A Numerical Solution for Solving Ruin Probability of the Classical Model with Two Classes of Correlated Claims

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A Thesis Submitted in Partial Fulfilment of the Requirements for the Degree of Master of Philosophy in Statistics

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DECLARATION

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.
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ABSTRACT

In risk theory, the classical model studies the ruin phenomenon caused by the claims of a single product. In reality, the insurers could have more than one products and their bankruptcy is related to the correlated features of the claim mechanism of many products. In this thesis, we focus at investigating the model proposed by Yuen, Guo and Wu (2002) in which the claim arrivals are generated by correlated renewal processes. The model could then be transformed to the case of having independent claim sizes with the number of claims being Poisson and Erlang processes. By using such transformation, Yuen, Guo and Wu (2002) derive the explicit formula of ruin probability under assumption that the claim sizes are exponentially distributed and some asymptotic results of the ruin probability under some relatively more general conditions. In this thesis, we make use of the hybrid numerical methodology proposed by Wong, Ho, Hu and Liu (2006) to compute the ruin probability under
various distribution assumptions. From our simulation study, we find that our pro-
posed methodology compares favorably with respect to the exact formula of Yuen,
Guo and Wu (2002) for exponential claims. Moreover, our methodology shows that
their asymptotic upper bound is not effective if the initial reserve is large.
摘要

Contents

1 Introduction 1
   1.1 Risk Theory ................................................. 1
   1.2 Hybrid Numerical Scheme ................................. 3

2 The Model 5
   2.1 Model ....................................................... 5
   2.2 Integro-Differential Equations .......................... 8
   2.3 Explicit Formulas and Asymptotic Properties ............ 13

3 Numerical Method 16
   3.1 From Integro-Differential Equations to Integral Equations .. 17
   3.2 From Integral Equations to Linear Equations ............... 19
   3.3 Boundary Conditions ...................................... 20
   3.4 Importance Sampling ..................................... 23
4 Numerical Study

4.1 Exponential Claims with Equal Means ................................ 28
  4.1.1 Importance Sampling ............................................. 28
  4.1.2 System of Linear Equations ................................. 31

4.2 Exponential Claims with Unequal Means ......................... 32

5 Conclusion .................................................................. 40

Bibliography ................................................................ 43
List of Figures

4.1 Ruin probability $\psi(u)$ under different method with $u_{max} = 20$  . . . 36
4.2 Absolute relative error under different method with $u_{max} = 20$  . . . 37
4.3 Asymptotic upper bound for $\psi(u) + \psi_1(u)$ with $u_{max} = 35$  . . . 38
4.4 Absolute relative error between the solution and the upper bound  
under the unequal means case  . . . . . . . . . . . . . . . . . . . . . . . . . 39
List of Tables

4.1 Ruin probability $\psi(u_n)$ obtained from explicit formula, importance sampling and crude monte carlo method of Yuen, Guo and Wu (2002) 30

4.2 Absolute relative error for importance sampling and the crude monte carlo method with the exact solution 30

4.3 Ruin probability $\psi(u)$ from importance sampling and the crude monte carlo methods by Yuen, Guo and Wu (2002) for different initial surplus 33
Chapter 1

Introduction

1.1 Risk Theory

The risk theory in actuarial science studies the ruin phenomenon of the insurers. The theoretical foundation of ruin theory was first laid down by Filip Lundberg in 1903. Lundberg's model investigates the probability that the insurer's surplus would eventually fall below zero. His model, which is also known as the classical risk model, assumes the insurer's surplus is generated by a single product only. While the premiums in the classical model are collected in a constant rate denoted by \( c \), the claims arrivals are described by a Poisson process and the claim sizes are a sequence of i.i.d. random variables. That is, the insurer's surplus at time \( t \) is given by:

\[
S(t) = u + ct - U(t)
\]  

(1.1)
where $u$ is the amount of the initial reserve, $c$ is the rate of premium and $U(t) = \sum_{i=1}^{N_1(t)} X_i$ is the claim process with $X_i$ being the claim size and $N(t)$ being the number of claims in $[0, t]$.

In reality, the insurer's overall surplus could be driven by the claims from more than 1 products. In particular, the claim mechanisms of various products could be correlated. For example, if an earthquake happens, the claims from homeowner insurance and those from automobile insurance cannot be considered independent. Recently, many authors studied the risk model of correlated claims. Wang (1998) presents a set of tools for modeling and combining correlated risks. Ambagaspitiya (1998) computes the aggregate distribution for the whole book of correlated business. Cossette and Marceau (1999) consider the case of correlated classes of business by using a discrete time risk model. Li and Lu (2004), Yuen and Wu (2003), and Wang and Yuen (2005) all study models under various assumptions of dependent claims.

In this thesis, we focus at the following claim model proposed by Yuen, Guo and Wu (2002):

$$U(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i$$  \hspace{1cm} (1.2)

where $X_i$ is the $i^{th}$ claim size from the first class of business and $Y_i$ is the $i^{th}$ claim size from the second class of business. Also, $N_1(t)$ and $N_2(t)$ are the number of claims in $[0, t]$ for the first class and for the second class, respectively. $X_i$ are assumed to be i.i.d. with a common distribution function $F_X$ and mean $\mu_X$ while $Y_i$ are i.i.d. from
CHAPTER 1. INTRODUCTION

$F_Y$ with mean $\mu_Y$. Besides, $\{X_i, i = 1,2,\ldots\}$ and $\{Y_i, i = 1,2,\ldots\}$ are independent to each other and they are independent of $N_1(t)$ and $N_2(t)$. Moreover, $N_1(t)$ and $N_2(t)$ are assumed to be dependent in the following way:

$$N_1(t) = K_1(t) + \overline{K}(t) \quad \text{and} \quad N_2(t) = K_2(t) + \overline{K}(t)$$

where $K_1(t)$, $K_2(t)$, and $\overline{K}(t)$ are three independent renewal processes with the exact specification stated in next chapter. Under the assumption of $F_X(x) = 1 - \exp\{-x/\mu_X\}$ for $x \geq 0$ and $F_Y(y) = 1 - \exp\{-y/\mu_Y\}$ for $y \geq 0$, Yuen, Guo and Wu (2002) obtain an explicit formula for the ultimate ruin probability (or simply ruin probability) which is defined as:

$$\psi(u) = 1 - \Pr(S(t) \geq 0 \text{ for all } t \geq 0).$$

Furthermore, they also study the behavior of $\psi(u)$ when $u \rightarrow \infty$ under less restrictive distribution assumptions.

1.2 Hybrid Numerical Scheme

The techniques employed in the derivation of the explicit formula for $\psi(u)$ in Yuen, Guo and Wu (2002) can hardly be generalized to tackle the cases of non-exponential claim size distribution. In this thesis, instead of pursuing the exact computation, we apply the numerical methodology proposed by Wong, Ho, Hu and Liu (2006) to compute the ruin probability under various claim distribution assumptions. Pre-
vious attempts in numerically evaluating ruin probability include Paulsen, Kasozi and Steigen (2005) who solve $\psi(u)$ from an integral equation under the classical risk model with stochastic return by using a block-by-block method. Note that the integral equation is obtained by integrating twice from an integro-differential equation. That is why the integral equation is in terms of some initial conditions which are difficult to be determined in general. For the integral equation in Paulsen, Kasozi and Steigen (2005), they suggest a novel way to "cancel" out the initial conditions. For other risk models, such cancellation procedure is not feasible.

Wong, Ho, Hu and Liu (2006) suggests using importance sampling to estimate the boundary conditions instead of just the initial conditions. The setup of boundary conditions aims at attaining a tolerable error for a practical range $u$ which means $\psi(u) \in [10^{-4}, 10^{-3}]$. Combining the importance sampling technique with the numerical recipe in solving the integral equation, Wong, Ho, Hu and Liu (2006) compute $\psi(u)$ for the classical model with $N(t)$ being an Erlang (2) process and the surplus earns a deterministic continuous-time compounding interest over time.

