerical results after 7.5 and 15 hours respectively; Dotted line: numerical result of water surface after 15 hours.
5.2.5 Sensitivity analysis

Bed slope effect

In order to study the bed slope effect, the morphological evolutions are simulated again using the present model without including the bed slope effect. The obtained results are compared with previous ones which have included the bed slope effect in the computations.

Figure 5.13~15 show the comparisons of the bed elevations at 7.5hr and 15hr between the numerical results simulated with and without bed slope effect. It is seen that the bed slope effect included in the numerical model has the effect of smoothing bed elevation with relatively high gradient. For the part downstream the trench which has relatively small gradient on the bed elevation, this effect is negligible on the sediment transport calculation.
Figure 5.13: Comparisons of bed elevations in Test 1 after 7.5 and 15 hours between numerical results simulated with and without bed slope effect. Solid line: initial bed profile; Circles and triangles: experimental measurements of bed elevation after 7.5 and 15 hours respectively; Dash-dot line and dashed line: numerical results of bed elevation after 7.5 and 15 hours from present model with bed slope effect; Line with plus and with cross: numerical results of bed elevation after 7.5 and 15 hours respectively from present model without bed slope effect; Dotted line: numerical result of water surface after 15 hours.
Figure 5.14: Comparisons of bed elevations in Test 2 after 7.5 and 15 hours between numerical results simulated with and without bed slope effect. Solid line: initial bed profile; Circles and triangles: experimental measurements of bed elevation after 7.5 and 15 hours respectively; Dash-dot line and dashed line: numerical results of bed elevation after 7.5 and 15 hours from present model with bed slope effect; Line with plus and with cross: numerical results of bed elevation after 7.5 and 15 hours respectively from present model without bed slope effect; Dotted line: numerical result of water surface after 15 hours.
Figure 5.15: Comparisons of bed elevations in Test 3 after 7.5 and 15 hours between numerical results simulated with and without bed slope effect. Solid line: initial bed profile; Circles and triangles: experimental measurements of bed elevation after 7.5 and 15 hours respectively; Dash-dot line and dashed line: numerical results of bed elevation after 7.5 and 15 hours from present model with bed slope effect; Line with plus and with cross: numerical results of bed elevation after 7.5 and 15 hours respectively from present model without bed slope effect; Dotted line: numerical result of water surface after 15 hours.
Angle of repose

In the previous numerical simulations, a normal value of angle of repose for submerged sediment was used, that is $\phi = 31^\circ$. In this section, the sensitivity of the angle of repose will be examined.

As an example, Test 3 among the three tests is used for study. Three different values of angle of repose, i.e., $\phi = 27^\circ$, $31^\circ$ and $35^\circ$, are chosen to be used in the simulations. Other parameters are kept identical with those used in the previous computations.

Figure 5.16 shows the bed elevations after 7.5 hours and after 15 hours calculated with different values of $\phi$. From the comparisons, it can be seen that there is only a very small difference on the simulations of the bed evolutions when different values of angle of repose are used.
Figure 5.16: Comparison of bed elevations after 7.5 and 15 hours in Test 3 predicted using different values of angle of repose. Solid line: initial bed profile; Circles and triangles: experimental measurements of bed elevation after 7.5 and 15 hours respectively; Dash-dot line: bed elevations after 7.5 and 15 hours using $\phi = 27^\circ$; Dashed line: bed elevations after 7.5 and 15 hours using $\phi = 31^\circ$; Line with plus: bed elevations after 7.5 and 15 hours using $\phi = 35^\circ$; Dotted line: numerical result of water surface after 15 hours.
5.3 Sediment Transport over a Dune

5.3.1 Experimental setup

The experiment of sediment transport over a dune was conducted in a rectangular recirculating flume in Hydraulic Engineering Laboratory at National University of Singapore.

The flume used in this study is 15m long, 0.6m wide and 0.6m deep. Water is recirculated through the flume by operating two centrifugal pumps and steady discharge can be observed through a flow meter. Moreover, a flow straightener is constructed in the inlet tank to straighten the inflow in the channel direction. At the downstream end of the flume, a tail gate allows the adjustment of the water level in the flume.

A sand dune was constructed artificially in the flume. The side slope of the dune is 1:5 and its initial profile is shown in Figure 5.17. The sand used to build the dune is well-sorted coarse sand and has the following averaged size after the sieve analysis for three samples (see Figure 5.18): $d_{10} = 411.88\mu m$, $d_{50} = 535.98\mu m$, $d_{90} = 952.84\mu m$.
Figure 5.37: Sketch of initial dune profile. All dimensions are in meter.

Figure 5.18: Particle size distribution curves of three sand samples.
The experiment of the 1D dune evolution was conducted under the condition of open-channel flow. During the experiment, the mean flow velocity and the water depth were kept constant as 0.32 m/s and 0.25 m respectively. Under these conditions, the dune evolved downstream with the sand transported in the mode of bed load according to the observation. However, since the flow rate was not large enough to erode the sand on the upstream flat bed, no sand was needed for adding upstream during the whole experiment. The whole process of the experiment lasted 2 hours and the flow was slowed down for the bed elevation measurement every half an hour. In addition, this experiment has been tested for 3 times in order to ensure the consistency and repeatability.

The PV-07 Electronic bed profile indicator developed by WL | Delft Hydraulics has been used to measure the sand bed elevations. It is sensitive to bed level variations of 0.2mm. To continuously measure the bed elevations, the profile indicator was moved with constant speed by a carriage installed on the top rails of the flume.

5.3.2 Experimental results

Figure 5.19 shows the bed elevations of the dune measured at 0.5, 1, 1.5 and 2 hours in Test 1, 2 and 3. Due to the present of the dune, the water depth decreased and the flow velocity increased on the part of the channel with the dune. Thus the dune was eroded with the sand being transported downstream. On the other hand, the water depth was recovered downstream the dune and the flow gradually slowed down there. The result was that the sand transported from the dune began to settle down. Therefore, the main phenomenon for the sediment transport over a dune under the open-channel condition was that as a result of the flow erosion, the dune became flatter and flatter. The eroded sand was transported and
deposited at the downstream of the dune, making the toe of the dune moving downstream gradually.

It also can be seen from Figure 5.19 that the experimental measurements from Test 1, 2 and 3 corresponding to the same time are very similar. This means that the repeatability of this experiment is good under the controlled flow condition. Therefore, the averaged values of the measurements from these three tests will be presented as the final experimental results, as shown in Figure 5.20.
Figure 5.19: Bed elevations of 1D dune measured at (a): $t=0.5\text{hr}$; (b): $t=1\text{hr}$; (c): $t=1.5\text{hr}$; and (d): $t=2\text{hr}$ in Test 1 (solid line), Test 2 (dashed line) and Test 3 (dash-dot line). Dotted line: initial profile.

Figure 5.20: Averaged bed elevations of Test 1, 2 and 3 at $t=0.5\text{hr}$ (solid line), 1hr (dashed line), 1.5hr (dash-dot line) and 2hr (crosses). Dotted line: initial profile.
5.3.3 Numerical simulation and results

In this section, the numerical study will be carried out for the morphological evolution of the 1D dune. In addition, the measurements from the experiment will be used for the comparison with numerical results.

The computational domain is 6m long with 1m upstream and 2m downstream the dune respectively. The initial morphology of the dune can be shown in the first panel of Fig. 5.21. Flow flux of $20.08 \text{m}^2/\text{s}$ and water depth of 0.25m are specified at the upstream and downstream boundaries respectively. Similar to the calculation of the previous trench case, the Manning’s coefficient in the simulation is chosen as $n = 0.0145 \text{s/m}^{1/3}$ according to $k_s = 3d_{so}$. Since only bed load transport was observed in the experiment, the suspended load will not be calculated in the simulation. In addition, the spatial step is $\Delta x = 0.1 \text{m}$ while the time steps are $\Delta t_{\text{Flow}} = 0.06s$ and $\Delta t_{\text{Sedi}} = 0.6s$ for the flow and sediment computations respectively using the approximate method.

Figure 5.21 shows the comparisons of the bed elevations at $t=0$, 0.5, 1, 1.5 and 2 hours between the numerical results and the experimental data. Overall, good agreement can be found. The present model reasonably simulates both the erosion and the deposition of the sediment happening in the process of the dune evolution. A small discrepancy is found in the prediction of the dune toe.

Figure 5.22 shows the time history of the total volume of the sand dune normalized by its initial value. It is seen that the volume conservation has been maintained quite well during the whole computation with the volume loss being only around 0.4%.
Figure 5.21: Comparisons of bed elevations of the dune at $t=0, 0.5, 1, 1.5$ and $2$ hours between numerical results (solid line) and experimental data (dashed line).
Figure 5.22: Time history of total volume of sand dune; total volume of the sand dune is normalized by its initial total volume.
5.3.4 Sensitivity analysis

Bed slope effect

To study the bed slope effect, the dune evolution is simulated again without including the bed slope effect. Figure 5.23 shows the comparisons of bed elevations at \( t=0, 0.5, 1, 1.5 \) and 2 hours between the numerical results simulated with and without bed slope effect. It can be seen that these two sets of the results are generally similar except at the front of the dune where some numerical wiggles occur. After including the bed slope effect in the simulation, the wiggles are smoothened and the dune profile appears more realistic.

