PROPAGATION AND INSTABILITY OF
ION-CYCLOTRON-WAVES IN PLASMAS

A Thesis

Presented to the Faculty of the Graduate School
of The Chinese University of Hong Kong
for the Degree of Master of Philosophy

by

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May, 1976
ACKNOWLEDGEMENTS

The research was carried out under the direction of my supervisor, Dr. K. F. Lee, and appreciation is hereby expressed for the advice and encouragement received from him at all times. I am also indebted to my supervisor for stimulating my interest in plasma physics.
ABSTRACT

The propagation and instability of low frequency electromagnetic waves in a fully-ionized collisional plasma situated in an external magnetic field is studied. Based on Maxwell's equations and the fluid equations, the conductivity tensor is calculated and the dispersion relation for obliquely propagating plane waves are obtained. An approximation which reduces the dispersion relation from a 3x3 determinant to a 2x2 determinant is introduced. This reduced dispersion relation is studied both for plasmas carrying no current and for plasmas carrying a field-aligned current. For current-carrying plasmas, it is found that waves with frequencies below the ion cyclotron frequency become unstable when the parallel phase velocity is smaller than the electron-ion relative drift velocity. The growth rate of the instability is directly proportional to the electron-ion collision frequency. The relevance of the results to laboratory plasmas and the ionosphere is mentioned. The validity of the reduced dispersion is discussed and the thesis ends with some suggestions for further work.
## CONTENTS

<table>
<thead>
<tr>
<th>ACKNOWLEDGEMENTS</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>II</td>
</tr>
<tr>
<td>CONTENTS</td>
<td>III</td>
</tr>
</tbody>
</table>

### CHAPTER 1 INTRODUCTION

1.1 THE STUDY OF PLASMA INSTABILITIES 1
1.2 ION CYCLOTRON INSTABILITIES 4
1.3 OUTLINE OF THESIS 7

### CHAPTER 2 BASIC EQUATIONS

2.1 MAXWELL'S EQUATIONS 8
2.2 CONDUCTIVITY TENSOR 10
2.3 DISPERSION RELATION FOR PLANE WAVES 11
2.4 PLASMA EQUATIONS 15

### CHAPTER 3 DISPERSION RELATION FOR LOW FREQUENCY WAVES

3.1 CALCULATION OF CONDUCTIVITY TENSOR 18
3.1.1 Linearized Equations of Motion 18
3.1.2 Conductivity Tensor 21
3.2 THE DISPERSION RELATION 29
3.3 THE REDUCED DISPERSION RELATION 32
6.2 COMPARISON WITH ELECTROSTATIC RESULT

6.3 SUGGESTIONS FOR FURTHER WORK

APPENDIX A

APPENDIX B

REFERENCES
CHAPTER 1
INTRODUCTION

1.1 THE STUDY OF PLASMA INSTABILITIES

The study of waves and instabilities has occupied a central role in the history of plasma physics research. The reason is that the majority of plasmas created in the laboratory and occurring in nature are in a highly non-equilibrium state, possessing free energies of various forms. For example, the E region of the equatorial ionosphere and much of the auroral ionosphere are known to carry currents. The earth's radiation belts and the high temperature plasmas created in the laboratory in fusion research have particle distribution functions which are very different from the equilibrium Maxwellian velocity distribution. Both electric currents and non-Maxwellian distributions are sources of free energy. Such non-equilibrium plasmas can be unstable against small amplitude perturbations. These instabilities can take on a
variety of forms. It is clear that in order to describe the behaviour of the plasma, it is necessary to know whether a given equilibrium is stable. If not, what types of instabilities the plasma is susceptible to and what are the ultimate fate of the instabilities.

As the dynamical equations governing the plasma are inherently nonlinear, the study of plasma instabilities is a subject of enormous complexity. The early development in this field was therefore concerned with linear theory. In linear theory, the perturbations around an equilibrium are assumed to be small so that terms involving products of two or more perturbed quantities are neglected. The resultant linearized equations can be analyzed by standard transform techniques to determine whether unstable solutions exist. The linear theory can thus answer the first question raised in the last paragraph, namely, what types of instabilities can a given equilibrium support. It can also determine the instability boundaries and the initial growth rate. However, it cannot answer the question of what is the ultimate state of the plasma, since the theory breaks down when the perturbations grow to amplitude levels which are no longer small. Thus the question of how the instabilities affect the gross property of the plasma must be answered within
the framework of nonlinear and turbulent theory, which is the mainstream of plasma research today, and is likely to remain so for the foreseeable future.

Despite the shift of emphasis from linear to nonlinear theory in present-day plasma physics research, linear theory is by no means a closed book. This is due to the fact that most laboratory and naturally occurring plasmas are situated in a magnetic field, and a magnetized plasma appears to support an endless list of instabilities, the search of which is still continuing. Mathematically, the general stability analysis for magnetized plasmas is very complicated even within the framework of linearized theory. For this reason, most of the early studies were based on certain simplifying assumptions. The most popular one is the electrostatic approximation. In this approximation, the perturbations are assumed to contain only electric field fluctuations but no magnetic field fluctuations. Mathematically, this means that instead of the full set of Maxwell's equations, only Poisson's equation is considered. This simplifies the problem considerably, and there is a large body of literature on electrostatic instabilities. On the other hand, the study of the stability of electromagnetic waves, i.e. waves with both electric and magnetic field fluctuations,
were confined mainly to the special cases of propagation parallel or perpendicular to the magnetic field.\textsuperscript{6,7}

1.2 \textbf{ION CYCLOTRON INSTABILITIES}

This thesis is concerned with one type of plasma instabilities, namely, instability around the ion cyclotron frequency. Such instabilities occur in plasmas in which there is a relative streaming motion between electrons and ions, i.e. current-carrying plasmas, which are found to exist in the ionosphere and in many laboratory devices. Instabilities around the ion cyclotron frequency are of importance in fusion research because they provide an efficient means of heating up the ions by means of direct currents. As such, they have been studies extensively in the literature. On the theoretical side, they were first treated by Drummond and Rosenbluth.\textsuperscript{8} Their analysis was based on the electrostatic approximation using the Vlasov kinetic equation to describe the dynamics of the plasma. As such, the results were applicable only to electrostatic waves in high temperature low density plasmas for which collisions were negligible. This instability is now referred to as the collisionless electrostatic ion cyclotron instability. The theory of Drummond and Rosenbluth \textsuperscript{8} has been extended by others to more
complicated plasmas, e.g. those involving several ion species and those containing anisotropic velocity distributions. Experimentally, the collisionless electrostatic ion cyclotron instability has been studied by several authors.

For high density and low temperature plasmas, collisions are important and the analysis of Drummond and Rosenbluth is not applicable. The theory of the ion cyclotron instability in plasmas for which collisions are important was recently undertaken by Lee and Luhmann. The main motivation for this study was the plasma-wave experiment conducted at the UCLA plasma engineering laboratory, which my supervisor, Dr. K. F. Lee, visited during the summer of 1975. In this device, a fully-ionized one meter long plasma column is produced in a four-stage differentially pumped arc-jet. The diameter of the column is approximately 1 cm and the background gas can be either helium or argon. Details concerning the device were described by Tang et al. It suffices for our present purposes to know that the resultant plasma has density $10^{13-15} \text{ cm}^{-3}$ and temperature $T_i \approx T_e = 2-7$ ev. and is confined by axial magnetic field $B_0$ of strength up to 9 kG. Typical values of electron-ion collision frequency and ion cyclotron frequency are $10^8$ Hz and $0.5 \times 10^6$ Hz.
respectively. By applying a voltage across the column, the plasma can be made to carry an axial current equivalent to a relative electron-ion drift velocity of $10^6$-$10^8$ cm/sec. Intense electromagnetic fluctuations were observed in the plasma column. These fluctuations contain both electric and magnetic fields, and occur at frequencies of the order of but below the ion cyclotron frequency. 14

The theory of Lee and Luhmann 12 recently published was based on the fluid equations and the electrostatic approximation. They found an instability for obliquely propagating ion cyclotron waves when the wave phase velocity parallel to the background magnetic field is smaller than the electron-ion drift velocity. As the growth rate is proportional to the electron-ion collision frequency, Lee and Luhmann referred to it as the resistive electrostatic ion cyclotron instability. This instability, however, cannot explain the observations of the UCLA arc jet on two basic points. Firstly, the predicted frequencies were above the ion cyclotron frequency while the observations were below the ion cyclotron frequency. Secondly, the instability predicted by Lee and Luhmann contains only electric field fluctuations, while both electric and magnetic field fluctuations were associated with the observations.
1.3 OUTLINE OF THESIS

The purpose of this thesis is to extend the analysis of Lee and Luhmann to include electromagnetic perturbations. The subject of the thesis, accordingly, is concerned with the linear theory of electromagnetic instabilities around the ion cyclotron frequency in collisional, fully-ionized plasmas carrying a field-aligned current. In Chapter 2, the basic equations upon which the subsequent analysis is based are presented. Chapter 3 is devoted to the derivation of the conductivity tensor and the dispersion relation. The dispersion relation is analyzed in the cold plasma limit in Chapter 4. The effect of temperature is treated in Chapter 5. Finally, in Chapter 6, we conclude with some discussion and suggestion for further work.
CHAPTER 2
BASIC EQUATIONS

2.1 MAXWELL'S EQUATIONS

Any electromagnetic field in any medium must satisfy Maxwell's equations, which in cgs electrostatic units read:

\[ \nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \]  \hspace{2cm} (2.1a)

\[ \nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{4\pi}{c} \mathbf{J}(\mathbf{r}, t) + \frac{1}{c} \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \]  \hspace{2cm} (2.1b)

\[ \nabla \cdot \mathbf{E}(\mathbf{r}, t) = 4\pi \rho(\mathbf{r}, t) \]  \hspace{2cm} (2.1c)

\[ \nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \]  \hspace{2cm} (2.1d)

In the above equations, \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \) are the electric and magnetic fields, respectively. \( \mathbf{J}(\mathbf{r}, t) \) is the conduction current density and \( \rho(\mathbf{r}, t) \) is the charge...
density. The quantity $c$ is the velocity of light.

Associated with (2.1a-d) is the charge conservation equation

$$\nabla \cdot \vec{J}(\vec{r},t) + \frac{\partial \rho(\vec{r},t)}{\partial t} = 0 \quad (2.1e)$$

The medium we are concerned with in this thesis is a plasma. The dynamical behavior of the plasma enters into the above equations through the quantity $\vec{J}(\vec{r},t)$, which results from the motion of the charged particles under the influence of the electromagnetic fields. Since $\vec{J}(\vec{r},t)$ in turn generates electromagnetic fields, neither $\vec{J}(\vec{r},t)$ nor $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ can be regarded as given. One must find the relation between $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ and then substitute this relation into Maxwell's equations to find the self-consistent solutions.

In this thesis, we employ the simplifying assumption that the plasma is infinite and homogeneous. Accordingly, equations (2.1a-e) can be Fourier-transformed in space and time. Each Fourier component of a field quantity varies in space and time according to

$$\exp i(\vec{k} \cdot \vec{r} - wt) \quad (2.2)$$

and the Fourier components obey the following set of equations:
Equation (2.2) is in the form of a travelling plane wave, with angular frequency $w$ and wavenumber $k$.

2.2 CONDUCTIVITY TENSOR

If we restrict ourselves to the study of small amplitude perturbations in the plasma, we can assume that $\bar{J}(\bar{k},w)$ is related to $\bar{E}(\bar{k},w)$ in a linear manner, i.e.

$$\bar{J}(\bar{k},w) = \text{constant} \times \bar{E}(\bar{k},w)$$

where the constant of proportionality is in general dependent not only on the parameters characterizing the steady state (e.g. number density, strength of the magnetostatic field, steady streaming velocity), but also on $w$ and $k$. It is a scalar for a
plasma in the absence of an external magnetostatic field and is a 3x3 matrix for a plasma in an external magnetostatic field. In general, therefore, we write

\[ \mathbf{j}(\mathbf{k}, \omega) = \mathbf{\sigma} \cdot \mathbf{E}(\mathbf{k}, \omega) \]  

(2.4)

where \( \mathbf{\sigma} \) is called the conductivity tensor of the plasma.

