CONTROL SYSTEMS ANALYSIS AND DESIGN

VIA

THE MOST CONTROLLABLE AND OBSERVABLE SUBSYSTEMS

By

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CHAPTER 1

GENERAL SURVEY

1. Introduction

The analysis and design of control systems are usually done on a mathematical model for the particular physical system concerned. The system model may turn out to be of high order. To reduce the computational burden and to minimize the cost for the particular analysis and design problems, sometimes it is desired to reduce the order of the system model. A list of various methods and a classification of the methods can be found in the paper (Genesio and Milanese 1976).

A common philosophy adopted in deriving a reduced order model is to consider the particular given system model as a composite system of a dominant subsystem and a non-dominant subsystem in a certain sense. It is a well-known fact that no reduced order model can meet all purposes therefore the dominant subsystem should retain the properties of the original system which are of fundamental importance in the particular analysis and design problems. For instance, if for the case that it is the steady state response of the given system which is of fundamental importance, the accuracy of the transient response of the reduced order model may be sacrified.

The above consideration is depicted in Fig. 1. And the dominant subsystem is taken as the reduced order model for the particular problem by discarding the non-dominant subsystem and ignoring the couplings between the subsystems.

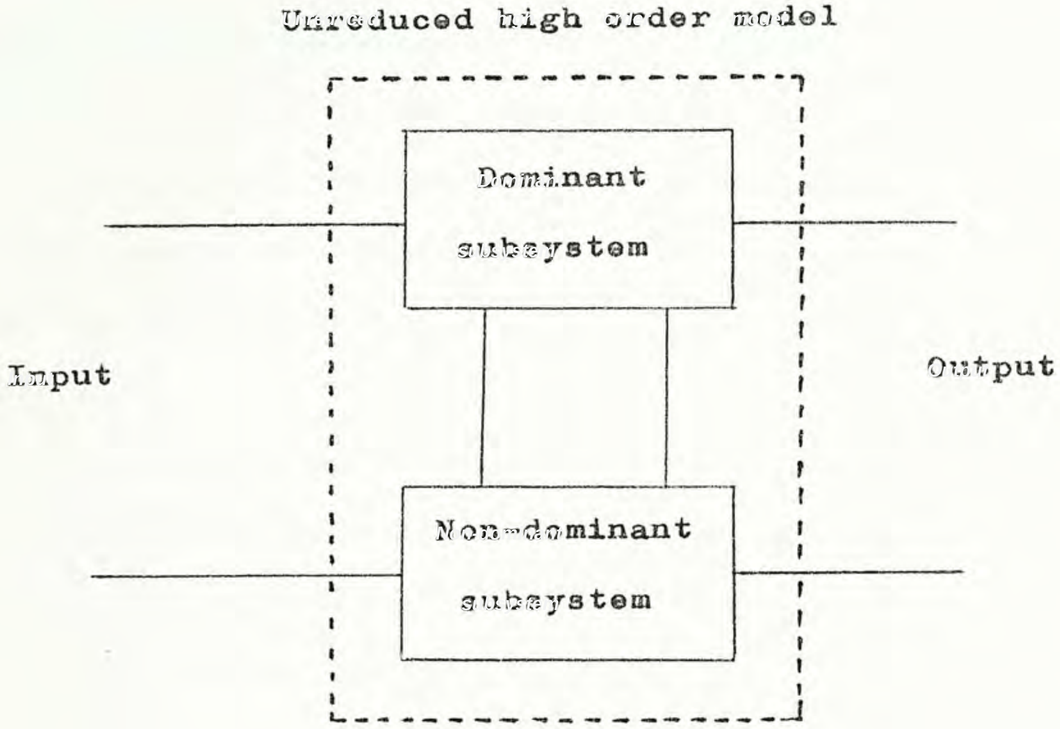


Fig. 1

It is known that when a subsystem of a composite system is completely uncontrollable or is completely unobservable, the subsystem will not contribute to the input-output characteristics of the composite system. Therefore, from the input-output point of view, a dominant subsystem can be defined as the subsystem which is completely controllable and completely observable.

If the given system is in fact both completely controllable and completely observable, the latter consideration may not be applied. However, when the original system is composed of a 'most controllable' and/or 'most observable' subsystem, a reduced order model can still be found. Moore (1981) derived a reduced order model which corresponds to the 'most controllable and most observable' subsystem of the given system. While, in the reduced order modelling procedure of Kwong (1982), called 'optimal chained aggregation', the most observable subsystem is extracted as the reduced order model. It is important to point out that in order that one can identify the 'most controllable' or the 'most observable' subsystem, one should have a measure of the 'degree of controllability' and a measure of the 'degree of observability'. Notice that the measure may have a determinant influence on the responses of the reduced order models. The measure of Moore (1981) will be shown to stress on the steady state response of the system, while that of Kwong (1982) stresses on the transient response.

In this thesis, a reduced order modelling procedure will be given in Chapter 2. The proposed procedure shares the same point of view as that of Kwong (1982) except that the most observable subsystem is extracted through an internal coordinate transformation of the given system to take into account of the input coupling effect in the reduced order model. The procedure of Kwong (1982) ignores the input coupling effect completely. In Chapter 3 we consider the problem of deriving a suboptimal control law for the linear regulator problem. Since it is the steady state response which is of fundamental importance here, the methodology of Moore (1981) is employed. In order that the effect of the weighting matrices in the cost functional may be taken into account in the suboptimal control law, an implicit output vector and a modified input vector are introduced.

2. The mathematical tool

Singular value decomposition is the main mathematical tool used to perform the order reduction. Singular value decomposition is recognized as a reliable technique to find the rank of a matrix. An excellent review of the properties of singular value decomposition can be found in (Klema and Laub 1980). For the purpose of completeness, the singular value decompostion theorem is given below.

Theorem (Klema and Laub 1980)

Let $A \in R^{m \times n}$, then there exists orthogonal matrices $U \in R^{m \times m}$ and $V \in R^{n \times n}$ such that

$$A = U \Sigma V' \tag{2.1}$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \tag{2.2}$$

and $S = \text{diag}(\sigma_1, \ldots, \sigma_k)$ with

$$\sigma_1 \geqslant \ldots \geqslant \sigma_k > 0$$

Note that the columns of U are called the left singular vectors of A while the columns of V are called the right singular vectors of A. The rank of the matrix A is k. If the orthogonal matrices U and V are partitioned compatibly as

$$U = \begin{array}{c} \begin{array}{cc} k & m-k \end{array} \\ \left[\begin{array}{c:c} U_1 & U_2 \end{array} \right] \end{array} \; m \tag{2.3}$$

and

$$V = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left[\begin{array}{c:c} V_1 & V_2 \end{array} \right] \end{array} \; n \tag{2.4}$$

then U_i, V_i provide orthonormal bases for the four fundamental subspaces: $\text{Im } V_1 = (\ker A)^{\perp}$, $\text{Im } U_2 = \text{Im } A$, $\text{Im } V_2 = \ker A$ and $\text{Im } U_2 = (\text{Im } A)^{\perp}$.

CHAPTER 2

REDUCED ORDER MODELLING

1. Introduction

In this chapter an algorithm is proposed for obtaining a reduced order model for a given system. The proposed method is based on extracting the most observable subspace of the given system. The extraction of the most observable subspace is done through an internal coordinate transformation of the given system model, called the 'controllability balanced representation'. The reasons for the need to transform the given system model is twofold: first, to take into account the input coupling effect and second, to make the algorithm independent of the internal coordinates of the original system model.

We shall first review the series expansions of the transfer function in Section 2. The expansions of the transfer function at $s=\infty$ and $s=0$ are considered. This is of help in interpreting the properties of the reduced order model. In Section 3, the 'controllability balanced representation' is introduced and some of the properties of the representation are discussed. In section 4, along with the proposed algorithm, the 'optimal chained aggregation' reduced order modelling procedure of Kwong (1982) will be outlined and discussed. The present work is in fact stimulated by the paper. In section 5, numerical examples are given to demonstrate the results.

2. Frequency domain analysis

Consider a linear, time-invariant and asymptotically stable system defined by

$$\dot{x} = Ax + Bu \tag{2.1a}$$

$$y = Cx \tag{2.1b}$$

where x is the nxl state vector, u is the mxl input vector and y is the pxn output vector. $\{A, B, C\}$ are constant matrices of compatible dimensions. It is also assumed that the system (2.1) is both completely controllable and completely observable.

The transfer function matrix of (2.1) is defined by

$$G(s) = C(sI-A)^{-1}B \tag{2.2}$$

The Laurent expansion of (2.2) about $s=\infty$ yields

$$G(s) = \sum_{i=0}^{\infty} M_i s^{-(i+1)} \tag{2.3}$$

where $$M_i = CA^iB \tag{2.4}$$

The coefficient M_i of (2.4) are termed the Markov parameters of (2.1). On the other hand, the Laurent expansion of (2.2) about s=0 yields

$$G(s) = \sum_{i=0}^{\infty} H_i s^i \tag{2.5}$$

where

$$H_j = -CA^{-(j+1)}B, \qquad j=0,1,\cdots \tag{2.6}$$

are called the time moments of (2.1).

It is easily proved that a simplified model having the Markov parameters close to that of the original model will yield accurate transient response. And when the time moments of the simplified model are close to that of the original one, the steady state part of the response will be retained.

3. Controllability balanced representation

In this section the controllability balanced representation is defined. Given the asymptotically stable system model (2.1) which is also assumed to be both controllable and observable, the controllability grammian is defined as

$$W_c = \int_0^{\infty} e^{At}BB'e^{A't}\,dt \tag{3.1}$$

and the observability grammian is defined as

$$W_o = \int_0^{\infty} e^{A't}C'Ce^{At}\,dt \tag{3.2}$$

It is shown (Brockett 1970) that the grammians W_c and W_o are the unique symmetric positive definite matrices which satisfy the following matrix Liapunov equations

$$AW_c + W_cA' = -BB' \tag{3.3}$$

$$A'W_o + W_oA = -C'C \tag{3.4}$$

The range spaces of W_c and W_o span the controllable and observable subspaces respectively.

The grammians are not invariant under the equivalent transformations, and the grammians for the transformed system which is given by

$$\left\{ T^{-1}AT,\ T^{-1}B,\ CT \right\} \tag{3.5}$$

are

$$W_c(T) = T^{-1} W_c (T^{-1})' \tag{3.6}$$

$$W_o(T) = T' W_o T \tag{3.7}$$

The notations $W_c(T)$, $W_o(T)$ denote the grammians of the original system $\{A, B, C\}$ under the transformation

$$x = T x_1 \tag{3.8}$$

To fascilitate future discussion, the eigenvalues for (3.1) are given by σ_{ci} and

$$\sigma_{c1} \geqslant \sigma_{c2} \geqslant . \ . \ . \geqslant \sigma_{cn} > 0 \tag{3.9}$$

and that the eigenvalues for (3.2) are given by σ_{oi} and

$$\sigma_{o1} \geqslant \sigma_{o2} \geqslant . \ . \ . \geqslant \sigma_{on} > 0 \tag{3.10}$$

Definition: The model $\{A, B, C\}$ is controllability balanced if the controllability grammian of the model W_c is such that

$$W_c = I_n \tag{3.11}$$

where I_n is the nth order identity matrix.

Definition: The model $\{A, B, C\}$ is observability balanced if the observability grammian of the model W_o is such that

$$W_o = I_n \tag{3.12}$$

Lemma 1

The controllability (observability) balanced representation is invariant to orthogonal transformation.

Proof

From (3.6), $W_c(T) = T^{-1}W_c(T^{-1})'$,

if $W_c = I_n$

and $TT' = T'T = I_n$

therefore $W_c(T) = T'I_nT = I_n$

and the lemma is proved. The proof for the observability balanced representation is similar.

One can observe from the proof of Lemma 1 that an algorithm which can transform the given system model to the controllability balanced representation is as follows.

Algorithm 1

Step 1 Find the controllability grammian W_c for $\{A, B, C\}$ by solving the matrix Liapunov equation (3.3).

Step 2 Find the decomposition of W_c, i.e.

$$W_c = U_c \Sigma_c^2 U_c'$$

where U_c is orthogonal and

$$\Sigma_c = \mathrm{diag}\ (\sigma_{c1}, \ldots, \sigma_{cn})$$

Step 3 Perform the transformation

$$x = Px_1$$

where $P = U_c\Sigma_c$

The transformed system $\left\{P^{-1}AP,\ P^{-1}B,\ CP\right\}$ will now in the controllability balanced representation. Note that the decomposition of W_c in Step 2 can be performed by means of singular value decomposition.

The feasibility of Algorithm 1 can be easily seen, since

$$W_c = U_c \Sigma_c^{\ 2} U_c' \\ = (U_c \Sigma_c)(U_c \Sigma_c)' \qquad (3.13)$$

therefore

$$(U_c \Sigma_c)^{-1} W_c (U_c \Sigma_c)'^{-1} = I_n \qquad (3.14)$$

The transformation of a given system to the observability balanced representation is similar to the procedure given in Algorithm 1, except that (3.7) is used. Notice that the controllability balanced representation has also been used by Moore (1981) though implicitly.

4. The proposed model reduction procedure

To obtain the simplified model by the present method the original system model $\{A, B, C\}$ in (2.1) is transformed to the controllability balanced representation $\{A_1, B_1, C_1\}$ by means of the transformation

$$x = Tx_1 \tag{4.1}$$

where the nonsingular transformation matrix T is obtained by employing Algorithm 1. A sequence of orthogonal transformation is then performed to $\{A_1, B_1, C_1\}$ to extract the most observable subsystem. An algorithm is given in Patel (1981) to perform the above transformation. The same algorithm was employed in the reduced order modelling procedure by Kwong (1982) which is called the optimal chained aggregation procedure. The optimal chained aggregation procedure will be further discussed later in this section.

4.1 Algorithm for obtaining the reduced model

In this subsection the algorithm through which the reduced order model is obtained is given.

Algorithm 2

Step 1 Transform $\{A, B, C\}$ to the controllability balanced representation $\{A_1, B_1, C_1\}$ by means of the transformation

$$x = T_1 x_1 \tag{4.2}$$

the transformation matrix T_1 is obtained by the Algorithm 1.

