A Hybrid Method for Solving the Ruin Functionals of the Classical Risk Model Perturbed by Diffusion

LEUNG, Kit Hung

A Thesis Submitted in Partial Fulfilment of the Requirements for the Degree of Master of Philosophy in Statistics

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Thesis / Assessment Committee

Professor P. L. Leung (Chairman)
Professor P. S. Wong (Thesis Supervisor)
Professor H. Y. Wong (Committee Member)
Professor H. L. Yang (External Examiner)
The undersigned certify that we have read a thesis, entitled "A hybrid method for solving the ruin functionals of the classical risk model perturbed by diffusion" submitted to the Graduate School by LEUNG, Kit Hung (梁潔紅) in partial fulfilment of the requirements for the degree of Master of Philosophy in Statistics. We recommend that it be accepted.

Prof. P. S. Wong
Supervisor

Prof. P. L. Leung

Prof. H. Y. Wong

Prof. Hailiang Yang
External Examiner
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A hybrid method for solving the ruin functionals of the classical risk model perturbed by diffusion

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ABSTRACT

The classical model in the risk theory describes the claim mechanism by using the compound Poisson process and assumes the premium is collected uniformly over time. Gerber (1970) modified the classical model by adding a diffusion term for modelling the uncertainty in surplus investment process. Such diffusion perturbed process has been studied by various researchers who investigated the theoretical properties of the ruin probability and the related ruin functionals. In particular, Dufresne and Gerber (1991) derived an explicit formula for the ruin probability under the assumption of the claim amounts are exponentially distributed. Note that no explicit formulas are available for general claim distribution. This thesis aims at providing a versatile analysis toolbox to study the ruin phenomena of the aforementioned model. We first adopt the integro-differential equation derived by other researchers. Then, we turn that into a Volterra integral equation of the second kind and solve that by using numerical method which is contingent on the availability of the initial conditions. By noting the fact that the integral equation is derived from a stochastic process, we propose using importance sampling to estimate the boundary
condition needed. Our scheme gives highly comparable results to those explicit formulas which are derived under restrictive assumptions. For settings without known explicit formulas, our scheme can still provide accurate approximation.
摘要

於風險理論中的典型模型以普阿松複合過程來描述索賠機制，並假定保險費均勻地隨著時間收集。Gerber（1970年）修改了該典型模型，於典型模型中增加了一擴散項目，為投資過程中的盈餘的不定性作模型建立。該擴散了擴散項目的過程曾被多位研究員作研究，而探討多為破產概率的理論性能及其與破產相關的泛函。尤其Dufresne 和Gerber（1991年）於索賠額為指數分佈的假定下為破產概率推導出一個顯式公式。我們注意到並沒有可利用的一般索賠分佈的顯式公式。本文旨在為上述模型提供一個靈活的分析方法以研究與破產相關的現象。我們先採用其他研究員所推導的積分微分方程，然後將其轉化至第二類Volterra積分方程，並以數值計算法求解。此方法乃取決於有可用的初始條件。注意到該積分方程乃透過隨機過程推導，我們建議以重要性抽樣方法來估計所需的邊界條件。我們的方案所得出的研究結果相對那些在有限制的假設下所推導的顯式公式有優勢。在沒有已知的顯式公式的情況下，我們的方案仍能提供準確的近似計。
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Chapter 1

Introduction

1.1 Classical Model

In classical risk theory, the claim mechanism is modelled by the compound Poisson process and the premium is assumed to be collected uniformly over time. That is, the surplus of an insurance company at time $t$ is described in the following model:

$$R(t) = u + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0$$

(1.1)

where $u = R(0) \geq 0$ is the initial reserve, $c > 0$ is the constant premium rate, $N_t$ is the number of claims in the time interval $[0, t]$ and $Z_i$ is the claim size of the $i^{th}$ claim. In particular, \{$N_t, t \geq 0$\} is assumed to be a Poisson process with intensity $\lambda > 0$ and \{\(Z_k, k \geq 1\)\} is a sequence of i.i.d. non-negative random variables. We further assume that \{\(N_t, t \geq 0\)\} and \{\(Z_k, k \geq 1\)\} are independent. Denote $P$ and $p$ as the distribution and density of $Z_k$ where $P(0) = 0$. 

1
CHAPTER 1. INTRODUCTION

Since the expected claim amount per unit time is \( \lambda \mu \), the surplus process increases by \( c - \lambda \mu \) per unit time. Thus \( c - \lambda \mu \) is called the safety loading and is assumed to be positive such that the trivial case of ultimate ruin being a sure event is excluded in the analysis of the classical model.

Define \( T = \inf\{t \geq 0 : R(t) < 0\} \) as the time of ruin. For the case of \( R(t) > 0 \ \forall \ t \geq 0 \), we adopt the notation of \( T = \infty \). The probability of ultimate ruin, or simply the ruin probability in this thesis, is defined as

\[
\psi(u) = E[1\{T < \infty\}|R(0) = u] = Pr\{T < \infty|R(0) = u\},
\]

where \( 1\{A\} \) is the indicator function of the event \( A \). From the insurer’s point of view, it is important to have sufficient initial reserve \( u \) such that \( \psi(u) \) is kept to be small to ensure the survival of the business. Other than the ruin probability, the insurer is also interested in analyzing the surplus immediately before ruin \( R(T-) \), and the deficit at ruin \( |R(T)| \) from risk management perspective. The functions of \( T \) which includes \( 1\{T < \infty\} \), \( R(T-) \) and \( |R(T)| \) are collectively called the risk functionals. In particular, Gerber and Shiu (1997) suggests a unified framework to study the following class of expected discounted risk functionals:

\[
W(u, \delta, w) = E[e^{-\delta T} w(R(T-), |R(T)|)1\{T < \infty\}|R(0) = u], \ u \geq 0
\]  

(1.2)

where \( \delta \) is a discount factor and \( w(\cdot, \cdot) \) can be viewed as a ruin penalty which depends on the surplus immediately before ruin \( R(T-) \) and the deficit at ruin \( |R(T)| \). Gerber and Shiu (1997) call (1.2) the ruin functions. It is easy to see that \( W(u, 0, 1) = \psi(u) \).

Also, taking \( w(x, y) = x \) and \( \delta = 0 \) in (1.2) makes \( W(u, 0, w) \) becoming the expected surplus immediately before ruin while the case of \( w(x, y) = y \) and \( \delta = 0 \) in (1.2)
refers to the expected deficit at ruin.

Issues related to the ruin functionals have also been studied by many authors. Lin and Willmot (1999, 2000) analyze the joint and marginal moments related to the time of ruin, the surplus before ruin and the deficit at ruin. For δ = 0, Asmussen (2000) lists many seminal results of ψ(u). Other variations of the model has also been investigated. One of the directions is assuming $N_t$ to be an Erlang(2) process instead of a Poisson process. Dickson and Hipp (1998) gives an explicit formula of ψ(u) for the case of the claim amount is exponentially distributed. Incorporating a continuous compounding interest mechanism into the Erlang(2)-driven model, Wong, Ho, Hu and Liu (2006) propose a hybrid numerical scheme for ψ(u) computation.

1.2 Diffusion-perturbed model

Gerber (1970) considers another variation of the classical model. Instead of assuming other point processes, Gerber (1970) adds a diffusion term into the classical surplus process:

$$R(t) = u + ct + \sigma B_t - \sum_{i=1}^{N_t} Z_i, t \geq 0$$

where $\sigma > 0$ is the volatility, $\{B_t, t \geq 0\}$ is the standard Brownian motion and is assumed to be independent with the processes $\{N_t, t \geq 0\}$ and $\{Z_k, k \geq 1\}$. The model is motivated by the fact that the surplus could be invested into some financial products which adds non-negligible randomness to the classical case (1.1). The diffusion term is used to model such additional uncertainty. The ruin functions for the (1.3) is defined similarly in the manner of (1.2) although we use $W_\sigma(u, \delta, w)$ to
highlight the dependence of \( \sigma \). There is no explicit ruin function formula published in the academic literature although some asymptotic results are obtained by Chiu and Yin (2003). In this thesis, we aim at applying the hybrid methodology proposed by Wong, Ho, Hu and Liu (2006) to study not only \( \psi(u) \) but a wider class of risk functions \( W_\sigma(u, \delta, \omega) \).

