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Transcendental Numbers

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Transcendental Numbers

Abstract
The numbers e and π are transcendental numbers, meaning each of them are not the root of any polynomial with rational coefficients. We prove that e and π are transcendental numbers. The original proofs use the Fundamental Theorem of Symmetric Polynomials and Stirling's Formula, which we develop and prove. Since π is not algebraic, neither is $\sqrt{\pi}$, which answers the ancient question of whether one can square a circle. The proof that π is transcendental is a beautiful example of how higher level mathematics can be used to answer ancient questions.

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Senior Thesis

Transcendental Numbers

by

Jacob Juillerat

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The report of the investigation undertaken as a
Senior Thesis, to carry two courses of credit in
the Department of Mathematics.

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Michael T. Orr
Krebs Provost and Dean of the Faculty

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Abstract

The numbers e and π are transcendental numbers, meaning each of them are not the root of any polynomial with rational coefficients. We prove that e and π are transcendental numbers. The original proofs use the Fundamental Theorem of Symmetric Polynomials and Stirling's Formula, which we develop and prove. Since π is not algebraic, neither is $\sqrt{\pi}$, which answers the ancient question of whether one can square a circle. The proof that π is transcendental is a beautiful example of how higher level mathematics can be used to answer ancient questions.

To the Math and Physics department for fostering my curiosity;

To Professor Treviño for his guidance and patience;

To those who offered their support;

Thank you.

1 Introduction

A number is **algebraic** over a field F, generally $\mathbb{Q}$, if it is the root of some polynomial with coefficients in F. We define a **transcendental** number to be a number that is not algebraic, i.e. the root of no polynomial in $\mathbb{Q}$.

The existence of transcendental numbers was proven before their was an example of a transcendental number. In 1844, Joseph Louiville proved the existence of transcendental numbers. In 1851, Louiville gave the first example of a transcendental number [7]. He proved that $\alpha = \sum_{k=0}^{\infty} 10^{-k!}$ is transcendental. From this number we get the infinitude of the transcendental numbers. It can be shown that if for each 1 in the decimal expansion of α we flip a coin and for heads we keep the 1, and for tails we replace it with a zero, the resulting number is also transcendental assuming we keep infinitely many ones. Since there are an infinite number of 1's in the decimal expansion of α, we get an infinite number of transcendental numbers.

Charles Hermite was the first to prove the transcendence of a naturally occurring number, e, in 1873. This inspired Ferdinand Lindemann to prove the transcendence of π in 1882 using a similar method. His proof can be generalized to show e^{α} is transcendental when α is algebraic and nonzero. There are still many unsolved problems when it comes to transcendental numbers. It is unknown if most combinations of e and π are transcendental, such as $e\pi, e + \pi, \pi^e$, etc [7].

In this paper, we will walk through the necessary steps to proving e and π are transcendental. We will start by showing that e is irrational in section 2. In section 3 we will take a look at Stirling's Formula, which will be used in later proofs. Section 4 starts with Lemma 1, the basis of our contradictions for later proofs. We will then prove that π is irrational in the same manner as the proofs for the transcendence of e and π. We will prove that e is transcendental in section 5. In section 6 we will finally prove that π is transcendental. To do this, we will start with some definitions, move to the Fundamental Theorem of Symmetric Polynomials and a relevant Corollary, then use all of it to prove that π is transcendental. We will then discuss a few applications in section 7.

2 Irrationality of e

The proof for the irrationality of e is quite simple compared to the proofs that follow. It requires only an understanding of Taylor series expansions and some simple algebra. This proof follows the observations given by Fourier in 1815, which are reproduced in [1].

Theorem. *The value* e *is irrational.*

Proof: By way of contradiction, assume that $e = \dfrac{n}{m}$ with $n, m \in \mathbb{Z}$ and $m \neq 0$. Consider taking the Taylor series expansion for e:

$$e = \frac{n}{m} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ \ldots \ = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

We first show that e is not an integer. Taking the first two terms of the Taylor expansion, we get $1+\dfrac{1}{1!} < e$, which is $2 < e$. Also, we notice:

$$\begin{aligned} 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ \ldots \ &= 2 + \frac{1}{2!} + \frac{1}{3!} + \ \ldots \\ &= 2 + \frac{1}{2}\left[1 + \frac{1}{3} + \frac{1}{3 \cdot 4} + \ \ldots \right] \\ &< 2 + \frac{1}{2}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \ \ldots \right] \\ &= 2 + \frac{1}{2}\left[\frac{1}{1 - \frac{1}{3}}\right] = 2 + \frac{3}{4} \ < \ 3. \end{aligned}$$

From this, we get the inequality $2 < e < 3$, showing that e is not an integer; therefore, $m \neq 1$. If we multiply the Taylor series expansion by $m!$ we can break up the sum as follows:

$$e(m!) = n(m-1)! = \sum_{k=0}^{\infty} \frac{m!}{k!} = \sum_{k=0}^{m} \frac{m!}{k!} + \sum_{k=m+1}^{\infty} \frac{m!}{k!}.$$

The left hand side is now an integer, which means the right hand side must also be an integer. The first term of the right hand side is clearly an integer since $m \geq k$. The second term takes some manipulation:

$$0 < \sum_{k=m+1}^{\infty} \frac{m!}{k!} = \frac{1}{m+1} + \frac{1}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} + \ \ldots$$

$$< \frac{1}{m+1} + \frac{1}{(m+1)(m+1)} + \frac{1}{(m+1)(m+1)(m+1)} + \ \ldots$$

$$= \frac{1}{m+1} + \frac{1}{(m+1)^2} + \frac{1}{(m+1)^3} + \ \ldots \ = \sum_{k=1}^{\infty} \left(\frac{1}{m+1} \right)^k$$

$$= \frac{1}{(m+1)} \left(\frac{1}{1 - \frac{1}{m+1}} \right) = \frac{1}{m} < 1.$$

This makes the right hand side an integer plus a value between 0 and 1, which is clearly not an integer; therefore, we have produced a contradiction. QED

3 Stirling's Formula

The proofs for the transcendence of e and π require taking limits of factorials. Originally stated by Abraham de Moivre in 1730, Stirling's Formula allows us to rewrite the factorials inside the limits making them easier to evaluate.

3.1 Wallis Formula

Stirling's Formula requires the Wallis Formula, originally discovered in 1650 by Englishman John Wallis. The derivation given, see [4], is a result of Euler's proof from 1734 for the Basel problem, or the sum of reciprocals of the squares.

