Survey on Heegaard Floer Homology

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Abstract

Heegaard Floer homology is an invariant of a compact 3-manifold equipped with a *spin*^{c} structure. It is conjecturally equivalent to Seiberg-Witten-Floer homology. A knot in a three-manifold induces a filtration on the homology groups, and the filtered homotopy type is a powerful knot invariant, which categorifies the Alexander polynomial.

It was defined and developed by Peter Ozsváth and Zoltán Szabó; the associated knot invariant was independently discovered by Jacob Rasmussen.

The aim of this dissertation is to give an exposition on Heegaard Floer homology for compact oriented 3-manifolds. We will also discuss some applications of this theory by illustrating examples.

摘要

Heegaard Floer 同調群是對三維緊致流形 (外加$spin^c$結構) 的一項拓樸不變量，它被猜想為等價於Seiberg-Witten-Floer同調群。一個在三維緊致流形上的結會對Heegaard Floer 同調群衍生一串代數過濾，而所得出的結果是一項結不變量，用以識別Alexander多項式。

Heegaard Floer 同調理論是由兩位數學家Peter Ozsváth及Zoltán Szabó所創立，而相關的結不變量則由Jacob Rasmussen分別發現。

本論文主要對Heegaard Floer 同調理論作簡介。我們會列舉一些例子以作討論。

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Contents

Chapter 1

Introduction

The study of Morse theory can be dated back to the last century. Before Morse, Arthur Cayley and James Clerk Maxwell had developed some of the ideas of Morse theory in the context of topography. Suppose we are given a mountainous landscape M. If f is the height function $f : M \to \mathbb{R}$ sending each point to its elevation, then the inverse image of a point in $\mathbb{R}$ is simply a contour line. Each connected component of a contour line can be either a single point, a simple closed curve, or a closed curve with a double point. Double points in contour lines occur at saddle points or passes (those extremum points). Saddle points are points where the surrounding landscape curves up in one direction and down in the other.

Imagine that *God* now pours water on this piece of landscape (i.e. flooding occurs). The region covered by water when the water reaches an elevation of x is $f^{-1}(-\infty, x]$. Consider how the topology of this region changes as the water rises. It appears, intuitively, that it does not change except when a passes the height of a critical point; that is, a point where the gradient of f is 0. To each critical point p we can associate a number called index, which counts the number of independent directions around p in which f decreases. By collecting data from critical points of f, one can roughly rebuild our original landscape.

In a formal way, Morse theory studies the relationship between a smooth manifold M and those real-valued smooth functions f defined on it. Typically, one can know much about the topology of M by analyzing the behavior of critical points around some f (e.g. height function as the above example). It can help us to find the

CW structures or the handle decompositions of the manifold for which to gain information about its homology.

One direction of this theory is Morse homology. Basically, it is a homology theory defined for any smooth manifold. It is constructed using the smooth structure and an auxiliary metric on the manifold, but turns out to be topologically invariant, and is in fact isomorphic to singular homology.

Given any smooth manifold M, let f be a Morse function (which is a real-valued function on M with non-degenerated critical points) and g a Riemannian metric on M. Such pair (f, g) gives rise to a gradient vector field ∇f. It is proven that the difference in index between any two critical points is equal to the dimension of the moduli space of gradient flows (with respect to the gradient vector filed) between those points. Therefore there is a one-dimensional moduli space of flows between a critical point of index i and one of index $i-1$. Each flow can be reparametrized by a one-dimensional translation in the domain. After modding out by these reparametrizations (which can be viewed as a $\mathbb{R}$-action on the moduli space), the quotient space is zero-dimensional, that is, a collection of oriented points representing unparametrized flow lines.

We are ready to define a chain complex $C^*(M,(f,g))$ as follows. The set of chains is the $\mathbb{Z}$-module generated by the critical points. The differential ∂ of this complex sends a critical point p of index i to a sum of index $(i-1)$ critical points, with coefficients corresponding to the number of unparametrized flow lines from p to those index $(i-1)$ critical points (to be more precise we should count with signs with respect to orientations, but this may be avoided if one uses $\mathbb{Z}_2$-coefficient instead). By careful studying of the compactification of the moduli spaces, it can be shown that $\partial^2 = 0$ and we get a homology group with respect to ∂.

While the classical Morse homology is defined only on finite dimension manifolds, many mathematicians hope to generalize this theory to the infinite dimensional case as well. Many theories on such generalization are thus developed, and some of them are due directly to Andreas Floer ([24] and [27]), while others are derived or inspired by his work (which is now known to be Floer homology). It studies some infinite-dimensional spaces where the index of critical points remains finite, such as the energy functional (which resembles the role of Morse function in classical Morse theory) for geodesics on a Riemannian manifold.

There are many branches on Floer homology. One is called Lagrangian Floer homology. The chain complex of this homology is generated by the intersection points of two Lagrangian submanifolds of a symplectic manifold and its differential counts pseudoholomorphic Whitney discs. Papers on this subject are due to Fukaya, Oh, Ono, and Ohta ([21] and [22]). It is important to note that the Floer homology of a pair of Lagrangian submanifolds may not always exist; when it does, it provides an obstruction to isotoping one Lagrangian away from the other using a Hamiltonian isotopy.

In 2001, Peter Ozsváth and Zoltán Szabó [2] invented a new version of Floer homology - the Heegaard Floer homology, which can be viewed as a variation of the Lagrangian Floer homology. It defines on an oriented 3-manifold Y with a Heegaard splitting of genus g. More precisely, Y is represented by a Heegaard diagram $(\Sigma_g, \alpha, \beta)$, where Σ_g is an oriented 2-manifold and $\alpha = \{\alpha_1, \cdots, \alpha_g\}$ and $\beta = \{\beta_1, \cdots, \beta_g\}$ are attaching circles for two handlebodies which bound Σ_g. A choice of complex structure on Σ_g induces one on its g-fold symmetric product $\mathrm{Sym}^g(\Sigma_g)$. Moreover, we obtain a pair of tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \text{ and } \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$$

embedded in $\mathrm{Sym}^g(\Sigma_g)$, which are totally real with respect to the induced complex structure on $\mathrm{Sym}^g(\Sigma_g)$.

In this set-up, one can define a new variant of Lagrangian Floer homology, with a chain complex whose generators are intersection points of $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and the boundary operator counts pseudoholomorphic disks (with some suitable boundary conditions) in $\mathrm{Sym}^g(\Sigma_g)$ with boundary lies in $\mathbb{T}_\alpha \cup \mathbb{T}_\beta$. But it turns out that the resulting group cannot bring in anything non-trivial. So Ozsváth-Szabó include a choice of reference points $z \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$ which defines a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ (this is also important in associating Spin^c structure over Y). This point z induces a subvariety $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ in $\mathrm{Sym}^g(\Sigma_g)$. Using this subvariety, one can obtain different variants of Heegaard Floer homology. For example, the simplest non-trivial version of Heegaard Floer homology, $\widehat{HF}(Y)$, counts pseudoholomorphic disks in $\mathrm{Sym}^g(\Sigma_g)$ which are disjoint from $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$. In fact, under some natural filtration on the complex, one can get various complexes $CF^-(Y)$, $CF^+(Y)$ and $\widehat{CF}(Y)$, which then gives rise to homology group HF^-, HF^+, HF^∞ and $\widehat{HF}$.

In this paper, we would like to give an exposition on Heegaard Floer homology step by step. In Chapter 2, we first review the classical Morse theory and give the definition of Morse function. We will then discuss the application of Morse theory to 3-manifold and see how Morse homology is defined. In Chapter 3, we will give some preliminaries on symplectic geometry. We then proceed to describe Lagrangian Floer homology, which relates to the version of Floer theory used by Ozsváth-Szabó. We end this chapter with discussions on the Floer complex as well as the obstructions to its existence. In the last two Chapters, we will define Heegaard Floer homology by explaining the topological prerequisite as well as its motivation. At the end we illustrate some examples and report some further developments on this subject.

Chapter 2

Morse Homology

2.1 Introduction

Apart from the theory of simplicial complex, Morse homology is another way of computing the homology of a manifold. The techniques and step needed to set the Morse complex and to exhibit their independence on the choices of data (Morse function and generic metric), are exactly the same needed to define the Floer homology group.

In this chapter we first review the classical Morse theory and give the definition of Morse function. We will then discuss the application of Morse theory to 3-manifold. Finally we will see how Morse homology is defined, which inspires the later discovery of Heegaard Floer homology. For more details refer to [16], [17], [18], [19] and [24].

2.2 Classical Morse Theory and Morse Functions

Morse theory is built for understanding the topology of a space X through real valued valued functions $f : X \to \mathbb{R}$. For the simplest case, we let X to be a smooth m-dimensional manifold, compact and without boundary, and f is assumed to be smooth and **generic**.

Definition 2.2.1 A subset of a Baire topological space is called **generic** if it contains a countable intersection of open and dense sets.

In our setting, we equip the space of smooth functions on a differentiable manifold X as well as the space of Riemannian metrics on X with some C^k **topology** for sufficiently large k. So by the Morse Lemma [16], for generic f , its critical points p are **isolated** and in some suitable local coordinates $x_1, \cdots, x_m$ near p (called the **local normal form** of p), the function f may be written as

$$f(x) = -x_1^2 - \ldots - x_i^2 + x_{i+1}^2 + \ldots + x_m^2.$$

The number of negative squares occurring here is independent of the choice of local coordinates and is called the **Morse index** ind(p) of the critical point. Functions $f : X \to \mathbb{R}$ that satisfy these conditions are called **Morse functions**. Let us look at some examples of Morse functions.

Example 2.2.1 (The height function on the sphere) Consider the unit sphere S^2 with the orthogonal coordinates (x, y, z) in $\mathbb{R}^3$; that is, S^2 is defined by the equation

$$x^2 + y^2 + z^2 = 1$$

Let $f : S^2 \to \mathbb{R}$ be a function on S^2 which assigns to each point $p = (x, y, z)$ on S^2 its third coordinate z. In other words, f is the "height function" for the sphere.

It is readily to see that f has only two critical points, namely the north pole p_0 $= (0, 0, 1)$ and the south pole $s_0 = (0, 0, -1)$. To show that f is a Morse function, it suffices to prove that p_0 and q_0 are both non-degenerated, which follows immediately by computing the Hessian of f with respect to the coordinate system (x, y). □

Let $f : X \to \mathbb{R}$ be a Morse function. One examines the topological structure of X by looking at the family of sublevel sets

$$\mathbb{X}^c := f^{-1}(-\infty, c].$$

These spaces act as the main ingredients in analyzing the topology of our ambient space X. The following propositions point out the usefulness of the function f.

Proposition 2.2.1 If f has no critical value in the interval [a,b], then $X_{[a,b]} :=$ $\{p \in X | a \leq f(p) \leq b\}$ is diffeomorphic to the product

$$f^{-1}(a) \times [0, 1]$$

Proof: Choose a generic metric $\langle\cdot,\cdot\rangle$ on X. By assumption, $\langle\nabla f, f\rangle > 0$ on [a,b], and hence we can define a new vector field φ on X by

$$\varphi = \frac{1}{\langle\nabla f, f\rangle}$$

Consider the integral curve $c_p(t)$ of φ which starts at a point p of $f^{-1}(a)$. Then we obtain,

$$\begin{aligned}
\frac{d}{dt}f(c_p(t)) &= \langle\frac{dc}{dt}(t), f\rangle \\
&= \langle\varphi_{c(t)}, f\rangle \\
&= \frac{1}{\langle\nabla f, f\rangle}\langle\nabla f, f\rangle \\
&= 1
\end{aligned}$$

Thus the integral curve $c_p(t)$ keeps an upward climb with the constant speed 1 with respect to the *height* defined by f. Since it starts at the level $f = a$ at the time $t = 0$, it will reach the level $f = b$ at the time $t = b - a$. Define a map $h : f^{-1}(a) \times [0,1] \to X_{[a,b]}$ by

$$h(p,t) = c_p(t).$$

By the facts that $c_p(t)$ depends smoothly on both p and t, and also that two distinct integral curves do not meet (uniqueness of intgral curves), h is a diffeomorphism. Therefore $X_{[a,b]} \cong f^{-1}(a) \times [0,1]$. Together with the observation $f^{-1}(a) \times [0,1] \cong f^{-1}(a) \times [0, b-a]$, we complete the proof. □

Corollary 2.2.2 The spaces X^c are diffeomorphic to each other as c varies in each interval of regular (i.e. noncritical) values. In other words if f has no critical values in the interval [a,b], then $X^a \cong X^b$.

Proof: It follows directly from proposition 1.1. Geometrically what we do is to let the manifold X^a flow along ∇f. Then after a certain period of time, X^a meets and conicides with X^b. □

2.3 Handlebody Decomposition for 3-manifold

The example below shows how one can use a Morse function to give a special kind of decomposition of an oriented 3-manifold Y that is known as a **Heegaard splitting**.

Example 2.3.1 (Heegaard decomposition of a 3-manifold) : Choose the Morse function $f : Y \to \mathbb{R}$ to be *self-indexing*, i.e. so that all the critical points of index i lie on the level $f^{-1}(i)$. Then the cut $f^{-1}(3/2)$ at the half way point is a Riemann surface Σ_g of genus g (here g is equal to the number of index 1 critical points of f), and the sublevel set $Y^{3/2}$ is a *handlebody of genus* g, i.e. the union of a closed 3-ball D^3 with g 1-handles. Before going any further let me first give the definitions for some terms discussed :

Definition 2.3.1 A **n-dimensional λ-handle** is defined to be the product $D^\lambda \times D^{n-\lambda}$, which is homeomorphic to the closed n-ball. A **handlebody** of type (n, λ) is an n-dimensional manifold that is obtained from the closed n-ball by attaching only λ-handles.

Proposition 2.3.1 With the notations used in Example 1.3.1, we have $f^{-1}(-\infty, \frac{3}{2}] \cong f^{-1}[\frac{3}{2}, 3]$.

Proof: Without loss of generality we can assume that f has only one index 0 critical point and one index 3 critical point (refer to Theorem 3.35 of [1]). Then the handle decomposition of Y is given by

$$Y = h^0 \cup (h_1^1 \sqcup \cdots \sqcup h_{k_1}^1) \cup (h_1^2 \sqcup \cdots \sqcup h_{k_2}^2) \cup h^3$$

By computing the Euler number of Y, we obtain

$$\chi(Y) = 1 - k_1 + k_2 - 1 = -k_1 + k_2$$

On the other hand, the Euler number of an odd-dimensional manifold is always 0, thus $k_1 = k_2$. Let this value be g, and it is now readily to see that $f^{-1}(-\infty, \frac{3}{2}]$ and $f^{-1}[\frac{3}{2}, 3]$ are both homeomorphic to the genus g 3-dimensional solid torus. $\square$

Therefore, Y can be built from a single copy of the surface $\Sigma = f^{-1}(3/2)$ by attaching handlebodies $U_\alpha := Y^{3/2}, U_\beta$ to its two sides. In other words, Y can be obtained by gluing the two handlebodies together along their common boundary Σ_g. In symbol we write $Y = U_\alpha \cup_{\Sigma_g} U_\beta$, and this is called a **Heegaard decomposition** for Y.

Theorem 2.3.2 (refer to [13])(existence of Heegaard decomposition) Let Y be an oriented compact 3-manifold. Then Y admits a Heegaard decomposition.

Proof: Start with a triangulation of Y. The union of the vertices and the edges gives a graph in Y. Let U_0 be a small neighborhood of this graph. In other words replace each vertex by a closed ball, and each edge by a solid cylinder. By definition U_0 is a handlebody. It is easy to see that Y - U_0 is also a handlebody, given by a regular neighborhood of a graph on the centers of the triangles and tetrahedra in the triangulation. □

The attaching map of U_α is determined by the **loops** in Σ_g that bound discs in U_α, which are called **attaching circles** :

Definition 2.3.3 A set of attaching circles $\{\alpha_1, \ldots, \alpha_g\}$ for a handlebody U is a collection of closed embedded curves in $\Sigma_g = \partial U$ with the following properties :

- The curves α_i are disjoint from each other.
- $\Sigma_g - \alpha_1 - \ldots - \alpha_g$ is connected.
- The curves α_i bound disjoint embedded disks in U.

So Y can be described by two collections of attaching circles, namely $\alpha := \{\alpha_1, \ldots, \alpha_g\}$ and $\beta := \{\beta_1, \ldots, \beta_g\}$, of disjointed circles on Σ_g. This description is known as the **Heegaard diagram** for Y.

Definition 2.3.4 Let $(\Sigma_g, U_\alpha, U_\beta)$ be a genus g Heegaard decomposition for Y. A Heegaard diagram is given by Σ_g together with a collection of attaching circles α and β respectively for U_α and U_β. In this case we say Y admits a **Heegaard diagram** $(\Sigma_g, \alpha, \beta)$.

Example 2.3.2 There is a well known decomposition of the 3-sphere $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ into two solid tori (handlebodies of genus 1), $U_1 := \{|z_1| \leqslant |z_2|\}$ and $U_2 := \{|z_1| \geqslant |z_2|\}$, and the corresponding circles in the 2-torus $\Sigma_1 = \{|z_1| = |z_2|\}$ are

$$\alpha_1 = \{\frac{1}{\sqrt{2}}(e^{i\theta}, 1) : \theta \in [0, 2\pi]\}, \beta_1 = \{\frac{1}{\sqrt{2}}(1, e^{i\theta}) : \theta \in [0, 2\pi]\}$$

with a single intersection point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In this example, α_1 and β_1 are also called **meridian** and **longitude** respectively. One way to see that both U_1 and U_2 are solid tori is the following. By introducing new coordinates, U_1 and U_2 can be written as

$$U_1 = \{(ae^{i\alpha}, be^{i\beta}) : a \leqslant \frac{1}{\sqrt{2}}; \alpha, \beta \in [0, 2\pi]\}, U_2 = \{(ae^{i\alpha}, be^{i\beta}) : b \leqslant \frac{1}{\sqrt{2}}; \alpha, \beta \in [0, 2\pi]\}$$

where $a, b \geqslant 0$ and $a^2 + b^2 = 1$. Now consider the standard solid torus whose axial sections are pairs of 2-disks of radius $\frac{1}{\sqrt{2}}$. The mapping $(ae^{i\alpha}, be^{i\beta}) \mapsto (a, \alpha, \beta)$ defines a homeomorphism of the manifold U_1 onto the standard solid torus. In a similar way $(ae^{i\alpha}, be^{i\beta}) \mapsto (b, \beta, \alpha)$ defines a homeomorphism of U_2 onto the solid torus.

2.4 Stable manifold and Unstable manifold

Let Y be an oriented 3-manifold and $(\Sigma_g, \alpha, \beta)$ be its Heegaard diagram. In [2], Peter Ozsváth and Zoltán Szabó capture information about the intersection points between the two families of attaching curves α and β. To do this we need to define some terms.

Definition 2.4.1 Again let X to be a smooth m-dimensional manifold, compact and without boundary, and $f : X \to \mathbb{R}$ be a Morse function. Suppose p is a critical point of f. The stable and unstable manifolds at p of the flow -∇f are defined respectively as

$$W_f^s := \{p\} \cup \{u(t) \in X : t \in \mathbb{R}, \dot{u}(t) = -\nabla f(u(t)), \lim_{t \to +\infty} u(t) = p\}$$

$$W_f^u := \{p\} \cup \{u(t) \in X : t \in \mathbb{R}, \dot{u}(t) = -\nabla f(u(t)), \lim_{t \to -\infty} u(t) = p\}$$

Of course, the question arises whether $W_f^s(p)$ and $W_f^u(p)$ are indeed manifolds (refer to Corollary 6.3.1 in [14]). And it is readily to see that $W_f^u(p)$ is diffeomorphic to $\mathbb{R}^d$, where $d = \text{ind}(p)$.

So what is the relationship between unstable manifolds and handles? We explain this by using the example of Morse decomposition of the torus $\mathbb{T}=S^1 \times S^1$. Let $\mathbb{T} \to \mathbb{R}$ be the height function, which is a Morse function. Assume that f has unique minimum and maximum. For $c \in \mathbb{R}$, if c is close to min$f$, then the sublevel set X^c is diffeomorphic to the closed ball $D^2 := \{x \in \mathbb{R}^2 : |x| \leqq 1\}$ of dimension 2. When c passes a critical point p of index 1, a 1-handle (homeomorphic to $[0,1]\times D^{2-1}=[0,1]\times D^1$) is added.

Therefore, one should think of this 1-handle as a neighborhood of the unstable manifold $W^u_f(p) \cong \mathbb{R}^1$, which is $D^1 \times D^{2-1}$. The D^1 part corresponds to the unstable manifold, and the boundary ∂D^1 gives the **attaching circle** of this 1-handle (while ∂D^{2-1} yields the attaching belt of the handle); see [16].

Now we get back to our 3-manifold Y and its Heegaard diagram $(\Sigma_g, \alpha, \beta)$. Similar to the above example, each $\alpha_j \in \alpha$ is the intersection $W^s_f(p_j) \cap \Sigma_g$ of the upward gradient trajectories from some index 1 critical point p_j of f with the level set Σ_g (viewing those α_j as $\partial D^{3-1} = \partial D^2 = S^1$). Similarly, $\beta_k \in \beta$ are the intersections with Σ_g of the downward gradient trajectories from the index 2 critical points q_k. Hence, each intersection point $\alpha_i \cap \beta_k$ corresponds to a gradient trajectory from q_k to p_j.

2.5 Trajectory flows and the Morse-Smale-Witten Complex

As suggested in [17], much of the geometric information contained in a self-indexing Morse function f can be expressed in terms of the Morse-Smale-Witten complex $(C_*(X; f), \partial)$, where X here is a closed smooth manifold of finite dimension. To define the complex, one needs some extra conditions on f, which is so-called **Morse-Smale-Floer conditions**:

Definition 2.5.1 An Morse function f is said to be of **Morse-Smale type** if every sequence $\{x_n\}$ with:

1. $|f(x_n)|$ bounded
2. $||df(x_n)|| \to 0$ for $n \to \infty$

contains a convergent subsequence.

Furthermore, if for every pair (x, y) of critical points of f the unstable submanifold $W_f^u(x)$ intersects the stable submanifold $W_f^s(y)$ transversally, then f is of **Morse-Smale-Floer type**.

Obviously, the Morse-Smale conditions are automatically satisfied if X is compact. It is also satisfied if f is proper, namely if for every $c \in \mathbb{R}$, the set

$$\{x \in X : |f(x)| \leq c\}$$

is compact. However, the Morse-Smale conditions are more general than that and it holds for example for the energy functional on the space of closed curves of *Sobolev class* $H^{1,2}$ on a compact Riemannian manifold.

It first appears that these conditions are somewhat restrictive on the class of Morse function on X. Yet according to Smale, Milnor and Witten ([14]), at least if X is finite dimensional and compact, the set of all functions satisfying the Morse condition as well as the set of all Riemannian metrics for which a given Morse function satisfies the Morse-Smale-Floer condition are **generic**.

From now on, unless otherwise stated, we will assume that our Morse functions are of Morse-Smale-Floer type.

Now we define our **trajectory space**.

Definition 2.5.2 For critical points p and q, we consider the set

$$\mathcal{M}(q,p) := \{u(t) \in X : t \in \mathbb{R}, \dot{u}(t) = -\nabla f(u(t)), \lim_{t\to+\infty} u(t) = p \lim_{t\to-\infty} u(t) = q\}$$

In other words, $\mathcal{M}(q,p)$ is the **trajectory space** containing flow lines flowing from q to p (with respect to $-\nabla f$), i.e. $\mathcal{M}(q,p) := W_f^u(q) \cap W_f^s(p)$. And we define the relative index of q and p as

$$\mu(q,p) := \dim(W_f^u(q) \cap W_f^s(p)).$$

Recall that two submanifold X_1, X_2 of X intersect transversally if for all $x \in X_1 \cap X_2$, the tangent spaces $T_x X$ is the linear span of the tangent spaces $T_x X_1$ and $T_x X_2$. If X is finite dimensional, then if X_1 and X_2 intersect transversally at x, we have

$$\dim(X_1) + \dim(X_2) = \dim(X_1 \cap X_2) + \dim(X).$$

For finite dimensional X, the Morse index $\text{ind}(p)$ of all critical points p of f are finite. Hence we have

$$\mu(q,p) = \text{ind}(q) - \text{ind}(p).$$

There is a group action of $\mathbb{R}$ on $\mathcal{M}(q,p)$, namely $(a \cdot u)(t) := u(t+a)$ for $a \in \mathbb{R}$ and $u \in \mathcal{M}(q,p)$. We denote the quotient by $\widehat{\mathcal{M}}(q,p) := \mathcal{M}(q,p)/\mathbb{R}$. And it has dimension

$$\dim(\widehat{\mathcal{M}}(q,p)) = \text{ind}(q) - \text{ind}(p) - 1.$$

From [20], it states that we can identify $\widehat{\mathcal{M}}(q,p)$ with $\mathcal{M}(q,p) \cap \{z \in X | f(z) = a\}$ for some a between $f(q)$ and $f(p)$. It follows that $\widehat{\mathcal{M}}(q,p)$ is compact.

We first prove several lemma, which will be useful to our later context. Details can also be founded in [14].

Lemma 2.5.3 f is decreasing along flow lines. In particular, there are no nonconstant flow lines with that

$$x(-\infty) = x(\infty).$$

Proof: We compute

$$\begin{aligned} \frac{d}{dt} f(x(t)) &= df(x(t))\dot{x}(t) \\ &= \langle \text{grad} f(x(t)), \dot{x}(t) \rangle \\ &= -||\dot{x}(t)||^2. \end{aligned}$$

and the result follows. □

Lemma 2.5.4 For any flow line, we have $\text{grad} f(x(t)) \to 0$ or $|f(x(t))| \to \infty$ whenever $t \to \pm\infty$.