Note that although there is no known explicit formula for the ruin function under the definition of \( T = \inf\{t : R(t) < 0\} \) being the time of ruin, an explicit formula for \( \psi(u) \) is available if the following time of ruin definition is adopted:

\[
S = \inf\{t : R(t) \leq 0\}
\]

and similarly, \( S = \infty \) if \( R(t) > 0 \) for all \( t \). Such definition enables Dufresne and Gerber (1991) to decompose the ultimate ruin probability \( \psi(u) = \Pr\{S < \infty | R(0) = u\} \) as:

\[
\psi(u) = \psi_d(u) + \psi_s(u)
\]

where \( \psi_d(u) \) is the probability of ruin that is caused by oscillation and \( \psi_s(u) \) is the probability of ruin that is caused by a claim. i.e.,

\[
\psi_d(u) = \Pr\{S < \infty, R(S) = 0 | R(0) = u\}
\]

\[
\psi_s(u) = \Pr\{S < \infty, R(S) < 0 | R(0) = u\}.
\]

Specifically, \( \psi_d(0) = 1 \) and \( \psi_s(0) = 0 \). The explicit formula for \( \psi(u) \) in this case (with exponentially distributed claims) is used as a benchmark to cross-validate the hybrid methodology employed in this thesis. The details of the comparison are in Chapter 5.
1.3 Hybrid computational scheme

The above review mainly focuses at the theoretical ruin analysis of the classical model and its variations. The main results in this area are usually presented in the form of an explicit solution for the case of exponentially distributed claim. To relax such restrictive distribution assumption, the researchers may also derive some asymptotic statements for various ruin functions when $u \to \infty$ under some more less stringent conditions. However, the performance of the asymptotic results for moderate $u$ is far from being clear. In fact, the insurer should not be interested to know the $u$ needed to attain $\psi(u) = 10^{-9}$ and a not-so-risk-averse company may even consider setting $\psi(u) = 10^{-2}$. Therefore, instead of addressing the above issue by asymptotic analysis, a number of researchers employ the numerical approach of computing $\psi(u)$ for a practical range.

Paulsen, Kasozi and Steigen (2005) is the pioneer in applying the block-by-block numerical method in solving integral equations of $\psi(u)$ for a diffusion driven model. Although their method is quite effective, the method is difficult to apply for other problems because the integral equation is numerically solvable only when the corresponding boundary conditions can be cancelled out. Interestingly, Paulsen and Rasmussen (2003) is also one of the first researchers using importance sampling to solve the ruin probability problem. In particular, they develop a methodology for choosing the trial process optimally for $\psi(u)$ estimation under semi-martingale framework.

In this thesis, we apply the recipe of Wong, Ho, Hu and Liu (2006) who combine
the idea of numerical method and importance sampling as follows. First, for the ruin function of interest, we make use of the integro-differential equation derived by other researchers and turn it into an integral equation which depends on the boundary conditions. Since it is not possible to "cancel" out the boundary conditions for many models as in Paulsen, Kasozi and Steigen (2005), we suggest using the importance sampling methodology to evaluate the boundary conditions. Unlike Wong, Ho, Hu and Liu (2006) who determine the trial process with intuition, our choice in this thesis is optimal in the sense of Paulsen and Rasmussen (2003). Based upon the estimated boundary conditions, the integral equation is solved by a novel application of numerical method which returns a relatively small error. Such approach could help not only approximating ruin functions under a more general framework but also provide a way to validate the performance of the asymptotic properties when $u$ is moderate.

The organization for the rest of this thesis is as follows. The connection of the integro-differential equations and the corresponding integral equations for various ruin functions are presented in Chapter 2. Chapter 3 discusses the numerical method employed in solving the integral equations. The technique of importance sampling for boundary condition determination is shown in Chapter 4. Chapter 5 gives some numerical examples. The thesis is concluded in Chapter 6.
Chapter 2

Integro-differential Equations

In this chapter, we first review the integro-differential equation of the ruin functions derived previously by Chiu and Yin (2003). Then, the technique of turning the integro-differential equation into a Volterra integral equation of the second kind is presented. Such integral equation is the input of the proposed hybrid numerical methodology. Similar discussion for the methodology of Dufresne and Gerber (1991) that makes use of $S = \inf\{t : R(t) \leq 0\}$ is in Section 2.2.

2.1 Integro-differential equation of Chiu and Yin (2003)

In the rest of this thesis, we consider only the diffusion-perturbed model (1.3). Also, for the sake of simplicity, we focus at the ruin functions with $\delta = 0$ only. Thus, to simplify the notation, $W(u)$ is used to denote $W_\sigma(u,0,w) = E[1(T <
Theorem 1 If $W(u)$ is twice continuously differentiable, then $W(u)$ satisfies the following integro-differential equation:

$$\frac{1}{2}\sigma^2 W''(u) + cW'(u) + \lambda \int_0^u W(u-z)p(z)dz + \omega(u) = \lambda W(u)$$  \hspace{1cm} (2.1)

where $u \geq 0$, $p$ is the common probability density function of the claims, $Z_k$, and 

$$\omega(u) = \int_u^\infty w(u,z-u)p(z)dz.$$ 

Theorem 1 is proved in Chiu and Yin (2003, Theorem 3.1). Thus, instead of detailing the proof, here we simply review the essential steps in their derivation and explain how renewal theory is used in the proof.

Let $T_1$ be the epoch of the first claim. Since $N_t$ is a Poisson process, $T_1$ is exponentially distributed with rate $\lambda$. Given $T_1$, the analysis is divided into the following scenarios:

1. Before $T_1$, there exists some $t < T_1$ such that $R(t) < 0$. That is, the Brownian motion causes the ruin.

2. Before $T_1$, $R(t) \geq 0$ for all $t < T_1$ and $R(T_1) < 0$. That is, the first claim causes the ruin.

3. Before $T_1$, $R(t) \geq 0$ for all $t < T_1$ and $R(T_1) \geq 0$.

One can immediately see from the scenario 3 that the surplus process restarts (or renew) at time $T_1$ and the new initial surplus level is $R(T_1) = u + cT_1 + \sigma B_{T_1} - Z_1 > 0$.
under the condition. This kind of argument of restarting the waiting process is called the renewal argument and is commonly employed in the derivation of integro-differential equation in risk theory.

The scenarios 1 and 2 are handled by the strong Markov property and Ito’s lemma and are described very clearly in Chiu and Yin (2003). Note that Theorem 1 is true only when $W(u)$ is twice continuously differentiable. The following theorem gives a general condition that leads to such consequence.

**Theorem 2** If $w(\cdot, \cdot)$ is a bounded non-negative continuous function and the common density function of the claim, $p(z)$, is continuous over the positive real line, then $W(u)$ is twice continuously differentiable.

The proof of Theorem 2 is in Wang and Wu (2000) and is not repeated here.

**Theorem 3** Given the condition of Theorem 1, $W(u)$ has to satisfy is twice continuously differentiable, then $W(u)$ satisfies the following integral equation:

$$W(u) = \int_0^u K(u, s)W(s)ds + \alpha(u) - \lambda \int_0^u \int_0^s \omega(v)dvds$$

where

$$K(u, s) = 2\sigma^{-2} [\lambda u - \lambda s - c - \lambda P_2(u - s)],$$

and

$$\alpha(u) = uW'(0) + (1 + \frac{2cu}{\sigma^2})W(0).$$

Here $P_2$ is defined as:

$$P_2(u) = \int_0^u \int_0^v p(s)ds$$
CHAPTER 2. INTEGRO-DIFFERENTIAL EQUATIONS

with \( u > 0 \) and \( v > 0 \) and \( p \) is the common probability density function of the claim amount.

**Proof.** We start from the integro-differential equation of Theorem 1:
\[
\frac{1}{2} \sigma^2 W''(v) + c W'(v) + \lambda \int_0^v W(v - z) p(z) dz = \lambda W(v) - \lambda \omega(v)
\]
for \( v \geq 0 \) and integrate both sides with \( v \) ranges from 0 to \( u > 0 \). Then, the right hand side becomes \( \lambda \int_0^u [W(v) - w(v)] dv \) while the left hand side is of the form:
\[
\frac{1}{2} \sigma^2 \int_0^u W''(v) dv + c \int_0^u W'(v) dv + \lambda \int_0^u \int_0^v W(v - z) p(z) dz dv
\]
\[
= \frac{1}{2} \sigma^2 \int_0^u dW'(v) + c \int_0^u dW(v) + \lambda \int_0^u W(v) P(u - v) dv
\]
\[
= \frac{1}{2} \sigma^2 W'(u) - \frac{1}{2} \sigma^2 W'(0) + c W(u) - c W(0) + \lambda \int_0^u P(u - v) W(v) dv
\]
where \( P \) is the cumulative distribution function of \( p \), i.e.,
\[
P(u) = \int_0^u p(v) dv.
\]

Use \( s \) to play the role of \( u \) and express the equation in the form of:
\[
\frac{1}{2} \sigma^2 W'(s) - \frac{1}{2} \sigma^2 W'(0) + c W(s) - c W(0) + \lambda \int_0^s P(s - v) W(v) dv = \lambda \int_0^s [W(v) - w(v)] dv.
\]