Wallis Formula.

$$\lim_{n \to \infty} \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{2n}} = \sqrt{\frac{\pi}{2}}.$$

Proof: Consider $\dfrac{\sin x}{x}$. We can factor this in a unique way by finding its roots. It has roots $x = \pm n\pi$ where $n \in \mathbb{N}$. Therefore we get:

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

$$= \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots .$$

Plugging in $x = \frac{\pi}{2}$ we get:

$$\frac{2}{\pi} = \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{16}\right)\left(1 - \frac{1}{36}\right)\cdots$$

$$= \left(\frac{3}{4}\right)\left(\frac{15}{16}\right)\left(\frac{35}{36}\right)\cdots$$

$$= \left(\frac{1 \cdot 3}{2 \cdot 2}\right)\left(\frac{3 \cdot 5}{4 \cdot 4}\right)\left(\frac{5 \cdot 7}{6 \cdot 6}\right)\cdots$$

$$= \prod_{k=1}^{\infty} \frac{(2k-1)(2k+1)}{(2k)(2k)} .$$

Taking the reciprocal and rewriting it yields:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n)(2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots (2n-1) \cdot (2n+1)}.$$

If we take the square root we achieve the desired result since $(2n+1) \sim 2n$ for n large:

$$\lim_{n\to\infty} \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{2n}} = \sqrt{\frac{\pi}{2}}.$$

QED

3.2 Stirling's Formula

Stirling's Formula is a very influential approximation. It allows us to evaluate limits involving factorials, as well as approximate factorials of large numbers. This is used extensively in other fields, such as thermodynamics.

Stirling's Formula. *As* $n \to \infty$, $n! \sim n^n e^{-n}\sqrt{2\pi n}$.

Proof: This proof follows that given by [3]. We want to show that $\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$. First, since the function $\log t$ is an increasing function for $t > 0$, $\log t < \log j$ for $t \in (j-1, j)$:

$$\int_{j-1}^{j} \log t \ \mathrm{d}t < \log j \int_{j-1}^{j} 1 \ \mathrm{d}t = \log j.$$

Likewise, $\log j < \log t$ for $t \in (j, j+1)$, so:

$$\log j = \log j \int_{j}^{j+1} 1 \ \mathrm{d}t < \int_{j}^{j+1} \log t \ \mathrm{d}t.$$

Putting these two together yields:

$$\int_{j-1}^{j} \log t \ \mathrm{d}t < \log j < \int_{j}^{j+1} \log t \ \mathrm{d}t.$$

Adding up these inequalities for $j = 1, 2, \ \ldots, n$ and using the properties of logarithms and integrals, we get:

$$\int_{0}^{n} \log t \ \mathrm{d}t < \log n! < \int_{1}^{n+1} \log t \ \mathrm{d}t.$$

While the first integral is improper, it converges to:

$$n \log n - n < \log n! < (n+1)\log(n+1) - n.$$

We can rearrange the last expression where $O(1) = (n+1)(\log(n+1) - \log n)$ to get:

$$n \log n - n < \log n! < n \log n - n + \log n + O(1).$$

Taking half the difference of the two sides of the inequality yields:

$$b_n = n \log n - n + \frac{1}{2}\log n,$$

which is a good approximation of $\log n!$. Define a_n as follows:

$$a_n = \log n! - b_n = \log n! - (n \log n - n + \frac{1}{2}\log n).$$

Consider the difference:

$$a_n - a_{n+1} = \left(n + \frac{1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1 = \left(\frac{2n+1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1.$$

Using simple algebraic manipulation, we write:

$$\frac{n+1}{n} = \frac{n+1}{n}\left(\frac{1-\dfrac{1}{2n+1}}{1-\dfrac{1}{2n+1}}\right) = \frac{\dfrac{n+1}{n}\left(\dfrac{2n}{2n+1}\right)}{1-\dfrac{1}{2n+1}}$$

$$= \frac{\dfrac{(2n+1)+1}{2n+1}}{1-\dfrac{1}{2n+1}} = \frac{1+\dfrac{1}{2n+1}}{1-\dfrac{1}{2n+1}}.$$

Setting $\alpha = \dfrac{1}{2n+1}$ and doing a Taylor Expansion about zero yields:

$$\frac{1}{2}\log\left(\frac{n+1}{n}\right) = \frac{1}{2}\log\left(\frac{1+\alpha}{1-\alpha}\right) = \frac{1}{2}[\log(1+\alpha) - \log(1-\alpha)]$$

$$= \frac{1}{2}\left[\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\alpha^j - \sum_{j=1}^{\infty}\frac{(-1)^{2j+1}}{j}\alpha^j\right]$$

$$= \frac{1}{2}\sum_{j=1}^{\infty}\frac{\alpha^j}{j}[(-1)^{j+1}+1] = \sum_{j=1}^{\infty}\frac{\alpha^{2j-1}}{2j-1}$$

$$= \alpha + \frac{1}{3}\alpha^3 + \frac{1}{5}\alpha^5 + \ \ldots \ .$$

Plugging this into our difference, we get:

$$a_n - a_{n+1} = \left(\frac{2n+1}{2}\right)\log\left(\frac{n+1}{n}\right) - 1$$

$$= \left(\frac{1}{\alpha}\right)\left(\frac{1}{2}\log\left(\frac{1+\alpha}{1-\alpha}\right)\right) - 1$$

$$= \frac{1}{\alpha}\left(\alpha + \frac{1}{3}\alpha^3 + \frac{1}{5}\alpha^5 + \ \ldots\right) - 1$$

$$= \left(1 + \frac{1}{3}\alpha^2 + \frac{1}{5}\alpha^4 + \ \ldots\right) - 1 = \frac{1}{3}\alpha^2 + \frac{1}{5}\alpha^4 + \ \ldots \ .$$

Since $\alpha > 0$ for all n, this shows that the difference $a_n - a_{n+1}$ is also greater than zero for all n. Continuing with this expansion of the difference:

$$0 < a_n - a_{n+1} = \frac{1}{3}\alpha^2 + \frac{1}{5}\alpha^4 + \ \ldots \ < \frac{1}{3}\ (\alpha^2 + \alpha^4 + \ \ldots\ )$$

$$= \frac{\alpha^2}{3(1-\alpha^2)} = \frac{1}{3\left(\dfrac{1}{\alpha^2} - 1\right)} = \frac{1}{3((2n+1)^2 - 1)} = \frac{1}{12n(n+1)}.$$

This shows that the sequence a_n is decreasing, and since the difference approaches zero as $n \to \infty$, the sequence converges to some number c. Therefore:

$$\lim_{n\to\infty} e^{a_n} = \lim_{n\to\infty} \frac{n!e^n}{n^n\sqrt{n}} = e^c,$$

and hence:

$$\lim_{n\to\infty} \frac{n!e^n}{e^c n^n \sqrt{n}} = 1.$$

Now we need to show $e^c = \sqrt{2\pi}$, which was first done by James Stirling in 1730. To do this, we use the Wallis formula for π:

$$\lim_{n\to\infty} \frac{2\cdot 4 \cdots (2n)}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{2n}} = \sqrt{\frac{\pi}{2}}.$$

Factoring out a 2 from every term in the numerator, the numerator becomes $2^n n!$. The denominator is $(2n)!$ divided by the numerator. Therefore the Wallis formula for π can be rewritten as:

$$\lim_{n\to\infty} \frac{(2^n n!)^2}{(2n)!}\frac{1}{\sqrt{2n}} = \sqrt{\frac{\pi}{2}}.$$

