On Prime Spectrum of Commutative Ring

by

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Abstract

The inclusion ordering and Zariski topology on prime spectrum is an example of sets in which both the ordering and topology are compatible. In this thesis, we presents a study on the interplay relationships between these two structures on the prime spectrums and its underlying rings.

The notion of spectral spaces and patch topology introduced by M. Hochster [16] are studied. Characterizations of some spectral posets were obtained by W. J. Lewis and J. Ohm in [27]. In this thesis, we include some interesting results obtained by them. Moreover, we also provide a partial solution to a question about ordered disjoint union proposed by W. J. Lewis and J. Ohm [27]. In addition, the Zariski topology and their relationships with the algebraic properties of a ring are studied. The result of G. DeMarco and A. Orsatti [6] that a ring is pm if and only if its maximal spectrum is a retract of the prime spectrum is introduced. Moreover, a dual statement concerning Baer rings and their minimal prime spectrums obtained by J. Kist [23] is included.

As a case study, the investigation of S. Fischer [8] [7] on prime spectrums of Bézout rings is mentioned. Besides, we also present a detailed proof of D. Lazard's characterization [25] concerning the property A(0) in terms of the D-closed subsets of prime spectrum, together with an example provided by W. J. Lewis and J. Ohm which shows that the property A(0) cannot be determined merely by the ordering structure on prime spectrum. Although the ordering structure determines the Zariski topology on prime spectrum for Noetherian rings, an example is supplied to illustrate that Noetherianness cannot be determined alone by the topology on prime spectrum. Applications of some results mentioned previously in the thesis are provided in order to emphasize that some algebraic results can indeed be obtained by means of topological arguments.
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Introduction

It is well-known that the set of all prime ideals of a commutative ring with unity is called the prime spectrum of the ring. In fact, the prime spectrum can be considered as a poset under the set inclusion as the ordering as well as a topological space under the Zariski topology. The prime spectrum considered as a topological space has many interesting and special properties. In his remarkable thesis [16], M. Hochster firstly studied some of these properties and arrived at the notion of spectral spaces, which characterize all those spaces that can be realized as the prime spectrums of some rings. In the literature, the notion of Zariski topology on the prime spectrum and their relations with some algebraic properties of the rings were studied by J. Ohm and R. L. Pendleton [29], G. DeMarco and A. Orsatti [6], D. Lazard [25] and others. W. J. Lewis [26] studied the possibility of two classes of posets to be spectral, in an attempt to answer a question raised by I. Kaplansky [20]. His results were then generalized by W. J. Lewis himself and J. Ohm in [27]. In their paper, they introduced the concept of C(m) topology which can be used to enhance the determination of whether a poset is spectral. Besides, it is also interested to study the algebraic properties of a ring from some subspace of its prime spectrum. For instance, J. Kist [23] found that a reduced ring is Baer if and only if the minimal prime spectrum is a retract of the prime spectrum. Other results about a ring and its minimal prime spectrum can also be found in [1], [28] and [33].
In this thesis, we present a study on the interplay relations between rings and their prime spectrums. In particular, we notice that the ordering structure and Zariski topology on the prime spectrum of a ring can influence the algebraic properties of the ring itself. Some applications are also provided in the thesis. We aim to emphasize that some algebraic results about a ring can be obtained via topological approach.

In chapter 1, the notion of spectral spaces and patch topology are introduced. Characterizations of spectral spaces and maxspectral spaces in terms of the patch topologies are mentioned. These results are mainly due to M. Hochster [17].

In chapter 2, some necessary conditions which lead to a poset be spectral are discussed. The classes of Bézout, Arithmetical and Prüfer rings are mentioned, due to the distinguished ordering structures of their prime spectrums. In particular, the work of W. J. Lewis and J. Ohm on this topic are presented. We shall sketch how they employ the notion of ordered disjoint unions in generalizing the theorem of W. J. Lewis [26] on spectral trees with finitely many minimal elements. In this connection, we also give a partial answer to a question posed in [27].

In chapter 3, the interplay relationships between the Zariski topology on prime spectrum and the ring are investigated. It was found by G. DeMarco and A. Orsatti [6] that a ring is pm is equivalent to its prime spectrum being normal. In this case, we observe that the maximal spectrum is a retract of the prime spectrum. A dual statement of the above result concerning the minimal prime spectrum was provided by G. Artico and U. Marconi in [1]. Moreover, the characterizations of minspectral spaces obtained by M. Hochster [17] will be elaborated in this chapter as well.
Introduction

In chapter 4, we aim to elaborate more about those results that were studied in the preceding chapters. Some applications of these results are provided as well. As a case study, the investigation about the prime spectrums of Bézout rings due to S. Fischer [8] will be mentioned. In particular, a counterexample supplied by S. Fischer to answer a question raised by A. V. Geramita is included. We demonstrate here that some algebraic objects can be constructed via topological means. The work of D. Lazard [25] about D-closed subsets and their relationships with the property A(0) are elaborated with detailed proofs. Furthermore, the notion of C(m) topology suggested by W. J. Lewis and J. Ohm [27] are discussed. On the other hand, we point out that for Noetherian rings, the ordering structure on prime spectrum determines the Zariski topology completely. An example of a non-Noetherian ring with Noetherian prime spectrum is provided. This example serves as an evidence that the Noetherian condition cannot be determined by using the Zariski topology alone on its prime spectrum. At the end of the chapter, some applications of several results discussed previously are provided in order to emphasize that some algebraic results can indeed be obtained by using topological arguments.
Chapter 1

Spectral spaces

In this chapter, we give some basic definitions and preliminary results of M. Hochster [16] that are required frequently in the subsequent work. For the sake of clarity and to avoid superfluity, those not yet required will be provided whenever it is appropriate.

1.1 Basic notions

1.1.1 Order compatible topology

Let $X$ be a set. It may come equipped with a partial ordering $\mathcal{O}$ or a topology $T$.

Given $(X, \mathcal{O})$ a poset $X$ with partial ordering $\mathcal{O}$ and $x \in X$. The generization $G_{X, \mathcal{O}}(x)$ and specialization $S_{X, \mathcal{O}}(x)$ of $x$ in $X$ are defined by

$$G_{X, \mathcal{O}}(x) = \{ y \in X | y \leq x \} \quad \text{and} \quad S_{X, \mathcal{O}}(x) = \{ y \in X | y \geq x \}.$$ 

More generally, if $A$ is a subset of $X$, then
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\[ G_{X,\mathcal{O}}(A) = \{ y \in X | y \leq a \text{ for some } a \in A \}; \]
\[ S_{X,\mathcal{O}}(A) = \{ y \in X | y \geq a \text{ for some } a \in A \}. \]

Let \( Y \) be a subset of \( X \). \( Y \) is said to be closed under generization (specialization respectively) if \( G_{X,\mathcal{O}}(y) \subseteq Y \) \( (S_{X,\mathcal{O}}(y) \subseteq Y \text{ resp.)} \) for all \( y \in Y \). Given \((X, T)\) a space \( X \) with topology \( T \), we write \( \{x\}_{X,T} \) for the \( T \)-closure of \( \{x\} \) in \( X \). Subscripts will be dropped whenever the notation is clear from the context.

Let \( X \) be a poset with partial ordering \( \mathcal{O} \). A topology \( T \) on \( X \) is called compatible with \( \mathcal{O} \) if \( \{x\} = S(x) \) for all \( x \in X \). It is easy to see that \( T \) is compatible with \( \mathcal{O} \) if and only if the following conditions are satisfied:

(i) \( S(x) \) is \( T \)-closed for all \( x \in X \), and

(ii) \( T \)-closed sets are closed under specialization.

Let \( Y \) be a subset of \( X \). Then \((Y, \mathcal{O}|_Y)\) means that \( Y \) is a poset with the induced ordering of \( \mathcal{O} \) restricted on \( Y \). Also \((Y, T|_Y)\) means a space \( Y \) with the subspace topology. If \( T \) and \( \mathcal{O} \) are compatible, then so are \( T|_Y \) and \( \mathcal{O}|_Y \).