Proof: If for example $f_\infty = \lim_{t \to \infty} f(x(t)) > -\infty$, then for $0 \leq t \leq \infty$, we have

$$f_0 := f(x_(0)) \geq f(x(t)) \geq f_\infty,$$

since f is decreasing. And for $t_1, t_2 \in \mathbb{R}$,

$$\begin{aligned} f(x(t_1)) - f(x(t_2)) &= -\int_{t_1}^{t_2} \frac{d}{dt} f(x(t)) dt \\ &= \int_{t_1}^{t_2} ||\dot{x}(t)||^2 dt \\ &= \int_{t_1}^{t_2} ||\text{grad} f(x(t))||^2 dt. \end{aligned}$$

So it implies

$$\int_0^\infty ||\dot{x}(t)||^2 := f_0 - f_\infty < \infty.$$

and hence $\lim_{t\to\infty} \dot{x}(t) = 0$. □

Lemma 2.5.5 Let $x(t)$ be a flow line for which $f(x(t))$ is bounded. Then the limits $x(\pm\infty) := \lim_{t\to\pm\infty} x(t)$ exist and are critical points of f.

Proof: By Lemma 2.5.4, $\text{grad}f(x(t)) \to 0$ for $t \to \pm\infty$. We analyze the situation as $t \to +\infty$ (the case for $t \to -\infty$ follows similarly). By the Morse-Smale condition, we can find a sequence $\{t_n\} \subset \mathbb{R}$ with $t_n \to +\infty$ for $n \to +\infty$, such that $x(t_n)$ converges to some critical point $x(+\infty)$ of f. We wish to show that $\lim_{t\to+\infty} x(t)$ exists, and it then has to coincide with the critical point $x(+\infty)$.

Notice that by the non-degeneracy of $x(+\infty)$, we can find a neighborhood U of $x(+\infty)$, with the property that any flow line in U containing $x(+\infty)$ as an accumulation point of some sequence $x(t_n)$, is contained in the stable manifold $W_f^s(x(+\infty))$ of $x(+\infty)$. Obviously, $x(t)$ satisfies the above condition and hence it lies in $W_f^s(x(+\infty))$. This implies $\lim_{t\to+\infty} x(t) = x(+\infty)$. □

Lemma 2.5.6 Suppose $||\text{grad}f(x(t))|| \geq \varepsilon$ for $t_1 \leq t \leq t_2$. Then

$$d(x(t_1), x(t_2)) \leq \frac{1}{\varepsilon}(f(x(t_1)) - f(x(t_2))).$$

where $d(\cdot, \cdot)$ is the Euclidean distance.

Proof:

$$\begin{aligned} d(x(t_1), x(t_2)) &\leq \int_{t_1}^{t_2} ||\dot{x}(t)||dt \\ &\leq \frac{1}{\varepsilon}\int_{t_1}^{t_2} ||\dot{x}(t)||^2 dt \\ &= \frac{1}{\varepsilon}(f(x(t_1)) - f(x(t_2))). \end{aligned}$$

□

Lemma 2.5.7 Suppose $\{x_n\} \subset X$ converges to x_0. Then for any $T > 0$, the flow lines $x_n(t)$ with $x_n(0) = x_n$ converge to the flow line $x_0(t)$ on $[-T, T]$.

Proof: This follows from the continuous dependence of solutions of ODEs on the initial data under the assumption of the Picard-Lindelöf theorem. The proof of that theorem is based on the Banach fixed point theorem, and the fixed point produced in that theorem depends continuously on a parameter, see [15] p.129. Thus the curves $x_n(t)$ converge uniformly to $x_0(t)$ on any finite interval $[-T, T]$. □

Lemma 2.5.8 Let q and p be critical points of f and let $\{x_n(t)\}$ be a sequence of flow lines in $\mathcal{M}(q,p)$ with $\lim_{t\to-\infty} x_n(t) = q$ and $\lim_{t\to+\infty} x_n(t) = p$ for all n. Then after choosing a subsequence, $\{x_n(t)\}$ converges to some flow line $x_0(t)$ on some compact interval in $\mathbb{R}$.

Proof: Let $t_0 \in \mathbb{R}$. If, for some subsequence, that

$$||\text{grad}f(x_n(t_0))|| \to 0 \text{ as } n \to \infty,$$

then by the Morse-Smale condition, we may assume that $x_n(t_0)$ converges, and the convergence of the flow lines on compact intervals then follows from Lemma 2.5.7. So we remain to show the case that

$$||\text{grad}f(x_n(t_0))|| \geq \varepsilon$$

for all n and some $\varepsilon > 0$. Since $f(x_n(t)$ is bounded between $f(q)$ and $f(p)$, $||\text{grad}f(x_n(t))|| \to 0$ when $t \to \infty$ by Lemma 2.5.4. It implies that we can find $t_n < t_0$ with

$$||\text{grad}f(x_n(t_n))|| = \varepsilon \text{ and } ||\text{grad}f(x_n(t))|| \geq \varepsilon$$

for $t_n \leq t \leq t_0$. Therefore,

$$\begin{aligned} f(x(t_n)) - f(x(t_0)) &= \int_{t_n}^{t_0} ||\text{grad}f(x(t))||^2 dt \\ &\geq \varepsilon^2 \int_{t_n}^{t_0} dt \\ &= \varepsilon^2 |t_n - t_0|. \end{aligned}$$

and we get $|t_n - t_0| \leq \frac{1}{\varepsilon^2}|f(x(t_n)) - f(x(t_0))| \leq \frac{1}{\varepsilon^2}|f(p) - f(q)|$. It implies $t_n \nrightarrow -\infty$. Since $\mathcal{M}(q,p)$ is compact, the sequence $y_n := x_n(t_n) \subset \mathcal{M}(q,p)$ converges to some x_0. According to Lemma 2.5.7., by choosing $y_n(t) = x_n(t)$, we see that $x_n(t)$ converges on any compact interval towards some flow line $x_0(t)$. □

Lemma 2.5.9 Let q, p be critical points of f. For any sequence $\{x_n(t)\} \subset \mathcal{M}(q,p)$, after selection of a subsequence, there exist critical points

$$q = q_1, q_2, \cdots, q_k = p,$$

flow lines $y_i \in \mathcal{M}(q_i, q_{i+1})$ and $t_{n,i} \in \mathbb{R}$ $(i = 1, \cdots, k-1, n \in \mathbb{N})$ such that the flow lines $x_n(t + t_{n,i})$ converge to y_i for $n \to \infty$. In this case, we say that the sequence $x_n(t)$ converges to the **broken trajectory** $y_1\#y_2\#\cdots\#y_{k-1}$.

Proof: By Lemma 2.5.8., $x_n(t)$ converges (after selection of a subsequence if necessary) to some flow line $x_0(t)$. $x_0(t)$ need not be in $\mathcal{M}(q,p)$, but the limit points (which exist and are critical points of f by Lemma 2.5.5.) must satisfy

$$f(q) \geq f(x_0(-\infty) \geq f(x_0(\infty) \geq f(p).$$

If, for example $f(q) = f(x_0(-\infty)$, then we have $\lim_{t\to-\infty} x_n(t) = \lim_{t\to-\infty} x_0(t)$ for all n, which implies $q = x_0(-\infty)$.

If $f(q) > f(x_0(-\infty)$, we choose $t_{n,i}$ such that

$$f(x_0(-\infty) > f(x_n(t_{n,i}) > f(p).$$

By Lemma 2.5.8., $x_(t + t_{n,i})$ converges to a limiting flow line, say $y_0(t)$. By our choice of $t_{n,i}$, we have

$$f(q) \geq f(y_0(-\infty) \text{ and } f(y_0(\infty) \geq f(x_0(-\infty),$$

since otherwise the flow line $y_0(t)$ would contain the critical point $x_0(-\infty)$ in its interior.

If $f(p) > f(y_0(-\infty)$ or $f(y_0(\infty) > f(x_0(-\infty)$, we repeat the same process. This must stop after a finite number of times, because the critical points of f are isolated by the non-degeneracy assumption on the critical points of f. Therefore the result follows. □

Lemma 2.5.10 In the situation of Lemma 2.5.9, we have

$$\sum_{i=1}^{k-1} \mu(p_i, p_{i+1}) = \mu(q,p)$$

Sketch of proof: It suffices to prove the case when $k = 3$, as the general case follows easily by induction.

We want to show the following: suppose p_1 is connected to p_2 and p_2 to p_3 by the flow. Then $\mu(p_1, p_3) = \mu(p_1, p_2) + \mu(p_2, p_3)$. We have that by this fact : in some small neighborhood U of p_2 in $W^u(p_1) \cap W^s(p_3)$, U can be made diffeomorphic to $U^1 \times U^2$, where U^1 and U^2 are some neighborhood of p_2 in $W^u(p_1) \cap W^s(p_2)$ and $W^u(p_2) \cap W^s(p_3)$ respectively.

Such local product structure is possible as one can show that $W^s(p_2)$ (and respectively for $W^u(p_2)$) is a **leaf** of the smooth stable (unstable) **foliation** of p_2 in U. The stable and unstable foliations yield a local product structure in the sense that each point near p_2 is the intersection of precisely one stable and one unstable leaf. In particular, by assumption, $W^u(p_1)$ intersects each leaf of stable foliation of p_2 transversally in some manifold of dimension $\mu(p_1, p_2)$, and similarly $W^s(p_3)$ intersects each leaf of unstable foliation p_2 transversally in some manifold of dimension $\mu(p_2, p_3)$. These altogether verifies the claim. □

Lemma 2.5.11 Suppose that q, p ($q \neq p$) are critical points of f, connected by the flow, with

$$\mu(q, p) = 1.$$

Then there exist only finitely many trajectories from q to p.

Proof: For any point x on such a trajectory, we have

$$f(q) \geq f(x) \geq f(p)$$

as f is decreasing along the flow line. We may assume that there exist $\varepsilon > 0$ such that on each flow line from q to p, we can find some x with $||\text{grad} f(x)|| = \varepsilon$. This is because otherwise we would have a sequence of flow lines $\{s_i\}$ from q to p with $\sup_{x \in s_i} ||\text{grad} f(x)|| \to 0$ as $i \to \infty$, by Lemma 2.5.4. And by the Morse-Smale condition, a subsequence of $\{s_i\}$ would converge to a flow line s with $\text{grad} f(x) \equiv 0$ on s. s would then be constant, which is impossible since s connects q and p by Lemma 2.5.9.

Therefore, if, contrary to our assumption, we have a sequence $\{s_i\}$ of trajectories from q to p, we can select $x_i \in s_i$ with $||\text{grad} f(x_i)|| = \varepsilon$. By the compactness of

$\mathcal{M}(q,p)$, we can find a convergent subsequence of $\{x_i\}$ which converge to some limiting point x, hence also of $\{s_i\}$ by Lemma 2.5.7. The limiting trajectory s of $\{s_i\}$ also has to connect q and p, because by assumption $\mu(q,p) = 1$ and Lemma 2.5.10 rules out that s is a broken trajectory connecting other critical points of f. The Morse-Smale-Floer condition implies that s is isolated in the 1-dimensional manifold $\mathcal{M}(q,p)$. This is not compatible with the assumption that there exists a sequence $\{s_i\}$ of different flow lines converging to s. □

Lemma 2.5.12 Let q, p be critical points of f connected by the flow with

$$\mu(q,p) = 2.$$

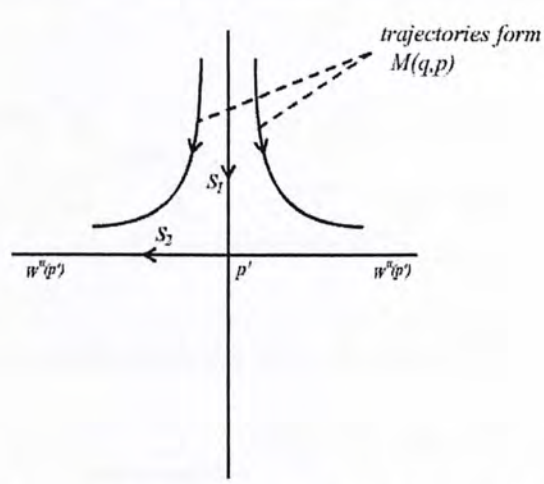
Then each component of the space of flow lines from q to p either is compact after including q and p (diffeomorphic to S^2), or its boundary consists of two different broken trajectories from q to p.

Proof: If a component C of $\mathcal{M}(q,p)$ is compact (after including p and q), then it is a 2-dimensional manifold that is a smooth family of curves, flow lines from q to p with common end points but disjoint interiors. Thus, such a component is diffeomorphic to S^2.

Assume that C is not of the above case, then by Lemma 2.5.9., there exist broken trajectories from q to p in the boundary of C. We want to show that such trajectories must occur in pairs. Let $s_1\#s_2$ be a trajectory in the boundary with $s_1(-\infty) = q$ and $s_2(\infty) = p$. We put $\acute{p} = s_1(\infty) = s_2(-\infty)$. In some suitable neighborhood of $\acute{p}$, we have that $\mathcal{M}(q,p)$ is a smooth surface containing s_1 in its interior.

$\mathcal{M}(q,p)$ intersects a smooth 1-dimensional family of leaves of the stable foliation in a 1-dimensional manifold. The family of those stable leaves intersected by $\mathcal{M}(q,p)$ then is parameterized by a smooth curve in $W^u(\acute{p})$ containing $\acute{p}$. Therefore, $\mathcal{M}(q,p)$ contains different flow lines originating from q in opposite directions, and these flow lines when passing to limit would give a trajectory from q to $\acute{p}$ in the boundary of C. We thus get another flow line corresponding to s_2 (similarly for s_1). □

Now we are ready to state the following compactification theorem, which is essential in defining the boundary operator between the Morse-Smale-Witten complexes (refer to [14], [17] and [18]).



[figure : s_1 and s_2 are the limiting flow lines]

Theorem 2.5.13 For generic Riemannian metrics, $\widehat{\mathcal{M}}(q,p)$ is a smooth manifold of dimension ind(q) - ind(p) - 1. Moreover,

1. If ind(q) - ind(p) - 1 $= 0$, then $\widehat{\mathcal{M}}(q,p)$ is compact.

2. If ind(q) - ind(p) - 1 $= 1$, then $\widehat{\mathcal{M}}(q,p)$ will be a union of circles and open intervals, and we have a suitable compactification of $\widehat{\mathcal{M}}(q,p)$ so that the boundary can be identified with the set

$$\bigcup_{ind(r)=ind(q)-1} \widehat{\mathcal{M}}(q,r) \times \widehat{\mathcal{M}}(r,p)$$

i.e. the set of once-broken flow lines from q to p.

We now start to define the Morse-Smale-Witten complex. The k-chains in this complex are finite sums of critical points of index k ($Crit_k(f)$)

$$C_k(X;f) := \{ \sum_{x \in Crit_k(f)} a_x x : a_x \in \mathbb{Z} \},$$

We then define the boundary operator $\partial : C_k(X;f) \to C_{k-1}(X;f)$ of the complex by

$$\partial x := \sum_{y \in Crit_{k-1}(f)} n(x,y) y$$

where $n(x,y)$ counts the number elements of $\widehat{\mathcal{M}}(x,y)$ with the orientation or sign.

We claim that $C_*(X;f)$ is a chain complex, i.e. $\partial^2 = 0$. To see this, we calculate

$$\begin{aligned}
\partial^2 x &= \sum_{y \in Crit_{k-1}(f)} n(x,y)\partial y \\
&= \sum_{y \in Crit_{k-1}(f)} \sum_{z \in Crit_{k-2}(f)} n(x,y)n(y,z)z
\end{aligned}$$

According to the compactification theorem, the space $\widehat{\mathcal{M}}(x,z)$ has boundary

$$\bigcup_{ind(r)=ind(q)-1} \widehat{\mathcal{M}}(q,r) \times \widehat{\mathcal{M}}(r,p)$$

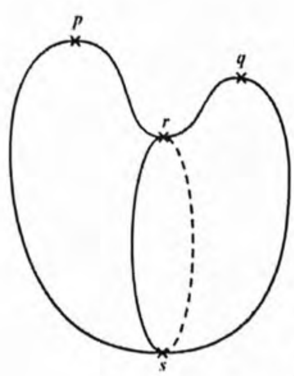
which is just the set of once-broken flow lines from x to z. By choosing a suitable orientation of $\widehat{\mathcal{M}}(x,z)$ compatible with the compactification, $\sum_y n(x,y)n(y,z)$ is zero since those once-broken flow lines occur in canceling pairs.

Therefore the homology group

$$H_*(X;f) := \frac{\ker\partial}{\mathrm{im}\partial}$$

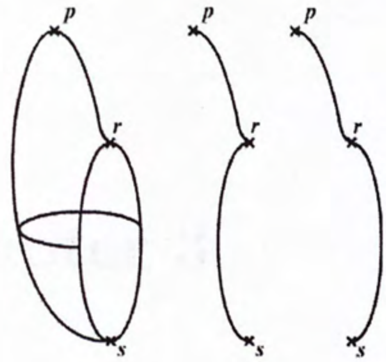
of this complex is defined. It is worth to point out that although the chain complex depends on the choice of a generic metric g and function f (since the chain groups depend on f and the boundary operator depends on g), the homology group is independent of the choice of both g and f. (refer to [16] and [17])

Example 2.5.1 (M. Akaho, [18]) Let X be a 2-sphere and f the height function as in the figure:



[figure : notice that it is homeomorphic to S^2]

The indices of the critical points are ind(p) = ind(q) = 2, ind(r) = 1 and ind(s) = 0. In this case $\widehat{\mathcal{M}}(p,r)$ and $\widehat{\mathcal{M}}(q,r)$ consist of one point and $\widehat{\mathcal{M}}(r,s)$ consists of



[figure : the two flow lines]

two points. We can identify $\widehat{\mathcal{M}}(p,s)$ with an open interval and compactify $\widehat{\mathcal{M}}(p,s)$ such that the boundary is $\widehat{\mathcal{M}}(p,r) \times \widehat{\mathcal{M}}(r,s)$, see the figure:

Then we have

$$\partial p = r, \partial q = -r, \partial r = 0, \partial s = 0$$

and the homology is $\mathbb{Z}[p+q] \oplus \mathbb{Z}[s]$, which is isomorphic to the singular homology of the 2-sphere.

Chapter 3

Lagrangian Floer Homology

3.1 Introduction

Floer theory can be interpreted as an infinite dimensional case for the classical Morse Theory. Probably inspired by the work of Witten, Conley and Gromov, Floer realised that there are some interesting infinite dimensional situations in which a similar approach makes sense. In these cases, the ambient manifold $\mathcal{X}$ is infinite dimensional and the critical points of the function (which resembles the Morse function) $\mathcal{F} : \mathcal{X} \to \mathbb{R}$ have infinite index and coindex. Therefore one usually cannot get much information from the sublevel sets $\mathcal{F}^{-1}(-\infty, c]$ of $\mathcal{F}$. Also, one may not be able to choose a metric on $\mathbb{X}$ such that the gradient flow of $\mathcal{F}$ is everywhere defined. However, Floer pointed out that in some important cases one can choose a metric so that the spaces $\mathcal{M}(x, y)$ of gradient trajectories between distinct critical points x, y of $\mathcal{F}$ have properties analogous to those in the finite dimensional case. Hence he defined the so-called **Floer chain complex** which will be explained below.

In this chapter we will first give some preliminaries on symplectic geometry. We then proceed to describe the version of Floer theory used by Ozsváth-Szabó. We end this chapter with discussions on the Floer complex as well as the obstructions to its existence. For more details refer to [17], [18] and [20].

3.2 Preliminaries on Symplectic Geometry

3.2.1 Basic Definitions

We need some definitions for terms in symplectics geometry (for details refer to [18]). We begin ourselves with finite dimensional vector space.

Definition 3.2.1 Let V be a finite dimensional real vector space. A **symplectic form** ω is a non-degenerate anti-symmetric bilinear form on V. A **symplectic vector space** is denoted by (V, ω).

For example, $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \cdots, x_n, y_n)$ endowing with the 2-form $\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i$, is a symplectic vector space. Of course the 2-form is compatible with the usual inner product in $\mathbb{R}^{2n}$, namely

$$\omega_0(u, v) = Ju \cdot v, \text{ where } J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here, J is skew-symmetric ($J^T = -J$) and it satisfies $J^2 = -I$. If we rewrite the basis of $\mathbb{R}^{2n}$ as $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, then J can be written as

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

where I is the n by n identity matrix. Sometime we will call ω_0 the **standard symplectic form** on $\mathbb{R}^{2n}$ and $(\mathbb{R}^{2n}, \omega_0)$ is called the **standard symplectic space.**

Definition 3.2.2 The ω**-orthogonal complement** of a linear subspace $W \subset V$ is defined as

$$W^{\perp \omega} := \{v \in V | \omega(v, w) = 0 \ \forall w \in W\}$$

Definition 3.2.3 A liner subspace W of a symplectic vector space (V, ω) is called

- **symplectic** if the restriction of ω to W is non-degenerate (equivalently $W \cap W^{\perp \omega} = 0$);
- **isotropic** if $W \subset W^{\perp \omega}$;
- **Lagrangian** if $W = W^{\perp \omega}$.

The dimension of a symplectic vector space V must be even, which follows easily from the non-degeneracy of ω.

Proposition 3.2.1 Let (V, ω) be a symplectic vector space. Then V is even-dimensional, say $\dim(V) = 2n$. In fact, there exist a basis $(e_1, f_1, \cdots, e_n, f_n)$ for V such that

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \ \forall i, j$$

$$\text{and } \omega(e_i, f_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad \forall i$$

Proof: Firsr we pick a vector e_1 in V. Then we can find another vector $v \in V$ such that $\omega(e_1, v) \neq 0$, as ω is assumed to be non-degenerate, and after normalization on v we can get a f_1 such that $\omega(e_1, f_1) = 1$. Observe that e_1 and f_1 must be linearly independent: if e_1 is some multiple of f_1, since ω is anti-symmetric, we then will get $\omega(e_1, f_1) = 0$. Hence e_1 and f_1 will span a 2-dimensional subspace V_1 of V. By construction, V_1 is a symplectic subspace of V.

If V has dimension 2, then $V = V_1$ and we are done. Otherwise we can consider the subspace

$$V_2 = \{v \in V | \omega(v, u) = 0 \ \forall u \in V_1\}$$

Now, V_2 is also a symplectic subspace of V and it is a complement of V_1 with $V_1 \cap V_2 = \{0\}$. So we can apply the same construction to V_2. By induction, the result follows. $\square$

Referring to the above proposition, a basis $(e_1, f_1, \cdots, e_n, f_n)$ which satisfies the suggested conditions is called a **symplectic basis** of (V, ω). For example, the standard basis of $\mathbb{R}^{2n}$ is symplectic with respect to the standard symplectic form ω_0.

We will also study mappings between symplectic vector space.

Definition 3.2.4 Let (V_1, ω_1) and (V_2, ω_2) be two symplectic vector spaces. A symplectic isomorphism from (V_1, ω_1) to (V_2, ω_2) is a bijection linear map $\varphi : V_1 \to V_2$ such that $\varphi^* \omega_2 = \omega_1$, meaning that

$$\omega_2(\varphi(u), \varphi(v)) = \omega_1(u, v), \ \forall u, v \in V_1$$

If $(V_1, \omega_1) = (V_2, \omega_2)$ then we will call φ a symplectic automorphism.

From the definition and proposition 3.2.1, we know that every symplectic vector space of dimension $2n$ is symplectically isomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

It is not difficult to image that the theory of symplectic geometry can be applied to manifolds.

Definition 3.2.5 Let M be a k-dimensional manifold and a ω 2-form on M. We call ω a **symplectic form** if and only if ω is closed and non-degenerate. If such ω exists, (M, ω) will be called a **symplectic manifold**. From the previous proposition we can conclude that the dimension of M is even. We have the following examples of symplectic manifolds:

- $\mathbb{R}^{2n}$ with $\omega_0 := \sum_{i=1}^{n} dx_i \wedge dy_i$
- Kähler manifold with Kähler forms
- Let X be a smooth manifold and $(x_1, \ldots, x_n)$ a local coordinate system. We have a local coordinate system of the cotangent bundle T^*X such that $\sum_{i=1}^{n} y_i dx_i$ corresponds to $(x_1, y_1, \ldots, x_n, y_n)$. Then the 2-form $\omega := \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic form on T^*X.

A particular type of submanifold of a symplectic manifold (M^{2n}, ω) is given as follow, which is useful for our later construction. Let L be a n-dimensional submanifold of (M^{2n}, ω). If $\omega|_{TL} = 0$, then we call L a **Lagrangian submanifold**. We have the following examples of Lagrangian submanifolds:

- 1-dimensional submanifolds of Riemann surfaces
- The zero-section 0_Y of T^*Y, where Y is a smooth manifold

Let (M, ω_M) and (N, ω_N) be two symplectic manifolds. A smooth mapping $f : M \to N$ is called **symplectic** if

$$f^*\omega_N = \omega_M.$$

If f is a symplectic diffeomorphism, then f^{-1} is also symplectic. In this case f is called a **symplectomorphism**.

3.2.2 The Symplectic Group

We back to our case of finite dimensional vector spaces. Now for the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, the matrices A associating with those symplectic automorphisms can be characterized by the equation

$$A^T J A = J$$

where A^T denotes the transpose of A. We call these matrices **symplectic matrices**, and denote this set of matrices by $Sp(2n)$.