In this thesis, we employ the hybrid methodology of Wong, Ho, Hu and Liu (2006) in computing $\psi(u)$ for the model (1.2). The rest of this thesis is organized as follows. Chapter 2 provides a description on the model and derives the integro-differential equations for $\psi(u)$. Chapter 3 derives the integral equations for $\psi(u)$ and presents the numerical method and the importance sampling techniques. The numerical study result is shown in Chapter 4. Chapter 5 concludes.
Chapter 2

The Model

2.1 Model

Consider the surplus process

\[ S(t) = u + ct - U(t) \tag{2.1} \]

where \( u \) is the amount of initial reserve, \( c \) is the rate of premium and \( U(t) \) is the aggregate claim amount process. Let \( X_i \) be the size of the \( i^{th} \) claim from the first class of business and \( Y_i \) be the size of the \( i^{th} \) claim from the second class of business. Also, suppose \( N_1(t) \) and \( N_2(t) \) are the number of claims in \([0, t]\) for the first and the second class, respectively. Then,

\[ U(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i. \tag{2.2} \]

In particular, we assume \( X_i \) are i.i.d. with a common distribution function \( F_X \) whose
mean is denoted by $\mu_X$. Similarly, $Y_i$ are also i.i.d. with a common distribution function $F_Y$ and mean $\mu_Y$. Also, \{X_i, i = 1, 2, \ldots\} and \{Y_i, i = 1, 2, \ldots\} are independent and they are independent of $N_1(t)$ and $N_2(t)$. $N_1(t)$ and $N_2(t)$ are dependent and they are related in the following way:

$$N_1(t) = K_1(t) + \tilde{K}(t) \quad \text{and} \quad N_2(t) = K_2(t) + \tilde{K}(t)$$

where $K_1(t)$, $K_2(t)$, and $\tilde{K}(t)$ are three independent renewal processes. In particular, $K_1(t)$ and $K_2(t)$ are independent Poisson processes with parameters $\lambda_1$ and $\lambda_2$, respectively, while $\tilde{K}(t)$ being an Erlang(2) process with parameter $\bar{\lambda}$. i.e., the inter-arrival times for $\tilde{K}(t)$ are i.i.d. gamma random variables with shape parameter being 2 and the rate parameter being $\bar{\lambda}$.

To simplify our calculation of the ruin probability, we adopt the transformation in Yuen, Guo and Wu (2002) and reduce the model to the case of two independent claim number processes by the following argument. Note that

$$U(t) = \sum_{i=1}^{N_1(t)} X_i + \sum_{i=1}^{N_2(t)} Y_i$$

$$= \sum_{i=1}^{K_1(t)} X_i + \sum_{i=1}^{K_2(t)} Y_i + \sum_{i=1}^{\tilde{K}(t)} (A_i + B_i)$$

$$= \sum_{i=1}^{K_1(t)} X_i + \sum_{i=1}^{K_2(t)} Y_i + \sum_{i=1}^{\tilde{K}(t)} Y'_i$$

where $A_i$ are i.i.d. with a common distribution function $F_X$ and $B_i$ are i.i.d. with a common distribution function $F_Y$. It is clear that $Y'_i = A_i + B_i$ and is i.i.d. with a common distribution function $F_X * F_Y$ where $*$ is the convolution operator. Note
that the ultimate ruin probability,

\[ \psi(u) = 1 - \phi(u) = 1 - \Pr(S(t) \geq 0 \text{ for all } t \geq 0|S(0) = u) \]
\[ = 1 - \Pr(u + ct \geq U(t) \text{ for all } t \geq 0|S(0) = u), \]

is the quantity of interest. Which class of business causes the ruin is not an issue to be addressed. Thus by making use of the fact that \( \sum_{i=1}^{K_1(t)} X_i + \sum_{i=1}^{K_2(t)} Y_i \) has the same distribution of \( \sum_{i=1}^{K_1(t)+K_2(t)} X_i' \) where \( X_i' \) are i.i.d. with the following distribution function:

\[ \Pr\{X_i' \leq x\} = F_X(x) \frac{\lambda_1}{\lambda_1 + \lambda_2} + F_Y(x) \frac{\lambda_2}{\lambda_1 + \lambda_2}, \]

the ruin probability can be computed by considering

\[ \psi(u) = 1 - \Pr(u + ct \geq \tilde{U}(t) \text{ for all } t \geq 0|S(0) = u), \]

where

\[ \tilde{U}(t) = \sum_{i=1}^{K_{12}(t)} X_i' + \sum_{i=1}^{\tilde{K}(t)} Y_i' \]

and \( K_{12}(t) = K_1(t) + K_2(t) \) is still a Poisson process with parameter \( \lambda_1 + \lambda_2 \).

Without using extra notation, we denote the surplus process by:

\[ S(t) = u + ct - \sum_{i=1}^{K_{12}(t)} X_i' - \sum_{i=1}^{\tilde{K}(t)} Y_i'. \quad (2.3) \]

Note that \( K_{12}(t) \) and \( \tilde{K}(t) \) are independent to each other. Let \( \mu_X \) and \( \mu_Y \) be \( E(X_i') \) and \( E(Y_i') \), respectively. Since

\[ E[\tilde{U}(t)] = E[\sum_{i=1}^{K_{12}(t)} X_i'] + E[\sum_{i=1}^{\tilde{K}(t)} Y_i'] \]
\[ \begin{align*}
= & \quad \mu_X E[K_{12}(t)] + \mu_Y E[\tilde{K}(t)] \\
= & \quad [(\lambda_1 + \lambda_2)\mu_Y + \frac{1}{2} \lambda Y]t,
\end{align*} \]

the expected claim amounts per unit time is \((\lambda_1 + \lambda_2)\mu_Y + \frac{1}{2} \lambda Y\), and the safety loading is \(c - [(\lambda_1 + \lambda_2)\mu_Y + \frac{1}{2} \lambda Y]\). To exclude the trivial case of \(\psi(u) = 1\) for all \(u \geq 0\), we assume the safety loading is positive, i.e., the surplus would increase over time on average, or equivalently,

\[ c > (\lambda_1 + \lambda_2)\mu_Y + \frac{1}{2} \lambda Y. \quad (2.4) \]

### 2.2 Integro-Differential Equations

This section presents a system of integro-differential equations related to \(\psi(u)\). Such system is pivotal not only in the derivation of the explicit formula in Yuen, Guo and Wu (2002) but is also the starting point of our proposed hybrid methodology.

To start with, let \(\phi(u)\) be the probability of ultimate survival, or simply the survival probability such that \(\psi(u) = 1 - \phi(u)\). i.e.,

\[ \phi(u) = \Pr(S(t) \geq 0 \text{ for all } t \geq 0|S(0) = u). \]

The following system of integro-differential equations is more natural to be expressed in terms of \(\phi(u)\) instead of \(\psi(u)\).

Let \(T_1, T_2, \ldots\) be the inter-arrival times for \(K_{12}(t)\). Since \(K_{12}(t)\) is a Poisson process with parameter \(\lambda_1 + \lambda_2\), \(T_i\) are i.i.d. exponential random variables with mean
(\lambda_1 + \lambda_2)^{-1}$. Let \( S_1, S_2, \ldots \) be the inter-arrival times for \( \tilde{K}(t) \) which is an Erlang(2) process with parameter \( \tilde{\lambda} \). That is, \( S_i \) are i.i.d Erlang(2, \( \tilde{\lambda} \)) random variables which means

\[ S_i = S_{i1} + S_{i2} \]

where \( S_{i1} \) and \( S_{i2} \) are i.i.d. exponential random variables with mean \( \tilde{\lambda}^{-1} \).

Using the above decomposition of \( S_i \), we would like to outline the general idea of the renewal argument employed in the derivation. By conditioning on a random time \( V = \min(T_i, S_{i1}) \), the probability of ruin at that instance is evaluated via the following scenario analysis:

1. \( V = T_i \) and ruin happens. That is, \( S(V) < 0 \).

2. \( V = T_i \) and ruin does not happen at \( V \). In that case, the surplus process restarts as if the initial surplus is \( S(V) = u + cT_1 - X_1 \).