Figure 5.23: Comparisons of bed elevations at \( t=0, 0.5, 1, 1.5 \) and 2 hours between numerical results simulated with (solid line) and without (dash-dot line) bed slope effect. Dashed line: measured bed elevations.
Angle of repose

Next, the sensitivity of the angle of repose in simulating the dune evolution will be examined. Similarly, three different values of angle of repose will be chosen in the computations, i.e., $\phi = 27^\circ, 31^\circ$ and $35^\circ$. Figure 5.24 shows the comparisons of the bed elevations at $t=0$, 0.5, 1, 1.5 and 2 hours predicted using different values of angle of repose. It is seen that the value of the angle of repose has negligible effect on the calculations of the dune evolution.

![Figure 5.24: Comparisons of bed elevations at $t=0$, 0.5, 1, 1.5 and 2 hours predicted using different values of angle of repose. Solid line: numerical results using $\phi = 27^\circ$; Dash-dot line: numerical results using $\phi = 31^\circ$; Dotted line: numerical results using $\phi = 35^\circ$; Dashed line: measured bed elevations.](image)
Chapter 6

Turbulent Flows and Morphological Evolution in Channels with Abrupt Cross-Section Change

6.1 Introduction

The turbulent flows and the morphological evolution in the channels with changed cross-section are the important objects of the study. On the one hand, the flow will be accelerated or decelerated due to the change of the cross-section and thus the turbulent characteristics will change their patterns significantly. On the other hand, due to the local change of the hydrodynamic condition, the deposition or erosion of the sediment will happen which makes the morphology evolve. The study of the sediment transport has been reviewed in Chapter 1 and only the experimental study of the turbulent flow will be briefed in the following paragraphs.

The Acoustic Doppler Velocimeter (ADV) is a helpful instrument in the pointwise measurement of the flow field in both laboratories and fields. Using acoustic Doppler technology, the ADV can measure instantaneous three-dimensional flow velocities. The acoustic sensor of an ADV consists of one acoustic transmitter and three acoustic receivers. The transmitter emits an acoustic signal that is reflected back by the sound-scattering particles in the water and received by the receivers. The processing module attached to the probe performs the signal processing to compute the Doppler shift from which the flow velocities are obtained.
One can acquire the velocities within the flow field with a ADV. From these measurements, the mean velocities and the turbulence characteristics can be calculated. For example, Liu et al. (2004) studied the flow field characteristics of hydraulic jumps in the laboratory flume. A SonTek MicroADV was used to measure the instantaneous velocity field. From the measured data, the mean velocity distributions as well as the turbulence kinetic energy were obtained for the whole flow field. With the help of the Taylor’s frozen turbulence hypothesis, the velocity power spectra were transferred from the frequency domain to the wavenumber domain and thus the dissipation rate of the turbulence kinetic energy could be estimated according to the Kolmogorov theory of local isotropic turbulence. Therefore, the turbulence characteristics of hydraulic jumps could be analyzed. Similar turbulence spectra were calculated to determine the inertial subrange with -5/3 slope by Sukhodolov et al. (2004) and Lien and D’Asaro (2006) to study the turbulent flows in the groin field and in the waterways respectively.

Collecting and analyzing the comprehensive experimental data of the turbulent flow is important for both understanding the flow characteristics and validating the numerical models. For example, Web et al. (2001) conducted the experiments on flow at a 90-degree open-channel junction. The velocity measurements were taken using an ADV and both the mean velocities and the turbulent kinetic energy were obtained to describe the flow field. These experimental results provided a relatively full data set for the flow at a 90-degree junction. Note that no further analyses for the experimental data have been done in their study to obtain the other turbulent characteristics.

In this chapter, the turbulent flow, the sediment transport and the morphological evolution in channels with changed cross-section are studied experimentally and numerically. Firstly, the turbulent flow field in a channel with an abrupt expansion in the
cross-section is measured in the laboratory flume. The experimental data is analyzed to obtain the mean flow field as well as the turbulent parameters. This flow field is simulated numerically using the hydrodynamic model and the numerical results are compared with the depth-averaged experimental measurements. After the study of the hydrodynamic condition, the experiment of the sediment transport in the channel with an abrupt expansion is conducted under the same flow conditions. The morphological evolution of the sand bed is recorded. Moreover, the numerical simulation for this scenario is carried out using the present model. The predicted bed evolution is compared with the measured one. In the second experimental setup, the channel has an abrupt contraction in the cross-section. Like the first setup, both the experimental and the numerical studies are conducted for the turbulent flow and the morphological evolution in this channel. The comparisons between the experimental measurements and the numerical results are made. Lastly, comparisons of the numerical results obtained with the present model for both the hydrodynamics and the bed evolution are made against the reported experimental measurements and the numerical results from the three-dimensional model of Duc and Rodi (2008) who carried out their study a channel consisting of a contraction and an expansion.
6.2 Turbulent Flow in a Channel with an Abrupt Expansion

6.2.1 Laboratory experiments

The experiment to study the turbulent flow in a channel with an abrupt expansion in the cross-section was conducted in Hydraulic Engineering Laboratory at National University of Singapore. The flume used in this experiment is same as the one used for the dune experiment. Sanded smooth plywood with epoxy coating was used for the transition as shown in Figure 6.1. The upstream and downstream widths of the channel are 30cm and 60cm respectively.

During the experiment, the flow rate and the water depth were controlled as 24 l/s and 15 cm respectively. The three-dimensional flow velocities were measured with a SonTek/YSI 16-MHz MicroADV (Acoustic Doppler Velocimeter) with the sampling rate of 50Hz. During the experiments, high concentration Kaolin powder was added to the water to increase the signal to noise ratio (SNR) and the correlation factor. The outputs of SNR and correlation factor from the instrument were monitored in real time and they were controlled around 30dB and 90% respectively which are much greater than their minimum requirements (15dB for SNR and 70% for correlation). The locations where the velocity profiles were taken are shown in Figure 6.1 and these are at points of intersection between the longitudinal coordinates \(x=-10, 25, 50, 75, 100, 125, 150, 175, 200, 225, 250, 275\) and 300cm and the lateral coordinates \(y=5, 10, 15, 20, 25, 30, 35, 40, 45, 50\) and 55cm. The location of \(x=0\) is at the position of the abrupt expansion as shown in Figure 6.1. In the vertical direction, the measurements were made at elevations 8, 7, 6, 5, 4, 3, 2 and 1cm from the flume bottom. At each single location, 5000 samples were collected. The
upstream approach velocities in the narrow channel were measured at $x=-10\text{cm}$, away from the influence of the transition.

Figure 6.1: Plan view sketch of channel with abrupt expansion; $x=0$ is expansion position. Dots represent horizontal locations of velocity measurement.
6.2.2 Analysis of experimental data

After the samples of the velocities have been acquired, further analysis will be carried out to obtain the mean flow and the turbulent field. The mean velocities are defined as follows:

\[
\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i, \quad \bar{v} = \frac{1}{n} \sum_{i=1}^{n} v_i, \quad \bar{w} = \frac{1}{n} \sum_{i=1}^{n} w_i
\]

(6.1)

where \( u, v \) and \( w \) are velocity components in \( x-, y- \) and \( z- \) directions, \( n \) is number of samples. The root-mean-square values of the velocity fluctuations or turbulent velocities are defined as the sample standard deviation:

\[
rms u' = \sqrt{\langle u'^2 \rangle} = \sqrt{\frac{\sum_{i=1}^{n} u_i^2 - (\sum_{i=1}^{n} u_i)^2}{n-1}}
\]

\[
rms v' = \sqrt{\langle v'^2 \rangle} = \sqrt{\frac{\sum_{i=1}^{n} v_i^2 - (\sum_{i=1}^{n} v_i)^2}{n-1}}
\]

(6.2)

\[
rms w' = \sqrt{\langle w'^2 \rangle} = \sqrt{\frac{\sum_{i=1}^{n} w_i^2 - (\sum_{i=1}^{n} w_i)^2}{n-1}}
\]

where \( u', v' \) and \( w' \) are velocity fluctuations in \( x-, y- \) and \( z- \) directions.

The turbulent kinetic energy has the following definition:

\[
k = \frac{1}{2} \left( \langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle \right)
\]

(6.3)

Therefore, it can be calculated from the r.m.s. turbulent velocities as follows

\[
k = \frac{1}{2} \left[ (rms \ u')^2 + (rms \ v')^2 + (rms \ w')^2 \right]
\]

(6.4)

Next, the method to estimate the dissipation rate of the turbulent kinetic energy, i.e., \( \varepsilon \), will be discussed in detail. The main idea is to obtain the inertial subrange in Kolmogorov
spectrum through the spectral analysis. Taylor’s frozen turbulence hypothesis assumes that the turbulent field is frozen in time and transported horizontally past the observer (Panofsky and Dutton, 1984). It is a conventional method to convert temporal data at a selected location into spatial record through $x = \bar{u}t$, where $\bar{u}$ is the mean velocity. Therefore, frequency scales $f$ become wave number scales $\tilde{k}$ through $\tilde{k} = 2\pi f / \bar{u}$, while the spectra remain unchanged in their shapes as well as their magnitudes (Kaimal and Finnigan, 1994).