For a symplectic vector space (V, ω) with a symplectic basis $\beta = (e_1, f_1, \cdots, e_n, f_n)$, by our construction, a linear automorphism φ of V is symplectic if and only if the associated matrix of φ with respect to β is symplectic. And the set of symplectic automorphism of (V, ω) (denoted by $Sp(V)$) forms a group, which is isomorphic to $Sp(2n)$. We will pay more attention to the matrix group $Sp(2n)$.

First of all, observe that since symplectic automorphisms preserve ω_0, they also preserve the $2n$-form $\sigma_0 = \omega_0 \wedge \cdots \wedge \omega_0$, the n-times wedge product of ω_0 by itself. But the $2n$-form σ_0 precisely coincides with the standard volume form in $\mathbb{R}^{2n}$, in other words, every symplectic automorphism preserves the standard volume form. Hence every symplectic matrix in $Sp(2n)$ has determinant 1. Moreover, for any $A \in Sp(2n)$, its inverse is given by

$$A^{-1} = J^{-1} A^T J$$

Next, the matrix group $Sp(2n)$ is, in fact, a Lie group. To show $Sp(2n)$ is a smooth manifold, notice that

$$Sp(2n) = \{A \in M_{2n\times 2n}(\mathbb{R}) | \phi(A) = J\}$$

where $M_{2n\times 2n}(\mathbb{R})$ is the vector space of real $2n$ by $2n$ matrices and

$$\phi(A) := A^T J A.$$

We can treat ϕ as a map from $M_{2n\times 2n}(\mathbb{R})$ to $Skew(2n)$, the vector space of all skew-symmetric $2n$ by $2n$ matrices. Since $M_{2n\times 2n}(\mathbb{R})$ and $Skew(2n)$ are diffeomorphic to $\mathbb{R}^{4n^2}$ and $\mathbb{R}^{n(2n-1)}$ respectively, ϕ can be viewed as a smooth map from $\mathbb{R}^{4n^2}$ to

$\mathbb{R}^{n(2n-1)}$. If we can show that J is a regular value of ϕ, then we will be done because $Sp(2n) = \phi^{-1}(J)$, which implies $Sp(2n)$ is a smooth submanifold of $\mathbb{R}^{4n^2}$.

We calculate the differential of ϕ at $A \in M_{2n\times 2n}(\mathbb{R})$, which is given by

$$d\phi_A(H) = H^T JA + A^T JH$$

where H is in the tangent space of $M_{2n\times 2n}(\mathbb{R})$ at A. Here, $d\phi_A$ is onto if A is invertible. More directly, if $B \in Skew(2n)$, we can take $H = \frac{1}{2}J(A^{-1})^T B^T$ such that

$$d\phi_A(H) = H^T JA + A^T JH = \frac{1}{2}B - \frac{1}{2}B^T = B.$$

Since every matrix in $Sp(2n)$ is invertible, we can deduce that $d\phi_A$ is onto whenever $A \in Sp(2n)$. This proves J is a regular value of ϕ and hence $Sp(2n)$ is a Lie group. The dimension of $Sp(2n)$ is $4n^2 - n(2n-1) = n(2n+1)$.

The Lie algebra of $Sp(2n)$ is the tangent space to $Sp(2n)$ at the identity matrix I, which is given by

$$\mathfrak{sp}(2n) := \mathrm{Ker} d\phi_I = \{H \in M_{2n\times 2n}(\mathbb{R}) | H^T J + JH = 0\}$$

The matrices in $\mathfrak{sp}(2n)$ are called **infinitesimally symplectic**. We know that the matrix defined by

$$e^H := \sum_{n=0}^{\infty} \frac{1}{n!} H^n$$

is in $Sp(2n)$ if $H \in \mathfrak{sp}(2n)$.

Example 3.2.1 We examine the group $Sp(2n)$ for $n = 1$ Since $\det(A)$=1 for all $A \in Sp(2)$, $Sp(2) \subset SL(2)$ where $SL(2)$ is the set of matrices with determinant 1. And by direct computation, $B^T JB = J$ for all $B \in SL(2)$. So $Sp(2) = SL(2)$.

For any invertible matrix A, we can decompose A into the following **polar form**:

$$A = PO,\ P := (AA^T)^{\frac{1}{2}},\ O := P^{-1}A,$$

here, P is a positive definite matrix with $P^2 = AAT$. And from this construction, P is symmetric and O is orthogonal. Since A has determinant 1, P also has determinant 1 and thus $P \in Sp(2)$. Similarly, $O \in Sp(2)$. In particular, $O \in SO(2)$, the orthogonal matrices with determinant 1.

Let H be the set of 2×2 symmetric and positive definite matrices with determinant 1. This set consist of matrices with the following properties:

$$P = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ where } a, b, c \in \mathbb{R} \text{ and } ac - b^2 = 1$$

Since $ac - b^2 = 1$ and all the entries are real, a and c are both non-zero. And because the matrices are positive definite, both a and c are positive real numbers. Therefore, H can be written as:

$$H = \{ \begin{pmatrix} a & b \\ b & \frac{b^2+1}{a} \end{pmatrix} \mid \text{where } a \text{ is positive and } b \text{ is arbitrary any real numbers} \}$$

Now we can identify H by $P \leftrightarrow (a, b)$ with the product space $\mathbb{R}^+ \times \mathbb{R}$ ($\mathbb{R}^+$ denotes the positive real numbers), and this space is readily seen to be homeomorphic to $\mathbb{R}^2$.

From [41], we have the following result:

Proposition 3.2.2 The map $f : H \times SO(2) \to SL(2)$ defined by $f(P, O) = PO$ is a homeomorphism.

Sketch of proof: f is a bijection because the polar decomposition of an invertible matrix is unique. And since f is defined by matrix multiplication, it is a continuous map. Similarly the inverse f^{-1} is also continuous. □

Since $SO(2)$ is a circle, we have proven that $Sp(2)$ is homeomorphic to the product space $S^1 \times \mathbb{D}^2$ ($\mathbb{D}^2$ is the open unit disk in $\mathbb{C}$), which is the interior of the full torus. In particular, $Sp(2)$ is homotopic equivalent to the unit circle $S^(1)$, and therefore

$$\pi_1(Sp(2)) \cong \pi_1(S^1) = \mathbb{Z}$$

3.2.3 Maslov index for non-degenerate paths in Sp(2n)

Although the concept of Maslov index is not used in this Chapter, it will become crucial in Chapter 4. Before defining the Maslov index for $Sp(2n)$, we will first examine the case for $Sp(2)$. The general case follows only with some modifications.

We are going to define a symplectically invariant map

$$\rho : Sp(2) \to SO(2)$$

which has the property that $\rho(A^n) = \rho(A)^n$ for every $A \in Sp(2)$ and all $n \in \mathbb{N}$. Although this map is not a group homomorphism, it turns out to be important and it will be called the **rotation function**.

Since $\det(A) = 1$ for $A \in Sp(2)$, the eigenvalues of A must be in the form of λ, $\frac{1}{\lambda}$, where $\lambda \in \mathbb{R} \cup SO(2)$. In particular, the eigenvalues 1 and -1 are always double.

Then we consider the matrix $G := -iJ$, which is Hermitian with respect to the standard Hermitian product $\langle \cdot, \cdot \rangle$ on $\mathbb{C}$. Observe that for $A \in Sp(2)$, we have $A^*GA = G$, where A^* is the conjugate transpose of A.

We pick $A \in Sp(2)$. Assume that A has eigenvalues $\lambda \neq \pm 1$ (i.e. λ and $\bar{\lambda}$ are both lying on the unit circle $SO(2)$ excluding the point ± 1) with that ξ and $\bar{\xi}$ are the corresponding eigenvectors. Then

$$\langle G\xi, \bar{\xi} \rangle = \langle A^*GA\xi, \bar{\xi} \rangle = \langle GA\xi, A\bar{\xi} \rangle = \lambda^2 \langle G\xi, \bar{\xi} \rangle.$$

By assumption $\lambda \neq \pm 1$, so $\langle G\xi, \bar{\xi} \rangle = 0$. And for $\langle G\xi, \xi \rangle$, because

$$\langle G\xi, \xi \rangle = \langle \xi, G^*\xi \rangle = \langle \xi, G\xi \rangle = \overline{\langle G\xi, \xi \rangle},$$

therefore $\langle G\xi, \xi \rangle$ is real. Moreover, $\langle G\xi, \xi \rangle$ is non-zero, otherwise by $\langle G\xi, \bar{\xi} \rangle = 0$, which implies $\bar{\xi} = c\xi$ for some non-zero constant. Now

$$\begin{cases} A\xi = \lambda\xi \\ A\bar{\xi} = \bar{\lambda}\bar{\xi} \end{cases} \Rightarrow c\lambda\xi = c\bar{\lambda}\xi \Rightarrow \lambda = \bar{\lambda}$$

which is impossible since $\lambda \neq \pm 1$. By the similar reason, $\langle G\bar{\xi}, \bar{\xi} \rangle$ is also real and non-zero. Moreover, by considering the real vector $\xi + \bar{\xi}$, we have

$$0 = \langle G(\xi + \bar{\xi}), \xi + \bar{\xi} \rangle = \langle G\xi, \xi \rangle + \langle G\xi, \bar{\xi} \rangle + \langle G\bar{\xi}, \xi \rangle + \langle G\bar{\xi}, \bar{\xi} \rangle = \langle G\xi, \xi \rangle + \langle G\bar{\xi}, \bar{\xi} \rangle$$

and hence $\langle G\xi, \xi \rangle = -\langle G\bar{\xi}, \bar{\xi} \rangle$. These facts lead to the following definition.

Definition 3.2.6 If $\lambda \in SO(2) \setminus \{-1, 1\}$ is an eigenvalue of $A \in Sp(2)$ and ξ is the corresponding eigenvector, the **Krein sign** of λ is the sign of $\langle G\xi, \xi \rangle$.

From the previous result, if the eigenvalue $\lambda \in SO(2) \setminus \{-1, 1\}$ is Krein-poistive, then the eigenvalue $\bar{\lambda}$ will be negative. We can now define the rotation function

$\rho : Sp(2) \to SO(2)$:

$$\rho(A) = \begin{cases} \lambda & \text{if } \lambda \in SO(2) \setminus \{-1, 1\} \text{ is the Krein-poistive eigenvalue of } A, \\ 1 & \text{if the eigenvalues of } A \text{ are real and positive,} \\ -1 & \text{if the eigenvalues of } A \text{ are real and negative.} \end{cases}$$

The rotation function ρ satisfies several properties. First of all, ρ is symplectically invariant, i.e. $\rho(M^*AM) = \rho(A)$ for all $M, A \in Sp(2)$. To prove this, suppose $\rho(A) = \lambda$ with $A\xi = \lambda\xi$. Then $\langle G\xi, \xi\rangle > 0$. If $\rho(M^*AM) = \kappa$ with $M^*AM\zeta = \kappa\zeta$, i.e. $\langle G\zeta, \zeta\rangle > 0$, then

$$M^*AM\zeta = \kappa\zeta \Rightarrow AM\zeta = \kappa M\zeta$$

which implies $M\zeta$ is an eigenvector of A with eigenvalue κ. We either have $\kappa = \lambda$ or $\kappa = \bar{\lambda}$. If $\kappa = \bar{\lambda}$, then $M\zeta = c\bar{\xi}$ for some constant c. Then

$$\langle GM\zeta, M\zeta\rangle = \langle cG\bar{\xi}, c\bar{\xi}\rangle = |c|^2\langle G\bar{\xi}, \bar{\xi}\rangle < 0$$

But

$$\langle GM\zeta, M\zeta\rangle = \langle M^*GM\zeta, \zeta\rangle = \langle G\zeta, \zeta\rangle > 0$$

which is a contradiction. Therefore we must have $\kappa = \lambda$.

Secondly, ρ is continuous, because

$$\rho(A) = \frac{\lambda}{|\lambda|},$$

where λ is any eigenvalue of A in the case $\lambda \in \mathbb{R}$, and it is the Krein-poistive eigenvalue in the case $\lambda \in SO(2) \setminus \{-1, 1\}$.

If λ is an eigenvalue of A^k with eigenvector ξ for $k \in \mathbb{N}$, then λ^k is an eigenvalue of A^k with the same eigenvector ξ. In other words,

$$\rho(A^k) = \rho(A)^k.$$

However, since the eigenvalues of the product of two matrices need not be the product of the eigenvalues, ρ is not a group homomorphism.

We want to express $Sp(2)$ in terms of ρ. Recall the fact that $Sp(2)$ is homeomorphic to $S^1 \times \mathbb{D}^2$. And there is a natural polar coordinates (θ, r, σ) defined on $S^1 \times \mathbb{D}^2$. To do this, we can associate polar coordinates $(r = |z|, \sigma = \arg z)$ on $\mathbb{D}^2$ and $(z = e^{i\theta})$ on S^1. Sometime, for computational purpose, we will let $r = \tanh^2 \tau$.

In terms of the above parametrization, for any $A \in Sp(2)$ with the decomposition $A = PO$, we have

$$P = \begin{pmatrix} \cosh\tau + \cos\sigma \sinh\tau & \sin\sigma\sinh\tau \\ \sin\sigma\sinh\tau & \cosh\tau - \cos\sigma\sinh\tau \end{pmatrix}, O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

These coordinates make the calculation of the eigenvalues of $A \in Sp(2)$ much simpler. Consider the equation:

$$\det(\lambda I - A) = \lambda^2 - (\mathrm{tr}A)\lambda + 1 = 0,$$

the discriminant Δ of the above polynomial is given by

$$\Delta = (\mathrm{tr}A)^2 - 4 = 4\cosh^2\tau\cos^2\theta - 4.$$

So A has a double eigenvalue ($\lambda = \pm 1$) if and only if $\Delta = 0$, which is equivalent to

$$r = \sin^2\theta.$$

The set of symplectic matrices with a double eigenvalues is

$$Sp(2)^\sharp = \{A \in Sp(2) | \det(I - A) = 0 \text{ or } \det(I + A) = 0\} = \{(\theta, r, \sigma) \in S^1 \times \mathbb{D}^2 | r = \sin^2\theta\}.$$

The above set $Sp(2)^\sharp$ consists of two connected components. One component contains I and thus it consists of matrices with eigenvalue 1. The other one contains $-I$ with eigenvalue -1.

We divide $Sp(2)$ into three subsets:

$$\begin{aligned} Sp(2)^+ &= \{A \in Sp(2) | \det(I - A) > 0\}, \\ Sp(2)^- &= \{A \in Sp(2) | \det(I - A) < 0\}, \\ Sp(2)^0 &= \{A \in Sp(2) | \det(I - A) = 0\}. \end{aligned}$$

Here, $Sp(2)^0$ is a subset of $Sp(2)^\sharp$, which is precisely the component of $Sp(2)^\sharp$ containing I.

$Sp(2)^0$ is a surface in $Sp(2)$ with a point singularity at I. It divides $Sp(2)$ into two connected components, namely $Sp(2)^+$ and $Sp(2)^-$. Since $-I \in Sp(2)^+$, $Sp(2)^+$ is the "upper-left" region, while $Sp(2)^-$ is the remaining "lower-right" region.

We want to define an integer to every continuous path $\gamma : [0, 1] \to Sp(2)$ for $\gamma(0) = I$ and $\gamma(1) \in Sp(2)^*$, and this integer will be called the **Maslov index** for

γ. Roughly speaking, one can view maslov index as the "pull back" of the winding number of $\rho(\gamma)$ on S^1 via ρ.

We first observe that the matrix $-I$ is lying in $Sp(2)^+$ with $\rho(-I) = -1$. We fix a matrix W in $Sp(2)^-$ such that $\rho(W) = 1$, say

$$W := \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

For every path $\gamma : [0,1] \to Sp(2)$, choose a continuous function (the angle function) $\theta : [0,1] \to \mathbb{R}$ such that $\rho(\gamma(t)) = e^{i\theta(t)}$ and define

$$\Delta_t(\gamma) := \frac{\theta(t) - \theta(0)}{\pi}.$$

For each $A \in Sp(2)^*$, we can take a path $\alpha_A : [0,1] \to Sp(2)^*$ such that $\alpha_A(0) = A$ and $\alpha_A(1) \in \{-I, W\}$. The real number $\Delta_1(\alpha) = \frac{\theta(1)-\theta(0)}{\pi}$ does not depend on the choice of α because $Sp(2)^+$ and $Sp(2)^-$ are both contractible and connected. We thus define a continuous function $f : Sp(2)^* \to \mathbb{R}$

$$f(A) := \Delta_1(\alpha).$$

Finally we can have the following definition:

Definition 3.2.7 Let $\gamma : [0,1] \to Sp(2)$ be a continuous path such that $\gamma(0) = I$ and $\gamma \in Sp(2)^*$. The **Maslov index** $\mu(\gamma)$ of γ is the integer

$$\mu(\gamma) := \Delta_1(\gamma) + f(\gamma(1)).$$

It is shown in [42] that $\mu(\gamma)$ is, in fact, an integer.

Example 3.2.2 Of course, for constant paths $\gamma(t) \equiv I$, $\mu(\gamma) = 0$.

Consider a path $\gamma : [0,1] \to Sp(2)$ defined by

$$\gamma(t) := \begin{pmatrix} 1+t & 0 \\ 0 & \frac{1}{1+t} \end{pmatrix}.$$

where $\gamma(0) = I$ and $\gamma(1) = W$. Since $\rho(\gamma(t)) \equiv 1$, so we take $\theta : [0,1] \to \mathbb{R}$ with $\theta(t) \equiv 0$. Then $\Delta_1(\gamma) = 0$.

Similarly, by taking $\alpha_{\gamma(1)} : [0,1] \to Sp(2)^*$ with $\alpha_\gamma \equiv \gamma(1) = W$, we get $f(\gamma(1)) = 0$. Again $\mu(\gamma) = 0$.

We look at a less trivial case. Consider another path $\gamma : [0,1] \to Sp(2)$ defined by

$$\gamma(t) := \begin{pmatrix} e^{it\pi} & 0 \\ 0 & e^{-it\pi} \end{pmatrix}.$$

where $\gamma(0) = I$ and $\gamma(1) = -I$. We can take $\theta : [0,1] \to \mathbb{R}$ with $\theta(t) = t\pi$. Then

$$\Delta_1(\gamma) := \frac{\theta(1) - \theta(0)}{\pi} = \frac{\pi - 0}{\pi} = 1.$$

By taking $\alpha_{\gamma(1)} : [0,1] \to Sp(2)^*$ with $\alpha_\gamma \equiv \gamma(1) = -I$, we get $f(\gamma(1)) = 0$. Thus $\mu(\gamma) = 1 + 0 = 1$.

In general, for any symplectic vector space (V, ω), we can have the notion of Maslov index in a similar way.

Again we first need to define the rotation function. The following is due to [40] and [20].

Proposition 3.2.3 For any symplectic vector space (V, ω) with $\dim(V) = 2n$, there exists a unique continuous mapping

$$\rho_V : Sp(V) \to S^1$$

which satisfies the following conditions:

1. If $M : (V_1, \omega_1) \to (V_2, \omega_2)$ is a symplectic isomorphism, then

$$\rho_{V_2}(MAM^{-1}) = \rho_{V_1}(A), \; A \in Sp(V_1, \omega_1).$$

2. Define $A \in Sp(V_1 \times V_2, \omega_1 \oplus \omega_2)$ to be $A(\xi_1, \xi_2) := (A_1\xi_1, A_2\xi_2)$, $A_1 \in Sp(V_1, \omega_1)$ and $A_2 \in Sp(V_2, \omega_2)$. Then we have

$$\rho_{V_1 \times V_2}(A) = \rho_{V_1}(A_1)\rho_{V_2}(A_2).$$

3. If $A \in Sp(2n) \cap O(2n) \cong U(n)$, then

$$\rho(A) = \det(A).$$

4. If $A \in Sp(V)$ has no eigenvalue on the unit circle S^1, then

$$\rho(A) = \pm 1.$$

5. If $A \in Sp(V)$ and $n \in \mathbb{N}$, then

$$\rho(A^n) = \rho(A)^n.$$

Similar to the case in $n = 1$, We can divide $Sp(2n)$ into three subsets:

$$\begin{aligned} Sp(2n)^+ &= \{A \in Sp(2n) | \det(I - A) > 0\}, \\ Sp(2n)^- &= \{A \in Sp(2n) | \det(I - A) < 0\}, \\ Sp(2n)^0 &= \{A \in Sp(2n) | \det(I - A) = 0\}. \end{aligned}$$

Define $Sp(2n)^* = \{A \in Sp(2n) | \det(I - A) \neq 0\}$. From [20], $Sp(2n)^*$ has two connected components, namely $Sp(2n)^+$ and $Sp(2n)^-$.

Definition 3.2.8 We call a path $\gamma : [0,1] \to Sp(2n)$ **admissible** if $\gamma(0) = I_{2n}$ and $\gamma(1) \in Sp(2n)^*$.

We associate every admissible symplectic arc $\gamma : [0,1] \to Sp(2n)$ the Maslov index with the following settings. Choose a continuous function $\theta : [0,1] \to \mathbb{R}$ such that $\rho(\gamma(t)) = e^{i\theta(t)}$ and set

$$\Delta(\gamma) = \frac{\theta(1) - \theta(0)}{\pi}.$$

Then, we connect $\gamma \in Sp(2n)^*$ with W^+ or W^- using a path $f : [0,1] \to Sp(2n)^*$:

$$f(0) = \gamma(1), \quad f(1) \in \{W^+, W^-\},$$

where W^+ and W^- are representatives from $Sp^+(2n)$ and $Sp^-(2n)$, say

$$W^+ = -I_{2n} \in Sp^+(2n) \quad W^- = \mathrm{diag}(2, -1, \cdots, -1, \frac{1}{2}, -1, \cdots, -1) \in Sp^-(2n)$$

Definition 3.2.9 The **Maslov index** $\mu(\gamma)$ is defined by

$$\mu(\gamma) = \Delta(\gamma) + \Delta(f).$$

3.2.4 Maslov index - the analytic aspect

In the previous section, we define the Maslov index algebraically. Yet such index is closely related to the index of a certain *Fredholm operator* [42]. We present the main idea of the theorem of Salamon and Zehnder; for details we refer to [20].

Definition 3.2.10 Let u be locally integrable in a bounded domain $\Omega \subset R^n$ and α any multi-index. Then a locally integrable function v is called the α-th **weak derivative** of u if it satisfies

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx \quad \text{for all } \phi \in C_0^{|\alpha|}(\Omega).$$

We call a function **weakly differentiable** if all its weak derivatives of first order exist and k **times weakly differentiable** if all its weak derivatives exist for orders up to and including k.

Denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$. Clearly $C^k(\Omega) \subset W^k(\Omega)$. So the concept of weak derivative can be viewed as an extension of the classical concept which maintains the validity of *integration by parts.*

Definition 3.2.11 The **Sobolev space** $H^{1,2}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ is the Banach space

$$\{\xi \in L^2(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n}) : \frac{\partial \xi}{\partial s}, \frac{\partial \xi}{\partial t} \in L^2(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})\},$$

where the partial derivatives are in the **weak sense** and the norm is given by

$$||\xi||^2_{H^{1,2}} = ||\xi||^2_{L^2} + ||\frac{\partial \xi}{\partial s}||^2_{L^2} + ||\frac{\partial \xi}{\partial t}||^2_{L^2}.$$

Here, $H^{1,2}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ is also a Hilbert space with the scalar product

$$(\xi, \eta)_{H^{1,2}} = \int_{\mathbb{R}\times\mathbb{R}/\mathbb{Z}} (\xi\eta + \frac{\partial \xi}{\partial s}\frac{\partial \eta}{\partial s} + \frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial t}) ds dt.$$

Define an operator $F : H^{1,2}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ by

$$F\xi = \frac{\partial \xi}{\partial s} + J\frac{\partial \xi}{\partial t} + S\xi$$

where $S(s,t)$ is a real symmetric $n \times n$ matrix continuous in (s,t) and 1-periodic in t. We further assume that $S(s,t)$ converges uniformly in t to the limit $S^+(t)$ and $S^-(t)$ as s tends to $+\infty$ and $-\infty$ respectively.

Definition 3.2.12 A bounded linear operator $F : X \to Y$, where X, Y are Banach space, is called **Fredholm** if $\ker F$ and $\mathrm{coker} F = Y/\mathrm{image}(F)$ are finite dimensional. The **index** of F is given by

$$\mathrm{ind}(F) = \dim \ker F - \dim \mathrm{coker} F.$$

The most important property of Fredholm operators is the invariance of their index under perturbations. Let $\mathcal{F}$ be the set of Fredholm operators $F : X \to Y$ with the topology induced by the operator norm. From [20], we have the following theorem:

Theorem 3.2.13 (Dieudonné) The index function ind: $\mathcal{F} \to \mathbb{Z}$ is locally constant.

We come back to our operator $F : H^{1,2}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ defined by

$$F\xi = \frac{\partial \xi}{\partial s} + J\frac{\partial \xi}{\partial t} + S\xi.$$

Because $(JS(s,t))^T J + J(JS(s,t) = S(s,t)J^T J + (-S(s,t)) = S(s,t) - S(s,t) = 0$, we have $JS(s,t) \in \mathfrak{sp}(2n)$. Therefore, we can define a path $\Psi(s,t)$ in $Sp(2n)$ which is given by

$$\frac{\partial}{\partial t}\Psi(s,t) = JS(s,t)\Psi(s,t)$$

with the initial condition $\Psi(s,0) = I$. As s tends to $\pm\infty$, $\Psi(s,t)$ converges to the matrices $\Psi^{\pm}(t)$ uniformly in t with

$$\frac{\partial}{\partial t}\Psi^{\pm}(t) = JS^{\pm}(t)\Psi^{\pm}(t).$$

We quote the following theorem given in [20].