### 2.3 Dispersion Relation for Plane Waves

The conductivity tensor assumes different forms, depending on the types of plasma and the types of waves under study. We shall calculate it for our problem in the next chapter. For the moment, let us assume that \( \mathbf{\sigma} \) has been calculated. If we substitute (2.3a) and (2.4) into (2.3b), we obtain the following equation for the electric field:

\[- \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) + \frac{w^2}{c^2} \mathbf{E} = 4\pi i \frac{w}{c^2} \mathbf{\sigma} \cdot \mathbf{E}\]  

(2.5)

since

\[- \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \mathbf{k} (\mathbf{k} \cdot \mathbf{E}) - \mathbf{k}^2 \mathbf{E}\]

The equation (2.5) can be written as

\[- \mathbf{k} \cdot (\mathbf{k} \mathbf{E}) + k^2 \mathbf{E} + \frac{w^2}{c^2} \mathbf{E} = 4\pi i \frac{w}{c^2} \mathbf{\sigma} \cdot \mathbf{E}\]  

(2.6)
Equation (2.6) is a system of three homogeneous algebraic equations for the three components of $\vec{E}$. Let us choose a Cartesian coordinate system such that the $z$-axis is along the direction of the external magnetic field $\vec{B}_0$. Without loss of generality, we take the propagation vector $\vec{k}$ to be in the $x$-$z$ plane so that it has $x$ and $z$ components but no $y$ component. In terms of this coordinate system, equation (2.6) becomes in component form

\begin{align}
- k_x (k_x E_x + k_z E_z) &+ (k^2 + \frac{w^2}{c^2})E_x = \frac{4\pi i w}{c^2} (\vec{k} \cdot \vec{E})_x \quad (2.7a) \\
(k^2 + \frac{w^2}{c^2})E_y & = \frac{4\pi i w}{c^2} (\vec{k} \cdot \vec{E})_y \quad (2.7b) \\
- k_z (k_x E_x + k_z E_z) &+ (k^2 + \frac{w^2}{c^2})E_z = \frac{4\pi i w}{c^2} (\vec{k} \cdot \vec{E})_z \quad (2.7c)
\end{align}

where

\[ k^2 = k_x^2 + k_z^2 \quad (2.8) \]

In this thesis, we shall be concerned with low-frequency waves with $w^2 \ll k^2c^2$. Making use of this assumption and noting that
\[ \mathbf{\underline{\sigma}} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \]  

(2.9)

Equations (2.7) can be written into the form

\[ R_{xx} E_x + R_{xy} E_y + R_{xz} E_z = 0 \]  

(2.10a)

\[ R_{yx} E_x + R_{yy} E_y + R_{yz} E_z = 0 \]  

(2.10b)

\[ R_{zx} E_x + R_{zy} E_y + R_{zz} E_z = 0 \]  

(2.10c)

where

\[ R_{xx} = k_z^2 - \frac{4\pi i w}{c^2} \sigma^-_{xx} \]  

(2.11a)

\[ R_{xy} = - \frac{4\pi i w}{c^2} \sigma^-_{xy} \]  

(2.11b)

\[ R_{xz} = - k_x k_z - \frac{4\pi i w}{c^2} \sigma^-_{xz} \]  

(2.11c)

\[ R_{yx} = - \frac{4\pi i w}{c^2} \sigma^-_{yx} \]  

(2.11d)

\[ R_{yy} = k_x^2 E_y - \frac{4\pi i w}{c^2} \sigma^-_{yy} \]  

(2.11e)

\[ R_{yz} = - \frac{4\pi i w}{c^2} \sigma^-_{yz} \]  

(2.11f)
Note that dropping the term \( w^2/c^2 \) compared to \( k^2 \) means that displacement current is negligible compared to conduction current, which is a good approximation at low frequencies.

For the system of equation (2.10) to have a nontrivial solution, we must have

\[
\begin{bmatrix}
R_{xx} & R_{xy} & R_{xz} \\
R_{yx} & R_{yy} & R_{yz} \\
R_{zx} & R_{zy} & R_{zz}
\end{bmatrix}
= 0 \quad (2.12)
\]

Equation (2.12) is the dispersion relation and it relates the angular frequency \( w \) to the wavenumber \( \vec{k} \).

Only for those values of \( w \) and \( \vec{k} \) satisfying this equation will the plane wave solution (2.2) be an admissible solution to the system of Maxwell's equations.

Once the dispersion relation is found, one can principle determine whether it admits solutions corresponding to
growing waves or instabilities. If so, the frequencies and the initial growth rates of the instabilities can be determined. On the other hand, if the dispersion relation admits only stable solutions, it is of interest to examine the various properties of the stable waves, for example, the refractive index of the waves.

2.4 PLASMA EQUATIONS

In order to obtain physical results from (2.12), it is necessary to know the conductivity tensor $\sigma$. To calculate this quantity, we must first decide on the mathematical model to use for describing the plasma. In this thesis, we are interested in studying low frequency electromagnetic waves in collisional, fully-ionized plasmas carrying a field-aligned current. As this is the first attempt at a fairly involved problem, we shall use the fluid equation to describe the dynamics of the plasma, instead of the more complicated kinetic equations. Since the plasma is fully ionized, there are no neutral particles and we have the following momentum transfer and continuity equations for the electrons and ions:
\[ N_e \frac{\partial}{\partial t} (\frac{\partial}{\partial \vec{r}} + \vec{V}_e \cdot \vec{\nabla}) \vec{V}_e = -eN_e (\vec{E} + \frac{\vec{V}_e \times \vec{B}}{c}) \]
\[ -m_e N_e \nu_{ei} (\vec{V}_e - \vec{V}_i) - \vec{\nabla} \cdot \vec{P}_e \]  
(2.13)

\[ \frac{\partial N_e}{\partial t} + \vec{\nabla} \cdot (N_e \vec{V}_e) = 0 \]  
(2.14)

\[ N_i \frac{\partial}{\partial t} (\frac{\partial}{\partial \vec{r}} + \vec{V}_i \cdot \vec{\nabla}) \vec{V}_i = eN_i (\vec{E} + \frac{\vec{V}_i \times \vec{B}}{c}) \]
\[ -m_i N_i \nu_{ie} (\vec{V}_i - \vec{V}_e) - \vec{\nabla} \cdot \vec{P}_i \]  
(2.15)

\[ \frac{\partial N_i}{\partial t} + \vec{\nabla} \cdot (N_i \vec{V}_i) = 0 \]  
(2.16)

where

- \( N_e \) = electron number density
- \( N_i \) = ion number density
- \( V_e \) = electron velocity
- \( V_i \) = ion velocity
- \(-e\) = electron charge
- \( e\) = ion charge (we assume singly charged ions)
- \( m_e \) = electron mass
- \( m_i \) = ion mass
- \( \nu_{ei} \) = electron-ion collision frequency
- \( \nu_{ie} \) = ion-electron collision frequency
\(- \vec{\nabla} P_e = \text{pressure force of electrons}\)
\(- \vec{\nabla} P_i = \text{pressure force of ions}\)

For the purpose of this thesis, we use the equations of state

\[ P_e = N_e K T_e \quad (2.17a) \]
\[ P_i = N_i K T_i \quad (2.17b) \]

where \( K \) is Boltzmann's constant, \( T_e \) and \( T_i \) are electron and ion temperatures respectively.

In the next chapter, we first calculate the conductivity tensor \( \vec{\sigma} \) for low frequency waves in magnetized, collisional plasmas based on the fluid equations. We then substitute the result in equation (2.12) to study the stability of such waves.
CHAPTER 3
DISPERSION RELATION
FOR LOW FREQUENCY WAVES

3.1 CALCULATION OF CONDUCTIVITY TENSOR

3.1.1 Linearized Equations of Motion

Let us consider an infinite, homogeneous, and fully-ionized plasma immerse in a magnetostatic field $\vec{B}_o$ aligned with the z direction. In the equilibrium state, the electrons are assumed to have a zero-order drift velocity $\vec{V}_d$ along $\vec{B}_o$ and the ions are assumed to be at rest. To obtain the equations governing perturbations about this equilibrium state, we write

$$N_e = N_o + n_e(\vec{r}, t)$$  \hspace{1cm} (3.1a)

$$N_i = N_o + n_i(\vec{r}, t)$$  \hspace{1cm} (3.1b)

$$\vec{V}_e = \vec{V}_d + \vec{v}_e(\vec{r}, t)$$  \hspace{1cm} (3.1c)

$$\vec{V}_i = \vec{v}_i(\vec{r}, t)$$  \hspace{1cm} (3.1d)
where \( N_0 \) is the equilibrium number density, \( \vec{B}_0 \) the external magnetic field, \( \vec{v}_d \) the electron drift velocity, all of which are treated as given. The rest of the quantities, i.e. \( n_e, n_i, \vec{v}_e, \vec{v}_i, \vec{b} \) and \( \vec{E} \) are perturbations from the equilibrium state and are functions of space and time. In the linear approximation, we assume all perturbed quantities to be small so that when equations (3.1)-(3.6) are substituted into equations (1.33)-(1.36), terms involving products of two or more perturbed quantities are neglected. Moreover, since the wave frequencies we are interested in are of the order of the ion cyclotron frequency, we neglect electron inertia for simplicity. Under these assumptions, the linearized equations of continuity and momentum transfer for electrons are

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}_d + N_0 \vec{v}_e) = 0 \tag{3.2}
\]

\[
0 \approx -e(\vec{E} + \vec{v}_e x \vec{B}_0 + \vec{v}_d x \vec{b} \overline{c}) - m_e \gamma_e \vec{v}_e - KT_e \nabla n_e / N_0 \tag{3.3}
\]
In equation (3.3), we have neglected the term $v_i^i$ in $m_e v_e (\vec{v}_e - \vec{v}_i)$ since it is much smaller than $\vec{v}_e$.

For the ions, we first note that the term $m_i n_i v_i (\vec{v}_i - \vec{v}_e)$ represents the loss of momentum of the ions as a result of collisions with electrons. Since ions are massive, they lose very little momentum to the electrons. This term can therefore be neglected.

Secondly, for perturbations around the ion cyclotron frequency, the motion of the ions parallel to the magnetic field is negligible compared to the perpendicular motion. Under these assumptions, the linearized equations of continuity and momentum transfer for the ions are

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (N_0 \vec{v}_i) = 0 \quad (3.4)$$

$$N_0 m_i \frac{\partial \vec{v}_i}{\partial t} = eN_0 (E + \frac{\vec{v}_i \times \vec{B}}{c}) - KT_i \vec{V}_i n_i \quad (3.5)$$

where subscript $\perp$ denotes the perpendicular (i.e. x of y) components.