This concept has important consequences for the statistical functions of turbulent flow. The means and variances measured in time must be equal to those measured in space. The autocovariances $R_\tau(x)$ in space and $R_\tau(t)$ in time can be related by

$$ R_\tau(x) = R_\tau(\bar{u}t) = R_\tau(t), \quad R_\tau(t) = R_\tau(x/\bar{u}) = R_\tau(x) $$

and the spectra are related according to

$$ \tilde{k}E_\tau(\tilde{k}) = fE_\tau(f) $$

where $E_\tau(\tilde{k})$ and $E_\tau(f)$ represent the total kinetic energy in wave number space and frequency space respectively. These relationships remain valid so long as Taylor’s hypothesis be valid and it is necessary that the turbulence be statistically stationary in time and homogeneous in the $x$-direction (Panofsky and Dutton, 1984).

For the Taylor’s hypothesis to be valid, the mean velocity should be at least 10 times larger than the root mean square value of the fluctuating velocities (Soloviev and Lukas, 2003). Instead of using the mean velocity, Lien and D’Asaro (2006) used the instantaneous velocity to achieve the conversion from time to space. This method is analogous to Taylor’s conversion, but avoids the assumption of a large ratio of mean
velocity to fluctuations. The spatial series is obtained based on the distance computed from

\[ x = x_j = \sum_{i=1}^{j-1} |u_i| \Delta t, \]

where the instantaneous velocity at \( x = x_j \) is \( u_j \). As the spectral analysis requires equal spatial steps \( \Delta x \), the wave number spectrum can be computed from spatial series with \( u(x) \) computed from the interpolations for the spatial positions as needed: the spatial distance should be interpolated linearly to a fixed interval which can be the mean interval of \( x \), while velocities at these positions should also be obtained by the linear interpolation of original velocity data. In this study, this method was adopted to determine the wave number spectra.

**Energy cascade**

In a fully turbulent flow at high Reynolds number, the turbulence can be considered to be composed of eddies of different sizes. The kinetic energy enters the turbulence through the production mechanism at the largest scales of motion (i.e., the smallest wavenumber). These eddies display strongly the configuration of the boundaries and the processes that generate the energy and thus are generally anisotropic. However, they are unstable and break up, transferring their energy, which is neither produced nor dissipated, to smaller and smaller scale eddies by the vortex stretching. During this process, often referred to as the energy cascade, the directional information of the large scales are lost, and they become progressively isotropic (Kolmogorov’s hypothesis of local isotropy). The energy will finally be dissipated into internal heat energy by the viscosity at the smallest scales. The process described above can be identified by three spectral regions, namely, the energy-containing range, the inertial range and the dissipation range (Pope, 2000).
Kolmogorov inertial subrange

According to Kolmogorov’s second similarity hypothesis, the flow motions in the inertial subrange are determined by the inertial effects, independent of viscosity. The energy in this subrange is neither produced nor dissipated, thus $E(\tilde{k})$ has a universal form entirely controlled by $\epsilon$, independent of molecular viscosity, $\nu_m$. Therefore, from dimensional arguments, the energy spectrum within the inertial subrange takes the following form

$$E(\tilde{k}) = C \epsilon^{2/3} \tilde{k}^{-5/3}$$  \hspace{1cm} (6.7)

This is the well known Kolmogorov $-5/3$ power law for the inertial subrange and $C$ is a universal Kolmogorov constant.

In the isotropic turbulence, the one-dimensional spectra can be determined by the energy spectrum $E(\tilde{k})$ and has the following forms for the longitudinal and transverse velocity spectra respectively

$$E_u(\tilde{k}) = C_1 \epsilon^{2/3} \tilde{k}^{-5/3}$$  \hspace{1cm} (6.8)

$$E_v(\tilde{k}) = E_w(\tilde{k}) = C_2 \epsilon^{2/3} \tilde{k}^{-5/3}$$  \hspace{1cm} (6.9)

where $C_1 = 0.53$ and $C_2 = 0.71$ as suggested by Sreenivasan (1995) after reviewing a large number of experimental data. Here, $\tilde{k}$ is the wave number in the mean flow, $E_u(\tilde{k})$, $E_v(\tilde{k})$ and $E_w(\tilde{k})$ are the one-dimensional wave number spectra of the velocities $u$, $v$ and $w$ respectively. The total kinetic energy spectrum can be obtained as follows (Lien and D’Asaro, 2006)
\[ E(\tilde{k}) = \frac{1}{2} \left[ E_u(\tilde{k}) + E_v(\tilde{k}) + E_w(\tilde{k}) \right] = C' \varepsilon^{2/3} \tilde{k}^{-5/3} \] (6.10)

where \( C' = \frac{1}{2} (C_1 + 2C_2) = 0.98 \). The dissipation rate of the turbulent kinetic energy can then be estimated from (6.10) to give

\[ \varepsilon = 1.04 \left( \frac{1}{3} \left[ E(\tilde{k}) \tilde{k}^{5/3} \right]^{3/2} \right) \] (6.11)

where \( \langle \rangle \) represents taking the ensemble average in the inertial subrange.

Figure 6.2 (a), (b) and (c) show the measured time series for velocity components \( u \), \( v \) and \( w \) respectively at the location (-10cm, 45cm, 5cm) shown in Figure 6.1. The sampling rate was 50Hz and each time series covered 5000 points. Figure 6.3 shows the spectrum for the total kinetic energy computed from (6.10). The velocity spectra (for \( u \), \( v \) and \( w \)) are computed using a multitaper spectral analysis with two tapers to avoid the aliasing problems (Lien and D’Asaro, 2006). Through calculating the spatial distance

\[ x = x_j = \sum_{i=1}^{j-1} |u_i| \Delta t, \quad j = 1, 2, ..., n \] where \( \Delta t = 0.02s \), the temporal domain is converted to the spatial domain. Correspondingly, the spatial interval is \( dx = x_a / n \). The new series of \( u_{\text{new}} \), \( v_{\text{new}} \), and \( w_{\text{new}} \) on the spatial domain should be interpolated linearly from original time series \( u \), \( v \) and \( w \) according to the fixed spatial interval \( dx \). After that, the power spectra of the velocity series, i.e., \( E_u(\tilde{k}) \), \( E_v(\tilde{k}) \) and \( E_w(\tilde{k}) \), can be calculated on the spatial domain using a multitaper spectral analysis with two tapers and the total kinetic energy spectrum \( E(\tilde{k}) \) can be obtained from (6.10) and depicted in Figure 6.3. From this figure, the inertial subrange can be identified according to the feature of -5/3 slope, i.e., [30, 300], for this measurement location. After finding the inertial subrange, the value of
$1.04 \left[ E \left( \hat{k} \right) \hat{k}^{5/3} \right]^{3/2}$ can be calculated for each single point within this subrange. Therefore, the ensemble average of $1.04 \left[ E \left( \hat{k} \right) \hat{k}^{5/3} \right]^{3/2}$ for all points within the inertial subrange will give the estimated $\varepsilon$ according to (6.11). For this measurement location, $\varepsilon$ is estimated to be $1.5 \times 10^{-4} \text{ m}^2/\text{s}^3$ through this method described above.

**Depth-averaged quantities**

After the mean velocities, $\bar{u}$ and $\bar{v}$, the TKE, $\hat{k}$ and its dissipation rate, $\varepsilon$, are obtained at each measurement location, the depth-averaged quantities, i.e., the depth-averaged velocities in $x$- and $y$-directions, $U$ and $V$, the depth-averaged TKE, $\hat{k}$, and the depth-averaged dissipation rate, $\hat{\varepsilon}$, can be obtained further by depth-averaging their vertical profiles. After obtaining both $\hat{k}$ and $\hat{\varepsilon}$, the depth-averaged turbulent viscosity can be calculated according to its definition

$$\hat{\nu}_t = C_\mu \frac{\hat{k}^2}{\hat{\varepsilon}}$$  \hspace{1cm} (2.33)

It is noted that, due to the operation constraint of the ADV measurement, the velocity measurement is taken from 1cm to 8cm from the flume bottom while the water depth is 15cm in the experiment. Therefore, the depth-averaged value obtained from this depth range will be taken as the depth-averaged value at the corresponding horizontal measurement location. This may be attributed to the experimental errors. For example, when the mean velocity follows the logarithm profile across the water depth, this depth-averaging method will lead to around 5% error for the depth-averaged mean velocity after the calculation.
Figure 6.2: Time series of velocity components (a): $u$, (b): $v$ and (c): $w$ at location (-10cm, 45cm, 5cm).
Figure 6.3: Wave number spectra of total kinetic energy at location (-10cm, 45cm, 5cm) and the inertial subrange.
6.2.3 Numerical simulation

The numerical simulation of the turbulent flow is performed in a domain of 9 m × 0.6 m (see Figure 6.4). The domain covers the narrow channel 2 m upstream from the expansion position. The computational domain is discretized by a 180×30 uniform grid system with Δx = 0.05 m and Δy = 0.02 m. The time step Δt = 0.008 s is used. Flow flux of 0.08 m²/s and water depth of 0.15m are specified at the upstream and downstream boundaries, respectively. Since the flume is made of glass, the Manning coefficient n is chosen as 0.01 s/m¹/³. Six minutes is all that is required for the numerical calculation to reach the steady solution from a cold start.