Theorem 3.2.14 (Salamon-Zehnder) Let the operator $F : H^{1,2}(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n}) \to L^2(\mathbb{R} \times \mathbb{R}/\mathbb{Z}; \mathbb{R}^{2n})$ be given by

$$F\xi = \frac{\partial \xi}{\partial s} + J\frac{\partial \xi}{\partial t} + S\xi.$$

Assume further that the paths $\Psi^{\pm}(t)$ are admissible. Then F is Fredholm.

After we know that F is a Fredholm operator, we can express its index in terms of Maslov index. By assumption, the paths $\Psi^{\pm}(t)$ are admissible, so we can associate a Maslov index each of them. The following theorem is proven in [42].

Theorem 3.2.15 Under the conditions of the preceding theorem the index of $F = \frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + S$ is given by

$$\text{ind } F = \mu(\Psi^-) - \mu(\Psi^+).$$

3.3 Definition of Floer Homology

Now we start to define the Lagrangian Floer Homology (basically follows from [17]). Let M be a $2n$-dimensional manifold with symplectic form ω. Choose two Lagrangian submanifolds L_0 and L_1. We assume that they intersect transversally and also that their intersection is non-empty, since otherwise the complex we are going to define is trivial. Define $\mathcal{P} := \mathcal{P}(L_0, L_1)$, the space of paths from L_0 to L_1, i.e.

$$\mathcal{P} = \{x : [0,1] \to M : x(0) \in L_0, x(1) \in L_1\}$$

Pick a base point $x_0 \in L_0 \cap L_1$, consider it as a constant path in $\mathcal{P}$. Let $\tilde{\mathcal{P}}$ be the universal cover of $\mathcal{P}$ based at x_0,

$$\tilde{\mathcal{P}} := \{\hat{x} : [0,1] \times [0,1] \to M | \hat{x}(0,t) = x_0, \hat{x}(s,i) \in L_i, \hat{x}(1,t) = x(t)\}/\text{homotopy}$$

So elements in $\tilde{\mathcal{P}}$ are pairs, $(x, [\hat{x}])$, where $[\hat{x}]$ is an equivalence class of maps $\hat{x} : [0,1] \times [0,1] \to M$ satisfying the required boundary conditions. Next we define our "Morse function" $\mathcal{F}$. The function is the action functional $\mathcal{F} : \tilde{\mathcal{P}} \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{F}(x, [\hat{x}]) &:= \int_0^1 \int_0^1 \hat{x}^*(\omega) \\ &= \int_0^1 ds \int_0^1 \omega(\frac{\partial \hat{x}}{\partial t}, \frac{\partial \hat{x}}{\partial s}) \end{aligned}$$

First we need to know whether $\mathcal{F}$ is well-defined or not. We state the following lemma :

Lemma 3.3.1 The functional $\mathcal{F}$ does not depend on the homotopy class of the map $\hat{x}$.

Proof: For any $\hat{x_0}$, $\hat{x_1} \in [\hat{x}]$, choose a homotopy $F : D^1 \times D^2 \to M$ between them. Then we have

$$\begin{aligned} 0 &= \int_{D^1 \times D^2} F^*(d\omega) \ (\because \omega \text{ is a closed form}) \\ &= \int_{\partial(D^1 \times D^2)} F^*(\omega) \ (\text{By Stokes' Theorem}) \\ &= \pm(\int_{D^2} \hat{x_1}^*(\omega) - \int_{D^2} \hat{x_0}^*(\omega) + \int_{\partial D^2 \cap L_1} F^*(\omega) - \int_{\partial D^2 \cap L_0} F^*(\omega)) \\ &= \pm(\int_{D^2} \hat{x_1}^*(\omega) - \int_{D^2} \hat{x_0}^*(\omega)) \ (\because \omega \text{ vanishes on } L_i) \end{aligned}$$

Hence the result follows. □

From this lemma, we can take $\mathcal{F}$ as a function on $\mathcal{P}$, which is our"Morse function" on this infinite dimensional space (refer to [17] and [19] for more details). In order to find out those critical points of $\mathcal{F}$, we need some more definitions.

Definition 3.3.2 Given a path x : [0,1]$\to M$, the **tangent space** $T_{x(t)}(\mathcal{P})$ consist of all smooth sections ξ of the pullback bundle $x^*(TM)$, i.e. $\xi(t) \in T_{x(t)}(M)$ for all $t \in$ [0,1], and satisfies the boundary conditions $\xi(i) \in T_{x(i)}(L_i)$ for i=0, 1. For two sections ξ, η, we set

$$\langle \xi, \eta \rangle := \int_0^1 g_J(\xi(t), \eta(t))dt$$

where g_J is a Riemannian metric on M and $J : TM \to TM$ is an ω-compatible almost complex structure in the sense $g_J(v, w) = \omega(v, Jw)$ for $v, w \in T_pM$, $\forall p \in M$.

Now the differential of $\mathcal{F}$ at $x(t)$ in the direction of ξ is defined by

$$d\mathcal{F}_{x(t)}(\xi) = \frac{\partial \mathcal{F}(\hat{x}(s,t))}{\partial s}|_{s=0}, \text{ where } \hat{x}(0,t) = x(t), \ \frac{\partial \hat{x}(s,t)}{\partial s}|_{s=0} = \xi$$

Therefore,

$$\begin{aligned} d\mathcal{F}_{x(t)}(\xi) &= \int_{D^2} \frac{\partial \hat{x}^* \omega}{\partial s}|_{s=0} \\ &= \int_{D^2} \hat{x}^* \mathcal{L}_\xi \omega|_{s=0} \ (\text{definition of Lie derivative}) \\ &= \int_{D^2} x^* d\iota_\xi \omega \ (\text{Cartan's Formula}) \\ &= \int_{D^2} dx^* \iota_\xi \omega \\ &= \int_{S^1} x^* \iota_\xi \omega \ (\text{Stokes' Theorem}) \\ &= \int_{S^1} \omega(\xi(t), \frac{dx(t)}{dt})dt. \end{aligned}$$

And we conclude that $d\mathcal{F}_x(\xi) = 0$ if and only if $\frac{dx(t)}{dt} = 0$, which implies that $x(t)$ is a constant map to $L_0 \cap L_1$. In other words, the critical points of $\mathcal{F}$ are the lift to $\tilde{\mathcal{P}}$ of the points of intersection $L_0 \cap L_1$.

Just similar to the classical Morse theory, we are also interested in studying the $\mathcal{F}$-gradient trajectories between the critical points. To achieve this aim, we need to know what "grad$\mathcal{F}$" is. By definition, the g_J-gradient of $\mathcal{F}$ is the unique vector field $\nabla\mathcal{F}$ satisfying the relation $d\mathcal{F}(\xi) = \langle \nabla\mathcal{F}, \xi \rangle$, where ξ is any tangent vector. Then we calculate,

$$\begin{aligned} d\mathcal{F}_{x(t)}(\xi) &= \int_0^1 \omega(\frac{dx(t)}{dt}, \xi(t))dt \\ &= \int_0^1 \omega(\frac{dx(t)}{dt}, J_{x(t)}(-J_{x(t)}\xi(t))dt \\ &= \langle \frac{dx(t)}{dt}, -J_{x(t)}\xi(t) \rangle \\ &= \langle J_{x(t)}\frac{dx(t)}{dt}, \xi(t) \rangle \end{aligned}$$

Hence $\nabla\mathcal{F}(x(t)) = J_{x(t)}\frac{dx(t)}{dt}$. Now we consider the following set, for q and $p \in L_0 \cap L_1$,

$$\mathcal{M}(q,p) := \left\{ u(s,t) : \mathbb{R} \times [0,1] \to M \quad \frac{\partial u}{\partial s} = -\nabla\mathcal{F}(u), \middle| \begin{array}{l} \lim_{s\to+\infty} u(s,t) = p \\ \lim_{s\to-\infty} u(s,t) = q \\ \|\nabla u\|_2 < \infty \end{array} \right\}$$

which is the space of bounded trajectories connecting p and q. Here, $\nabla u = (\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t})$ and $\|\cdot\|_2$ is the L^2-norm with respect to the metric g. It thus turns out that the gradient flow lines are given by **J-holomorphic strips** in M with boundary conditions

$$\partial_s u + J_t \partial_t u = 0, \ u(s,0) \in L_0, \ u(s,1) \in L_1.$$

Here we allow J_t to vary smoothly with $t \in [0,1]$. We quote the following proposition from [24].

Proposition 3.3.1 (A. Floer, [24]) If L_0 intersects L_1 transversely, then for each $p, q \in L_0 \cap L_1$, there exist smooth Banach manifolds

$$\mathcal{P}(q,p) = \mathcal{P}_k^p(q,p) \subset \mathcal{P}_{k;\mathrm{loc}}^p$$

where $\mathcal{P}_{k;\mathrm{loc}}^p$ is given by the space of L_k^p-paths

$$\mathcal{P}_{k;\mathrm{loc}}^p(L_0, L_1) = \{u \in L_{k;\mathrm{loc}}^p(\mathbb{R} \times M | u(\mathbb{R} \times \{0\} \subset L_0, u(\mathbb{R} \times \{1\} \subset L_1\}$$

so that the map F given by

$$F(u) = \partial_s u + J_t \partial_t u$$

defines a smooth section of a smooth Banach space bundle over $\mathcal{P}(q,p)$ with fibers $\mathcal{L}_u = L^p(u^*T\mathcal{P})$, and $\mathcal{M}(q,p)$ is the zero set of F. The linearization of F

$$DF(u) : T_u\mathcal{P} \to \mathcal{L}_u$$

are Fredholm operators for $u \in \mathcal{M}(q,p)$.

Applying the notion of Maslov index, we can calculate,

Proposition 3.3.2 (A. Floer, [24]) There exist a map

$$\mu : \mathcal{P} \to \mathbb{Z}$$

which is defined up to an additive integer, so that for $u \in \mathcal{M}(q,p)$,

$$\mathrm{ind}(DF(u)) = \mu(q) - \mu(p),$$

which is the dimension of $\mathcal{M}(q,p)$

According to [26], the space $\mathcal{M}(q,p)$ has much in common with trajectory spaces in finite-dimensional case, the sets of trajectories connecting two critical points can be described as the intersection of the stable manifold of the one with the unstable manifold of the other point. This intersection is transversal in the "generic case" case, so that then the space of bounded trajectories decomposes into finite-dimensional manifolds. It turns out that one can prove very similar results for the set $\mathcal{M}(q,p)$, where intersection theory of finite-dimensional manifolds is replaced by techniques of the theory of elliptic partial differential equations. For details, see the main theorem in [26].

Making use of the above propositions, we have the following result:

Proposition 3.3.3 With the notation used in Proposition, $\hat{\mathcal{M}}(q,p)$ is a manifold of dimension $\mu(q) - \mu(p) - 1$, where $\hat{\mathcal{M}}(q,p)$ is the quotient of $\mathcal{M}(q,p)$ by $\mathbb{R}$, with the action :

$$(a \cdot u)(s,t) := u(s+a,t) \text{ for } a \in \mathbb{R}.$$

If $\mu(q) = \mu(p) + 1$, then $\hat{\mathcal{M}}(q,p)$ is a finite set.

The above result justifies the following construction.

Definition 3.3.3 Let C_k be the vector space over $\mathbb{Z}$ generated by $\{x \in L_0 \cap L_1 | \mu(x) = k\}$ for $k \in \mathbb{N}$. For $\mu(q) = \mu(p) + 1$, we define $\partial : C_{k+1} \to C_k$ by

$$\partial q = \sum_{p \in L_0 \cap L_1, \mu(p)=k} n(q,p)p,$$

where $n(q,p) = \#\hat{\mathcal{M}}(q,p)$.

Proposition 3.3.4 (A. Floer, [24]) As defined above, $\partial^2 = 0$.

Proof: We have for $r \in C_{k-1}$ and $q \in C_{k+1}$,

$$n(q,r) = \sum_{\mu(p)=k} n(q,p)n(p,r).$$

This is the number of pairs of adjacent trajectories joining r and q with appropriate signs (if we are using $\mathbb{Z}_2$ coefficient rather than $\mathbb{Z}$, the sign can be neglected). As similar to the case in classical Morse complex, these pairs of adjacent trajectories between q and r are in 1-1 correspondence with the ends of $\widehat{\mathcal{M}}(q,p)$, so they cancel each others. □

Therefore, (C_*, ∂) is a chain complex. Its homology

$$HF_*(M) = \frac{\ker\partial}{\mathrm{im}\partial}$$

is called the **Floer homology** of M.

3.4 Some Remarks

1. Apparently, the Floer homology group depends on the complex structures. In [24], however, Floer proved that the homology group is independent of J.

   Theorem 3.4.1 (A. Floer, [24]) There exists a natural chain homomorphism between two chain complexes $C_*(M,J)$ and $C_*(M,J')$, which induces an isomorphism of Floer homology

   $$HF_*(M,J) \cong HF_*(M,J').$$

Moreover, there is a natural isomorphism between the Floer homology and the singular homology of M

$$HF_*(M, J) \cong H_*(M; \mathbb{Z}).$$

2. According to [17], one cannot always define a Floer complex in some other *irregular cases* (e.g. the trajectories do not have finite energy) because ∂^2 may not always vanish. Different from the case in Morse complex, it may be impossible to define a good compactification of the 1-dimensional trajectory space $\hat{\mathcal{M}}(q, p)$ simply by adding once-broken flow lines. (In [21], Fukaya-Oh-Ohta-Ono sets up a framework in which to measure the obstructions to the existence of the Floer complex.)

3. In the next chapter, we will see that by considering a very special case of the Lagrangian Floer homology construction, Ozsváth-Szabó can be able to define a homology group (known as Heegaard Floer homology) for 3-manifold from the geometry of the Heegaard diagram.

Chapter 4

Heegaard Floer Homology

4.1 Introduction

Beginning in 2001, Ozsváth-Szabó introduced a new version of Floer homology - the heegaard Floer homology - based on Heegaard splittings of genus g of an oriented 3-manifold Y^3. With an overwhelming amount of calculational evidence, it is conjectured in [2] that the Seiberg-Witten theory and Ozsváth-Szabó theory are isomorphic. Each theory has it own advantages and disadvantages, for example Heegaard Floer homology is more combinatorial in flavor than Seiberg-Witten theory. (refer to [28])

In this chapter we first recall some basic topological preliminaries. Afterwards we define the Heegaard Floer complex. Then we briefly describe some calculations and applications. For details refer to [2] and [17].

4.2 Basic Set-Up

Let Y be a connected, compact, oriented 3-manifold. As explained in the previous chapter, Y is completely determined by a triple $(\Sigma_g, \alpha, \beta)$ (the **Heegaard diagram**) where Σ is a Riemannian surface of genus g and α, β are sets of attaching circles for two handlebodies U_α, U_β which bound Σ.

The basic idea is to use this data to construct a symplectic manifold (M, ω) together with a pair of Lagrangian submanifolds $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ and then to consider the corresponding Floer complex. However, as explained in [2] and [17], $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are in fact not *a priori* Lagrangian. Nevertheless, certain constructions from Floer theory can still be applied, so that one can define a chain complex $CF^\infty(Y)$.

Before we give the definition for the complex, let us recall some topological preliminaries which are needed in the setting.

4.3 Topological Preliminaries

4.3.1 Symmetric Product

Definition 4.3.1 For a topological space X, the **n-fold symmetric product** $\mathrm{Sym}^n(X)$ is the quotient space of the product of n copies of X obtained by factoring out the action of the symmetric group on n letters, i.e.

$$\mathrm{Sym}^n(X) := \overbrace{X \times \cdots \times X}^{n \text{ copies}} / S_n$$

The **diagonal** D in $\mathrm{Sym}^n(X)$ consists of those g-tuples of points in X, where at least two entries coincide.

Example 4.3.1 Let's see the case for $X = S^2$. We claim that $\mathrm{Sym}^n(X) = \mathbb{C}\mathrm{P}^n$ (A. Hatcher, [29]). First observe that $\mathbb{C}\mathrm{P}^n$ can be identify with the set of nonzero polynomials of degree at most n with coefficients in $\mathbb{C}$, modulo scalar multiplication, i.e. we have a bijection $\mathbb{C}\mathrm{P}^n \leftrightarrow \{a_0 + \cdots + a_n z^n | a_i \in \mathbb{C} \text{ with } a_j \neq 0 \text{ for some } j\}/\text{scalar multiplication}$. And for the sphere S^2 we view it as $\mathbb{C} \cup \{\infty\}$ by stereographic projection.

We define $\Phi : (S^2)^n \to \mathbb{C}\mathrm{P}^n$ by setting $\Phi(a_1, \cdots, a_n) = (z + a_1) \cdots (z + a_n)$ with factors $z + \infty$ omitted, in particular $\Phi(\infty, \cdots, \infty) = 1$. To check that Φ is continuous, suppose for some a_i approaches ∞, say a_n, and all other a_j's are finite. Since

$$\begin{aligned}&(z + a_1) \cdots (z + a_n)\\ &= z^n + (a_1 + \cdots + a_n) z^{n-1} + \cdots + \textstyle\sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k} z^{n-k} + \cdots + a_1 \cdots a_n,\end{aligned}$$

We see that, dividing through by a_n and letting a_n approach to ∞, this polynomial approaches to $z^{n-1} + (a_1 + \cdots + a_{n-1}) z^{n-2} + \cdots + a_1 \cdots a_{n-1} = (z + a_1) \cdots (z +$

a_{n-1}). The same argument would still apply if several a_i's approach ∞ simultaneously.

The value $\Phi(a_1, \cdots, a_n)$ is clearly unchanged under permutation of the a_i's, so there is an induced map $\tilde{\Phi} : \mathrm{Sym}^n(S^2) \to \mathbb{C}\mathrm{P}^n$ which is a continuous bijection, hence a homeomorphism since both spaces are compact and Hausdorff.

Example 4.3.2 The map $\varphi : \mathrm{Sym}^n(\mathbb{C}) \to \mathbb{C}^n$ defined by

$$\varphi([a_1, \cdots, a_n]) = (z + a_1) \cdots (z + a_n)$$

is a diffeomorphism (here we identify $\mathbb{C}^n$ with the set of monic polynomials of degree n). This shows that for any two dimensional (oriented) manifold M (in particular a Riemann surface), $\mathrm{Sym}^n(M)$ is an n-dimensional complex manifold.

It is worth to point out that $\mathrm{Sym}^n(M)$ has no *natural* smooth structure; it inherits a complex structure J_j from the choice of a complex structure j on M, but different choices of j give rise to different smooth structures on $\mathrm{Sym}^n(M)$ (two complex structures J_{j_1} and J_{j_2} are different in the sense that the identity map $(\mathrm{Sym}^n(M), J_{j_1}) \to (\mathrm{Sym}^n(M), J_{j_2})$ is not smooth) . Here the J_i is specified by the property that the natural quotient map

$$\pi : M \times \cdots \times M \to \mathrm{Sym}^n(M)$$

is holomorphic (where the product space is endowed with a product complex structure). (refer to [31])

Back to our case of 3-manifolds Y. For a Heegaard diagram $(\Sigma_g, \alpha, \beta)$, we consider the g-fold symmetric product $M_g := \mathrm{Sym}^g(\Sigma_g)$ of Σ_g. As explained in the above example, M_g is a complex manifold with $\dim_{\mathbb{C}}(M_g) = g$.

The manifold M_g has rather simple homotopy and cohomology. We refer the following lemma given in [2].

Lemma 4.3.2 (Ozaváth-Szabó, [2]) Let Σ_g be a genus g surface. Then

$$\pi_1(\mathrm{Sym}^g(\Sigma_g)) \cong H_1(\mathrm{Sym}^g(\Sigma_g);) \cong H_1(\Sigma_g).$$

Sketch of proof: We begin by proving the isomorphism on the level of homology. There is a map

$$H_1(\Sigma_g) \to H_1(\mathrm{Sym}^g(\Sigma_g))$$

induced from the inclusion $\Sigma \times \{x\} \times \cdots \times \{x\} \subset \mathrm{Sym}^g(\Sigma_g)$. To invert this, for a closed curve in $\mathrm{Sym}^g(\Sigma_g)$, by moving it to the "general position" (i.e. to miss out the diagonal D in $\mathrm{Sym}^n(X)$), we get a map of g-fold cover of S^1 to Σ_g. This corresponds to a homology class in $H_1(\Sigma_g)$, and hence a map $H_1(\mathrm{Sym}^g(\Sigma_g)) \to H_1(\Sigma_g)$.

By similar argument, one can show $\pi_1(\mathrm{Sym}^g(\Sigma_g)) \cong H_1(\mathrm{Sym}^g(\Sigma_g))$ (refer to [2] for the rest) □

The cohomology of $\mathrm{Sym}^g(\Sigma_g)$ was studied in [31] and [32]. As is usual in the study of Gromov invariants and Lagrangian Floer theory, we have to understand the **holomorphic spheres** (which will be discussed later) in $\mathrm{Sym}^g(\Sigma_g)$. To do this, we study how the **first Chern class** c_1 (of the tangent bundle $T\mathrm{Sym}^g(\Sigma_g)$) evaluates on homology classes which are representable by spheres, which provides a linkage between the homotopy and the cohomology of $\mathrm{Sym}^g(\Sigma_g)$.

We are going to state the following proposition in [2]. To this end, we introduce some notations. If X is a connected space endowed with a basepoint $x \in X$, let $\pi_2'(X)$ denote the quotient of $\pi_2(X, x)$ by the action of $\pi_1(X, x)$. This action is independent of the choice of x, and the **Hurewicz homomorphism** from $\pi_2(X, x)$ to $H_2(X)$ factors through $\pi_2'(X)$.

Proposition 4.3.1 (Ozaváth-Szabó, [2]) Let Σ_g be a Riemann surface of genus $g > 1$, then

$$\pi_2'(\mathrm{Sym}^g(\Sigma_g)) \cong \mathbb{Z}$$

Furthermore, if $\{A_i, B_i\}$ is a symplectic basis for $H_1(\Sigma_g)$, then there is a generator of $\pi_2'(\mathrm{Sym}^g(\Sigma_g))$, denoted by S, whose image under the Hurewicz homomorphism is Poincaré dual to

$$(1-g)U^{g-1} + \sum_{i=1}^{g} \mu(A_i)\mu(B_i)U^{g-2}$$

In the case where $g > 2$, $\pi_1(\mathrm{Sym}^g(\Sigma_g))$ acts trivially on $\pi_2(\mathrm{Sym}^g(\Sigma_g))$ and thus

$$\pi_2(\mathrm{Sym}^g(\Sigma_g)) \cong \mathbb{Z}$$

here, U is a two-dimensional cohomology class which is Poincaré dual to the submanifold

$$\{x\} \times \mathrm{Sym}^{g-1}(\Sigma_g) \subset \mathrm{Sym}^g(\Sigma_g)$$

and $\mu : H_1(\Sigma_g) \to H^1(\mathrm{Sym}^g(\Sigma_g))$ is an isomorphism.

We omit the proof. Basically, the isomorphism $\pi_2'(\mathrm{Sym}^g(\Sigma_g)) \cong \mathbb{Z}$ is given by the intersection number with the submanifold $\{x\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ for generic x. And for $g > 2$, the generator of $S \in \pi_2(\mathrm{Sym}^g(\Sigma_g))$ can be constructed in the following way: let $\tau : \Sigma_g \to \Sigma_g$ be a **hyperelliptic involution** with the property that $\Sigma_g/\tau = S_0$, where $S^2 \cong S_0 \subset \mathrm{Sym}^2(\Sigma_g)$. Then $S_0 \times \{x\} \times \cdots \times \{x\} \subset \mathrm{Sym}^g(\Sigma_g)$ is a sphere representing S.

The evaluation of the first Chern class on the generator S is given in the following:

Proposition 4.3.2 (Ozaváth-Szabó, [2]) The first Chern class of $\mathrm{Sym}^g(\Sigma_g)$ is given by

$$c_1 = U - \sum_{i=1}^{g} \mu(A_i)\mu(B_i).$$

In particular, $\langle c_1, [S] \rangle = 1$.

See [2] for the calculation of c_1. The rest follows from the previous proposition.

4.3.2 The Tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$

For a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ of a 3-manifold Y^3, since the attaching circles $\alpha = \{\alpha_1, \ldots, \alpha_g\}$ are mutually disjointed, it naturally induces a smoothly embedded g-dimensional torus

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g \subset \mathrm{Sym}^g(\Sigma_g)$$

More precisely, $\mathbb{T}_\alpha$ consists of those g-tuples of points $\{x_1, \cdots, x_g\}$ for which $x_i \in \alpha_i$ for $i = 1, \cdots, g$. Similarly, we define the torus $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g \subset \mathrm{Sym}^g(\Sigma_g)$.

As suggested in [17], these two tori enjoy a certain compatibility with any complex structure on $\mathrm{Sym}^g(\Sigma_g)$ induced from Σ_g.

Definition 4.3.3 Let $(Z,\ J)$ be a complex manifold, and $L \subset Z$ be a submanifold. Then L is called **totally real** if none of its tangent spaces contains a J-**complex line**, i.e. $T_\lambda L \cap JT_\lambda L = \{0\}$ for each $\lambda \in L$.