Since $n! \sim n^n\sqrt{n}e^{-n}e^c$ we get:

$$\frac{2^{2n}(n!)^2}{(2n)!}\frac{1}{\sqrt{2n}} \sim \frac{2^{2n}(n^n\sqrt{n}e^{-n}e^c)^2}{(2n)^{2n}(\sqrt{2n})e^{-2n}e^c}\frac{1}{\sqrt{2n}} = \frac{e^c}{2} \sim \sqrt{\frac{\pi}{2}}.$$

Therefore:

$$e^c \sim \sqrt{2\pi}.$$

QED

3.3 Bounds for Stirling's Formula

When using Stirling's Formula it is sometimes necessary to consider the bounds of the approximation. In particular, we use the lower bound to help with the limits in the next few sections. With some simple manipulation we can determine bounds for our approximation:

$$0 < a_n - a_{n+1} < \frac{1}{12n(n+1)} = \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Using the fact that this is a telescoping sequence:

$$\begin{aligned} 0 < \ a_n - a_{n+1} &< \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ a_{n+1} - a_{n+2} &< \frac{1}{12}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ a_{n+2} - a_{n+3} &< \frac{1}{12}\left(\frac{1}{n+2} - \frac{1}{n+3}\right) \\ &\ldots \\ a_{n+k-1} - a_{n+k} &< \frac{1}{12}\left(\frac{1}{n+k-1} - \frac{1}{n+k}\right). \end{aligned}$$

Adding these up we get:

$$0 < a_n - a_{n+k} < \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+k}\right).$$

Let $k \to \infty$ and since a_n converges to c:

$$0 < a_n - c < \frac{1}{12n}.$$

Take the anti-logarithm of this inequality and we get our bounds:

$$1 \leq \frac{n!}{e^{-n}n^n\sqrt{2\pi n}} \leq \ e^{\frac{1}{12n}}.$$

4 Irrationality of π

The proof for the irrationality of π is done by way of contradiction. Using the rational number acquired from the initial assumption, we construct a polynomial that maps integers to integers. We show that this function is never zero, but converges to zero, giving us a contradiction by violating Lemma 1.

4.1 Lemma 1

This Lemma is the basis for the proofs for the irrationality of π, as well as the transcendence of π and e. It is a direct consequence from the epsilon-delta definition of limits.

Lemma 1. *Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a function such that $f(n) \to 0$ as $n \to \infty$. Then there exists an $N \in \mathbb{Z}$ such that $f(n) = 0$ for all $n \geq N$.*

Proof: Since $f(n) \to 0$ as $n \to \infty$, there exists an $N \in \mathbb{Z}$ such that $\mid f(n) - 0 \mid < \epsilon$ for all $n \geq N$. Let $\epsilon = \frac{1}{2}$ then $\mid f(n) - 0 \mid < \frac{1}{2}$ meaning $f(n) = 0$ for all $n \geq N$ since $f(n)$ is an integer. QED

4.2 The Irrationality of π

This proof follows Ivan Niven's "simple" proof from 1947 [6]. There are many ways to prove that π is irrational, but this proof follows the same outline for proving the transcendence of e and π. It is important to note that the proof for the transcendence of e came before Niven's proof for the irrationality of π. For the purpose of this paper, since proving π is irrational is easier, we look at it first.

Theorem. *The value π is irrational.*

Proof: By way of contradiction, assume that $\pi = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Define:

$$f(x) = \frac{x^n(a - bx)^n}{n!}, \tag{1}$$

$$F(x) = f(x) - f^{(2)}(x) + \ . \ . \ . \ = \sum_{i=0}^{\infty} \ (-1)^i \ f^{(2i)}(x). \tag{2}$$

Equation (1) has degree $2n$, so equation (2) can be written as a sum to n instead of infinity. Consider:

$$\begin{aligned} &\frac{d}{dx}[F'(x)\sin(x) - F(x)\cos(x)] \\ &\qquad = F''(x)\sin(x) + F'(x)\cos(x) - F'(x)\cos(x) + F(x)\sin(x) \\ &\qquad = [F''(x) + F(x)]\sin(x) = f(x)\sin(x). \end{aligned}$$

Integrating both sides, using the Fundamental Theorem of Calculus, we get:

$$\int_0^\pi f(x)\sin(x)\ \mathrm{d}x = [F'(x)\sin(x) - F(x)\cos(x)]\Big|_0^\pi = F(\pi) + F(0).$$

Claim. $F(\pi) + F(0)$ is a non-negative integer.

Proof: First look at $F(0)$. Consider expanding $f(x)$ as a polynomial. Let the coefficient of x^k be $\frac{c_k}{n!}$. Since $a, b \in \mathbb{Z}$, so is c_k. Consider $f^{(k)}(0)$. Since the degree of each term of the polynomial is at least n, $f^{(k)}(0) = 0$ for $k < n$. For $n \le k \le 2n$, the coefficient of x^0, i.e. the coefficient of x^k differentiated k times, is $\frac{c_k}{n!}k!$ because we differentiated the original term k times; therefore, $f^{(k)}(0) = \frac{c_k}{n!}k!$ for $n \le k \le 2n$ must be an integer. So $F(0) = \sum_{i=0}^{n} (-1)^i\ f^{(2i)}(0)$ is an integer.

Next consider $F(\pi)$.

$$f(\pi - x) = \frac{\left(\frac{a}{b} - x\right)^n \left(a - b\left(\frac{a}{b} - x\right)\right)^n}{n!} = \frac{\left(\frac{a}{b} - x\right)^n (a - (a - bx))^n}{n!}$$

$$= \frac{\left(\frac{a}{b} - x\right)^n (bx)^n}{n!} = \frac{(a - bx)^n x^n}{n!}$$

$$= f(x).$$

Differentiate both sides k times, we get:

$$(-1)^k f^{(k)}(\pi - x) = f^{(k)}(x).$$

Plugging in zero for x:

$$(-1)^k f^{(k)}(\pi) = f^{(k)}(0).$$

Thus, $f^{(k)}(\pi)$ is also an integer. So $F(\pi) = \sum_{i=0}^{n} (-1)^i\ f^{(2i)}(\pi)$ is an integer.

Therefore, $F(0) + F(\pi) \in \mathbb{Z}$.