Step 2 Find the singular value decomposition of C_1, i.e.

$$C_1 = U_1 \sum V_1' \tag{4.3}$$

Transform $\left\{A_1, B_1, C_1\right\}$ to $\left\{A_2, B_2, C_2\right\}$ by means of the orthogonal transformation

$$x_1 = V_1 x_2 \tag{4.4}$$

where A_2, B_2 and C_2 is given by

$$A_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} \tag{4.5a}$$

$$B_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \tag{4.5b}$$

$$C_2 = \begin{bmatrix} C_{21} & 0 \end{bmatrix} \tag{4.5c}$$

Step 3 Define the residue subsystem as $\left\{A_{R2}, B_{R2}, C_{R2}\right\}$ where $A_{R2}=A_{24}$, $B_{R2}=A_{23}$ and $C_{R2}=A_{22}$. If the norm of C_{R2} is small as compared with the norm of A_1 then the reduced order model is the one defined by $\left\{A_{21}, B_{21}, C_{21}\right\}$. If not, repeat Step 2 to the residue subsystem $\left\{A_{R2}, B_{R2}, C_{R2}\right\}$ until at $k > 2$, the norm of the output matrix C_{Rk} of the residue subsystem $\left\{A_{Rk}, B_{Rk}, C_{Rk}\right\}$ is small as compared with the norm of A_1. Perform the sequence of orthogonal transformations

$$P_1,\ P_2,\ \ldots,\ P_{k-1} \tag{4.6}$$

to $\left\{A_1, B_1, C_1\right\}$; where $P_1 = V_1$ and

$$P_i = \begin{bmatrix} I_i & 0 \\ 0 & V_i \end{bmatrix} \qquad (4.7)$$

V_i composes of the right singular vectors of C_{Ri}, $3 \leq i < k$. I_i is an identity matrix of appropriate order. After performing the sequence of orthogonal transformation (4.6), the system model would be in a form

$$A_k = \left[\begin{array}{cccc:c} A_k^{11} & A_k^{12} & 0 & \cdots & 0 \\ A_k^{21} & A_k^{22} & A_k^{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A_k^{k-1,1} & A_k^{k-1,2} & A_k^{k-1,3} & \cdots & A_k^{k-1,k} \\ \hdashline A_k^{k1} & A_k^{k2} & A_k^{k3} & \cdots & A_k^{kk} \end{array}\right] \qquad (4.8a)$$

$$B_k = \left[\begin{array}{c} B_k^1 \\ B_k^2 \\ \vdots \\ \hdashline B_k^k \end{array}\right] \qquad (4.8b)$$

$$C_k = \left[\begin{array}{ccc:c} C_{21} & 0 & \cdots & 0 \end{array}\right] \qquad (4.8c)$$

And the reduced order model is the one defined by the top left hand block shown in (4.8a), (4.8b) and (4.8c).

If there exists no integer k such that the output matrix C_{Rk} of the residue subsystem $\{A_{Rk}, B_{Rk}, C_{Rk}\}$ is small as compared with the norm of A , then no reduced order model can be claimed.

When the block matrix $A_k^{k-1,k}$ is exactly a null matrix, the system $\left\{A_k, B_k, C_k\right\}$ is not completely observable, the subsystem corresponds to the matrix A_k^{kk} is completely unobservable. Since we have assumed that the given system $\left\{A, B, C\right\}$ is completely controllable and completely observable, $A_k^{k-1,k}$ cannot be a null matrix. However, when the norm of the block matrix $A_k^{k-1,k}$ is small, the subsystem corresponds to A_k^{kk} is said to be almost unobservable. The norm of $A_k^{k-1,k}$ is therefore a measure of the degree of observability of the almost unobservable subsystem. It is now seen that Algorithm 2 corresponds to exploring the almost unobservable subsystem, and the reduced order model is obtained by discarding the almost unobservable subsystem.

4.2 Properties of the reduced order model

We shall investigate some of the properties of the reduced order model obtained by employing the algorithm 2 given in the previous subsection. The procedure by Kwong (1982) will also be considered here. Notice that the only difference between the optimal chained aggregation of Kwong and the present one is that Step 1 in Algorithm 2 is omitted.

4.2.1 Minimality of the reduced order model

When a given realization of a rational transfer function matrix is minimal we mean that the state space of the realization is of minimal order. It is well known that a realization is minimal if and only if it is jointly completely controllable and completely observable. Since the observability of the reduced model is guarenteed by the algorithm of Patel (1981), we now consider the controllability of the reduced model.

Lemma 2

If the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad (4.9)$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is in the controllability balanced representation, the subsystems $\{A_{11}, B_1, C_1\}$ and $\{A_{22}, B_2, C_2\}$ would also be in the controllability balanced representation, under the conditions that the input matrices of the subsystems are not null matrices, and A_{11} , A_{22} are asymptotically stable.

Proof

Since the controllability grammian is the unique symmetrical matrix which satisfies

$$AW_c + W_cA' = -BB'$$

and for the controllability balanced representation, W_c is an identity matrix, i.e. $W_c = I_n$, therefore

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} A_{11}' & A_{21}' \\ A_{12}' & A_{22}' \end{bmatrix} = - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1' & B_2' \end{bmatrix}$$

implies

$$A_{11} + A_{11}' = -B_1B_1'$$

$$A_{22} + A_{22}' = -B_2B_2'$$

and so the lemma is proved.

By means of Lemma 2, we can claim the following.

Property 1

Under the condition that the input matrix of the reduced order model is a non-null matrix, the reduced order model is minimal, if the reduced order model is asymtotically stable.

4.2.2 Effect of scaling on the procedure

Since the system model is generally a mathematical model of a realistic plant, during the process of modelling different units (scaling) or even different system of unit may be employed in modelling the physical plant. Therefore it is important to investigate whether the derived reduced order model would depend upon the scaling used.

Assume that $\left\{\bar{A},\bar{B},\bar{C}\right\}$ and $\left\{A,B,C\right\}$ are two controllability balanced representation of a given system model.
By Lemma 1,

$$\bar{A} = P'AP,\ \bar{B} = P'B,\ \bar{C} = CP \tag{4.10}$$

where P is an orthogonal transformation matrix.

Refer to Algorithm 2, the singular value decompositions of $\bar{C}$ and C are respectively,

$$\bar{C} = (\ \bar{U}\Sigma \,\vdots\, 0\ )\bar{V}',\ \bar{V} = (\ \bar{V}_1 \,\vdots\, \bar{V}_2\ ) \tag{4.11}$$

$$C = (\ U\Sigma \,\vdots\, 0\ )V',\ V = (\ V_1 \,\vdots\, V_2\ ) \tag{4.12}$$

Σ is now considered as a diagonal matrix with the singular values of $\bar{C}$ or C as elements, since $\bar{C}$ and C are related by an orthogonal matrix, they have the same singular values. The corresponding transformed systems after completing the second step of Algorithm 2 (see (4.5)) are given by

$$\begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \\ \bar{A}_{23} & \bar{A}_{24} \end{bmatrix}, \begin{bmatrix} \bar{B}_{21} \\ \bar{B}_{22} \end{bmatrix}, \begin{bmatrix} \bar{C}_{21} & 0 \end{bmatrix} \tag{4.13}$$

$$\begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \begin{bmatrix} C_{21} & 0 \end{bmatrix} \tag{4.14}$$

Since the columns of $\bar{U}$ are the eigenvectors of $\bar{C}\bar{C}'=CPP'C'=CC'$ therefore the columns of $\bar{U}$ are also the eigenvectors of CC'. That is $\bar{U}$ and U are related by

$$\bar{U} = U\, I_{\pm 1} \tag{4.15}$$

where $I_{\pm 1}$ is a diagonal matrix of compatible dimension and the diagonal elements are either +1 or -1, to take into account that the eigenvectors may have opposite directions. Without loss of generality, it is assumed that

$$\bar{U} = U \tag{4.16}$$

This would imply that

$$\bar{C}_{21} = C_{21} \tag{4.17}$$

Moreover, since the columns of $\bar{V}_1$ are the eigenvectors of the matrix $\bar{C}'\bar{C} = P'C'CP$, therefore

$$P\bar{V}_1 = V_1 \tag{4.18}$$

And,

$$\bar{A}_{21} = \bar{V}_1{}'\,\bar{A}\,\bar{V}_1$$

$$= (P'V_1)'(P'AP)(P'V_1)$$

$$= V_1{}'\,A\,V_1$$

$$= A_{21} \qquad (4.19)$$

Similarly,

$$\bar{B}_{21} = \bar{V}_1{}'\,\bar{B}$$

$$= (P'V_1)'PB$$

$$= V_1{}'B$$

$$= B_{21} \qquad (4.20)$$

Therefore we have

$$\bar{A}_{21} = A_{21}\ ,\ \bar{B}_{21} = B_{21} \ \text{ and } \ \bar{C}_{21} = C_{21} \qquad (4.21)$$

If the above consideration is repeated for the subsystems $\left[\bar{A}_{24},\ (\bar{A}_{23} \vdots \bar{B}_{22}),\ \bar{A}_{22}\right]$ and $\left[A_{24},\ (A_{23} \vdots B_{22}),\ A_{22}\right]$ we can assert the following.

Property 2

The reduced order model derived by employing the Algorithm 2 is unique to within a similarity transformation and is independent of the internal coordinate system of the given unreduced system model.

Property 2 implies that we can always obtain the same reduced order model irrespective of the units being used to represent the state variables. While the optimal chained aggregation procedure of reduced order modelling can be easily demonstrated to be dependent on the internal coordinate system of the given system model. An example will be given in the next section to illustrate this point.

4.2.3 The dual of the reduced order modelling procedure

The underlying philosophy in the reduced order modelling procedure given in Algorithm 2 is that the almost unobservable subsystem is assumed to be completely unobservable and is deleted from the state space. But it may so happen that the almost unobservable subsystem may be strongly controllable. The reason for first transforming the given system model to the controllability balanced representation is that, from Lemma 2 every subsystems of the controllability balanced representation is also controllability balanced. That is from the point of view of (3.1), the subsystems have the same degree of controllability. The subsystem which is almost unobservable can then be deleted, since we do not have the risk that the almost unobservable subsystem is strongly controllable.

It is obvious that the dual of the Algorithm 2 can also be used to derive a reduced order model. The reduced order model is obtained by first transforming the system model to the observability balanced representation, and then the most controllable subsystem is extracted by using the dual of algorithm given in (Patel 1981).

4.2.4 The impulse response of the reduced model

From the structure of the transformed system, see (4.8), we see that the Markov parameters of the reduced order model will approximate that of the unreduced ones if the norm of $A_k^{k-1,k}$ is small as compared with that of A_k. From the analysis given in Section 2, we may confirm that the transient response of the reduced order model will approximate that of the unreduced one. If the steady state part of the response of the reduced order model should also approximate that of the unreduced one, the time moments of the reduced order model should also approximate that of the unreduced one. If the transformed system (4.8) is rewritten as $\left\{A_T, B_T, C_T\right\}$,

$$A_T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B_T = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C_T = \begin{bmatrix} C_1 & 0 \end{bmatrix} \tag{4.22}$$

where $\left\{A_{11}, B_1, C_1\right\}$ is the reduced order model obtained by employing Algorithm 2. From (2.6) we see that in order that the time moments of the reduced order model should approximate that of the unreduced one, the inverse of A_T, that is

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} \tag{4.23}$$

should be of good approximation to that of

$$\bar{A}^{-1} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix} \tag{4.24}$$

where

$$\bar{A} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

For $A_T^{-1} \doteqdot \bar{A}^{-1}$, it requires that (Stewart 1973) the norm of

$$\left\| \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} 0 & A_{12} \\ 0 & 0 \end{bmatrix} \right\| \ll 1 \tag{4.25}$$

which is equivalent to the condition

$$\left\| \left(A_{11}^{-1}A_{12} \;\middle|\; -A_{22}^{-1}A_{21}A_{11}^{-1}A_{12} \right) \right\| \ll 1 \tag{4.26}$$

Notice that if $A_T^{-1} \doteqdot \bar{A}^{-1}$, it implies that the almost unobservable subspace is an approximate invariant subspace of the original state space, this is in consistent with that the unobservable subspace is the largest invariant subspace contained in the kernel of the output matrix.

5. Examples

We consider the example used by Kwong (1982). The system is given by

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (5.1)$$

where

$$A = \begin{bmatrix} -0.21053 & -0.10526 & -0.0007378 & 0 & 0.0706 & 0 \\ 1.0 & -0.03537 & -0.000118 & 0 & 0.0004 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & -605.16 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & -3906.25 & -12.5 \end{bmatrix}$$

$$B' = \begin{bmatrix} -7.211 & -0.05232 & 0 & 794.7 & 0 & -448.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.0 & 0 & 0 & 0.000334 & 0 & -0.007728 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(5.2)

and $x' = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}'$ is the state vector.

To demonstrate how the effect of the units (scaling) employed in modelling a physical plant may affect the responses of the reduced order models, the state variable x_4 is substituted by $\hat{x}_4$, where

$$100000\hat{x}_4 = x_4 \qquad (5.3)$$

Notice that (5.3) corresponds to using different units to represent the state variable x_4. With the substitution of

x_4 by $\hat{x}_4$, the system model (5.1) is transformed to

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$$
$$y = C\hat{x} \qquad (5.4)$$

where $\hat{A} = P^{-1}AP$, $\hat{B} = P^{-1}B$ and $\hat{C} = CP$. The transformation matrix P is given by

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.5)$$

and

$$\hat{A} = \begin{pmatrix} -0.21053 & -0.10526 & -0.0007378 & 0 & 0.0706 & 0 \\ 1 & -0.03537 & -0.000118 & 0 & 0.0004 & 0 \\ 0 & 0 & 0 & 100000 & 0 & 0 \\ 0 & 0 & -0.006052 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3906.25 & -12.5 \end{pmatrix}$$

$$\hat{B}' = \begin{pmatrix} -7.211 & -0.05232 & 0 & 0.007947 & 0 & -448.5 \end{pmatrix}$$

$$\hat{C} = \begin{pmatrix} 1 & 0 & 0 & 33.4 & 0 & -0.007728 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (5.6)$$

$\left\{\hat{A}, \hat{B}, \hat{C}\right\}$ is called the scaled system model of (5.1).

If now Algorithm 2 is applied to the scaled system $\left\{\hat{A},\hat{B},\hat{C}\right\}$, we get the transformed system $\left\{\bar{A},\bar{B},\bar{C}\right\}$, where

$$\bar{A}=\left[\begin{array}{cccc|cc} -0.00002 & 0.3336 & -0.06266 & 0.0222 & 0 & 0 \\ -0.3537 & -0.0568 & -4.303 & -0.00032 & 0 & 0 \\ 0.06868 & 4.630 & -0.4702 & -59.48 & 5.923 & 0.364 \\ -0.05495 & -1.777 & 64.59 & -13.9 & 1.4 & 0.0802 \\ \hline -0.01549 & -0.8407 & -3.504 & -14.55 & -3.111 & 25.54 \\ 0.00316 & 0.1715 & -0.8575 & 2.602 & -24.27 & -0.1294 \end{array}\right]$$

$$\bar{B}'=\left[\begin{array}{cccc|cc} 0.006211 & 0.337 & -0.9697 & 5.272 & 2.494 & -0.5088 \end{array}\right]$$

$$\bar{C}=\left[\begin{array}{cccc|cc} -1.224 & -10.30 & 0 & 0 & 0 & 0 \\ -30.71 & 0.4107 & 0 & 0 & 0 & 0 \end{array}\right] \tag{5.7}$$

The fourth order reduced model obtained by coordinate truncation of $\left\{\bar{A},\bar{B},\bar{C}\right\}$ has the eigenvalues $-7.089 \pm j61.769$, $-0.123 \pm j0.312$. The initial part of the impulse response y_1 of the reduced order model is given in Fig. 1.