Notice that $T_\lambda L \cap JT_\lambda L = T^{0,1}_\lambda(Z) \cap \mathbb{C}T_\lambda(L)$, where $T^{0,1}_\lambda(Z)$ is the **antiholomorphic tangent vectors** of Z at λ and $\mathbb{C}T_\lambda(L)$ is the **complexified tangent space** of L at λ (refer to [33]).

Lemma 4.3.4 (Ozaváth-Szabó, [11])Let $\mathbb{T}_\alpha \subset \mathrm{Sym}^g(\Sigma_g)$ be the torus induced from a set of attaching circles. Then $\mathbb{T}_\alpha$ is a totally real submanifold of $\mathrm{Sym}^g(\Sigma_g)$. (for any complex structure induced from Σ_g)

Proof: Note that the projection map $\pi : \Sigma_g \times \cdots \times \Sigma_g \to \mathrm{Sym}^g(\Sigma_g)$ is a holomorphic local diffeomorphism away from the diagonal subspace (consisting of those g-tuples for which at least two of the points coincide). Since $\mathbb{T}_\alpha \subset \mathrm{Sym}^g(\Sigma_g)$ misses the diagonal, the lemma follows immediately from the fact that $\alpha_1 \times \cdots \times \alpha_g \subset \Sigma_g \times \cdots \times \Sigma_g$ is a totally real submanifold (for the product complex structure), which follows easily from the definitions. □

The dimensions of these two tori are both equal to g. Note that if all the α_i curves meet all the β_j curves transversally, then the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ meet transversally. Each intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ can be written as

$$\mathbf{x} := (x_1, \cdots, x_g), \quad x_k \in \alpha_k \cap \beta_{\sigma(k)}, \text{ where } k = 1, \cdots, g, \ \sigma \in S_g.$$

These intersection points are important in defining the Heegaard Floer complexes. Similar to the Lagrangian Floer theory, we are going to study the **holomorphic disks** connecting those intersection points.

4.3.3 Intersection Points and Disks

Fix a complex structure j on Σ_g and consider the corresponding complex structure J on the symmetric product $\mathrm{Sym}^g(\Sigma_g)$. For two intersection points $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, as suggested in Chapter 2, the elements in $\mathcal{M}(\mathbf{x}, \mathbf{y})$ are the J-holomorphic strips $u : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g(\Sigma_g)$ from $\mathbf{x}$ to $\mathbf{y}$ satisfying

$$\partial_s u + J_u \partial_t u = 0, \ u(s,0) \in \mathbb{T}_\alpha, \ u(s,1) \in \mathbb{T}_\beta.$$

Since the domain $\mathbb{R} \times [0,1]$ is conformally equivalent to the closed unit disk $\mathbb{D}$ in $\mathbb{C}$ with the two points $\pm i$ removed, so we can think of the strips as continuous maps

$$u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$$

which take the left boundary $\partial\mathbb{D} \cap \{\mathfrak{Re}(z) < 0\}$ to $\mathbb{T}_\alpha$ and the right boundary $\partial\mathbb{D} \cap \{\mathfrak{Re}(z) > 0\}$ to $\mathbb{T}_\beta$. More precisely, we can define:

Definition 4.3.5 Given a pair of intersection points $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a **Whitney disk** connecting $\mathbf{x}$ and $\mathbf{y}$ is a continuous map

$$u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$$

with the properties that $u(-i) = \mathbf{x}$, $u(i) = \mathbf{y}$, $u(e_1) \subset \mathbb{T}_\alpha$, $u(e_2) \subset \mathbb{T}_\beta$. Here e_1 and e_2 are the arcs in the boundary of $\mathbb{D}$ with $\mathfrak{Re}(z) < 0$, $\mathfrak{Re}(z) > 0$ respectively. Let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of Whitney disk.

For arbitrary $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, it is *not always possible* to find such a map u connecting them. To see the obstruction, we first need the following proposition.

Proposition 4.3.3 (Ozaváth-Szabó, [11]) With the same notations used, we have the following isomorphisms:

$$\frac{H_1(\mathrm{Sym}^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong \frac{H_1(\Sigma_g)}{\langle [\alpha_1], \cdots, \alpha_g], [\beta_1], \cdots, [\beta_g] \rangle} \cong H_1(Y^3; \mathbb{Z})$$

Proof: The first "$\cong$" follows directly from Lemma 3.3.2. To get the second "$\cong$", we first define a surjective homomorphism $f : H_1(\Sigma_g) \to H_1(Y, \mathbb{Z})$. For a closed curve γ on Σ_g, it gives a closed curve in Y by inclusion (treat γ as a curve on the U_α handlebody). And for any curve ζ in Y, by applying homotopy on ζ if necessary, we can define another curve $\tilde{\zeta}$ lying on Σ_g with $[\tilde{\zeta}] = [\zeta] \in H_1(Y)$. Passing to homology, we have just defined our desired surjective homomorphism $f : H_1(\Sigma_g) \to H_1(Y, \mathbb{Z})$. By First Isomorphism Theorem, it remains to show $\ker(f) = \langle [\alpha_1], \cdots, \alpha_g], [\beta_1], \cdots, [\beta_g] \rangle$. This follows since each of the attaching circles $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ bounds a disk in Y, giving trivial classes in $H_1(Y; \mathbb{Z})$. □

Now, let $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a pair of intersections. Choose a pair of paths $a : [0, 1] \to \mathbb{T}_\alpha$, $b : [0, 1] \to \mathbb{T}_\beta$ from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The difference is a loop in $\mathrm{Sym}^g(\Sigma_g)$.

Let $\varepsilon(\mathbf{x}, \mathbf{y})$ denote the image of $a - b$ in $H_1(Y^3; \mathbb{Z})$ under the isomorphism given in Proposition 3.3.3. It is clear that $\varepsilon(\mathbf{x}, \mathbf{y})$ is independent of the choice of the paths a and b. Here $\varepsilon(\mathbf{x}, \mathbf{y})$ is *additive*, in the sense that

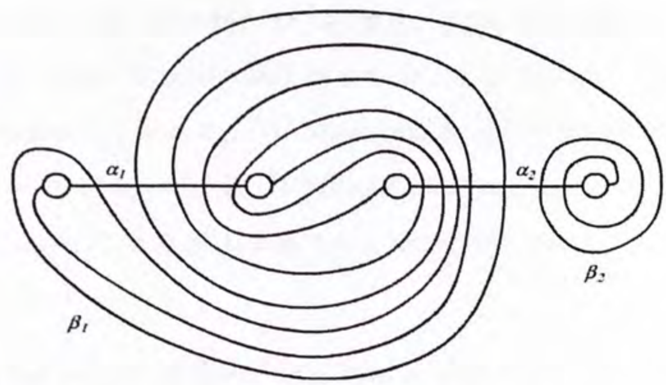
$$\varepsilon(\mathbf{x}, \mathbf{y}) + \varepsilon(\mathbf{y}, \mathbf{z}) = \varepsilon(\mathbf{x}, \mathbf{z}), \text{ for } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

This allows us to partition the intersection points into **equivalent classes**, where $\mathbf{x} \sim \mathbf{y}$ iff $\varepsilon(\mathbf{x}, \mathbf{y}) = 0$.

As suggested in [11], the value of ε can be calculated in Σ_g, using the identification between $\pi_1(\mathrm{Sym}^g(\Sigma_g))$ and $H_1(\Sigma_g)$. By writing $\mathbf{x} = \{x_1, \cdots, x_g\}$ and $\mathbf{y} = \{y_1, \cdots, y_g\}$, we can treat the path $a : [0,1] \to \mathbb{T}_\alpha$ as a collection of arcs in $\alpha_1 \cup \cdots \cup \alpha_g \subset \Sigma_g$, with boundary $\partial a = y_1 + \cdots + y_g - x_1 - \cdots - x_g$. Similarly, the path $b : [0,1] \to \mathbb{T}_\beta$ can be viewed as a collection of arcs in $\beta_1 \cup \cdots \cup \beta_g \subset \Sigma_g$ with boundary $\partial b = y_1 + \cdots + y_g - x_1 - \cdots - x_g$. So the difference is a closed one-cycle in Σ_g, whose image in $H_1(Y^3; \mathbb{Z})$ is $\varepsilon(\mathbf{x}, \mathbf{y})$ as defined above.

What is the *significance* of this equivalent relation? On one hand, if $\pi_2(\mathbf{x}, \mathbf{y})$ is non-empty, then $\varepsilon(\mathbf{x}, \mathbf{y}) = 0$ (i.e. $\mathbf{x} \sim \mathbf{y}$). The reason is simple: if there exist a Whitney disk $u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$ between $\mathbf{x}$ and $\mathbf{y}$, then we can take the path $a : [0,1] \to \mathbb{T}_\alpha$ connecting $\mathbf{x}$ and $\mathbf{y}$ *via* the boundary $\partial\mathbb{D} \cap \{\mathfrak{Re}(z) < 0\}$. Similarly, we can take $b : [0,1] \to \mathbb{T}_\beta$ connecting $\mathbf{x}$ and $\mathbf{y}$ *via* $\partial\mathbb{D} \cap \{\mathfrak{Re}(z) > 0\}$. Now $a - b$ is just $\partial\mathbb{D}$, and therefore $\varepsilon(\mathbf{x}, \mathbf{y}) = 0$. So if $\varepsilon(\mathbf{x}, \mathbf{y}) \neq 0$, then we can deduce that there *does not* exist a disk connecting $\mathbf{x}$ and $\mathbf{y}$.

Example 4.3.3 We now quote the following example in [11] to illustrate the above construction.



`[figure : the +2-surgery on (2,3) torus knot]`

One can interpret this as the 3-manifold obtained by *+2 surgery on (2,3) torus knot* (refer to [6] for the general case). Let Y to be this 3-manifold. Here we think of S^2 as the plane together with the point at infinity. In the picture the two circles on the left are identified, or equivalently we glue a handle to S^2 along these circles.

Similarly we identify the two circles on the right side of the picture. After this identification the two horizontal lines become closed circles α_1 and α_2. As for the two β curves, β_1 lies in the plane and β_2 goes through both handles once.

By direct counting,

$$\sharp(\alpha_1 \cap \beta_1) = 3,\ \sharp(\alpha_1 \cap \beta_2) = 2,\ \sharp(\alpha_2 \cap \beta_1) = 3,\ \sharp(\alpha_2 \cap \beta_2) = 4$$

So $\sharp(\mathbb{T}_\alpha \cap \mathbb{T}_\beta) = 3 \times 4 + 2 \times 3 = 18$. And for simplicity, we label $\alpha_1 \cap \beta_1 = \{x_1, x_2, x_3\}$, $\alpha_2 \cap \beta_1 = \{v_1, v_2, v_3\}$, $\alpha_2 \cap \beta_1 = \{y_1, y_2\}$, $\alpha_2 \cap \beta_2 = \{w_1, w_2, w_3, w_4\}$.

From the picture, for some appropriate orientation of $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$, we have

$$[\alpha_1] \cdot [\beta_1] = -1,\ [\alpha_2] \cdot [\beta_1] = -1,\ [\alpha_1] \cdot [\beta_2] = 2,\ [\alpha_2] \cdot [\beta_2] = 4$$

Thus, if $\{[\alpha_1], B_1, [\alpha_2], B_2\}$ is a standard symplectic basis for $H_1(\Sigma_2)$ (for example, choose B_1 and B_2 to be the classes represented by the **longitudes** in Σ_2), then

$$[\beta_1] \equiv -B_1 - B_2,\ [\beta_2] \equiv 2B_1 + 4B_2$$

in $H_1(\Sigma_2)/\langle[\alpha_1], [\alpha_2]\rangle$. Therefore, $H_1(\Sigma_2)/\langle[\alpha_1], [\alpha_2], [\beta_1], [\beta_2]\rangle = \langle B_1, B_2\rangle/\{-B_1 - B_2 \equiv 0,\ 2B_1 + 4B_2 \equiv 0\}$. By Proposition 3.3.3., $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$, which is generated by $B_1 = -B_2 = h$.

We can calculate, for example, $\varepsilon(\{x_1, w_1\}, \{x_2, w_1\})$ as follows. We choose a closed loop in Σ_2 which is composed of one arc $a \subset \alpha_1$, and another in $b \subset \beta_1$, both of which connect x_1 and x_2. We then calculate the intersection number $(a - b) \cap \alpha_1 = 0$, $(a - b) \cap \alpha_2 = -1$. It follows that $[a - b] = -B_2 = h$ in $H_1(Y)$. So, $\varepsilon(\{x_1, w_1\}, \{x_2, w_1\}) = h \neq 0$ and we deduce that there is no disk connecting $\{x_1, w_1\}$ and $\{x_2, w_1\}$.

Remark: With the notion of **Spin^c structure** (which will be discussed in the later section), the equivalent classes of the intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are determined by $H_1(Y)$. In our example, the intersection points can be exactly partitioned into 2 equivalence classes.

4.3.4 Domains

In order to have a better understanding of $\pi_2(\mathbf{x}, \mathbf{y})$, we need to study the domain $D(\phi)$ associated to each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Definition 4.3.6 Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For each point $w \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$, denote

$$n_w : \pi_2(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}$$

the algebraic intersection number

$$n_w = \sharp\phi^{-1}(\{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g)).$$

Because $V_w = \{w\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ is disjoint from $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ (by the definition of w), n_w is well-defined and is only dependent on the homotopy class of ϕ. (refer to [34])

There is a natural operation on $\pi_2(\mathbf{x}, \mathbf{y})$. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if we take a Whitney disk connecting $\mathbf{x}$ to $\mathbf{y}$, and another one connecting $\mathbf{y}$ to $\mathbf{z}$, then we can get a Whitney disk connecting $\mathbf{x}$ to $\mathbf{z}$ by **splicing**. Such operation gives rise to a "multiplication" on $\pi_2(\mathbf{x}, \mathbf{y})$,

$$* : \pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{z}) \to \pi_2(\mathbf{x}, \mathbf{z})$$

And this operation is readily seen to be associative. Hence for the case $\mathbf{x}=\mathbf{y}$, $(\pi_2(\mathbf{x}, \mathbf{x}), *)$ is a group.

Now we can define the domain belonging to a Whitney disk.

Definition 4.3.7 Let $D_1, \cdots, D_m$ denote the closures of the components of $\Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. Given a Whitney disk $\phi : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$, the **domain** associated to ϕ is the formal linear combination of $\{D_i\}_{i=1}^m$,

$$D(\phi) = \sum_{i=1}^{m} n_{z_i}(\phi) D_i$$

where $z_i \in D_i$ are points in the interior of D_i.

There are several properties for domain.

Proposition 4.3.4 (Ozaváth-Szabó, [11]) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{z})$. Then we have

$$D(\phi_1 * \phi_2) = D(\phi_1) + D(\phi_2)$$

Similarly,

$$D(S * \phi) = D(\phi) + \sum_{i=1}^{m} D_i$$

where S denotes the positive generator of $\pi_2(\mathrm{Sym}^g(\Sigma_g))$.

Proof: These follow readily from the fact that $n_w(\phi_1 * \phi_2) = n_w(\phi_1) + n_w(\phi_2)$, where $\phi_1 \in \pi_2(\mathbf{x}, \mathbf{y})$, $\phi_2 \in \pi_2(\mathbf{y}, \mathbf{z})$ and $w \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$ □

Definition 4.3.8 For a pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ (i.e. $z \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$), a **periodic domain** is a two-chain $P = \sum_{i=1}^{m} a_i D_i$ whose boundary is a sum of $\alpha-$ and β-curves with $n_z(P) = 0$. For any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a class $\phi \in \pi_2(\mathbf{x}, \mathbf{x})$ with $n_z(\phi) = 0$ is called a **periodic class**.

Remark: The set $\Pi_{\mathbf{x}}(z)$ of periodic classes is naturally a subgroup of $\pi_2(\mathbf{x}, \mathbf{x})$. And the domain belonging to a periodic class is a periodic domain.

We are now readily to give the following proposition, which describes the algebraic topology of $\pi_2(\mathbf{x}, \mathbf{y})$.

Proposition 4.3.5 (Ozaváth-Szabó, [2]) Suppose $g > 1$. For all $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, there is an isomorphism

$$\pi_2(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$$

which identifies the subgroup of periodic classes

$$\Pi_{\mathbf{x}}(z) \cong H^1(Y; \mathbb{Z}).$$

In general, for each $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, if $\varepsilon(\mathbf{x}, \mathbf{y}) \neq 0$, then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty; otherwise

$$\pi_2(\mathbf{x}, \mathbf{y}) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z}).$$

Sketch of proof: Fix $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. We can identify $\pi_2(\mathbf{x}, \mathbf{x})$ with the fundamental group of $\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ based at the constant $(\mathbf{x})$ path, i.e. $\pi_1(\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta))$. Moreover, observe that we have a natural **Serre fibration**:

$$\Omega(\mathrm{Sym}^g(\Sigma_g)) \to \mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta) \to \mathbb{T}_\alpha \times \mathbb{T}_\beta$$

where $\Omega(\mathrm{Sym}^g(\Sigma_g))$ denotes the loop-spaces of $\mathrm{Sym}^g(\Sigma_g)$ and $\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)$ denotes the space of paths in $\mathrm{Sym}^g(\Sigma_g)$ joining $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$. The short exact sequence gives rise to a homotopy long exact sequence:

$$0 \to \pi_2(\mathrm{Sym}^g(\Sigma_g)) \to \pi_1(\mathcal{P}(\mathbb{T}_\alpha, \mathbb{T}_\beta)) \to \pi_1(\mathbb{T}_\alpha \times \mathbb{T}_\beta) \to \pi_1(\mathrm{Sym}^g(\Sigma_g)).$$

Compare with the Mayer-Vietoris sequence of Y,

$$\cdots \to H^0(\Sigma_g) \to H^1(Y) \to H^1(U_\alpha) \oplus H^1(U_\beta) \to H^1(\Sigma_g) \to \cdots$$

where $U_\alpha \cap U_\beta = \Sigma_g$ and $U_\alpha \cup U_\beta = Y$, we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_2(\mathrm{Sym}^g(\Sigma_g)) \arrow[r] \arrow[d, "f_1"] & \pi_2(\mathbf{x}, \mathbf{x}) \arrow[r] \arrow[d, "f_2"] & \pi_1(\mathbb{T}_\alpha) \times \pi_1(\mathbb{T}_\beta) \arrow[r] \arrow[d, "f_3"] & \pi_1(\mathrm{Sym}^g(\Sigma_g)) \arrow[r] \arrow[d, "f_4"] & \cdots \arrow[d] \\
\cdots \arrow[r] & H^0(\Sigma_g) \arrow[r] & H^1(Y) \arrow[r] & H^1(U_\alpha) \oplus H^1(U_\beta) \arrow[r] & H^1(\Sigma_g) \arrow[r] & \cdots
\end{tikzcd}$$

Under the identification $\pi_1(\mathrm{Sym}^g(\Sigma_g)) \cong H_1(\Sigma_g) \cong H^1(\Sigma_g)$ (Lemma 3.3.2), the images of $\pi_1(T_\alpha)$ and $\pi_1(T_\beta)$ corresponds to $H^1(U_\alpha)$ and $H^1(U_\beta)$ respectively. So f_4 is an isomorphism and f_3 is surjective. And by Proposition 3.3.1, $\pi_2(\mathrm{Sym}^g(\Sigma_g)) \cong \mathbb{Z} \cong H^0(\Sigma_g)$, so f_1 is an isomorphism too. Therefore, by straightforward diagram chasing, we can conclude that f_2 is a surjective homomorphism and we get a short exact sequence:

$$0 \to \mathbb{Z} \to \pi_2(\mathbf{x}, \mathbf{x}) \to H^1(Y; \mathbb{Z}) \to 0$$

which implies $\pi_2(\mathbf{x}, \mathbf{x}) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z})$. And because the homomorphism $n_z : \pi_2(\mathbf{x}, \mathbf{x}) \to \mathbb{Z}$ is surjective, so $\pi_2(\mathbf{x}, \mathbf{x}) \cong \mathbb{Z} \oplus \ker(n_z)$, i.e. $\pi_2(\mathbf{x}, \mathbf{x}) \cong \mathbb{Z} \oplus \Pi_{\mathbf{x}}(z)$. Hence $\Pi_{\mathbf{x}}(z) \cong H^1(Y; \mathbb{Z})$. The proposition for $g > 2$ thus follows. The case for $g = 2$ follows similarly, only now one must divide by the action of $\pi_1(\mathrm{Sym}^g(\Sigma_g))$.

For $\mathbf{x} \neq \mathbf{y}$ and $\varepsilon(\mathbf{x}, \mathbf{y}) = 0$, $\pi_2(\mathbf{x}, \mathbf{y})$ is non-empty. By similar arguments as the above, the remaining part follows. □

4.3.5 Spin^c Structures

The reason for introducing the notion of Spin^c structures is to refine the discussion about the equivalence classes on $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. To do this we need some definitions.

Definition 4.3.9 Let Y be a smooth manifold of dimension n. A **frame** at a point $x \in Y$ is a basis of the tangent space at x. The **frame bundle** is the fibre bundle consisting of all frames on Y. This fibre bundle is a **principal bundle** under the action of the **general linear group** $GL(n)$.

Definition 4.3.10 Let Y be a closed oriented 3-manifold with a Riemannian metric g. We can consider the associated **principal** $SO(3)$**-bundle** Fr on Y (the oriented orthonormal frames)

$$SO(3) \to Fr \to Y$$

A **Spin^c structure** on Y can be viewed as a lift of the structure group $SO(3)$ of the tangent bundle TY to the group Spin$^c(3)$, where

$$\mathrm{Spin}^c(3) := \mathrm{Spin}(3) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 = SU(2) \times_{\mathbb{Z}/2\mathbb{Z}} S^1 = U(2).$$

So it gives us a **principal** $U(2)$**-bundle** on Y, say $F \to Y$. We get the following commutative diagram:

$$\begin{tikzcd}
F \arrow[r, "\phi'"] \arrow[d, "\pi"'] & Y \\
Fr \arrow[ur, "\phi"'] &
\end{tikzcd}$$

here, $\pi : F \to Fr$ gives a $U(1)$-bundle on Fr, and there is an isomorphism α between the two $SO(3)$-bundles on Y, namely $F/U(1) \to Y$ and $\phi : Fr \to Y$. In other words, each Spin^c structure corresponds to an isomorphism class $(F \to Y, \alpha)$.

Definition 4.3.11 We can have an equivalent definition of Spin^c structure in terms of cohomology: a Spin^c structure on Y is an element of $H^2(Fr)$ whose restriction on every fiber $f^{-1}(x)$ is the non-zero element of $H^2(SO(3)) = \mathbb{Z}/2\mathbb{Z}$. To show that the two definitions are indeed equivalent, recall that the isomorphism classes of circle bundles over a manifold X are in one-to-one correspondence with the elements of $H^2(X)$. And now, it is sufficient to associate with any pair $(F \to Y, \alpha)$ the element of $H^2(Fr)$ corresponding to the circle bundle $\alpha \circ \tilde{\pi} : F \to F/U(1) \cong Fr$. The set of Spin^c structure on Y is denoted by **Spin^c(Y)**, and Spin$^c(Y) \subset H^2(Fr)$.

One can define a group action of $H_1(Y) \cong H^2(Y)$ on $H^2(Fr)$ via the pull-back homomorphism $\phi^* : H^2(Y) \to H^2(Fr)$ and addition, i.e.

$$g \cdot a := \phi^*(g) + a$$

where $g \in H^2(Y)$ and $a \in H^2(Fr)$. From the definition of Spin^c(Y), we can see that this action preserves Spin^c(Y)$\subset H^2(Fr)$. And that induced action of $H_1(Y)$ on Spin^c(Y) is in fact **free** and **transitive**. The action is free follows readily from its definition, that if $g \cdot a = a$ then $g = 0 \in H^2(Y)$. To show that the action is also transitive, observe that because Y is parallelisable, so $Fr = Y \times SO(3)$. By Künneth formula, $H^2(Fr) = H^2(Y) \oplus (\mathbb{Z}/2\mathbb{Z})$, and therefore Spin^c(Y) consists of only one orbit with respect to the group action.

According to [35], in order to study the Spin^c structure on Y we need to have a better understanding of the "non-vanishing vector fields" on Y (in [36] it is called **Euler structure**). We are going to explain it in detail.

Definition 4.3.12 By the Poincaré-Hopf theorem, for a connected closed smooth manifold Y with $\chi(Y) = 0$, there exists a non-vanishing vector field on Y. Since we are working on 3-dimensional Y, the condition $\chi(Y) = 0$ is satisfied and it is reasonable to deal with non-vanishing fields.

Vector fields u and v on Y are called **homologous** if they are homotopic in the complement of finitely many disjoint 3-balls in Y. It is clear that being homologous is an equivalence relation, and the class of a vector field u on Y is denoted by $[u]$ and called an **Euler structure** on Y.

Definition 4.3.13 An equivalent definition of Euler structures on Y is given in terms of the S^2 fiber bundle of unit tangent vectors, say

$$S^2 \longrightarrow SY \xrightarrow{\varphi} Y$$

This S^2-bundle is defined as below. For any Euler structure $[u]$ on Y, we can consider the 2-dimensional vector space $u^\perp$ formed by the tangent vectors orthogonal to u, i.e. $T_xY = u(x) \oplus u(x)^\perp$ for any $x \in Y$. This gives a sphere S^2 at each $x \in Y$ containing vectors $\frac{u(x)}{|u(x)|}$, e_1 and e_2, where e_1 and e_2 lie in $u^\perp$ with lengths both equal to 1.