We now use (3.2)-(3.5) to calculate the relationship between the conduction current $\vec{J}$ and the perturbed electric field $\vec{E}$. 
3.1.2 Conductivity Tensor

The current density \( \vec{J} \) is given by

\[
\vec{J} = eN_i \vec{v}_i - eN_e \vec{v}_e
\]  
(3.6)

Substitute (3.1)-(3.4) into (3.10), we obtain, after linearization:

\[
\vec{J} = eN_0 \vec{v}_i - eN_0 \vec{v}_e - eN_e \vec{v}_d
\]  
(3.7)

In order to calculate the conductivity tensor \( \vec{\sigma} \) defined by (2.4), we must first express the perturbed velocities \( \vec{v}_i, \vec{v}_e \) and the perturbed density \( n_e \) in terms of the components of the electric field. This can be accomplished by means of equations (3.2)-(3.5). For \( \exp\{ik_x x + k_z z - wt\} \) variation, they read

\[
-iwn_e + ik_z v_d n_e + N_0 (ik_x v_{ex} + ik_z v_{ez}) = 0
\]  
(3.8)

\[
0 = -e(E + \frac{\vec{v}_e x B_0}{c} + \frac{\vec{v}_d x b}{c}) - m_e \nu_e \nu_e
\]

\[
\frac{-KT_e}{N_0} (ik_x \hat{a}_x + ik_z \hat{a}_z) n_e
\]  
(3.9)

\[
-iwn_i + N_0 ik_x v_{ix} = 0
\]  
(3.10)
where \( \hat{a}_x \) and \( \hat{a}_z \) are unit vectors along the x and z axis respectively.

Let us first solve \( \vec{v}_e \) in terms of \( \vec{E} \) by means of equations (3.12) and (3.13). Solving (3.8) for \( n_e \) yields

\[
n_e = \frac{N(k_x \vec{v}_e x + k_z \vec{v}_e z)}{w - k_z \vec{d}}
\]

(3.12)

In (3.9), the term \( \frac{\vec{V}_d x \vec{b}}{c} \) involves the perturbed magnetic field \( \vec{b} \). This can be eliminated in favor of the electric field \( \vec{E} \) using Maxwell's equation

\[-ikx \vec{E} = \frac{iw}{c} \vec{b}\]

The result is

\[
\frac{\vec{V}_d x \vec{b}}{c} = \frac{\vec{V}_d x (kx \vec{E})}{w}
\]

\[
= \hat{a}_x \frac{V_d (k_x E_z - k_z E_x)}{w} - \hat{a}_y \frac{V_d k_z E_y}{w}
\]

(3.13)

Substituting equations (3.12) and (3.13) into equation (3.9) and writing the resultant equation in
component form, we have the following system of algebraic equations for $v_{ex}$, $v_{ey}$ and $v_{ez}$:

\begin{align}
    a_{11}v_{ex} + a_{12}v_{ey} + a_{13}v_{ez} &= c_1 
    \tag{3.14a} \\
    a_{21}v_{ex} + a_{22}v_{ey} + a_{23}v_{ez} &= c_2 
    \tag{3.14b} \\
    a_{31}v_{ex} + a_{32}v_{ey} + a_{33}v_{ez} &= c_3 
    \tag{3.14c}
\end{align}

where

\begin{align}
    a_{11} &= m_e \frac{\nu_{ei}}{e} + iKT_e \frac{k^2}{\omega - k_z \omega_d} 
    \tag{3.15a} \\
    a_{12} &= \frac{eB_o}{c} 
    \tag{3.15b} \\
    a_{13} &= \frac{iKT_e k_z}{\omega - k_z \omega_d} 
    \tag{3.15c} \\
    a_{21} &= - \frac{eB_o}{c} 
    \tag{3.15d} \\
    a_{22} &= m_e \frac{\nu_{ei}}{e} 
    \tag{3.15e} \\
    a_{31} &= \frac{iKT_e k_z}{\omega - k_z \omega_d} 
    \tag{3.15f} \\
    a_{33} &= m_e \frac{\nu_{ei}}{e} + \frac{iKT_e k^2}{\omega - k_z \omega_d} 
    \tag{3.15g}
\end{align}
\[ C_1 = - (eE_x + \frac{eV}{d} \left( k_x E_z - k_z E_x \right)) \]  
\[ (3.15h) \]

\[ C_2 = -(eE_y - \frac{eV}{d} k_z E_y) \]  
\[ (3.15i) \]

\[ C_3 = -eE_z \]  
\[ (3.15j) \]

Equations (3.14a-c) can be solved to yield \( v_{ex} \), \( v_{ey} \), and \( v_{ez} \) by the method of determinant. The result is

\[ \Delta v_{ex} = m_e^2 (\mu_{ei}^2 + \frac{iV}{2} \frac{k_z}{V_d} \left( \frac{V_d k_z}{w} - 1 \right) \mu_{ei} e E_x \]

\[ -m_e^2 (\mu_{ei}^2 + \frac{iV}{2} \frac{k_z}{V_d} \left( \frac{V_d k_z}{w} - 1 \right) \mu_{ei} e E_y \]

\[ -m_e^2 (\left( \frac{V_{Te} k_x}{w} \right) \left( \frac{V_d k_z}{w} - 1 \right) + \frac{V_{k_x}}{w} \mu_{ei} ) \mu_{ei} e E_z \]  
\[ (3.16a) \]

\[ \Delta v_{ey} = m_e^2 (\mu_{ei}^2 + \frac{iV}{2} \frac{k_z}{V_d} \left( \frac{V_d k_z}{w} - 1 \right) \mu_{ei} e E_x \]

\[ +m_e^2 (\mu_{ei}^2 + \frac{iV}{2} \frac{(k_x^2 + k_z^2)}{w} \left( \frac{V_d k_z}{w} - 1 \right) \mu_{ei} e E_y \]

\[ -m_e^2 (\left( \frac{V_{Te} k_x}{w} \right) \left( \frac{V_d k_z}{w} - 1 \right) + \frac{V_{k_x}}{w} \mu_{ei} ) \mu_{ei} e E_z \]  
\[ (3.16b) \]
\[ \Delta v_{ez} = -m_e^2 \left( i \frac{V_{Te}^2 k_z k_x}{w-k_z V_d} \left( \frac{V_{Te}^2 k_z}{w-k_z V_d} - 1 \right) \right) \nu_{ei} eE_x \]

\[ + m_e^2 \left( i \frac{V_{Te}^2 k_z k_x}{w-k_z V_d} \left( \frac{V_{Te}^2 k_z}{w-k_z V_d} - 1 \right) \right) \nu_{ei} eE_y \]

\[ + m_e^2 \left( \frac{V_{Te}^2 k_x^2}{w-k_z V_d} \left( \frac{V_{Te}^2 k_z}{w-k_z V_d} - 1 \right) \right) \nu_{ei}^2 eE_z \]

(3.16c)

where

\[ \Delta = m_e^2 (\nu_{ei}^2 + \omega_e^2 ) \left( \frac{i V_{Te} k_z}{w-k_z V_d} + \nu_{ei} \right) + \frac{i V_{Te}^2 k_x^2}{w-k_z V_d} \nu_{ei} \]

(3.17)

and \( V_{Te} \) & \( \omega_e \) are the electron thermal velocity & electron cyclotron frequency defined by

\[ V_{Te} = \left( \frac{K e}{m_e} \right)^{1/2} \]

(3.18a)

\[ \omega_e = \frac{eB_0}{m_e c} \]

(3.18b)

In a similarly manner, equations (3.10) and (3.11) can be solved for the perturbed ion velocities \( v_{ix} \) and \( v_{iy} \) to yield:
where \( V_{Ti} \) and \( \Omega_i \) are the ion thermal velocity and ion cyclotron frequency defined by

\[
V_{Ti} = \left( \frac{K T_{Ti}}{m_i} \right)^{1/2} \quad (3.20a)
\]

\[
\Omega_i = \frac{e B_o}{m_i c} \quad (3.20b)
\]

If we substitute equations (3.16) and (3.19) into equation (3.7), we obtain the following equations for the components of the current density \( \mathbf{J} \):
\[ J_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z \] (3.21a)
\[ J_y = \sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z \] (3.21b)
\[ J_z = \sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z \] (3.21c)

or in matrix form

\[
\begin{bmatrix}
J_x \\
J_y \\
J_z
\end{bmatrix} =
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz}
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\] (3.22)

The components of the conductivity tensor are given by:

\[ \sigma_{xx} = -\frac{iW^2_{\text{pi}} w}{4\pi\Delta_1} + \frac{W^2_{pe} (\tilde{w}/\epsilon_i + iV^2_{\text{Te}k_z})/\epsilon_i}{4\pi w\Delta'} \] (3.23a)

\[ \sigma_{xy} = \frac{W^2_{\text{pi}} h_i}{4\pi\Delta_1} - \frac{W^2_{pe} (\tilde{w}/\epsilon_i + iV^2_{\text{Te}k_z})/\epsilon_i}{4\pi w\Delta'} \] (3.23b)

\[ \sigma_{xz} = \frac{W^2_{pe} (-iV^2_{\text{Te}k_z} + V_d k_z/\epsilon_i)}{4\pi w\Delta'} \] (3.23c)
\[
\begin{align*}
\Sigma_{y^c} &= -\frac{w^2}{4\pi\Delta_i} + \frac{w^2}{4\pi\Delta_i} \frac{\psi_{\text{ei}} + iV_{\text{Te}}^2 k_x^2 \psi_{\text{ei}}}{4\pi w \Delta'} \\
\Sigma_{y_\gamma} &= \frac{i w^2}{4\pi \Delta_i} \left( \frac{V_{\text{Ti}}^2 k_x^2}{w} - w \right) + \frac{w^2}{4\pi w \Delta'} \frac{\psi_{\text{ei}} + iV_{\text{Te}}^2 (k_x^2 + k_z^2) \psi_{\text{ei}}}{4\pi w \Delta'} \\
\Sigma_{y_\zeta} &= \frac{w^2}{4\pi w \Delta'} \frac{(-iV_{\text{Te}}^2 k_x k_z + V_d k_x \psi_{\text{ei}}') \psi_{\text{ei}}}{4\pi w \Delta'} \\
\Sigma_{z_\times} &= \frac{w^2}{4\pi w \Delta'} \frac{(-iV_{\text{Te}}^2 k_x k_z + V_d k_x \psi_{\text{ei}}') \psi_{\text{ei}}}{4\pi w \Delta'} \\
\Sigma_{z_y} &= -\frac{w^2}{4\pi w \Delta'} \frac{(-iV_{\text{Te}}^2 k_x k_z + V_d k_x \psi_{\text{ei}}') \psi_{\text{ei}}}{4\pi w \Delta'} \\
\Sigma_{z_z} &= \frac{w^2}{4\pi \Delta'} \left( \frac{iV_{\text{Te}}^2 k_x \psi_{\text{ei}}}{w} + \frac{V_d k_x \psi_{\text{ei}}}{w w} + \frac{w(\psi_{\text{ei}}^2 + \psi_{\text{ei}}')^2}{w} \right)
\end{align*}
\] 

where

\[
\begin{align*}
\tilde{w} &= w - k_z V_d \\
\Delta_x &= V_{\text{Ti}}^2 k_x - w^2 + \psi_{\text{ei}}^2
\end{align*}
\]
\[ \Delta' = (\nu_{ei}^2 + \nu_e^2)(\frac{V_{Te}^2 k_z^2}{w-k_z V_d} + \nu_{ei}) + \frac{V_{Te}^2 k_x^2}{w-k_z V_d} \nu_{ei} \]

where \( W_{pe} \) and \( W_{pi} \) are the electron and ion plasma frequencies defined by

\[ W_{pe}^2 = 4\pi N_0 e^2 / m_e \]

\[ W_{pi}^2 = 4\pi N_0 e^2 / m_i \]

We also note that the following symmetries exist:

\[ \varpi_{xy} = -\varpi_{yx} \]

\[ \varpi_{xz} = \varpi_{zx} \]

\[ \varpi_{yz} = -\varpi_{zy} \]

3.2 THE DISPERSION RELATION

The general form of the dispersion relation is given by equation (2.12), i.e.

\[
\begin{vmatrix}
R_{xx} & R_{xy} & R_{xz} \\
R_{yx} & R_{yy} & R_{yz} \\
R_{zx} & R_{zy} & R_{zz}
\end{vmatrix} = 0
\]  