3. \( V = S_{i1} \). Thus the surplus process restarts at the initial surplus \( S(V) = u + cS_{i1} \). However, although this new restarting process can still be expressed as in the form of (2.3), the new \( \tilde{K}(t) \) is not exactly an Erlang(2, \( \tilde{\lambda} \)) process: Its arrival times are still i.i.d. Erlang(2, \( \tilde{\lambda} \)) except the first one is exponentially distributed with mean \( \tilde{\lambda} \). We denote the ultimate survival probability of this new restarting process as \( \phi_1(u) \) where \( u \) is the initial surplus.

What we called renewal argument is the "restarting" idea. Note that the argument can be applied similarly to the new restarting process in scenario 3 and
another integro-differential equation for $\phi_1(u)$ can be obtained.

The following is the formal derivation: Since $T_1$ and $S_{11}$ are independent and exponentially distributed with mean $(\lambda_1 + \lambda_2)^{-1}$ and $\tilde{\lambda}^{-1}$, $V$ is exponentially distributed with mean $\lambda^{-1}$ where $\lambda = \lambda_1 + \lambda_2 + \tilde{\lambda}$. Also,

$$\Pr\{V = S_{11}\} = \Pr\{T_1 < S_{11}\} = \frac{\lambda_1 + \lambda_2}{\lambda}$$

and

$$\Pr\{V = S_{11}\} = \Pr\{T_1 > S_{11}\} = \frac{\tilde{\lambda}}{\lambda}.$$ 

Therefore,

$$\phi(u) = \Pr(S(t) \geq 0 \text{ for all } t \geq 0 | S(0) = u)$$

$$= \int_0^\infty \Pr(V = S_{11}, V = t)\phi_1(u + ct)dt$$

$$+ \int_0^\infty \Pr(V = T_1, V = t)\int_0^{u + ct} \phi(u + ct - x)dF_{X'}(x)dt. \quad (2.5)$$

Using similar argument for the new process whose first arrival time of new $\tilde{K}(t)$ is exponentially distributed with mean $\tilde{\lambda}^{-1}$, we obtain the following equation:

$$\phi_1(u) = \int_0^\infty \Pr(V = t, V = S_{12})\int_0^{u + ct} \phi(u + ct - x)dF_{Y'}(x)dt$$

$$+ \int_0^\infty \Pr(V = t, V = T_1)\int_0^{u + ct} \phi_1(u + ct - x)dF_{X'}(x)dt \quad (2.6)$$

Note that

$$Pr(V = t, V = S_{11}) = Pr(V = t | V = S_{11})Pr(V = S_{11}) = f_V(t)\frac{\tilde{\lambda}}{\lambda}$$
and

\[
Pr(V = t, V = T_1) = f_V(t)\frac{\lambda_1 + \lambda_2}{\lambda}
\]

where \( f_V(t) = \lambda \exp\{-\lambda t\} \) for \( t \geq 0 \). Using the above results, (2.5) becomes

\[
\phi(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_0^\infty \lambda \exp\{-\lambda t\} \phi_1(u + ct) dt
\]

and

\[
\phi_1(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} \phi(u + ct - x) dF_X(x) dt
\]

and (2.6) becomes

\[
\phi_1(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_0^\infty \lambda \exp\{-\lambda t\} \int_0^{u+ct} \phi(u + ct - x) dF_X(x) dt
\]

Put \( s = u + ct \) into \( \phi(u) \) and \( \phi_1(u) \) becomes

\[
\phi(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_u^\infty \exp\{-\lambda \left( \frac{s - u}{c} \right)\} \phi_1(s) \frac{ds}{c}
\]

\[
+ (\lambda_1 + \lambda_2) \int_u^\infty \exp\{-\lambda \left( \frac{s - u}{c} \right)\} \int_0^{s - x} \phi(s - x) dF_X(x) \frac{ds}{c}
\]

\[
\Rightarrow c\phi(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_u^\infty \phi_1(s) \exp\{-\lambda \left( \frac{s - u}{c} \right)\} ds
\]

\[
+ (\lambda_1 + \lambda_2) \int_u^\infty \exp\{-\lambda \left( \frac{s - u}{c} \right)\} \int_0^{s - x} \phi(s - x) dF_X'(x) ds
\]  \(2.7\)

and

\[
c\phi_1(u) = \frac{\lambda}{\lambda_1 + \lambda_2} \int_u^\infty \exp\{-\lambda \left( \frac{s - u}{c} \right)\} \int_0^{s - x} \phi(s - x) dF_Y'(x) ds
\]

\[
+ (\lambda_1 + \lambda_2) \int_u^\infty \exp\{-\lambda \left( \frac{s - u}{c} \right)\} \int_0^{s - x} \phi_1(s - x) dF_X'(x) ds.
\]  \(2.8\)
Differentiate (2.7) and (2.8) with respect to $u$ gives the system of integro-differential equations:

$$c_0^i(u) = \lambda \exp \left( \frac{\lambda u}{c} \right) \left( \frac{\lambda}{c} \right) \int_u^\infty \phi_1(s) \exp \left( -\frac{\lambda s}{c} \right) ds - \bar{\lambda} \exp \left( \frac{\lambda u}{c} \right) \phi_1(u) \exp \left( -\frac{\lambda u}{c} \right)$$

$$+ (\lambda_1 + \lambda_2) \exp \left( \frac{\lambda u}{c} \right) \left( \frac{\lambda}{c} \right) \int_u^\infty \exp \left( -\frac{\lambda s}{c} \right) \int_0^s \phi(s-x)dF_{X'}(x)ds$$

$$- (\lambda_1 + \lambda_2) \exp \left( \frac{\lambda u}{c} \right) \exp \left( -\frac{\lambda u}{c} \right) \int_0^u \phi(u-x)dF_{X'}(x)$$

$$= \left( \frac{\lambda}{c} \right) (c\phi(u)) - \bar{\lambda} \phi_1(u) - (\lambda_1 + \lambda_2) \int_0^u \phi(u-x)dF_{X'}(x)$$

$$\Rightarrow c_0^1(u) = -\bar{\lambda} \phi_1(u) - (\lambda_1 + \lambda_2) \int_0^u \phi(u-x)dF_{X'}(x) + \lambda \phi(u) \quad (2.9)$$

$$c_0^1(u) = \lambda \exp \left( \frac{\lambda u}{c} \right) \left( \frac{\lambda}{c} \right) \int_u^\infty \exp \left( -\frac{\lambda s}{c} \right) \int_0^s \phi(s-x)dF_{X'}(x)ds$$

$$- \bar{\lambda} \exp \left( \frac{\lambda u}{c} \right) \exp \left( -\frac{\lambda u}{c} \right) \int_0^u \phi(u-x)dF_{X'}(x)$$

$$+ (\lambda_1 + \lambda_2) \exp \left( \frac{\lambda u}{c} \right) \left( \frac{\lambda}{c} \right) \int_u^\infty \exp \left( -\frac{\lambda s}{c} \right) \int_0^s \phi_1(s-x)dF_{X'}(x)ds$$

$$- (\lambda_1 + \lambda_2) \exp \left( \frac{\lambda u}{c} \right) \exp \left( -\frac{\lambda u}{c} \right) \int_0^u \phi_1(u-x)dF_{X'}(x)$$

$$= \left( \frac{\lambda}{c} \right) (c\phi_1(u)) - \bar{\lambda} \int_0^u \phi(u-x)dF_{X'}(x) - (\lambda_1 + \lambda_2) \int_0^u \phi_1(u-x)dF_{X'}(x)$$

$$\Rightarrow c_0^1(u) = -\bar{\lambda} \int_0^u \phi(u-x)dF_{X'}(x) - (\lambda_1 + \lambda_2) \int_0^u \phi_1(u-x)dF_{X'}(x)$$

$$+ \lambda \phi_1(u) \quad (2.10)$$

Note that the boundary conditions are $\phi(\infty) = \phi_1(\infty) = 1$. 
2.3 Explicit Formulas and Asymptotic Properties