QED (Claim)

For $0 < x < \pi$, $\sin(x) > 0$ and $f(x) > 0$. This implies:

$$0 < \int_0^\pi f(x)\sin(x)\ \mathrm{d}x = F(0) + F(\pi) \in \mathbb{Z}. \tag{3}$$

Lets find the x value that achieves the maximum value for $f(x)$ by setting $f'(x) = 0$:

$$f'(x) = \frac{nx^{n-1}(a-bx)^n}{n!} - \frac{bnx^n(a-bx)^{n-1}}{n!} = 0$$
$$x^{n-1}(a-bx)^n = bx^n(a-bx)^{n-1}$$
$$a - bx = bx.$$

The maximum value for $f(x)$ is achieved when $x = \frac{\pi}{2}$. From this we get the inequality:

$$0 < f(x) \leq \left(\frac{a\pi}{4}\right)^n \frac{1}{n!}.$$

Plugging this into our integral we acquire the inequality:

$$0 \; < \int_0^\pi f(x)\sin(x)\;\mathrm{d}x \;\leq\; \int_0^\pi \left(\frac{a\pi}{4}\right)^n \frac{1}{n!}\;\mathrm{d}x \;\leq\; \pi\left(\frac{a\pi}{4}\right)^n \frac{1}{n!}.$$

Using Stirling's Formula:

$$\lim_{n\to\infty} \pi\left(\frac{a\pi}{4}\right)^n \frac{1}{n!} \;\leq\; \lim_{n\to\infty} \pi\left(\frac{a\pi}{4}\right)^n \frac{e^n}{n^n\sqrt{2\pi n}}$$

$$= \lim_{n\to\infty} \frac{\pi}{\sqrt{2\pi n}}\left(\frac{a\pi e}{4n}\right)^n = 0.$$

By the Squeeze Theorem, $\int_0^\pi f(x)\sin(x)\;\mathrm{d}x$ converges to zero; therefore, we have constructed a function that maps integers to integers, and converges to zero. By (3), this function never equals zero, which contradicts Lemma 1. Therefore, the assumption that $\pi \in \mathbb{Q}$ is false. QED

5 The Transcendence of e

The proof for the transcendence of e is done by way of contradiction similar to the proof for the irrationality of π. Using the polynomial acquired from the assumption we made hoping to get a contradiction, we construct a new polynomial that maps integers to integers. We show that this function is never zero, but converges to zero, giving us a contradiction by violating Lemma 1.

Theorem. *The value* e *is transcendental.*

Proof: By way of contradiction, assume e is not transcendental. This means e satisfies the equation:

$$\sum_{j=0}^{m} a_j e^j = a_m e^m + \ \ldots \ + a_1 e + a_0 = 0, \tag{4}$$

for $a_j \in \mathbb{Q}$ for all j, some $m \in \mathbb{N}$, and $a_0 \neq 0$. We can multiply by a suitable integer to make $a_j \in \mathbb{Z}$ for all j. Consider:

$$f(x) = \frac{x^{p-1}(x-1)^p(x-2)^p \ \ldots \ (x-m)^p}{(p-1)!}, \tag{5}$$

with p a prime number. Notice that $f(x)$ is a polynomial of degree $mp+p-1$. Also consider:

$$F(x) = \sum_{i=0}^{mp+p-1} f^{(i)}(x) = f(x) + f^{(1)}(x) + \ \ldots \ + f^{(mp+p-1)}(x). \tag{6}$$

Multiply (6) by e^{-x} and differentiate with respect to x:

$$\begin{aligned} \frac{d}{dx}[e^{-x}F(x)] &= e^{-x}F'(x) - e^{-x}F(x) = e^{-x}[F'(x) - F(x)] \\ &= e^{-x}[-f(x) + f^{(mp+p)}(x)] = -e^{-x}f(x). \end{aligned}$$

If we integrate from zero to j and multiply by a_j for all j we get:

$$a_j \int_0^j e^{-x} f(x) \, \mathrm{d}x = -a_j[e^{-x}F(x)]\Big|_0^j = a_j\left[-e^{-j}F(j) + e^0 F(0)\right] = a_j F(0) - a_j e^{-j} F(j).$$

Multiplying by e^j and summing over j gives us:

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) \ \mathrm{d}x = \sum_{j=0}^{m} \left[a_j e^j F(0) - a_j F(j)\right] = F(0) \sum_{j=0}^{m} a_j e^j - \sum_{j=0}^{m} F(j) a_j.$$

By the initial assumption (4) and since $F(j) = \sum_{i=0}^{mp+p-1} f^{(i)}(j)$:

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) \mathrm{d}x = -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^{(i)}(j). \tag{7}$$

Claim. Equation (7) is not zero for large enough p.

Proof: We consider two cases.

Case 1: For $j \neq 0$ we will show $f^{(i)}(j) \in \mathbb{Z}$ and is divisible by p. The only nonzero terms arise when the term $(x-j)^p$ is differentiated exactly p times. For anything less than p derivatives, a power of $(x-j)$ survives, making the expression zero. Differentiating this term p times gets $\frac{p!}{(p-1)!} = p$ times $x^{p-1}(x-1)^p(x-2)^p \cdots (x-(j-1))^p(x-(j+1))^p \cdots (x-m)^p$. When we take more derivatives, we use the product rule on this extra term. Since the extra term is just integers raised to an integer power, it will remain an integer; thus, we acquire $p\left(\sum_{k=0}^{l} b_k\right)$ for some $l, b_k \in \mathbb{N}$. This is pk_1 for some $k_1 \in \mathbb{Z}$.

Case 2: For $j = 0$ we have to differentiate $x^{p-1}, (p-1)$ times in order for $f^{(i)}(0) \neq 0$. This first happens when $i = (p-1)$, which then gives us a $(p-1)!$ in the numerator canceling out the denominator. Therefore for $i = (p-1)$,

$$a_0 f^{(p-1)}(0) = a_0(x-1)^p(x-2)^p \cdots (x-m)^p|_{x=0} = \; a_0(-1)^p(-2)^p \cdots (-1)^p.$$

For $i > (p-1)$ in order for $f^{(i)}(0) \neq 0$ we must use $(p-1)$ derivatives as above. When we differentiate the middle expression again, a p will be brought down, making the expression a multiple of p. Overall, the $j = 0$ case results in $k_2 p + a_0(-1)^p \cdots (-m)^p$ for some $k_2 \in \mathbb{Z}$.

Putting the three cases together, equation (7) becomes $Kp + a_0(-1)^p \cdots (-m)^p$ with $K = k_1 + k_2$. If $p > \max(m!, \mid a_0 \mid)$, then p does not divide $a_0(-1)^p \cdots (-m)^p$ since p does not divide $1, 2, ..., m$ or a_0, and p is prime. Therefore, $Kp + a_0(-1)^p \cdots (-m)^p \neq 0$ for p large enough.

QED (Claim)

Thus, for p large enough we have:

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x) \mathrm{d}x = -Kp - a_0(-1)^p \cdots (-m)^p \neq 0.$$

However, if $0 \leq x \leq m$:

$$\mid f(x) \mid \leq \frac{m^{p-1} m^p \cdots m^p}{(p-1)!} = \frac{m^{mp+p-1}}{(p-1)!}.$$

Using this and the fact that for $x \geq 0$, $|e^{-x}| \leq 1$, we conclude:

$$\begin{aligned}\left|\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)\mathrm{d}x\right| &\leq \sum_{j=0}^{m} |a_j e^j| \int_0^j |e^{-x}||f(x)|dx \\ &\leq \sum_{j=0}^{m} |a_j e^j| \int_0^j \frac{m^{mp+p-1}}{(p-1)!}\mathrm{d}x \\ &= \sum_{j=0}^{m} |a_j e^j| \, j\frac{m^{mp+p-1}}{(p-1)!}.\end{aligned}$$

Claim. This expression approaches zero as $p \to \infty$.