(3.24)
For our problem, the elements of the determinant (3.24) are obtained by substituting the components of the conductivity tensor, given by equations (3.23a-i) into equations (2.11a-i). Introducing the following symbols:

\[ \tilde{w} = w - k_z V_d \]  \hspace{1cm} (3.25a)

\[ Q = \frac{V_T^2 k_x k_z}{x_{ei}} + iV_d k_x y_{ei} \]  \hspace{1cm} (3.25b)

\[ G = \frac{V_T^2}{x_{ei}} \tilde{w} + iV_T^2 k_z \]  \hspace{1cm} (3.25c)

\[ \Delta_e = \frac{V_T^2}{k_x x_{ei}} + \frac{\gamma_e^2 + i \frac{V_T^2 k_x y_{ei}}{G}}{c^2} \]  \hspace{1cm} (3.25d)

\[ \Delta_i = k_x^2 V_T^2 - w^2 + \gamma_i^2 \]  \hspace{1cm} (3.25e)

We have the following expressions:

\[ R_{xx} = - \frac{k_z^2}{w^2} + \frac{w^2}{c^2 \Delta_i} + \frac{w^2 y_{ei}}{c^2 \Delta_e} \]  \hspace{1cm} (3.26a)

\[ R_{xy} = \frac{i \frac{w^2}{c^2 \Delta_i} \tilde{w}}{x_{ei}} - \frac{i \frac{w^2 y_{ei}}{c^2 \Delta_e}}{x_{ei}} \tilde{w} \]  \hspace{1cm} (3.26b)
\[ R_{xz} = k_x k_z + \frac{W_{pe}^2}{c^2 \Delta e} e_i \tilde{w} Q \]  
\[ R_{yx} = - \frac{i W_{pe}^2 e_i}{c^2 \Delta i} + i \frac{W_{pe}^2 \tilde{w}}{c^2 \Delta e} \]  
\[ R_{yy} = - k_x^2 - k_z^2 - \frac{W_{pi} (V_{T_i}^2 k_x^{-2} - w^2)}{c^2 \Delta i} + \frac{W_{pe}^2 \tilde{w}}{c^2 \Delta e} (1 - \frac{\gamma_e^2}{\Delta e}) \]  
\[ R_{yz} = \frac{W_{pe}^2 \tilde{w} Q}{c^2 \Delta e} \]  
\[ R_{zx} = k_x k_z + \frac{W_{pe}^2}{c^2 \Delta e} e_i \tilde{w} Q \]  
\[ R_{zy} = - \frac{W_{pe}^2 e_i \tilde{w} Q}{c^2 \Delta e} \]  
\[ R_{zz} = - k_x^2 + i \frac{W_{pe}^2 (iV_{Te}^2 k_x^{-2} e_i \tilde{w} + V_{T_i}^2 k_x^2 e_i + W^2 (\gamma_e^2 + \eta_e^2))}{c^2 \Delta e} \]  

Note that the following symmetries exist:

\[ R_{xy} = - R_{yx} \]  
\[ R_{xz} = R_{zx} \]  
\[ R_{yz} = - R_{zy} \]
Equation (3.24), with the elements given by equations (3.26a-i), is the dispersion relation describing the propagation of low-frequency waves in a fully-ionized collisional plasma within the framework of a two-fluid model and under the assumption that both electron inertia and displacement current can be neglected.

3.3 THE REDUCED DISPERSION RELATION

The dispersion relation (3.24), with the elements given by equations (3.26a-i) is extremely complicated. This is to be expected, since we are considering a fairly general problem, viz. electromagnetic waves propagating obliquely with respect to the background magnetic field in a current-carrying collisional plasma. Most stability studies concerning electromagnetic waves dealt with various limiting cases, e.g. waves propagating parallel or perpendicular to the magnetic field. On the other hand, obliquely propagating waves were studied mainly under the electrostatic approximation, i.e. the perturbed magnetic field is assumed to be zero.

Equation (3.24) in its entirely is so complex that it is difficult to analyze not only analytically but also numerically. In what follows, we shall not
study it in its full form. Rather, we shall reduce it to a simpler form using an approximation to be discussed below.

Neglecting displacement current, the \( y \) component of the wave equation, i.e. (2.10b) is

\[
k^2 E_y = \frac{4\pi iw}{c^2} (\nabla_{yx} E_x + \nabla_{yy} E_y + \nabla_{yz} E_z)
\]

or

\[
E_y (k^2 - \frac{4\pi iw}{c^2} \nabla_{yy}) = \frac{4\pi iw}{c^2} (\nabla_{yx} E_x + \nabla_{yz} E_z)
\]  \( (3.27) \)

Using equations (2.11), the above equation can be written as

\[
E_y = \frac{- R_{yx} E_x - R_{yz} E_z}{k^2 - \frac{4\pi iw}{c^2} \nabla_{yy}}
\]  \( (3.28) \)

If the condition

\[
k^2 \gg \frac{4\pi iw}{c^2} \nabla_{yy}
\]  \( (3.29) \)

is satisfied, equation (3.28) becomes

\[
E_y \approx \frac{- R_{yx} E_x - R_{yz} E_z}{k^2}
\]  \( (3.30) \)
Substituting equation (3.30) into the x and z components of the wave equation, i.e. (2.10a) and (2.10c), we have

\[ (R_{xx} - \frac{R_{xy}R_{yx}}{k^2})E_x + (R_{xz} - \frac{R_{xy}R_{yz}}{k^2})E_z = 0 \]  \hspace{1cm} (3.31a)

\[ (R_{zx} - \frac{R_{zy}R_{yx}}{k^2})E_x + (R_{zz} - \frac{R_{zy}R_{yz}}{k^2})E_z = 0 \]  \hspace{1cm} (3.31b)

Under the conditions

(i) \( k^2R_{xx} \gg R_{xy}R_{yx} \) \hspace{1cm} (3.32a)

(ii) \( k^2R_{xz} \gg R_{xy}R_{yz} \) \hspace{1cm} (3.32b)

(iii) \( k^2R_{zx} \gg R_{zy}R_{yx} \) \hspace{1cm} (3.32c)

(iv) \( k^2R_{zz} \gg R_{zy}R_{yz} \) \hspace{1cm} (3.32d)

Equations (3.31a) and (3.31b) reduce to

\[ R_{xx}E_x + R_{xz}E_z = 0 \]  \hspace{1cm} (3.33a)

\[ R_{zx}E_x + R_{zz}E_z = 0 \]  \hspace{1cm} (3.33b)
The reduced dispersion relation is then

\[
\begin{vmatrix}
R_{xx} & R_{xz} \\
R_{zx} & R_{zz}
\end{vmatrix} = 0 \quad (3.34)
\]

Physically, the assumption leading to the reduced dispersion relation is that under conditions (3.29) and (3.32a-d), the wave has negligible $E_y$. This $E_y \approx 0$ approximation was first introduced by Perkins in his study of electromagnetic instabilities in counterstreaming ion beams. Recently, the results he obtained appeared to have experimental confirmation.

In what follows, we shall analyze the reduced dispersion relation (3.34) with the expressions for $R_{xx}$, $R_{xz}$, $R_{zx}$, and $R_{zz}$ we derive earlier. The conditions (3.29) and (3.32a-d) will be examined subsequently to make sure that they are consistent with the results obtained.
CHAPTER 4
COLD PLASMA MODEL

In section 3.3, we have seen that under the conditions (3.29) and (3.32), the original dispersion relation in the form of a 3x3 determinant is reduced to a 2x2 determinant given by equation (3.34). Even in this simplified form, it is an algebraic equation of considerable complexity. In this chapter, we study equation (3.34) in the cold plasma limit in order to establish the essential features of the waves in the simplest possible manner. The effects of temperature will be discussed in the next chapter.

Setting $T_e = T_i = 0$ in equations (3.26a), (3.26c), (3.26g) and (3.26i), we obtain the following expressions for $R_{xx}$, $R_{xz}$, $R_{zx}$ and $R_{zz}$ for a cold plasma:

$$R_{xx} = -k_z^2 + \frac{\omega_p^2 w^2}{\omega_i^2 - w^2} + i \frac{\omega_p \omega_i}{c^2 (\omega_e^2 + \omega_i^2)}$$

(4.1a)
The dispersion relation is obtained by substituting the above elements in the equation

\[ R_{xz} = R_{zx} = k_x k_z + \frac{i \nu_e i k_x V_d w^2}{c^2 (\nu_e^2 + \nu_i^2)} \]  

\[ R_{zz} = -k_x^2 + i \frac{w^2 \nu_e^2}{c^2 \nu_i w} + i \frac{\nu_e w^2 \nu_i^2 V_d^2}{c^2 w (\nu_e^2 + \nu_i^2)} \]  

The dispersion relation is obtained by substituting the above elements in the equation

\[ R_{xx} R_{zz} - R_{xz}^2 = 0 \]  

4.1 REFRACTIVE INDEX FOR A PLASMA CARRYING NO CURRENT

Although the main concern of this thesis is on the stability of electromagnetic ion cyclotron waves in a current-carrying collisional plasma, it is also of interest to study the propagation of these waves in the case when the plasma carries no current, i.e. \( V_d = 0 \).

In such a case, there is no free energy in the plasma and there is no possibility for the wave to be generated internally by an instability. The wave is now regarded as externally imposed, and the problem becomes one of determining the refractive index of the wave as a function of frequency \( w \), which is regarded as a real
variable. Setting \( V_d = 0 \) in equation (4.1) and substituting the resultant expressions in (4.2), the dispersion relation can be written as

\[
(-k_z^2 + \frac{w^2 \pi_i w^2}{c^2(n_i^2 - w^2)} + \frac{i w^2 \nu_{ei} w}{c^2(\nu_{ei}^2 + n_e^2)})(-k_x^2 \frac{i w^2 \nu_{pe}}{c^2 \nu_{ei}})
- k_x^2 k_z^2 = 0
\]  

(4.3)

We now define the refractive index

\[
\mu = \frac{k c}{w}
\]  

(4.4)

and let \( \theta \) be the angle between \( \mathbf{k} \) and the \( z \) axis. Then

\[
k_x = k \sin \theta
\]  

(4.5a)

\[
k_z = k \cos \theta
\]  

(4.5b)

Solving (4.3) for \( \mu \) and using the relations (4.5), we obtain the result

\[
\mu^2 = \frac{w^2 \pi_i}{(n_i^2 - w^2)} + \frac{\nu_{ei}^2 w^2}{w(n_e^2 + \nu_{ei}^2)}
\frac{\cos^2 \theta + \frac{\nu_{ei}^2 \sin^2 \theta}{(n_e^2 + \nu_{ei}^2)} - \frac{i \nu_{ei} w^2 \nu_{pi} \nu_{sei}^2}{w^2 \nu_{pe}(n_i^2 - w^2)}}
\]  

(4.6)
For most plasmas encountered in practice, e.g. in Q Machines and in the UCLA arc-jet, \( \nu_{ei}^2 \ll \mathcal{L}_e^2 \). Making this assumption, we can re-arrange (4.6) into the following form

\[
\mathcal{M}^2 = A + i B \tag{4.7}
\]

where

\[
A = \frac{w_{pi}^2 \cos^2 \Theta - \nu_{ei}^2 w_{pi}^2 \sin^2 \Theta}{(\mathcal{L}_i^2 - w^2)} - \frac{\nu_{ei}^2 w_{pi}^2 \sin^2 \Theta}{\mathcal{L}_e^2 (\mathcal{L}_i^2 - w^2)} \tag{4.8a}
\]

\[
B = \frac{\nu_{ei} w_{pe}^2 \cos^2 \Theta + \frac{w_{pi}^2 \nu_{ei} w_{pi} \sin^2 \Theta}{w_{pe} (\mathcal{L}_i^2 - w^2)^2}}{\cos^4 \Theta + \frac{\nu_{ei} w_{pi}^2 w_{pi}^2 \sin^4 \Theta}{w_{pe}^2 (\mathcal{L}_i^2 - w^2)}} \tag{4.8b}
\]