For the case of $X_i$ and $Y_i$ being i.i.d. exponentially distributed with common mean $\mu$, Yuen, Guo and Wu (2002) solve the above system of integro-differential equations explicitly as:

$$\phi(u) = C_1 + C_2 q(z_2) \exp\{z_2 u\} + C_3 q(z_3) \exp\{z_3 u\} + C_4 q(z_4) \exp\{z_4 u\}$$

$$\phi_1(u) = C_1 + C_2 \exp\{z_2 u\} + C_3 \exp\{z_3 u\} + C_4 \exp\{z_4 u\}$$

where

$$q(z) = 1 + \frac{\mu}{\lambda} \left( \lambda + \bar{\lambda} - \frac{c}{\mu} \right) z + \frac{\mu^2}{\lambda} \left( \lambda - \frac{2c}{\mu} \right) z^2 - \frac{c\mu^2}{\lambda} z^3$$

and

$$z_1 = 0, \quad z_2 = -\frac{1}{\mu}, \quad z_3 = \frac{\lambda\mu - c}{c\mu}$$

$$z_4 = \frac{1}{2c\mu} (\lambda\mu - c - (8c\mu\bar{\lambda} + (c - \lambda\mu)^2)^{1/2})$$

$$z_5 = \frac{1}{2c\mu} (\lambda\mu - c + (8c\mu\bar{\lambda} + (c - \lambda\mu)^2)^{1/2})$$

are the roots of the following “characteristic equation”. Note that the characteristic equation in Yuen, Guo and Wu (2002) contains some typo errors. The correct characteristic equation should be:

$$c^3 \mu^2 z^5 + c\mu (3c - 2\lambda\mu) z^4 + ((\lambda\mu - c)(\lambda\mu - 3c) - 2c\mu\bar{\lambda}) z^3$$

$$+ (\lambda\mu - c) (\lambda + \bar{\lambda} - \frac{c}{\mu}) + \bar{\lambda}(\lambda\mu - 3c) \left( \lambda - \frac{c}{\mu} \right) z^2 + 2\bar{\lambda} \left( \lambda - \frac{c}{\mu} \right) z = 0.$$
CHAPTER 2. THE MODEL

Using the boundary conditions, \( C_1 = 1 \) and \( (C_2, C_3, C_4) \) is solved from:

\[
\begin{align*}
\lambda &= (c_{z_2} - \lambda)C_2 + (c_{z_3} - \lambda)C_3 + (c_{z_4} - \lambda)C_4 \\
\frac{\bar{\lambda}}{\mu} &= \left(\frac{c_{z_2}^2 - \lambda z_2 - \frac{\lambda - c_{z_2}}{\mu}}{\mu}\right)C_2 + \left(\frac{c_{z_3}^2 - \lambda z_3 - \frac{\lambda - c_{z_3}}{\mu}}{\mu}\right)C_3 + \left(\frac{c_{z_4}^2 - \lambda z_4 - \frac{\lambda - c_{z_4}}{\mu}}{\mu}\right)C_4
\end{align*}
\]

\[
\lambda_1 + \lambda_2 = \left((c_{z_2} - \lambda)q(z_2) + \bar{\lambda}\right)C_2 + \left((c_{z_3} - \lambda)q(z_3) + \bar{\lambda}\right)C_3 + \left((c_{z_4} - \lambda)q(z_4) + \bar{\lambda}\right)C_4.
\]

The above equations of \( (C_2, C_3, C_4) \) are also different from those in Yuen, Guo and Wu (2002) and could be verified to be the correct ones.

Relaxing the restrictive distribution assumption of \( X_i \) and \( Y_i \), Yuen, Guo and Wu (2002) derive the following asymptotic statement:

\[
\lim_{a \to \infty} \frac{\exp\{Ru\} (\psi(u) + \psi_1(u))}{2} \leq \frac{\rho}{1 + \rho (\lambda_1 + \lambda_2) h_1^{(1)}(R) + (1/2) \bar{\lambda} h_2^{(1)}(R) - c},
\]

where \( \rho \) is the relative security loading

\[
\rho = \frac{c}{(\lambda_1 + \lambda_2) \mu X' + (1/2) \bar{\lambda} \mu Y'} - 1
\]

and \( R \) is the positive solution of the equation

\[
(\lambda_1 + \lambda_2) h_1(r) + \frac{1}{2} \bar{\lambda} h_2(r) = cr
\]

with

\[
\begin{align*}
h_1(r) &= \int_0^\infty \exp\{rx\} dF_{X'}(x) - 1 \\
h_2(r) &= \int_0^\infty \exp\{rx\} dF_{Y'}(x) - 1
\end{align*}
\]
and $h_1^{(1)}$ and $h_2^{(1)}$ being their first derivatives, respectively. The upper bound holds if there exists $r_1 > 0$ and $r_2 > 0$ such that

\[ \lim_{r \to r_1} h(r) = 0 \quad \text{and} \quad \lim_{r \to r_2} h(r) = 0. \]

The asymptotic bound as well as the aforementioned explicit solution are going to be investigated in Chapter 4.
Chapter 3

Numerical Method

The explicit solution presented in the last chapter is derived under the restrictive assumption of $X_i$ and $Y_i$ are both i.i.d. exponential. Instead of using the asymptotic techniques, we adopt the hybrid approach proposed by Wong, Ho, Hu and Liu (2006) in tackling the ruin probability problem in this thesis. First, the system of integro-differential equations are turned to integral equations. Instead of solving the integral equations explicitly, we employ trapezoidal approximation of integrals and solve the corresponding system of integral equations correspondingly. Note that in order to make the numerical scheme performs reasonably well in the practical zone of $\{u : \psi(u) \in [10^{-4}, 10^{-3}]\}$, the boundary conditions have to be estimated with great care. Here, we apply the importance sampling to estimate the boundary conditions and our choice of the trial process is determined by using the heuristic argument in Wong, Ho, Hu and Liu (2006). It should be highlighted that unlike Wong, Ho,
Hu and Liu (2006) who solved the ruin probability problem of single business, the model of this thesis consists of two correlated businesses and its ruin probability problem is substantially more difficult.

3.1 From Integro-Differential Equations to Integral Equations

Integral equations are obtained by integrating the integro-differential equation.

Integrating (2.9) both sides from 0 to u

$$\int_0^u c \phi^{(1)}(s) ds = \int_0^u \left[ -\lambda \phi_1(s) - (\lambda_1 + \lambda_2) \int_0^s \phi(s-x) dF_X(x) + \lambda \phi(s) \right] ds$$

$$\phi(u) - \phi(0) = -\frac{\lambda}{c} \int_0^u \phi_1(s) ds + \frac{\lambda_1 + \lambda_2}{c} \int_0^u \int_0^s \phi(s-x) d(1-F_X(x)) ds + \frac{\lambda}{c} \int_0^u \phi(s) ds$$

$$\phi(u) = \phi(0) - \frac{\lambda}{c} \int_0^u \phi_1(s) ds + \frac{\lambda_1 + \lambda_2}{c} \int_0^u \phi(s) ds$$

$$+ \frac{\lambda_1 + \lambda_2}{c} \int_0^u \int_0^s (s-x) d(1-F_X(x)) ds$$

$$\phi(u) = \phi(0) + \frac{\lambda}{c} \int_0^u \phi(x) dx - \frac{\lambda}{c} \int_0^u \phi_1(x) dx$$

$$+ \frac{\lambda_1 + \lambda_2}{c} \left[ \int_0^u \phi(u-x)(1-F_X(x)) dx - \int_0^u \phi(x) dx \right]$$

$$\phi(u) = \phi(0) + \frac{\lambda}{c} \int_0^u (\phi(x) - \phi_1(x)) dx + \frac{\lambda_1 + \lambda_2}{c} \int_0^u \phi(u-x)(1-F_X(x)) dx$$

Integrating (2.10) both sides from 0 to u

$$\int_0^u c \phi^{(1)}(s) ds = \int_0^u \left[ -\lambda \int_0^s \phi(s-x) dF_Y(x) - (\lambda_1 + \lambda_2) \int_0^s \phi_1(s-x) dF_X(x) + \lambda \phi_1(s) \right] ds$$

$$\phi_1(u) - \phi_1(0) = -\frac{\lambda}{c} \int_0^u \int_0^s \phi(s-x) dF_Y(x) dx ds + \frac{\lambda_1 + \lambda_2}{c} \int_0^u \int_0^s \phi_1(s-x) dF_X(x) dx ds$$
CHAPTER 3. NUMERICAL METHOD