![Figure 6.4: Computational domain and grid arrangement in channel with abrupt expansion; lines are plotted every two grid nodes for easier visibility.](image-url)
6.2.4 Results and discussions

Figure 6.5 ~ Figure 6.9 show the mean flow and the turbulent field in the channel with an abrupt expansion. The x- and y- component velocities, $U$ and $V$, are normalized by $U_0$, the turbulent kinetic energy, $\hat{k}$, by $U_0^2$, the dissipation rate, $\hat{\varepsilon}$, by $U_0^3/H_0$, and the turbulent viscosity $\hat{\nu}_t$, by $U_0H_0$, where $U_0 = 0.53$ m/s is the mean upstream velocity and $H_0 = 0.15$ m is the water depth. Figure 6.5 shows the spatial distribution of the longitudinal velocity. This is also the major feature of the flow pattern in the expanded channel since the longitudinal velocities have predominantly large magnitudes compared to the transverse velocities. It is seen that a uniform-distributed flow coming from upstream narrow channel enters a wider channel. Due to the abrupt expansion of the cross-section, the recirculating flow appears in the region just downstream of the expansion. This recirculating region affects the downstream water up to some distance (around $x=200$ cm) after which the flow will start recovery to the mainstream. Encouragingly, the numerical model simulates the whole flow pattern very successfully, including in the featured recirculating region.
Figure 6.5: Depth-averaged velocity $U$ (Crosses: experimental data; Solid lines: numerical results); $U$ is normalized by $U_0 = 0.53$ m/s; $x=0$ is the expansion position.
Figure 6.6: Depth-averaged velocity $V$ (Crosses and pluses: experimental data; Solid lines: numerical results); $V$ is normalized by $U_0 = 0.53$ m/s; $x=0$ is the expansion position.
Figure 6.7: Depth-averaged TKE $\hat{k}$ (Crosses: experimental data; Solid lines: numerical results); $\hat{k}$ is normalized by $U_0^2$; $x=0$ is the expansion position.
Figure 6.8: Depth-averaged dissipation rate $\hat{\varepsilon}$ (Crosses: experimental data; Solid lines: numerical results); $\hat{\varepsilon}$ is normalized by $\frac{U_0^3}{H}$; $x=0$ is the expansion position.
Figure 6.9: Depth-averaged turbulent viscosity \( \hat{\nu}_t \) (Crosses: experimental data; Solid lines: numerical results); \( \hat{\nu}_t \) is normalized by \( U_0 H_0 \); \( x = 0 \) is the expansion position.
Figure 6.6 shows the distribution of the transverse velocities, $V$, whose magnitudes are only about 5% of $U$. As shown in the figure, the numerical model can predict reasonably well the main trend of $V$ in the channel. However, quite large discrepancies are found within the recirculating region, i.e., $0 < x < 175$ cm and $0 < y < 30$ cm, between the numerical results and the measurements. From the experimental measurements, it is seen that the flow pattern of the recirculating flow with the feature of the positive $V$ will extend up to $x=175$cm. However, the numerical computation exhibits this feature over a shorter length, i.e., only up to $x=50$cm. More encouragingly, the depth-averaged model is also capable of giving reasonably good predictions for the turbulent quantities, $\hat{k}$, $\hat{\varepsilon}$ and $\hat{\nu}_* \hat{\nu}$, as shown in Figure 6.7 ~ Figure 6.9. It is found that strong turbulence is generated in the region with high velocity gradients and the highest turbulence may be generated in a small region downstream the obstacle corner. On the other hand, the turbulence is low in the main flow with uniform velocity distribution. The numerical computations can capture these turbulent characteristics fairly accurately.

Overall, from the comparisons between the numerical results and experimental data, it can be seen that the SWE model with the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence closure can provide reasonably accurate numerical simulation for both the mean flow and the turbulent field in a channel with an abrupt expansion.
6.3 Morphological Evolution in a Channel with an Abrupt Expansion

6.3.1 Laboratory experiments

On the basis of the turbulence measurements, the sediment transport and the morphological evolution were also studied experimentally in the channel with an abrupt expansion. A 15cm thick layer of uniform sand bed was laid in the flume and the initial bed was flat. The sand used in this study was coarse sand with $d_{10} = 411.88 \mu m$, $d_{50} = 535.98 \mu m$ and $d_{90} = 952.84 \mu m$, which is the same as the one used in the dune experiment. The hydrodynamic conditions in the flow experiment, i.e., the flow rate of 24 l/s and the water depth of 15cm, were maintained in the sand experiment and were kept steady during the bed evolution. Under these flow conditions, only bed load transport was observed during the experiment. Since the bed erosion and the sediment transport occurred in the upstream narrow channel, it was necessary to feed sand upstream to maintain the bed condition. After trial and error in the experiment, the speed of sand feeding was determined and it was equivalent to the rate of 0.0146 kg/s/m. The whole process of the bed evolution was set for 8 hours and bed elevations were recorded using the PV-07 Electronic bed profile indicator every hour. Before taking the measurements, the flow was gradually slowed down, until no sand moved and bed stopped evolving, before bed profiling was conducted. After taking the bed measurements, the flow in the flume was gradually resumed to the constant conditions and the bed evolution continued. These were done carefully to ensure that the bed features did not change when the flow was reduced or increased gradually. In order to assess repeatability, the entire experiment was repeated 3 times.
6.3.2 Experimental results

The experimental results from the three tests have been compared against each another. The patterns of the bed evolution observed from three tests are similar and thus the repeatability of this experiment has been found under the controlled flow condition as evident in Figure 6.10 (a) ~ (h) showing the bed elevations measured from Test 1, 2 and 3 at $t=1, 2, 3, \ldots, 7$ and 8 hours respectively. Therefore, the averaged values of the measurements from these three tests are used for the final presentation of the experimental results.
Figure 6.10: Measurements of bed profiles along \( y = 5, 10, \ldots, 50 \) and 55 cm at (a) \( t = 1 \) hr; (b) \( t = 2 \) hr; (c) \( t = 3 \) hr; (d) \( t = 4 \) hr; (e) \( t = 5 \) hr; (f) \( t = 6 \) hr; (g) \( t = 7 \) hr; and (h) \( t = 8 \) hr in Test 1, 2 and 3; \( x = 0 \) is at the abrupt expansion.
Figure 6.10: Cont’d.
(c) $t=3$ hr

Figure 6.10: Cont’d.
(d) $t=4$ hr

Figure 6.10: Cont’d.
(e) $t=5 \text{ hr}$

Figure 6.10 (e): Cont’d.
(f) $t=6\text{ hr}$

Figure 6.10: Cont’d.
(g) $t = 7\ \text{hr}$

Figure 6.10: Cont’d.
(h) $t=8$ hr

Figure 6.10: Cont’d.
Figure 6.11 (a) ~ (h) show the contours of bed elevation at \( t=1, 2, \ldots, 7 \) and 8 hours. Note that for the experimental results, it is the averaged value of the measurements from three repeated tests. From this figure, the morphological evolution of the sand bed in the channel with an abrupt expansion can be shown in detail. Since the bed of the upstream channel is kept from erosion with the sand feeding, this amount of the sand added outside is carried by the flow to the expanded channel. Due to the abrupt expansion of the channel’s cross-section, the flow velocity is slowed down after entering the wide channel thus the sand starts to deposit on the bed in the vicinity of the expansion position (see Figure 6.11 (a)). With time, more and more sand coming from upstream deposits and a sand hump appears in the second hour of the experiment (see Figure 6.11 (b)). Subsequently, the process of the bed evolution is mainly the evolution of the hump (see Figure 6.11 (c) ~ (h)). As the sand keeps entering the wide channel and then depositing, the hump becomes bigger and higher. At the same time, the hump is transported under the flow condition. Note that the front of the hump continues to progress downstream. Meanwhile, the hump is also spread transversely due to the transverse component of the flow. As a result, the hump comes to the right side wall of the flume with the sand building up behind the expansion. When the experiment is continued up to 8 hours, a quite large sand hump lies across the flume with its front aligned obliquely and the maximum elevation of the hump is about 10cm (see Figure 6.11 (h)).

In summary, the main feature of the morphological evolution in the channel with an abrupt expansion is the continuous deposition of the sand in the expanded region of the channel. As a result, a sand hump forms and evolves with increasing volume as it is being transported both longitudinally and transversely.
Figure 6.11: Contour of bed elevations at: (a) $t=1$ hr; (b) $t=2$ hr; (c) $t=3$ hr; (d) $t=4$ hr; (e) $t=5$ hr; (f) $t=6$ hr; (g) $t=7$ hr; and (h) $t=8$ hr. Upper panel: averaged values of measurements from three tests; lower panel: numerical results.
(c) $t=3\text{hr}$

Experimental:

Numerical:

(d) $t=4\text{hr}$

Experimental:

Numerical:

Figure 6.11: Cont’d.
(e) $t=5\text{hr}$

Experimental:

Numerical:

(f) $t=6\text{hr}$

Experimental:

Numerical:

Figure 6.11: Cont’d.
(g) $t=7\text{hr}$

Experimental:

Numerical:

(h) $t=8\text{hr}$

Experimental:

Numerical:

Figure 6.11: Cont’d.
6.3.3 Numerical simulation

Based on the numerical simulation of the turbulent flow, the sediment transport and morphological evolution are also modeled numerically using the present model. The computational domain and numerical discretization are kept the same as those used in the flow computation. Moreover, the same flow conditions are specified. Based on the grain diameter, the Manning coefficient $n$ is computed as $n = 0.0145 \text{ s/m}^{1/3}$ using

$$n = \frac{H^{1/6}}{18 \cdot \log_{10} \frac{12H}{k_s}}$$

with $k_s = 3d_{90}$. According to the observations in the experiment, only bed load transport is calculated in the simulation. The approximate method to calculate the sediment transport is adopted with the time steps for the flow and sediment computations of $\Delta t_{\text{Flow}} = 0.008 \text{ s}$ and $\Delta t_{\text{Sed}} = 0.08 \text{ s}$ respectively. In addition, the effect of bed slope is included in the calculations.