If \( \nu_{ei} = 0 \), \( B = 0 \) and equation (4.7) shows that \( \mathcal{M} \) is purely real. This means that the wave propagates unattenuated through the plasma. If \( \nu_{ei} > 0 \), \( \mathcal{M} \) is a complex number, the real part being a measure of its velocity of propagation and the imaginary part
indicating that the wave is damped by collisions. If we let

\[ \mu = \mu_r + i \mu_i \]  \hspace{1cm} (4.9)

and equate the real and imaginary parts of (4.7) separately to zero, we obtain the following two equations

\[ \mu_r^2 - \mu_i^2 = A \]  \hspace{1cm} (4.10a)

\[ 2 \mu_r \mu_i = B \]  \hspace{1cm} (4.10b)

Solving the above equations for \( \mu_r \) and \( \mu_i \), we obtain

\[ \mu_r = \left[ \frac{1}{2} \left( A + \sqrt{A^2 + B^2} \right) \right]^{1/2} \]  \hspace{1cm} (4.11a)

\[ \mu_i = \frac{B}{2 \mu_r} \]  \hspace{1cm} (4.11b)

Let us compute \( \mu_r \) and \( \mu_i \) for a set of parameters as representative of the UCLA arc-jet:

- \( I_i = 0.5 \times 10^6 \text{ Hz} \)
- \( W_{pe} = 5.66 \times 10^{11} \text{ Hz} \)
- \( m_e/m_i = 1/7344 \) (helium gas)
- \( \nu_{ei} = 10^7 \text{ Hz} \)

The results are shown in Table 3.1
Table 4.1

Values of Refractive Index for the Set of Parameters Given on Page 40

<table>
<thead>
<tr>
<th>$\frac{\omega}{v_0}$</th>
<th>$\theta = 85^\circ$</th>
<th>$\theta = 80^\circ$</th>
<th>$\theta = 70^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\mu_r$</td>
<td>$\mu_i$</td>
<td>$\mu_r$</td>
</tr>
<tr>
<td>0.1</td>
<td>$1.52 \times 10^5$</td>
<td>$4.79 \times 10^3$</td>
<td>$7.65 \times 10^4$</td>
</tr>
<tr>
<td>0.3</td>
<td>$1.59 \times 10^5$</td>
<td>$9.95 \times 10^3$</td>
<td>$7.98 \times 10^4$</td>
</tr>
<tr>
<td>0.5</td>
<td>$1.76 \times 10^5$</td>
<td>$2.10 \times 10^4$</td>
<td>$8.79 \times 10^4$</td>
</tr>
<tr>
<td>0.7</td>
<td>$2.18 \times 10^5$</td>
<td>$5.05 \times 10^4$</td>
<td>$1.07 \times 10^5$</td>
</tr>
</tbody>
</table>
4.2 CURRENT CARRYING PLASMA

4.2.1 General Remarks

We now investigate a plasma carrying a field-aligned current, i.e. \( V_d > 0 \) in equations (4.1a-c). The plasma now contains a source of free energy, and it is necessary to determine whether this equilibrium state is stable against small amplitude perturbations. If there exist solutions which grow with time, an instability is said to exist. Mathematically the wavenumber \( k \) is now regarded as a real number, and frequency \( w \) is treated as a complex variable. If for a certain wavenumber, one of the solutions for \( w \) contains a positive imaginary part, this solution would correspond to an instability. The real part of \( w \) gives the frequency at which the unstable oscillations occur while the imaginary part gives the rate at which the amplitude increases, i.e. the growth rate of the instability. Physically, an instability corresponds to a phenomenon in which the directed energy in the form of a dc current is converted into growing oscillations of a certain frequency and wavelength. In a sense, the plasma behaves like an oscillator.

As mentioned in chapter 1, linear theory can only
predict whether an equilibrium is stable and if not, what types of instabilities it can support and what are the initial growth rates of the instabilities. Obviously, the perturbation amplitudes cannot grow without limit. They would be limited by nonlinear effects, which are beyond the scope of linear theory.

4.2.2 Analytical Results

The dispersion relation (4.2), with the elements given by equations (4.1a-c), is a 5th order algebraic equation of complex coefficients in the variable $w$. Such an equation can only be studied numerically. However, simple analytical results, which show the essential features and give much insight into the nature of the instability, can be obtained if we simplify the dispersion relation in the following manner. Let us denote the ratio of the 2nd term to the 1st term in $R_{xz}$ by $\gamma_{xz}$ and the ratio of the 3rd term to the 2nd term in $R_{zz}$ by $\gamma_{zz}$. From equations (4.1), we have

$$|\gamma_{xz}| = \frac{\nu_{ei} V_d w^2}{c^2 (\nu_e^2 + \nu_{ei}^2) k_z}$$  \hspace{1cm} (4.12a)

$$|\gamma_{zz}| = \frac{\nu_{ei}^2 v^2 k^2}{w^2 (\nu_e^2 + \nu_{ei}^2)}$$  \hspace{1cm} (4.12b)
If the conditions

\[
\left| \gamma_{xz} \right| \ll 1 \quad (4.13a)
\]
\[
\left| \gamma_{zz} \right| \ll 1 \quad (4.13b)
\]

are satisfied, the elements \( R_{xz} \) and \( R_{zz} \) simplify to

\[
R_{xz} = R_{zx} \sim k_x k_z \quad (4.14a)
\]

\[
R_{zz} \sim - k_x^2 + i \frac{w^2 w^2_{pe}}{c^2 \nu \tilde{w}} \quad (4.14b)
\]

We shall examine the circumstances under which conditions (4.13) are valid subsequently. For the moment, let us use the above simplified forms for \( R_{xz} \) and \( R_{zz} \). The dispersion relation (4.2) then takes the form:

\[
\left\{ k_z^2 \frac{w^2}{\nu^2} - \frac{w^2_{pe}}{c^2 (\lambda_i^2 - w^2)} - i \frac{\nu_{ei} w^2_{pe}}{c^2 \tilde{\lambda}_e^2} \right\} (k_x^2 \tilde{w} - i \frac{w^2_{pe}}{c^2 \nu_{ei}})
\]

\[- k_x^2 k_z^2 \tilde{w} = 0 \quad (4.15)\]

where we have neglected \( \nu_{ei}^2 \) compared to \( \lambda_e^2 \).
Under the conditions

\[ \frac{Y_{ei} w^2 \omega (\omega_i^2 - w^2)}{\omega_i^2 w^2 \omega_i} \ll 1 \]  \hspace{1cm} (4.16a)

and

\[ \left( \frac{w^2 \omega_i^4}{c^4 \omega_i^2} \right) \left( \frac{w^2 \omega_i^2 \omega_i}{c^2 \omega_i^2} \right) \ll 1 \]  \hspace{1cm} (4.16b)

The third term in the square bracket of (4.15) can be neglected and the dispersion relation reduces to a simple quadratic equation in \( w \):

\[
(k_z c^2 + w_{pi}^2)w^2 + i \mu_{ei} k_x c^2 w - k_z c^2 \omega_i^2 \\
- i \mu_{ei} \frac{m_e}{m_i} k_x c^2 k_z V_d = 0
\]  \hspace{1cm} (4.17)

If we set \( w = x + iy \) in (4.17) and assume \( |y| \ll x \), we obtain, on equating the real and imaginary parts of the equation separately to zero, the following expressions for the real and imaginary parts of the frequency:
\[ x = \text{Re} \, w = \frac{k_z c \ell_i}{(k_z c^2 + \omega_{pi}^2)^{1/2}} \]  

\[ y = \text{Im} \, w = -\frac{\sqrt{\epsilon_e k_x c^2 \omega_{pe}}}{2m_i(k_z c^2 + \omega_{pi}^2)} (1 - \frac{k_z V_d}{x}) \]  

From (4.18b), we see that, if \( V_d < x/k_z \), \( y \) is negative and the wave is damped by collisions as it propagates through the plasma. However, when the condition

\[ V_d > x/k_z = \frac{c \ell_i}{(k_z c^2 + \omega_{pi}^2)^{1/2}} \]  

is satisfied, \( y \) is positive, corresponding to instability. The condition \( V_d > x/k_z \) is simply the mathematical statement that the parallel component of the phase velocity of the wave must be smaller than the electron drift velocity if energy is to be converted from the kinetic energy of the drifting electrons to the wave energy. This condition is the same as the condition for instability for electrostatic ion cyclotron waves predicted by Lee and Luhmann. However, while the frequency of the electrostatic ion cyclotron instability...
is always above the ion cyclotron frequency \( \Omega_i \), the frequency given by equation (4.18a) is always smaller than \( \Omega_i \). This in turns means that for the same axial wavenumber \( k_z \), the minimum drift velocity \( V_d \) required to excite the instability is smaller for the present instability.

Equation (4.18b) shows that when the instability condition (4.19) is satisfied, the growth rate is directly proportional to the electron-ion collision frequency \( \nu_{ei} \). Hence both collisions and electron drift are required for the instability. It is also important to point out that this instability has both a perturbed electric field \( \vec{E} \) and a perturbed magnetic field \( \vec{B} \). It is therefore an electromagnetic instability, in contrast to that of Lee and Luhmann,\(^{12}\) which is an electrostatic instability.

The analytical results (4.18a) and (4.18b) are arrived at by a perturbation technique, and are valid as long as the assumption \(|y| \ll x\) is satisfied. This would certainly be the case when \( V_d \) is slightly greater than \( x/k_z \), i.e. just above the threshold for instability. However, for a general set of parameters, it is possible for \( y \) to be comparable to \( x \). In these circumstances, we need to solve equation (4.17) as it stands. This is
best done numerically. To do this, it is convenient to cast equation (4.15) in non-dimensional form, i.e.

\[ Az^2 + Bz + C = 0 \]  

(4.20)

where

\[ z = \frac{w}{\Lambda_i} \]  

(4.21a)

\[ A = \frac{k_z^2 c^2}{W^2_{pe}} + \frac{m_e}{m_i} \]  

(4.21b)

\[ B = \frac{i m_e k_x^2 c^2}{m_i \Lambda_i W^2_{pe}} \]  

(4.21c)

\[ C = -\frac{k_z^2 c^2}{W^2_{pe}} - i \frac{\nu_e e k_x^2 c k_z V_d}{\Lambda_i m_i W^2_{pe} \Lambda_i} \]  

(4.21d)

For a given set of values of the parameters \( k_z, k_x, \nu_e, V_d, W^2_{pe}, m_e/m_i \) and \( \Lambda_i \), equation (4.20) can be solved numerically for the real and imaginary parts of \( z \). A computer program for equations (4.18a), and (4.18b) and (4.20) has been written. This is given in Appendix A. Some of the results obtained are shown in Table 4.2.
### Table 4.2