3.2 From Integral Equations to Linear Equations

The two integral equations will be reduced to

\[
\phi(u) = \phi(0) + \int_0^u \frac{\lambda}{c} \phi(s)ds + \frac{\lambda_1 + \lambda_2}{c} \int_0^u \phi_1(s)x(u-x)dx
\]

\[
= \phi(0) + \int_0^u K_00(u, x)\phi(x)dx + \int_0^u K_10(u, x)\phi_1(x)dx \tag{3.1}
\]

and

\[
\phi_1(u) = \phi_1(0) + \frac{\lambda}{c} \int_0^u \phi_1(x)dx - \frac{\lambda}{c} \int_0^u \phi(x)dx + \int_0^u F_{x'}(u-x)dx
\]

\[
= \phi_1(0) + \int_0^u K_{11}(u, x)\phi_1(x)dx + \int_0^u K_{01}(u, x)\phi(x)dx \tag{3.2}
\]
3.2 From Integral Equations to Linear Equations

To solve \( \phi(u) \) and \( \phi_1(u) \) numerically over \([0, u_{\text{max}}]\), we approximate the integral by cutting \([0, u_{\text{max}}]\) into \( n \) intervals with equal length \( h \) such that \( h = u_{\text{max}}/n \).

Let \( u_i = ih \) for \( i = 0, \ldots, n \). Thus, for \( i = 1, \ldots, n \), equation (3.1) and (3.2) are approximated by

\[
\phi(u_i) = \phi(0) + \sum_{j=0}^{i} k_{00}(u_i, u_j)\phi(u_j) + \sum_{j=0}^{i} k_{10}(u_i, u_j)\phi_1(u_j)
\]

\[
\phi_1(u_i) = \phi_1(0) + \sum_{j=0}^{i} k_{01}(u_i, u_j)\phi(u_j) + \sum_{j=0}^{i} k_{11}(u_i, u_j)\phi_1(u_j)
\]

where for \( a, b = 0 \) or \( 1 \)

\[
k_{ab}(u_i, u_j) = \begin{cases} 
    hK_{ab}(u_i, u_j) & \text{if } i \neq j \\
    hK_{ab}(u_i, u_j)/2 & \text{if } j = 0 \text{ or } j = i.
\end{cases}
\]

\( \phi(u_i) \) and \( \phi_1(u_i) \) are further rearranged as follows:

\[
\sum_{j=1}^{i} h_{00}(u_i, u_j)\phi(u_j) + \sum_{j=1}^{i} h_{10}(u_i, u_j)\phi_1(u_j) = a_i \quad (3.3)
\]

\[
\sum_{j=1}^{i} h_{01}(u_i, u_j)\phi(u_j) + \sum_{j=1}^{i} h_{11}(u_i, u_j)\phi_1(u_j) = b_i \quad (3.4)
\]

where

\[
a_i = \phi(0) (1 + k_{00}(u_i, 0)) + k_{10}(u_i, 0)\phi_1(0)
\]

\[
b_i = \phi(0) k_{01}(u_i, 0) + (1 + k_{11}(u_i, 0))\phi_1(0)
\]
and

\[ h_{00}(u_i, u_j) = \begin{cases} -k_{00}(u_i, u_j) & \text{if } i \neq j \\ 1 - k_{00}(u_i, u_j) & \text{if } j = i. \end{cases} \]

\[ h_{11}(u_i, u_j) = \begin{cases} -k_{11}(u_i, u_j) & \text{if } i \neq j \\ 1 - k_{11}(u_i, u_j) & \text{if } j = i. \end{cases} \]

\[ h_{10}(u_i, u_j) = -k_{10}(u_i, u_j), \text{ and } h_{01}(u_i, u_j) = -k_{01}(u_i, u_j). \]

In matrix form, the system of linear equations becomes

\[
\begin{bmatrix}
H_{00} & H_{10} \\
H_{01} & H_{11}
\end{bmatrix}
\begin{bmatrix}
\phi \\
\phi_1
\end{bmatrix}
= 
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]

where each \( H_{ij} \) is a lower triangular \( n \times n \) matrix and \( a = (a_1, \ldots, a_n)^T \) and \( b = (b_1, \ldots, b_n)^T \) and

\[ \phi = (\phi(u_1), \ldots, \phi(u_n))^T \text{ and } \]

\[ \phi_1 = (\phi_1(u_1), \ldots, \phi_1(u_n))^T \]

### 3.3 Boundary Conditions

Although the above system is neatly presented in an blockwise lower triangular form, it can only be solved if \( \phi(0) \) and \( \phi_1(0) \) are known. The boundary conditions of the integral equations \( \phi(\infty) = \phi_1(\infty) = 1 \) do not help us to determine these
initial conditions. In fact, in Chapter 4, we will show that even if \( \phi(0) \) and \( \phi_1(0) \) are known up to a certain precision, ignoring the trailing digits could make huge errors in \( \phi(u) \) in the practical zone where \( \psi(u) \in [10^{-4}, 10^{-3}] \). To circumvent the issue of high sensitivity towards the initial conditions, we transform the above system such that the input needed would be \( (\phi(0), \phi(u_{\text{max}})) \) instead of \( (\phi(0), \phi_1(0)) \) and \( \phi_1(0) \) is treated as one of the unknowns. Note that \( \phi(u) \) is the function of interest while \( \phi_1(u) \) is only an auxiliary variable. After such transformation, the inputs are then estimated by using importance sampling which is going to be discussed in the next section. In the followings, we describe the recipe of the transformation.

By turning \( \phi_1(0) \) as one of the unknowns and moving from the left hand side \( \phi(u_{\text{max}}) = \phi(u_n) \) to the right hand side, equations (3.3) and (3.4) are of the form:

For \( i = 1, \ldots, n - 1 \),

\[
\sum_{j=1}^{i} h_{00}(u_i, u_j)\phi(u_j) + \sum_{j=1}^{i} h_{10}(u_i, u_j)\phi_1(u_j) - k_{10}(u_i, 0)\phi_1(0) = c_i
\]

\[
\sum_{j=1}^{i} h_{01}(u_i, u_j)\phi(u_j) + \sum_{j=1}^{i} h_{11}(u_i, u_j)\phi_1(u_j) - (1 + k_{11}(u_i, 0))\phi_1(0) = d_i
\]

with

\[
c_i = \phi(0)(1 + k_{00}(u_i, 0)), \quad d_i = \phi(0)k_{01}(u_i, 0).
\]

For \( i = n \),

\[
\sum_{j=1}^{n-1} h_{00}(u_n, u_j)\phi(u_j) + \sum_{j=1}^{n} h_{10}(u_n, u_j)\phi_1(u_j) - k_{10}(u_n, 0)\phi_1(0) = c_n
\]

\[
\sum_{j=1}^{n-1} h_{01}(u_n, u_j)\phi(u_j) + \sum_{j=1}^{n} h_{11}(u_n, u_j)\phi_1(u_j) - (1 + k_{11}(u_n, 0))\phi_1(0) = d_n
\]
\[ c_n = \phi(0)(1 + k_{00}(u_n, 0)) - h_{00}(u_n, u_n)\phi(u_n) \]
\[ d_n = \phi(0)k_{01}(u_n, 0) - h_{01}(u_n, u_n)\phi(u_n). \]