The transformed system obtained by the optimal chained aggregation procedure of Kwong (1982) is $\left\{\tilde{A},\tilde{B},\tilde{C}\right\}$, where

$$\tilde{A}=\left[\begin{array}{cccc|cc} -4.916 & -0.00315 & -0.5245 & -7.515 & 0 & 0 \\ -0.02993 & -0.03537 & -0.8980 & -0.4389 & 0 & 0 \\ 304 & 0.09451 & 6.563 & 6.864 & 0.3648 & 0.02142 \\ -595.3 & 0.04624 & -13.1 & -13.99 & -0.7519 & -0.0433 \\ \hline -1.293 & 0 & -1717 & 3509 & -4.939 & 26.12 \\ 99950 & -0.00001 & 2688 & 1315 & -23.14 & -0.3453 \end{array}\right]$$

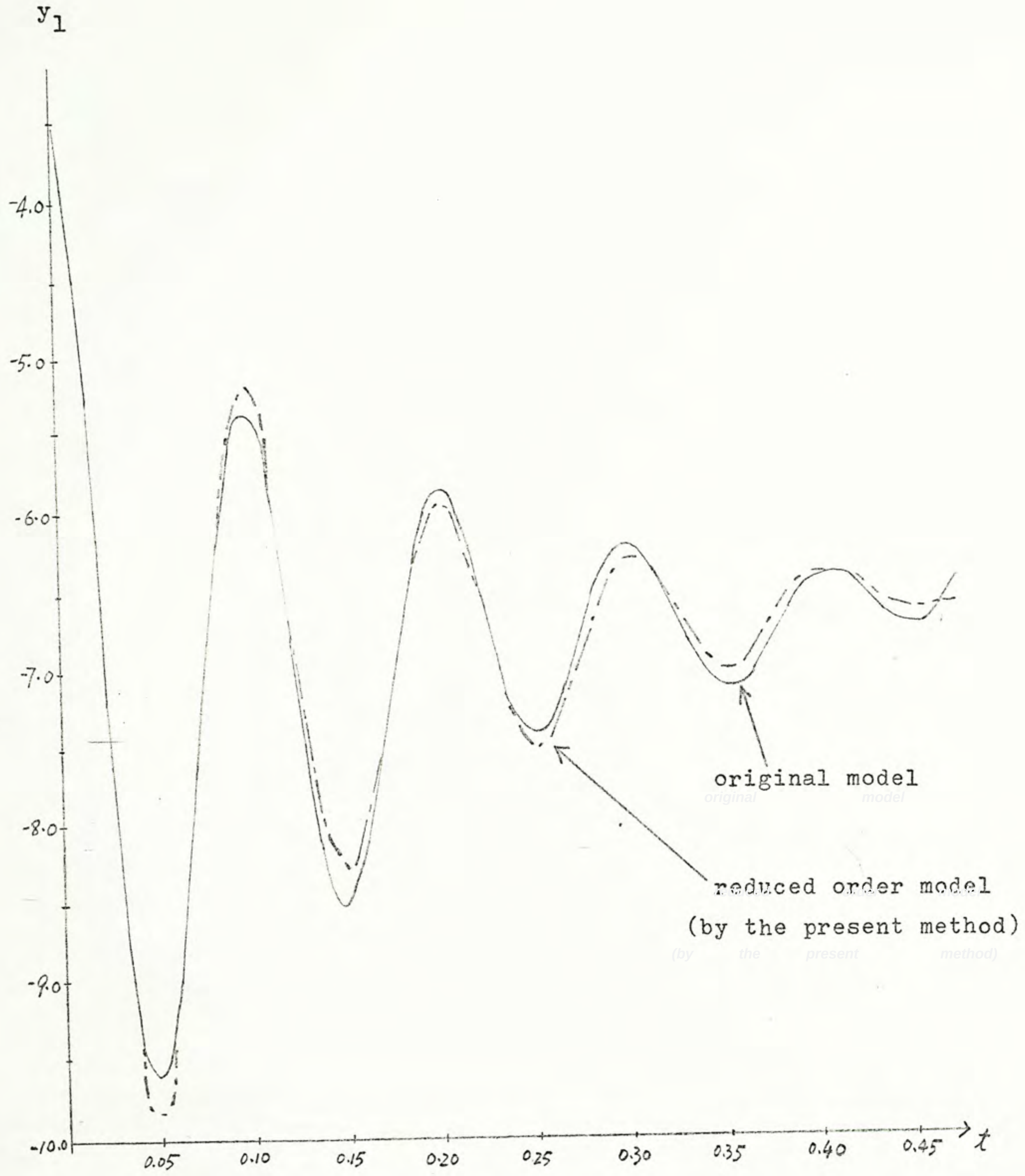
$$\tilde{B}' = \left[-0.1041 \quad -0.05232 \quad 6.859 \quad 2.389 \;\middle|\; 448.5 \quad -0.0008 \right]$$

$$\tilde{C} = \left[\begin{array}{cccc|cc} 33.41 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right] \tag{5.8}$$

The initial part of the impulse response y_1 of the fourth order reduced obtained by optimal chained aggregation is shown in Fig. 2. It is seen that the response of the reduced model deviates greatly from that of the original system model. This example demonstrates that the reduced order modelling procedure by Kwong (1982) may be affected by the scaling (units) employed during the modelling process, since when the procedure is applied to the unscaled system model in the original paper (Kwong 1982) the response of the reduced model is shown to a good fit with that of the unreduced model. While, irrespective to the scaling of the system model the same reduced order model is obtained when the procedure given in Algorithm 2 is employed. In Table 1 the eigenvalues of the different models are shown.

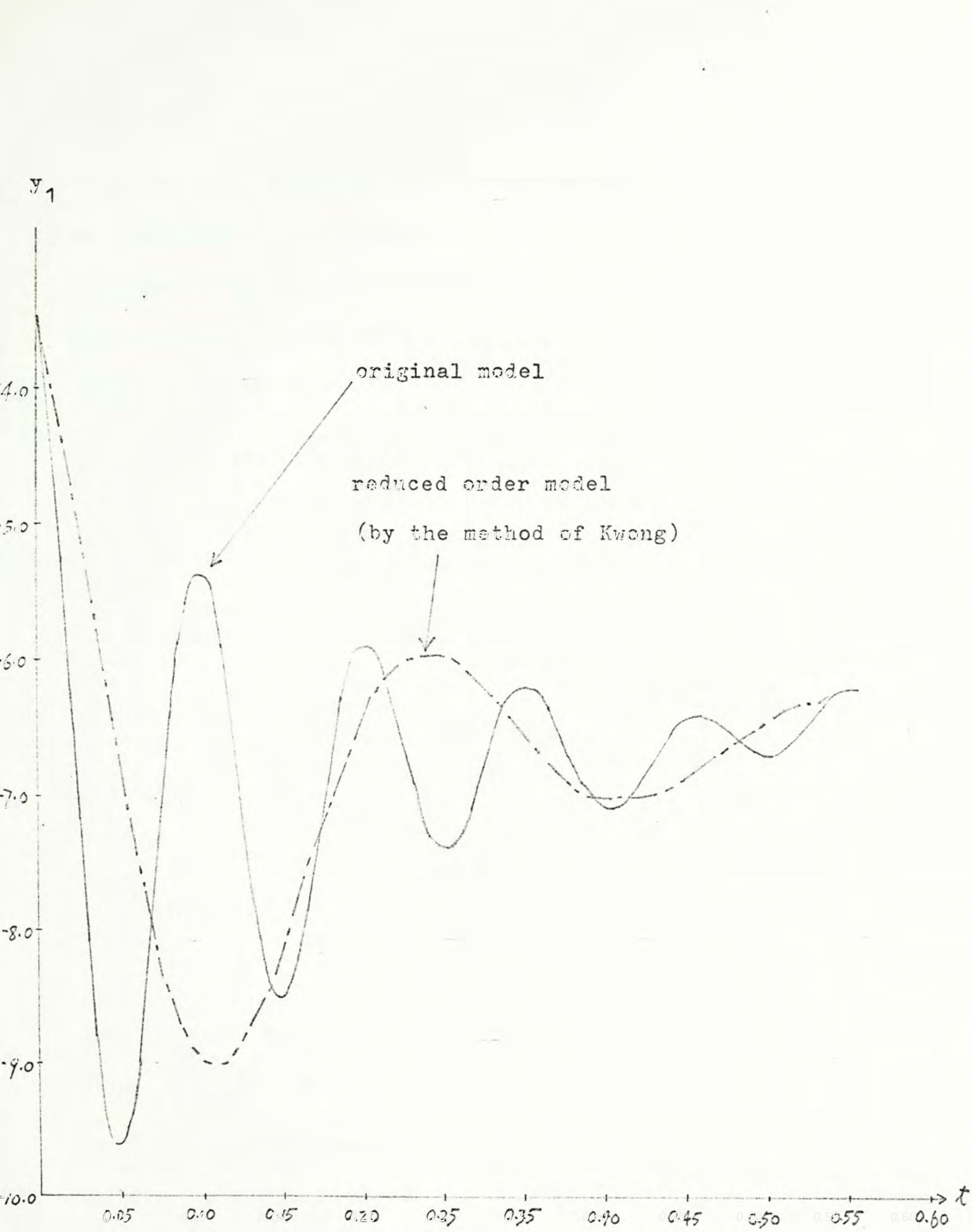
TABLE 1

ORIGINAL MODEL	REDUCED MODEL BY KWONG	REDUCED MODEL BY THE PRESENT METHOD
$-6.25 \pm j62.187$	$-6.068 \pm j24.54$	$-7.089 \pm j61.769$
$-2.46 \pm j24.477$	$-0.123 \pm j0.3124$	$-0.123 \pm j0.312$
$-0.123 \pm j0.3124$		



y_1
-4.0
-5.0
-6.0
-7.0
-8.0
-9.0
-10.0
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
t
original model
reduced order model
(by the present method)

Fig. 1



y_1
original model
reduced order model
(by the method of Kwong)
4.0
5.0
6.0
7.0
8.0
9.0
10.0
0.05
0.10
0.15
0.20
0.25
0.35
0.40
0.45
0.50
0.55
0.60
t

Fig. 2

Example 2

In this example the system model (5.2) given in the previous example is reworked by applying the dual of Algorithm 2 to derive a reduced order model. The transformed system model $\left\{\bar{A},\bar{B},\bar{C}\right\}$ when the dual of Algorithm 2 (section 4.2.3) is applied to the system (5.2) is:

$$\bar{A}=\left[\begin{array}{cccc|cc} -0.00576 & 1.512 & -0.5328 & 0.01583 & -0.324 & -0.04984 \\ -1.41 & -0.4574 & 62.9 & -0.04912 & 2.844 & 0.425 \\ 0 & -58.21 & -12.32 & -15.08 & -14.99 & -2.312 \\ 0 & 0 & 15.86 & -0.209 & -3.091 & 1.263 \\ \hline 0 & 0 & 0 & 3.583 & -4.559 & 23.65 \\ 0 & 0 & 0 & -1.132 & -25.06 & -0.1126 \end{array}\right]$$

$$\bar{B}'=\left[\begin{array}{cccc|cc} -32.42 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\bar{C}=\left[\begin{array}{cccc|cc} 0.1073 & -0.9434 & 4.964 & -0.1569 & 3.019 & 0.4657 \\ 0.00161 & -0.1577 & 0.00073 & 0.6272 & -0.0286 & -0.09133 \end{array}\right] \qquad (5.9)$$

The initial part of the impulse response y_1 of the fourth order reduced model is given in Fig. 3.

The transformed system $\left\{\tilde{A},\tilde{B},\tilde{C}\right\}$ obtained by the dual of the optimal chained aggregation procedure is given by

$$\tilde{A}=\left[\begin{array}{cccc|cc}-6.751 & -4100 & -1352 & -1400 & 84.75 & -0.6987\\ 3.395 & 387.9 & 1317 & 2952 & -220.3 & -14\\ 0 & -119.6 & -405.9 & -910.5 & 67.98 & 4.321\\ 0 & 0 & 3.551 & 8.414 & -0.6589 & -0.0533\\ \hline 0 & 0 & 0 & 12.89 & -1.3209 & -0.2115\\ 0 & 0 & 0 & 0.1334 & 0.9938 & -0.0426\end{array}\right]$$

$$\tilde{B}'=\left[\begin{array}{cccc|cc}912.6 & 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$\tilde{C}=\left[\begin{array}{cccc|cc}-0.003813 & -0.02149 & -0.02028 & 0.08621 & 0.9958 & -0.0079\\ -0.000057 & -0.00244 & -0.00759 & 0.00896 & 0.0069 & 0.9999\end{array}\right] \quad (5.10)$$

The response of the fourth order reduced model obtained by the above procedure is also shown in Fig. 3. We see that the response of the reduced order model deviates greatly from the original one.

From this example and the previous one, we see that merely extracting the most observable or the most controllable subsystem as the reduced order model may not be a reliable method of model reduction. This is due to the possibility that the most observable subsystem so extracted may be least controllable, and the most controllable subsystem may be least observable.

In Table 2 the eigenvalues of the reduced order models are shown.