Proof: By Stirling's Approximation, since $1 \leq \dfrac{p!}{p^p e^{-p}\sqrt{2\pi p}}$ we get

$$\frac{1}{p!} \leq \frac{1}{p^p e^{-p}\sqrt{2\pi p}}.$$

Therefore since m and p are both positive:

$$\begin{aligned}0 &\leq \frac{m^{mp+p-1}}{(p-1)!} = \frac{p\ m^{mp+p-1}}{p!} \\ &\leq \frac{p\ m^{mp+p-1}}{p^p e^{-p}\sqrt{2\pi p}} = \frac{1}{\sqrt{2\pi p}}\frac{e^p m^{mp+p-1}}{p^{p-1}} = \frac{1}{\sqrt{2\pi p}}\left(\frac{e\ m^{m+1-\frac{1}{p}}}{p^{1-\frac{1}{p}}}\right)^p.\end{aligned}$$

As $p \to \infty$, the expression in the parenthesis approaches zero since both e and m are fixed. This makes the entire expression approach zero as $p \to \infty$.

QED (Claim)

Therefore, since equation (7) approaches zero as $p \to \infty$, and is a function that maps $\mathbb{Z} \to \mathbb{Z}$, Lemma 1 states that there exists a prime P such that for all $p \geq P$:

$$\left|\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)dx\right| = 0.$$

Contradicting the claim that equation (7) does not equal zero for large enough p. QED

This proof follows Charles Hermite's original proof from 1873, which is reproduced in [7].

6 The Transcendence of π

The proof for the transcendence of π is very analogous to that of e, except it requires Galois Theory to set up the function of interest; therefore, we have a few definitions to establish and theorems to prove.

6.1 Definitions

Definition: We can order the elements of a Cartesian product of any two sets A and B using **Lexicographic Order** by $(a_1, b_1) < (a_2, b_2)$ both in $A \times B$ if and only if either:

1. $a_1 < a_2$, or
2. $a_1 = a_2$ and $b_1 < b_2$.

This can be extended to a Cartesian product of an arbitrary number of sets. For example, if $x_1 < x_2 < \cdots < x_n$ then $(x_1, x_2, ..., x_n) < (x_1, x_2, ..., x_{i+1}, x_i, ..., x_n)$ for $1 < i < n$ under the lexicographic ordering.

Definition: Let $f = \sum_\alpha a_\alpha x^\alpha$ be a nonzero polynomial in $K[x_1, x_2, ..., x_n]$ where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. $K[x_1, x_2, ..., x_n]$ is the ring of all polynomials with variables $x_1, x_2, ..., x_n$ and coefficients in the field K. The **multidegree** of f is $\text{multideg}(f) = \max(\alpha : a_\alpha \neq 0)$ under lexicographic ordering. For example, the polynomial $f = 3x^2y^3 + 4x^3y^1$ has multidegree $(3, 1)$, written $\text{multideg}(f) = (3, 1)$.

Definition: The **leading coefficient** of f is $\text{LC}(f) = a_{\text{multideg}(f)} \in K$. Likewise, the **leading monomial** of f is $\text{LM}(f) = x^{\text{multideg}(f)}$. Putting this together, the **leading term** of f is $\text{LT}(f) = \text{LC}(f) \cdot \text{LM}(f)$. That is, the leading term is the coefficient and variables of the term with the highest multidegree.

Definition: A polynomial $f \in K[x_1, x_2, ..., x_n]$ is **symmetric** if:

$$f(x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(n)}) = f(x_1, x_2, ..., x_n)$$

for all possible permutations $x_{\tau(1)}, x_{\tau(2)}, ..., x_{\tau(n)}$ on the variables $x_1, x_2, ..., x_n$. That is, a polynomial is symmetric if relabeling the variables does not change the function itself. For example, consider $f(x,y) = x^2 + y^2$. Relabeling the function as $f(y,x) = y^2 + x^2$ does not change the function, i.e. $f(y,x) = f(x,y)$; therefore, $f(x,y)$ is a symmetric polynomial.

Definition: We define the **elementary symmetric polynomials** $\sigma_0, \sigma_1, ..., \sigma_n \in K[x_1, x_2, ..., x_n]$ by:

$$\begin{aligned}
\sigma_0 &= 1 \\
\sigma_1 &= x_1 + x_2 + \cdots + x_n \\
\sigma_2 &= x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n \\
&\cdots \\
\sigma_n &= x_1x_2 \cdots x_n
\end{aligned}$$

with each σ_j is a symmetric function. That is, σ_1 is all the ways to choose 1 variable from the n variables, σ_2 is all the ways to choose 2 variables from the n variables, and so on. These can be derived from the expansion of a function f about its roots $\alpha_1, \alpha_2, ..., \alpha_n :$

$$\begin{aligned}
f(x) &= \prod_{i=1}^{n}(x - \alpha_i) \\
&= x^n(1) - x^{n-1}(\alpha_1 + \alpha_2 + \cdots + \alpha_n) + \cdots + (-1)^n x^0(\alpha_1\alpha_2 \cdots \alpha_n) \\
&= x^n(\sigma_0) - x^{n-1}(\sigma_1) + \cdots + (-1)^n x^0(\sigma_n) \\
&= \sum_{j=0}^{n}(-1)^j x^{n-j}\sigma_j.
\end{aligned}$$

6.2 The Fundamental Theorem of Symmetric Polynomials

As with the proof for the transcendence of e, we will need to construct a function, $f(x)$, that will give us the contradiction needed. To do this, we will need to show that certain derivatives of $f(x)$ evaluated at various values are integers. The easiest way to do this will be to look at the elementary symmetric polynomials and using the Fundamental Theorem of Symmetric Polynomials.

Theorem. (The Fundamental Theorem of Symmetric Polynomials). *Every symmetric polynomial in $K[x_1, x_2, ..., x_n]$ can be written uniquely as a polynomial in the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$.*

Proof: We will use a lexicographic ordering with $x_1 > x_2 > \cdots > x_n$. Let the leading term of a nonzero symmetric polynomial $f \in K[x_1, x_2, ..., x_n]$, be $\mathrm{LT}(f) = ax^\alpha$ where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, we claim that
$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

Proof of Claim: Suppose by way of contradiction that there exists some i such that $\alpha_i < \alpha_{i+1}$. Let $\beta = (\alpha_1, ..., \alpha_{i+1}, \alpha_i, ..., \alpha_n)$ i.e. let β be α with α_{i+1} and α_i switched. Notice that ax^α is a term of f, ax^β is a term of $f(..., x_{i+1}, x_i, ...)$. Since f is symmetric, $f(..., x_{i+1}, x_i, ...) = f$, meaning that ax^β is a term of f. Since $\beta > \alpha$ under the lexicographic ordering, ax^β is the leading term of f, a contradiction. Therefore $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