Integrating both sides with respect to \( s \) from 0 to \( u \) again turns the right hand side into
\[
\lambda \int_0^u \int_0^s [W(v) - w(v)] dv ds
\]
\[
= \lambda \int_0^u \int_v^u ds [W(v) - w(v)] dv
\]
\[
= \lambda u \int_0^u [W(v) - w(v)] dv - \lambda \int_0^u v [W(v) - w(v)] dv
\]
And the left hand side becomes:
\[
\frac{1}{2} \sigma^2 \int_0^u W'(s) ds - \frac{1}{2} \sigma^2 W'(0) \int_0^u ds + c \int_0^u W(s) ds - cW(0) \int_0^u ds \\
+ \lambda \int_0^u \int_0^s P(s-v)W(v) dv ds \\
= \frac{1}{2} \sigma^2 \int_0^u dW(s) - \frac{1}{2} \sigma^2 W'(0) \int_0^u ds + c \int_0^u W(s) ds - cW(0) \int_0^u ds \\
+ \lambda \int_0^u \int_v^u P(s-v)dsW(v) dv \\
= \frac{1}{2} \sigma^2 W(u) - \frac{1}{2} \sigma^2 W(0) - \frac{1}{2} \sigma^2 uW'(0) + c \int_0^u W(s) ds - cuW(0) \\
+ \lambda \int_0^u P_2(u-v)W(v) dv
\]
where \( P_2 \) is the primitive function of \( P \). That is,
\[
P_2(u) = \int_0^u P(v) dv.
\]
The integral equation is obtained by re-arranging the terms. ■

**Corollary 4** Based upon Theorem 3, the integral equation for (A) the ultimate ruin probability, (B) the expectation of the surplus immediately before ruin and (C) the expectation of the deficit at ruin are given by
\[
W(u) = \int_0^u K(u,s)W(s) ds + \alpha_j(u)
\]
where case (A), (B) and (C) corresponds to \( j = 1, 2 \) and 3, respectively with specific \( \alpha_j \) in the form of:
\[
\alpha_1(u) = uW'(0) + (1 + \frac{2cu}{\sigma^2})W(0) - \frac{\lambda u^2}{\sigma^2} + \frac{2\lambda}{\sigma^2} \int_0^u P_2(s) ds, \quad (2.2)
\]
\[
\alpha_2(u) = uW'(0) + (1 + \frac{2cu}{\sigma^2})W(0) \\
- \frac{\lambda u}{\sigma^2} \int_0^u s^2 p(s) ds + \frac{\lambda}{\sigma^2} \int_0^u s^3 p(s) ds - \frac{\lambda u^3}{3\sigma^2} + \frac{\lambda}{\sigma^2} \int_0^u s^2 P(s) ds, \quad (2.3)
\]
and
\[
\alpha_3(u) = uW'(0) + \left(1 + \frac{2cu}{\sigma^2}\right)W(0)
- \frac{\lambda u}{\sigma^2} \int_0^u s^2 p(s) ds - \frac{\lambda u^2}{\sigma^2} \int_u^\infty s p(s) ds + \frac{\lambda u^3}{3\sigma^2} - \frac{\lambda}{\sigma^2} \int_0^u s^2 P(s) ds.
\]

(2.4)

**Proof.** For the ultimate ruin probability, we just take \(w(\cdot, \cdot) = 1\) and thus
\[
\int_0^u \omega(v) dv = \int_0^u \int_v^\infty w(v, z - v)p(z) dz dv
= \int_0^u \int_v^\infty p(z) dz dv
= \int_0^u [1 - P(v)] dv
= u - P_2(u).
\]

Consequently,
\[
\int_0^u \int_0^s \omega(v) dv ds = \int_0^u s ds - \int_0^u P_2(s) ds
= \frac{u^2}{2} - \int_0^u P_2(s) ds.
\]

Plugging into the integral equation gives the corresponding \(\alpha_1\). For the expectation of the surplus immediately before ruin, one can simply take \(w(x, y) = x\). In that
case, integrating $\omega(u)$ from 0 to $u$ can be proceeded as follows:

$$\int_0^u \omega(v)dv = \int_0^u \int_v^\infty w(v, z - v)p(z)dzdv$$

$$= \int_0^u \int_v^\infty vp(z)dzdv$$

$$= \int_0^u \int_0^z vdp(z)dz + \int_0^u \int_v^\infty vdp(z)dz$$

$$= \int_0^u \frac{z^2}{2}p(z)dz + \frac{u^2}{2} \int_0^\infty p(z)dz$$

$$= \frac{1}{2} \int_0^u z^2p(z)dz + \frac{u^2}{2}[1 - P(u)]$$

$$= \frac{1}{2} \int_0^u z^2p(z)dz + \frac{u^2}{2} - \frac{u^2}{2} P(u)$$

and

$$\int_0^u \int_0^s \omega(v)dvds$$

$$= \frac{1}{2} \int_0^u \int_0^s z^2p(z)dzds + \frac{1}{2} \int_0^u s^2ds - \frac{1}{2} \int_0^u s^2P(s)ds$$

$$= \frac{1}{2} \int_0^u \int_0^s dz^2p(z)dz + \frac{u^3}{6} - \frac{1}{2} \int_0^u s^2P(s)ds$$

$$= \frac{1}{2} \int_0^u (u - z)z^2p(z)dz + \frac{u^3}{6} - \frac{1}{2} \int_0^u s^2P(s)ds$$

$$= \frac{u}{2} \int_0^u z^2p(z)dz - \frac{1}{2} \int_0^u z^3p(z)dz + \frac{u^3}{6} - \frac{1}{2} \int_0^u s^2P(s)ds$$

$$= \frac{u}{2} \int_0^u s^2p(s)ds - \frac{1}{2} \int_0^u s^3p(s)ds + \frac{u^3}{6} - \frac{1}{2} \int_0^u s^2P(s)ds.$$
following manner:

\[
\int_0^u \omega(v)dv \\
= \int_0^u \int_v^\infty w(v, z - v)p(z)dzdv \\
= \int_0^u \int_v^\infty (z - v)p(z)dzdv \\
= \int_0^u \int_v^\infty zp(z)dzdv - \int_0^u \int_v^\infty vp(z)dzdv \\
= \int_0^u \int_0^z dvzp(z)dz + \int_u^\infty \int_0^z dvzp(z)dz - \int_0^u \int_v^\infty vp(z)dzdv \\
= \int_0^u z^2p(z)dz + u \int_v^\infty zp(z)dz - \int_0^u \int_v^\infty vp(z)dzdv \\
= \int_0^u z^2p(z)dz + u \int_v^\infty zp(z)dz - \left[ \int_0^u \frac{z^2}{2}p(z)dz + \frac{u^2}{2} - \frac{u^2}{2}P(u) \right]
\]
and

\[ \int_0^u \int_0^s \omega(v)dvds = \int_0^u \int_0^s z^2 p(z)dzds + \int_0^u \int_s^\infty szp(z)dzds \]
\[ - \int_0^u \left\{ \int_0^s \frac{z^2}{2} p(z)dz + \frac{s^2}{2} - \frac{s^2}{2} P(s) \right\} ds \]
\[ = \int_0^u dss^2 p(z)dz + \int_0^u \int_0^s sdp(z)dz + \int_u^\infty \int_0^s sdp(z)dz \]
\[ - \int_0^u \left\{ \int_0^s \frac{z^2}{2} p(z)dz + \frac{s^2}{2} - \frac{s^2}{2} P(s) \right\} ds \]
\[ = \int_0^u (u - z)z^2 p(z)dz + \frac{1}{2} \int_0^u z^3 p(z)dz + \frac{u^2}{2} \int_u^\infty zp(z)dz \]
\[ - \frac{1}{2} \int_0^u (u - z)z^2 p(z)dz - \frac{u^3}{6} + \frac{1}{2} \int_0^u s^2 P(s)ds \]
\[ = \frac{1}{2} \int_0^u (u - z)z^2 p(z)dz + \frac{1}{2} \int_0^u z^3 p(z)dz + \frac{u^2}{2} \int_u^\infty zp(z)dz \]
\[ - \frac{u^3}{6} + \frac{1}{2} \int_0^u s^2 P(s)ds \]
\[ = \frac{u}{2} \int_0^u z^2 p(z)dz + \frac{u^2}{2} \int_u^\infty zp(z)dz - \frac{u^3}{6} + \frac{1}{2} \int_0^u s^2 P(s)ds \]
\[ = \frac{u}{2} \int_0^u s^2 p(s)ds + \frac{u^2}{2} \int_u^\infty sp(s)ds - \frac{u^3}{6} + \frac{1}{2} \int_0^u s^2 P(s)ds. \]

\( \alpha_3 \) is obtained by grouping the terms appropriately. \( \blacksquare \)