Let \((X, T)\) be a \( T_0 \) space. Then \( X \) can be partially ordered by a partial order \( O(T) \) induced by \( T \), where \( O(T) \) is defined by specifying \( x \leq y \) if and only if \( y \in \{x\} \). Conversely, if \((X, O)\) is a poset, then \( X \) can be topologized by a topology \( T(O) \) which is induced by \( O \), where \( T(O) \) is the topology which has \( \{S(x) | x \in X\} \) as a subbase for its closed sets. We shall call this topology the closure of points (COP for short) topology. Clearly, the COP-topology is \( T_0 \) and is the weakest topology compatible with \( \mathcal{O} \) i.e. another topology \( T' \) is \( \mathcal{O} \)-compatible if and only if \( T' \) is stronger than the COP-topology \( T(O) \). (Recall that for any two topologies \( T_1, T_2 \) on a space \( X \), \( T_1 \)

\(^1\)We use \( \subseteq \) to denote inclusion and \( \subset \) to denote proper inclusion.
is weaker than $\mathcal{T}_2$ if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.) Also, it is easy to see that $\mathcal{T}$ is compatible with $\mathcal{O}$ if and only if $\mathcal{O}(\mathcal{T}) = \mathcal{O}$. Thus, it is possible to recover the ordering from any given order compatible topology. Moreover, if $(X, \mathcal{T}) \approx (X', \mathcal{T}')$, then $(X, \mathcal{O}(\mathcal{T})) \simeq (X, \mathcal{O}(\mathcal{T}'))$ (We use $\approx$ and $\simeq$ to denote homeomorphism and order isomorphism respectively). However, any two topological spaces $(X, \mathcal{T})$ and $(X', \mathcal{T}')$ satisfying $(X, \mathcal{O}(\mathcal{T})) \simeq (X, \mathcal{O}(\mathcal{T}'))$ does not necessarily imply that $(X, \mathcal{T}) \approx (X', \mathcal{T}')$. It is because that there exists poset $(X, \mathcal{O})$ which has more than one $\mathcal{O}$-compatible topology.

### 1.1.2 Prime spectrums and Zariski topology

We always mean $R$ a commutative ring with unity. A non-empty subset $I$ of a ring $R$ is an ideal of $R$ if the following conditions are satisfied.

(i) $I \neq R$;

(ii) $a, b \in I$ implies $a + b \in I$;

(iii) $r \in R, a \in I$ implies $ra \in I$.

We always partially order the set of all ideals $\mathcal{I}(R)$ of $R$ by set inclusion (known as inclusion ordering) in the following sense: For any $I, J \in \mathcal{I}(R)$, $I \subseteq J$ if and only if $I \subseteq J$. An ideal $P$ of $R$ is said to be prime if $ab \in P$ implies $a \in P$ or $b \in P$. An ideal $M$ of $R$ is said to be maximal if and only if $M$ is maximal in the poset $\mathcal{I}(R)$ with respect to set inclusion as its ordering.

The prime spectrum $\text{Spec}(R)$ (maximal spectrum $\text{Max}(R)$ resp.) of a ring $R$ is the set of all prime (maximal resp.) ideals of $R$ which is partially ordered by set inclusion. Clearly $\text{Max}(R) \subseteq \text{Spec}(R) \subseteq \mathcal{I}(R)$.

---

2 An order-preserving bijective mapping is called an order isomorphism.
For any subset $I$ of $R$, let $V(I) = \{ P \in \text{Spec}(R) | I \subseteq P \}$ and $D(I) = \text{Spec}(R) \setminus V(I)$. Then the topology $\{ D(I) | I \subseteq R \}$ is called the Zariski topology on $\text{Spec}(R)$. Equivalently, a subset $F$ of $\text{Spec}(R)$ is closed in the Zariski topology if and only if there exists $I \in \mathcal{I}(R)$ such that $F = V(I)$. $\text{Spec}(R)$ with Zariski topology and inclusion ordering is an example where the topology and ordering are compatible. We will always consider $\text{Spec}(R)$ as a topological space endowed with Zariski topology and as a poset with set inclusion ordering. As $\text{Max}(R) \subseteq \text{Spec}(R)$, we always consider $\text{Max}(R)$ as a topological space with the subspace topology.

### 1.1.3 Lattice-ordered groups

By an ordered group, we mean an (additive) abelian group $G$ which contains a subset $G^+$ satisfying the following conditions:

(i) $G^+ + G^+ \subseteq G^+$;

(ii) $G^+ \cap (-G^+) = \{0\}$.

We shall call $G^+$ the positive elements (or positive cone) of $G$ and an order relation $\leq$ is defined on $G$ by: $f \leq g$ if and only if $g - f \in G^+$ for all $f, g \in G$.

A lattice-ordered group is an ordered group $G$ in which every pair of elements has an infimum. We shall use the notation $\inf_G(f, g)$ for the infimum of $f$ and $g$ for any $f, g \in G$. The subscript will be dropped whenever the notation is clear from its context. By a lattice isomorphism between two lattice-ordered groups $G$ and $G'$, we mean a group isomorphism $\phi$ such that $\phi(\inf_G(g, h)) = \inf(\phi(g), \phi(h))$ for all $g, h \in G$. It is easy to see that every lattice isomorphism must be an order isomorphism. We say that $G$ is lattice isomorphic to $G'$ if such mapping $\phi$ exists.
Let $G$ be a lattice-ordered group. A proper subset $Q$ of $G^+$ is called a prime $V$-segment provided the following conditions are satisfied:

(i) $Q + G^+ = Q$;

(ii) $G^+ \setminus Q$ is closed under addition;

(iii) $\inf(f, g) \in Q$ whenever $f, g \in Q$.

By the prime spectrum $\text{Spec}(G)$ of $G$, we mean the set of all prime $V$-segments of $G$ which is partially ordered by set inclusion. For any $f \in G^+$, let $V(f) = \{Q \in \text{Spec}(G) | f \in Q\}$ and $D(f) = \text{Spec}(G) \setminus V(f)$. Then $\text{Spec}(G)$ can be topologized by taking $\{V(f) | f \in G^+\}$ as its closed subbase. Hence, $\text{Spec}(G)$ is another example of set in which the topology and order are compatible.

1.1.4 Spectral spaces and patch topology

For any topological space $(X, T)$, $X$ is said to be:

(a) quasi-compact if every open covering of $X$ has a finite subcovering.

(b) compact if $X$ is quasi-compact and Hausdorff.

(c) sober if $X$ is $T_0$ and each closed irreducible subset $F$ (i.e. not the union of two proper closed subsets of $F$) has a generic point. By a generic point of $F$, we mean a point $p$ of $F$ such that $\{p\} = F$.

(d) coherent if $X$ has an open base $B$ consisting of quasi-compact sets and $B$ is closed under finite intersection (such open base $B$ of $X$ is called a QCI-base for $X$).
Then a space \((X, T)\) is said to be \textit{spectral} if the space \(X\) is quasi-compact, sober and coherent. Under such conditions, \(T\) is called a \textit{spectral topology} on the space \(X\).

Under the Zariski topology, we have \(p \subseteq q\) if and only if \(q \in \{p\}\) for any \(p, q \in \text{Spec}(R)\). Hence, \(\text{Spec}(R)\) clearly satisfies the \(T_0\) separation axiom. In addition, it is also easy to observe that an open set \(U\) in \(\text{Spec}(R)\) is quasi-compact if and only if \(U = D(I)\) for some finitely generated ideal \(I\). Therefore \(\text{Spec}(R)\) is quasi-compact and has an open base \(B\) consisting of quasi-compact open sets. Moreover, the intersection of any two quasi-compact open sets is still quasi-compact, i.e. \(B\) is a QCI-base. It can be easily shown that every irreducible closed subset \(F\) of \(\text{Spec}(R)\) is of the form \(V(P)\) for some \(P \in \text{Spec}(R)\), in other words, \(F\) has a generic point. We then obtain the result that \(\text{Spec}(R)\) is a spectral space.

Let \(X\) be a spectral space. By the \textit{patch topology} on \(X\) we mean the topology which is obtainable by taking, as a closed subbase, those sets that are either closed or quasi-compact open in the original spectral topology (or to say, the patch topology has the quasi-compact open sets and their complements in the original topology as an open subbase). A subset of \(X\) is called a \textit{patch} if it is closed in the patch topology on \(X\).

\subsection*{1.2 Properties of patches and the patch topology}

In general, a spectral space \(X\) needs not be compact (Note that a spectral space must be quasi-compact), those which are Hausdorff corresponds to a particular subclass of spectral spaces (See chapter 3). However, \(X\) would turn out to be a compact space if it is retopologized by the patch topology due to the following

\footnote{To emphasize that \(p, q\) are also points of the space \(\text{Spec}(R)\) rather than just the prime ideals of \(R\), sometimes we write the prime ideal in small letter.}
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proposition given in [16].