In matrix form, the equations become

\[
\begin{bmatrix}
  L_{00} & L_{10} \\
  L_{01} & L_{11}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
c \\
d
\end{bmatrix}
\]

where \( x = (\phi(u_1), \ldots, \phi(u_{n-1}))^T \) and \( y = (\phi_1(0), \ldots, \phi_1(u_n))^T \). Note that \( c \) and \( d \) are of dimension \( n \), \( L_{00} \) is of dimension \((n - 1) \times (n - 1)\) and \( L_{11} \) is of dimension \((n + 1) \times (n + 1)\). Also, the dimensions of \( L_{10} \) and \( L_{01} \) are \((n - 1) \times (n + 1)\) and \((n + 1) \times (n - 1)\), respectively. Specifically, the entries are

\[
l_{00}(u_i, u_j) = \begin{cases} 
h_{00}(u_i, u_j) & \text{if } 0 < j \leq i \\
0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, \ldots, n - 1, j = 1, \ldots, n - 1
\]

\[
l_{10}(u_i, u_j) = \begin{cases} 
-k_{10}(u_i, 0) & \text{if } j = 0 \\
h_{10}(u_i, u_j) & \text{if } 0 < j \leq i \\
0 & \text{otherwise} \end{cases} \quad \text{for } i = 1, \ldots, n - 1, j = 0, \ldots, n
\]

\[
l_{01}(u_i, u_j) = \begin{cases} 
h_{00}(u_n, u_j) & \text{if } i = 0 \\
h_{01}(u_i, u_j) & \text{if } 0 < j \leq i \\
0 & \text{otherwise} \end{cases} \quad \text{for } i = 0, \ldots, n, j = 1, \ldots, n - 1
\]
CHAPTER 3. NUMERICAL METHOD

\[ l_{11}(u_i, u_j) = \begin{cases} 
-k_{10}(u_n, 0) & \text{if } i = 0, j = 0 \\
h_{10}(u_n, u_j) & \text{if } i = 0, j > 0 \\
(1 + k_{11}(u_i, 0)) & \text{if } i > 0, j = 0 \text{ for } i = 0, \ldots, n, j = 0, \ldots n \\
h_{11}(u_i, u_j) & \text{if } 0 < j \leq i \\
0 & \text{otherwise} 
\end{cases} \]

3.4 Importance Sampling

In Yuen, Guo and Wu (2002), crude Monte Carlo simulations are used in validating the explicit formulas obtained under the assumption of i.i.d. exponential claim sizes. The simulations are performed by drawing random sample paths of the surplus process and compute the relative proportion of ruins before a fixed time \( \tau \). The choice of \( \tau \) is crucial in such experiment. If \( \tau \) is too small, the estimated ruin probability would be seriously biased. On the other hand, if \( \tau \) is too large, the computational effort needed becomes very demanding.

In this thesis, other than validating the formulas, \( \phi(u_0) \) and \( \phi(u_n) \) are the necessary ingredients for the above systems of equations. As we have already commented, the crude Monte Carlo method is not an adequate tool in estimating the ruin probability. Here we adopt the approach of Wong, Ho, Hu and Liu (2006) and makes use of the idea of importance sampling to determine the boundary conditions.

The procedure of importance sampling starts with choosing a set of parameter for
the surplus process such that the ruin with surely happens within a finite time. Such process is called the trial process and its corresponding parameters are superscripted by *. To ruin in finite time, the trial security loading has to be negative. That is,

$$c^* < (\lambda_1^* + \lambda_2^*)\mu_{X^*} + \frac{1}{2}\lambda^*\mu_{Y^*},$$

(3.5)

with

$$\psi(u) = \Pr\{T < \infty\} = \int \frac{f_T(t)}{f_n^*(t)} dt$$

where $f_T$ and $f_n^*$ are the probability density functions of $T$ under the original process and the trial process, respectively. One of the ways of choosing trial parameter for ruin almost surely within a finite time is setting $(c, \lambda_1, \lambda_2, \lambda) = (c^*, \lambda_1^*, \lambda_2^*, \lambda^*)$ and pick $\mu_{X^*}$ and $\mu_{Y^*}$, substantially greater than $\mu_{X^*}$ and $\mu_{Y^*}$, respectively such that the inequality (3.5) holds. To simplify the notation, denote $\mu_{X^*} = \eta_{X^*}$ and $\mu_{Y^*} = \eta_{Y^*}$.

Then, under such setting,

$$\psi(u) = \Pr_{\mu}\{T < \infty\}$$

$$= E_{\eta}
\left[
\left(\Pi_{i=1}^{N(T)-1} I\{S_i > 0\}\right) I\{S_{N(T)} < 0\}\right]
\frac{f_{\mu}(U_1, \ldots, U_{N(T)})}{f_{\eta}(U_1, \ldots, U_{N(T)})}
$$

$$= E_{\eta}
\left[
\left(\Pi_{i=1}^{N(T)-1} I\{S_i > 0\}\right) I\{S_{N(T)} < 0\}\right]
\Pi_{i=1}^{N(T)} f_{\mu}(U_i)\frac{f_{\mu}(U_i)}{f_{\eta}(U_i)}
$$

where $S_i$ is the trial surplus process at the $i^{th}$ claim epoch, $U_i$ is the trial claim size at that time epoch, $N(t)$ is the number of claims in $[0, t]$ for the trial process; and $f_{\mu}$ and $f_{\eta}$ are the probability density functions of the claim sizes under the original process and the trial process, respectively. Suppose now the trial process is
simulated $M$ times and the $k^{th}$ sample path ruin at $T_k$ with $N(T_k) = J_k$ of which $I_k$ of the claims are from density $f_{\eta_N}$. Then,

$$
\psi(u) \approx \frac{1}{M} \sum_{k=1}^{M} \left[ \prod_{i=1}^{I_k} \frac{f_{\mu_k}(U_{i,k})}{f_{\eta}(U_{i,k})} \right] = \frac{1}{M} \sum_{k=1}^{M} \left[ \prod_{i=1}^{I_k} \frac{f_{\mu_N}(X_{i,k})}{f_{\eta_N}(X_{i,k})} \times \prod_{i=I_k+1}^{I_{k+1}} \frac{f_{\mu_Y}(Y_{i,k})}{f_{\eta_Y}(Y_{i,k})} \right].
$$

To be more specific, the simulation of the surplus process paths is performed in the following manner:

1. Set $N_X = 0$, $N_Y = 0$, $S = u$, $\omega = 1$.

2. Simulate $T_X$ from exponential density with mean $(\lambda_1 + \lambda_2)^{-1}$ and $T_Y$ from $Erlang(2, \lambda)$ distribution independently.

3. (a) If $T_X < T_Y$, then

i. set $N_X$ to be $N_X + 1$,

ii. set $S$ to be $S + cT_X - X_{N_X}$ where the claim size $X$ is simulated from $f_{\eta_X}$, and

iii. set $\omega$ to be $\omega \times \frac{f_{\mu_N}(X_{N_X})}{f_{\eta_N}(X_{N_X})}$.

(b) If $T_Y < T_X$, then

i. set $N_Y$ to be $N_Y + 1$,

ii. set $S$ to be $S + cT_Y - X_{N_Y}$ where the claim size $Y$ is simulated from $f_{\eta_Y}$, and

iii. set $\omega$ to be $\omega \times \frac{f_{\mu_Y}(X_{N_Y})}{f_{\eta_Y}(X_{N_Y})}$. 

4. If $S < 0$, then save $\omega$ and repeat step 1 until there are $M$ replicates. Otherwise, go to step 2.

The estimate of $\psi(u)$ is given by $\sum_{k=1}^{M} \omega_k / M$. 

Chapter 4

Numerical Study
Chapter 4

Numerical Study

The hybrid methodology detailed in the previous chapters is put into test in this chapter. As we have mentioned, Yuen, Guo and Wu (2002) derive an explicit formula for ruin probability under the assumption of the claim sizes are i.i.d. exponentially distributed. They also validate their explicit formula by using crude importance sampling. Thus, in Section 1 of this chapter, their result is used as a benchmark for comparison for our proposal. In Section 2, the hybrid methodology is applied to the case of $X_i$ and $Y_i$ are independently distributed but with different means. Our result compares favorably to the asymptotic bounds of Yuen, Guo and Wu (2002).
CHAPTER 4. NUMERICAL STUDY

4.1 Exponential Claims with Equal Means

4.1.1 Importance Sampling

In this section, we assume the claim sizes $X_i$ and $Y_i$ are i.i.d. exponentially distributed with common mean $\mu$. Since Yuen, Guo and Wu (2002) choose the following set of parameters, we use the same set for the purpose of easy comparison.

$$\lambda_1 = \lambda_2 = 1.5, \quad \bar{\lambda} = 1, \quad c = 6, \mu = 1.$$ 

In particular, since $X_i$ and $Y_i$ are i.i.d. exponential, $X_i'$ is also i.i.d. exponential with the same mean and $Y_i'$ follows Erlang distribution with mean $2\mu$. Moreover, the safety loading in this case:

$$c - (\lambda_1 + \lambda_2)\mu_{X'} - \frac{1}{2}\bar{\lambda}\mu_{Y'}$$

$$= c - (\lambda_1 + \lambda_2)\mu - \frac{1}{2}\bar{\lambda}(2\mu)$$

$$= 2 > 0.$$ 

Thus, the trivial case of $\psi(u) = 1$ for all $u$ is excluded.