**Remark:** The ruin probability is tied to the survival probability \( \phi(u) = 1 - \psi(u) \)
which can be also be shown to satisfy a similar integral equation:

\[ \phi(u) = \int_0^u K(u, s)\phi(s)ds + \gamma(u) \quad (2.5) \]

where

\[ \gamma(u) = u\phi'(0) + \left(1 + \frac{2cu}{\sigma^2}\right)\phi(0). \]

Note that unlike \( \alpha_j(u) \), \( \gamma(u) \) is independent of the claim distribution \( P(z) \).
2.2 Integro-differential equations for $\psi_s(u)$ and $\psi_d(u)$

As we have already discussed in Chapter 1, the time of ruin can also be defined as $S = \inf\{t \geq 0 : R(t) \leq 0\}$ from which the explicit formula for $\psi(u)$ is derived by Dufresne and Gerber (1991). Their technique requires decomposition of $\psi(u)$ into $\psi_d(u)$ and $\psi_s(u)$ which are the ruin probability caused by Brownian motion and that caused by the compound Poisson claim process, respectively. Using a similar renewal argument for Theorem 1, it can be shown that $\psi_d(u)$ and $\psi_s(u)$ both satisfy some integro-differential equations (2.1) under the assumption that both the claim size distribution $p(z)$ and the integrated penalty function $\omega(u) = \int_0^\infty w(u, z - u)p(z)dz$ are twice continuously differentiable. See Li and Garrido (2005, Theorem 1). In particular,

\[ \frac{1}{2}\sigma^2\psi''_s(u) + cp'_s(u) + \lambda \int_0^u \psi_s(u-z)p(z)dz + \lambda \omega(u) = \lambda \psi_s(u) \]  \hspace{1cm} (2.6)

with the boundary condition $\psi_s(0) = 0$ while

\[ \frac{1}{2}\sigma^2\psi''_d(u) + cp'_d(u) + \lambda \int_0^u \psi_d(u-z)p(z)dz = \lambda \psi_d(u) \]  \hspace{1cm} (2.7)

with the boundary condition $\psi_d(0) = 1$. Note that (2.6) is of the form of (2.1) and can be turned into an integral equation according to Theorem 3. Moreover, (2.7) corresponds to the case of (2.1) with $\omega(u) = 0$. Thus Theorem 3 also guarantees that (2.7) can be transformed into the corresponding integral equation. Both integral equations are used as the input of the hybrid methodology. Most importantly, since the explicit form is known for $\psi_d(u)$ and $\psi_s(u)$ (with exponentially distributed claims), they are used as a benchmark for the proposed computational scheme.
Chapter 3

Numerical Method

Equations (2.5), (2.3) and (2.4) are Volterra integral equations of second kind with the following form:

$$\rho(u) - \int_0^u K(u, s)\rho(s)ds = \nu(u). \quad (3.1)$$

Note that the analytic solution of the integral equation for general $K$ and $\nu$ can rarely be obtained explicitly. As we have reviewed in Chapter 1, a popular approach in risk theory research is to investigate the behavior of $\rho(u)$ when $u \to \infty$. However, the asymptotic result (usually in terms of bounds) may not be useful in practice. For example, in evaluating the ultimate probability, one would consider $\psi(u)$ only in the range of $10^{-3}$ to $10^{-4}$ but not smaller. Thus we suggest a more versatile yet relatively accurate and practically useful numerical method.
3.1 Trapezoidal approximation

For the integral equation (3.1), we consider the trapezoidal approximation to the integral \( \int_0^u K(u, s)p(s)ds \). i.e., for \( i = 1, \ldots, n \),

\[
\int_0^{u_i} K(u_i, s)p(s)ds \approx \frac{hK(u_i, u_0)p(u_0)}{2} + h \sum_{j=1}^{i-1} K(u_i, u_j)p(u_j) + \frac{hK(u_i, u_i)p(u_i)}{2}
\]

where \( h = u/n, \ u_i = ih \). Taking such approximation as exact, we arrive at the following system of equations:

\[
\rho(u_i) - h \left[ \frac{K(u_i, u_0)p(u_0)}{2} + \sum_{j=1}^{i-1} K(u_i, u_j)p(u_j) + \frac{K(u_i, u_i)p(u_i)}{2} \right] = \nu(u_i)
\]

for \( i = 1, \ldots, n \).

Using the notation of \( v_i, \rho_0, \rho_i \) and \( K_{ij} \) for \( v(u_i), \rho'(0), \rho(u_i) \) and \( K(u_i, u_j) \), respectively, the above system can be rewritten as:

\[
-h \sum_{j=1}^{i-1} K_{ij} \rho_j + (1 - \frac{h}{2} K_{ii}) \rho_i = \nu_i + \frac{h}{2} K_{i0} \rho_0
\]

\[
= u_i \rho'_0 + (1 + \frac{2c}{\sigma^2}) \rho_0 + \tau_i + \frac{h}{2} K_{i0} \rho_0
\]

\[
= u_i \rho'_0 + d_i \rho_0 + \tau_i
\]

where \( d_i = (h/2) K_{i0} + 1 + 2cu_i/\sigma^2 \) and

\[
\tau_i = \begin{cases} 
0 \text{ in (2.5)} \\
-\frac{\lambda u^2}{\sigma^2} \int_0^{u_i} s^2 p(s)ds + \frac{\lambda}{\sigma^2} \int_0^{u_i} s^3 p(s)ds - \frac{\lambda u^3}{3\sigma^2} + \frac{\lambda}{\sigma^2} \int_0^{u_i} s^2 P(s)ds \text{ in (2.3)} \\
-\frac{\lambda u^2}{\sigma^2} \int_0^{u_i} s^2 p(s)ds - \frac{\lambda u^2}{3\sigma^2} \int_0^{u_i} s p(s)ds + \frac{\lambda u^3}{3\sigma^2} - \frac{\lambda}{\sigma^2} \int_0^{u_i} s^2 P(s)ds \text{ in (2.4)}
\end{cases}
\]

(3.2)
CHAPTER 3. NUMERICAL METHOD

Note that all \( \tau_i \) can be determined by the model specification. The matrix form of the above system of linear equations is

\[
Ax = b\rho_0 + d\rho_0 + \tau, \tag{3.3}
\]

where \( x = (\rho_1, \ldots, \rho_n)^T \) is the unknown vector,

\[
A = \begin{pmatrix}
1 - \frac{h}{2}K_{11} & & & \\
-\frac{h}{2}K_{21} & 1 & & \\
-\frac{h}{2}K_{31} & -\frac{h}{2}K_{32} & 1 & \\
& \vdots & \ddots & \ddots \\
-\frac{h}{2}K_{n1} & -\frac{h}{2}K_{n2} & \ldots & -\frac{h}{2}K_{n,n-1} + (1 - \frac{h}{2}K_{nn})
\end{pmatrix} \tag{3.4}
\]

which is lower triangular and

\[
b = \begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_n
\end{pmatrix}, \quad d = \begin{pmatrix}(h/2)K_{10} + 1 + 2c\rho_1/\sigma^2 \\
(h/2)K_{20} + 1 + 2c\rho_2/\sigma^2 \\
\vdots \\
(h/2)K_{n0} + 1 + 2c\rho_n/\sigma^2
\end{pmatrix}, \quad \tau = \begin{pmatrix}\tau_1 \\
\tau_2 \\
\vdots \\
\tau_n
\end{pmatrix}. \tag{3.5}
\]

Such lower triangular system can be easily solved if \( \rho_0 \) and \( \rho_0' \) are given.

3.2 Boundary Conditions

Although the standard Monte Carlo method or importance sampling method can be employed in estimating \( \rho_0 \) which is the ruin functionals for \( u = 0, \rho_0' \) is far from being trivial. One may try to tackle this by simulating \( \rho_\Delta \) with small positive \( \Delta \)
and use the approximation of $\rho'_0 \approx (\rho_\Delta - \rho_0)/\Delta$. Another way is simply making use of the above numerical scheme and turning $\rho'_0 = (\rho_1 - \rho_0)/h$. Both approaches could work nicely if the range of $u$ needed is close to zero. However, substantial error for $\rho(u)$ is seen in Wong, Ho, Hu and Liu (2006) when $u$ is close to the zone with $\psi(u) \in [10^{-4}, 10^{-3}]$. In fact, for many cases, such zone of interest could make $u$ significantly away from zero. To address such issue, we follow the methodology of Wong, Ho, Hu and Liu (2006) to bound the error when $u = 0$ and at the zone of interest (which is close to $u_n$) by setting $\rho'_0$ as unknown and simulating $\rho_0$ and $\rho_n$ by importance sampling scheme in the next chapter.