QED (Claim)

Consider:

$$h = \sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \cdots \sigma_{n-1}^{\alpha_{n-1} - \alpha_n} \sigma_n^{\alpha_n}.$$

The function h is symmetric since it is the product of symmetric polynomials, namely the elementary symmetric polynomials. We will find the leading term of h. To do this we note that $\mathrm{LT}(\sigma_r) = x_1 x_2 \cdots x_r$ for $1 \leq r \leq n$. Hence:

$$\begin{aligned}
\mathrm{LT}(h) &= \mathrm{LT}(\sigma_1^{\alpha_1 - \alpha_2} \sigma_2^{\alpha_2 - \alpha_3} \cdots \sigma_n^{\alpha_n}) \\
&= \mathrm{LT}(\sigma_1)^{\alpha_1 - \alpha_2} \mathrm{LT}(\sigma_2)^{\alpha_2 - \alpha_3} \cdots \mathrm{LT}(\sigma_n)^{\alpha_n} \\
&= x_1^{\alpha_1 - \alpha_2} (x_1 x_2)^{\alpha_2 - \alpha_3} \cdots (x_1 \cdots x_n)_n^\alpha \\
&= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = x^\alpha.
\end{aligned}$$

From this we see that f and ah have the same leading term, thus:

$$\mathrm{multideg}(f - ah) < \mathrm{multideg}(f)$$

for $f - ah \neq 0$. Define $f_1 = f - ah$. Since f and ah are symmetric, f_1 is also symmetric; therefore, we can repeat this process if $f_1 \neq 0$ to get $f_2 = f_1 - a_1 h_1$, where a_1 is a constant and h_1 is some product of $\sigma_1, \sigma_2, ..., \sigma_n$ to various powers. When we do this, we will also get the same relationship:

$$\text{multideg}(f_2) < \text{multideg}(f_1).$$

This means $\text{LT}(f_2) < \text{LT}(f_1)$ when $f_2 \neq 0$. We can continue this process to get a sequence of polynomials $f, f_1, f_2, ...$ with:

$$\text{multideg}(f) > \text{multideg}(f_1) > \text{multideg}(f_2) > \cdots .$$

The multidegree must terminate since the multidegree decreases at each step. Since f has at most n variables and the multidegree is a coordinate of non-negative integers, the multidegree eventually falls to $(0, 0, ..., 0)$ making it terminate. This means $f_{t+1} = 0$ for some t. Combining all the f's, we get:

$$\begin{aligned} f &= ah + f_1 = ah + a_1 h_1 + f_2 \\ &= ah + a_1 h_1 + \cdots + a_t h_t. \end{aligned}$$

Finally, since each h is a product of elementary symmetric polynomials, f is a polynomial in the elementary symmetric functions. We now need to prove uniqueness. Suppose:

$$f = g_1(\sigma_1, \sigma_2, ..., \sigma_n) = g_2(\sigma_1, \sigma_2, ..., \sigma_n)$$

for f a symmetric polynomial with g_1 and g_2 its representations in the elementary symmetric functions $\sigma_1, \sigma_2, ..., \sigma_n$. Here, g_1 and g_2 are polynomials in n variables, say $y_1, y_2, ..., y_n$, so they are both in $K[y_1, y_2, ..., y_n]$. Let $g = g_1 - g_2$. We first notice that $g(\sigma_1, \sigma_2, ..., \sigma_n) = 0$ in $K[x_1, x_2, ..., x_n]$ since each σ_i is in $K[x_1, x_2, ..., x_n]$. To prove uniqueness, we need to show that $g = 0$ in $K[y_1, y_2, ..., y_n]$. By way of contradiction, suppose $g(\sigma_1, \sigma_2, ..., \sigma_n) \neq 0$. Let $g = \sum_\beta a_\beta y^\beta$, that is let $g(\sigma_1, \sigma_2, ..., \sigma_n)$ be the sum of $g_\beta = a_\beta \sigma_1^{\beta_1} \sigma_2^{\beta_2} \cdots \sigma_n^{\beta_n}$ such that $\beta = (\beta_1, \beta_2, ..., \beta_n)$. It follows that:

$$\begin{aligned} \text{LT}(g_\beta) &= \text{LT}(a_\beta \sigma_1^{\beta_1} \sigma_2^{\beta_2} \cdots \sigma_n^{\beta_n}) \\ &= a_\beta \text{LT}(\sigma_1)^{\beta_1} \text{LT}(\sigma_2)^{\beta_2} \cdots \text{LT}(\sigma_n)^{\beta_n} \\ &= a_\beta x_1^{\beta_1} (x_1 x_2)^{\beta_2} \cdots (x_1 \cdots x_n)^{\beta_n} \\ &= a_\beta x_1^{\beta_1 + \cdots + \beta_n} x_2^{\beta_2 + \cdots + \beta_n} \cdots x_n^{\beta_n}. \end{aligned}$$

At the same time, we know that $\mathrm{LT}(g_\beta) = a_\beta \sigma_1^{\beta_1} \sigma_2^{\beta_2} \cdots \sigma_n^{\beta_n}$ in $K[\sigma_1, \sigma_2, ..., \sigma_n]$. The leading term is unique, so these must be equal; therefore, there must be a map:

$$(\beta_1, ..., \beta_n) \mapsto (\beta_1 + \cdots + \beta_n, \beta_2 + \cdots + \beta_n, \cdots, \beta_n).$$

We need to show this mapping is injective. Suppose $(\beta_1 + \cdots + \beta_n, \beta_2 + \cdots + \beta_n, \cdots, \beta_n) = (\gamma_1 + \cdots + \gamma_n, \gamma_2 + \cdots + \gamma_n, \cdots, \gamma_n)$, we need to show $(\beta_1, ..., \beta_n) = (\gamma_1, ..., \gamma_n)$. Working backwards, it is clear that $\beta_n = \gamma_n$. Looking at the next term $\beta_{n-1} + \beta_n = \gamma_{n-1} + \gamma_n$, using the previous relation, we get $\beta_{n-1} = \gamma_{n-1}$. Continuing this we get all the way down to $\beta_1 = \gamma_1$. Putting it all together yields the desired result $(\beta_1, ..., \beta_n) = (\gamma_1, ..., \gamma_n)$.

Since this map is injective, the g_β's have distinct leading terms. Consider choosing a β such that $\mathrm{LT}(g_\beta) > \mathrm{LT}(g_\gamma)$ for all $\gamma \neq \beta$. By the definition of the leading term, $\mathrm{LT}(g_\beta)$ is greater than all of the terms of the g_γ's. This means nothing cancels the $\mathrm{LT}(g_\beta)$, and since g is the sum of the g_β's, $g(\sigma_1, \sigma_2, ..., \sigma_n)$ cannot be zero in $K[x_1, x_2, ..., x_n]$. This provides the contradiction, meaning this representation in the elementary symmetric polynomials is unique. QED

This proof, see [2], was given by Gauss to help provide a second proof of the Fundamental Theorem of Algebra in 1816. The proof gives rise to an algorithm for writing a symmetric polynomial in terms of the elementary symmetric polynomials $\sigma_1, \sigma_2, ..., \sigma_n$.