**Proposition 1.2.1** Let \( X \) be a spectral space, then the patch topology on \( X \) is compact.

**Proof.** Certainly, the patch topology is Hausdorff. The quasi-compactness will follow if every family of closed and quasi-compact open sets maximal with respect to having the finite intersection property (FIP, for short) intersects. Suppose \( \mathcal{F} \) is such a family and \( F \) is the intersection of all closed sets in \( \mathcal{F} \), then \( F \in \mathcal{F} \) for otherwise \( \mathcal{F} \subset \mathcal{F} \cup \{F\} \) contradicts to the maximality of \( \mathcal{F} \). Moreover, \( F \) must be irreducible. For otherwise there exists a proper non-empty closed subset \( K \) of \( F \), it then follows that \( \mathcal{F} \subset \mathcal{F} \cup \{K\} \) with the latter having the FIP, which contradicts to the maximality of \( \mathcal{F} \). Since \( X \) is spectral, \( F \) contains a generic point \( p \), whence \( p \in \cap \mathcal{F} \) and \( \cap \mathcal{F} \neq \emptyset \).

The next proposition gives some information about patches.

**Proposition 1.2.2** Let \( Y \) be a patch in a spectral space \( X \). Then the following statements hold.

(a) \( Y \) is a spectral space and the patch topology on \( Y \) is the relative patch topology inherited from \( X \).

(b) If \( \overline{Y} \) is the spectral closure of \( Y \) (i.e. the closure of \( Y \) with respect to the original spectral topology) in \( X \), then

\[
\overline{Y} = \{ x \in X | x \geq y \text{ for some } y \in Y \}.
\]

**Proof.**

(a) It is easy to see that \( Y \) is a quasi-compact \( T_0 \) space having a QCI-base, using the fact that \( Y \) is closed in the patch topology. Suppose \( F \) is an irreducible closed
subset in \( Y \), then so is it in \( X \), whence contains a generic point \( p \), which closure in \( Y \) is just \( F \), i.e. \( F \) has a generic point in \( Y \). Therefore \( Y \) is a spectral space.

Let \( \mathcal{P}(Y), \mathcal{P}(X)|_Y \) be the patch topology on \( Y \) and the relative patch topology inherited from \( X \) respectively. Clearly \( \mathcal{P}(X)|_Y \) is Hausdorff and is weaker than \( \mathcal{P}(Y) \). Since \( \mathcal{P}(Y) \) is compact, the patch topology on \( Y \) and the relative patch topology agree on \( Y \).

(b) Suppose \( x \in \overline{Y} \). Consider the family \( \mathcal{F} \) containing \( Y \) and all quasi-compact spectral open neighbourhoods of \( x \). Since the patch topology on \( X \) is quasi-compact and \( \mathcal{F} \) is a collection of closed sets in the patch topology having the FIP, there exists \( y \in \cap \mathcal{F} \), in particular \( y \in Y \). Now \( y \) belongs to all quasi-compact spectral open neighbourhoods of \( x \) implies \( x \in \{y\} \). The converse part is trivial. \( \square \)

1.3 Properties of spectral spaces

It has been mentioned that the prime spectrum \( \text{Spec}(R) \) of any ring \( R \) must necessarily be spectral. The converse statement still holds by using the following remarkable theorem of Hochster [16, Theorem 6].

Theorem 1.3.1 Every spectral space is homeomorphic to \( \text{Spec}(R) \) for some commutative ring \( R \) with unity.

The proof of this theorem is rather technical and lengthy. The basic idea is this: if a ring having a given prime spectrum \( X \) can be found, then such a reduced ring can be found. As a reduced ring can be represented as a ring of functions on its spectrum and the values at a given point will be taken in the residue class domain at that point, Hochster then partially axiomatized this situation and eventually he
arrived at the notion of a spring. Not every spring comes from a ring, those that do is called affine spring. One of the key points in the proof of Hochster is to introduce the notion of index (In the precise formulation, an index is actually a family of valuations indexed by the pairs of points \((y, x)\) of the underlying space such that \(x \in \{y\}\)), an extra structure which enables us to modify a given spring into an affine spring. Together with a family of suitably chosen indeterminates, Hochster described how an indexed spring (i.e. a spring with an index) can be constructed from a given spectral space. He then showed that the indexed spring obtained can be simple, which is a sufficient condition that enables one to convert an indexed spring into an affine spring. Then the underlying ring of the affine spring obtained will be the ring in which its prime spectrum is homeomorphic to the given spectral space. It should be remarked here that Hochster worked with not merely objects but a category at every stage.

Hence, it is now clear that every topological space is homeomorphic to the prime spectrum of some ring \(R\) if and only if it is spectral.

Now we mention some interesting facts about spectral spaces given in [16].

**Proposition 1.3.2** Let \(X\) be a quasi-compact \(T_0\) space which has a QCI-base. The following conditions are equivalent:

(a) \(X\) is spectral.

(b) Every non-empty irreducible closed subspace of \(X\) has a generic point.

(c) Every family of quasi-compact open subsets of a closed subspace of \(X\) with the FIP has non-empty intersection.

(d) \(X\) with the patch topology is a compact space with a base of open and closed sets.
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(e) $X$ with the patch topology is quasi-compact.

(f) A family of patches in $X$ with FIP has non-empty quasi-compact intersection.

Proof.

(e) $\Rightarrow$ (d) It follows from the fact that $X$ with the patch topology is a Hausdorff space with a base of closed and open sets.

(d) $\Rightarrow$ (e) Trivial.

(a) $\Rightarrow$ (e) By proposition 1.2.1.

(e) $\Rightarrow$ (f) Trivial.

(f) $\Rightarrow$ (c) Since every quasi-compact open subset of a closed subspace of $X$ is a patch in $X$, the result follows.

(c) $\Rightarrow$ (b) Let $Y$ be a non-empty irreducible closed subspace of $X$, then the family $\mathcal{F}$ of all non-empty quasi-compact open subsets of $Y$ has FIP, hence there exists a point belongs to $\cap \mathcal{F}$, which is then a generic point of $Y$.

(b) $\Rightarrow$ (a) Trivial. $\square$

We call a space quasi-Hausdorff if any two points in it either have disjoint open neighbourhoods or are in the closure of a point in the space. We then have:

Proposition 1.3.3 Every spectral space is quasi-Hausdorff.

Proof. Suppose $x, y$ are two distinct points in $X$ which cannot be separated by two disjoint open neighbourhoods. Consider the family $\mathcal{F}$ of all quasi-compact open neighbourhoods of $x$ and $y$. Then $\mathcal{F}$ has FIP implies that there exist $z \in \cap \mathcal{F}$, by the quasi-compactness of the patch topology. Hence $x$ and $y$ belong to the closure
(89x829) of z, as desired.

In fact, this result has a partial converse (See proposition 1.3.5 below). First, we recall that any $T_0$ space induce a natural partial ordering, namely $y \leq x$ if $x \in \{y\}$ (See section 1.1.1). We then have:

**Proposition 1.3.4** Every lower directed set $^4 Y$ in a spectral space $X$ has a unique infimum $y$ in $Y$.

**Proof.** Since $Y$ is lower directed, $\overline{Y}$ is irreducible and hence $\overline{Y}$ contains a generic point $y$, which is the required unique infimum.

**Proposition 1.3.5** Suppose $X$ is a quasi-compact $T_0$ space having a QCI-base. Then $X$ is spectral if and only if every closed subspace is quasi-Hausdorff and every lower directed set has an infimum in its closure.

**Proof.**

$\Rightarrow$) Follows by proposition 1.3.3 and proposition 1.3.4.

$\Leftarrow$) We check that (b) of proposition 1.3.2 holds. Suppose $Y$ is a non-empty closed irreducible subspace of $X$. Then $Y$ is closed implies $Y$ is quasi-Hausdorff. Together with irreducibility of $Y$, it follows that the set $Y$ is lower directed. By assumption, $Y$ then has an infimum which is a generic point of $Y$.

Observing that the notions of patch topology and patch (See section 1.1.4) make sense whether the space considered is spectral or not. Now, we extend them to arbitrary spaces. For convenience, we call a space $X$ equipped with the patch topology the patch space of $X$. Before giving the topological duality of spectral spaces obtained by M. Hochster, the following result is needed.

$^4$A poset is lower directed if every finite subset has a lower bound.
Proposition 1.3.6 Let $X$ be a compact space and $\mathcal{U}$ be a family of open and closed (or equivalently, compact open) subsets of $X$. Then $X$ with the topology which has $\mathcal{U}$ as an open subbase is spectral if and only if it is $T_0$, in which case $X$ with its original topology is its patch space. Conversely, every spectral space arises from its patch space in this way.