Values of Complex Frequency for the Set of Parameters Given on P.50

<table>
<thead>
<tr>
<th>$k_z$</th>
<th>Root for Eq. (4.20)</th>
<th>Values for EOS. (4.18A-B)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>REAL $\left( \frac{X}{k_z} \right)$</td>
<td>IMAGINARY $\left( \frac{X}{k_z} \right)$</td>
</tr>
<tr>
<td>0.08</td>
<td>0.47006E 00</td>
<td>0.23862E 00</td>
</tr>
<tr>
<td>0.09</td>
<td>0.50275E 00</td>
<td>0.24844E 00</td>
</tr>
<tr>
<td>0.10</td>
<td>0.53330E 00</td>
<td>0.25620E 00</td>
</tr>
<tr>
<td>0.11</td>
<td>0.56185E 00</td>
<td>0.26219E 00</td>
</tr>
<tr>
<td>0.12</td>
<td>0.58853E 00</td>
<td>0.26666E 00</td>
</tr>
<tr>
<td>0.13</td>
<td>0.61344E 00</td>
<td>0.26983E 00</td>
</tr>
<tr>
<td>0.14</td>
<td>0.63669E 00</td>
<td>0.27188E 00</td>
</tr>
<tr>
<td>0.15</td>
<td>0.65837E 00</td>
<td>0.27297E 00</td>
</tr>
<tr>
<td>0.16</td>
<td>0.67858E 00</td>
<td>0.27324E 00</td>
</tr>
<tr>
<td>0.17</td>
<td>0.69741E 00</td>
<td>0.27281E 00</td>
</tr>
<tr>
<td>0.18</td>
<td>0.71494E 00</td>
<td>0.27179E 00</td>
</tr>
<tr>
<td>0.19</td>
<td>0.73126E 00</td>
<td>0.27028E 00</td>
</tr>
<tr>
<td>0.20</td>
<td>0.74645E 00</td>
<td>0.26835E 00</td>
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<td>0.21</td>
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<td>0.26607E 00</td>
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<td>0.22</td>
<td>0.77377E 00</td>
<td>0.26351E 00</td>
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<td>0.23</td>
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<td>0.26071E 00</td>
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<tr>
<td>0.24</td>
<td>0.79746E 00</td>
<td>0.25773E 00</td>
</tr>
<tr>
<td>0.25</td>
<td>0.80811E 00</td>
<td>0.25461E 00</td>
</tr>
<tr>
<td>0.26</td>
<td>0.81804E 00</td>
<td>0.25137E 00</td>
</tr>
<tr>
<td>0.27</td>
<td>0.82731E 00</td>
<td>0.24804E 00</td>
</tr>
<tr>
<td>0.28</td>
<td>0.83596E 00</td>
<td>0.24466E 00</td>
</tr>
<tr>
<td>0.29</td>
<td>0.84404E 00</td>
<td>0.24124E 00</td>
</tr>
<tr>
<td>0.30</td>
<td>0.85160E 00</td>
<td>0.23780E 00</td>
</tr>
</tbody>
</table>
In the Table 4.2 on Page 49, we have assumed the following values of parameters:

\[ \frac{m_e}{m_i} = \frac{1}{7344} \]

\[ k_x = 2 \text{ cm}^{-1} \]

\[ f_i = 0.5 \times 10^6 \text{ Hz} \]

\[ V_d = 10^6 \text{ cm/sec.} \]

\[ \nu_{ei} = 10^7 \text{ Hz} \]

\[ c = 3 \times 10^{10} \text{ cm/sec.} \]

\[ \Omega_{pe} = 5.66 \times 10^{11} \text{ Hz} \]
4.2.3 Conditions for the validity of the analytical results.

The analytical results of the previous section are extremely simple. They establish the existence of an electromagnetic ion cyclotron instability in collisional plasmas. For the results to be valid, conditions (4.13a), (4.13b), (4.16a) and (4.16b) must be satisfied. Let us examine these conditions for parameters representative of the UCLA arc-jet plasma, i.e. $\omega_i = 0.5 \times 10^6$ Hz

$\omega_{pe} = 5.66 \times 10^{11}$ Hz

$m_e/m_i = 1/7344$ (helium gas)

$\nu_{ei} = 10^7$ Hz

$V_d = 10^7$ cm/sec.

From (4.18a), we see that the wave frequency is of the order of $\omega_i$ and it is clear that inequality (4.16a) is easily satisfied. Using (4.18a), condition (4.16b) can be written as

$$\frac{k_x^2c^2}{W_{pi}} \left( \frac{k_y^2c^2}{W_{pi}} + 1 \right) \gg 1 \tag{4.22}$$
For the parameters given above, (4.13a), (4.13b), and (4.22) correspond to, respectively,

\[ k_z^2 \gg 2 \times 10^{-3} \quad (4.23a) \]
\[ k_x^2 \ll 3 \times 10^2 \quad (4.23b) \]
\[ k_x^2 (22k_z^2 + 1) \gg 4.5 \times 10^2 \quad (4.23c) \]

It is clear that the above conditions cover a broad range of values for \( k_x \) and \( k_z \).

4.2.4 Validity of the reduced dispersion relation

The results we have arrived at are based on the reduced dispersion relation (3.34), rather than the original dispersion relation (3.24). We must now examine under what circumstances the results obtained are consistent with the inequalities (3.29) and (3.32a-d), upon which the derivation of (3.24) are based. For this purpose, we first note from (4.18a) that

\[ \frac{1}{\Delta_i} \equiv \frac{1}{(\lambda_{2-2w}^2)} \approx \frac{k_z^2c^2 + \frac{w^2}{w_{pi}^2}}{w_{pi}^2 \lambda_{2-2w}^2} \quad (4.24) \]

For values of \( k_z^2c^2 \) of the same order as \( \frac{w^2}{w_{pi}} \), \( w \) would be some fraction of \( \lambda_{2-2w} \) and we can write
\[ w = O(\Lambda_i) \]

\[ \frac{1}{\Lambda_i} = O\left( \frac{1}{\Lambda_i^2} \right) \]

where \( O \) denotes "of the order of". With this in mind, it is deduced from equations (3.26) (specializing to cold plasma) that for inequalities (3.29) and (3.32a-d) to hold, the wavenumbers must be such that they satisfy

\[ k^2 \gg \frac{W_{pi}^2}{c^2} \quad (4.25a) \]

\[ k^2k_z \gg \frac{W_{pi}^4 V_d}{c^4 \Lambda_i} \quad (4.25b) \]

and

\[ k^2 \gg \frac{W_{pi}^4 V_d^2}{c^4 \Lambda_i^2} \quad (4.25c) \]

where

\[ k^2 = k_x^2 + k_z^2 \]

For parameters representative of the UCLA arc-jet plasma as listed on p.51

\[ \frac{W_{pi}^2}{c^2} = 4.5 \times 10^{-2} \text{ cm}^{-2} \]

\[ \frac{V_d}{\Lambda_i} = 20 \text{ cm} \]
and (4.25a)-(4.25c) read

\[ k^2 \gg 4.5 \times 10^2 \text{ cm}^{-2} \]  \hspace{1cm} (4.26a)

\[ k^2 k_z \gg 4 \times 10^{-2} \text{ cm}^{-3} \]  \hspace{1cm} (4.26b)

and

\[ k^2 \gg 8 \times 10^{-1} \text{ cm}^{-2} \]  \hspace{1cm} (4.26c)
5.1 GENERAL REMARKS

In the previous chapter, we have shown that electromagnetic waves below the ion cyclotron frequency propagating obliquely with respect to the background magnetic field in a current-carrying plasma are unstable when the parallel phase velocity is smaller than the electron drift velocity. The results were based on the cold plasma model. In this chapter, we study the instability when temperature is included in the analysis.

The elements appearing in the reduced dispersion relation, namely $R_{xx}$, $R_{xz}$ and $R_{zz}$, are given by equations (3.62a), (3.26c) and (3.26i). Examination of these expressions show that ion temperature enters only in the quantity

$$\Delta_i \equiv k_x^2 v_T^2 - w^2 + \lambda_i^2$$
Since \((\lambda_i^2 - w^2)\) is some fraction of \(\lambda_i^2\), the term \(k_x^2 V_{Ti}^2\) is usually negligible unless \(k_x^2\) takes on large values. For example, for the UCLA arcject plasma, \(V_{Te}\) is typically \(10^7\) cm/sec. For helium ions, this would correspond to an ion thermal velocity \(V_{Ti}\) of the order of \(10^5\) cm/sec and since \(\lambda_i\) is of the order of \(10^6\) Hz, the ratio \(k_x^2 V_{Ti}^2 / \lambda_i^2\) is \(10^{-2} k_x^2\). For simplicity, we neglect the \(k_x^2 V_{Ti}^2\) term in \(\Delta_i\), although including it does not introduce any mathematical complications, since it does not increase the order of the dispersion relation in the variable \(w\).

Turning now to the effect of electron temperature, we see that it enters each of the elements \(R_{xx}\), \(R_{xz}\) and \(R_{zz}\) and greatly increases the order and complexity of the dispersion relation, which can only be studied numerically. In order to obtain analytical results, we note that the term

\[ G = \mu_{ei} \tilde{\omega} + iV_{Te} k_z^2 \]

which occurs in elements

\[ R_{xz} = R_{zx} \]

and \(R_{zz}\), consists of a temperature dependent term \(V_{Te}^2 k_z^2\) and a temperature independent term \(\mu_{ei} \tilde{\omega}\). The elements, and hence the dispersion relation, simplifies if either the inequality
or the inequality

(b) \( \nu_{ei} \langle \omega \rangle \gg k_z^2 v_T^2 \)

are satisfied. Case (a) applies when \( w \) is small, i.e. just above or below the instability threshold, or when the electrons are very hot. Case (b) corresponds to conditions in which thermal effects are small, and hence the results would differ only slightly from those of the cold plasma theory. Let us now pursue these cases in detail.

5.2 HIGH TEMPERATURE APPROXIMATION

When the inequality \( \nu_{ei} \langle \omega \rangle \ll k_z^2 v_T^2 \) is satisfied, which we shall refer to as the high temperature approximation, we obtain the following expressions for \( R_{xx} \), \( R_{xz} = R_{zx} \) and \( R_{zz} \):

\[
R_{xx} = -k_z^2 + \frac{w_{pi}^2}{c^2(\gamma_i^2 - w^2)} - \frac{\nu_{ei} w_{pe}^2}{c^2(k_z^2 \gamma_e^2 + k_x^2 \gamma_{ei})} \tag{5.1a}
\]

\[
R_{xz} = R_{zx} = k_x k_z + \frac{\nu_{ei} k_x \gamma_v w_{pe}^2}{c^2 \gamma_e^2} + \frac{w_{pe} \nu_{ei} k_x k_z}{ic^2(k_z^2 \gamma_e + k_x^2 \gamma_{ei})} \tag{5.1b}
\]
As in the cold plasma case, we shall neglect the third term in $R_{xx}$ and the second term in $R_{xz}$. Furthermore, one can show that the third term in $R_{xz}$ is of the same order as the second term and hence can be dropped as well. Under these simplifications, the dispersion relation

$$R_{xx} R_{zz} - R^2_{xz} = 0$$

can be put into the form

$$-c^2 k_z w^2 \ln \frac{\mu^2}{\mu_1^2} w^2 - c^2 k_x V_e \frac{w^2}{\mu^2} (k_z \mu_e + k_x \mu_i) w^2$$

$$+ w^2 \frac{w^2}{\mu^2} + i \left( V^2 \frac{w^2}{\mu^2} k^2 \ln \frac{\mu^2}{\mu_1^2} w^2 \right)_0 = 0$$

(5.2)

Equation (5.2) is a fourth order algebraic equation in $w$. It can be further simplified if the inequality

$$k^2 c^2 (\mu_1^2 - w^2) \ll \frac{w^2}{\mu_i}$$

(5.3)
is satisfied. This condition is ensured if the axial wavenumber is such that

\[ k_z^2 c^2 / w_{\text{pi}}^2 << 1 \quad (5.4) \]

In this limit, equation (5.2) reduces to a quadratic equation, namely,

\[ -c^2 k_z^2 W_{\text{pe}}^2 \eta_e^2 (\eta_i^2 - w^2) - c^2 k_x^2 V_{\text{Te}}^2 w_{\text{pi}}^2 (k_z^2 \eta_e^2 + k_x^2 \nu_e^2) \]

\[ + W_{\text{pi}}^2 w_e^2 \eta_e^2 w^2 + i V_{\text{Te}}^2 w_{\text{pi}}^2 k_x^2 \nu_e^2 w_{\text{pi}}^2 (w - k_z V_d) = 0 \quad (5.5) \]