6.3.4 Results and discussions

Figure 6.11 shows the numerical results of the sand bed contour at $t=1, 2, \ldots, 7$ and 8 hours with the corresponding experimental measurements as the comparisons. Note that all the bed heights are referenced from an initially flat bed, i.e., $z=0$. For the numerical simulation, the model is able to capture the main changes happening in the channel. However, it is not able to predict the bed changes in the area immediately behind the obstacle as evident from Figure 6.11 which shows the evolution of the bed feature in the sheltered region immediately downstream of the expansion. This may be due to the lower transverse velocity component predicted by the model. The magnitude of the transverse component determines the principal direction of the sediment transport. The numerical
model gives reasonably good predictions for the bed evolutions near the main flow, i.e., $y>30\text{cm}$. However, the numerical results underestimate the heights of the bed features in the area behind the obstacle, i.e., $y<30\text{cm}$.

Overall, the depth-averaged model captures the main features of the sediment transport and gives reasonable predictions for the morphological evolution in the channel with an abrupt expansion.

### 6.4 Turbulent Flow in a Channel with an Abrupt Contraction

#### 6.4.1 Laboratory experiments

As a counterpart of the experiment in a channel with an abrupt expansion in cross-section, the experiment in a channel with an abrupt contraction in cross-section was also conducted. The same flume in Hydraulic Engineering Laboratory at National University of Singapore was used and the upstream and downstream widths are 60cm and 30cm respectively (see Figure 6.12). Similar to the last experiment, both the turbulent flow and the sand bed evolution have been studied. In the experimental study of the turbulent flow, the flow rate and the water depth were controlled as 18 l/s and 15 cm respectively. The flow velocities were measured with the SonTek/YSI 16-MHz MicroADV at a sampling rate of 50Hz. Figure 6.12 shows the horizontal locations of velocity measurements which were along $x=-100$, -50, -40, -30, -20, -10, 0, 10, 20, 30, 40 and 50cm and along $y=5$, 10, 15, 20, 25, 30, 35, 40, 45, 50 and 55cm. The vertical elevations where the measurements were made are at the 8, 7, 6, 5, 4, 3, 2 and 1cm to the flume bottom. 5000 samples were collected at each location. In addition, the velocities measured at the cross-section of $x=-100\text{cm}$ are taken as the upstream velocities. After the velocity data have been acquired,
they are analyzed using the method discussed in §6.2.2 to obtain the depth-averaged velocities, $U$ and $V$, the depth-averaged TKE, $\hat{k}$, the depth-averaged dissipation rate, $\hat{\varepsilon}$, and the depth-averaged turbulent viscosity, $\hat{\nu}_t$.

Figure 6.12: Plan view sketch of channel with abrupt contraction; $x=0$ is contraction position. Dots represent locations of velocity measurement.
6.4.2 Numerical simulation

The turbulent flow in the channel with an abrupt contraction is also investigated numerically. The computational domain is 6 m × 0.6 m and covers the wide channel 4 m upstream from the contraction position (see Figure 6.13). The computational domain is discretized by a 120 × 30 uniform grid system with $\Delta x = 0.05$ m and $\Delta y = 0.02$ m. The time step $\Delta t = 0.008$ s is used. Flow flux of 0.03 m$^3$/s and water depth of 0.15 m are specified at the upstream and downstream boundaries respectively. The flume has glass walls and bottom and the plywood sections for the contraction are rendered smooth and coated with epoxy. The Manning coefficient $n$ is chosen as 0.01 s/m$^{1/3}$. The numerical calculations took 6 minutes to reach the steady solution from an initially cold start.

Figure 6.13: Computational domain and grid arrangement in channel with abrupt contraction; lines are plotted every two grid nodes for easier visibility.
6.4.3 Results and discussions

Figures 6.14 ~ 6.18 show the hydrodynamic conditions, including both the mean flow and the turbulent field, in the channel with an abrupt contraction. The velocities, \( U \) and \( V \) are normalized by \( U_0 \), the turbulent kinetic energy, \( \hat{k} \) by \( U_0^2 \), and the dissipation rate, \( \hat{\epsilon} \) by \( U_0^3 / H_0 \), and the turbulent viscosity, \( \hat{\nu}_t \) by \( U_0 H_0 \), where \( U_0 = 0.2 \text{ m/s} \) is the upstream velocity and \( H_0 = 0.15 \text{ m} \) is the water depth. Figure 6.14 shows the longitudinal flow field and it can be seen that a uniform-distributed flow is coming from the upstream channel. Due to the obstruction of half of the channel, one side of the upstream flow is gradually retarded while the other side is gradually accelerated. When the flow enters the narrow channel, it is speeded up significantly. However, the velocity distributions in the narrow channel are strongly influenced by the presence of the obstacle and show strong non-uniformity. The upstream velocities near the obstacle are much smaller than those away from the obstacle. In the far downstream of the contraction, the trend of recovering to uniform flow can be observed. The numerical results predict these flow patterns well. Even the non-uniformity and the recovery of the flow in the narrow channel can be captured quite well except some underestimation on the retardance of the flow close to the obstacle.

From Figure 6.15, it can be seen that the transverse flow is much smaller than the longitudinal flow in the contracted channel (note the scale difference between Figure 6.14 and Figure 6.15). For the approaching upstream flow, it is almost completely longitudinal without transverse velocity component. However, due to the change of the channel cross-section, the flow gradually changes its direction and the transverse flow becomes increasingly greater when the flow approaches the contraction. After the flow enters the
contracted channel, the maximum transverse flow appears around the contraction position and the flow will run downstream with some transverse orientation in the area near the obstacle. The numerical results agree with the measurements very well in the upstream wide channel. However, big disagreement is observed in the narrow channel immediately downstream of the contraction position. In the numerical simulation, the flow recovers quickly to the longitudinal flow and misses its transverse feature.

Figures 6.16 ~ 6.18 show the turbulent field in the contracted channel. It can be found that the flow has generally low turbulent intensity before it comes into the downstream narrow channel. After the flow enters the contracted channel, however, all of the turbulent characteristics show very high magnitudes in the vicinity of the contraction obstacle. This means the flow has intensive turbulence in this region due to the abrupt contraction. Generally, the numerical results predict the distribution of the turbulent field reasonably well. However, some underestimations from the numerical model are also found in this high turbulence region. This may be due to the weakness of the Boussinesq eddy viscosity assumption as well as the two-equation models which is not valid in the flows with strong curvature for the Reynolds stress tensor proportional to the rate of strain. From the above comparisons, it is found that the overall agreement of the flow field between the experimental measurements and the numerical computations is reasonably good although some underestimations are found in the region adjacent to the obstacle in the downstream narrow channel.
Figure 6.14: Depth-averaged velocity $U$ (Crosses: experimental data; Solid lines: numerical results); $U$ is normalized by $U_0 = 0.2$ m/s; $x=0$ is at the abrupt contraction.
Figure 6.15: Depth-averaged velocity $V$ (Crosses: experimental data; Solid lines: numerical results); $V$ is normalized by $U_0 = 0.2$ m/s; $x=0$ is at the abrupt contraction.
Figure 6.16: Depth-averaged TKE $\hat{k}$ (Crosses: experimental data; Solid lines: numerical results); $\hat{k}$ is normalized by $U_0^2$; $x=0$ is at the abrupt contraction.
Figure 6.17: Depth-averaged dissipation rate $\dot{\varepsilon}$ (Crosses: experimental data; Solid lines: numerical results); $\dot{\varepsilon}$ is normalized by $U_0^3 / H_0$; $x=0$ is at the abrupt contraction.
Figure 6.18: Depth-averaged turbulent viscosity $\hat{\nu}_t$ (Crosses: experimental data; Solid lines: numerical results); $\hat{\nu}_t$ is normalized by $U_0H_0$; $x=0$ is at the abrupt contraction.
6.5 Morphological Evolution in a Channel with an Abrupt Contraction

6.5.1 Laboratory experiments

Following the turbulent flow experiment, the sediment transport and the morphological evolution were studied experimentally in succession in the channel with an abrupt contraction. A 20 cm thick layer of sand bed was laid in the flume and the initial bed was flat throughout. The sand used in this experiment is same as those used in the previous experiments. The same flow conditions in the hydrodynamic experiment were adopted, i.e., the flow rate of 18 l/s and the water depth of 15 cm, and the sediment was observed to be transported in the mode of bed load. In addition, no erosion or deposition happened on the upstream bed thus no sand was needed to feed and protect upstream bed.

The bed evolution over a 2-hour period was recorded using the PV-07 Electronic bed profile indicator every half an hour. For the bed profiling during the experiment, similar measures were taken to slow down and resume the flow before and after the measurement. Three tests were conducted to ensure the repeatability.