Proof. Let $T$ be the topology with $\mathcal{U}$ as its open subbase. Suppose $(X, T)$ is $T_0$. Since $T$ is weaker than the original topology, $(X, T)$ is quasi-compact. It is easy to see that the base generated by $\mathcal{U}$ is a QCI-base for $(X, T)$. Therefore, the hypothesis of proposition 1.3.2 is satisfied. To show that $(X, T)$ is spectral, it suffices to show that the patch space of $(X, T)$ is the space $X$ with original topology, by condition (e) of proposition 1.3.2. But the patch topology is Hausdorff and weaker than the original topology on $X$. Since $X$ is compact, and so they agree on $X$. The converse part is trivial. \[\Box\]

Corollary 1.3.7 A space $X$ with a base of quasi-compact open sets is spectral if and only if its patch space is compact.

Proof. Let $X$ be a space with a base of quasi-compact open sets. If the patch space of $X$ is compact, then $X$ must be $T_0$. Applying proposition 1.3.6 to the patch space of $X$ with $\mathcal{U}$ as the family of all quasi-compact open subsets of $X$, then the result follows. \[\Box\]

We can now obtain the topological duality mentioned before.

Proposition 1.3.8 Let $X$ be spectral. Retopologize $X$ by taking as a base for the closed sets the quasi-compact open sets of $X$. Then $X$ endowed with this new topology is spectral and the new order induced on $X$ by this topology is precisely the reverse of the original order.
Chapter 1. Spectral spaces

Proof. The patch space of $X$ with this new topology is clearly identical with the patch space of $X$ with the original topology, and $X$ in the new topology has a subbase for its open sets consisting of sets which are compact open in the patch topology. The new topology is clearly $T_0$, and it follows from proposition 1.3.6 that $X$ endowed with this new topology is a spectral space. The fact that the order is reversed is obvious.

\[\square\]

1.4 Another characterization of spectral spaces

We define a continuous map of spectral spaces to be spectral if inverse images of quasi-compact open sets are quasi-compact. It is easy to see that if $X, Y$ are spectral spaces, then a mapping $f : X \to Y$ is spectral if and only if it is continuous in both the original and patch topologies. Moreover, $f(X)$ is a patch.

Following that in [16], we now introduce a very simple spectral space $W$. Let $W$ be the space \{0, 1\} endowed with the topology \{\emptyset, \{0\}, \{0, 1\}\}. Note that the ordering induced by the topology on $W$ is the usual one, $0 < 1$. Moreover, it is easy to see that $W$ is spectral.

Let $\mathcal{F}$ be a family of mappings on a topological space $X$. $\mathcal{F}$ is said to separate points if for each pair of distinct points $x$ and $y$ in $X$, there is $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. $\mathcal{F}$ is said to separate points and closed sets if for each closed subset $A$ of $X$ and each $x$ in $X\setminus A$, there is $f \in \mathcal{F}$ such that $f(x) \notin \overline{f(A)}$.

In proving proposition 1.4.2, the following lemma (See [22, pages 116-117]) is crucial.

Lemma 1.4.1 (Embedding Lemma) Suppose that $\mathcal{F}$ is a family of continuous
mappings from a topological space $X$ to another topological space $Y$. Then

(a) The evaluation map $e$ from $X$ to $Y^F$ is continuous.

(b) The mapping $e$ is an open mapping from $X$ onto $e(X)$ if $F$ separates points and closed sets.

(c) The mapping $e$ is one-to-one if and only if $F$ separates points.

We can then give another characterization of spectral spaces obtained by Hochster [16].

**Proposition 1.4.2** A topological space $X$ is spectral if and only if it is homeomorphic with a patch in a product of copies of $W$.

**Proof.** Suppose $X$ is a spectral space. Let $V$ be the set of all spectral maps from $X$ to $W$. Let $e : X \to W^V$ be the evaluation map. Since $X$ is a $T_0$ space having a QCI-base, $V$ separates (1) points, as well as (2) points and closed sets. By lemma 1.4.1, $X$ is homeomorphic with $e(X)$, while the latter is a patch in the spectral space $W^V$. The converse part is trivial. \( \square \)

### 1.5 The maxspectral spaces

Along similar directions, it is natural to ask whether there are some conditions which can be used to characterize those spaces which are homeomorphic to the maximal spectrum of some ring $R$. The answer is affirmative and was given by Hochster [16, Proposition 11]:

**Theorem 1.5.1** A topological space is homeomorphic with the maximal spectrum of some ring $R$ (with the subspace topology inherited from $\text{Spec}(R)$), i.e. with the
subspace of closed points of some spectral space if and only if the space is $T_1$ and quasi-compact.

**Proof.** Suppose $M$ is $T_1$ and quasi-compact. Let $V$ be the set of all continuous mappings from $M$ to $W$. Let $f$ be the evaluation map from $M$ to $W^V$. Since $M$ is $T_1$, it is easy to see that $f$ is an embedding, by the lemma 1.4.1. Let $X$ be the closure of $f(M)$ in the patch topology on $W^V$, but endowed with the relative product topology, whence $X$ is a spectral space. Now $M$ is $T_1$ guarantees that every point of $f(M)$ is closed in $X$ clearly. On the other hand, the quasi-compactness of $M$ ensures that every closed point in $X$ is actually in $f(M)$. The converse part is trivial.

We shall call a topological space satisfying the condition in the above theorem a *maxspectral space*. In fact, some algebraic properties of a ring could be reflected from some topological properties of its maximal spectrum. Details about these will be provided in chapter 3.

**Remark** In case that if the space $X$ is compact (i.e. quasi-compact and Hausdorff), then not only is it a maxspectral space, but also it can be obtained neatly as the maximal spectrum of the ring $C(X)$ of real-valued continuous functions on $X$ (See [11, Chapter 4]).
Chapter 2

The ordering on $\text{Spec}(R)$

It would be interested to explore the conditions for a commutative ring $R$ which can be obtained from its $\text{Spec}(R)$ by just regarding it as a poset under the inclusion ordering, although the ordering on $\text{Spec}(R)$ provides less information than its endowed topology (The discussion about the relationship between a ring and the topology on $\text{Spec}(R)$ will be postponed to chapter 3). It is natural to ask whether there are some necessary or sufficient conditions for a poset to be spectral, that is, it is order isomorphic to the prime spectrum of a commutative ring. A complete solution to this problem remains open, but a number of results related to these topics were obtained in the literature. These results enhance by contracting the class of all posets to some much smaller subclasses that could allow us to discuss the problem in a setting which is easier to be handled than before. No matter the approach used by the predecessors are functorial or not, they all aimed to tackle the problem by reducing the size of posets considered in some sense, like the limitation on the cardinality or the dimension of a poset. In this chapter, some pioneers’ works will be mentioned and some other directions which aim to tackle the problem will also be discussed.
Chapter 2. The ordering on $\text{Spec}(R)$

2.1 Two distinguished properties of a spectral poset

Before trying to tackle those problems concerning $\text{Spec}(R)$ as a poset, it is desirable to have some information about the size of the class of all spectral posets, since it could result in saving a lot from being not to consider those useless non-spectral posets in our setting. In estimating the size of the class of all spectral posets, it is quite reasonable to ask whether every poset is spectral or not, i.e. whether the class of posets we considered is nice enough to contain all posets or not. The answer to this question is negative due to the observation of I. Kaplansky. Kaplansky noticed that there are two distinguished properties of spectral poset which could not be shared by all posets.

Irving Kaplansky observed the following two properties of $\text{Spec}(R)$, in [20]:

(K1) Every totally ordered set in $\text{Spec}(R)$ has a supremum and an infimum.

(K2) If $P, Q \in \text{Spec}(R)$ and $P \subseteq Q$, then there exists $P_1, Q_1 \in \text{Spec}(R)$ with $P \subseteq P_1 \subseteq Q_1 \subseteq Q$ such that there is no prime ideal lying properly between $P_1$ and $Q_1$.

Notice that not every poset is spectral by the properties (K1) and (K2), Kaplansky then asked whether or not a poset $X$ satisfying these two properties is spectral or not. In other words, whether these two necessary conditions (K1) and (K2) are sufficient as well for a poset to be spectral or not. W. J. Lewis [26] showed that the answer to the question is affirmative if (1) $X$ is finite or (2) $X$ is a tree with finitely many minimal elements. By a tree, we mean a poset $X$ such that $G(x)$ is a totally ordered set for all $x \in X$. 
Chapter 2. The ordering on Spec(R)

2.2 Finite partially ordered sets

In this section, we always assumed that Spec(R) is finite. In fact, it is not so surprising that Lewis tried to tackle Kaplansky's problem in this case, as all finite posets clearly satisfy both conditions (K1) and (K2). In establishing the result in this case, Lewis [26] made extensive use of the following two propositions:

**Proposition 2.2.1** Let $R$ be an integral domain (i.e. a ring without zero-divisors) containing a field, and suppose that $R$ contains a finite number of maximal ideals. If a poset $X$ is the result of tying together the maximal ideals of Spec($R$) in some pattern, then there is an integral domain $S \subseteq R$ such that Spec($S$) $\cong X$.