If we set

\[ w = w_r + i w_i \]

in equation (5.5) and assume

\[ |w_i| << w_r \]

we obtain

\[ w_r^2 = \frac{c^2 k_x^2 V_{\text{Te}}^2 w_{\text{pi}}^2 (k_z^2 \eta_e^2 + k_x^2 \nu_e^2) + c^2 k_z^2 w_{\text{pi}}^2 \eta_e^2 \nu_e^2}{w_{\text{pi}}^2 \eta_e^2 w_{\text{pi}}^2} \quad (5.6a) \]

\[ w_i = -\frac{V_{\text{Te}}^2 k_x^2 \nu_e^2 w_{\text{pi}}^2}{2 \eta_e^2 w_{\text{pi}}^2} (1 - k_z V_d / w_r) \quad (5.6b) \]
Equation (5.6b) shows that instability occurs if the condition

\[ V_d > w_r / k_z \]

is satisfied, i.e. the parallel phase velocity is smaller than the electron drift velocity. This criterion is the same as the result obtained from the cold plasma model. If the additional condition

\[ k_z^2 \nu_e^2 \gg k_x^2 \nu_i^2 \]

is satisfied, equations (5.6) further simplifies to

\[ w_r = k_z V_A \left(1 + \frac{k_x^2 \nu_s^2}{\nu_i^2}\right)^{1/2} \]

\[ w_i = -\frac{\nu_e k_x^2 \nu_s^2}{2 \nu_i \nu_i} \left(1 - \frac{V_d}{V_A (1 + k_x^2 \nu_s^2)^{1/2}} \right) \]

where \( V_A \) and \( V_s \) are respectively the Alfven and ion sound velocities defined by

\[ V_A = \frac{c_i \nu_i}{W_{pi}} \]

\[ V_s = \frac{k T_e}{m_i} \]
Equation (5.8a) is essentially the Alfven frequency with a thermal correction term.

The results (5.6) and (5.8) are valid only in the limit

\[ \frac{k_Z^2 c^2}{\omega_{pi}^2} \ll 1 \]

In this limit, equation (5.8a) shows that the ion cyclotron wave goes over to the Alfven wave. For values of \( k_z \) not satisfying this inequality, we must go back to equation (5.2). Again setting

\[ w = w_r + i w_i \]

and assuming

\[ |w_i| \ll w_r \]

we can solve for \( w_r \) and \( w_i \). If we introduce the dimensionless variables

\[ x = \frac{w_r}{\omega_i} \]
\[ y = \frac{w_i}{\omega_i} \]

we find that \( x \) and \( y \) are given by

\[ x^2 = \frac{k_Z^2 + (k_Z^2 + k_x^2 \omega_i^2 / \omega_e^2) k_x v_s / \omega_i}{k_Z^2 + m_e / m_i} \]

\[ y = N/D \]
where

\[ K_z^2 = \frac{k_z^2 c^2}{w_{pe}^2} \quad (5.11a) \]

\[ K_x^2 = \frac{k_x^2 c^2}{w_{pe}^2} \quad (5.11b) \]

\[ D = K_z^2 (2x - 4x^3) / w_{pe}^2 + (K_z^2 + K_x^2 / \eta_e^2) 2x k_x^2 V_s / \eta_i^2 \quad (5.11c) \]

\[ N = \left( x^2 (K_z^2 + m_e / m_i) - K_z^2 \right) (x - k_z V_d / \eta_i) k_x^2 V_s / \eta_i \eta_i \eta_e \quad (5.11d) \]

Some numerical results based on equations (5.10) have been obtained. They are illustrated in Table 5.1, in which we assume the following values of parameters:

\[ w_{pe} = 5.66 \times 10^{11} \text{ Hz} \]

\[ w_{pi} = 6.6045 \times 10^{9} \text{ Hz} \]

\[ m_e / m_i = 1/7344 \]

\[ \eta_e = 3.672 \times 10^{9} \text{ Hz} \]

\[ \eta_i = 0.5 \times 10^6 \text{ Hz} \]

\[ c = 3 \times 10^{10} \text{ cm/sec.} \]

\[ k_x = 2 \text{ cm}^{-1} \]

\[ V_{Te} = 10^7 \text{ cm/sec.} \]

The computer program for solving (5.10) is given in Appendix B.
Table 5.1
Values of Complex Frequency for the Set of Parameters Given on P.62

<table>
<thead>
<tr>
<th>$\nu_e$</th>
<th>$V_d$</th>
<th>$k_z$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^7$</td>
<td>$10^7$</td>
<td>0.05</td>
<td>0.2446</td>
<td>0.1654$x10^{-3}$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$10^7$</td>
<td>0.10</td>
<td>0.4565</td>
<td>0.1798$x10^{-3}$</td>
</tr>
<tr>
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<td>$10^7$</td>
<td>0.15</td>
<td>0.6214</td>
<td>0.2033$x10^{-3}$</td>
</tr>
<tr>
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<td>$10^7$</td>
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<td>0.7421</td>
<td>0.2331$x10^{-3}$</td>
</tr>
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<td>$10^7$</td>
<td>0.25</td>
<td>0.8282</td>
<td>0.2673$x10^{-3}$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$10^7$</td>
<td>0.30</td>
<td>0.8897</td>
<td>0.3048$x10^{-3}$</td>
</tr>
<tr>
<td>$10^7$</td>
<td>$10^7$</td>
<td>0.35</td>
<td>0.9341</td>
<td>0.3446$x10^{-3}$</td>
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<tr>
<td>$10^7$</td>
<td>$5\times10^7$</td>
<td>0.05</td>
<td>0.2446</td>
<td>0.1041$x10^{-2}$</td>
</tr>
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<td>$5\times10^7$</td>
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<td>0.4565</td>
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<td>0.9341</td>
<td>0.1935$x10^{-2}$</td>
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*To be cont'd*
<table>
<thead>
<tr>
<th>$\gamma_{ei}$</th>
<th>$V_d$</th>
<th>$k_z$</th>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
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5.3 LOW TEMPERATURE CORRECTION

In this section, we are interested in obtaining the first order thermal correction to the cold plasma result. For this purpose, we assume that the inequality

$$k_z^2 V_e^2 \ll \left/ \varepsilon_i \tilde{w} \right.$$ 

is satisfied, so that the elements of the reduced dispersion relation assume the following forms:

$$(5.12a)$$

$$R_{xx} \approx -k_z^2 + \frac{w^2 \omega_{pe}^2}{c^2 \sqrt{\omega_{pe}^2 - \omega^2}}$$

$$(5.12b)$$

$$R_{xz} = R_{zx} \approx k_x k_z$$

$$(5.12c)$$

$$R_{zz} \approx -k_x^2 + i \frac{w^2 \omega_{pe}^2}{c^2 \sqrt{\omega_{pe}^2 - \omega^2}} + \frac{2w^2 \omega_{pe} k_z^2 V_e^2 k_x V_d}{c^2 \omega_{pe}^2 \tilde{w}^2}$$

Using the above expressions, the dispersion relation becomes

$$w^4 \left( -k_z^2 \right)^2 + w^3 (k_z V_d k_z^2 \omega_{pe}^2 - i \varepsilon_i \varepsilon_i \varepsilon_i k_x^2 + \varepsilon_i \varepsilon_i k_z V_d )$$

$$+ w^2 (k_z V_d k_z^2 \omega_{pe}^2 + 2i \varepsilon_i \varepsilon_i \varepsilon_i k_x^2 V_d + 2i \varepsilon_i \varepsilon_i \varepsilon_i k_z^2 k_x V_d)$$

$$+ 2i \varepsilon_i \varepsilon_i k_x^2 V_d + w (-k_z^2 \omega_{pe}^2 k_z V_d - i \varepsilon_i \varepsilon_i \varepsilon_i k_x^2 V_d)$$

$$= 0$$
Since we are interested in the first order thermal correction to the cold plasma result, we solve equation (5.13) by means of a perturbation technique. Let

\[ w = w_0 + \xi w_1 \]  

(5.14)

where \( \xi \) is the smallness parameter characterizing \( V_{Te}^2 \), \( w_0 \) is the solution independent of \( V_{Te}^2 \) and

\[ |w_1| \ll |w_0| \]

Substituting (5.14) in (5.13) and equating the coefficients of each power of \( \xi \) separately to zero, we have, to order \( \xi^0 \),

\[
\begin{align*}
& w_o^2(K_z^2 + m_e/m_i) + i\gamma_{e}(m_e/m_i)k_x w_o - K_z^2 \gamma_i^2 \\
& - i\gamma_{e}(m_e/m_i)k_z V_d = 0
\end{align*}
\]

(5.15)

The above equation is the same as equation (4.20) and the solution for \( w_0 \) is the cold plasma result given by equations (4.18).

The order \( \xi \), equation (5.13) yields the following solution to \( w_1 \):

\[ w_1 = N_1/D_1 \]  

(5.16a)
where

\[ N_1 = \frac{k^2 V^2}{V_{Te}} \left( 2iK^2 \kappa \nu d \mu_i \eta_1^2 - 2i \nu_e \nu_i w_0^2 k_z \nu_d (K^2 \mu_e / m) \right) \]  

(5.16b)

\[ D_1 = 4w_0^3 (-K^2 \eta_e^2 - \eta_e \eta_i) \]

\[ + 3w_0^2 (K_z \nu_d K^2 \eta_e^2 - i \nu_e \nu_i \eta_1 K_x^2 + \eta_e \eta_i k_z \nu_d) \]

\[ + 2w_0 (K^2 \eta_e \eta_i^2 + 2i \nu_e \nu_i \eta_1 K_x k_z \nu_d) \]

\[ - K_z \eta_e \eta_i^2 k_z \nu_d - i \nu_e \nu_i \eta_1 K_x^2 k_z^2 \nu_d \]  

(5.16c)

The first order thermal correction \( w_1 \) vanishes in the limit \( V^2_{Te} = 0 \), as expected. Since both \( N \) and \( D \) are complex, \( w_1 \) will also be complex, with the real and imaginary parts being complicated functions of a number of parameters. However, since the results would not be much different from the cold plasma results, we shall not pursue this correction further.
CHAPTER 6

SUMMARY AND DISCUSSION

6.1 SUMMARY

In this thesis, we have studied the propagation and instability of obliquely propagating low frequency electromagnetic waves in a collisional, magnetized plasma. The equations we use are Maxwell's equations and the fluid equations, neglecting displacement current in the former and electron inertia in the latter. For easy reference, the results obtained are summarized below.