6.5.2 Experimental results

The experimental results from the three tests showed strong repeatability. For example, Figure 6.19 shows the comparisons of the bed measurements from Test 1, 2 and 3 at \( t=2 \) hours. Therefore, the averaged values of the measurements from these three tests were used as the final presentation of the experimental results.
Figure 6.19: Measurements of bed heights along $y=5, 10, \ldots, 50$ and 55 cm at $t=2$ hours in Test 1, 2 and 3; $x=0$ is contraction position.
Figure 6.20 (a) ~ (d) show the contours of bed elevations at \( t = 0.5, 1, 1.5 \) and 2 hours. Note that the averaged values of the measurements from the three tests were used to represent the experimental results. From this figure, the morphological evolution of the sand bed in the contracted channel can be seen clearly. The upstream channel was relatively wide and the flow was not strong enough to carry any sand from the bed and no erosion was observed in the wide channel during the whole process of the experiment. Under the strong contraction effect on the flow, erosion of the sand bed was observed around the contraction. Sand was entrained in this region and a cone-shape scour hole formed with the centre at the contraction corner (see Figure 6.20 (a)). The sand eroding from the scour hole began piling up temporarily immediately downstream the scour hole. Therefore, within the first hour of the experiment, in addition to the scour hole around the contraction corner, some partial siltation was also observed right downstream of the scour hole in the narrow channel (see Figure 6.20 (a) and (b)).

Under the continuous flow condition, the sand kept on being carried away from the bed around the contraction position and this resulted in a deep the scour hole. The piled sand just downstream of the scour hole was carried further downstream and the bed in the narrow channel flattened gradually further downstream. When the experiment was conducted up to 2 hours, the maximum scour hole was around 15cm and almost no siltation can be seen in the narrow channel (see Figure 6.20 (d)). Therefore, the main feature of the morphological evolution in the channel with an abrupt contraction in cross-section was the erosion of the sand bed with a scour hole around the contraction position.
Figure 6.20: Contour of bed elevations at: (a) \( t=0.5 \text{hr} \); (b) \( t=1 \text{hr} \); (c) \( t=1.5 \text{hr} \); and (d) \( t=2 \text{hr} \). Upper panel: averaged values of measurements from three tests; lower panel: numerical results with correction for bed shear stress.
(b) $t=1\text{hr}$

Experimental:

Numerical (with correction for bed shear stress):

Figure 6.20: Cont’d.
(c) $t=1.5\text{hr}$

Experimental:

Numerical (with correction for bed shear stress):

Figure 6.20: Cont’d.
(d) $t=2\text{hr}$

Experimental:

Numerical (with correction for bed shear stress):

Figure 6.20: Cont’d.
6.5.3 Numerical simulation

The sediment transport and morphological evolution are modeled numerically with the same computational domain and numerical discretization used in the flow computation. Other numerical settings follow those used in the numerical computation of the sediment transport and bed change in the expanded channel. For the convenience of the reference, some important parameters will be repeated here. The Manning coefficient is $n = 0.0145 \text{s/m}^{1/3}$ according to the sand diameter. The time steps for the flow and sediment computations are $\Delta t_{\text{Flow}} = 0.008 \text{s}$ and $\Delta t_{\text{Sedi}} = 0.08 \text{s}$ respectively.

6.5.4 Results and discussions

Figure 6.21 (a) ~ (d) show the detailed comparisons of bed elevations at $t=0.5, 1, 1.5$ and 2 hours among experimental measurements, numerical results with and without correction for bed shear stress. Note that all the bed changes are referenced from initially flat bed, i.e., $z=0$. Under the flow condition, a scour hole forms near the contraction position and becomes deeper and deeper as the time goes (see the bed changes around $x=0$ in Figure 6.21). The sediment eroded from the hole is transported downstream. For the numerical simulation without including the correction for bed shear stress in the modeling, the model significantly underestimates the bed shear stress in the vicinity of the contraction obstacle and thus is not able to simulate the scour hole. The sand bed eroded in the experiment has been remarkably underestimated when the bed shear stress has been corrected in the region with the strong secondary flow effect (see the dashed lines along $y=20, 25$ and $30\text{cm}$ in Figure 6.21). This means in the region where the flow turns its direction strongly, the bed shear stress has very large magnitude due to the strong...
secondary flow pattern. Under this flow condition, the original Manning’s formula which is derived with the assumption of uniform flow will not be able to accurately predict the local bed shear stress and need to be corrected accordingly. Using the correction method presented in Chapter 2, the prediction of the bed shear stress can be greatly improved in this region. Therefore, the scour hole around the contraction position has been predicted reasonably well (see the solid lines along \(y=20, 25\) and \(30\)cm in Figure 6.21). Moreover, in the main channel where the flow has very strong longitudinal velocity and very little secondary flow effect, the difference between the numerical results with and without the bed shear stress correction is negligible, as shown in Figure 6.21. On the other hand, some underestimation for the deepest scouring depth (around \(x=0\)) is found in the simulation. This is because the model underestimates the flow velocity and thus underestimates the bed shear stress in the region very close to the wall.

The contours of the bed elevations at \(t=0.5, 1, 1.5\) and 2 hours predicted by the model are shown in Figure 6.20 and compared against the experimental measurements. It can also be seen that the numerical model can reasonably predict the scour hole evolution in the contracted channel. In a word, with the help of the correction for the bed shear stress, the present depth-averaged model gives reasonably good predictions for the evolution of the scour hole in the channel with abrupt contraction. Without this correction, the model will underestimate the bed shear stress in the vicinity of the contraction position and therefore significantly underestimate the scour hole.
Figure 6.21: Comparisons of bed elevations among experimental measurements, numerical results with and without correction for bed shear stress at: (a) \( t=0.5 \text{hr} \); (b) \( t=1 \text{hr} \); (c) \( t=1.5 \text{hr} \); and (d) \( t=2 \text{hr} \). \( x=0 \) is contraction position.
(b) $t=1$ hr

Figure 6.21: Cont’d.
(c) $t=1.5\text{hr}$

Figure 6.21: Cont’d.
(d) $t=2\text{hr}$

Figure 6.21: Cont’d.
6.6 Morphological Evolution in a Channel Consisting of a Contraction and an Expansion

6.6.1 Laboratory experiments

As reported by Duc and Rodi (2008), a series of experiments were conducted in the laboratory of the Federal Waterways Engineering and Research Institute Karlsruhe (BAW) to investigate the bed deformation in a channel consisting of a contraction and an expansion. The flume used in the experiments was 16.5m long and 1.0m wide. The cross section of the channel was contracted from a width of 1.0m to a width 0.5m and then was expanded back to 1.0m. Instead of abrupt contraction or abrupt expansion, the contraction and expansion widths were achieved linearly as shown in Figure 6.22 for the shape of the channel. The side walls of the channel were vertical. When viewed from the upstream, the right side wall, i.e., the straight one, was made of smooth glass while the left side wall was rough concrete with estimated roughness $k_s = 3.0$ mm. A layer of sand with a thickness of approximately 20cm was constructed on the flume bed and was initially flat. The sand used in the experiments had the median diameter of $d_{50} = 5.5$ mm and a standard deviation of 1.47. The experiments were conducted for three different hydraulic conditions shown in Table 6.1. In Run 1, neither sediment transport nor bed deformation occurred according to the laboratory observations. However, laboratory observations showed the bed deformation in both Run 2 and Run 3. No sand was supplied from upstream since no sediment transport occurred in the channel upstream the contracted part under these hydraulic conditions. At the end of each run of the experiments, measurements were made for the water surface along the centerline of the contracted channel, the velocity profiles at some locations (see Figure 6.22) and the bed elevations.
Figure 6.22: Plan view sketch of channel consisting of a contraction and an expansion. Dots represent horizontal locations of velocity measurement (Duc and Rodi, 2008). All dimensions are in meter.

Table 6.1: Hydraulic conditions employed in the experiments (Duc and Rodi, 2008)

<table>
<thead>
<tr>
<th>Run</th>
<th>$Q$ (l/s)</th>
<th>$H$ (m)</th>
<th>$U$ (m/s)</th>
<th>Bed deformation</th>
<th>Duration (min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80</td>
<td>0.268</td>
<td>0.2985</td>
<td>No</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>130</td>
<td>0.3</td>
<td>0.4333</td>
<td>Yes</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>150</td>
<td>0.312</td>
<td>0.4808</td>
<td>Yes</td>
<td>125</td>
</tr>
</tbody>
</table>
6.6.2 Numerical simulation

The numerical study is carried out for the morphological evolution in the channel consisting of a contraction and an expansion by using the present model. The simulation is conducted in a domain of $16.5 \, \text{m} \times 1.0 \, \text{m}$. This domain is discretized by a $330 \times 50$ uniform grid system with $\Delta x = 0.05 \, \text{m}$ and $\Delta y = 0.02 \, \text{m}$. The Manning coefficient is chosen as $n = 0.0195 \, \text{s/m}^{1/3}$ according to the estimated bottom roughness $k_s = 3d_{so}$. Due to the availability of the experimental data, Run 1 and Run 3 are chosen for the simulation in this study. Since only flow field is calculated for Run 1, the time step $\Delta t = 0.005 \, \text{s}$ is used. For Run 3, the time steps for the flow and sediment computations are $\Delta t_{\text{Flow}} = 0.004 \, \text{s}$ and $\Delta t_{\text{Sedi}} = 0.04 \, \text{s}$ respectively, when using the approximate calculation method.