**Proposition 2.2.2** Let $R$ be an integral domain containing a field $K$, and let $R$ have $n$ maximal ideals. If $D_1, \ldots, D_n$ are all integral domains with quotient field $k$, then there is an integral domain $S \subseteq R$ such that Spec($S$) is the result of attaching the minimal element of each Spec($D_i$) to one of the maximal elements of Spec($R$).

Using the two propositions above and some other preliminary works, Lewis [26] showed that every finite poset must be spectral. Although the original proof of Lewis' result is quite constructive and dependent on a lot of preliminaries, it does give a way to build up a ring with a desired prime spectrum. To get rid of the dependency on a number of preliminary works, we provides here a much shorter proof by using a different approach, pointed out by Lewis himself.

**Theorem 2.2.3** Every finite poset is spectral.

**Proof.** Let $(X, \mathcal{O})$ be a finite poset. Then $X$ endowed with the COP-topology $T(\mathcal{O})$ is a finite $T_0$ space. Observe that $X$ is finite implies that every subset of $X$
Chapter 2. The ordering on \( \text{Spec}(R) \)

is quasi-compact, and every irreducible closed subset in the finite space \( X \) contains just one minimal element with respect to \( \mathcal{O} \). It is easy to see that \( (X, T(\mathcal{O})) \) is a spectral space. By theorem 1.3.1, we know that there is a ring \( R \) whose prime spectrum is homeomorphic to \( X \). Hence \( X \) is a spectral poset. \( \square \)

Theorem 2.2.3 does tell us that given any ring \( R \) with finitely many prime ideals, any ordering structure could be possible on \( \text{Spec}(R) \). In other words, nothing about the ordering structure on \( \text{Spec}(R) \) could be obtained from just knowing that \( \text{Spec}(R) \) is finite.

\section{2.3 Several classes of special rings}

Before continuing to elaborate Lewis' work, we mention here several classes of rings that will appear in the subsequent sections.

\textbf{Definition 2.3.1} A valuation ring is a ring in which the set of all ideals of \( R \) is totally ordered under the inclusion ordering.\(^1\)

\textbf{Definition 2.3.2} A Bézout ring is a ring in which every finitely generated ideal is principal.

\textbf{Definition 2.3.3} An Arithmetical ring is a ring \( R \) in which its ideals form a distributive lattice,

\[ A \cap (B + C) = (A \cap B) + (A \cap C) \text{ for all ideals } A, B, C \text{ of } R. \]

Before defining the class of Prüfer rings, we need some additional terminologies.

\(^1\)In this definition, we do not assume that the ring must be an integral domain, though some authors imposed this restriction in the literature (See [24]).
Let \( S \) be a multiplicatively closed subset (i.e. \( ab \in S \) for all \( a, b \in S \)) of a ring \( R \), \( S^{-1}R \) will denote the ring of quotients (or ring of fractions) of \( R \) with respect to \( S \) as defined in [2]. If \( P \) is a prime ideal of \( R \), then we follow the convention by using \( R_P \) to represent \( S^{-1}R \) where \( S = R \setminus P \), \( R_P \) is called the localization of \( R \) at \( P \). If \( S_0 \) is the set of all non-zero-divisors in \( R \), then \( S^{-1}R \) is called the total ring of quotients of \( R \) and will be denoted by \( T(R) \). Note that when \( R \) is an integral domain, \( T(R) \) is just the field of quotients of \( R \).

An element of a ring \( R \) is regular if it is a non-zero-divisor. An ideal of \( R \) is said to be regular if it contains a regular element. Let \( I \) be an ideal of \( R \), define \( I^{-1} = \{ z \in T(R) | zI \subseteq R \} \). We shall call \( I \) invertible if \( II^{-1} = R \).

**Definition 2.3.4** A Prüfer ring is a ring in which every finitely generated regular ideal is invertible.\(^3\)

A valuation domain is just a valuation ring without zero-divisors. Bézout, Arithmetical and Prüfer domains are similarly defined.

The following results conclude some relationships between the four classes of rings defined above.

**Theorem 2.3.5 (Jensen, [18])** A ring \( R \) is Arithmetical if and only if \( R_M \) is a valuation ring for any maximal ideal \( M \) of \( R \).

**Corollary 2.3.6 (Jensen, [18])** If \( R \) is an Arithmetical ring, then \( R_P \) is a valuation ring for any prime ideal \( P \) of \( R \).\(^2\)

\(^2\)The regular element defined here is not the usual Von-Neumann regular element.

\(^3\)In the literature, some authors defined a Prüfer ring to be the one that we call Arithmetical ring (See [3]).
Corollary 2.3.7 (Jensen, [18]) In an Arithmetical ring $R$, any two prime ideals $P_1$ and $P_2$ none of which is contained in the other, will be comaximal, i.e. $P_1 + P_2 = R$.

Proposition 2.3.8 (Kaplansky, [20]) Every valuation ring is Bézout.

Proposition 2.3.9 (Gilmer, [12, Theorem 18.6])

(1) Every Bézout ring is Arithmetical.

(2) Every Arithmetical ring is Prüfer.

Diagramatically, we have the following inclusion relationships:

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Prüfer rings

Arithmetical rings

Bézout rings

Valuation rings
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However, we can observe that two of these classes could coincide if some restrictions are imposed on the ring itself.
Chapter 2. The ordering on $\text{Spec}(R)$

**Proposition 2.3.10 (Jensen, [18])** A semi-local \(^4\) Arithmetical ring is a Bézout ring.

The following result is a consequence of proposition 2.3.10 and is well-known.

**Corollary 2.3.11** If $R$ is a local ring \(^5\), then the following are equivalent.

1. $R$ is a valuation ring.
2. $R$ is a Bézout ring.
3. $R$ is an Arithmetical ring.

The following proposition is a characterization for Arithmetical integral domains.

**Proposition 2.3.12 (Jensen, [18])** An integral domain is Arithmetical if and only if it is a Prüfer ring.

In addition to the containment relationship between valuation, Bézout and Arithmetical rings, these three classes of rings share a common property that their prime spectrums are trees when they are viewed as posets. Since there is an one-to-one order-preserving correspondence between all the prime ideals in $R_P$ and all the prime ideals of $R$ contained in a prime ideal $P$ of $R$, the property described above can be inferred from proposition 2.3.8, proposition 2.3.9 and theorem 2.3.5. It is this property that attracts us to focus our study on the relationship between these kinds of rings and those rings with prime spectrum that are trees.

\(^4\)A ring with finite number of maximal ideals is said to be semi-local.

\(^5\)A ring with a unique maximal ideal is said to be local.
Chapter 2. The ordering on $\text{Spec}(R)$

2.4 Spectral trees

Being known that the prime spectrum of an Arithmetical ring must be a tree with properties (K1) and (K2), it is natural to ask whether every tree $X$ with properties (K1) and (K2) is a spectral poset or not. In [26], Lewis gave an affirmative answer with the additional condition that $X$ has just a unique minimal element.

Theorem 2.4.1 (Lewis, [26]) Let $X$ be a poset. The following statements are equivalent.

(a) $X$ is a tree with properties (K1), (K2) and a unique minimal element.

(b) There exists a Bézout domain $R$ such that $\text{Spec}(R) \simeq X$.

(c) There exists a Prüfer domain $R$ such that $\text{Spec}(R) \simeq X$.

Now we describe briefly how Lewis’ proof was worked out. In fact, the implications $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ follow easily from the results in section 3. To show the remaining part $(a) \Rightarrow (b)$, Lewis first made the following observations.

For the sake of convenience in the discussion, we first give a few definitions. Let $R$ be an integral domain with field of quotients $K$. Let $R^* = R \setminus \{0\}$; $K^* = K \setminus \{0\}$ and $U(R)$ be the multiplicative group of units of $R$. Suppose $G$ is the quotient group $K^*/U(R)$ (written additively) and $\mu : K^* \rightarrow G$ is the canonical map, then we can obtain an ordered group by letting $G^+ = \mu(K^*)$. Such a group is called the group of divisibility of the integral domain $R$.

By the definition of Bézout domain, it is easy to see that the group of divisibility of a Bézout domain must be lattice-ordered. Moreover, we have the following theorem:
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**Theorem 2.4.2 (Jaffard-Ohm, [13])** If $G$ is a lattice-ordered group, then there exists a Bézout domain whose group of divisibility is lattice isomorphic to $G$.

In fact, theorem 2.4.2 is a fundamental tool for building a Bézout domain with a prescribed group of divisibility.