TABLE 2

Original model	Reduced model by the method of Kwong	Reduced model by the present method
$-6.25 \pm j62.187$	$-5.903 \pm j62.096$	$-6.38 \pm j62.19$
$-2.46 \pm j24.477$	$-2.248 \pm j24.442$	$-0.118 \pm j0.341$
$-0.123 \pm j0.3124$		

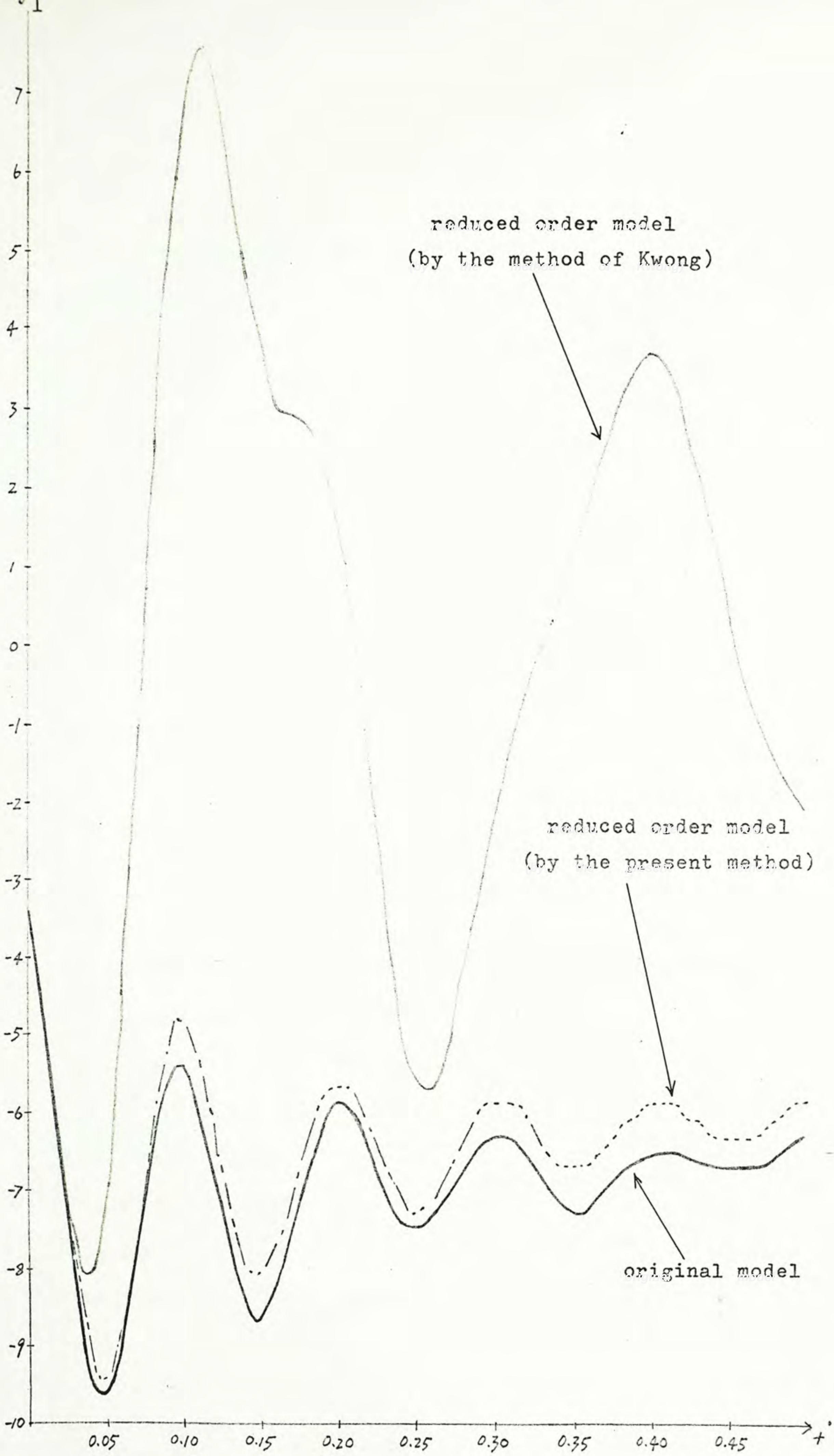

reduced order model
(by the method of Kwong)
reduced order model
(by the present method)
original model
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
t

Fig. 3

Example 3

In this example, the unreduced system model $\left\{ A, B, C \right\}$ is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -33.6 & -155.94 & -209.46 & -102.42 & -18.3 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 & 16 & 20 & 8 & 1 \\ 1 & 2 & 7 & 8 & 1 & 1 \end{bmatrix} \tag{5.11}$$

The transformed system model obtained by applying the Algorithm 2 is given by

$$\bar{A} = \left[\begin{array}{cccc|cc} -9.351 & -0.431 & -6.446 & -1.148 & 0 & 0 \\ -14.057 & -5.613 & -9.263 & 0.799 & 0 & 0 \\ -2.681 & 2.192 & -2.227 & -0.991 & -0.1506 & -0.1123 \\ -1.865 & -3.134 & -0.480 & -0.243 & -0.0138 & -0.1416 \\ \hline -3.48 & 2.696 & 1.849 & -0.547 & -0.3238 & -0.1489 \\ 4.506 & -3.491 & -2.087 & 0.8676 & 0.9814 & -0.5428 \end{array}\right]$$

$$\bar{B}' = \left[\begin{array}{cccc|cc} 4.325 & 3.350 & 2.111 & 0.697 & 0.805 & -1.042 \end{array}\right]$$

$$\bar{C} = \left[\begin{array}{cccc|cc} 0.270 & -0.049 & 0 & 0 & 0 & 0 \\ 0.171 & 0.078 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (5.12)$$

While the transformed system model obtained by means of the optimal chained aggregation procedure is given by

$$\tilde{A} = \left[\begin{array}{cccc|cc} -11.78 & -0.706 & 0.714 & 0.391 & 0 & 0 \\ 67.58 & 6.469 & -6.266 & 0.044 & 0 & 0 \\ 171.5 & 16.42 & -14.62 & -1.444 & -1.61 & -0.1505 \\ -174.3 & -17.08 & 14.37 & 1.544 & 0.56 & -0.5303 \\ \hline -119.7 & -11.65 & 9.964 & 1.178 & 0.2062 & 0.8213 \\ 4.735 & 1.313 & -0.3656 & -0.031 & -0.236 & -0.123 \end{array}\right]$$

$$\tilde{B}' = \left[\begin{array}{cccc|cc} 0.045 & -2.24 & -0.611 & 0.62 & 0.426 & -0.0188 \end{array}\right]$$

$$\tilde{C} = \left[\begin{array}{cccc|cc} 27.08 & 0.855 & 0 & 0 & 0 & 0 \\ 10.74 & -2.155 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (5.13)$$

In Table 3 the eigenvalues of the fourth order reduced order models are shown. And the initial part of the impulse responses are shown in Fig. 4 and Fig. 5. It is seen that the responses of the reduced model obtained by the present method fit so well with that of the unreduced one that the two cannot be distinguished. From this example we can also see that the present method can also be used to reduce the order of a given transfer function.

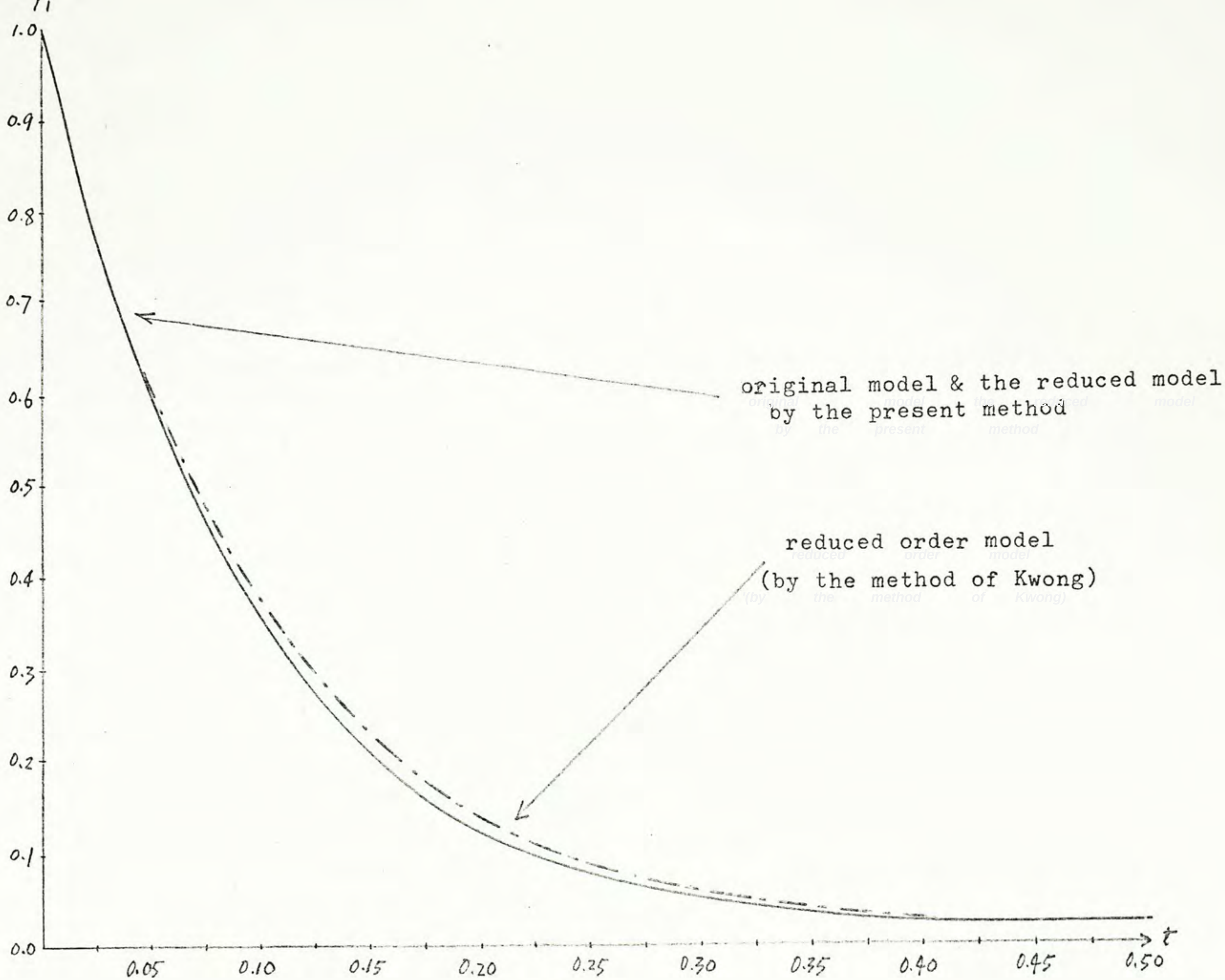

h
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
original model & the reduced model
by the present method
reduced order model
(by the method of Kwong)
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
t

Fig. 4

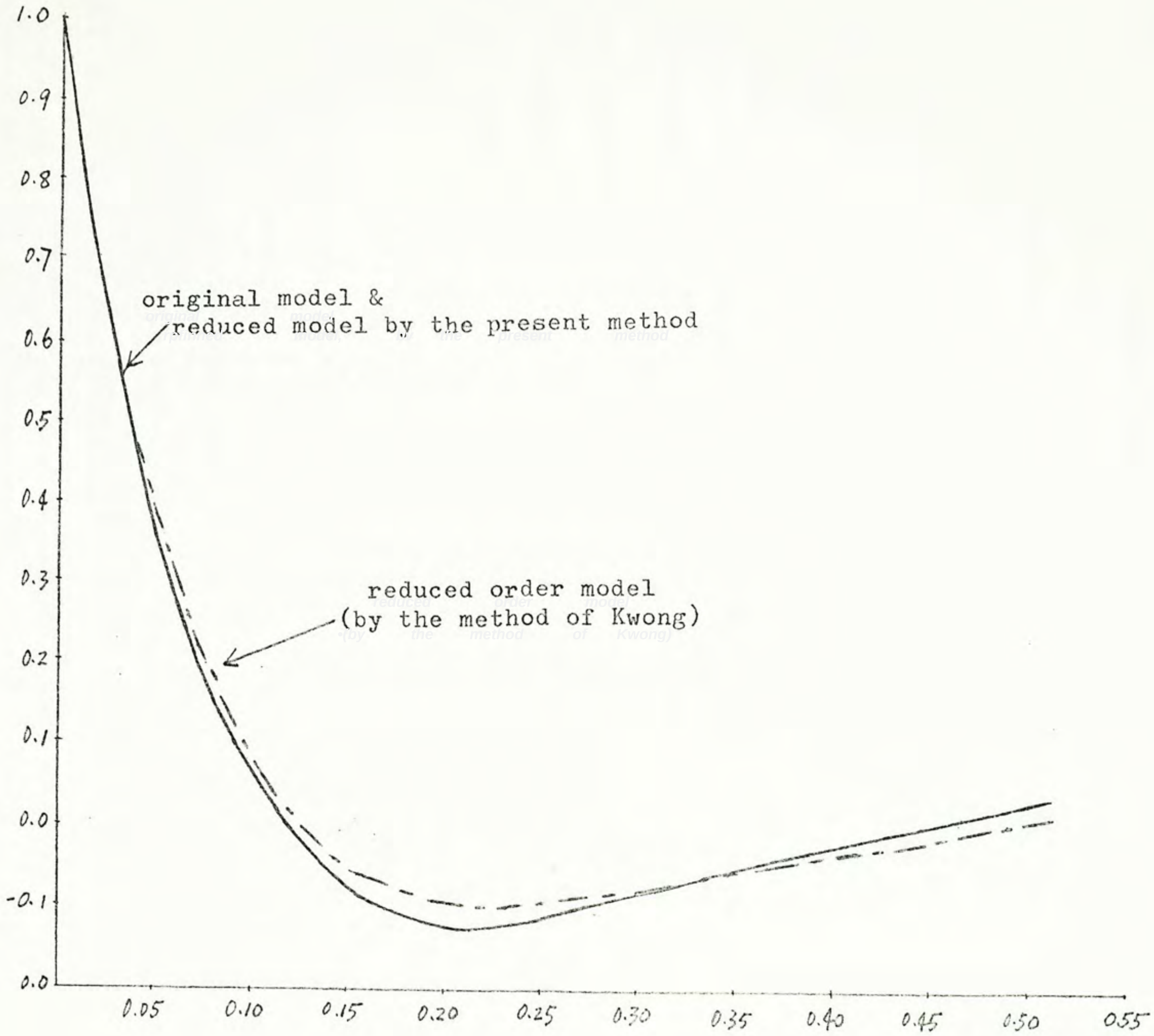

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-0.1
0.0
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
original model &
reduced model by the present method
reduced order model
(by the method of Kwong)

Fig. 5

TABLE 3

Original model	Reduced model by method of Kwong	Reduced model by the present method
-10	-11.73	-10.04
-5	-4.37	-4.73
-2	-2.08	-2.66
-1	-0.2005	-0.0001
-0.2		
-0.1		

6. Discussion

We have demonstated in this chapter that by applying the algorithm of Patel (1981) to the controllability balanced representation (or to its dual) we can obtain a reduced order model whose transient response is close to that of the unreduced given system model. In addition, the present proposed procedure has some very useful properties. It should be noted that the present reduced order modelling procedure cannot in general guarentee the stability of the reduced model. Since the reduced model is in fact an aggregated model (Kwong 1982, Aoki 1968) of the original system, if the original system is stable and if the discarded subsystem is really 'almost unobservable' the stability of the reduced order model can usually be fulfilled.

CHAPTER 3

SUBOPTIMAL CONTROL

1. Introduction

In this chapter we shall consider the near optimal solution of the optimal linear regulator problem. It is well known that the determination of the optimal state feedback control law involves the solution of the matrix Riccati equation, which corresponds to the solution of a set of scalar nonlinear equations. The number of equations in the set increases with the square of the order of the given system to be regulated. This would mean a large amount of computational effort and cost, hence there is a need to derive a suboptimal control law for the regulator problem through an approximate model of the given system.

To fascilitate future discussion, the regulator problem is first defined in section 2. In section 3 we shall discuss how a suboptimal control law can be derived through an almost uncontrollable model. In section 4 we shall review the balanced representation of Moore (1981) which we shall make use of. The proposed method is given in section 5. And numerical examples are given in section 6. In section 7 another procedure is proposed which can further improve the performance of the suboptimal control law for systems which have dominant fast modes.