The boundary conditions needed at the upstream and downstream boundaries are the flow flux and the water depth specified respectively. The flow flux and water depth used as the boundary conditions in Run 1 are $0.08 \, \text{m}^2/\text{s}$ and $0.268 \, \text{m}$ respectively, while in Run 3, they are $0.15 \, \text{m}^2/\text{s}$ and $0.312 \, \text{m}$ respectively. For the side walls, different roughness conditions should be considered when using the logarithmic law-of-the-wall. For the side wall which was made of smooth glass, hydraulically smooth regime is used while for the side wall which was made of rough concrete, hydraulically rough regime with $k_s = 3.0 \, \text{mm}$ is used.
6.6.3 Results and discussions

Run 1:

Figure 6.23 shows the resultant velocity field in the channel consisting of a contraction and an expansion. It is noted that Figure 6.23 (a) is the depth-averaged velocity field calculated using the present model while Figure 6.23 (b) and (c) are the velocity fields at the free surface calculated using FAST3D (Duc and Rodi, 2008) and measured in the experiment (Duc and Rodi, 2008) respectively. It should be mentioned that FAST3D is a fully three-dimensional numerical model developed at the Institute for Hydromechanics, University of Karlsruhe, to calculate the flow and the sediment transport. In this model, the Reynolds-averaged Navier-Stokes equations with the $k-\varepsilon$ turbulence closure are solved for the hydrodynamics. The bed load transport is calculated using the concept of the nonequilibrium adaptation length. And the bed deformation is obtained from an overall mass balance equation for sediment transport.

From the comparisons in Figure 6.23, it is found that the present model reproduces the flow pattern in this channel very well, agreeing with the ones calculated using 3D model and measured in the flume. The flow intensity increases significantly after entering the contracted channel. This effect will maintain even the flow has already left the contracted part and entered the downstream wide channel. The flow recovers to weak intensity only in the further downstream channel. As shown in Figure 6.23, the length of this intensive flow obtained using FAST3D is about 1m longer than the one measured in the flume as well as the one calculated using the present model.

Figure 6.24 shows the depth-averaged resultant velocity profiles at cross sections of $x=2.5, 5.1, 7.5, 11.0, 13.0$ and $16.0$ m calculated using the present model. The comparison
among them shows the good agreement. Since the contraction of the channel cross section is gradual rather than abrupt in the previous case, there is smaller retardance effect from the left side wall on the flow than that happening in the abruptly-contracted channel. In addition, in the part of channel expanding from contracted part to downstream wide channel, a recirculating region can be found. Since Duc and Rodi (2008) also provided the vertical profiles of the resultant velocity at the intersections of these cross sections with the centerline of the contracted channel (see Figure 6.22) obtained from the flume measurements and using FAST3D, their depth-averaged values can be computed for the comparisons with the numerical results calculated using the present model, as shown in Figure 6.24. Very good agreement can be found from the point of view of the depth-averaged velocities.

Figure 6.25 shows the comparisons of water surface along the centerline of the contracted channel, i.e., $y=0.25\text{m}$, among the experimental measurements (Duc and Rodi, 2008), the numerical results using the present model and using FAST3D (Duc and Rodi, 2008). From the comparisons, good agreement is found again in terms of the water surface. A depression of the water surface has been found in the contracted part of the channel and the present model can predict it very well.
Figure 6.23: Resultant velocity field in Run 1: (a) Depth-averaged velocity field calculated using the present model; (b) Velocity field at free surface calculated using FAST3D (Duc and Rodi, 2008); (c) Measured velocity field at free surface (Duc and Rodi, 2008). Velocities are in m/s.
Figure 6.24: Depth-averaged resultant velocities. Solid lines: numerical results calculated using present model; Crosses: numerical results calculated using FAST3D (Duc and Rodi, 2008); Circles: experimental measurements (Duc and Rodi, 2008). Velocities are in m/s.

Figure 6.25: Comparisons of water surface along centerline of contracted channel (y=0.25m) among experimental measurements (circles; Duc and Rodi, 2008), numerical results using present model (solid line) and numerical results using FAST3D (dashed line; Duc and Rodi, 2008).
Run 3:

The bed evolution has lasted for 125 minutes in Run 3 and Figure 6.26 shows the contour of the bed elevations at the end of the experiment. Note that Figure 6.26 (a) is the numerical results of the present model; (b) is the numerical results obtained using FAST3D (Duc and Rodi, 2008); and (c) is the experimental measurements (Duc and Rodi, 2008). It can be seen in Figure 6.26 that the scour occurs in the contracted channel at the end of the experiment. There is a long scour ditch occurring along the whole contracted channel and the predicted scour pattern is qualitatively similar to that predicted using FAST3D and that observed in the experiment. After detailed comparisons, however, some differences can be found among these three results. The scour ditch appeared in Figure 6.26 (a) is along the smooth side wall with the maximum scour depth in the middle of the ditch. In Figure 6.26 (b), the scour ditch extends along the centerline of the contracted channel and the maximum scour hole occurs at the beginning of the contracted channel. Nevertheless, according to the experimental observation, as shown in Figure 6.26 (c), the scour ditch is along the left side wall and the maximum scour hole is in the vicinity of the wall at the beginning of the contracted part. In Figure 6.26, it can also be seen that the sand eroded from the contracted channel deposits in the downstream wide channel due to the slowdown of the flow thus a dune with the crescent shape occurs. Although there is some underestimation for the width of the dune using the present, the present numerical results agree reasonably with the results from 3D model and the experimental measurements.

More detailed comparisons of bed elevations among the experimental measurements (Duc and Rodi, 2008), the present numerical results and the numerical results using FAST3D are shown in Figure 6.27. It is seen that the present model can predict quite well
the scour and deposition due to the presence of the contracted channel although some overestimation for the scour depth and some underestimation for the dune height are found along the centerline of the contracted channel. Figure 6.28 shows the water surface along the centerline of the contracted channel, i.e., $y=0.25m$, at the end of the experiment. Similar to the situation on the flat bed, an apparent depression of the water surface is found in the contracted channel. The present model can predict this pattern very well regardless of the scatter experimental data.

Remarkably, the present two-dimensional model has significantly better computational efficiency than the three-dimensional models. Dealing with $330 \times 50 = 16,500$ meshes in this case, for example, it takes only about 3 hours for 125 minutes simulation on a PC with CPU of 2.4GHz. However, the time required for 150 minutes simulation using FAST3D, as reported by Duc and Rodi (2008), is about 17 hours on a PC with CPU of 2.8GHz for the coarse grid ($124 \times 24 \times 12 = 35,712$ grids) and 221 hours on a PC with CPU of 3.2GHz for the fine grid ($247 \times 47 \times 23 = 267,007$ grids).
Figure 6.26: Contour of bed elevations at the end of Run 3 (at $t=125$ min): (a) Numerical results using present model; (b) Numerical results using FAST3D (Duc and Rodi, 2008); (c) Experimental measurements (Duc and Rodi, 2008). Bed elevations are in meter.
Figure 6.27: Comparisons of bed elevations along centerline of contracted channel (y=0.25m) at the end of Run 3 (at t=125 min) among experimental measurements (circles; Duc and Rodi, 2008), numerical results using present model (solid line) and using FAST3D (dashed line; Duc and Rodi, 2008).
Figure 6.28: Comparisons of water surface along centerline of contracted channel \((y=0.25\text{m})\) at the end of Run 3 \((t=125\text{ min})\) among experimental measurements (circles; Duc and Rodi, 2008), numerical results using present model (solid line) and using FAST3D (dashed line; Duc and Rodi, 2008).
6.7 Summaries

In this chapter, the turbulent flow field as well as the sediment transport and morphological evolution in the channels with the changed cross-sections are studied. Both experimental studies and numerical simulations are carried out. The major findings are summarized as follows.

**Turbulent flow in a channel with an abrupt expansion**

- The main flow in this channel is longitudinal. Due to the presence of the abrupt expansion in cross-section, the recirculating flow forms behind the expansion. The depth-averaged model can simulate this flow field excellently.
- The transverse component of the flow has much smaller magnitude, only 5% of the longitudinal component. It is difficult for the model to accurately predict the transverse flow in the recirculating region although it can give good predictions outside this region.
- In terms of the distributions of the turbulent characteristics, the turbulence intensity is much stronger within the recirculating region than the rest regions. The peak turbulence is found right downstream the obstacle corner and is generated from the high velocity gradients in this area. The numerical results calculated with depth-averaged turbulent closure have the encouraging agreement with the experimental measurements.

**Sediment transport and morphological evolution in a channel with an abrupt expansion**

- The sediment coming upstream deposits on the bed after entering the wide channel and a sand hump forms across the flume obliquely. The hump is built up when
• The hump is transported by the flow in both longitudinal and transverse directions with the maximum elevation within the recirculating region. The numerical results predict the evolution trend of the hump quite well. However, the underestimation for the bed elevations within the recirculating region has always been found during the whole evolution process.

**Turbulent flow in a channel with an abrupt contraction**

• Due to the presence of the abrupt contraction in cross-section, the longitudinal flow will be retarded by the obstruction on the one side and accelerated on the other side. The present model gives accurate prediction for this flow pattern with the exception of some underestimations on the retardance effect adjacent to the wall in the contracted channel.