**Proposition 2.4.3 (Lewis, [26])** Let $R$ be a Bézout domain with field of quotients $K$ and group of divisibility $G$. Let $\mu : K^* \to G$ be the canonical map. Then the mapping $f : \text{Spec}(R) \to \text{Spec}(G)$ defined by $f(p) = \mu(p \setminus \{0\})$ is an order isomorphism (The zero prime ideal of $R$ is associated with the empty set which is vacuously a prime $V$-segment of $G$).

In view of the theorem 2.4.2 and the preceding proposition, Lewis then completed his proof by showing that:

**Proposition 2.4.4 (Lewis, [26, Theorem 3.4])** Let $X$ be a tree with a unique minimal element such that (K1) and (K2) hold. Then there exists a lattice-ordered group $G$ such that $\text{Spec}(G)$ is order isomorphic to $X$.

**Corollary 2.4.5** Let $X$ be a poset. Then $X$ is a tree with $n$ minimal elements for which (K1) and (K2) hold if and only if $X \cong \text{Spec}(R)$ for a ring $R$ such that $R$ is the direct sum of $n$ non-trivial Bézout (Prüfer) domains.

**Proof.** Observe that a prime ideal $P$ of a direct sum $\bigoplus_{i=1}^n R_i$ must be of the form $P = \bigoplus_{i=1}^n I_i$, with $I_i = R_i$ for all $i \neq j$ and $I_j \in \text{Spec}(R_j)$ for some $j$. As $X$ decomposes into disjoint union of $n$ trees with the desired properties, the result then follows from theorem 2.4.1. \qed
The following corollary shows that the conditions (K1) and (K2) are indeed sufficient for a totally ordered set to be spectral.

**Corollary 2.4.6** Let $X$ be a poset. Then there is a valuation ring $R$ such that $\text{Spec}(R) \cong X$ if and only if $X$ is a totally ordered set satisfying properties (K1) and (K2).

**Proof.** Since a local Prüfer domain must be a valuation ring, the result follows directly from theorem 2.4.1.

Although a totally ordered set $X$ satisfying conditions (K1) and (K2) is order isomorphic to $\text{Spec}(R)$ for some valuation ring $R$, a ring with totally ordered prime spectrum is not necessarily a valuation ring. In fact, it is equivalent to the condition that every radical ideal in the ring is a prime ideal, due to the following result of Prekowitz (See [21]).

**Proposition 2.4.7** The following statements for a ring $R$ are equivalent:

1. The prime ideals in $R$ are totally ordered.
2. Every radical ideal in $R$ is prime.
3. For any $a$ and $b$ in $R$, either $a$ divides a power of $b$ or $b$ divides a power of $a$.

**Proof.**

(1) $\Rightarrow$ (2) If the prime ideals in $R$ are totally ordered, then a radical ideal, as an intersection of prime ideals, is clearly a prime one.

(1) $\Rightarrow$ (3) Since (1) implies (2), $r((a))$ and $r((b))$ are prime and hence comparable by (1), thus either $b \in r((a))$ or $a \in r((b))$, the result follows.
Chapter 2. The ordering on $\text{Spec}(R)$

(3) Suppose there exist prime ideals $P, Q$ that are incomparable, then we can find $a, b \in R$ such that $a \in P \setminus Q$ and $b \in Q \setminus P$. Then neither $a$ nor $b$ divides the power of the other, a contradiction.

(2) If $P$ and $Q$ are incomparable, then $I = P \cap Q$ is a radical ideal which is not prime, a contradiction. □

2.5 Ordered disjoint unions

If a poset $X$ is the disjoint union of posets $\{X_\alpha\}$, denoted by $X = \bigsqcup X_\alpha$, we say that $X$ is the ordered disjoint union of the $X_\alpha$'s if

$x \leq_X y$ if and only if there is an $\alpha$ such that $x, y \in X_\alpha$ and $x \leq_{X_\alpha} y$.

Let $\Lambda$ be an indexed set containing an element $o$, and let $\Lambda' = \Lambda \setminus \{o\}$. Suppose we are given a collection of rings $\{R_\lambda| \lambda \in \Lambda\}$ such that each $R_\lambda$, $\lambda \in \Lambda'$, is an $R_o$-algebra via a homomorphism $\phi_\lambda : R_\alpha \rightarrow R_\lambda$; and let $R$ be the subring of $\prod_{\lambda \in \Lambda} R_\lambda$ defined by

$(r_\lambda) \in R$ if and only if $\phi_\lambda(r_\lambda) = r_\lambda$ for all but a finite number of $\lambda \in \Lambda'$.

Let us now examine $\text{Spec}(R)$. For each $\alpha \in \Lambda$ let $A_\alpha = \{(a_\lambda) \in R| a_\alpha = 0\}$. The $A_\alpha$'s are ideals of $R$; and if $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ and $\alpha \neq o$, then $(0, 0, \ldots, 0, 1_\alpha, 0, \ldots) \in A_\beta$ and $(1, 1, \ldots, 1, 0_\alpha, 1, \ldots) \in A_\alpha$. It follows that $A_\alpha + A_\beta = R$. Let $P$ be any prime ideal of $R$ such that $P \not\supseteq A_\alpha$. Choose $z = (a_\lambda) \in A_\alpha \setminus P$. Then $zA_\alpha = 0$ for all but a finite number of $\lambda \in \Lambda'$, say $\alpha_1, \ldots, \alpha_n$. Thus $zA_{\alpha_1} \cdots A_{\alpha_n} = 0$.

As $z \not\in P$, it follows that $A_{\alpha_1} \subseteq P$ for some $\alpha_1$. Next note that $R/A_{\alpha} \simeq R_{\alpha}$, $\alpha \in \Lambda$, since $A_\alpha$ is the kernel of the projection homomorphism onto $R_\alpha$. Thus, if $X_\alpha = \{P \in \text{Spec}(R)| P \supseteq A_\alpha\}$, then $X_\alpha$ is order isomorphic to $\text{Spec}(R_{\alpha})$. It follows that $\text{Spec}(R)$ is the ordered disjoint union of the sets $X_\alpha$, $\alpha \in \Lambda$, where
Chapter 2. The ordering on \(\text{Spec}(R)\)

\(X_\alpha \simeq \text{Spec}(R_\alpha)\).

W. J. Lewis and J. Ohm [27] proved the following theorem:

**Theorem 2.5.1** Let \(\{X_\lambda \mid \lambda \in \Lambda\}\) be a collection of spectral posets and \(X\) be the ordered disjoint of the \(X_\lambda\)'s. Then there is a ring \(R\) such that \(\text{Spec}(R) \simeq X\).

**Proof.** Choose an element \(o \in \Lambda\), and let \(\Lambda' = \Lambda \setminus \{o\}\). Since \(X_o\) is spectral, there is a ring \(R_o\) such that \(\text{Spec}(R_o) \simeq X_o\). Let \(M\) be a maximal ideal of \(R_o\) and \(k = R_o/M\). For each \(\lambda \in \Lambda'\), we can choose a ring \(R_\lambda\) such that \(\text{Spec}(R_\lambda) \simeq X_\lambda\) and \(R_\lambda\) is a \(k\)-algebra, by the construction in the original proof of theorem 1.3.1 (See [16]). Thus, we have composite ring homomorphisms \(\phi_\lambda : R_o \to k \to R_\lambda, \lambda \in \Lambda'\). The theorem now follows from the discussion above. \(\square\)

In short, the preceding theorem states that the ordered disjoint union of spectral posets is still spectral. It tells us that a new spectral poset can always be constructed from a collection of spectral posets already in hand. Using similar technique as that in proving theorem 2.5.1, Lewis and Ohm [27] gave a modified version of theorem 2.4.1.

**Theorem 2.5.2** A poset \(X\) is a tree satisfying (K1), (K2) if and only if \(X \simeq \text{Spec}(R)\) for some Bézout ring \(R\).

**Proof.** For any Bézout ring \(R\), \(\text{Spec}(R)\) is well known to be a tree; so let us assume \(X\) is a tree satisfying (K1) and (K2). If \(x, y\) are two distinct minimal elements of \(X\), then \(S(x) \cap S(y) = \emptyset\). Thus, \(X\) can be written as the ordered disjoint union of trees \(X_\lambda, \lambda \in \Lambda\), where each \(X_\lambda\) has a unique minimal element. As \(X\) satisfies (K1) and (K2), so does each \(X_\lambda\). Pick an element \(o \in \Lambda\), let \(\Lambda' = \Lambda \setminus \{o\}\). By theorem 2.4.1, there is a Bézout domain \(R_o\) such that \(\text{Spec}(R_o) \simeq X_o\). Now let \(K\) be the quotient field of \(R_o\). Using theorem 2.4.1 as well as Ohm's proof of theorem 2.4.2
[13, page 215], we can choose a Bézout domain $R_\lambda$ such that $\text{Spec}(R_\lambda) \simeq X_\lambda$ and $K \subseteq R_\lambda$, for each $\lambda \in \Lambda'$. Thus, for $\lambda \in \Lambda'$, we have composite ring homomorphism $\phi_\lambda : R_\alpha \rightarrow K \rightarrow R_\lambda$. The construction at the start of this section provides a ring $R$ such that $\text{Spec}(R) \simeq X$.