Wong, Ho, Hu and Liu (2006) essentially moves everything related to $\rho_n$ to the right hand side and everything related to $\rho'_0$ to the left hand side of the equations. That is, for $i = 1, \ldots, n - 1$,

\[-h \sum_{j=1}^{i-1} K_{ij} \rho_j + (1 - \frac{h}{2} K_{ii}) \rho_i - u_i \rho'_0 = (1 + \frac{2cu_i}{\sigma^2}) \rho_0 + \tau_i + \frac{h}{2} K_{i0} \rho_0 \]

\[= d_i \rho_0 + \tau_i \]

and for $i = n$,

\[-h \sum_{j=1}^{n-1} K_{nj} \rho_j - u_n \rho'_0 = (1 + \frac{2cu_n}{\sigma^2}) \rho_0 + \tau_n + \frac{h}{2} K_{n0} \rho_0 - (1 - \frac{h}{2} K_{nn}) \rho_n \]

\[= d_n \rho_0 + e_n \rho_n + \tau_n \]

where $e_n = - (1 - \frac{h}{2} K_{nn})$. Putting them into matrix form leads to

$$By = d\rho_0 + e\rho_n + \tau$$  \hspace{1cm} (3.6)
where
\[
B = \begin{pmatrix}
1 - \frac{h}{2} K_{11} & -u_1 \\
-hK_{21} + (1 - \frac{h}{2} K_{22}) & -u_2 \\
-hK_{31} - hK_{32} + (1 - \frac{h}{2} K_{33}) & -u_3 \\
\vdots & \vdots \\
-hK_{n1} - hK_{n2} + \ldots -hK_{n,n-1} & -u_n
\end{pmatrix}, \quad (3.7)
\]

\(y = (\rho_1, \ldots, \rho_{n-1}, \rho_0)^T\) is the unknown vector, \(d\) is shown in Equation (3.5) and \(e = (0, 0, \ldots, e_n)^T\).

Although \(B\) is no longer lower triangular in the new system, (3.6) can still be solved very efficiently for \(n = 500\) and \(u_{\text{max}} = 30\) in Chapter 5. In solving the Volterra integral equations (2.5), (2.3) and (2.4), the unknown vectors \(y\) to be solved in the numerical system (3.6) are summarized in Table (3.1), whereas \(B\) is of the form in Equation (3.7), \(d\), \(e\) and \(\tau\) are of the form in Equation (3.5).

<table>
<thead>
<tr>
<th>Equation to be solved</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.5) (H/n), (H/n_i)</td>
<td>((\phi_1, \ldots, \phi_{n-1}, \phi_0)^T)</td>
</tr>
<tr>
<td>(2.3) (W_1, \ldots, W_{n-1}, W_0)</td>
<td>((W_1, \ldots, W_{n-1}, W_0)^T)</td>
</tr>
<tr>
<td>(2.4) (W_1, \ldots, W_{n-1}, W_0)</td>
<td>((W_1, \ldots, W_{n-1}, W_0)^T)</td>
</tr>
</tbody>
</table>

Table 3.1: The unknown vector \(y\) for the system of linear equations.
Chapter 4

Importance Sampling

To solve the systems of linear equation (3.6) in Chapter 3, the boundary values, $\rho_0$ and $\rho_n$, are the only missing links because $B$, $d$, $e$ and $\tau$ are known. In fact, unlike the case of using $T = \inf\{t \geq 0 : R(t) < 0\}$ as the time of ruin which requires both $\rho_0$ and $\rho_n$, $\rho_0$ is known for the case of using $S = \inf\{t \geq 0 : R(t) \leq 0\}$ as the time of ruin. In this chapter, a detailed discussion of how to estimate these boundary conditions is delivered.

The most primitive way is to employ crude Monte Carlo (CMC) method which means the sample paths of the risk surplus process (1.3) with corresponding initial reserve is simulated repeatedly with $t \in [0, \tau]$ where $\tau$ is "sufficiently" large. For the ultimate ruin probability estimation, the estimates of $\rho_0$ is the corresponding proportion of simulated paths with $R(0) = 0$ and $R(t) < 0$ with $t \leq \tau$. For other ruin functionals, the estimate of $\rho_0$ is the corresponding average. Similarly, $\rho_n$ can be estimated by setting $R(0) = u_n$.

Note that since the safety loading is set to be positive, $\Pr\{T = \infty| R(0) = u\} > 0$.
CHAPTER 4. IMPORTANCE SAMPLING

for any $u$. Thus one can see that the choice of any finite $\tau$ is difficult and critical. If $\tau$ is too small, the estimates of $\rho_0$ and $\rho_n$ are severely biased. On the other hand, if $\tau$ is too large, the computational time could be extremely long. To address the issue, we suggest using the idea of importance sampling. That is, instead of simulating the original risk process, another trial process with finite time of ruin is simulated. The estimate is then the average of weights which are the Randon-Nikodym derivatives of the original density with respect to the trial density computed at the simulated value. However, although the importance sampling method gives unbiased estimate, the standard error is inflated by a factor which is a function of similarity between the original process and the trial process. An optimal choice of trial process would ensure such factor to be closest to 1.

There are many academic articles discussing various possible choices in trial process. For example, in Chapter 9 of Rolski, Schmidli, Schmidt and Teugels (1999), the method of using exponential martingale is studied and some analysis of variance reduction is presented. In this thesis, we employ the technology developed by Paulsen and Rasmussen (2003). Their theory is developed under the semi-martingale framework and their optimality in the choice of trial process is measured via a concept called “strongly efficient” which means the magnification in standard error is always within a tolerable range no matter how small the target ruin probability is. Since their argument is clearly stated in their paper, here we again only outline the recipe of Paulsen and Rasmussen (2003).

Let $Z$ be a random variable having the same distribution as the claim and let

$$M_Z(\beta) = E[e^{\beta Z}] = \int_0^\infty e^{\beta z} p(z)dz$$

be the corresponding moment generating function.
CHAPTER 4. IMPORTANCE SAMPLING

Consider the trial surplus process $R'(t)$ of the form

$$R'(t) = u + c't + \sigma B'_t - \sum_{i=1}^{N'_t} Z'_i, \ t \geq 0 \quad (4.1)$$

where $c' > 0$ is the premium rate, $\{B'_t, t \geq 0\}$ is the standard Brownian motion under the trial measure, $\{N'_t, t \geq 0\}$ is a Poisson counting process with intensity $\lambda'$ and $N'_t$ denotes the number of claims in the time interval $[0, t]$, and $Z'_i$ denotes the $i$th claim size whose density is given by $p'(z)$ with $E[Z'_i] = \eta > 0$. $Z'_i$ are i.i.d. and $\{B'_t, t \geq 0\}$, $\{N'_t, t \geq 0\}$ and $\{Z'_k, k \geq 1\}$ are independent.

According to Section 4 of Paulsen and Rasmussen (2003), the optimal choice of parameters are given by $c' = c - \sigma^2 \beta'$ and $\lambda' = \lambda + c\beta' - \frac{1}{2}\sigma^2 \beta'^2$ with $\beta'$ being the positive solution of the equation

$$\lambda[M_Z(\beta') - 1] - c\beta + \frac{1}{2}\sigma^2 \beta'^2 = 0. \quad (4.2)$$

Also, the claim size density of the optimal trial process is “twisted” from that of the original process in the form of:

$$p'(z) = \frac{e^{\beta' z}}{M_Z(\beta')} p(z). \quad (4.3)$$

One can easily check that the safety loading under the trial process is given by $c' - \lambda' \eta \leq 0$. Such fact ensures $Pr'(T' < \infty) = 1$ where $Pr'$ is the probability measure under the trial process. That implies ultimate ruin occurs with certainty for the optimal trial process. By using the result in Section 5 of Paulsen and Rasmussen (2003), one can show that the optimal Radon-Nikodym derivative is given by $e^{-\beta'u} e^{\beta'R(T)}$. That is, the ultimate probability of ruin is given by:

$$\psi(u) = Pr(T < \infty | R(0) = u) = E'[\frac{e^{\beta'R(T')}}{e^{\beta'u}}]$$
where \( E' \) is the expected value under the optimal trial measure.

Thus, theoretically, one can simulate from the trial model (4.1) repeatedly \( M \) times where \( M \) is a large positive integer and estimate \( \hat{\psi}(u) \) by

\[
\hat{\psi}(u) = \frac{1}{M} \sum_{r=1}^{M} \frac{e^{\beta R(T'_r)}}{e^{\beta u}}
\]

where \( T'_r \) is the time of ruin of \( r \)th simulated replicate of the trial process. Also, the central limit theorem ensures the following holds:

\[
\sqrt{M}(\hat{\psi}(u) - \psi(u)) \xrightarrow{d} N(0, \text{Var}(\frac{e^{\beta R(T'_r)}}{e^{\beta u}}))
\]

which could help us to assess the approximation error.