An Euler structure on Y is an element of $H^2(SY)$ whose restriction on every fiber $S_xY(=Y)$ is the generator of $H^2(S^2)=\mathbb{Z}$ determined by the orientation of Y at x.

To establish the equivalence, for any non-vanishing vector field u on Y, we define a map $f: Y \to SY$ with $f(x)=\frac{u(x)}{|u(x)|}$. This f gives a **3-cycle** in SY, i.e. an element in $H_3(SY)$ (we can treat it as a section on SY). We orient SY so that the intersection number of this cycle with every oriented fiber equals $+1$. Thus the element of $H^2(SY)\cong H_3(SY)$ represented by this cycle is an Euler structure on Y in the sense of the second definition. We denote the set of Euler structures on Y by **Vect**(Y), and Vect$(Y)\subset H^2(SY)$.

Similar to Spin$^c(Y)$, we can also define a group action of $H_1(Y)\cong H^{m-1}(Y)$ on Vect(Y) via the pull-back homomorphism $\varphi^*: H^{m-1}(Y)\to H^{m-1}(SY)$ and addition, i.e.

$$g\cdot b := \varphi^*(g)+b$$

where $g\in H^{m-1}(Y)$ and $b\in H^{m-1}(SY)$. By the same argument as for Spin$^c(Y)$, this action is free and transitive and preserves Vect(Y).

Example 4.3.4 For n being odd, $\chi(S^n)=0$, and it is easy to give an example of a continuous non-vanishing vector field on S^n, simply take

$$u(x_1,\cdots,x_{n+1})=(-x_2,x_1,-x_4,x_3,\cdots,-x_{n+1},x_n)$$

Let's see the associated Euler structures on S^3. By Adams' theorem [38], the tangent bundle of S^n is trivial if and only if $n=0,1,3,7$, corresponding to the unit spheres in the 4 real division algebras. Therefore, $SS^3=S^3\times S^2$. By Künneth formula,

$$H^2(S^3\times S^2)=H^0(S^3)\otimes H^2(S^2)\oplus H^1(S^3)\otimes H^1(S^2)\oplus H^2(S^3)\otimes H^0(S^2)=\mathbb{Z}$$

We choose the standard orientation on S^3 (which is compatible to the orientation $+1\in H^2(S^2)\cong\mathbb{Z}$). Hence from the above definition, we know that there are exactly one Euler structure on S^3 corresponding to the generators $+1\in H^2(S^2)$. In fact, it is just $[u]$, where $u: S^3\to S^3$ is given by

$$u(x_1,x_2,x_3,x_4)=(-x_2,x_1,-x_4,x_3).$$

Remark: The other one $[-u]$ corresponds to $-1 \in H^2(S^2)$, which is the opposite orientation of S^2 (compatible to the opposite orientation of S^3).

The next lemma will illustrate the relation between $\mathrm{Spin}^c(Y)$ and $\mathrm{Vect}(Y)$.

Lemma 4.3.14 (Turaev, [36]) For a closed oriented 3-manifold Y, there is a canonical bijection $\mathrm{Spin}^c(Y) \leftrightarrow \mathrm{Vect}(Y)$.

Proof: Notice that we have a natural map $p : SO(3) \to S^2$ sending an orthonormal triple (e_1, e_2, e_3) to its first vector $e_1 \in S^2$. Now,

$$p^{-1}(e_1) = \{(e_2, e_3) : e_2, e_3 \in \mathbb{R}^3 \text{ and } (e_1, e_2, e_3) \in SO(3)\},$$

which can be identified with $SO(2) \cong S^1$. So p can be viewed as a circle fiber bundle. Each fiber represents a 1-cycle in $SO(3)$, and its homology class is the non-zero element of $H_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$.

We examine the pull-back homomorphism $p^* : H^2(S^2) \to H^2(SO(3))$. For each generator $g \in H^2(S^2) = \mathbb{Z}$, its Poincaré dual is represented by a point $x \in S^2$. So the Poincaré dual of $p^*(g)$ is represented by the circle $p^{-1}(x)$. In other words, p^* sends any generator of $H^2(S^2)$ to the non-zero element of $H^2(SO(3))$.

By endowing Y with a Riemannian metric, we can consider the principal $SO(3)$-bundle $Fr \to Y$ and the S^2-bundle $SY \to Y$. Let p be the bundle morphism $Fr \to SY$ sending an orthonormal frame (e_1, e_2, e_3) at some point $x \in Y$ to $e_1 \in S^2$. It follows from the previous paragraph and the definitions of $\mathrm{Vect}(Y)$ and $\mathrm{Spin}^c(Y)$ that the pull-back homomorphism $p^* : H^2(SY) \to H^2(Fr)$ sends $\mathrm{Vect}(Y)$ to $\mathrm{Spin}^c(Y)$.

To show that the resulting map $p^* : \mathrm{Vect}(Y) \to \mathrm{Spin}^c(Y)$ is a bijection, it suffices to prove that it is **$H_1(Y)$-equivariant**. By definition, we have the following commutative diagram:

$$\begin{tikzcd} Fr \arrow[r, "\phi"] \arrow[d, "p"'] & Y \\ SY \arrow[ur, "\varphi"'] & \end{tikzcd}$$

passing to the pull-back maps, we get $\phi^* = p^* \circ \varphi^*$. Since for any $\alpha \in \mathrm{Vect}(Y)$,

$$g \cdot p^*(\alpha) = \phi^*(g) + p^*(\alpha)$$

Therefore

$$\begin{aligned} p^*(g \cdot \alpha) &= p^*(\varphi^*(g) + \alpha) \\ &= p^*\varphi^*(g) + p^*(\alpha) \\ &= \phi^*(g) + p^*(\alpha) \\ &= g \cdot p^*(\alpha) \end{aligned}$$

and the result follows □

Now we are ready to make use of the Spin^c-structure to refine our notion of equivalence classes in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The main idea is given in [2]. To do this we define a map:

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y)$$

where $z \in \Sigma_g - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. We choose a Morse function f on Y which is *compatible* with the attaching circles (i.e. f induces the pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$ for Y), and assume that f has only one maximum point and one minimum point. As suggested in Chapter 1, each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ determines a g-tuple of trajectories with respect to the negative gradient flow of f connecting the index two critical points to index one critical points. Similarly, we can treat the point z as a trajectory connecting the index three critical point to the index zero critical point.

Next, we delete the tubular neighborhoods of these $g+1$ trajectories, and it gives a subset of Y (the complement of disjoint union of 3-balls in Y) where the negative gradient vector field $-\nabla f$ does not vanish.

Because $\mathbf{x}$ and z altogether pass through all critical points, $-\nabla f$ has index zero on all the boundary spheres, so $-\nabla f$ can be extended to a nowhere vanishing vector field $\sigma_\mathbf{x}$ on Y. Now $\sigma_\mathbf{x} = -\nabla f$ outside a union of 3-ball, and thus

$$[\sigma_\mathbf{x}] = [-\nabla f] \in \mathrm{Vect}(Y) = \mathrm{Spin}^c(Y).$$

Finally, we set

$$s_z(\mathbf{x}) = [\sigma_\mathbf{x}].$$

Proposition 4.3.6 (Ozaváth-Szabó, [11]) For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$,

$$s_z(\mathbf{y}) - s_z(\mathbf{x}) = \mathbf{PD}[\varepsilon(\mathbf{x}, \mathbf{y})],$$

where $\mathbf{PD}[\varepsilon(\mathbf{x}, \mathbf{y})]$ = Poincaré dual of $[\varepsilon(\mathbf{x}, \mathbf{y})]$. In particular, $s_z(\mathbf{x}) = s_z(\mathbf{y})$ if and only if $\pi_2(\mathbf{x}, \mathbf{y})$ is non-empty.

Proof: For each $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we can get the g trajectories $\gamma_\mathbf{x}$ for $-\nabla f$ connecting the index 2 to the index 1 critical points which contain the g-tuple $\mathbf{x}$. And let γ_z to be the corresponding trajectory from the index 0 to the index 3 critical point.

So $\gamma_\mathbf{x} - \gamma_\mathbf{y}$ is a closed loop in Y. As previously stated, we obtain $s_z(\mathbf{x}) = [\sigma_\mathbf{x}]$ by modifying $-\nabla f$ in a neighborhood of $\gamma_\mathbf{x} \cup \gamma_\mathbf{z}$. Since

$$\sigma_\mathbf{x} = -\nabla f = \sigma_\mathbf{y}$$

outside some neighborhoods of $\gamma_\mathbf{x} \cup \gamma_\mathbf{z}$ and $\gamma_\mathbf{y} \cup \gamma_\mathbf{z}$, $s_z(\mathbf{x}) - s_z(\mathbf{y})$ can be represented by a cohomology class, namely $[\sigma_\mathbf{x} - \sigma_\mathbf{y}]$, which is compactly supported in a neighborhood of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$. Therefore, $s_z(\mathbf{x}) - s_z(\mathbf{y})$ is just some multiple of the Poincaré dual of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$, say

$$s_z(\mathbf{x}) - s_z(\mathbf{y}) = C \cdot \mathbf{PD}[\gamma_\mathbf{x} - \gamma_\mathbf{y}]$$

for some constant C.

Next we want to show that $C = 1$. To do this we try to find a disk D_0 such that it is transverse to $\gamma_\mathbf{x} - \gamma_\mathbf{y}$. To find such D_0, we first take some $x_i \in \mathbf{x}$ which $x_i \notin \mathbf{y}$ (if no such x_i exists then $\mathbf{x} = \mathbf{y}$ and the lemma follows trivially), and let D_0 to be a small neighborhood of x_i in Σ_g which $\mathbf{y} \cap D_0 = \varnothing$. So we can pick $\sigma_\mathbf{x}$ which is equal to $-\nabla f$ near ∂D_0, and pick $\sigma_\mathbf{y}$ which is equal to $-\nabla f$ over D_0 (here, we can choose the tubular neighborhood of $\gamma_\mathbf{y} \cup \gamma_\mathbf{z}$ which is disjointed from D_0) .

Since the Poincaré dual of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$ and the Thom class of the normal bundle of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$ can be represented by the same forms (see [30]), by viewing D_0 as a fiber of the normal bundle of $\gamma_\mathbf{x} - \gamma_\mathbf{y}$ at the point x_i, we have

$$(s_z(\mathbf{x}) - s_z(\mathbf{y}))(D_0) = C \cdot \mathbf{PD}[\gamma_\mathbf{x} - \gamma_\mathbf{y}](D_0) = C$$

Because $\sigma_{\mathbf{x}}|_{D_0}$ and $\sigma_{\mathbf{y}}|_{D_0}$ are both maps from the disk D_0 to S^2, so we get

$$\begin{aligned} C &= (s_z(\mathbf{x}) - s_z(\mathbf{y}))(D_0) \\ &= \int_{D_0} \sigma^*_{\mathbf{x}}(\mu) - \int_{D_0} \sigma^*_{\mathbf{y}}(\mu) \\ &= \deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}}) \end{aligned}$$

where μ is the generator of $H^2(S^2)$. Therefore,

$$s_z(\mathbf{x}) - s_z(\mathbf{y}) = (\deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}})) \cdot \mathbf{PD}[\varepsilon(\mathbf{x}, \mathbf{y})]$$

We calculate the value of $\deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}})$. Take another disk D_1 with the same boundary of D_0, such that $D_0 \cup D_1$ bounds a 3-ball B containing the index 1 critical point p_i of $-\nabla f$ *corresponding to* x_i (i.e. p_i and x_i lie on the same trajectory line in $\gamma_{\mathbf{x}}$) and no other critical point. So we can assume $\sigma_{\mathbf{x}} \equiv -\nabla f$ over D_1 (here we can choose D_1 to be disjointed from $\gamma_{\mathbf{x}} \cup \gamma_{\mathbf{z}}$). Now,

$$\begin{aligned} 0 &= \int_B d\sigma^*_{\mathbf{x}}(\mu) \\ &= \int_{D_0} \sigma^*_{\mathbf{x}}(\mu) + \int_{D_1} \sigma^*_{\mathbf{x}}(\mu) - \int_{D_0 \cap D_1} \sigma^*_{\mathbf{x}}(\mu) \\ &= \deg_{D_0}(\sigma_{\mathbf{x}}) + \deg_{D_1}(\sigma_{\mathbf{x}}) - 0 \\ &= \deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_1}(\nabla f) \end{aligned}$$

and hence

$$\deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}}) = \deg_{D_1}(\nabla f) + \deg_{D_0}(\nabla f).$$

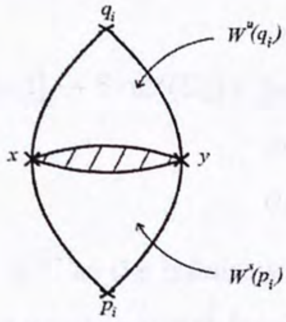
Because the winding number of ∇f around p_i is -1, so

$$\deg_{D_1}(\nabla f) + \deg_{D_0}(\nabla f) = -1.$$

Finally, it remains to show $\mathbf{PD}[\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}] = \mathbf{PD}[\varepsilon(\mathbf{x}, \mathbf{y})]$. Let $a \subset \alpha_1 \cup \cdots \cup \alpha_g$ be a collection of arcs with $\partial a = \mathbf{y} - \mathbf{x}$, and $b \subset \beta_1 \cup \cdots \cup \beta_g$ be such collection with $\partial b = \mathbf{y} - \mathbf{x}$. By the definition of $\varepsilon(\mathbf{x}, \mathbf{y})$, we know that $a - b$ gives a representative for $\varepsilon(\mathbf{x}, \mathbf{y})$.

Define $x_i \in \gamma_{x_i} \subset \gamma_{\mathbf{x}}$ to be the trajectory line in Y joining some index 2 critical point q_i to an index 1 critical point p_i (similar for γ_{y_i}). For each $a_i \subset a$ which connects x_i to y_i, we have a homotopy (relative to the end points x_i and y_i) between

a_i and $\tilde{\gamma}_{y_i} - \tilde{\gamma}_{x_i}$, where $\tilde{\gamma}_{x_i}$ is the segment in γ_{x_i} joining x_i to p_i (similar for $\tilde{\gamma}_{y_i}$). Such homotopy exists because a_i and $\tilde{\gamma}_{y_i} - \tilde{\gamma}_{x_i}$ bound a *triangle* (with vertices y_i, x_i and p_i) in the **stable manifold** $W^s(p_i)$. Similarly, each $b_i \subset b$ is homotopic to $\tilde{\tilde{\gamma}}_{y_i} - \tilde{\tilde{\gamma}}_{x_i}$, where $\tilde{\tilde{\gamma}}_{x_i}$ is the segment in γ_{x_i} joining q_i to x_i (similar for $\tilde{\tilde{\gamma}}_{y_i}$), with b_i and $\tilde{\tilde{\gamma}}_{y_i} - \tilde{\tilde{\gamma}}_{x_i}$ bound a *triangle* in the **unstable manifold** $W^u(q_i)$.



the whitney disk

`[figure :  the shaded region refers to the Whitney disk]`

Therefore, $a - b$ is homologous to $\gamma_{\mathbf{x}} - \gamma_{\mathbf{y}}$ and the result follows. □

Remark: In calculating the value of $\deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}})$, other than using the index 1 critical point, one can also use index 2 critical point. In this case, $D_0 \cup D_1$ will bound a three-ball in Y containing the index 2 critical point q_i corresponding to x_i. Because now the direction of $-\nabla f$ is reversed (compare with the previous case) with respect to both D_0 and D_1, we get

$$s_z(\mathbf{x}) - s_z(\mathbf{y}) = (-\deg_{D_0}(\sigma_{\mathbf{x}}) + \deg_{D_0}(\sigma_{\mathbf{y}})) \cdot \mathbf{PD}[\varepsilon(\mathbf{x}, \mathbf{y})]$$

Following the previous steps,

$$\deg_{D_0}(\sigma_{\mathbf{x}}) - \deg_{D_0}(\sigma_{\mathbf{y}}) = \deg_{D_1}(\nabla f) + \deg_{D_0}(\nabla f).$$

Now the winding number of ∇f around q_i is $+1$, so

$$\deg_{D_1}(\nabla f) + \deg_{D_0}(\nabla f) = 1,$$

and the same result follows.

According to the above proposition, we deduce that for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{x} \sim \mathbf{y}$ if and only if $s_z(\mathbf{x}) = s_z(\mathbf{y})$.

4.3.6 Holomorphic Disks and Maslov Index

Recall that a complex structure on Σ_g induces a complex structure on $\mathrm{Sym}^g(\Sigma_g)$. Given a homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let $\mathcal{M}(\phi)$ be the moduli space of holomorphic representatives of ϕ. To be precise, we have

$$\mathcal{M}(\mathbf{x}, \mathbf{y}) = \left\{ u : \mathbb{R} \times [0,1] \to \mathrm{Sym}^g(\Sigma_g) \;\middle|\; \begin{array}{c} u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha \\ u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta \\ \lim_{t \to -\infty} u(s+it) = \mathbf{x} \\ \lim_{t \to +\infty} u(s+it) = \mathbf{y} \\ \partial_s u + J(s)\partial_t u = 0 \end{array} \right\}$$

where we treat the unit disk $\mathbb{D}$ in $\mathbb{C}$ as the infinite strip $\mathbb{R} \times [0,1]$ by using Riemann mapping theorem, and $J(s)$ is a one-parameter family of almost-complex structure. For $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, the space $\mathcal{M}(\phi)$ is defined to be the subset of $\mathcal{M}(\mathbf{x}, \mathbf{y})$ consisting of maps which represent the given homotopy class ϕ.

Consider the group of complex automorphism of the unit disk $\mathbb{D}$ which preserve i and $-i$. By continuity we know that these automorphism must also preserve e_1 and e_2, where e_1 and e_2 are the arcs in the boundary of $\mathbb{D}$ with $\mathfrak{Re}(z) < 0$, $\mathfrak{Re}(z) > 0$ respectively. By viewing $\mathbb{D}$ as $\mathbb{R} \times [0,1]$, the automorphisms preserving e_1 and e_2 correspond to the vertical translation, which is isomorphic to $\mathbb{R}$. Now for any $u \in \mathcal{M}(\phi)$, we can precompose u with any of these automorphisms and get another holomorphic disk. By dividing the above $\mathbb{R}$-action, the unparametrized moduli space $\widehat{\mathcal{M}}(\phi)$ is given by

$$\widehat{\mathcal{M}}(\phi) = \frac{\mathcal{M}(\phi)}{\mathbb{R}}.$$

Elements in $\widehat{\mathcal{M}}(\phi)$ are called **holomorphic disks**. The word "disk" is used is simply because of the the identification between $\mathbb{R} \times [0,1]$ and $\mathbb{D}$.

To calculate the dimension of the moduli space $\widehat{\mathcal{M}}(\phi)$, we need to apply the **Fredholm theory**.

Definition 4.3.15 Let E be a vector bundle over $[0,1] \times \mathbb{R}$ equipped with a metric and compatible connection ∇, p, δ be positive real numbers, and k be a non-negative integer. The **δ-weighted Sobolev space of sections of** E, denoted by $L^p_{k,\delta}([0,1] \times$

$\mathbb{R}, E)$, is the space of sections σ for which the norm

$$||\sigma||_{L^p_{k,\delta}(E)} = \sum_{l=0}^{k} \int_{[0,1]\times\mathbb{R}} |\nabla^{(l)}\sigma(s,t)|^p e^{\delta\tau(t)} ds \wedge dt$$

is finite. Here, $\tau : \mathbb{R} \to \mathbb{R}$ is a smooth function with $\tau(t) = |t|$ provided that $|t| \geq 1$.

Now we fix some $p > 2$, and consider the case when $E=\mathrm{Sym}^g(\Sigma_g)$. Let $\mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$ be the space of maps

$$u : [0,1] \times \mathbb{R} \to \mathrm{Sym}^g(\Sigma_g)$$

satisfying the boundary conditions

$$u(\{1\} \times \mathbb{R}) \subset \mathbb{T}_\alpha, u(\{0\} \times \mathbb{R}) \subset \mathbb{T}_\beta$$

with limits $\mathbf{x}$ and $\mathbf{y}$ as $t \to -\infty$ and $+\infty$ respectively. By equipping $\mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$ with the norm given in the above definition, $\mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$ is a Banach manifold, whose tangent space at any $u \in \mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$ is

$$L^p_{1,\delta}(u) := \left\{ \xi \in L^p_{1,\delta}([0,1] \times \mathbb{R}, u^*(T\mathrm{Sym}^g(\Sigma_g))) \mid \begin{array}{l} \xi(1,t) \in T_{u(1,t)}(\mathbb{T}_\alpha), \forall t \in \mathbb{R} \\ \xi(0,t) \in T_{u(0,t)}(\mathbb{T}_\beta), \forall t \in \mathbb{R} \end{array} \right\},$$

which consists of all smooth sections ξ of the pullback bundle $u^*(T\mathrm{Sym}^g(\Sigma_g)))$ (refer to the Definition 3.3.2). And we have the space of sections

$$L^p_{1,\delta}(\wedge^{0,1}(u)) := L^p_\delta([0,1] \times \mathbb{R}, u^*(T\mathrm{Sym}^g(\Sigma_g))).$$

Clearly $L^p_{1,\delta}(u) \subset L^p_{1,\delta}(\wedge^{0,1}(u))$. For each $u \in \mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$, we define $\bar{\partial}_{J(s)} : \mathcal{B}_\delta(\mathbf{x}, \mathbf{y}) \to L^p_{1,\delta}(\wedge^{0,1}(u))$ by

$$\bar{\partial}_{J(s)} u = \frac{\partial u}{\partial s} + J(s)\frac{\partial u}{\partial t}.$$

Note that $\bar{\partial}_{J(s)}$ is zero exactly when u is a holomorphic disk.

We linearize the above equations. For a curve $u \in \mathcal{B}_\delta(\mathbf{x}, \mathbf{y})$ and a section $\nu \in L^p_{1,\delta}(u)$, the linearization is

$$\begin{aligned} \nabla_\nu\left(\frac{\partial}{\partial s} + J(s)\frac{\partial}{\partial t}\right) &= \nabla_\nu \frac{\partial}{\partial s} + J(s)\nabla_\nu \frac{\partial}{\partial t} + (\nabla_\nu J(s))\frac{\partial}{\partial t} \\ &= \frac{\partial \nu}{\partial s} + J(s)\frac{\partial \nu}{\partial t} + (\nabla_\nu J(s))\frac{\partial}{\partial t}. \end{aligned}$$

So we get a map

$$D_u : L^p_{1,\delta}(u) \to L^p_{1,\delta}(\wedge^{0,1}(u))$$

which is given by the formula

$$D_u(\nu) = \frac{\partial \nu}{\partial s} + J(s)\frac{\partial \nu}{\partial t} + (\nabla_\nu J(s))\frac{\partial u}{\partial t}.$$

According to [20] and [2], since the intersection of $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ is transverse, D_u is a Fredholm operator for all sufficiently small non-negative δ. This defines a map $\mu : \mathcal{B}_\delta(\mathbf{x}, \mathbf{y}) \to \mathbb{Z}$ given by

$$\mu(u) = \text{ind} D_u$$

and this μ is nothing other than the Maslov index of u (see [25] and [20]). The index descends to $\pi_2(\mathbf{x}, \mathbf{y})$, and for each $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with $\mu(\phi) = n$, the dimension of $\mathcal{M}(\phi)$ is equal to n (=indD_ϕ). Therefore,

$$\dim(\widehat{\mathcal{M}}(\phi)) = n - 1 \text{ for each } \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \text{ with } \mu(\phi) = n.$$

We quote the following lemma which we will use it later

Lemma 4.3.16 (from [2]) Let $S \in \pi_2(\text{Sym}^g(\Sigma_g))$ be the positive generator. Then for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$,

$$\mu(\phi + k[S]) = \mu(\phi) + 2k.$$

Proof: (also from [2]) It follows from the excision principle for the index that attaching a topological sphere Z to a disk changes the Maslov index by $2\langle c_1, [Z]\rangle$, where c_1 is the first Chern Class of $T\text{Sym}^g(\Sigma_g)$. On the other hand for the positive generator we have $\langle c_1, [S]\rangle = 1$. □

4.4 Definition of Heegaard Floer Homology

We are ready to define various Heegaard Floer chain complexes, namely $\widehat{CF}$, CF^+, CF^- and CF^∞.

We assume that our ambient 3-manifold Y is a homology three sphere with $b_1(Y) = 0$. Let $(\Sigma_g, \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, z)$ be a pointed Heegaard diagram of genus $g > 0$ for Y. Fix a spin^c structure $t \in \text{Spin}^c(Y)$.

4.4.1 The chain complex $\widehat{CF}$

Let $\widehat{CF}(\alpha, \beta, t)$ be the free Abelian group generated by the points in $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(\mathbf{x}) = t$. We introduce a **relative grading** on this group, which is defined by

$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_z(\phi),$$

where ϕ is any element in $\pi_2(\mathbf{x}, \mathbf{y})$ and μ is the Maslov index.