2. The regulator problem

We shall briefly review the linear quadratic regulator problem in this section.

Consider the system

$$\dot{x} = Ax + Bu, \quad x(0)=x_0 \tag{2.1}$$

where x is an nx1 state vector, u is a mx1 control vector. The matrices A and B are constant matrices of compatible dimensions. The quadratic regulator problem is to establish a state feedback control law

$$u = -Fx \tag{2.2}$$

where F is a constant mxn matrix, such that it is required to control the system at the set point $x_d=0$ under the constraint that the cost functional

$$J = \int_0^\infty (x'Qx + u'Ru)\, dt \tag{2.3}$$

is minimized. The input weighting matrix R is a positive definite symmetric matrix, and the state weighting matrix Q is a positive semidefinite symmetric matrix. Since Q is positive semidefinite we can decompose Q as

$$Q = H'H \tag{2.4}$$

If the pair (A,B) is completely controllable and (A,H) is completely observable, the feedback matrix F is given by

$$F = R^{-1}B'M \tag{2.5}$$

where M is the unique positive definite symmetric matrix solution of the Riccati equation

$$0 = A'M+MA-MBR^{-1}B'M+Q \tag{2.6}$$

and the optimal cost is given by

$$J = x_0{}'Mx_0 \tag{2.7}$$

If we define

$$y = Hx \tag{2.8}$$

as the implicit output vector for the system (2.1), we see that the cost functional (2.3) can be rewritten as

$$J = \int_0^{\infty} (y'y + u'Ru)\, dt \tag{2.9}$$

Notice that the implicit output vector offers a machinery through which we can take the effect of the state weighting matrix Q in the cost functional (2.3) in our suboptimal control law.

3. Almost uncontrollable system and suboptimal control

In this section we shall investigate how we can obtain a suboptimal control law for an almost uncontrollable system. Kwong (1983) employed a similar method to derive the suboptimal control law, though with different setting. We shall first consider how the optimal control law can be obtained for a system which is not completely controllable.

3.1 Optimal control and not completely controllable system

It is known that if the given system (2.1) is not controllable it may not be possible to obtain M by solving (2.6). It was however shown that (Dressler and Larson 1969) under some additional assumptions one can obtain the optimal control.

We assume here that the system (2.1) is not completely controllable, then there exists a transformation

$$x = T\hat{x} \tag{3.1}$$

for some nonsingular matrix T, such that the transformed system given by $\left\{ \hat{A},\ \hat{B} \right\}$, where

$$\hat{A} = T^{-1} A\ T, \quad \hat{B} = T^{-1} B \tag{3.2}$$

and $\hat{A}$, $\hat{B}$ have the forms

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \qquad B = \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix} \tag{3.3}$$

Dressler and Larson (1969) showed that if $\{\hat{A}_{11}, \hat{H}_1\}$ is observable, and $\hat{A}_{22}$ is a stable matrix, then the solution of the regulator problem for the not completely controllable system given in the form (3.3) corresponds to the solution of the Riccati equation

$$0 = \hat{A}_{11}'p_1 + p_1\hat{A}_{11} - p_1\hat{B}_{11}R^{-1}\hat{B}_{11}'p_1 + \hat{Q}_1 \quad (3.4)$$

for p_1, where $\hat{Q}_1 = \hat{H}_1'\hat{H}_1$

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ \hat{Q}_2' & \hat{Q}_4 \end{bmatrix}, \qquad \hat{Q} = T'Q\,T \quad (3.5)$$

And, at the same time, p_2 is solved in the following equation,

$$0 = (\hat{A}_{11} - \hat{B}_{11}R^{-1}\hat{B}_{11}'p_1)'p_2 + p_2\hat{A}_{22} + p_1\hat{A}_{12} + \hat{Q}_2 \quad (3.6)$$

where p_1 is the solution of the Riccati equation (3.4). The feedback matrix for the optimal control law (2.2) is then given by

$$F = R^{-1}\hat{B}_{11}\left[\, p_1 \,\vdots\, p_2 \,\right] T^{-1} \quad (3.7)$$

3.2 Almost uncontrollable system and suboptimal control

We now consider the case where the given system (2.1) is completely controllable. It is not possible to find the transformation matrix T to transform the system to the form (3.3). But if there exists a transformed system of S, $\hat{S}$, such that the states are so arranged that we can partition the system into two subsystems $\hat{S}_1$ and $\hat{S}_2$ whereas one of the subsystems say, $\hat{S}_2$, is 'almost uncontrollable' then we can ignor the input coupling of the subsystem $\hat{S}_2$ and at the same time neglect the coupling $\hat{A}_{21}$ between the subsystems $\hat{S}_1$ and $\hat{S}_2$. The notion of 'almost uncontrollability' means that a system is completely controllable, but small perturbation of the system parameters may render the system uncontrollable. A system is said to be almost uncontrollable if the almost uncontrollable subsystem exists. The above consideration is depicted in Fig. 1. After setting $\hat{B}_{22}=0$, and $\hat{A}_{21}=0$ we can then apply the result of Section 3.1 to derive a suboptimal control law for the original regulator problem. In order that the above discussion be a sensible one, it is important that we should have a measure of the degree of controllability to enable us to identify the 'almost uncontrollable subsystem', this is given in the following section.

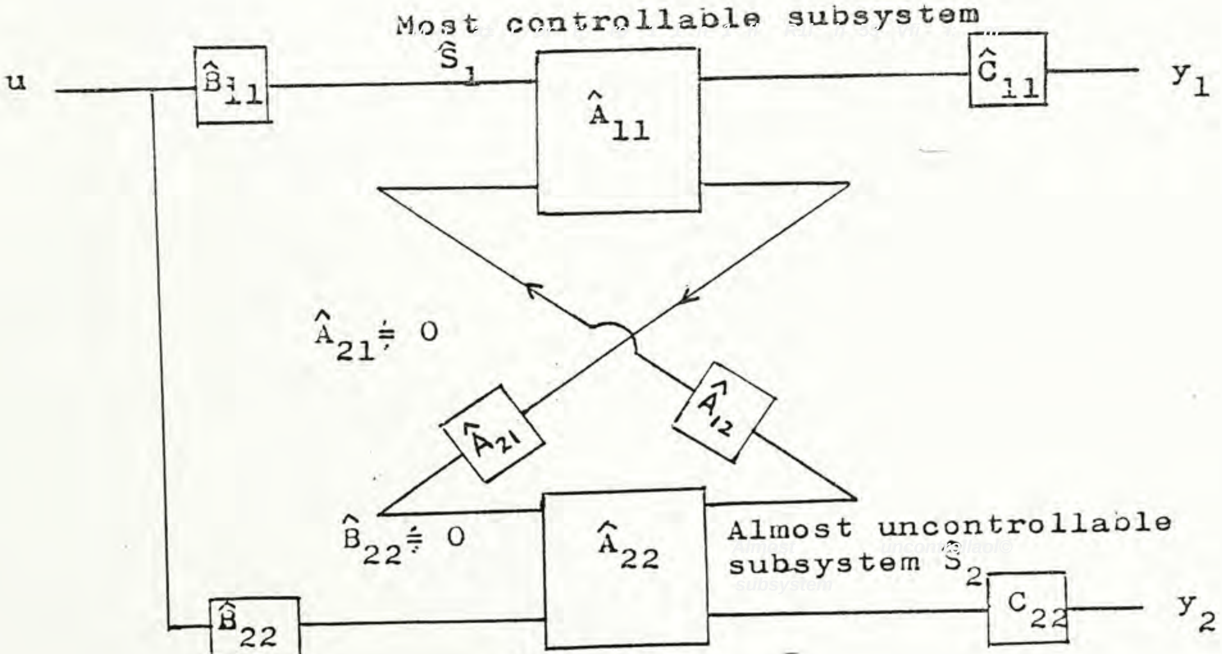


Fig. 1

4. The balanced representation (Moore 1981)

We shall first define in the following the balanced representation.

Definition

A completely controllable and completely observable asymptotically stable system is said to be in its balanced representation if the controllability grammian and the observability grammian of the system are equal and are diagonal.

Consider the asymptotically stable system $\{A,B,C\}$ which is assumed to be both completely controllable and completely observable, we shall demonstrate how to transform $\{A,B,C\}$ to its balanced representation in the following algorithm. Noting that the notations $W_c(T)$, $W_o(T)$ denote the grammians of the given system $\{A,B,C\}$ under the transformation $x=Tx_1$, where x is the state vector of $\{A,B,C\}$ and x_1 is that for the transformed system. And

$$W_c(T) = T^{-1}W_c(T^{-1})' \tag{4.1}$$

$$W_o(T) = T'W_oT \tag{4.2}$$

W_c, W_o are the controllability grammian and observability for the original system $\{A,B,C\}$.

Algorithm 1

Step 1 Solve

$$AW_c + W_cA' = -BB' \tag{4.3}$$

for W_c.

Step 2 Find the singular value decomposition of W_c, that is

$$W_c = V_c \Sigma_c^2 V_c' \qquad (4.4)$$

Step 3 Perform the transformation $x=T_1 x_1$, where

$$T_1 = V_c \Sigma_c \qquad (4.5)$$

and the transformed system is given by $\{A_1, B_1, C_1\}$ where $A_1=T_1^{-1}AT_1$, $B_1=T^{-1}B$ and $C_1=CT_1$.

Step 4 Solve

$$W_o(T_1)A_1 + A_1'W_o(T_1) = -C_1'C_1 \qquad (4.6)$$

for $W_o(T_1)$.

Step 5 Find the singular value decomposition of $W_o(T_1)$, that is,

$$W_o(T_1) = V_o \Sigma^2 V_o' \qquad (4.7)$$

where

$$\Sigma = \text{diag } (\sigma_1 , \ldots ,\sigma_n) \qquad (4.8)$$

Step 6 Perform the transformation $x_1=T_2x_2$, where

$$T_2 = V_o \Sigma^{-\frac{1}{2}} \qquad (4.9)$$

and the transformed system $\{\hat{A}, \hat{B}, \hat{C}\}$, where $\hat{A}=T_2^{-1}A_1T_2$, $\hat{B}=T_2^{-1}B_1$ and $\hat{C}=C_1T_2$, is now in the balanced representation.

Notice that step 1 to step 3 in the above algorithm correspond to transforming the system $\left\{A,B,C\right\}$ to the controllability balanced representation defined in Chapter 2.

The controllability grammian and observability grammian for the balanced representation $\left\{\hat{A},\hat{B},\hat{C}\right\}$ is now given by

$$\int_0^\infty e^{\hat{A}t}\hat{B}\hat{B}'e^{\hat{A}'t}\,dt = \int_0^\infty e^{\hat{A}'t}\hat{C}'\hat{C}e^{\hat{A}t}\,dt = \sum \qquad (4.10)$$

where $\sum = \mathrm{diag}\,(\sigma_1, \ldots, \sigma_n)$, $\sigma_i > \sigma_{i+1} > 0$.

After introducing the balanced representation, we are now ready to give our proposed method of suboptimal control for the regulator problem in the next section.

5. THE PRESENT METHOD

In this section the computation of the suboptimal control law is proposed. The system

$$\dot{x} = Ax + Bu, \; x(0) = x_o \tag{5.1}$$

is assumed to be asymptotically stable. The cost functional is given by

$$\int_0^\infty (x'Qx + u'Ru) \, dt \tag{5.2}$$

Before the proposed method is given in section 5.3, we shall show how the effect of the weighting matrices Q and R can be taken into account in the system model, and thus simplifies the linear quadratic regulator problem.

5.1 State weighting matrix and the implicit output vector

It has been pointed out in section 2 that the state weighting matrix Q in the cost functional (5.2) is to reflect the relative importance of keeping the states near the origin of the state space. It has been shown that an implicit output vector is defined as

$$y = Hx \tag{5.3}$$

where

$$Q = H'H \tag{5.4}$$

With the help of the implicit output vector (5.3) the cost functional can then be given as

$$\cdot \int_0^\infty (y'y + u'Ru) \, dt \tag{5.5}$$

and the system is to be defined as

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Hx \end{aligned} \tag{5.6}$$

It is worth noting the decomposition Q = H'H in (5.4) is not unique (Strang 1980). But we shall show in a later section that the nonuniqueness of the decomposition of Q does not affect our future consideration in any way. In fact the implicit output vector or the output matrix is not involved in our computation of the suboptimal control law. The implicit output vector is merely a conceptual tool.

5.2 Input weighting matrix and the modified input matrix

In this subsection we shall show how the linear quadratic regulator can be further simplified.

Consider (5.5), since R is a symmetric positive definite matrix, we can decompose R as

$$R = D'D \tag{5.7}$$

where the matrix D is nonsingular. By putting

$$\bar{u} = Du \tag{5.8}$$

$$\bar{B} = BD^{-1} \tag{5.9}$$

we can rewrite (5.5) and (5.6) as

$$\begin{aligned}\dot{x} &= Ax + BD^{-1}Du \\ &= Ax + \bar{B}\bar{u}\end{aligned} \tag{5.10}$$

$$y = Hx \tag{5.11}$$

and the cost functional is

$$\int_0^\infty (y'y + \bar{u}'\bar{u}) \, dt \tag{5.12}$$

We shall call $\bar{u}$ and $\bar{B}$ as the modified input vector and the modified input matrix respectively.

One can interpret (5.8) as merely using different units for the input variables. Since D is nonsingular, the optimal control law for (5.10) - (5.12) corresponds to that of (5.1) and (5.2) by the relation

$$u = D^{-1}\bar{u} \tag{5.13}$$

Therefore once the control law for (5.10) - (5.12) is obtained, the control law for (5.1) and (5.2) is readily found by using the relation (5.13). In (5.10) - (5.12) the weighting of the input variables R in the cost functional is taken into account by the modified input matrix $\bar{B}$.

To end this subsection we shall show how the nonsingular transformation for the system (5.10) - (5.11) may affect the control law. It is easily seen that the cost functional (5.12) is invariant to the nonsingular transformation

$$x = Tx_1 \tag{5.14}$$

If the control law for the regulator problem of the transformed system $\left\{ T^{-1}AT,\ T^{-1}B,\ HT \right\}$ is given by

$$u = -F_1x_1 \tag{5.15}$$

F_1 is related to that of the control law for the original system $u = -Fx$ by the relation

$$F = F_1T^{-1} \tag{5.16}$$

(5.16) is obtained by putting (5.14) to (5.15).

5.3 Derivation of the suboptimal control law

The results of section 5.1 and 5.2 show that the original regulator problem can be switched to that of the form given by (5.10) - (5.12). If the suboptimal control law is derived by approximating the almost uncontrollable subsystem of the original system (if it exists) as being completely uncontrollable, the performance degradation can then be seen to be minimized if the almost uncontrollable subsystem of the original system is also almost unobservable.