6.1.1 Cold Plasma Model

(1) Refractive index for plasma carrying no current:

\[ \mu = \frac{\kappa c}{w} = \mu_r + i\mu_i \]
\[ \mathcal{M}_r = \left( \frac{1}{2} \left( A + (A^2 + B^2)^{1/2} \right) \right)^{1/2} \]

\[ \mathcal{M}_i = \frac{B}{2 \mathcal{M}_r} \]

\[ A = \frac{W_{pi}^2 \cos^2 \theta - \nu_{ei}^2 W_{pi}^2 \sin^2 \theta}{(\Lambda_i^2 - w^2)} - \frac{\nu_{ei}^2 W_{pi}^2 \sin^2 \theta}{\Lambda_e (\Lambda_i^2 - w^2)} \]

\[ A = \frac{\nu_{ei}^2 W_{pe}^2 \cos^2 \theta + \frac{W_{pi}^2 \nu_{ei}}{W_{pe}} \sin^2 \theta}{\cos^4 \theta + \frac{\nu_{ei}^2 W_{pi}^2 \sin^2 \theta}{W_{pe} (\Lambda_i^2 - w^2)^2}} \]

(2) Current-Carrying Plasma

\[ w = w_r + i w_i \]

\[ w_r = \frac{k_z c \Lambda_i}{(k_z c^2 + W_{pi}^2)^{1/2}} \]

\[ w_i = -\frac{\nu_{ei} k_x c^2 m_e}{2 m_i (k_z c^2 + W_{pi}^2)} \left( 1 - \frac{k_z V_d}{x} \right) \]

Instability condition: \[ V_d > \frac{w_r}{k_z} \]
6.1.2 High-Temperature Approximation

Under the condition

\[ \nu_e (w - k_z v_d) \ll k_z^2 v_e^2 \]

We obtain

\[ x^2 \equiv \left( \frac{w_i}{\lambda_i} \right)^2 = \frac{k_x^2 + (K_x^2 + K_e^2 / \lambda_e^2)k_x^2 v_e^2 / \lambda_i^2}{K_z^2 + m_e / m_i} \]

\[ y \equiv \left( \frac{w_i}{\lambda_i} \right) = N/D \]

where

\[ K_z^2 = k_z^2 c^2 / \omega_{pe}^2 \]

\[ K_x^2 = k_x^2 c^2 / \omega_{pe}^2 \]

\[ D = K_z^2 (2x - 4x^3) \lambda_e^2 / \omega_{pe}^2 + (K_z^2 + K_e^2 / \lambda_e^2) 2xk_x^2 v_e^2 / \lambda_i^2 \]

\[ N = (x^2 (K_z^2 + m_e / m_i) - K_x^2) / (x - k_z v_d / \lambda_i) k_x^2 \lambda_e^2 / \lambda_i^2 \lambda_e^2 \]

In the limit \( k_z^2 c^2 \ll \omega_{pe}^2 \), we have

\[ w_r^2 = \frac{c^2 k_x^2 v_e^2 \omega_{pe}^2 (k_x^2 \lambda_e^2 + k_x^2 v_e^2) + c^2 k_z^2 \omega_{pe}^2 \lambda_i^2}{\omega_{pe}^2 \lambda_e^2 \omega_{pe}^2} \]

\[ w_i = - \frac{V_d^2}{k_z^2} \frac{1}{\omega_{pe}^2} (1 - k_z v_d / w_r) \]

Instability condition: \( V_d > \frac{w_r}{k_z^2} \)
We have also obtained the first order thermal correction to the cold plasma result based on the approximation

\[ \nu_{ei}(w - k_z V_d) \gg k_z^2 V^2 \]

The results are only slightly different from the cold plasma results and are given on Section 5.3.

6.2 COMPARISON WITH ELECTROSTATIC RESULT

As mentioned in the introduction, the stability of electrostatic waves in the same type of plasma considered in this thesis has been studied in reference 12. Mathematically, the analysis of reference 12 consists of using only Poisson's equation in conjunction with the fluid equations, rather than the whole set of Maxwell's equations. The results are

\[ w = w_r + iw_i \]

\[ w_r = (\mathcal{L}_i^2 + k_x^2 V_s^2)^{1/2} \]

\[ w_i = -\frac{\nu_{ei} k_x^2}{k_z^2 m_i} (1 - k_z V_d / w_r) \]

\[ V_s = k T_e / m_i \]

Comparing with the results for the electromagnetic instability, we first note that the condition for
instability for both cases are

\[ V_d \geq \frac{w_r}{k_z} \]

and the growth rate is proportional to \( \nu_{ei} \). However, the real frequency for the electrostatic instability is above the ion cyclotron frequency \( \nu_i \). For the electromagnetic instability, it is below \( \nu_i \) and can, in fact, be as low as the Alfvén frequency. The predictions of the electromagnetic instability are thus consistent with the observations of the UCLA arc-ject experiment mentioned in chapter 1. A detailed comparison between theory and experiment is not possible at this time because of the lack of data.

6.3 SUGGESTIONS FOR FURTHER WORK

There are many areas of further investigation possible, and we shall limit to a few obvious ones. Firstly, besides the UCLA experiment, the results obtained should have some relevance to the dynamics of the topside ionosphere, where field-aligned currents exist. The significance of the electrostatic ion cyclotron instability in this region of the ionosphere has been pointed out by the authors of reference 9.
Whether the instability studied in this thesis is important or not awaits a detailed investigation.

Secondly, the results obtained are based on the reduced dispersion relation. Qualitatively, one can say that it is a good approximation for waves with large values of $k$ and for frequencies not too close to the ion cyclotron frequency (so that $1/(\Omega_i^2 - \omega^2)$ is not large). Quantitatively, however, it is unclear as to the precise values of the parameters for which the approximation is a good or a poor one. Such a quantitative criterion can only be established numerically, and it would be interesting to carry it out for a few cases.

Finally, we must bear in mind that the whole analysis is based on the fluid equations. It is well-known that use of the fluid equations would miss the effects due to wave-particle resonance. Such effects may or may not be important for the present instability. This can only be answered after an analysis based on the kinetic equation is carried out.
APPENDIX A

********** COLD PLASMA **********

THIS PROGRAM IS TO CALCULATE THE COMPLEX ROOTS OF EQUATION (4.20)
VIZ A*Z*Z + B*Z + C = 0
AND ALSO TO CALCULATE THE VALUES OF X, Y FOR EQUATIONS
(4.18A), (4.18B) RESPECTIVELY
ALL THE ABOVE THREE EQUATIONS ARE FOUND IN THESIS

DIMENSION RVD(100), RVEI(100)
COMPLEX A, B, C, D, DR, Z1, Z2
READ(1,10) RME, RMI, RKX, WI, CV, WPE
READ(1,10) (RVD(J), J=1,5)
READ(1,10) (RVEI(J), J=1,5)
10 FORMAT (E14.5)
DO 90 IJ=1,5
   VD=RVD(IJ)
   VEI=RVEI(IJ)
WRITE (2,80)
80 FORMAT (1H1 14X,'ROOT FOR EQ.(4.20)',10X,
   9'VALUES FOR EQS.(4.18A-B)'
WRITE (2,280)
280 FORMAT (/14X,'REAL',10X,'IMAGINARY',5X,'REAL',10X,'IMAGINARY')
DO 40 K=1,100
   R=FLOAT(K)
   AR=((R/100.0)*CV/WPE)**2 + RME/RMI
   AI=0.0
   A=CMPLX(AR, AI)
   BR=0.0
   BI=((RKX*CV/WPE)**2)*VEI/WI*RME/RMI
   B=CMPLX(BR, BI)
   CR=-(R/100.0*CV/WPE)**2
   CI=-(R/100.0*CV/WPE)**2*(RME/RMI)*(VEI/WI)*(R/100.0)*VD/WI
   C=CMPLX(CR, CI)

TO FIND THE ROOTS OF QUADRATIC EQUATION WITH COMPLEX COEFFICIENTS
A*Z*Z + B*Z + C = 0
THIS EQUATION IS REFERRED TO EQUATION (4.20)
Z, A, B AND C ARE REFERRED TO EQUATIONS (4.21A-D)
D = B*B - 4*A*C
DR = CSQRT(D)
Z1 = (-B + DR) / (2*A)
Z2 = (-B - DR) / (2*A)
X = ((R/100.0)*CV/WPE)/SQRT(AR)
Y = ((RKX*CV/WPE)**2)*VEI/2.0/WI*RME/RMI*(R/100.0*VD/X/WI-1)/AR

THE X AND Y EQUATIONS REFERRED TO EQUATIONS (4.18A) AND (4.18B)

RK = R/100.0
WRITE (2, 20) RK, Z1, X, Y
20 FORMAT (/4X, 'KZ=1, F4.2, 4(2X, E12.5))
DO 999 N = 1, 4
IF (K.EQ.20*N) GO TO 111
999 CONTINUE
GO TO 40
111 WRITE (2, 112)
112 FORMAT (/14X, 'ROOT FOR EQ. (4.20)', 10X, 'VALUES FOR EOS. (4.18A-B)'
WRITE (2, 380)
380 FORMAT (/14X, 'REAL', 10X, 'IMAGINARY', 5X, 'REAL', 10X, 'IMAGINARY')
40 CONTINUE
WRITE (2, 160)
160 FORMAT (/1H1, 'THE ABOVE RESULTS ARE BASED ON', //, 15X, 'THE FOLLOWING VALUES OF PARAMETERS)
WRITE (2, 60) RME, RMI, RKX, WI, VD, VEI, CV, WPE
60 FORMAT (/,
922X, SHME =, 2X, E14.5, //, 22X, SHMI =, 2X, E14.5, //,
922X, SHKX =, 2X, E14.5, //, 22X, SHWI =, 2X, E14.5, //,
922X, SHVD =, 2X, E14.5, //, 22X, SHVF =, 2X, E14.5, //,
922X, SHC =, 2X, E14.5, //, 22X, SHWPE =, 2X, E14.5)
90 CONTINUE
STOP
END

LENGTH 409, NAME CHANKATFU
APPENDIX B

HOT PLASMA

This program is to calculate the values of X and Y for equations (5.10A) and (5.10B) respectively. These equations are found in Thesis of p.61.

DIMENSION RVD(100), RKZ(100), RVEI(100)
REAL KX, KZ
READ (1,100) WPE, WPI, WE, WI, CV, KX, VTE
READ (1,100) (RVD(J), J=1,16)
READ (1,100) (RKZ(J), J=1,16)
READ (1,100) (RVEI(J), J=1,16)

100 FORMAT (E14.5)
DO 135 IJ=1,16
VD=RVD(IJ)
KZ=RKZ(IJ)
VEI=RVEI(IJ)
R1=KX*KX
R2=KZ*KZ
R3=R1*R2
R4=CV/WPE*CV/WPE
R5=WE/WPE*WE/WPE
R6=VEI/WPE*VEI/WPE
R7=WPI/WPE*WPI/WPE
R8=VTE/WPE*VTE/WPE
R9=VD/WI
R10=R9*R9
R11=VTE/WI*VTE/WI
R12=VEI/WI
R13=KZ*VD/WI
R14=VEI/WPE
R15=WI/WPE
R16=R2*R5 + R1*R6
T1=R2*R4*R5*R16
T2=R7*R5*R16
T3=R3*R8*R6
T4 = -2*R3*KZ*R8*R6*R9
T5 = -R2*R4*R5*R16
T6 = -R1*R4*R11*R7*R16*R16
T7 = -R3*R8*R6
T8 = R3*R2*R8*R10*R6
T9 = 2*R3*KZ*R9*R8*R6
T10 = -R3*R2*R10*R8*R6
T11 = -2*R3*R4*R8*R12*R16
T12 = R1*R8*R7*R12*R16
T13 = R2*R5*R14*R15
TTA = T1 + T2 + T3
TTB = T4
TTC = T5 + T6 + T7 + T8
TTD = T9
TTE = T10
XX1 = R2*R4*R5 + R1*R4*R11*R7*R16
XX2 = R2*R4*R5 + R7*R5
XX = SORT(XX1/XX2)
YY1 = R1*R6*R12*(XX - R13)
YY2 = XX*XX*(R7 + R2*R4) - R2*R4
YN = YY1 + YY2
YY3 = R2*R4*R5*(2*XX - 4*XX*XX*XX)
YY4 = 2*R1*R4*R11*R7*R16*XX
YY5 = -4*R7*R5*XX*XX*XX
YYD = YY3 + YY4 + YY5
YY = YYN/YYD
WRITE (2, 97) WPE, WPI, WE, WI, CV, VD, KX, KZ, VEI, VTE
97 FORMAT(///, 7X, 'ASSUMING THE FOLLOWING VALUES OF PARAMETERS', ///,
912X, 'VTE = ', E14.5)
WRITE (2, 96) XX, YY
96 FORMAT(///, 13X, 'X = ', E17.10, ///, 13X, 'Y = ', E17.10)
135 CONTINUE
STOP
END
REFERENCES

5. See for example, P. C. Clemmow and J. P. Dongherty, Electrodynamics of Particles and Plasmas, Addison-Wesley, Readings, Massachusetts, 1969, Chapter 10.