We show that the grading is, in fact, independent of the homotopy class of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. It suffices to show

$$\mu(\phi + k[S]) - 2n_z(\phi + k[S]) = \mu(\phi) - 2n_z(\phi)$$

for $[S] \in \pi_2(\mathrm{Sym}^g(\Sigma_g))$ and $k \in \mathbb{N}$ (here $[S]$ represents a topological 2-sphere, and two homotopic elements in $\pi_2(\mathbf{x}, \mathbf{y})$ only differ by some number of spheres). By Lemma 4.3.16, we calculate

$$\begin{aligned} \mu(\phi + k[S]) - 2n_z(\phi + k[S]) &= \mu(\phi) + 2k - 2(n_z(\phi) + k) \\ &= \mu(\phi) + 2k - 2n_z(\phi) - 2k \\ &= \mu(\phi) - 2n_z(\phi). \end{aligned}$$

Moreover, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\tilde{\phi} \in \pi_2(\mathbf{y}, \mathbf{z})$,

$$\begin{aligned} \mathrm{gr}(x, y) + \mathrm{gr}(y, z) &= \mu(\phi) - 2n_z(\phi) + \mu(\tilde{\phi}) - 2n_z(\tilde{\phi}) \\ &= \mu(\phi * \tilde{\phi}) - 2n_z(\phi * \tilde{\phi}) \\ &= \mathrm{gr}(x, z) \end{aligned}$$

as $\phi * \tilde{\phi} \in \pi_2(\mathbf{x}, \mathbf{z})$ and $\mu(\phi * \tilde{\phi}) = \mu(\phi) + \mu(\tilde{\phi})$ (see [11]).

Now, for for $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we define $n(\mathbf{x}, \mathbf{y}; \phi)$ to be the number of points (of appropriate signs with respect to orientation, see [23] and [22]) in $\widehat{\mathcal{M}}(\phi)$ if $\mu(\phi) = 1$, and $n(\mathbf{x}, \mathbf{y}; \phi) = 0$ if $\mu(\phi) \neq 0$ (recall that $\widehat{\mathcal{M}}(\phi)$ is just 0-dimensional for $\mu(\phi) = 1$).

Let $\partial : \widehat{CF}(\alpha, \beta, t) \to \widehat{CF}(\alpha, \beta, t)$ be the map defined by

$$\partial \mathbf{x} = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \phi \in \pi_2(\mathbf{x}, \mathbf{y}) \mid s_z(\mathbf{y}) = t, \mu(\phi) = 1, n_z(\phi) = 0\}} n(\mathbf{x}, \mathbf{y}; \phi)\mathbf{y}.$$

In this way, we are counting those ϕ which do not meet $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ (recall that $n_z(\phi)$ is the intersection number between ϕ and $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$). Note that for $\mathbf{x}$, $\mathbf{y}$ in this boundary map, $\mathrm{gr}(\mathbf{x}, \mathbf{y}) = 1$.

The following theorem is proven in [2].

Theorem 4.4.1 (Ozsváth-Szabó, [2]) When $b_1(Y) = 0$, the pair $(\widehat{CF}(\alpha, \beta, t), \partial)$ is a chain complex; i.e. $\partial^2 = 0$.

Sketch of proof: (From [2]) Basically, the idea of the proof is to analyze the Gromov compactification of $\widehat{\mathcal{M}}(\phi)$ for $n_z(\phi) = 0$ and $\mu(\phi) = 2$. To do this, we study the "ends" of $\widehat{\mathcal{M}}(\phi)$, and there are three possible cases:

1. those corresponding to "broken flow-lines", i.e. a pair $u \in \mathcal{M}(\mathbf{x}, \mathbf{y})$ and $v \in \mathcal{M}(\mathbf{y}, \mathbf{w})$ with $\mu(u) = \mu(v) = 1$,

2. those which correspond to a sphere "bubbling off", i.e. another $v \in \mathcal{M}(\mathbf{x}, \mathbf{w})$ and a holomorphic sphere $S \in \mathrm{Sym}^g(\Sigma_g)$ which meets v,

3. those which correspond to "boundary bubbling", i.e. there are a $v \in \mathcal{M}(\mathbf{x}, \mathbf{w})$, and a holomorphic map u from the disk, whose boundary is mapped into $\mathbb{T}_\alpha$ or $\mathbb{T}_\beta$, which meet in a point on the boundary.

Because $n_z(\phi) = 0$, counting those holomorphic disks in $\mathrm{Sym}^g(\Sigma_g)$ is equivalent to counting holomorphic disks in $\mathrm{Sym}^g(\Sigma_g - z)$. Since there are no spheres in $\mathrm{Sym}^g(\Sigma_g - z)$ or degenerate holomorphic disks, so the cases (2) and (3) can be eliminated and the only boundary components in the compactification consist of broken flow lines. The remaining case reduces to the usual Floer's compactness argument, and ∂^2 vanishes accordingly. $\square$

Hence we get the following:

Definition 4.4.2 The Heegaard Floer homology group $\widehat{HF}(Y, t)$ is the homology group of the complex $(\widehat{CF}(\alpha, \beta, t), \partial)$.

4.4.2 The chain complex CF^∞

In the last section we only count holomorphic disks which are disjoint from $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$. Now define another complex where all the holomorphic disks are used.

Let $CF^{\infty}(\alpha, \beta, t)$ be the free abelian group generated by pairs $[\mathbf{x}, i]$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $i \in \mathbb{Z}$. We define a grading on the generators by

$$\mathrm{gr}([\mathbf{x}, i], [\mathbf{y}, j]) = \mathrm{gr}(\mathbf{x}, \mathbf{y}) + 2i - 2j.$$

Of course, this grading is independent of the homotopy class of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Let $\partial^{\infty} : CF^{\infty}(\alpha, \beta, t) \to CF^{\infty}(\alpha, \beta, t)$ be the map defined by

$$\partial^{\infty}[\mathbf{x}, i] = \sum_{\{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, \phi \in \pi_2(\mathbf{x}, \mathbf{y}) | s_z(\mathbf{y}) = t, \mu(\phi) = 1\}} n(\mathbf{x}, \mathbf{y}; \phi)[\mathbf{y}, i - n_z(\phi)].$$

where $n(\mathbf{x}, \mathbf{y}; \phi)$ is the number of points in $\widehat{\mathcal{M}}(\phi)$ for $\mu(\phi) = 1$. And for $\mathbf{x}, \mathbf{y}$ in ∂^{∞}, $\mathrm{gr}(\mathbf{x}, \mathbf{y}) = 1$.

Again we have the following theorem proven in [2].

Theorem 4.4.3 (Ozsváth-Szabó, [2]) When $b_1(Y) = 0$, the pair $(CF^{\infty}(\alpha, \beta, t), \partial^{\infty})$ is a chain complex; i.e. $(\partial^{\infty})^2 = 0$.

Sketch of proof: Similar to the case in $(\widehat{CF}(\alpha, \beta, t), \partial)$, we also have three possible kinds of "ends" for the moduli space $\widehat{\mathcal{M}}(\phi)$.

Although the situation becomes much more complicated (see [2]), the cases (2) and (3) can still be eliminated by dimension counts and transversality theorem. So finally it reduces to the remaining case (1), which again by the usual Floer's compactness argument, $(\partial^{\infty})^2$ vanishes accordingly. □

The following is a consequence:

Definition 4.4.4 The Heegaard Floer homology group $HF\infty(Y, t)$ is the homology group of the complex $(CF^{\infty}(\alpha, \beta, t), \partial^{\infty})$.

4.4.3 The chain complexes CF^+ and CF^-

Ozsváth-Szabó discover that there is a chain map

$$U : CF^{\infty}(\alpha, \beta, t) \to CF^{\infty}(\alpha, \beta, t),$$

which lowers the grading by two, defined by

$$U[\mathbf{x}, i] = [\mathbf{x}, i - 1].$$

Let CF^- be the subgroup of CF^∞ which is freely generated by pairs $[\mathbf{x}, i]$ with $i < 0$. Let CF^+ be the quotient group CF^∞/CF^-.

Lemma 4.4.5 The group $CF^-(\alpha, \beta, t)$ is a subcomplex of $CF^\infty(\alpha, \beta, t)$, so there is a short exact sequnece of chain complexes:

$$0 \longrightarrow CF^-(\alpha,\beta,t) \xrightarrow{i} CF^\infty(\alpha,\beta,t) \xrightarrow{\pi} CF^+(\alpha,\beta,t) \longrightarrow 0$$

Proof: If $[\mathbf{y}, j]$ appears in $\partial^\infty([\mathbf{x}, i])$, then there is a homotopy class ϕ with $\mathcal{M}(\phi)$ non-empty, and $n_z(\phi) = i - j$ by the definition of ∂^∞. Since $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ is a subvariety, the holomorphic disk ϕ is either contained in $\{z\} \times \mathrm{Sym}^{g-1}(\Sigma_g)$ (which is excluded by the boundary condition of $\mathcal{M}(\phi)$), or it must meet the subvariety non-negatively. In other words, $n_z(\phi) \geq 0$ and thus $i \geq j$. It implies $\partial^\infty(CF^-(\alpha, \beta, t)) \subset CF^-(\alpha, \beta, t)$ and the result follows. □

By the definition of U, we easily see that U restricts to a endomorphism of $CF^-(\alpha, \beta, t)$, and hence it descends to an endomorphism of the quotient $CF^+(\alpha, \beta, t)$. We denote the induced action of U on $CF^-(\alpha, \beta, t)$ and $CF^+(\alpha, \beta, t)$ to be U^- and U^+ respectively.

In view of the action U, there is a short exact sequence

$$0 \longrightarrow \widehat{CF}(\alpha,\beta,t) \xrightarrow{j} CF^+(\alpha,\beta,t) \xrightarrow{U^+} CF^+(\alpha,\beta,t) \longrightarrow 0$$

where $j(\mathbf{x}) = [\mathbf{x}, 0]$ (here it means that the action of U on $\widehat{CF}$ is trivial).

Since $CF^-(\alpha, \beta, t)$ and $CF^+(\alpha, \beta, t)$ are just subcomplexes of $CF^\infty(\alpha, \beta, t)$, we have:

Definition 4.4.6 The Heegaard Floer homology group $HF^-(Y, t)$ and $HF^+(Y, t)$ are the homology group of the complex $(CF^-(\alpha, \beta, t), \partial^\infty)$ and $(CF^+(\alpha, \beta, t), \partial^\infty)$ respectively.

Because of the short exact sequences as given above, the Heegaard Floer homology groups are related by the exact sequences

$$\cdots \longrightarrow HF^-(Y,t) \xrightarrow{i} HF^\infty(Y,t) \xrightarrow{\pi} HF^+(Y,t) \longrightarrow \cdots$$

$$\cdots \longrightarrow \widehat{HF}(Y,t) \xrightarrow{j} HF^+(Y,t) \xrightarrow{U^+} HF^+(Y,t) \longrightarrow \cdots$$

As induced by the relative $\mathbb{Z}$-grading on the chain groups, it is clear that $HF^o(Y,t)$ is a relatively $\mathbb{Z}$-graded abelian group (here $HF^o(Y,t)$ is any of $HF^-(Y,t)$, $HF^\infty(Y,t)$, $HF^+(Y,t)$ or $\widehat{HF}(Y,t)$). In [4], Ozsváth-Szabó showed that when Y is an oriented rational homology three-sphere and t is a Spin^c-structure over Y, the relative $\mathbb{Z}$-grading on the Heegaard Floer homology can be lift to an **absolute $\mathbb{Q}$-grading.**

Theorem 4.4.7 (Ozsváth-Szabó, [4]) Let t be a torsion Spin^c-structure. Then, the homology groups $HF^o(Y,t)$ can be endowed with an absolute grading

$$\widetilde{\mathrm{gr}}: \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Q}$$

satisfying the following properties:

1. the homogeneous elements of least grading in $HF^+(S^3,s)$ have absolute grading zero (here s is the unique Spin^c-structure on S^3)
2. the absolute grading lifts the relative grading, in the sense that if $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then
$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y})$$
3. the natural maps i and π in the above long exact sequence preserve the absolute grading, while the coboundary map decreases absolute degree by one, and the U action decreases it by two.

Roughly speaking, the term "leasting grading" in property 1 means the least k in $HF_k^+(S^3,s)$ such that $HF_k^+(S^3,s)$ is non-trivial (in [8], it is proven that this least value is well-defined, i.e. it gives a lower bounded on k for $HF_k^+(Y,s)$ being non-trivial). Therefore, we can write

$$HF^o(Y,t) = \bigoplus_{d\in\mathbb{Q}} HF_d^o(Y,t),$$

while in each grading, $d \in \mathbb{Q}$, $HF_d^o(Y,t)$ is a finitely generated abelian group.

4.4.4 Some Remarks

1. At first, it seems not very clear why one needs such a variety of homology. However, as Ozsváth-Szabó have shown in [2], if we ignore the action of U and

consider only $HF^\infty(Y,t)$, we get very little information (In the next chapter, we will try to see that if Y is just a rational homology 3-sphere, then $HF^\infty(Y,t)$ has a rather simple structure).

2. It is proven in [2] that the homology groups as defined are all independent of the Heegaard splitting, the choice of attaching circles, the base point z and the complex structures used in the definitions. This can be summarized in the following theorem:

   Theorem 4.4.8 (Ozsváth-Szabó, [2]) The homology groups $\widehat{HF}(Y,t)$, $HF^\infty(Y,t)$, $HF^-(Y,t)$ and $HF^+(Y,t)$ are topological invariants of the 3-manifold (Y,t), where t is a Spin^c structure on Y.

3. For 3-manifold Y with $b_1(Y) > 0$, there is a technical problem due to the fact that $\pi_2(\mathbf{x}, \mathbf{y})$ is large. In the definition of the boundary map, we then have infinity many homotopy classes with Maslov index 1. In order to get a finite sum, Ozsváth-Szabó prove that only finitely many of these homotopy classes support holomorphic disks. They do this by using some "special" Heegaard diagrams, and with the constructions from the previous section, it also gives those similar Heegaard Floer homology groups. We refer [2] for details.

4. The action U on the complex $CF^\infty(Y,t)$ (for Y being a homology sphere) is an isomorphism, that when decends to the homology groups, yields an isomorphism
$$HF_k^\infty(Y,t) \to HF_{k-2}^\infty(Y,t)$$
   for $k \in \mathbb{Z}$.

5. In fact, with respect to Theorem 4.4.7, there is one further property of the absolute grading of the homology group, which concerns *cobordism* between manifolds. We postpone this property to the next Chapter.

Chapter 5

Examples and Applications

5.1 Introduction

There are many uses of Heegaard Floer theory. On one side, it is well adapted to certain natural geometric constructions in 3-manifold theory, such as adding a handle or performing a Dehn surgery on a knot. It is mainly because all these have descriptions in terms of Heegaard diagrams.

In this chapter, we will first examine the case when a 3-manifold Y is homology three-sphere. We define a large class of homology three-spheres and try to calculate the absolutely graded Heegaard Floer homology groups of them. Lastly, we will quote the recent developments and applications of the theory to different aspects.

5.2 The homology three-spheres

5.2.1 The sphere S^3

A homology three sphere Y is a 3-manifold having the homology groups of the 3-sphere S^3. So we have

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, Y is a connected space with $b_1(Y) = 0$.

The simplest example of a homology three sphere Y is, of course, $Y = S^3$. We will now calculate the Heegaard Floer homology of S^3. As we saw in Chapter 2, S^3 has a Heegaard Splitting consisting of a torus $\mathbb{T}^2$, with a single α and a single β curve intersecting once transversally at a single point $\mathbf{x}$. Let t be the unique Spin^c structure on S^3 (recall the example given in Chapter 3) and pick $z \in \Sigma - \alpha - \beta$.

We are going to define the Heegaard Floer chain complexes for S^3. For simplicity, we assume that all chain groups have coefficients in $\mathbb{Z}$, although one may choose some other ring as coefficient.

Now, the chain group $\widehat{CF}(S^3, t)$ has a single generator, so $\widehat{CF}(S^3, t) \cong \mathbb{Z}$. Choose ϕ to be the constant disk connecting $\mathbf{x}$ itself. Since $\{z\} \times \mathrm{Sym}^g(\Sigma) \cong \Sigma$ for $g = 1$, we have $n_z(\phi) = 1$ (the intersection is just $\mathbf{x}$). Therefore, ∂ is a trivial map, and hence $\widehat{HF}(S^3, t) \cong \mathbb{Z}$.

For $CF^\infty(S^3, t)$, it has the generator $[\mathbf{x}, i]$ where $i \in \mathbb{Z}$. Since $\mathrm{gr}(\mathbf{x}, \mathbf{x}) = 0$, we have $\mu(\phi) = 2$. Hence $\partial^\infty = 0$. Correspondingly, the induced boundary map on $CF^-(S^3, t)$ and $CF^+(S^3, t)$ are all trivial. So we get the following identification:

$$\begin{aligned} &HF^\infty(S^3, t) \cong \mathbb{Z}[U, U^{-1}] \\ &HF^-(S^3, t) \cong \mathbb{Z}[U] \\ &HF^-(S^3, t) \cong \mathbb{Z}[U, U{-}1]/\mathbb{Z}[U], \end{aligned}$$

where all three groups are viewed as $\mathbb{Z}$-module. Here, U^{-1} is the inverse map of U (as we have seen in Chapter 4),

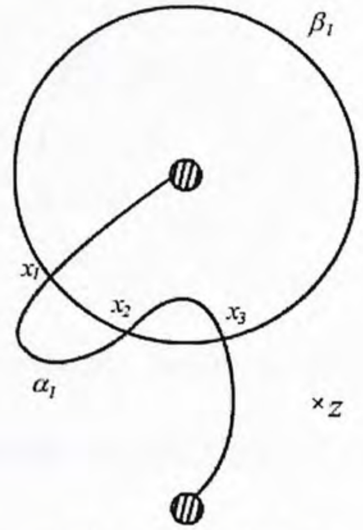
$$U^{-1} : CF^\infty(\alpha, \beta, t) \to CF^\infty(\alpha, \beta, t),$$

which increases the grading by two, defined by

$$U[\mathbf{x}, i] = [\mathbf{x}, i + 1].$$

What happens if we "move" the Heegaard diagram a little bit (more precisely, we move the Heegaard diagram **isotopically**, see [3])?

In this Heegaard diagram of S^3, there are three intersections between α and β, namely x_1, x_2 and x_3. By the Riemann mapping theorem, we can see that there exists holomorphic disks connecting x_1 to x_2 and x_2 to x_3. So with respect to the chain complex $\widehat{CF}(S^3, t)$, we have $\partial x_1 = x_2 = \partial x_3$. Hence, $\partial(x_1 + x_3) = 2x_2 = 0$



In this diagram, the two shaded circles are to be identified.

[figure : after an isotopic move of the Heegaard diagram of S^3]

in $\mathbb{Z}_2$ coefficient, and $x_1 + x_3$ generates the group $\widehat{HF}(S^3, t)$. So $\widehat{HF}(S^3, t)$ remains unchanged and so as the all other Heegaard Floer homology groups.

5.2.2 The Poincaré sphere and the Brieskorn spheres

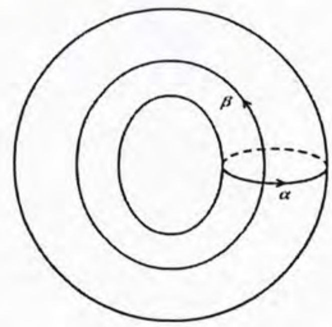
Yet there are a lot of examples of homology spheres other than S^3. These can be obtained by an intersecting method, namely the **rational surgery** of S^3.

We can obtain the standard 3-sphere S^3 by gluing two solid torus together with the identification between α (the meridian curve) and β (the longitude curve). So if we remove a tubular ε-neighborhood of the **trivial knot** (which is just a solid torus) from S^3, we will get another solid torus correspondingly.

We shall agree that the orientation of the meridian and longitude are chosen as shown in the figure.

We perform a **surgery** on S^3 by removing a tubular neighborhood of the trivial knot and re-pasting it back with the identification between α and $p\alpha + q\beta$, where p, q are integers.

We quote the following proposition. The first part asserts that in this case the integers p and q have no common divisors, while the second part asserts that the



[figure : how to orient the meridian and the longitude]

numbers p and q determine the curve $p\alpha + q\beta$ uniquely up to isotopy (see [13] for the details and proof).

Proposition 5.2.1 With the same notations as above, we have the following:

1. If the curve $p\alpha + q\beta$ is closed and has no self-intersections, then either the integers p and q are co-prime, or one of them is 0 and the other is ± 1.

2. If two closed curves without self-intersections on the torus are homotopic, then they are isotopic.

Now from the above proposition, it follows that the surgery of S^3 is totally determined by the rational number $r = \frac{p}{q}$. We call this number the **framing index** (or just **framing**) of the trivial knot, and the corresponding operation the **rational surgery** with **framing index** r.

Example 5.2.1 For identical surgery, we have $r = \frac{1}{0} = \infty$, while for rational surgery of index $r = 0$, we have a "torus switch" (interchanging the longitudes and meridians). The manifold obtained after performing a torus switch will be equal to $S^1 \times S^2$. To see this observe that such surgery is same as gluing two solid tori T_1 and T_2 together along the identical homeomorphism of their boundaries. Since T_i is homeomorphic to $S^1 \times D^2$ (D^2 is the 2-dimensional closed disk) and gluing together D^2 and D^2 along the identity map of their boundary circle produces S^2, gluing together T_1 and T_2 along the identity map of their boundaries produces $S^1 \times S^2$.

Example 5.2.2 For $r = \frac{p}{q}$, where $p \neq \pm 1$ and $q \neq 0$, the resulting manifold after performing the surgery is the lens space $L(p, q)$. The lens space $L(p, q)$ is obtained by gluing together two solid tori along the homeomorphism of the boundaries which takes the meridian α of one torus to the curve $q\alpha + p\beta$ on the other torus. On the other hand, S^3 can be obtained by identifying the boundaries of these tori along the homeomorphism take takes α to β and β to α. Hence the result follows.

Example 5.2.3 For $r = \frac{\pm 1}{n}$, where $n \neq 0$, the resulting manifold is just S^3. To see this, note that $L(p,q)$ can be defined as the quotient of the unit sphere S^3 in $\mathbb{C}^2$ by the equivalence relation:

$$(z,w) \sim (\exp(2\pi i/p)z, \exp(2\pi iq/p)w).$$

This definition implies that for $p = \pm 1$, no identifications of points occur, so $L(\pm 1, n) = S^3$.

Apart from the trivial knot, we can also do surgery on S^3 along some nontrivial knots. In fact, there is a theorem relating surgeries and homology sphere.

Theorem 5.2.1 (From [13]) Surgery of S^3 along any knot with framing ± 1 always produces a homology sphere.

Hence it is not surprising to see that there are infinitely many non-homeomorphic homology spheres, despite of the fact that surgery along different knots with framing ± 1 may produce the same homology sphere.

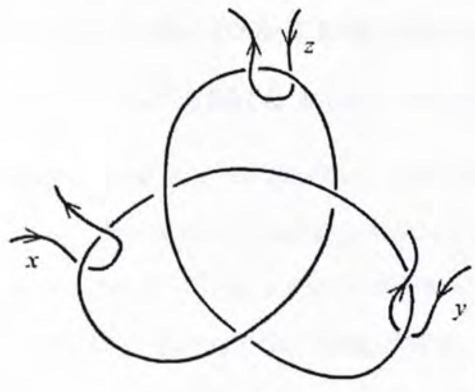
The manifold obtained by surgery on the sphere S^3 along the **right trefoil** with framing $+1$ is called the **Poincaré homology sphere**.

To prove that the Poincaré sphere is not S^3, we can compute its fundamental group. Assume the base point O is at infinity, and we denote the right trefoil by K. Any loop from O in $S^3 - K$ can be represented as the composition of the loops x, y and z and their inverses.

By studying the relation between x, y and z at each crossings of K, the fundamental group of $S^3 - K$ is given by

$$\pi_1(S^3 - K) = \{x, y, z : xy = yz = zx\} = \{x, y : xyx = yxy\}.$$

When we perform surgery on S^3 along the right trefoil with framing $+1$ to the curve $\alpha + \beta$, we identify $\alpha + \beta$ with the boundary of the meridional disk of the solid torus.



[figure : the right-handed trefoil]

So the fundamental group of the Poincaré sphere is obtained from the group $\pi_1(S^3 - K)$ by adding the relation $\alpha + \beta = 1$ Thus the fundamental group of the Poincaré sphere is isomorphic to the group

$$I = \{x, y : xyx = yxy,\ yx^2y = x^3\}.$$

which is the group of self-isometries of the icosahedron.

Remark: We recall the example 4.3.3. in Chapter 4. This time we can define a one-parameter family of Heegaard diagrams by changing the right side of the diagram. For $n > 0$, instead of twisting around the right circle twice as in the picture, twist n times. When $n < 0$, twist $-n$ times in the opposite direction.

When $n = 3$, it gives a 3-manifold (denoted by $\Sigma(2, 3, 5)$), which turns out to be the Poincaré sphere.

In fact, we can define a larger class of homology spheres in the following setting. We consider the locus $V(p, q, r)$ defined by

$$V(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 : x^p + y^q + z^r = 0\}$$

where p, q, r are pairwise relatively prime integers. Note that $V(p, q, r)$ is a codimension 2 subset in $\mathbb{C}^3$.

Definition 5.2.2 The **Brieskorn sphere** $\Sigma(p, q, r)$ is the homology sphere obtained by $V(p, q, r) \cap S^5$, where $S^5 \subset \mathbb{C}^3$ is the standard 5-sphere.

One can show that the Poincaré sphere is just $\Sigma(2, 3, 5)$.

5.2.3 Long exact surgery sequence and the absolutely graded Heegaard Floer homology

We can now try to compute the Heegaard Floer homology for these homology spheres. However, if one gets starts from the very beginning definition of the homology groups, it will be a very difficult task. So we introduce some long exact sequences of homology group (the **long exact surgery sequence**) which will facilitate our work.

To this end, we first give some definitions:

Definition 5.2.3 We say that two manifolds M and N are **cobordant** if their union is the complete boundary of a third manifold X; X is then called a **cobordism** between M and N.