### 4.1 Simulation Recipe

The following is the recipe of simulating the sample path of the trial process. To avoid complex notation, all parameters and random variables are assumed to be under the trial measure although no superscript “prime” is used.

1. For a large \( n \), simulate the i.i.d. interarrival time \( T_i \) from exponential distribution with mean \( 1/\lambda \) for \( i = 1, \ldots, n \).

2. For \( i = 1, \ldots, n \), simulate i.i.d. \( Z_i \) from the “twisted” density \( p(z) \).

3. For a fixed small positive number \( dt \), let \( n_i = \lfloor T_i/dt \rfloor \) which is the floor of \( T_i/dt \).

Let \( t'_i = T_i - n_i(dt) \). For \( k = 1, \ldots, n_i \), we simulate i.i.d. \( D_{ik} \sim N(0, dt) \) and
CHAPTER 4. IMPORTANCE SAMPLING

$E_i \sim N(0, t^*_i)$. The Brownian motion between the $(i - 1)^{th}$ and the $i^{th}$ claims is approximated by

$$\Delta B_i = \sum_{k=1}^{n_i} D_{ik} + E_i.$$ 

Thus the surplus process is approximated by

$$R(T_1 + \cdots + T_i) = u + c(T_1 + \cdots + T_i) + \sigma \sum_{j=1}^{i} \Delta B_i - \sum_{j=1}^{i} Z'_i$$

at the time of claims and is approximated by

$$R(T_1 + \cdots + T_i + l(dt))$$

$$= u + c(T_1 + \cdots + T_i + l(dt)) + \sigma \sum_{j=1}^{i} \Delta B_i - \sum_{j=1}^{i} Z'_i + \sigma \sum_{k=1}^{l} D_{i+1,k},$$

if $l < n_i$. $R(t)$ is approximated by linear interpolation for other cases.

4. The time of ruin is returned as $\inf \{ t : R(t) < 0 \}$ or $\inf \{ t : R(t) \leq 0 \}$ depends on the choice of model.

4.2 Discussion

Note that the trial density of Paulsen and Rasmussen (2003) is optimal only for the computation of the ruin probability. For other ruin functionals, the optimal choice is far from being clear. Nevertheless, based upon our experience which is going to be summarized in the next chapter, we find that the aforementioned trial density actually works reasonably well for ruin functionals other than the ruin probability. That is, the expected surplus before ruin and the expected deficit at ruin are estimated by
\[ \hat{\rho}(u_n) = \frac{1}{M} \sum_{r=1}^{M} R(T'_r) \frac{e^{\beta u_n} R(T'_r)}{e^{\beta u_n}} \]

and

\[ \hat{\rho}(u_n) = \frac{1}{M} \sum_{r=1}^{M} |R(T'_r)| \frac{e^{\beta u_n} R(T'_r)}{e^{\beta u_n}} \]

with initial reserve being set at \( u = u_n \).
Chapter 5

Numerical Examples

In this chapter, we illustrate the performance of our methodology in computing various ruin functions as a function of initial reserve $u$. The first section is devoted to the study of the probability of ruin caused by the Brownian oscillation and that caused by the claim. The results of the hybrid methodology are compared with the explicit formulas which are known in this case. In particular, we highlight the impact of boundary conditions on the estimate of ruin probabilities and how to apply the hybrid approach in a novel way to minimize the error. The second section computes different ruin functions by hybrid method and comments on how to validate the asymptotic results derived in Chiu and Yin (2003) by using the estimates.

5.1 Probabilities of ruin: Oscillation and claim

To evaluate our hybrid methodology, the diffusion-perturbed model in this section is assumed to have exponential claims because the explicit formulas of Dufresne and
Gerber (1991) for the ruin probability can be used as a benchmark. As in Chapter 1, for the time of ruin $S = \inf\{t \geq 0 : R(t) \leq 0\}$, we denote the corresponding ruin probabilities caused by Brownian oscillation and that caused by claim as $\psi_d(u)$ and $\psi_s(u)$, respectively. The ruin probability is simply $\psi(u) = \psi_s(u) + \psi_d(u)$.

Suppose the claims are exponentially distributed with common mean $\mu$. According to Dufresne and Gerber (1991, equation (6.4) and (6.17))

$$\psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u}$$  \hspace{1cm} (5.1)

$$\psi_d(u) = C'_1 e^{-r_1 u} + C'_2 e^{-r_2 u}$$  \hspace{1cm} (5.2)

where $r_1, r_2$ are the solutions obtained by solving $\beta$ in equation (4.2). Using the boundary conditions, the constants are determined as:

$$C_1 = \frac{r_1 - \frac{1}{\mu} \frac{r_2}{r_1 - r_2}}{\frac{1}{\mu} \frac{r_2}{r_1 - r_2}}$$

$$C_2 = \frac{r_2 - \frac{1}{\mu} \frac{r_1}{r_2 - r_1}}{\frac{1}{\mu} \frac{r_1}{r_2 - r_1}}$$

$$C'_1 = r_1 C_1 \frac{\sigma^2/2}{c - \lambda \mu}$$

$$C'_2 = r_2 C_2 \frac{\sigma^2/2}{c - \lambda \mu}$$

Note that $\psi_s(x)$ can be computed easily by $\psi(x) - \psi_d(x)$.

To compare our approach with the above explicit formulas, the parameters are set as: $\sigma = 1, c = 3, \lambda = 2$ and $\mu = 1$ such that the safety loading $c - \lambda \mu = 1$ is positive and it ensures $\psi(u) < 1$ for all $u$. Since the claims are assumed to be exponentially distributed with mean $\mu$, $M_z(\beta) = 1/(1 - \mu \beta)$. The parameters of the optimal trial process can be obtained by solving the equation (4.2) which is required
in getting the explicit solution. For our configuration of parameters, \( \beta' = 0.2984 \).

The corresponding twisted density for claim is then given by:

\[
p'(z) = \frac{e^{\beta'z}}{1/(1 - \mu \beta')} \left( \frac{1}{\mu} e^{-z/\mu} \right) = (1/\mu - \beta) e^{-(1/\mu - \beta)z}
\]

which is the density of an exponential random variable with mean \( \eta = \frac{1}{\mu - \beta} \). For our set of parameter values, \( \eta = 1.4254 \).

Using the formulas presented in Chapter 4, the other trial parameters are computed as:

\[
c' = c - \sigma^2 \beta' = 2.7016,
\]

\[
\lambda' = \lambda + c \beta' - \frac{1}{2} \sigma^2 \beta'^2 = 2.8508.
\]

Note that the safety loading under trial measure is \( c' - \lambda' \eta = -1.3619 < 0 \) and it ensures \( \Pr'(S < \infty | R(0) = u) = 1 \) for all \( u \geq 0 \). In this study, \( M \) is set to be \( 10^5 \) and \( dt \) is set to be 0.01 for the Brownian motion simulation. We perform importance sampling for ruin probabilities computation at \( u = 30 \). The results are listed Table in (5.1) where we also compare the estimate with exact probabilities by using the absolute relative error. The absolute relative error is defined as:

\[
\text{Absolute Relative Error} = \left| \frac{\text{Estimate} - \text{Exact value}}{\text{Exact value}} \right|.
\]

For our parameter configuration with \( u = 30 \), one can see that the compound Poisson claim is a much stronger cause for ruin than the Brownian oscillation. Thus even though the importance sampling estimate of \( \psi_d \) does not perform as well as that of \( \psi_s \), the relative error of the overall ruin probability \( \psi \) is still very small.
Table 5.1: Estimates of ruin probabilities at $u = 30$ obtained from importance sampling, exact values from Equations (5.1) and (5.2) and the corresponding absolute relative error, $S = \inf \{ t \geq 0 : R(t) \leq 0 \}$

(1.3%). We expect the performance of importance sampling estimate of $\psi_d$ could be improved by choosing a smaller $dt$ or a more effective discrete-time approximation of Brownian motion.

Since $\psi(30) = 9.496 \times 10^{-5}$ for our parameter configuration and $\psi$ is decreasing in $u$, $[0, 30]$ should cover the range of $\psi(u) \in [10^{-3}, 10^{-4}]$ which is the zone of interest. Thus, setting $u_{\text{max}} = 30$ for the numerical integration routine presented in Chapter 3 is legitimate. $h$ is specified to be $30/500 = 0.06$ for the trapezoidal approximation. Since $\psi_d(0) = 1$ and $\psi_s(0) = 0$, the routine of Section 3.1 suggests estimating $\psi_d'(0)$ and $\psi_s'(0)$ by

$$
\hat{\psi}_d'(0) = \frac{\hat{\psi}_d(\Delta) - \psi_d(0)}{\Delta} = \frac{\psi_d(\Delta) - 1}{\Delta}
$$

$$
\hat{\psi}_s'(0) = \frac{\hat{\psi}_s(\Delta) - \psi_s(0)}{\Delta} = \frac{\psi_s(\Delta)}{\Delta},
$$

respectively, with a small positive $\Delta$ and use them as the input of the lower triangular system (3.3). In Section 3.2, such approach is criticized and our proposed approach is using the importance sampling estimates of $\psi_d(30)$ and $\psi_s(30)$ and solving the
non-triangular system (3.6). For the first approach, we set $\Delta = 10^{-2}$ and obtain $\hat{\psi}_d'(0) = -36.62$ and $\hat{\psi}_s'(0) = 25.46$. For our proposed approach, we simply use the importance sampling estimates in Table (5.1) as input.