Example: Consider the function:

$$f = x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2 \in K[x, y, z].$$

Using the algorithm used in the above proof, we can write f in terms of the elementary symmetric polynomials. We start by eliminating the leading term of f, $\mathrm{LT}(f) = x^2y = \mathrm{LT}(\sigma_2\sigma_1)$, which gives:

$$f_1 = f - \sigma_2\sigma_1 = -3xyz.$$

The leading term is now the only term $-3xyz = -3\mathrm{LT}(\sigma_3)$, which gives:

$$f_2 = f_1 + 3\sigma_3 = 0.$$

Working backwards we get f in terms of the elementary symmetric polynomials:

$$f = \sigma_2\sigma_1 + f_1 = \sigma_2\sigma_1 - 3\sigma_3.$$

6.3 Corollary 1

This theorem gives rise to a Corollary, which will be used in the proof of the transcendence of π.

Corollary 1. *Consider a polynomial $f(x)$ in F. Let $F \subseteq K$ be a field extension containing the roots of $f(x)$, $\alpha_1, \alpha_2, ..., \alpha_n$. If $g(\alpha_1, \alpha_2, ..., \alpha_n) \in K$ is symmetric, then $g(\alpha_1, \alpha_2, ..., \alpha_n) \in F$.*

Proof: By assumption, we can write $f(x) \in K[x]$ as:

$$f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n).$$

Using the definition of the elementary symmetric polynomials $\sigma_0, \sigma_1, \sigma_2..., \sigma_n$:

$$f(x) = \prod_{i=1}^{n}(x - \alpha_i) = \sum_{j=0}^{n}(-1)^j\sigma_j x^{n-j}$$

where $\sigma_1, \sigma_2..., \sigma_n$ is evaluated at $(\alpha_1, \alpha_2, ..., \alpha_n)$ and $\sigma_0 = 1$. Since $f(x) \in F[x]$, its coefficients are in F, meaning the elementary symmetric polynomials evaluated at $(\alpha_1, \alpha_2, ..., \alpha_n)$ are in F. Let $g \in F[x_1, x_2, ..., x_n]$ be a symmetric polynomial. By the Fundamental Theorem of Symmetric Polynomials, we can write g in terms of $\sigma_1, \sigma_2, ..., \sigma_n$ with coefficients in F. Therefore, evaluating g at $(\alpha_1, \alpha_2, ..., \alpha_n)$ gets us $g \in F$. QED

6.4 The Proof for the Transcendence of π

Theorem. *The value π is transcendental.*

Proof: By way of contradiction, let π be the root of some nonzero polynomial over $\mathbb{Q}$. Since i is algebraic, then so is $i\pi$. Let $\theta_1(x) \in \mathbb{Q}[x]$ be a polynomial with zeros $\alpha_1 = i\pi, \alpha_2, \alpha_3, ..., \alpha_n$. Since $e^{i\pi} + 1 = 0$, we know:

$$(e^{\alpha_1} + 1)(e^{\alpha_2} + 1) \cdots (e^{\alpha_n} + 1) = 0. \tag{8}$$

Imagine multiplying this out, we would get:

$$e^{\gamma_1} + e^{\gamma_2} + \cdots + e^{\gamma_h} = 0,$$

where γ_i is some linear combination of the α_j's. Since there is at least one $\gamma_j = 0$, achieved by multiplying all the ones together, we can rewrite this as:

$$e^{\gamma_1} + e^{\gamma_2} + \cdots + e^{\gamma_r} + k = 0, \tag{9}$$

where $k \in \mathbb{N}$. We now construct a polynomial whose zeros are $\gamma_1, \gamma_2, ..., \gamma_h$. For example, consider the terms of the form:

$$e^{\alpha_s} e^{\alpha_t} \cdot 1 \cdots 1.$$

These lead to exponents of the form $\alpha_s + \alpha_t$. Multiplying (8) out would result in all such combinations of two α's: $\alpha_1 + \alpha_2, ..., \alpha_{n-1} + \alpha_n$. Let all these combinations of two α's be the roots of a polynomial $\theta_2(x)$. This means we can write:

$$\theta_2(x) = \prod_{1 \le i < j \le n} (x - (\alpha_i + \alpha_j)).$$

Since we get *all* combinations of two α's, $\theta_2(x)$ is symmetric in $\alpha_1, \alpha_2, ..., \alpha_n$. By the Fundamental Theorem of Symmetric Polynomials, $\theta_2(x)$ can be expressed in terms of the elementary symmetric polynomials in $\alpha_1, \alpha_2, ..., \alpha_n$. Since the α's are zeros of $\theta_1(x)$, which is in $\mathbb{Q}[x]$, their elementary symmetric polynomials can be written in terms of the coefficients of $\theta_1(x)$ as seen in the definition of elementary symmetric polynomials. This means the elementary symmetric polynomials are in $\mathbb{Q}[\alpha_1, \alpha_2, ..., \alpha_n]$. Therefore, by expressing $\theta_2(x)$ in terms of the elementary symmetric polynomials, $\theta_2(x)$ must also have rational coefficients.

By the same argument, the sums of k α_j's satisfy the polynomial $\theta_k(x) = 0 \in \mathbb{Q}[x]$ for $2 \le k \le n$. Using this, $\theta_1(x)\theta_2(x) \cdots \theta_k(x)$ is a polynomial over the rationals with zeros $\gamma_1, \gamma_2, ..., \gamma_h$. Since there is at least one $\gamma_j = 0$, we can divide by some power of x and multiply by a suitable integer to get a polynomial $\theta(x) \in \mathbb{Z}[x]$ with nonzero roots $\gamma_1, \gamma_2, ..., \gamma_r$ with $r < h$. We can write $\theta(x)$ as a polynomial:

$$\theta(x) = cx^r + c_1 x^{r-1} + \cdots + c_r$$

where c_r is not zero since zero is not a root of $\theta(x)$. Define:

$$f(x) = \frac{c^s x^{p-1}[\theta(x)]^p}{(p-1)!}$$

with $s = rp - 1$ and p a prime. Notice that $f(x)$ has degree $rp + p - 1 = s + p$. Define another function:

$$F(x) = f(x) + f'(x) + \cdots + f^{(s+p)}(x).$$

As with the proof for e transcendental:

$$\frac{d}{dx}\left[e^{-x}F(x)\right] = e^{-x}\left[F'(x) - F(x)\right] = -e^{-x}f(x).$$

Integrate from zero to x, changing the dummy variable to y:

$$\begin{aligned}\int_0^x \frac{d}{dy}\left[e^{-y}F(y)\right] \mathrm{d}y = e^{-y}F(y)\Big|_0^x &= e^{-x}F(x) - F(0)\\ &= -\int_0^x e^{-y}f(y)\mathrm{d}y.\end{aligned}$$

Making the substitution $y = \lambda x$ and multiplying by e^x to clean things up yields:

$$F(x) - e^x F(0) = -x\int_0^1 f(\lambda x)e^{x(1-\lambda)}\mathrm{d}\lambda.$$

Summing over x ranging from $\gamma_1, \gamma_2, ..., \gamma_r$ acquires:

$$\sum_{j=1}^r F(\gamma_j) - F(0)\sum_{j=1}^r e^{\gamma_j} = -\sum_{j=1}^r \gamma_j \int_0^1 f(\lambda\gamma_j)e^{\gamma_j(1-\lambda)}\mathrm{d}\lambda.$$

Using equation (9) gives us the expression of interest:

$$\sum_{j=1}^r F(\gamma_j) + kF(0) = -\sum_{j=1}^r \gamma_j \int_0^1 f(\lambda\gamma_j)e^{\gamma_j(1-\lambda)}\mathrm{d}\lambda. \tag{10}$$

Claim. The left hand side of equation (10) is a nonzero integer for large p.