On the other hand, the trial process needs a negative safety loading to ensure it ruins in finite time. Thus, the trial surplus process is assumed to be identical to the original process except that the claim sizes are i.i.d. exponential with common mean $\eta = 1.9$. That implies the trial safety loading:

$$c - (\lambda_1 + \lambda_2)\eta - \frac{1}{2}\bar{\lambda}(2\eta) = -1.6 < 0.$$
CHAPTER 4. NUMERICAL STUDY

Note that other than making the trial safety loading negative, we also require the trial process to be sufficiently similar to the original process such that the estimate of $\psi(u)$ is stable. In this experiment, any $\eta$ between 1.8 and 2.0 gives reasonable performance in estimating $\psi(u)$ for $M = 10^5$ which is our choice.

In Table 4.1, the importance sampling estimates of $\psi(u)$ with various levels of $u$ are compared with the exact values and the crude Monte Carlo estimates of Yuen, Guo and Wu (2002). The exact solutions are calculated by the explicit formula given by Yuen, Guo and Wu (2002). In particular, we would like to highlight that $\psi(u) \leq 0.004$ for $u \geq 15$. In that range, Yuen, Guo and Wu (2002) do not provide any Monte Carlo estimate of $\psi(u)$. As we have discussed previously, the practical zone of ruin probability is $\{u : \psi(u) \in [10^{-4}, 10^{-3}]\}$ and the performance of any method in that range is of extreme importance. As different stopping time values in the crude Monte Carlo method could give high variation of ruin probability estimate in that zone, we suspect that is the reason of no estimate of $\psi(u)$ reported in Yuen, Guo and Wu (2002).

To compare with the reported values (i.e., $u = 0, 2, 4, 6, 8, 10$), we employ the absolute relative error which is defined as

$$\text{Absolute Relative Error} = \frac{|\text{Estimate} - \text{Exact Value}|}{\text{Exact Value}}.$$

For those large ruin probabilities, our importance sampling estimates are very close to the exact values and compare favorably with the crude Monte Carlo esti-
CHAPTER 4. NUMERICAL STUDY

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
\textbf{u} & 0 & 2 & 4 & 6 & 8 & 10 \\
\hline
Exact solution & $6.072 \times 10^{-1}$ & $3.023 \times 10^{-1}$ & $1.531 \times 10^{-1}$ & $7.82 \times 10^{-2}$ & $4.00 \times 10^{-2}$ & $2.05 \times 10^{-2}$ \\
Importance Sampling & $6.138 \times 10^{-1}$ & $2.992 \times 10^{-1}$ & $1.524 \times 10^{-1}$ & $7.48 \times 10^{-2}$ & $3.75 \times 10^{-2}$ & $2.00 \times 10^{-2}$ \\
Crude Monte Carlo & $6.158 \times 10^{-1}$ & $3.103 \times 10^{-1}$ & $1.580 \times 10^{-1}$ & $8.09 \times 10^{-2}$ & $4.15 \times 10^{-2}$ & $2.13 \times 10^{-2}$ \\
\hline
\end{tabular}
\caption{Ruins probability $\psi(u_n)$ obtained from explicit formula, importance sampling and crude monte carlo method of Yuen, Guo and Wu (2002)}
\end{table}

However, for those within the zone of interest $\{u : \psi(u) \in [10^{-4}, 10^{-3}]\}$, the absolute relative difference of the importance sampling estimates is between $[1\%, 34\%]$. Through the simulation, we aim at finding out the minimum amount of the initial reserve with the conditions on the ruin probabilities lay within the practical zone. In particular, we observe that $\psi(20) < 10^{-3}$ and would satisfy the value-at-risk require-
ment discussed in chapter 1. Thus, we set $u_{\text{max}} = 20$ and solve the corresponding system of linear equations in the next subsection.

### 4.1.2 System of Linear Equations

Since the estimate of $\psi(20)$ is roughly $10^{-4}$, $[0, 20]$ should cover more than sufficient values of $u$ for any reasonable insurer to consider. Using $\psi(0) = 0.6138$ (or equivalently, $\phi(0) = 0.3862$) and $\psi(20) = 0.0007$ (or equivalently, $\phi(20) = 1 - 0.0007 = 0.9993$), the curve of $\psi(u)$ in Figure 4.1 is obtained by using $n = 1000$ which is equivalent to $h = 0.02$. Note that we also compare our solution with that being solved from the blockwise triangular system in Section 3.1 with $\phi(0)$ and $\phi_1(0)$ as input.

Here one can easily see that the lower triangular version in Section 3.1 produces substantial error when $u$ is reasonably large while the error of our hybrid solution is remarkably small. From Figure 4.2, we can see that the maximum absolute relative error of Section 3.1 methodology is 18 and is at the zone of interest (i.e., $\psi(u) < 10^{-3}$) while that of hybrid solution is only 3.

Since the error in the blockwise triangular system is overwhelming, we conclude that the performance of our hybrid methodology is satisfactory.
4.2 Exponential Claims with Unequal Means

Although Yuen, Guo and Wu (2002) claim that $\psi(u)$ is also explicit for the case of $X_i$ are i.i.d. exponential with mean $\mu_X$ and $Y_i$ are i.i.d. exponential with mean $\mu_Y$ with $\mu_X \neq \mu_Y$, they, instead of putting down the formulas, encourage the readers to use Mathematica or any other symbolic algebraic software to obtain the solution. Nevertheless, they provide an asymptotic upper bound for average ruin probabilities $(\phi(u) + \phi_1(u))/2$ under very mild conditions. Such upper bound is the benchmark of comparison in this subsection. Besides, the results of hybrid estimation are also compared with those of the crude Monte Carlo results in Yuen, Guo and Wu (2002).

Again, we fix the following parameter values by following Yuen, Guo and Wu (2002).

$$\lambda_1 = 2, \lambda_2 = 1.5, \bar{\lambda} = 1, \mu_X = 1, \mu_Y = 1/1.8, c = 4$$

In particular,

$$\mu_Y' = \mu_X + \mu_Y = 1.567$$

$$\mu_X' = \frac{\lambda_1}{\lambda_1 + \lambda_2} \mu_X + \frac{\lambda_2}{\lambda_1 + \lambda_2} \mu_Y = 0.809$$

and the safety loading:

$$c - (\lambda_1 + \lambda_2)\mu_X' - \frac{1}{2} \bar{\lambda}\mu_Y' = 0.385 > 0.$$ 

Thus, $\psi(u) < 1$ for all $u \geq 0$. Now for the trial process, we inflate $\eta_X$ and $\eta_Y$ to the extent such that the trial safety loading becomes negative. In particular, we set
\( \eta_X = 1.5 \) and \( \eta_Y = 0.8 \) and the safety loading becomes

\[
c - (\lambda_1 + \lambda_2)\eta_X - \frac{1}{2} \lambda \eta_Y
\]

\[
= c - (\lambda_1 + \lambda_2) \left[ \frac{\lambda_1}{\lambda_1 + \lambda_2} \eta_X + \frac{\lambda_2}{\lambda_1 + \lambda_2} \eta_Y \right] - \frac{1}{2} \lambda (\eta_X + \eta_Y)
\]

\[
= -1.35 < 0.
\]

i.e., the trial process ruins in finite time almost surely.

Table 4.3 compares our results with those of Yuen, Guo and Wu (2002). It is clear that both are quite close to each other and again, one cannot compare them in the zone of interest because Yuen, Guo and Wu (2002) do not provide those values. Moreover, since explicit formula in this case is not available, we try to compare our results with the asymptotic bounds derived by Yuen, Guo and Wu (2002).