Figures 5.1, 5.2 and 5.3 are the results for $\psi_s$, $\psi_d$ and $\psi$, respectively. The corresponding absolute relative errors are plotted on Figures 5.4, 5.5 and 5.6. The right panel is the comparison between the proposed approach and the exact solution. It is very difficult to see the difference between them without reading the graphs of the absolute relative errors. The maximum relative errors of all three ruin probabilities are at $u = 30$ and among them, the largest is 29.53% which corresponds to $\psi_d(30)$. The left panel is used to compare the approach of using $\hat{\psi}_d'(0)$ and $\hat{\psi}_s'(0)$ with the exact solution. The performance of that is substantially worse than that of our proposal and justifies our remarks on the issues of boundary conditions in Section 3.2.

5.2 Comparison with the asymptotic results

The diffusion-perturbed surplus model with $T = \inf\{t : R(t) < 0\}$ as the time of ruin has no known explicit formula even when the claim amounts are exponentially distributed. Under this framework, Chiu and Yin (2003) not only derive the integro-differential equation that we reviewed in Chapter 2 but also obtain the following asymptotic results.

Theorem 5 Consider the diffusion-perturbed surplus model. Let $\tilde{p}(\beta) = M_2(-\beta)$ be the Laplace transform of the claim distribution and let $\beta_1$ and $-\beta_2$ respectively be
CHAPTER 5. NUMERICAL EXAMPLES

1. Use $V_{n}$ as input
2. Use $V_{n}$ as input
3. Explicit

Figure 5.1: Probabilities of ruin caused by claims for different initial reserve $u$,
$S = \inf\{t \geq 0 : R(t) \leq 0\}$

1. Use $V_{n}$ as input
2. Use $V_{n}$ as input
3. Explicit

Figure 5.2: Probabilities of ruin caused by oscillation for different initial reserve $u$,
$S = \inf\{t \geq 0 : R(t) \leq 0\}$
Figure 5.3: Total ruin probabilities for different initial reserve $u$, $S = \inf\{t \geq 0 : R(t) \leq 0\}$

Figure 5.4: Absolute relative error of probabilities of ruin caused by claims between the hybrid method and the explicit formula for different initial reserve $u$, $S = \inf\{t \geq 0 : R(t) \leq 0\}$
Figure 5.5: Absolute relative error of probabilities of ruin caused by oscillation between the hybrid method and the explicit formula for different initial reserve $u$, $S = \inf\{t \geq 0 : R(t) \leq 0\}$

Figure 5.6: Absolute relative error of total ruin probabilities between the hybrid method and the explicit formula for different initial reserve $u$, $S = \inf\{t \geq 0 : R(t) \leq 0\}$
the unique positive and negative roots of

\[ \frac{1}{2} \sigma^2 \beta^2 + \lambda \hat{p}(\beta) + c \beta - \lambda = 0. \]  

(5.3)

If \( \hat{p}'(-\beta_2) < \infty, c + \frac{1}{2} \sigma^2 (\beta_1 - \beta_2) > 0 \) and \( \lim_{y \to \infty} e^{\beta_2 y} \int_y^\infty w(y, z - y)p(z)dz = 0 \), then

\[
\lim_{u \to \infty} e^{\beta_2 u} W_c(u, 0, w)
= \frac{-(1/2)\sigma^2 w(0,0)(\beta_1 + \beta_2)}{\lambda \hat{p}'(-\beta_2) + c - \sigma^2 \beta_2}
- \frac{\lambda \int_0^\infty \int_0^\infty (e^{\beta_2 z_1} - e^{-\beta_1 z_1})w(z_1, z_2)p(z_1 + z_2)dz_2 dz_1}{\lambda \hat{p}'(-\beta_2) + c - \sigma^2 \beta_2}.
\]  

(5.4)

The proof is listed in Chiu and Yin (2003, Theorem 3.3) and is not going to be reproduced here. Nevertheless, we would like to highlight that the result enables us to determine the asymptotic behavior of the ruin probability, the expected surplus before ruin and the expected deficit at ruin by setting \( w(x, y) = 1, x \) or \( y \) for all \( (x, y) \) in the first quadrant of \( \mathbb{R}^2 \), respectively. In particular, for exponential claim \( p(z) = \exp(-z/\mu)/\mu \) for all \( z > 0 \), the Laplace transform is given by \( \hat{p}(\beta) = M_2(-\beta) = 1/(1 + \mu \beta) \). Taking \( w(x, y) = 1 \) for all \( (x, y) \) gives

\[
\lim_{u \to \infty} e^{\beta_2 u} \psi(u) = \frac{-\frac{1}{2} \sigma^2 (\beta_1 + \beta_2) - \lambda \mu (\frac{1}{1 - \mu \beta_2} - \frac{1}{1 + \mu \beta_1})}{\lambda \mu (1 - \mu \beta_2)^2 + c - \sigma^2 \beta_2},
\]

or when \( u \) is large,

\[
\psi(u) \approx e^{-\beta_2 u} \frac{-\frac{1}{2} \sigma^2 (\beta_1 + \beta_2) - \lambda \mu (\frac{1}{1 - \mu \beta_2} - \frac{1}{1 + \mu \beta_1})}{-\lambda \mu (1 - \mu \beta_2)^2 + c - \sigma^2 \beta_2} = C_\psi e^{-\beta_2 u}
\]
Similarly, taking \( w(x, y) = x \) in Theorem 5 gives the asymptotic behavior of the expected surplus before ruin in the following form:

\[
\lim_{u \to \infty} e^{\beta_2 u} \rho(u) = \frac{-\frac{\lambda}{\mu - \beta_2} + \frac{\lambda}{\beta_1 + \frac{1}{\mu}}}{-\frac{\lambda \mu}{(1 - \mu \beta_2)^2} + c - \sigma^2 \beta_2}
\]

where \( \rho(u) \) is the expected surplus before ruin whose initial reserve is \( u \). That is equivalent to

\[
\rho(u) \approx e^{-\beta_2 u} \frac{-\frac{\lambda}{\mu - \beta_2} + \frac{\lambda}{\beta_1 + \frac{1}{\mu}}}{-\frac{\lambda \mu}{(1 - \mu \beta_2)^2} + c - \sigma^2 \beta_2} = C e^{-\beta_2 u}
\]  

(5.5) when \( u \) is large.

Finally, taking \( w(x, y) = y \) in Theorem 5 gives the asymptotic behavior of the expected deficit after ruin in the following form:

\[
\lim_{u \to \infty} e^{\beta_2 u} \xi(u) = \frac{-\frac{\lambda \mu}{\mu - \beta_2} + \frac{\lambda \mu}{\beta_1 + \frac{1}{\mu}}}{-\frac{\lambda \mu}{(1 - \mu \beta_2)^2} + c - \sigma^2 \beta_2}
\]

where \( \xi(u) \) is the expected deficit after ruin whose initial reserve is \( u \). Again, that is equivalent to

\[
\xi(u) \approx e^{-\beta_2 u} \frac{-\frac{\lambda \mu}{\mu - \beta_2} + \frac{\lambda \mu}{\beta_1 + \frac{1}{\mu}}}{-\frac{\lambda \mu}{(1 - \mu \beta_2)^2} + c - \sigma^2 \beta_2} = C \xi e^{-\beta_2 u}
\]

when \( u \) is large.