It remains to show that $R$ is a Bézout ring. Let $z^i = (a_\lambda^i) \in R$ for $i = 1, \ldots, n$. Since $R_\alpha$ is Bézout, there is a $y_\alpha \in R_\alpha$ such that $y_\alpha R_\alpha = (a_\alpha^0, \ldots, a_\alpha^n)R_\alpha$. For all but a finite number of $\lambda \in \Lambda'$, $\phi_\lambda(a_\alpha^i) = a_\lambda^i$ for all $i = 1, 2, \ldots, n$, we denote the set of all such $\lambda$'s by $\Delta$. Now for each $\lambda \in \Delta$, we let $y_\lambda = \phi(y_\alpha)$. Moreover, the equations expressing the equality $y_\alpha R_\alpha = (a_\alpha^0, \ldots, a_\alpha^n)R_\alpha$ hold for each $\lambda \in \Delta$ when $\phi_\lambda$ is applied. Let $\Lambda' \setminus \Delta = \{\lambda_1, \ldots, \lambda_t\}$. For each $i = 1, \ldots, t$, there is $y_{\lambda_i}$ such that $y_{\lambda_i} R_{\lambda_i} = (a_{\lambda_i}^0, \ldots, a_{\lambda_i}^n) R_{\lambda_i}$. Let $y = (y_j)$, then clearly $y \in R$ and $yR = (z^1, \ldots, z^n)R$.

In connection with theorem 2.5.1, Lewis and Ohm then raised the following open question in [27]:

**Question** Let $X$ be the ordered disjoint union of posets $\{X_\lambda | \lambda \in \Lambda\}$. If $X$ is spectral, then are the $X_\lambda$'s also spectral?

We now give a partial answer to the above question.

**Theorem 2.5.3** Let $X$ be the ordered disjoint union of posets $\{X_\lambda | \lambda \in \Lambda\}$. If $X$ is spectral, then $X_\lambda$'s are also spectral in the following cases:

1. $X$ has finite number of minimal elements.
2. $X$ has finite number of maximal elements.
3. The COP-topology on $X$ is a spectral topology.
Lemma 2.5.4 Let $X$ be an ordered disjoint union of $\{X_\lambda | \lambda \in \Lambda\}$. Let $T$ and $T_\lambda$ be the COP-topologies on $X$ and $X_\lambda$ respectively. Then $Y \cap X_\lambda$ is $T_\lambda$-closed ($T_\lambda$-open resp.) for any $T$-closed ($T$-open resp.) subset $Y$ of $X$.

Proof. Suppressing the notation for ordering, the assertion that $Y$ is $T$-closed implies $Y \cap X_\lambda$ is $T_\lambda$-closed follows directly from $S_X(a) \cap X_\lambda = S_{X_\lambda}(a)$ for any $a \in X$. Now suppose $Y$ is $T$-open in $X$, then $X \setminus Y$ is $T$-closed implies that $(X \setminus Y) \cap X_\lambda$ is $T_\lambda$-closed. Hence $Y \cap X_\lambda = X_\lambda \setminus ((X \setminus Y) \cap X_\lambda)$ is $T_\lambda$-open. \qed

Proof of theorem 2.5.3. Let $X$ be an ordered disjoint union of posets $\{X_\lambda | \lambda \in \Lambda\}$. Suppose that $X$ is spectral. Then we may assume that $X \simeq Spec(R)$ for some ring $R$. Consider the following cases separately:

(1) Assume $X$ has finite number of minimal elements. Then for any $\lambda \in \Lambda$, $X_\lambda$ also has finite number of minimal prime ideals, namely, $p_1, \ldots, p_n$. Our proof will be completed if we can show that $X_\lambda = V(\bigcap_{i=1}^n p_i)$ since it would follow that $X_\lambda \simeq Spec(R/\bigcap_{i=1}^n p_i)$. In fact, for any $p \in V(\bigcap_{i=1}^n p_i)$, $\bigcap_{i=1}^n p_i \subseteq p$ implies that $p_j \subseteq p$ for some $j$. Hence, $p \in X_\lambda$ since $p_j \in X_\lambda$ and $X$ is an ordered disjoint union of $\{X_\lambda | \lambda \in \Lambda\}$. Conversely, any $p \in X_\lambda$ must contain one of the $p_1, \ldots, p_n$ and hence $p$ belongs to $V(\bigcap_{i=1}^n p_i)$.

(2) This follows immediately from case (1) by invoking proposition 1.3.8 on posets.

(3) Let $T$ be the spectral COP-topology on $X$. For any $\lambda \in \Lambda$, let $T_\lambda$ be the COP-topology on $X_\lambda$. It is easy to see that every $T_\lambda$-closed set in $X_\lambda$ must also be $T$-closed, hence any family of $T_\lambda$-closed subsets of $X$ with finite intersection property
must have non-empty intersection, by the quasi-compactness of $T$. This implies that $T_{\lambda}$ is quasi-compact. To show $(X_{\lambda}, T_{\lambda})$ is sober, we suppose that $F$ is a $T_{\lambda}$-irreducible closed subset of $X_{\lambda}$. Then $F$ is clearly a $T$-closed subset of $X$. To show that $F$ is also a $T$-irreducible closed subset of $X$, let $F = C \cup D$ for some $T$-closed proper subsets $C, D$ of $F$. But then, we have $F = C \cap X_{\lambda} = C$ or $F = D \cap X_{\lambda} = D$ by lemma 2.5.4 and $F$ is $T_{\lambda}$-irreducible. Since $T$ is a spectral topology on $X$, there exists $a \in X$ such that $F = \{a\}_{X,T}$. It follows that $F = \{a\}_{X,T} \cap X_{\lambda} = S_X(a) \cap X_{\lambda} = S_{X_{\lambda}}(a) = \{a\}_{X_{\lambda},T_{\lambda}}$. Therefore $(X_{\lambda}, T_{\lambda})$ is sober.

Suppose $\mathcal{B}$ is a QCI-base for $(X, T)$. Then, $\mathcal{B} \cap X_{\lambda}$ is a collection of $T_{\lambda}$-open sets in $X_{\lambda}$ by lemma 2.5.4. Moreover, $\mathcal{B} \cap X_{\lambda}$ is a $T_{\lambda}$-open base for $X_{\lambda}$, which is clearly closed under finite intersection. For any $T_{\lambda}$-open subset $U$ of $X_{\lambda}$, the $T_{\lambda}$-closeness of $X_{\lambda} \setminus U$ implies that $U \cup (X \setminus X_{\lambda})$ is $T$-open. Hence, $U \cup (X \setminus X_{\lambda}) = \bigcup_{i \in I} B_i$ for some $\{B_i\}_{i \in I} \subseteq \mathcal{B}$. This implies that $U = \bigcup_{i \in I} (B_i \cap X_{\lambda})$. In addition, if $B \in \mathcal{B}$ satisfying $B \cap X_{\lambda} \subseteq \bigcup_{j \in J} (B_j \cap X_{\lambda})$ for some $\{B_j \cap X_{\lambda}\}_{j \in J} \subseteq \mathcal{B} \cap X_{\lambda}$, then $B \subseteq (B \cap X_{\lambda}) \cup (X \setminus X_{\lambda}) \subseteq \bigcup_{j \in J} (B_j \cap X_{\lambda}) \cup (X \setminus X_{\lambda})$. Since $(B_j \cap X_{\lambda}) \cup (X \setminus X_{\lambda})$ is $T$-open for all $j \in J$, $B \subseteq \bigcup_{j \in K} (B_j \cap X_{\lambda})$ for some finite subset $K$ of $J$ by the quasi-compactness of $\mathcal{B}$. Hence, $\mathcal{B} \cap X_{\lambda} \subseteq \bigcup_{j \in K} (B_j \cap X_{\lambda})$. This shows that $\mathcal{B} \cap X_{\lambda}$ is a QCI-base for $(X_{\lambda}, T_{\lambda})$ and $(X_{\lambda}, T_{\lambda})$ is spectral.