Proof. We first deal with $F(0)$. We consider three cases.

Case 1: For $t < p-1$, $f^{(t)}(0) = 0$ since the x^{p-1} term must survive.

Case 2: For $t = p-1$, the only term that is nonzero is the term that differentiates x^{p-1} exactly $p-1$ times. All other terms arising from the product rule have a power of x, and are thus zero. Therefore $f^{(p-1)}(0) = \frac{c^s\theta(0)^p}{(p-1)!}(p-1)!$ where $\theta(0) = c_r$, yielding $f^{(p-1)}(0) = c^s c_r^p$.

Case 3: For $t > p-1$, again the only terms that will be nonzero are those that differentiate x^{p-1} exactly $p-1$ times. The other derivatives go to $[\theta(x)]^p$. Since $\theta(x) \in \mathbb{Z}[x]$, differentiating $[\theta(x)]^p$ n times pulls a p down, and results in a new function still with integer coefficients. Evaluating it at zero yields p times an integer l. Therefore, due to the product rule, $f^{(t)}(0)$ creates an integer number of these terms all multiplied with $c^s \in \mathbb{Z}$. Adding all these terms together, and lumping all the integers into one produces $f^{(t)}(0) = pl_t$ for some $l_t \in \mathbb{Z}$.

We now consider $\sum_{j=1}^{r} F(\gamma_j) = \sum_{j=1}^{r}\sum_{t=0}^{s+p} f^{(t)}(\gamma_j)$, this time with two cases.

Case 1: For $0 \leq t < p$, a power of $\theta(x)$ will survive in each term. Since γ_j is a zero of $\theta(x)$ by construction, $\sum_{j=1}^{r} F(\gamma_j) = 0$.

Case 2: For $t \geq p$, we must differentiate $[\theta(x)]^p$ p times to eliminate the $\theta(x)$ since γ_j is a root of $\theta(x)$. Differentiating in this way pulls a $p!$ down, canceling the $(p-1)!$ in the denominator resulting in a factor of p. Notice that $\sum_{j=1}^{r} f^{(t)}(\gamma_j)$ is a symmetric polynomial in $\gamma_1, \gamma_2, ..., \gamma_r$ since $\sum_{j=1}^{r} f^{(t)}(\gamma_j)$ is some polynomial evaluated at each γ_j. Using Corollary 1, since $f(x) \in \mathbb{Z}$ with roots $\gamma_1, \gamma_2, ..., \gamma_r$, $\sum_{j=1}^{r} f^{(t)}(\gamma_j) \in \mathbb{Z}$. As argued, it must also be a multiple of p. Hence, $\sum_{j=1}^{r} F(\gamma_j) = pk_t$ for some $k_t \in \mathbb{Z}$.

Therefore the left hand side of equation (10) becomes:

$$pk_t + kc^s c_r^p + kpl_t = mp + kc^s c_r^p$$

for $m \in \mathbb{Z}$. As discussed earlier k, c, and c_r must be nonzero. Let $p > \max(|k|, |c|, |c_r|)$, then $mp + kc^s c_r^p$ is not divisible by p and is therefore nonzero.

QED (Claim)

Now consider the right hand side of equation (10):

$$-\sum_{j=1}^{r} \gamma_j \int_0^1 \frac{c^s(\lambda\gamma_j)^{p-1}[\theta(\lambda\gamma_j)]^p}{(p-1)!} e^{\gamma_j(1-\lambda)} \mathrm{d}\lambda.$$

Each term of the finite sum can be made as small as desired by making p very large.

Using Stirling's Formula as in the proof for the transcendence of e,

$$\left|\frac{c^s(\lambda\gamma_j)^{p-1}[\theta(\lambda\gamma_j)]^p}{(p-1)!}\right| = \left|\frac{\left(c^{r-\frac{1}{p}}(\lambda\gamma_j)^{1-\frac{1}{p}}[\theta(\lambda\gamma_j)]\right)^p}{(p-1)!}\right| \leq \left|\frac{q^p}{(p-1)!}\right| \to 0$$

for some constant q as $p \to \infty$. Therefore, the right hand side approaches zero as p gets large. Putting it all together, equation (10) creates the usual contradiction since it converges to zero for large p but is a nonzero integer for large p. QED

This proof follows Ferdinand Lindemann's original proof from 1882, reproduced in [7].

7 Applications

7.1 Constructibility

Definition: A real number x is **constructible** if we can create a line segment of length x using a straightedge and compass in a finite number of steps given a segment of unit length.

The ancient Greeks were intrigued by the notion of constructibility. They proved many theorems about what numbers are constructible, but one theorem eluded them. One of the unsolved problems of the ancient world was whether one could square a circle. That is, given a circle, can we construct a square with the same area. For a circle of radius 1, the area is π, and thus the sides of the square would be $\sqrt{\pi}$. So this question reduces to whether we can construct $\sqrt{\pi}$.

Theorem. *If α is constructible and not in $\mathbb{Q}$, then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^r$ for some $r \geq 0$ an integer.*

A proof for this theorem can be seen in Fraleigh's *A First Course in Abstract Algebra* [5]. This theorem states that for a number α to be constructible, the degree of the irreducible polynomial in $\mathbb{Q}$ with root α must be a power of two. More importantly, the degree must be finite. A transcendental number, by definition, is the root of no polynomial in $\mathbb{Q}$. This means its irreducible polynomial does not have finite degree, or degree equal to a power of two; therefore, transcendental numbers are not constructible. By proving the transcendence of π, we also get that $\sqrt{\pi}$ is transcendental, and is not constructible; thus answering the ancient question: we can not square a circle.

7.2 "Nice" Use of Galois Theory

As mathematicians we deal with highly complex and theoretical concepts. The proof for the transcendence of π uses many of these complicated results of higher level mathematics in an understandable way. It provides a nice application of Galois Theory in the usage of the symmetric polynomials to construct the equation of interest for proving π is transcendental. This helps mathematicians bring their results back down to the real world.

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