To explore the almost uncontrollable subsystem which is also almost unobservable, the system (5.10)-(5.11) is first transformed to the balanced representation by means of the algorithm given in section 4. The controllability grammian (observability grammian) of the balanced representation is given by

$$W_o = W_c = \text{diag}\ (\sigma_1, \ldots, \sigma_n) \qquad (5.17)$$

where $\sigma_i > \sigma_{i+1} > 0$. If there exists k, $1<k<n$ such that

$$\sigma_k \gg \sigma_{k+1} \qquad (5.18)$$

we say that the almost uncontrollable subsystem exists, and is of order n-k. Notice that (5.18) is a measure of the degree of controllability (and observability). The procedure to derive the suboptimal control law is given below.

Algorithm 2

Step 1 Find the implicit output matrix H by decomposing

$$Q = H'H \qquad (5.19)$$

Step 2 Find the modified input matrix $\bar{B}$ by decomposing

$$R = D'D \qquad (5.20)$$

and the modified input matrix is given by

$$\bar{B} = BD^{-1} \qquad (5.21)$$

Step 3 Perform the algorithm 1 to transform $\{A, \bar{B}, H\}$ to its balanced representation $\{\hat{A}, \hat{B}, \hat{C}\}$ by the nonsingular transformation

$$x = Tx_1 \qquad (5.22)$$

Step 4 Identify the presence of the almost uncontrollable subsystem. If there exists k, such that (5.18) is satisfied, then the almost uncontrollable subsystem is that defined by

$$\left\{\hat{A}_{22}, \hat{B}_{22}, \hat{C}_{22}\right\} \qquad (5.23)$$

where $\{\hat{A}, \hat{B}, \hat{C}\}$ is partitioned as

$$\begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array}\right] \end{array} \qquad \begin{array}{cc} \left[\begin{array}{c} \hat{B}_{11} \\ \hline \hat{B}_{22} \end{array}\right] & \begin{array}{c} k \\ n-k \end{array} \end{array} \qquad \begin{array}{c} \left[\begin{array}{c|c} \hat{C}_{11} & \hat{C}_{22} \end{array}\right] \\ \begin{array}{cc} k & n-k \end{array} \end{array}$$

Step 5 Use the result given in Section 3 to derive a suboptimal feedback law $\bar{u}=-F_1x_1$ for the balanced representation $\{\hat{A},\hat{B},\hat{C}\}$ by setting $\hat{A}_{21}=0$, $\hat{B}_{22}=0$.

Step 6 By means of the result given in the subsection 5.2, the suboptimal feedback control law $u=-Fx$ for the original regulator problem is that defined by

$$F=D^{-1}F_1T^{-1} \tag{5.28}$$

In Step 3, the transformation of the modified system to its balanced representation, neither the matrix $\bar{B}$ nor the matrix H need to be known explicitly. Rather it is the matrices

$$\bar{B}\bar{B}'=BR^{-1}B' \tag{5.29}$$

and

$$H'H=Q \tag{5.30}$$

which are involved in the transformation, see (4.3) and (4.6) in Algorithm 1. This confirms that the nonuniqueness of the decompositions (5.19) and (5.20) do not affect the resulting suboptimal control law.

Notice that the subsystems $\{\hat{A}_{11},\hat{B}_{11},\hat{C}_{11}\}$ and $\{\hat{A}_{22},\hat{B}_{22},\hat{C}_{22}\}$ are stable subsystems, see (Moore 1980) and (Pernebo and Silverman 1981).

6. EXAMPLES

Example 1

Consider the voltage regulator problem (Lamba and Rao 1974) for which the system is given by:

$$A = \begin{bmatrix} -0.2 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1.6 & 0 & 0 \\ 0 & 0 & -14.28 & 85.71 & 0 \\ 0 & 0 & 0 & -25 & 75 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix} \qquad (6.1a)$$

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 & 30 \end{bmatrix} \qquad (6.1b)$$

and the state weighting matrix is given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6.1c)$$

therefore the implicit output matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6.1d)$$

and the input weighting matrix R is given by

$$R = 1 \qquad (6.1e)$$

The balanced representation of $\{A, B, H\}$ is given by

$$\hat{A} = \begin{bmatrix} -0.0838 & 0.2159 & -0.0688 & 0.0199 & -0.0043 \\ -0.2159 & -0.5862 & 0.4541 & -0.1299 & 0.0282 \\ -0.0688 & -0.4541 & -6.933 & 4.19 & -0.8706 \\ -0.0199 & -0.1299 & -4.19 & -14.23 & 6.321 \\ -0.0043 & -0.0282 & -0.8706 & -6.321 & -28.15 \end{bmatrix} \quad (6.2a)$$

$$\hat{B}' = [\; 6.58 \quad 7.073 \quad 2.706 \quad 0.7842 \quad 0.1701 \;] \quad (6.2b)$$

$$\hat{C} = [\; 6.58 \quad -7.073 \quad 2.706 \quad -0.7842 \quad 0.1701 \;] \quad (6.2c)$$

The observability grammian and controllability grammian of the balanced representation is

$$W_o = W_c = \text{diag}\,(\; 258.2 \quad 42.67 \quad 0.528 \quad 0.02161 \quad 0.0005137 \;) \quad (6.3)$$

We see from the above that the most controllable subsystem may be defined as the third order subsystem given by

$$\hat{A}_{11} = \begin{bmatrix} -0.0838 & 0.2159 & -0.0688 \\ -0.2159 & -0.5862 & 0.4541 \\ -0.0688 & -0.4541 & -6.933 \end{bmatrix} \quad (6.4a)$$

$$\hat{B}_{11} = [\; 6.58 \quad 7.073 \quad 2.706 \;] \quad (6.4b)$$

$$\hat{C}_{11} = [\; 6.58 \quad -7.073 \quad 2.706 \;] \quad (6.4c)$$

and the almost uncontrollable subsystem is then given by

$$\hat{A}_{22} = \begin{bmatrix} -14.23 & 6.321 \\ -6.321 & -28.15 \end{bmatrix} \qquad (6.5a)$$

$$\hat{B}_{22} = \begin{bmatrix} 0.7842 & 0.1701 \end{bmatrix} \qquad (6.5b)$$

$$\hat{C}_{22} = \begin{bmatrix} -0.7842 & 0.1701 \end{bmatrix} \qquad (6.5c)$$

By assuming $\hat{A}_{21}=0$ and $\hat{B}_{22}=0$ and following the procedure given in algorithm 2, the suboptimal control law is given by $\hat{u}=-\hat{F}x$, where

$$\hat{F} = \begin{bmatrix} 0.9248 & 0.1707 & 0.01528 & 0.04637 & 0.2786 \end{bmatrix} \qquad (6.6)$$

and the optimal feedback law is given by $u=-Fx$, where

$$F = \begin{bmatrix} 0.9049 & 0.1705 & 0.01611 & 0.04931 & 0.2656 \end{bmatrix} \qquad (6.7)$$

To compare the performance of the suboptimal feedback control law with that of the optimal one, the performance index given in reference 10 (Levine and Athans 1970) is used. That is if P_s is the solution

$$0 = (A-B\hat{F})'P_s + P_s(A-B\hat{F}) + \hat{F}'R\hat{F} + Q \qquad (6.8)$$

and P_o is the solution of

$$0 = (A-BF)'P_o + P_o(A-BF) + F'RF + Q \qquad (6.9)$$

then the closeness of the trace of P_s as compared with the trace of P_o assesses the performance degradation due to the application of the suboptimal control law to the original regulator problem.

For the present example,

$$\text{trace } P_o = 0.3777 \tag{6.10}$$

and

$$\text{trace } P_s = 0.3778 \tag{6.11}$$

which shows that the performance degradation with the suboptimal control law $\hat{u}(t)$ is less than 0.026 per cent.

From the controllability grammian we see that considering the second order subsystem

$$\hat{A}_{11'} = \begin{bmatrix} -0.0838 & 0.2159 \\ -0.2159 & -0.5862 \end{bmatrix} \tag{6.12a}$$

$$\hat{B}_{11'}' = \begin{bmatrix} 6.58 & 7.073 \end{bmatrix} \tag{6.12b}$$

$$\hat{C}_{11'} = \begin{bmatrix} 6.58 & -7.073 \end{bmatrix} \tag{6.12c}$$

as the most controllable subsystem is also acceptable. The suboptimal feedback matrix is given by

$$\begin{pmatrix} 0.9165 & 0.1862 & 0.01996 & 0.06661 & 0.4382 \end{pmatrix} \tag{6.13}$$

the performance index for the suboptimal control law by considering the second order most controllable susbsystem is:

$$\text{trace } P_{s'} = 0.3887 \; . \tag{6.14}$$

The performance degradation for the present case is 2.91 per cent.

Notice that if the procedure of Kwong (1983) is being used to extract the suboptimal control law, no acceptable suboptimal control law can be obtained, since the transformed system model by using the procedure is

$$\bar{A} = \begin{bmatrix} -10 & 0 & 0 & 0 & 0 \\ 75 & -25 & 0 & 0 & 0 \\ 0 & 85.71 & -14.28 & 0 & 0 \\ 0 & 0 & -1.6 & -0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -0.2 \end{bmatrix} \qquad (6.15a)$$

$$\bar{B}' = \begin{bmatrix} 30 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6.15b)$$

and the state weighting matrix for the transformed system is

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6.15c)$$

It can be seen from the state weighting matrix $\bar{Q}$ for the transformed system that any 'most controllable' subsystem of order less than 5 in the sense of Kwong (1983) has a corresponding state weighting matrix which is a null matrix. Therefore, it is obvious that no approximate system can be obtained by the procedure of Kwong (1983) for the present example.

It is well known that the optimal linear regulator should correspond to an optimal set of eigenvalues. In Table 1 the eigenvalues for the optimal and suboptimal regulators are shown.

TABLE 1

	Optimal regulator	Suboptimal regulator (with 2nd order most controllable subsystem)	Suboptimal regulator (with 3rd order most controllable subsystem)
Eigenvalues	-3.82+j4.85 -3.82-j4.85 -25 -13.82 -11.48	-3.004+j3.403 -3.004-j3.403 -26.17 -15.47+j6.834 -15.47-j6.834	-3.634+j5.132 -3.634-j5.132 -24.02 -18.05 -9
Performance index (degradation)	0.3777	0.3887 (3 %)	0.3778 (0.03 %)

(Kwong's procedure fails in this example)

Example 2

Here we consider the example given in (Medanic et al. 1978). The system model is given by

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

where

$$A = \begin{bmatrix} -0.21053 & -0.10526 & -0.0007378 & 0 & 0.0706 & 0 \\ 1.0 & -0.03537 & -0.000118 & 0 & 0.0004 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & -605.16 & -4.92 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & 0 & 0 & 0 & -3906.25 & -12.5 \end{bmatrix} \tag{6.16a}$$

$$B' = \begin{bmatrix} -7.211 & -0.05232 & 0 & 794.7 & 0 & -448.5 \end{bmatrix} \tag{6.16b}$$

$$C = \begin{bmatrix} 1.0 & 0 & 0 & 0.000334 & 0 & -0.007728 \\ 0 & 1.0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.16c}$$

and the input weighting matrix is given by

$$R = 1 \tag{6.16d}$$

For the present example it is the output y of the given system that is to be regulated, and the weighting matrix for the output variables is an identity matrix.

The balanced representation of $\{A, B, C\}$ is given by

$$\hat{A} = \begin{bmatrix} -0.0821 & 0.303 & 0.0251 & 0.0007 & 0.001 & -0.0064 \\ -0.3275 & -0.1638 & 0.0752 & 0.0026 & 0.0028 & -0.0192 \\ -0.0949 & -0.185 & -12.67 & -62.78 & -1.235 & 5.523 \\ 0.0031 & 0.006 & 62.78 & -0.0139 & -0.0278 & 0.2664 \\ 0.0037 & 0.0073 & 1.235 & -0.0279 & -0.1024 & 24.49 \\ 0.0244 & 0.0476 & 5.523 & -0.2665 & -24.49 & -4.629 \end{bmatrix}$$

(6.17a)

$$\hat{B}' = \begin{bmatrix} -3.247 & -3.326 & -1.875 & 0.0617 & 0.0738 & 0.4827 \end{bmatrix}$$

(6.17b)

$$\hat{C} = \begin{bmatrix} 0.8505 & 1.34 & -1.875 & -0.0616 & -0.0738 & 0.4827 \\ 3.134 & -3.044 & 0 & 0.0023 & -0.0015 & 0 \end{bmatrix}$$

(6.17c)

The observability grammian and the controllability grammian of the balanced system (6.17) is given by

$$W_o = W_c = \text{diag}\ (64.2,\ 33.77,\ 0.139,\ 0.137,\ 0.0266,\ 0.0252)$$

(6.18)

The most controllable subsystem is defined to be the fourth order subsystem given by

$$\hat{A}_{11} = \begin{bmatrix} -0.0821 & 0.303 & 0.0251 & 0.0007 \\ -0.3275 & -0.1638 & 0.0752 & 0.0026 \\ -0.0949 & -0.185 & -12.67 & -62.78 \\ 0.0031 & 0.006 & 62.78 & -0.0139 \end{bmatrix} \qquad (6.19a)$$

$$\hat{B}_{11} = \begin{bmatrix} -3.247 & -3.326 & -1.875 & 0.0617 \end{bmatrix} \qquad (6.19b)$$

$$\hat{C}_{11}=\begin{bmatrix} 0.8505 & 1.34 & -1.875 & -0.0616 \\ 3.134 & -3.044 & 0 & 0.0023 \end{bmatrix} \qquad (6.19c)$$

and the almost uncontrollable subsystem is that given by

$$\hat{A}_{22}=\begin{bmatrix} -0.1024 & 24.49 \\ -24.49 & -4.629 \end{bmatrix} \qquad (6.20a)$$

$$\hat{B}_{22}{}'=\begin{bmatrix} 0.0738 & 0.4827 \end{bmatrix} \qquad (6.20b)$$

$$\hat{C}_{22}=\begin{bmatrix} -0.0738 & 0.4827 \\ -0.0015 & 0 \end{bmatrix} \qquad (6.20c)$$

By assuming $\hat{A}_{21}=0$ and $\hat{B}_{22}=0$ and following the procedure given in algorithm 2, the suboptimal control law is $\hat{u}=-\hat{F}x$, where

$$\hat{F} = \begin{bmatrix} -1.0888 & -0.9474 & 0.0004 & 0 & -0.0605 & -0.0010 \end{bmatrix} \quad (6.21)$$

And the optimal feedback control law $u=-Fx$ is defined by

$$F = \begin{bmatrix} -1.0851 & -0.9432 & 0.0023 & 0 & -0.0604 & -0.0014 \end{bmatrix} \quad (6.22)$$

We see the suboptimal feedback law (6.21) is very close to that of the optimal one. The performance index for the suboptimal regulator is given by

$$\text{trace } P_s = 1.2388 \qquad (6.23)$$

while that for the optimal one is given by

$$\text{trace } P_o = 1.2387 \qquad (6.24)$$

Therefore, the performance degradation of the suboptimal regulator by employing the suboptimal control law $\hat{u}(t)$ is less than 0.008%.