Following from [17], we make use of the idea of cobordism to relate 3-manifolds and 4-manifolds via Heegaard diagram. Suppose we are given three sets α, β, γ of g disjoint curves on the Riemann surface Σ_g that are the attaching circles for the handlebodies U_α, U_β and U_γ respectively. Then there are three associated manifolds:

$$Y_{\alpha,\beta} = U_\alpha \cup U_\beta,\ Y_{\alpha,\gamma} = U_\alpha \cup U_\gamma,\ Y_{\beta,\gamma} = U_\beta \cup U_\gamma.$$

We want to find a cobordism between $Y_{\alpha,\beta}$, $Y_{\alpha,\gamma}$ and $Y_{\beta,\gamma}$. In other words, we construct a 4-manifold $X = X_{\alpha\beta\gamma}$ with these three manifolds as boundary components. Let Δ be triangle with vertices v_α, v_β, v_γ and edges e_α, e_β, e_γ (e_i lies opposite v_i for $i = \alpha, \beta, \gamma$). We form X by the identification as the following:

$$X = (\Delta \times \Sigma_g) \sqcup (e_\alpha \times U_\alpha) \sqcup (e_\beta \times U_\beta) \sqcup (e_\gamma \times U_\gamma)/\sim$$

where $e_\alpha \times \Sigma_g \sim (e_\alpha \times \partial U_\alpha)$, $e_\beta \times \Sigma_g \sim (e_\beta \times \partial U_\beta)$ and $e_\gamma \times \Sigma_g \sim (e_\gamma \times \partial U_\gamma)$. Over the vertices of Δ, this space has corners, which can be naturally smoothed out. The resulting manifold X has three boundary components, one corresponds to each vertex, with $Y_{\alpha,\beta}$ lying over $v_\gamma = e_\alpha \cap e_\beta$ for example. We orient X so that

$$\partial X = -Y_{\alpha,\beta} + Y_{\alpha,\gamma} - Y_{\beta,\gamma}.$$

This cobordism is called a **pair of pants** cobordism (imagine the shape of X). In this case, we can define a map f^∞

$$f^\infty \to CF^\infty(Y_{\alpha,\beta}, t_{\alpha,\beta}) \otimes CF^\infty(Y_{\beta,\gamma}, t_{\beta,\gamma}) \otimes CF^\infty(Y_{\alpha,\gamma}, t_{\alpha,\gamma})$$

by counting **holomorphic triangles** in $\mathrm{Sym}^g(\Sigma_g)$ on the three tori $\mathbb{T}_\alpha$, $\mathbb{T}_\beta$ and $\mathbb{T}_\gamma$ given by:

$$f^\infty([\mathbf{x}, i] \otimes [\mathbf{y}, j], t) = \sum_{\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma} \sum_{\{\psi \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{w}) | t_z(\psi)=t, \mu(\psi)=0\}} (\#M(\psi)) \cdot [\mathbf{w}, i+j-n_z(\psi)].$$

Here we assume that there is a Spin^c-structure t on X and $t_{\alpha,\beta}$, $t_{\beta,\gamma}$, $t_{\alpha,\gamma}$ are respectively the restricted Spin^c-structure on $Y_{\alpha,\beta}$, $Y_{\alpha,\gamma}$, $Y_{\beta,\gamma}$. Under suitable hypothesis on the Heegaard diagrams, the above is finite. And the precise definition of holomorphic triangle is as follows.

Definition 5.2.4 Fix $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\gamma$ and $\mathbf{w} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. Consider the map

$$u : \Delta \to \mathrm{Sym}^g(\Sigma_g)$$

with the boundary conditions $u(v_\gamma) = \mathbf{x}$, $u(v_\alpha) = \mathbf{y}$, $u(v_\beta) = \mathbf{w}$ and $u(e_\alpha) \subset \mathbb{T}_\alpha$, $u(e_\beta) \subset \mathbb{T}_\beta$, $u(e_\gamma) \subset \mathbb{T}_\gamma$. Such a map is called a **Whitney triangle** connecting $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$. We let $\pi_2(\mathbf{x},\mathbf{y},\mathbf{w})$ denote the space of homotopy classes of Whitney triangles connecting $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$.

We back to our situation of knot surgeries in 3-manifolds. Let Y_1 and Y_2 be two closed oriented 3-manifolds. Suppose that Y_2 is obtained from Y_1 by doing a **0-surgery along a knot** K (not necessary trivial). This means we choose an identification of a neighborhood $N(K)$ of K with $S^1 \times D^2$, attach one part of $\partial(D^2 \times D^2)$ (the boundary of the 4-ball $D^2 \times D^2$), via the map

$$\varphi : \partial D^2 \times D^2 \to S^1 \times D^2 \cong N(K) \subset Y_1.$$

Now Y_2 is defined to be the smoothed union

$$Y_2 = (Y_1 - \text{interior of } N(K)) \sqcup_\varphi (D^2 \times S^1)$$

where φ identifies the boundary torus $S^1 \times S^1$ in $D^2 \times S^1$ with $\partial N(K)$.

We can find a cobordism between Y_1 and Y_2 explicitly. The 4-manifold X defined by

$$X = ([0,1] \times Y_1) \sqcup_{\varphi} (D^2 \times D^2)$$

is a cobordism from Y_1 to Y_2, as we can check that the boundary of X are exactly Y_1 and Y_2. Geometrically, X is constructed by adding a 2-handle $D^2 \times D^2$ to $[0,1] \times Y_1$ along $N(K) \subset \{1\} \times Y_1$.

How are the Heegaard diagrams of Y_1 and Y_2 related to each other? In fact, there is a nice interpretation of knot surgery in terms of Heegaard diagrams. Given a knot K in Y_1, we can choose a Heegaard diagram $(\Sigma_g, \alpha, \beta)$ for Y_1, in such a way that K lies in the surface $\Sigma_g - \beta_2 - \cdots - \beta g$ and intersects β_1 once transversely. By pushing K into U_β, we see that K is disjointed from the discs D_j (with boundary β_j for $j = 2, 3, \cdots, g$) and meets D_1 transversely at a single point.

In this case, when we do 0-surgery along a knot K on Y_1, we are, in fact, changing the Heegaard diagram of Y_1. To obtain Y_2, we attach α_1 to K on U_β. So the Heegaard diagram of Y_2 becomes $(\Sigma_g, \alpha, \gamma)$ where γ is given by $\gamma = \{K, \beta_2, \ldots, \beta_g\}$. And we write

$$Y_1 = Y_{\alpha,\beta} = U_\alpha \cup U_\beta, \ Y_2 = U_\alpha \cup U_\gamma = Y_{\alpha,\gamma},$$

What about the 3-manifold $Y_{\beta,\gamma}$? Since U_γ is the same as U_β, the identification between U_γ and U_β is just identity, so $Y_{\beta,\gamma}$ is a connected sum of g-copies of $S^2 \times S^1$, and its Heegaard Floer homology can be calculated in a comparatively easier way. Here we outline the calculation of $HF^+(\#^g(S^2 \times S^1))$. In [4], the Heegaard Floer groups are given by:

$$HF^+(\#^g(S^2 \times S^1, t_{\#^g(S^2\times S^1)})) \cong \wedge^* H^1(\#^g(S^2 \times S^1; \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}[U^{-1}],$$

where $t_{\#^g(S^2\times S^1)}$ is a Spin^c-structure over $\#^g(S^2 \times S^1)$ and U^{-1} increases the grading by 2.

To get the above result, we consider the pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$, where each α_i meets each β_i in exactly two canceling transverse intersection points (such Heegaard diagram is called the **standard Heegaard diagram** of $\#^g(S^2 \times S^1)$). To such a Heegaard diagram of $\#^g(S^2 \times S^1)$, the tori $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ meet in 2^g

intersection points, which correspond to the generators of

$$HF^+(\#^g(S^2\times S^1,t))\cong\wedge^*H^1(\#^g(S^2\times S^1;\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Z}[U^{-1}].$$

By pairing the map f^∞ with a canonical element in $HF^\infty(Y_{\beta,\gamma},t_{\beta\gamma})$, we get the map

$$F^\infty : HF^\infty(Y_1,t_1)\to HF^\infty(Y_2,t_2)$$

where t_i are the restriction of $t\in \mathrm{Spin}^c(X)$. There are also corresponding maps for the groups like $HF^\pm$ and $\widehat{HF}$. In fact, this is the main result in [7] which gives us surgery long exact sequences.

Theorem 5.2.5 (Ozsváth-Szabó, [7]) Let Y be a homology three-sphere, and let $K\subset Y$ be a knot. Let Y_0 be the manifold obtained by 0-surgery on K, and Y_1 be obtained by $(+1)$-surgery. Then there is a U-equivariant exact sequence of relatively $\mathbb{Z}$-graded complexes:

$$\cdots\longrightarrow HF^+(Y,t)\xrightarrow{F_1}HF^+(Y_0,t_0)\xrightarrow{F_2}HF^+(Y_1,t_1)\longrightarrow\cdots$$

The maps F_1 and F_2 are constructed in the same way as F^∞ (induced on HF^+). More precisely, these maps are defined by counting holomorphic triangles on a compatible Heegaard diagram for all three manifolds Y, Y_0 and Y_1.

In order to get a better understanding of the Heegaard Floer homology groups of these 3-manifolds, it is important to associate an absolute grading on the groups (see Chapter 4). In fact, there is a relation between those F_i and the grading. We give the following refined version of Theorem 4.4.7:

Theorem 5.2.6 (Ozsváth-Szabó, [4]) Let t be a torsion Spin^c-structure. Then, the homology groups $HF^o(Y,t)$ can be endowed with an absolute grading

$$\widetilde{\mathrm{gr}}:\mathbb{T}_\alpha\cap\mathbb{T}_\beta\to\mathbb{Q}$$

satisfying the following properties:

1. the homogeneous elements of least grading in $HF^+(S^3,s)$ have absolute grading zero (here s is the unique Spin^c-structure on S^3)

2. the absolute grading lifts the relative grading, in the sense that if $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then
$$\mathrm{gr}(\mathbf{x}, \mathbf{y}) = \widetilde{\mathrm{gr}}(\mathbf{x}) - \widetilde{\mathrm{gr}}(\mathbf{y})$$

3. the natural maps i and π in the above long exact sequence preserve the absolute grading, while the coboundary map decreases absolute degree by one, and the U action decreases it by two.

4. if W is a cobordism from Y_1 to Y_2 endowed with a Spin^c-structure whose restriction t_i to Y_i is torsion for $i = 1, 2$, then
$$\widetilde{gr}(F_{W,s}(\xi)) - \widetilde{gr}(\xi) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4} = \Delta,$$
where $\xi \in HF^o(Y_1, t_1)$, $F^o_{W,s}$ is the map induced by the cobordism map
$$F^o_{W,s} : HF^o_k(Y_1, t_1) \to HF^o_{k+\Delta}(Y_2, t_2)$$
and $t_i = s|Y_i$ for $i = 1, 2$.

Although the term Δ seems complicated, yet it is simple enough for us to deal with the case of homology three-sphere. The result is proven in [8].

Theorem 5.2.7 (Ozsváth-Szabó, [8]) Let $K \subset Y$ be a knot in an integral homology three-sphere, and let Y_0 and Y_1 be the three-manifolds obtained by 0-surgery and +1-surgery on Y along K. In the exact sequence
$$\cdots \longrightarrow HF^+(Y, t) \xrightarrow{F_1} HF^+(Y_0, t_0) \xrightarrow{F_2} HF^+(Y_1, t_1) \longrightarrow \cdots$$
the component of F_1 mapping into $HF^+(Y_0, t_0)$ (thought of as absolutely $\mathbb{Q}$-graded) has degree -1/2, the restriction of F_2 to $HF^+(Y_0, t_0)$ has degree -1/2. In other words, we get exact sequences
$$\cdots \longrightarrow HF^+_k(Y, t) \xrightarrow{F_1} HF^+_{k-\frac{1}{2}}(Y_0, t_0) \xrightarrow{F_2} HF^+_{k-1}(Y_1, t_1) \longrightarrow \cdots$$
for $k \in \mathbb{Q}$.

Let's back to our case of Brieskorn sphere $\Sigma(p, q, r)$. It is known that the Poincaré sphere $-\Sigma(2, 3, 5)$ can be obtained by doing +1-surgery on the right-handed trefoil knot (the (2,3)-torus knot) in S^3 (the orientation of $\Sigma(2,3,5)$ is inherited from the boundary of $V(2,3,5) \cap B^6$). To calculate $HF_k^+(-\Sigma(2,3,5), t_1)$, we need to find $HF_k^+(S^3, t)$ and $HF_k^+(Y_0, t_0)$.

For $HF_k^+(S^3, t)$, we know that $HF_0^+(S^3, t) = \mathbb{Z}$ and $HF_0^+(S^3, t) = 0$ for $k < 0$ by Theorem 5.2.6. In view of the isomorphism

$$U^{-1} : HF_k^+(S^3, t) \to HF_{k+2}^+(S^3),$$

we have that

$$HF_k^+(S^3, t) = \begin{cases} \mathbb{Z} & \text{k is even and } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

For $HF_k^+(Y_0, t_0)$, in general it is hard to calculate explicitly. Luckily, under some assumptions, the group HF^+ has rather simple structure.

Theorem 5.2.8 (Ozsváth-Szabó, [8]) Suppose that $K \subset S^3$ is a knot with the property that $+p$ surgery on K gives the lens space $L(p, 1)$ for some p. Then $HF^\infty(Y_0) \cong HF^\infty(S^2 \times S^1)$ as absolutely graded group. In particular,

$$HF_k^+(Y_0, t_0) = \begin{cases} \mathbb{Z} & \text{if } k \equiv 1/2 \pmod 2 \text{ and } k \geq -3/2 \\ \mathbb{Z} & \text{if } k \equiv -1/2 \pmod 2 \text{ and } k \geq -1/2 \\ 0 & \text{otherwise} \end{cases}$$

Now, it is easy to calculate the group $HF_k^+(-\Sigma(2,3,5)$ with the above results. For each $k \in \mathbb{Q}$, we have exact sequence

$$\cdots \longrightarrow HF_k^+(S^3, t) \xrightarrow{F_1^{(k)}} HF_{k-\frac{1}{2}}^+(Y_0, t_0) \xrightarrow{F_2^{(k)}} HF_{k-1}^+(-\Sigma(2,3,5), t_1) \longrightarrow \cdots$$

which immediately implies that

$$HF_k^+(-\Sigma(2,3,5), t_1) = \begin{cases} \mathbb{Z} & \text{k is even and } k \geq -2 \\ 0 & \text{otherwise} \end{cases}$$

This suggests that as a relatively graded $\mathbb{Z}[U]$-module, $HF^+(-\Sigma(2,3,5))$ is isomorphic to $HF^+(S^3)$, but the absolute grading still distinguishes them.

We do one more example about Brieskorn sphere. For the Brieskorn sphere $\Sigma(2,3,7)$, it is obtained as -1 surgery on the right-handed trefoil. The exact sequence for -1 surgery is given by

$$\cdots \longrightarrow HF_k^+(\Sigma(2,3,7), t_{-1}) \xrightarrow{F_1^{(k)}} HF_{k-\frac{1}{2}}^+(Y_0, t_0) \xrightarrow{F_2^{(k)}} HF_{k-1}^+(S^3, t) \longrightarrow \cdots$$

Since $HF_k^+(S^3) = 0$ for all $k < 0$, the generator of $HF_{-3/2}^+(Y_0, t_0)$ must come from a generator of $HF_{-1}^+(\Sigma(2,3,7), t_{-1})$. So we get

$$HF_k^+(-\Sigma(2,3,7), t_{-1}) = \begin{cases} \mathbb{Z} & \text{k is even and } k \geq 0 \\ \mathbb{Z} & \text{if } k = -1 \\ 0 & \text{otherwise.} \end{cases}$$

5.3 More Application

5.3.1 Knot Floer homology

As we may guess, that there is a close relationship between Heegaard Floer theory and knot theory. In [6], Ozsváth-Szabó give another version of Heegaard Floer homology that can be applied to knots in three-manifolds. Here we give an introduction to this theory.

Definition 5.3.1 From now on, a **knot** will consist of a pair (Y, K), where Y is an oriented three-manifold, and $K \subset Y$ is an embedded, oriented, null-homologous circle.

For simplicity, we restrict ourselves to $Y = S^3$. And we can associate a knot (S^3, K) a Heegaard diagram in the following sense. Let's consider the Heegaard diagram for S^3

$$(\Sigma_g, \alpha, \beta_0 \cup \{\gamma\})$$

where α is an unordered g-tuple of pairwise disjoint attaching circles $\alpha = \{\alpha_1, \cdots, \alpha_g\}$, β_0 is a $(g-1)$-tuple of pairwise disjoint attaching circles $\{\beta_2, \cdots, \beta_g\}$, γ is an embedded, oriented circle in Σ_g which is disjoint from β_0. This data is chosen so that $(\Sigma_g, \alpha, \beta_0)$ specifies the knot-complement $S^3 - N(K)$ ($N(K)$ is some ε-neighborhood of K), in other words if we attach disks (1-handles and 2-handles) to Σ_g along α and β_0, and then add a three-ball (the 3-handle), we obtain $S^3 - N(K)$ (γ can be viewed as the "meridian" for the knot in S^3).

We fix two base point z and w from $\Sigma_g - \alpha - \beta_0 - \gamma$. We call the data $(\Sigma_g, \alpha, \beta_0, \gamma, z, w)$ a **two pointed Heegaard diagram** compatible with the knot K (for simplicity we write $\beta = \beta_0 \cup \gamma$).

In a similar way, we can define a chain complex for a knot K. Details can be found in [6].

Definition 5.3.2 Let K be a knot in S^3 and $(\Sigma_g, \alpha, \beta, z, w)$ be a compatible two-pointed Heegaard diagram. Let $\widehat{C}(K)$ be the free abelian group generated by the intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For a generic choice of almost-complex structure J, let $\widehat{\partial}_K : \widehat{C}(K) \to \widehat{C}(K)$ be given by

$$\widehat{\partial}_K(\mathbf{x}) = \sum_{\mathbf{y}} \sum_{\{\phi \in \pi_2(\mathbf{x},\, \mathbf{y}) | \mu(\phi)=1, n_z(\phi)=n_w(\phi)=0\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot \mathbf{y}$$

where $\widehat{\mathcal{M}}(\phi)$ denote the quotient of the moduli space of J-holomorphic disks representing the homotopy type of ϕ, divided by the natural action of $\mathbb{R}$ on this space.

Similar to the previous Heegaard Floer homology theory, we can also define those corresponding chain groups $C^\infty(K)$, $C^-(K)$ and $C^-(K)$. Specifically, we let $C^\infty(K)$ to be the free abelian group generated by triples $[\mathbf{x}, i, j]$ with $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $i, j \in \mathbb{Z}$, with the differential

$$\partial_K^\infty([\mathbf{x}, i, j]) = \sum_{\mathbf{y}} \sum_{\{\phi \in \pi_2(\mathbf{x},\, \mathbf{y}) | \mu(\phi)=1\}} \#(\widehat{\mathcal{M}}(\phi)) \cdot [\mathbf{y}, i - n_z(\phi), j - n_w(\phi)].$$

Moreover, we can endow the chain group with the structure of a $\mathbb{Z}$-module, by defining

$$U \cdot [\mathbf{x}, i, j] = [\mathbf{x}, i-1, j-1].$$

And the following is the main result in [6]

Theorem 5.3.3 The above chain groups with the corresponding differentials are all chain complexes. The homology groups as obtained are independent of the choice of the two-pointed Heegaard diagrams and the almost-complex structures.

As an example, for the case when K is unknotted, we can use the standard genus 1 Heegaard diagram of S^3, and get $\widehat{HF}(K) = \mathbb{Z}$.

There are many ways to refine the notion of Knot Floer homology. For example, we can associate Spin^c-structure and absolute gradings on the homology groups. And of course, other than just considering knots in S^3, one can also give similar definitions on knots in an arbitrary oriented three-manifold Y. All these details can be found in [6].

5.3.2 Invariants on 4-manifolds

As we have seen, Heegaard Floer homology invariants can be associated to cobordism between 3-manifolds. Such connection can be used to construct an invariant for smooth, closed 4-manifolds.

Suppose X is a 4-manifold with $b_2^+(X) > 0$. We can delete 4-ball neighborhoods of two points in X. It gives us a 4-manifold X' which is a cobordism from S^3 to S^3. We further subdivide X' along a separating hypersurface N into a union $W_1 \cup_N W_2$, with the properties that

1. W_1 is a cobordism from S^3 to N with $b_2^+(W_1) > 0$
2. W_2 is a cobordism from N to S^3 with $b_2^+(W_2) > 0$
3. restriction map $H^2(W_1 \cup_N W_2) \to H^2(W_1) \oplus H^2(W_2)$ is injective.

In this case, we call the hypersurface N an **admissible cut** for X.

Recall that for a smooth cobordism W from Y_0 to Y_1, we have the commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & HF^-(Y_0,t_0) \arrow[r, "i"] \arrow[d, "F^-_{W,t}"'] & HF^\infty(Y_0,t_0) \arrow[r, "\pi"] \arrow[d, "F^\infty_{W,t}"'] & HF^+(Y_0,t_0) \arrow[r, "\delta"] \arrow[d, "F^+_{W,t}"'] & \cdots \\
\cdots \arrow[r] & HF^-(Y_1,t_1) \arrow[r, "i"] & HF^\infty(Y_1,t_1) \arrow[r, "\pi"] & HF^+(Y_1,t_1) \arrow[r, "\delta"] & \cdots
\end{tikzcd}$$

Let $HF^-_{\mathrm{red}}(Y)$ denote the kernel of i, and $HF^+_{\mathrm{red}}(Y)$ be the cokernel of π. By Proposition 4.8 in [2], there is an isomorphism

$$\delta' : HF^+_{\mathrm{red}}(Y) \to HF^+_{\mathrm{red}}(Y).$$

induced by δ from the exact sequence.

Back to our case, since $b_2^+(W_i) > 0$, by Lemma 8.2 in [4], the maps on HF^∞ induced by cobordism are trivial. Our aim is to define a map

$$\Phi_{X,t} : HF^-(S^3) \to HF^+(S^3)$$

to be the composite:

$$\Phi_{X,t} = F^+_{W_2,t|W_2} \circ (\delta')^{-1} \circ F^-_{W_1,t|W_1}.$$

To do this, observe that

$$i \circ F^-_{W_1,t|W_1} = F^\infty_{W_1,t|W_1} \circ i = 0,$$

so the image of $F^-_{W_1,t|W_1}$ lies in the kernel of i, hence in $HF^-_{\text{red}}(N,t|N)$. Moreover,

$$F^+_{W_2,t|W_2} \circ \pi = \pi \circ F^\infty_{W_2,t|W_1} = 0$$

so $F^+_{W_2,t|W_2}$ maps the image of π to 0, which means $F^+_{W_2,t|W_2}$ factors through the projection of $HF^+(N,t|N)$ to $HF^+_{\text{red}}(N,t|N)$.

Therefore, the map $\Phi_{X,t}$ is well-defined. From [4], this map is independent of the choice of N, giving a well-defined 4-manifold invariant. For more details please refer to [3], [4] and [7].

5.4 Further developments

We end this Chapter by quoting some problems and developments as suggested by Ozsváth-Szabó in [4].

1. Can one establish the conjectured relationship between Heegaard Floer homology and Seiberg-Witten theory?

   There are two approaches one might take to this problem. One direct, analytical approach would be to analyze moduli spaces of solutions to the Seiberg-Witten equations over a three-manifold equipped with a Heegaard decomposition. There is also a branch of Heegaard Floer theory which gives invariants on symplectic 4-manifolds (see [5]), and the technique of plumbing can be used to calculate the Heegaard Floer homology of some class of rational spheres (see [9] and [12]). And many mathematicians are giving efforts to relate the 4-manifold invariant as suggested in the previous section to the Seiberg-Witten invariant.

   Another approach leads to the next question:

2. Is there an axiomatic characterization of Heegaard Floer homology?

A **Floer functor** is a map which associates to any closed, oriented three-manifold Y a $\mathbb{Z}_2$-graded abelian group $\mathcal{H}(Y)$ and to any cobordism W from Y_1 to Y_2 a homomorphism $\mathcal{D}_W : \mathcal{H}(Y_1) \to \mathcal{H}(Y_2)$, which is natural under composition of cobordisms, and which induce exact sequences for triples of three-manifolds (Y_0, Y_1, Y_2). It is "interesting" to observe that if $\mathcal{T}$ is a natural transformation between Floer functors $\mathcal{H}$ and $\mathcal{D}$ to $\mathcal{H}'$ and $\mathcal{D}'$, then if $\mathcal{T}$ induces an isomorphism $\mathcal{T}(S^3) : \mathcal{H}(S^3) \to \mathcal{H}'(S^3)$, then $\mathcal{T}$ induces isomorphisms for all 3-manifolds Y, $\mathcal{T}(Y) : \mathcal{H}(Y) \to \mathcal{H}'(Y)$. This can be proved from **Kirby calculus**, see [43]. Unfortunately, this still falls short of giving an axiomatic characterization: one needs axioms which are sufficient to assemble a natural transformation $\mathcal{T}$.

3. For a given 3-manifold, is there an explicit relationship between the Heegaard Floer homology and the fundamental group of Y?

   This question is co-related to the question:

   If $K \subset S^3$ is a knot, is there an explicit relationship between the fundamental group of $S^3 - K$ and the knot Floer homology $\widehat{HF}(K)$?

   These questions are studied specifically by knot theorists, such as [10], [44] and [45].

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