• Compared to the longitudinal flow, the transverse flow has much smaller magnitude and can be simulated accurately by the present model.

• The turbulence intensity is generally low in most of the flow field with extraordinarily strong turbulence in the regions adjacent to the contraction wall in the narrow channel. The numerical simulation provides generally good prediction for the turbulence distribution although the high turbulence at those positions is still underestimated.
Sediment transport and morphological evolution in a channel with an abrupt contraction

- Due to the abrupt contraction of the flume cross-section, the sand bed around the contraction position is eroded by the flow. A cone-shaped scour hole forms and becomes deeper and deeper under the flow scouring. With the correction for the bed shear stress, the present model can improve the prediction for the bed shear stress when the flow shows the secondary flow effect. Therefore, the bed erosion around the contraction position can be well simulated using the present model.

Morphological evolution in a channel consisting of a contraction and an expansion

- The water surface depression in the contracted channel and velocity field predicted using the present model agree well with both the experimental measurements and the numerical results from the three-dimensional model.

- The sand bed deformations including the scour ditch occurring along the contracted channel and the dune depositing in the expanded channel downstream the contraction can be reasonably predicted using the present model. The numerical results show reasonable agreement with the 3D numerical results and the laboratory measurements.
Chapter 7

Conclusions and Future Work

In this chapter, the works that have been done in this study are summarized and the conclusions based on the study are given. Recommendations for the possible future work are also provided.

7.1 Conclusions

In this study, two objectives have been accomplished. One is that a two-dimensional depth-averaged numerical model to simulate sediment transport and morphological evolution has been developed. The other is the turbulent flows and sand bed evolutions under the open channel flow conditions have been experimentally studied and the valuable database has been set up.

The numerical model developed in this study consists of three modules. The hydrodynamic module is based on the shallow-water equations and the depth-averaged $\hat{k} - \hat{\varepsilon}$ turbulence closure is adopted to model the turbulence. In the sediment transport module, the suspended load transport is described by solving the convection-diffusion equation including the entrainment and deposition of the sediment from the bed while the bed load transport is defined from an empirical equation. The morphological module that is solved using fifth-order accurate WENO (Weighted Essentially Non-Oscillatory) scheme is used to calculate the bed evolution. In order to improve the prediction for the bed shear stress, the bed shear stress obtained from traditional Manning’s formula is
corrected according to the secondary flow effect. To simulate the sediment transport on the sloping bed more realistically, the effect of the bed slope is incorporated into the model. Both the critical shear stress for the sediment incipient motion and the sediment transport direction are corrected according to the local bed slope. In addition, an approximate method is proposed for the gradually varied beds to improve the computational efficiency. After the numerical implementation of the model, the numerical testing has been conducted very carefully for the hydrodynamic module, the convection-diffusion equation for the suspended load transport and the morphological evolution equation. Compared to the analytical solutions and the experimental measurements, satisfactory results have been obtained using the present model.

The model was first used to study the sediment transport and morphological evolution in the one-dimensional situations. As the first case, the morphological evolutions of the trenches under the open channel flow conditions were studied. Three tests with the only difference on the side slope were simulated. Both suspended load and bed load transport occurred in the experiments. The numerical results matched well with the experimental measurements in terms of the bed profiles after long experimental time. Secondly, the evolution of a sand dune under the open channel flow conditions was studied as a counterpart of the trench. The experimental study was carried out in the laboratory flume and only bed load transport occurred in the experiment. The bed profile of the dune at the different evolution time has been recorded. Using the present model, the evolution of the sand dune was also simulated with good agreement between the numerical results and the measured bed profiles.

The bed slope effect has also been investigated in the one-dimensional simulations. The numerical results in both situations of the trenches and dune have shown that the bed
slope effect will smooth the bed with relatively high gradient and make the simulation more realistic. In addition, the approximate calculation method proposed for the gradually varied beds has been validated in the trench case. The numerical results have shown that this approximate method will significantly improve the computational efficiency and at the same time keep the results with almost same accuracy. Therefore, it is a very promising method for simulating the gradually varied beds.

After the studies of the one-dimensional situations, the studies of the two-dimensional situations covering the turbulent flows, the sediment transport and the morphological evolution in the channels with changed cross-section were carried out. Firstly, the flow field in a channel with an abrupt expansion has been studied in the laboratory flume. Three-dimensional velocity components were measured using MicroADV throughout the whole flow field. Based on the collected velocity data, the three-dimensional mean velocity components and the turbulent kinematic energy were calculated and the dissipation rate of TKE was estimated using the spectrum analysis to find out the Kolmogorov inertial subrange. Furthermore, the turbulent viscosity was calculated according to its definition. After obtaining all these mean and turbulent flow quantities for the whole three-dimensional space in the flow field, their depth-averaged values were calculated based on the data within the measurement depth. From the measurements, it was found that the main flow is in longitudinal direction with much smaller transverse component. Due to the presence of the obstacle corner, a recirculating region forms behind the expansion position and the turbulence has very strong intensity within this region. In addition to the experimental study, the numerical simulation was conducted for the flow filed in the expanded channel using the present model. The numerical results have shown
to be very encouraging when compared with the experimental data in terms of both mean flow and turbulent flow fields.

Secondly, the bed load transport and morphological evolution in the same expanded channel were investigated experimentally under the same flow conditions. The main phenomenon observed in the expanded channel was the sand deposition due to the slowdown of the flow and a sand hump forming across the flume obliquely. The hump evolved in both longitudinal and transverse directions and the whole procedure which lasted for eight hours has been recorded by profiling the bed elevations hourly. In addition to the laboratory study, the present model was also applied for modeling the morphological evolution in the sudden-expanded channel. The numerical results gave the reasonably good prediction for the evolution trend although the deposited hump height was underestimated.

Thirdly, the flow field in a channel with an abrupt contraction has been studied in the laboratory flume. Similarly to the one in the channel with an abrupt expansion, detailed measurements have been carried out in the sudden-contracted channel for the three-dimensional velocities from which the mean velocity and the turbulence quantities as well as their depth-averaged values were obtained. The longitudinal component of the flow was found to be retarded by the obstruction on the one side and accelerated on the other side while the transverse one has very small magnitude. The turbulence field has extraordinarily strong intensity in the regions adjacent to the contraction wall in the narrow channel. The numerical results from the present model agree well with the measurements in terms of both mean and turbulent flow fields except some underestimations on those high turbulences.
Fourthly, the morphological evolution in the contracted channel was investigated in the laboratory flume. Under the flow condition, the sand bed was scoured due to the sudden contraction of the flume cross-section and a cone-shaped hole formed around the contraction position. In the modeling, the evolution of the scour hole was simulated accurately with the help of the bed shear stress correction.

Lastly, the hydrodynamic conditions and the morphological evolution in a channel consisting of a contraction and an expansion were studied numerically using the present model. The available experimental data and the numerical results from 3D model were used for comparisons. The present model gives successful predictions for the water surface depression in the contracted channel and the velocity field. In addition, reasonable agreements have been shown for simulation of the sand bed deformations including both a scour ditch and a dune.

7.2 Recommendations for Future Work

7.2.1 Cohesive sediment transport

In the present model, only cohesionless sediment transport has been considered. In the natural environment, fine-grained and cohesive suspended sediment plays a very important role in the water quality and the growth of ocean creatures. In order to increase the model’s applicability, cohesive sediment transport can be included.

For the sediment with grain-size less than 63 \( \mu m \), a single particle has very low settling velocity and can be suspended in the water for a long time. However, the cohesive properties of the fine sediments can make them flocculate and form large aggregates or flocs which have much larger settling velocity than a single particle. There are many
factors affecting the settling velocity of flocs, including sediment particle size, water salinity, turbulence intensity and sediment concentration. On the other hand, after the sediment settles down onto the bed, the consolidation of the bed will occur associated with different time scales. With accurate description of the settling velocity, the processes of the flocculation and aggregation, deposition, consolidation and re-suspension are necessary to be represented in the numerical simulation of the cohesive sediment transport.

### 7.2.2 Bed evolution in channel bends

Flow field and morphological evolution in the straight channels with changed cross-section have been studied in Chapter 6. The situation will become much more complex in channel bends as the water flows along the curve. Flow shows strong spiral characteristic due to the superposition of the secondary flow on the forward movement of the water. Therefore, the sediment transport is determined to be much more complex than in the straight channels. Due to the similarity to the alluvial natural rivers, the morphological evolution in the channel bends is of importance to be studied.

Quite abundant experiments with different layout of bend have been carried out to study the bed evolution in channel bends, e.g., Zeng (1982) and Yen (1970). These provide valuable data base for the validation of the numerical model.

### 7.2.3 Bed evolution in dam-break problems

In Chapter 4, some numerical testing for the flows in the ideal dam-break and the laboratory partial dam-break has been conducted using the present model. When the channel bed is movable, large amount of the sediment will be eroded and then transported
within a short time and the morphological change will become very significant. Due to the strong interaction between flow and sediment during the dam-break, the present model which is the uncoupled one with simplified governing equations will not be able to make satisfactory prediction for the problem. Therefore, the model coupling flow motion and sediment transport is necessary for the accurate description of the physical phenomenon. In addition, due to the constructions of the dams, this kind of model will have extensive application in predicting the environmental impact of the dam-break and should be developed in the future study.
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