If instead, the procedure of Kwong (1983) is used, at the 5th step of the procedure, the transformed system model is given by

$$\bar{A} = \begin{bmatrix} -6.751 & -410 & -1352 & -1400 & 8.475 & 0.699 \\ 3.395 & 387.9 & 1317 & 2952 & -220.4 & -140 \\ 0 & -119.6 & -405.9 & -910.5 & 67.98 & 4.321 \\ 0 & 0 & 3.551 & 8.414 & -0.659 & -0.0533 \\ 0 & 0 & 0 & 12.89 & -1.321 & -0.212 \\ 0 & 0 & 0 & 0.133 & 0.994 & -0.0426 \end{bmatrix} \qquad (6.25a)$$

$$\bar{B}' = \begin{bmatrix} 912.6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (6.25b)$$

$$\bar{C} = \begin{bmatrix} -0.003813 & -0.02149 & -0.02028 & 0.08621 & 0.9958 & -0.00785 \\ -0.0000573 & -0.002441 & -0.007593 & 0.008963 & 0.006901 & 0.9999 \end{bmatrix} \qquad (6.25c)$$

Using the method proposed by Kwong (1983), the suboptimal control law $\bar{u}=-\bar{F}x$ thus obtained is

$$\bar{F} = \begin{bmatrix} -0.06931 & -0.01447 & -0.01059 & 0.005597 & -0.1147 & -0.0019 \end{bmatrix} \qquad (6.26)$$

The performance index for the suboptimal control law (6.26) is

$$\text{trace } P_{s'} = 5.77 \qquad (6.27)$$

Comparing this with the performance index for the optimal regulator, we see that the suboptimal control law (6.26) is not acceptable.

In Table 2 the eigenvalues for the optimal regulator and the suboptimal regulators are shown.

TABLE 2

	Optimal regulator (6.22)	Suboptimal regulator by the present method (6.21)	Suboptimal regulator by the method of Kwong (6.26)
Eigenvalues	$-6.59 \pm j62.33$ $-2.465 \pm j24.51$ -7.046 -1.008	$-6.48 \pm j62.36$ $-2.453 \pm j24.49$ -7.089 -1.014	$-6.706 \pm j62.51$ $-4.651 \pm j23.95$ $-0.3747 \pm j0.3086$
Performance index	1.2387	1.2388	5.77

Example 3

As this last example in this section, we use the example taken from Harvey and Stein (1978). The system model is defined by

$$A = \begin{bmatrix} -0.746 & 0.387 & -12.9 & 0 & 0.952 & 6.05 \\ 0.024 & -0.174 & 4.31 & 0 & -1.76 & -0.416 \\ 0.006 & -0.999 & -0.0578 & 0.0369 & 0.0092 & -0.0012 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} \quad (5.28a)$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 10 \end{bmatrix} \quad (5.28b)$$

The state weighting matrix is given by $Q = H'H$, where

$$H = \begin{bmatrix} -0.131 & -0.612 & 1.64 & 0.0175 & 1 & 0 \\ 0.567 & 0.160 & -2.39 & 0.0303 & 0 & 1 \end{bmatrix} \quad (5.28c)$$

and the input weighting matrix is

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (5.28d)$$

The balanced representation of $\left\{ A, B, H \right\}$ is given by

$$\hat{A} = \begin{bmatrix} -0.0173 & 0.0134 & 0.1933 & -0.0809 & -0.0245 & 0.0365 \\ 0.0317 & -0.0521 & -1.996 & 0.1282 & 0.2249 & -0.0642 \\ -0.168 & 2.0251 & -0.1708 & 0.401 & 0.0508 & -0.1389 \\ -0.0802 & 0.1152 & -0.2232 & -0.8927 & -0.5684 & 0.6983 \\ -0.0799 & 0.2587 & -0.5298 & -1.1019 & -9.6573 & 1.8927 \\ 0.0329 & -0.0367 & 0.010 & 0.6626 & 0.6062 & -5.1876 \end{bmatrix} \tag{6.29a}$$

$$\hat{B} = \begin{bmatrix} -0.7741 & 0.7149 \\ 1.5008 & -0.2184 \\ -2.5420 & -0.7494 \\ -1.7528 & 2.2146 \\ -4.0449 & -0.7536 \\ 0.3887 & -1.0684 \end{bmatrix} \tag{6.29b}$$

$$\hat{C} = \begin{bmatrix} -0.2323 & 1.2803 & 0.2029 & -0.9146 & -4.1133 & 0.4461 \\ 1.0278 & -0.8129 & -2.6424 & 2.6721 & -0.0961 & -1.0458 \end{bmatrix} \tag{6.29c}$$

The observability grammian and controllability grammain for the balanced system (6.29) is given by

$$W_o = W_c = \mathrm{diag}\ (32.2,\ 22.1,\ 20.6,\ 4.5,\ 0.9,\ 0.1) \tag{6.30}$$

The most controllability subsystem is defined to be the fourth order subsystem given by

$$\hat{A}_{11}=\begin{bmatrix} -0.0173 & 0.0134 & 0.1933 & -0.0809 \\ 0.0317 & -0.0521 & -1.996 & 0.1282 \\ -0.168 & 2.0251 & -0.1708 & 0.401 \\ -0.0802 & 0.1152 & -0.2232 & -0.8927 \end{bmatrix} \qquad (6.31a)$$

$$\hat{B}_{11}=\begin{bmatrix} -0.7741 & 0.7149 \\ 1.5008 & -0.2184 \\ -2.5420 & -0.7494 \\ -1.7528 & 2.2146 \end{bmatrix} \qquad (6.31b)$$

$$\hat{C}_{11}=\begin{bmatrix} -0.2323 & 1.2803 & 0.2029 & -0.9146 \\ 1.0278 & -0.8129 & -2.6424 & 2.6721 \end{bmatrix} \qquad (6.31c)$$

and the almost uncontrollable subsystem is defined by

$$\hat{A}_{22}=\begin{bmatrix} -9.6573 & 1.8927 \\ 0.6062 & -5.1876 \end{bmatrix} \qquad (6.32a)$$

$$\hat{B}_{22}=\begin{bmatrix} -4.0449 & -0.7536 \\ 0.3887 & -1.0684 \end{bmatrix} \qquad (6.32b)$$

$$\hat{C}_{22}=\begin{bmatrix} -4.1133 & 0.4461 \\ -0.0961 & -1.0458 \end{bmatrix} \qquad (6.32c)$$

By following the procedure given in algorithm 2, the suboptimal control law $\hat{u}=-\hat{F}x$ is obtained, where

$$\hat{F} = \begin{bmatrix} -0.0752 & -0.8813 & 0.9369 & 0.0286 & 0.8116 & 0.1263 \\ 0.5337 & 0.5606 & -3.054 & 0.0162 & -0.02 & 0.9613 \end{bmatrix} \tag{6.33}$$

and the optimal feedback control law u=-Fx is given by

$$F = \begin{bmatrix} -0.1293 & -0.8792 & 1.5651 & 0.0263 & 0.6664 & -0.0249 \\ 0.5181 & 0.4156 & -2.8133 & 0.0208 & -0.0124 & 0.8507 \end{bmatrix} \tag{6.34}$$

The performance index for the suboptimal regulator (6.33) is given by

$$\text{trace } P_s = 2.19 \tag{6.35}$$

while that for the optimal regulator is

$$\text{trace } P_o = 2.03$$

If instead the procedure of Kwong (1983) is employed, the feedback matrix and the performance index are given by

$$\bar{F} = \begin{bmatrix} -0.1116 & -0.5983 & 1.43 & 0.0234 & 0.6603 & -0.0043 \\ 0.5269 & 0.1922 & -3.035 & 0.0178 & -0.0021 & 0.8683 \end{bmatrix} \tag{6.37}$$

$$\text{trace } P_{s'} = 2.45 \tag{6.38}$$

We see that the performance of the suboptimal regulator (6.35) is superior than that of (6.37). The performance degradation for the suboptimal regulator (6.35) is 7 per cent, while that for (6.37) is 20 per cent. In Table 3 the eigenvalues for the respective regulators are shown.

TABLE 3

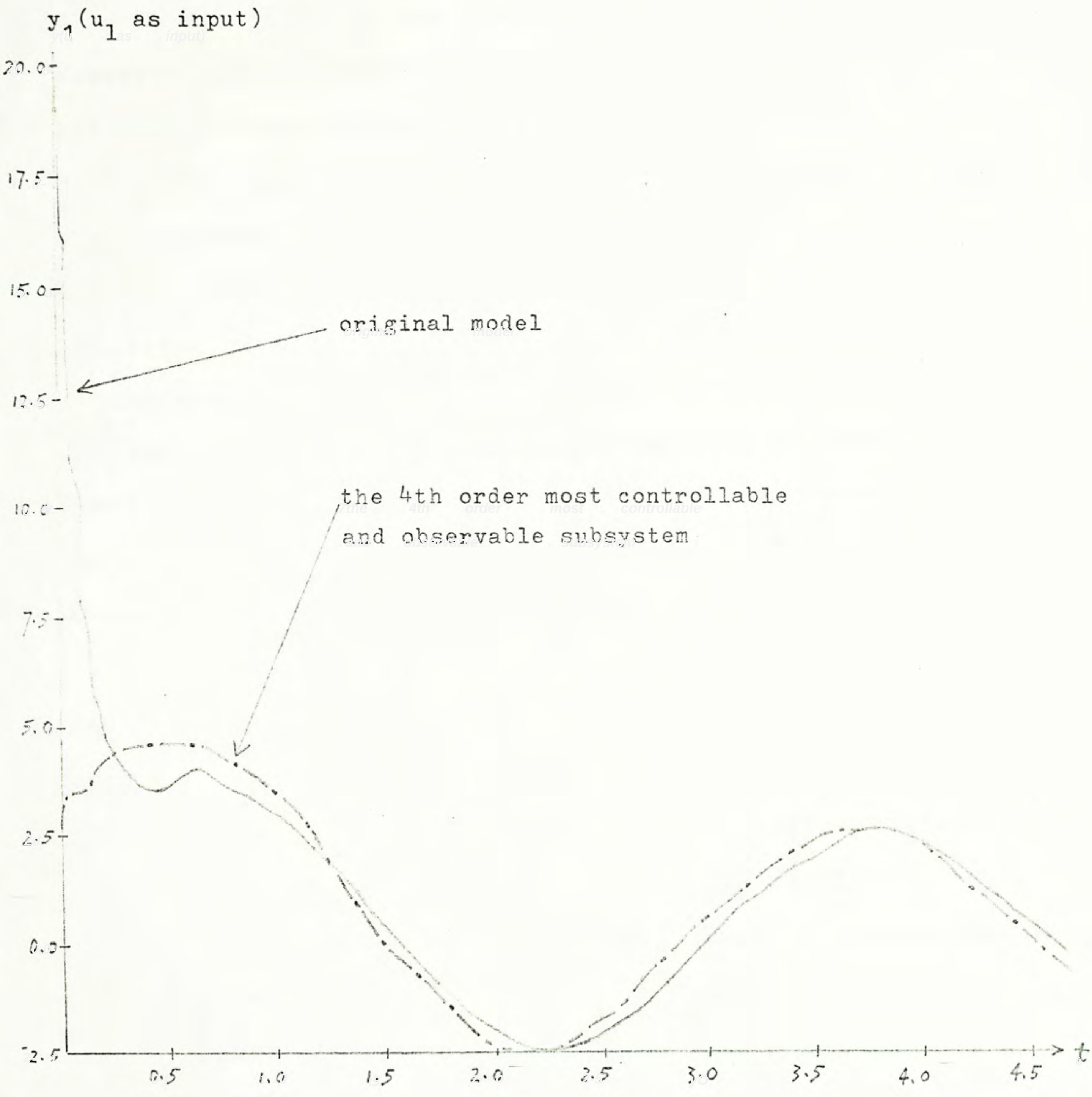
	Optimal regulator (6.34)	Suboptimal regulator by the present method (6.33)	Suboptimal regulator by the method of kwong (6.37)
Eigenvalues	-21.7 -10.21 -3.808 -0.737+j2.313 -0.737-j2.313 -0.049	-25.1 -11.9 -3.487 -0.674+j2.159 -0.674-j2.159 -0.047	-22.46 -10.43 -3.851 -0.557+j2.298 -0.557-j2.298 -0.033
Performance index	2.03	2.19	2.45

Remarks:

1. It is obvious that from the definition of the linear quadratic regulator problem that it is the steady state behavior of the unregulated system which is of major importance. Since it is the transient behavior which is emphasized in the procedure of Kwong (1983), it may seem that the suboptimal control law obtained by the method may not be satisfactory. This is in fact the situation, as can be seen from the previous examples.

2. Although it is the steady state behavior which is of fundamental importance for the regulator problem, the performance of the suboptimal control law may be improved if the transient behavior of the unregulated system model is also taken into account in deriving the suboptimal control law. In Fig.1 the transient response of the approximate model (6.31) is shown. We see that the initial part of the transient response of the approximate model (6.31) deviates greatly from that of the original one (6.29). This is due to the fact that the present method emphasized the steady state behavior.

   In the next section another procedure is proposed which in addition to the steady state behavior, the behavior of the transient is also taken into account.



y1(u1 as input)
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
-2.5
original model
the 4th order most controllable
and observable subsystem
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
t

Fig. 1

7. Improved algorithm to derive the suboptimal control law

In Algorithm 2, the controllability grammian and the observability grammian (5.17) are employed to explore the almost uncontrollable subsystem. The grammians are defined by the integrals (4.10), thus the fast modes which are important only during the initial portion of the response may be considered as the almost uncontrollable modes in the sense of (5.18). However, we see that if the performance of the suboptimal control law is to be further improved, the initial transient behavior should also be considered. The key here is to match the first Markov parameter of $\left\{A, \bar{B}, H\right\}$ in the approximate model, where $\bar{B}$, H is the modified input matrix and the implicit output matrix respectively.

Algorithm 3

Step 1 Employ Step 1 to Step 4 of Algorithm 2 to $\left\{A, \bar{B}, H\right\}$ to explore the almost uncontrollable subsystem $\left\{\hat{A}_{22}, \hat{B}_{22}, \hat{C}_{22}\right\}$ defined in (5.23). If the norms $\hat{B}_{22}$ and $\hat{C}_{22}$ is small, Algorithm 2 is carried out to complete the suboptimal design problem.(Since when the norm of $\hat{B}_{22}$ and $\hat{C}_{22}$ is small, the subsystem $\left\{\hat{A}_{22}, \hat{B}_{22}, \hat{C}_{22}\right\}$ contributes little to the input-output response of the original system.) Otherwise, go to Step 2.