Although all of the above results provide simple asymptotic formulas for the corresponding ruin functions, how would these results perform for moderate \( u \) is unknown. Besides, as we have iterated many times, the practical range is where \( \psi(u) \in [10^{-4}, 10^{-3}] \). One need to know if the asymptotic results hold in that zone. Furthermore, if the insurer could afford risk such that \( \psi(u) \in [10^{-2}, 10^{-1}] \) is considered, the above asymptotic statements may not be able to offer much help.
5.2.1 Ruin Probability

In this section, the above listed issues are addressed by comparing our numerical results with the asymptotic properties for the diffusion-perturbed surplus model with the parameters specified in the last section. To begin with, we first compare the performances in terms of ruin probability. As we have already illustrated in the last section, the numerical scheme that requires estimation of $\psi'(0)$ gives huge error. Thus, we propose the non-triangular numerical scheme which requires the input of $\psi(0)$ and $\psi(u_{\text{max}})$. By using the optimal choice of trial process presented in the last section, the importance sampling estimates of $\psi(0)$ and $\psi(u_{\text{max}})$ are $\hat{\psi}(0) = 0.8997$ and $\hat{\psi}(u_{\text{max}}) = 9.362 \times 10^{-5}$ where $u_{\text{max}} = 30$ and the number of replicates is $M = 10^5$. Also, note that $C_\psi = 0.7343$ and $\beta_2 = 0.2984$ under current configuration.
Figure 5.8: Absolute relative error of ruin probabilities of Exponential claims between the hybrid method and the asymptotic behaviour (Upper) $0 \leq u \leq 30$, (Middle) $5 \leq u \leq 20$, (Lower) $20 \leq u \leq 30$, $T = \inf\{t : R(t) < 0\}$

Taking the importance sampling estimates as the starting values and setting the grid size $h = 0.005$, the estimate of $\psi(u)$ for $u \in [0, 30]$ is computed by our numerical scheme and is plotted in Figure 5.7 together with the asymptotic ruin probability. It is easy to see that the curves overlap except for a small zone close to 0. To check the difference more carefully, the absolute relative error is plotted in Figure 5.8 where

$$\text{Absolute relative error} = \frac{|\text{Estimate} - \text{Asymptotic value}|}{\text{Asymptotic value}}.$$
Thus for \( u \) close to 0, the estimate, which is denoted by \( \hat{\psi} \), is close to be 15\% of the asymptotic value. While for \( u > 1 \), the difference would be less than 5\%. All these show that for the chosen parameter values, the asymptotic ruin probability offers a satisfactory approximation for \( u > 1 \) or \( 0 < \psi(u) < 0.6 \). That is, the initial reserve for \( \psi(u) = \pi \) with \( \pi < 0.6 \) under our current model could be approximated by:

\[
\log C_\psi - \beta_2 u = \log \pi \Rightarrow u = (\log C_\psi - \log \pi)/\beta_2
\]

Note that such assertive statement can only be verified by the application of our proposed methodology.

### 5.2.2 Surplus before ruin

Under the current parameter configuration, \( C_p = 1.5151 \) and the asymptotic behavior is given by

\[
\rho(u) \approx 1.5151 e^{-0.2984u}.
\]

Such asymptotic statement is monotonically decreasing and is legitimate when \( u \) is sufficiently large. To check the zone of validity for the above statement, we employ our proposed hybrid methodology and apply importance sampling for surplus before ruin based upon the trial parameter computed in the previous section. The boundary estimate of \( \rho(0) \) and \( \rho(30) \), which are denoted by \( \hat{\rho}(0) \) and \( \hat{\rho}(30) \), are 0.2347 and \( 1.9836 \times 10^{-4} \), respectively. Note that the absolute relative differences at 0 and 30 can be computed as:

Absolute relative difference at 0 = \[
\frac{|0.2347 - 1.5151|}{1.5151} = 84.51\%
\]

Absolute relative difference at 30 = \[
\frac{|1.9836 \times 10^{-4} - 1.5151 e^{-0.2984 \times 30}|}{1.5151 e^{-0.2984 \times 30}} = 1.23\%
\]
Figure 5.9: (Upper) Expected surplus immediately before ruin, (Lower) Expected deficit at ruin for different initial reserve $u$, $T = \inf\{t : R(t) < 0\}$

From Figure 5.9, one again see that $\hat{\rho}(u)$ and $C_\rho e^{-\beta_2 u}$ differs from each other mainly when $u$ is close to 0. Figure 5.10 shows the absolute relative difference for $u \in [0, 30]$ and gives essentially the same message.

To further analyze the approximation, consider the special case of $\psi(u) = 10^{-3}$. $u$ is roughly 22 from the curve of $\hat{\psi}$. The asymptotic result gives the expected surplus immediately before ruin $C_\rho e^{-\beta_2 \times 22} = 0.00214$ while our methodology gives $\hat{\rho}(22) = 0.00217$. The corresponding absolute relative difference at 22 is $|0.00217 - 0.00214|/0.00214 = 1.65\%$. The analysis again confirms that the asymptotic result $C_\rho e^{-\beta_2 u}$ is a good approximation to $\rho(u)$ for the range of $u$ where $\psi(u) < 10^{-3}$. 
5.2.3 Deficit after ruin

$C_\xi = 0.6247$ for the given parameter configuration. Thus the asymptotic behavior of the expected deficit after ruin is given by

$$\xi(u) \approx 0.6247e^{-0.2984u}.$$ 

The importance sampling estimate of $\hat{\xi}(0)$ and $\hat{\xi}(30)$ are

$$\hat{\xi}(0) = 0.2359 \text{ and } \hat{\xi}(30) = 8.3928 \times 10^{-5},$$

respectively. The absolute relative differences at 0 and 30 can then be computed as:

- Absolute relative difference at 0 = \frac{|0.2359 - 0.6247|}{0.6247} = 62.24%
- Absolute relative difference at 30 = \frac{|8.3928 \times 10^{-5} - 0.6247e^{-0.2984 \times 30}|}{0.6247e^{-0.2984 \times 30}} = 3.88%

By observing Figure 5.9, $\hat{\xi}(u)$ and $C_\xi e^{-\beta_2 u}$ differs from each other when $u$ is close to 0. Similar message is delivered by Figure 5.11 which shows the absolute relative difference for $u \in [0, 30]$.

Again we consider $u = 22$ which makes $\psi(u) = 10^{-3}$. The asymptotic result gives the expected deficit immediately after ruin as $C_\xi e^{-\beta_2 \times 22} = 8.8084 \times 10^{-4}$ while our methodology gives $\hat{\xi}(22) = 9.1708 \times 10^{-4}$. The corresponding absolute relative difference at 22 is $|\hat{\xi}(22) - 8.8084 \times 10^{-4}| / 8.8084 \times 10^{-4} = 4.11\%$. All these reinforce the idea that the asymptotic formulas work satisfactorily within the practical zone.
Figure 5.10: Absolute relative error of expected surplus immediately before ruin of Exponential claims between the hybrid method and the asymptotic behaviour (Upper) $0 \leq u \leq 30$, (Middle) $5 \leq u \leq 20$, (Lower) $20 \leq u \leq 30$, $T = \inf\{t : R(t) < 0\}$
Figure 5.11: Absolute relative error of expected deficit at ruin of Exponential claims between the hybrid method and the asymptotic behaviour (Upper) $0 \leq u \leq 30$, (Middle) $5 \leq u \leq 20$, (Lower) $20 \leq u \leq 30$, $T = \inf\{t : R(t) < 0\}$
Chapter 6

Conclusion

While Wong, Ho, Hu, Liu (2006) only focus at ruin probability for an Erlang(2) driven model, this thesis applies the hybrid methodology to evaluate a wider class of ruin functions which includes the ruin probability, the expected surplus before ruin and the expected deficit at ruin for the diffusion-perturbed model. Moreover, instead of choosing the trial process for the importance sampling step by intuition, our choice is optimal by Paulsen and Rasmussen (2003). The simulation results in Chapter 5 confirm that our methodology is almost as good as the exact formulas. We also validate the use of the asymptotic results derived by Chiu and Yin (2003) when $u$ is only moderate such that $\psi(u) \in [10^{-4}, 10^{-3}]$.

An important by product of this thesis is the effective simulation of the distribution of surplus before ruin and the deficit at ruin. It should be noted that although the previous literature focuses at the expected values of the ruin functionals, the percentiles of the functionals are more relevant from the risk management perspective. The integro-differential equation approach is surely not the most effective way
in computing the percentiles and the importance sampling procedure in this thesis has been shown to be computationally efficient.

Other than the above contributions, there are many future research directions generated from this thesis. So far in our example we obtain \( u_{\text{max}} = 30 \) such that \( \psi(30) < 10^{-4} \) by trial and error. Finding \( u_{\text{max}} \) by a systematic approach is an essential step in making the methodology applicable in the insurance area. Another essential topic is the error analysis. In Chapter 4, we study the errors created by importance sampling by using central limit theorem. A full analysis should also include the discretization error from the trapezoidal approximation. Apart from issues related to our model, the method in this thesis can also be applied to more general model, such as the renewal model perturbed by diffusion. The key step of the method in this thesis is to solving the integro-differential equation. For many other models such as renewal models, the integro-differential equation satisfied by the ruin related functions can be obtained in a similar way and the corresponding ruin functionals could be studied in a similar manner. Even though there are many loose ends left to be tied up by other researchers, we still hope this thesis could shed some lights on how to apply the hybrid methodology in risk theory.
References