<table>
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<tr>
<th>( u )</th>
<th>0</th>
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<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
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<td>Importance sampling</td>
<td>0.8778</td>
<td>0.6702</td>
<td>0.5357</td>
<td>0.4265</td>
<td>0.3266</td>
<td>0.2454</td>
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<tr>
<td>Crude Monte Carlo</td>
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<td>0.7009</td>
<td>0.5575</td>
<td>0.4444</td>
<td>0.3545</td>
<td>0.2828</td>
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<table>
<thead>
<tr>
<th>( u )</th>
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<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0565</td>
<td>0.0356</td>
<td>0.0162</td>
<td>0.0068</td>
</tr>
<tr>
<td>Crude Monte Carlo</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
</tr>
</tbody>
</table>

Table 4.3: Ruin probability \( \psi(u) \) from importance sampling and the crude monte carlo methods by Yuen, Guo and Wu (2002) for different initial surplus

Let

\[
h_1(r) = \int_0^\infty \exp\{rx\}dF_{X'}(x) - 1
\]

\[
h_2(r) = \int_0^\infty \exp\{rx\}dF_{Y'}(x) - 1.
\]
CHAPTER 3. NUMERICAL METHOD

Assume there exists $r_1 > 0$ and $r_2 > 0$ such that

$$\lim_{r \to r_1} h_1(r) = \infty \text{ and } \lim_{r \to r_2} h_2(r) = \infty.$$ 

Then

$$\lim_{u \to -\infty} \exp\left( Ru \right) \left( \psi(u) + \psi_1(u) \right) \leq \frac{\rho}{1 + \rho (\lambda_1 + \lambda_2) h_1^{(1)}(R) + (1/2) \lambda h_2^{(1)}(R) - c},$$

where $\rho$ is the relative security loading

$$\rho = \frac{c}{(\lambda_1 + \lambda_2) \mu_X + (1/2) \lambda \mu_Y} - 1,$$

$R$ is the positive solution of the equation

$$(\lambda_1 + \lambda_2) h_1(r) + \frac{1}{2} \lambda h_2(r) = cr,$$

and $h_1^{(1)}$ and $h_2^{(1)}$ are the first derivatives of $h_1$ and $h_2$, respectively.

The proof of the above asymptotic bound is in Yuen, Guo and Wu (2002) and is not going to be repeated here. What we would like to study in this subsection is the behavior of the bound. That is, we would like to investigate the difference

$$\exp\left\{ -Ru \right\} \frac{\rho}{1 + \rho (\lambda_1 + \lambda_2) h_1^{(1)}(R) + (1/2) \lambda h_2^{(1)}(R) - c} \left( \psi(u) + \psi_1(u) \right) - c$$

where $u$ is large. For the case of $X_i$ being i.i.d. exponential with mean $\mu_X$ and $Y_i$ being i.i.d. exponential with mean $\mu_Y$ and $\mu_X \neq \mu_Y$, it is easy yet tedious to compute the following functions:
\[ h_1(r) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \left( \frac{1}{1 - r \mu_X} \right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \left( \frac{1}{1 - r \mu_Y} \right) - 1 \]

\[ h_1^{(1)}(r) = \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \left( \frac{\mu_X}{(r \mu_X - 1)^2} \right) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \left( \frac{\mu_Y}{(r \mu_Y - 1)^2} \right) \]

\[ h_2(r) = \frac{\mu_X}{\mu_X - \mu_Y} \left( \frac{1}{1 - r \mu_X} \right) - \frac{\mu_Y}{\mu_X - \mu_Y} \left( \frac{1}{1 - r \mu_Y} \right) - 1 \]

\[ h_2^{(1)}(r) = \frac{\mu_X^2}{\mu_X - \mu_Y} \left( \frac{1}{(r \mu_X - 1)^2} \right) - \frac{\mu_Y^2}{\mu_X - \mu_Y} \left( \frac{1}{(r \mu_Y - 1)^2} \right) \cdot \]

Thus, \( r_1 = r_2 = 1/\mu_X \) ensures

\[ \lim_{r \to r_1} h_1(r) = \infty \quad \text{and} \quad \lim_{r \to r_2} h_2(r) = \infty. \]

Under our parameter configuration, \( R = 0.1031 \) and \( \rho = 9.2857 \). By using \( \psi(0) = 0.8799 \), \( \psi(35) = 0.0068 \), \( n = 1000 \) and \( M = 10^5 \). Our estimates \( [\psi(u) + \psi_1(u)]/2 \) is plotted against \( u \) in Figure 4.3. In the same figure, we also plot the bound

\[ \exp\{-Ru\} \frac{\rho}{1 + \rho (\lambda_1 + \lambda_2) h_1^{(1)}(R) + (1/2) \lambda h_2^{(1)}(R) - c}. \]

One can see that the difference between the bound and the estimate seems to be an increasing function of \( u \).

The absolute relative error in Figure 4.4 confirms that is the case. In fact, the relative error could go up to 6 when \( u = 35 \). Thus the bound is not very effective when \( u \) is large. Note that such bound for \( [\psi(u) + \psi_1(u)]/2 \) is mainly of theoretical interest and does not make any practical sense because \( \psi_1(u) \) is only an auxiliary variable.
Figure 4.1: Ruin probability $\psi(u)$ under different method with $u_{\text{max}} = 20$
Figure 4.2: Absolute relative error under different method with $u_{\text{max}} = 20$
Upper bound for $\psi(u) + \psi_1(u)$, $u = 35$

Figure 4.3: Asymptotic upper bound for $\psi(u) + \psi_1(u)$ with $u_{\text{max}} = 35$
Figure 4.4: Absolute relative error between the solution and the upper bound under the unequal means case.
Chapter 5

Conclusion

This thesis applies the hybrid methodology developed by Wong, Ho, Hu and Liu (2006) to the ruin probability estimation of a class of surplus process driven by two correlated claim processes in the sense of Yuen, Guo and Wu (2002). In Wong, Ho, Hu and Liu (2006), the ruin probability for an Erlang(2)-driven with continuous compounding model is the focus. Their idea is to make use of the integro-differential equation and turn that into an integral equation in terms of the initial conditions. By making use of the numerical techniques, the integral equation is approximated by a system of linear equations with unknown boundary conditions. The boundary conditions then are estimated by the importance sampling method.

In this thesis, the ruin probability problem of two correlated claim processes is the solution of two linear integral equations. Applying the hybrid methodology, the approximate solution is given by a substantially more complex system of linear
equations. According to the presentation in Chapter 4, our result is remarkably close to the exact solution under the assumption that all claims are i.i.d. exponentially distributed. For the case while no explicit formula is available, our solution shows that the upper asymptotic bound derived in Yuen, Guo and Wu (2002) may not be very effective. In fact, the bound in Yuen, Guo and Wu (2002) derived is mainly of mathematical interest because that is a bound for \( \psi(u) + \psi_1(u) / 2 \) and \( \psi_1(u) \) is only an auxiliary variable.

In our numerical study, the case for exponential claims with equal and unequal means are considered. To further demonstrate our hybrid methodology, it is possible to consider other claim distribution such as gamma distribution. Nevertheless, this may leave to a future improvement under this topic.

Many possible future research directions are generated from this thesis. One important topic is the error analysis. As we have discussed, the importance sampling estimate of the form:

\[
\psi(u) = E_n[I\{T < \infty\} \times \frac{f(T)}{f^*(T)}] \approx \hat{\psi}(u) = \frac{1}{M} \sum_{k=1}^{M} \frac{f(T_k)}{f^*(T_k)}
\]

where \( f \) and \( f^* \) are the density of the time of ruin under the original process and the trial process, respectively. \( T_k \) are i.i.d. sampled from the trial process. The error of \( \hat{\psi}(u) \) can be quantified by using the central limit theorem

\[
\sqrt{M}(\psi(u) - \hat{\psi}(u)) \overset{d}{\rightarrow} N(0, \text{Var} \frac{f(T_k)}{f^*(T_k)}).
\]

On top of such error, one should also study the error created by the trapezoidal
approximation. Although all these loose ends are left to the other researchers, we still hope this thesis could help popularizing the use of hybrid methodology in risk theory.

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