Step 2 Find the singular value decomposition of $\hat{B}$, the input matrix of the balanced representation $\left\{\hat{A},\hat{B},\hat{C}\right\}$ of $\left\{A,\bar{B},H\right\}$, that is

$$\hat{B} = V_1 \Sigma U_1' \, , \quad \Sigma = \begin{pmatrix} D \\ 0 \end{pmatrix} \tag{7.1}$$

D is the diagonal matrix with its elements the singular values of $\hat{B}$.

Step 3 Transform the system $\left\{\hat{A},\hat{B},\hat{C}\right\}$ to S_1

$$S_1: \quad \left\{V_1'\hat{A}V_1, V_1'\hat{B}, \hat{C}V_1\right\} \tag{7.2}$$

And the transformed system S_1 is now given in the form

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \tag{7.3}$$

Notice now that x_{11} is an m×l state vector, where m is the rank of the input matrix $\bar{B}$.

Step 4 Define the residue subsystem S_{R1} as

$$\left\{ A_{14}, A_{13}, \begin{bmatrix} C_{12} \\ A_{12} \end{bmatrix} \right\} \tag{7.4}$$

And use Algorithm 2 to explore the almost uncontrollable subsystem of the residue subsystem. If the balanced representation of S_{R1} after performing Algorithm 2 is given by

$$\left\{ \left[\begin{array}{c|c} A_{21} & A_{22} \\ \hline A_{23} & A_{24} \end{array}\right], \left[\begin{array}{c} A_{131} \\ \hline A_{132} \end{array}\right], \left[\begin{array}{cc} C_{121} & C_{122} \\ \hline A_{121} & A_{122} \end{array}\right] \right\} \tag{7.5}$$

The subsystem

$$\left\{ A_{24},\ A_{132},\ \begin{bmatrix} C_{122} \\ A_{122} \end{bmatrix} \right\} \tag{7.6}$$

is the almost uncontrollable subsystem for the residue subsystem S_{R1}. And now $\left\{ A, \bar{B}, H \right\}$ is transformed to the form

$$\left\{ \left[\begin{array}{cc|c} A_{11} & A_{121} & A_{122} \\ A_{131} & A_{21} & A_{22} \\ \hline A_{132} & A_{23} & A_{24} \end{array}\right],\ \left[\begin{array}{c} B_{11} \\ 0 \\ \hline 0 \end{array}\right],\ \left[\begin{array}{cc|c} C_{11} & C_{121} & C_{122} \end{array}\right] \right\} \tag{7.7}$$

The block matrices are all of compatible orders. Thus the most controllable subsystem for $\left\{ A, \bar{B}, H \right\}$ is given by

$$\left\{ \begin{bmatrix} A_{11} & A_{121} \\ A_{131} & A_{21} \end{bmatrix},\ \begin{bmatrix} B_{11} \\ 0 \end{bmatrix},\ \begin{bmatrix} C_{11} & C_{121} \end{bmatrix} \right\} \tag{7.8}$$

And the uncontrollable system $\left\{ A_{24}, 0, C_{122} \right\}$ is considered as the almost uncontrollable subsystem for $\left\{ A, \bar{B}, H \right\}$. Follow the procedure outlined in algorithm 2, the suboptimal control law can then be found.

Remarks:

1. In algorithm 3, step 2 corresponds to matching the first markov parameter of the original system model $\left\{A, \bar{B}, H\right\}$ in the approximate model. That is why the procedure would give an approximate model having a initial transient response which is close to that of the original one. And after step 2, the system model can be depicted as shown in Fig. 2.

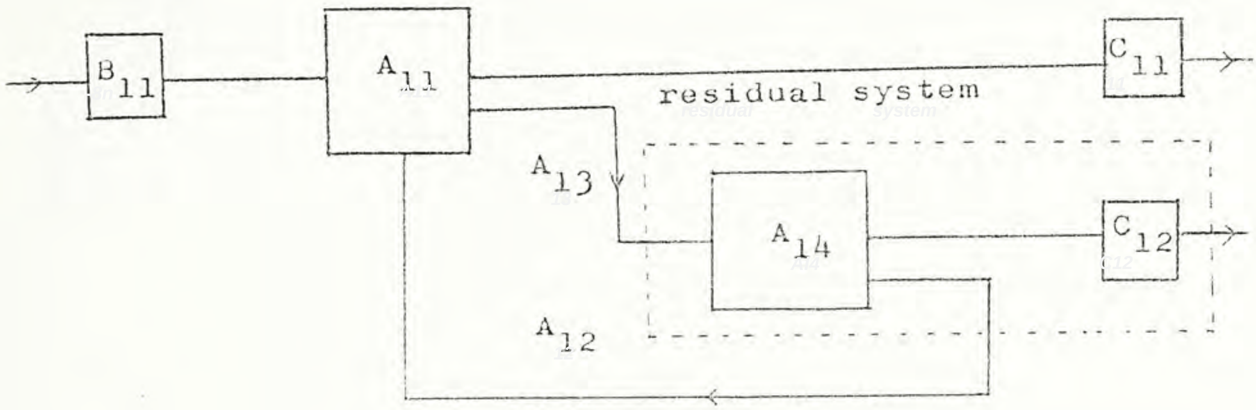


Fig. 2

2. In the residual system defined in (7.4), A_{13} and A_{12} are considered as the additional input matrix and output matrix to take into account the coupling effects between the subsystems shown in Fig. 2. In Fig. 2 the residue system is also shown.

In the next example, Example 3 is reworked by using the algorithm 3.

Example 4

Here we use the system model and weighting matrices shown in Example 3. The transformed system after employing algorithm 3 is given by

$$\hat{A} = \begin{bmatrix} -6.488 & -0.436 & -0.6784 & 4.055 & -0.7511 & -1.549 \\ -1.373 & -2.015 & -0.5523 & 0.803 & -0.3562 & 1.348 \\ 1.445 & 0.7335 & -0.1603 & 1.622 & 0.06548 & -0.0022 \\ 4.835 & 0.8814 & -1.698 & -3.12 & -0.6791 & 1.1903 \\ 0.6226 & -0.6185 & -0.06552 & -0.3941 & -0.2782 & 0.8791 \\ -0.8642 & 1.902 & -0.01768 & 0.475 & 0.8846 & -3.916 \end{bmatrix} \tag{7.9a}$$

$$\hat{B} = \begin{bmatrix} -5.376 & 0.0571 \\ -0.0296 & -2.78 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{7.9b}$$

$$\hat{C} = \begin{bmatrix} -3.72 & -0.0764 & -0.8845 & 2.144 & 0.264 & -0.116 \\ 0.0198 & -3.596 & 1.039 & -1.572 & 0.0969 & -0.367 \end{bmatrix} \tag{7.9c}$$

The most controllable subsystem is defined to be the fourth order subsystem given by

$$\hat{A}_{11} = \begin{bmatrix} -6.488 & -0.436 & -0.6784 & 4.055 \\ -1.373 & -2.015 & -0.5523 & 0.803 \\ 1.445 & 0.7335 & -0.1603 & 1.622 \\ 4.835 & 0.8814 & -1.698 & -3.12 \end{bmatrix} \quad (7.10a)$$

$$\hat{B}_{11} = \begin{bmatrix} -5.376 & 0.0571 \\ -0.0296 & -2.78 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (7.10b)$$

$$\hat{C}_{11} = \begin{bmatrix} -3.72 & -0.0764 & -0.8845 & 2.144 \\ 0.0198 & -3.596 & 1.039 & -1.572 \end{bmatrix} \quad (7.10c)$$

and the almost uncontrollable subsystem for the original system model is now the uncontrollable system

$$\hat{A}_{22} = \begin{bmatrix} -0.2782 & 0.8791 \\ 0.8846 & -3.916 \end{bmatrix} \quad (7.11a)$$

$$\hat{C}_{22} = \begin{bmatrix} 0.264 & -0.116 \\ 0.0969 & -0.367 \end{bmatrix} \quad (7.11b)$$

$\hat{B}_{22}$ is now a null matrix.

Following the procedure given in algorithm 3, the suboptimal control law is defined by the feedback matrix

$$\hat{F} = \begin{bmatrix} -0.1379 & -0.9417 & 1.564 & 0.0289 & 0.6829 & -0.0064 \\ 0.5544 & 0.4553 & -2.6879 & 0.0228 & -0.0032 & 0.8226 \end{bmatrix} \quad (7.12)$$

The performance index for the suboptimal regulator is given by

$$\text{trace } P_s = 2.05 \qquad (7.13)$$

and the performance degradation is 1.0% . It is seen that the performance degradation for the suboptimal regulator obtained by the algorithm given in this section is further reduced as compared with that obtained by algorthm 2. In Table 4 the results for the varies suboptimal control laws are shown for comparsion.

In Fig. 2 the response of the fourth order reduced model obtained by means of Algorithm 3 is shown together with the fourth order reduced model obtained by means of Algorithm 2 for comparsion. It can be seen that the fourth order reduced model obtained by Algorithm 3 has a better initial transient response.

TABLE 4

	Optimal regulator	Suboptimal regulator by method of Section 7	Suboptimal regulator by method of Section 6	Suboptimal regulator by method of Kwong
Eigenvalues	-21.7 -10.21 -3.808 $-0.737 \pm j2.313$ -0.049	-22.39 -9.55 -4.335 $-0.766 \pm j2.317$ -0.052	-25.1 -11.9 -3.487 $-0.674 \pm j2.159$ -0.047	-22.46 -10.43 -3.851 $-0.557 \pm j2.298$ -0.033
Performance index	2.03	2.05	2.19	2.45

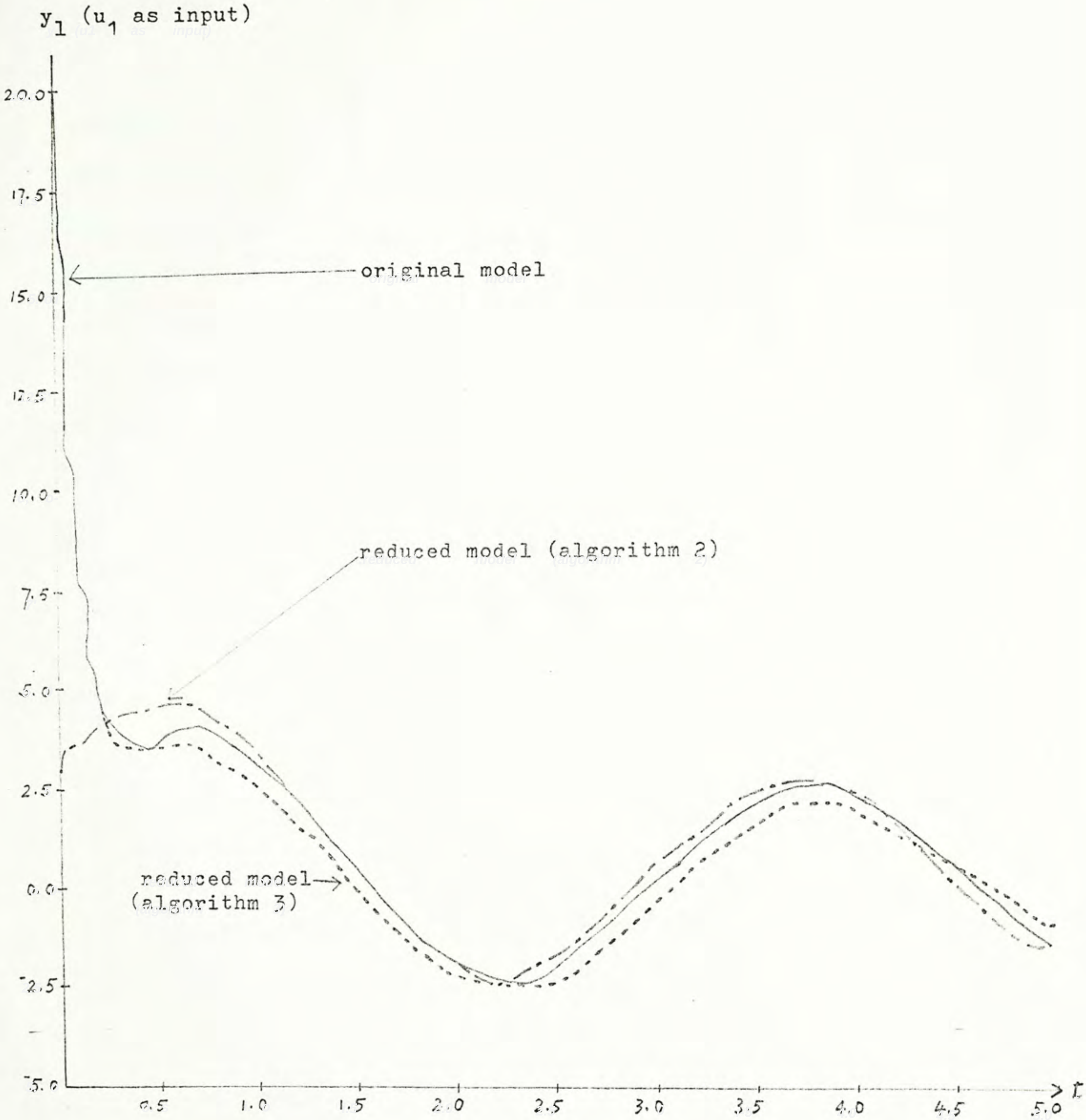

y_1 (u_1 as input)
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
-2.5
-5.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
t
original model
reduced model (algorithm 2)
reduced model (algorithm 3)

Fig. 2

8. DISCUSSIONS

In this chapter a procedure to derive a suboptimal control law is proposed using an internal cooridinate transformation of the given system, so that the almost uncontrollable subsystem is identified. The computational burden is reduced. The coordinate transformation also takes into account the weighting matrices in the cost functional of the regulator problem. This is important since the control law depends on both the system dynamics and also the weighting matrices in the cost functional. It is also demonstrated in the chapter how the dynamics of the unregulated system are related to the suboptimal control law.

CHAPTER 4

CONCLUSIONS

In this thesis we have demonstrated that how the notion of 'almost uncontrollability' and 'almost unobservability' can be employed to derive a simplified model to fascilitate the analysis and design of control systems. In the procedures only orthogonal and diagonal matrices are involved, which from the computation point of view, are convenient. It is also shown that in the examples that the simplified model thus obtained fit the tasks of analysis and design well.

Although we have some well established criteria to test the exact controllability and observability of a given system (Paige 1981), it is demonstrated in this thesis that this is not the case of 'almost controllability' or 'almost observability'.When one employ different measures for the above notions of 'almost controllability' and 'almost observability', one may obtain different simplified model which can fulfill different predetermined tasks. It is believed by the present author that the above consideration can be applied to the more general nonlinear and time-variant system to fascilitate the analysis and design of the more general system models. But, in order that the above consideration be a fruitful one, the correspoding notions of 'almost controllability' and 'almost observability' should be re-established for the more general system models. The latter would be a very interesting field of further investigation.

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