In view of our theorem 2.5.3(3), we have a strengthen version of the question raised by Lewis and Ohm:

**Question** If $X$ is an ordered disjoint union of $\{X_{\lambda}|\lambda \in \Lambda\}$ and $X$ is a spectral poset, is the COP-topology on $X$ a spectral topology?
2.6 Another necessary condition for a poset to be spectral

It was mentioned in [26] that Hochster did observe the conditions (K1) and (K2) are not sufficient for a poset to be spectral. In fact, his observation is due to the following example.

Example 2.1 Consider $X = \{p_m|m \in \mathbb{N}\} \cup \{q_n|n \in \mathbb{N}\}$ which is partially ordered by defining $p_m \leq q_n$ for all $m \leq n$ (See the figure below).

\begin{center}
\begin{tikzpicture}
\node (p1) at (0,0) {$p_1$};
\node (p2) at (1,0) {$p_2$};
\node (p3) at (2,0) {$p_3$};
\node (p4) at (3,0) {$p_4$};
\node (p5) at (4,0) {$\ldots$};
\node (q1) at (0,1) {$q_1$};
\node (q2) at (1,1) {$q_2$};
\node (q3) at (2,1) {$q_3$};
\node (q4) at (3,1) {$q_4$};
\node (q5) at (4,1) {$q_5$};
\node (q6) at (5,1) {$\ldots$};
\draw (p1) -- (q1);
\draw (p2) -- (q2);
\draw (p3) -- (q3);
\draw (p4) -- (q4);
\draw (p5) -- (q5);
\draw (p6) -- (q6);
\end{tikzpicture}
\end{center}

Consider the collection of subsets $\mathcal{C} = \{V(p_m)|m \in \mathbb{N}\}$. Let $T$ be the COP-topology on $X$. Then $\mathcal{C}$ is a collection of $T$-closed subsets satisfying the finite intersection property but with empty intersection. It follows that $(X,T)$ is not quasi-compact. Hence, $X$ is not spectral, as any spectral poset must be quasi-compact in its COP-topology. Therefore, $X$ is a non-spectral poset satisfying conditions (K1) and (K2).
From the above example, Lewis and Ohm [26] derive another necessary condition for a poset to be spectral.

**Proposition 2.6.1** Every spectral poset \((X, \mathcal{O})\) must satisfy the following condition:

(H) Let \(S = \{S(x) | x \in X\}\) and \(\mathcal{G} = \{G(x) | x \in X\}\). If \(\mathcal{F}\) is a collection of subsets of \(X\) such that \(\mathcal{F} \subseteq S\) or \(\mathcal{F} \subseteq \mathcal{G}\), then \(\cap \{F | F \in \mathcal{F}\} = \emptyset\) implies that there is a finite collection of sets from \(\mathcal{F}\) whose intersection is empty.

**Proof.** As \((X, \mathcal{O})\) is spectral, there exists a \(\mathcal{O}\)-compatible spectral topology \(T\) on \(X\). Since the COP-topology \(T_0\) is contained in \(T\), \(T_0\) is quasi-compact. If \(\mathcal{F} \subseteq S\), then \(\mathcal{F}\) is a collection of \(T_0\)-closed subsets of \(X\) and hence \(\mathcal{F}\) satisfies the stated condition because \(T_0\) is quasi-compact. By proposition 1.3.8, we know that \((X, \mathcal{O})\) is spectral implies \((X, \mathcal{O}^*)\) is spectral, where \(\mathcal{O}^*\) is the reverse ordering of \(\mathcal{O}\). Then the result for \(\mathcal{F} \subseteq \mathcal{G}\) follows easily by the fact that \(\mathcal{G}\) is the closed subbase for the COP-topology on \((X, \mathcal{O}^*)\). \(\square\)

In fact, the property (H) reflects the fact that \(Spec(R)\) is quasi-compact in the Zariski topology. However, the conditions (K1), (K2) together with (H) still cannot form a sufficient set of conditions for a poset to be spectral. This fact is illustrated by the following example given by Lewis and Ohm in [27].

**Example 2.2** Let \(Y\) be a compact space which is not spectral; for example, the closed unit interval will do, since it does not possess a QCI-base. Because \(Y\) is compact, there does not exist a properly stronger compact topology for \(Y\). Let \(Z = \{X_C | C \subseteq Y\}\) and \(C\) is closed in \(Y\). Let \(X = Y \cup Z\) and order \(X\) by specifying that if \(y \in Y\), \(y \geq x_C\) if and only if \(y \in C\). Thus, for every closed set \(C \subseteq Y\) we create an element \(x_C\) and “place” it below each element of \(C\). The set \(X\) is one-dimensional\(^6\), \(Y\)

\(^6\)The dimension \(dim(X, \mathcal{O})\) of a poset \((X, \mathcal{O})\) is defined by \(dim(X, \mathcal{O}) = \sup \{n \in \{0\} \cup \mathbb{N} | \text{there is a chain } x_0 < x_1 < \ldots < x_n, x_i \in X \text{ for all } i\}\). We always use \(\mathbb{N}\) to denote the set of all positive
being the set of maximal elements and \( Z \) the set of minimal elements. Let \( T \) be any order compatible topology for \( X \). Choose proper closed subsets \( C_1, C_2 \) of \( Y \) such that \( Y = C_1 \cup C_2 \). Now \( Y = S(x_Y) \cap (S(x_{C_1}) \cup S(x_{C_2})) \), so \( Y \) is closed in \( T \). Similarly, if \( C \) is any closed subset of \( Y \) then \( C = Y \cap S(x_C) \) is also \( T \)-closed. If \( T \) were a spectral topology, then the topology \( T|_Y \) would also be spectral and would be stronger than the original topology for \( Y \). Our choice for \( Y \) makes this case impossible, so \( X \) is not spectral.

It remains for us to show that \( X \) satisfies the conditions (K1), (K2) and (H). Because \( X \) is one-dimensional, (K1) and (K2) are trivially satisfied. Let \( F \) be a collection of subsets of \( X \) such that \( \cap \{F \mid F \in F\} = \emptyset \). If \( F \subseteq S \), then for \( x \neq y \in X \), we have \( S(x) \cap S(y) \subseteq Y \). Since each \( F \cap Y \) is closed in \( Y \) and \( \cap \{F \cap Y \mid F \in F\} = \emptyset \), the quasi-compactness of \( Y \) allows the choice of a finite set of \( F \)'s whose intersection is empty. If \( F \subseteq G \), we notice that \( x_Y \in G(y) \) for each \( y \in Y \), so there must be an \( F \in F \) and a \( z \in Z \) such that \( F = G(z) = \{z\} \). Choose any \( F' \in F \) such that \( z \notin F' \). Then, \( F \cap F' = \emptyset \). This shows that \( X \) satisfies the condition (H) as well.

## 2.7 Possible partial orderings for spectral posets

We have seen that the conditions (K1), (K2) together with (H) still cannot form a sufficient condition set for a poset to be spectral. However, Hochster [16] did provide a characterization of possible partial orderings on a spectral poset.

**Proposition 2.7.1** Let \((X, \mathcal{O})\) be a poset. Let \( W = \{0, 1\} \) as in section 1.4. Then there is a spectral topology on \( X \) inducing the given partial order \( \mathcal{O} \) if and only if there is a family of order-preserving maps \( V \) from \( X \) to \( W \) such that the following integers.
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conditions hold:

(1) If $x, y \in X$ and $x \not> y$, then for some $v \in V$, $v(x) < v(y)$.

(2) If $h : V \rightarrow W$ and for each finite subset $V_0$ of $V$ there is an $x \in X$ such that $h(v) = v(x)$ for all $v \in V_0$, then there is an $x \in X$ such that for all $v \in V$, $h(v) = v(x)$.

**Proof.** Suppose there is such a family of order-preserving maps $V$ from $X$ to $W$ such that conditions (1) and (2) hold. Let $f$ be the evaluation map from $X$ to $W^V$, where $W$ is topologized as that in section 1.4. Then (1) guarantees that $f$ is injective, while (2) guarantees that the image is a patch. (1) then gives that the order on this patch agrees with that on $X$. The converse part is obvious. □

The following result is an easily verified consequence.

**Corollary 2.7.2** A totally ordered set arises from a spectral topology if and only if the set is compact and totally disconnected in its COP-topology. In this case, there is only one spectral topology for the space inducing its order, and the corresponding patch topology is identical with the order topology on the set.

Hence, theorem 2.7.1 does provide a path for us to search other necessary condition for a poset to be spectral. Along the direction of this path, our first task should be trying to get some conditions on the poset $X$ related with the space $W$. 
Chapter 3

The topology on \( \text{Spec}(R) \)

In this chapter, we study mainly the Zariski topology on the spectrum of a ring \( R \). Some usual topological notions on \( \text{Spec}(R) \) such as Hausdorffness, irreducibility and connectedness which are related with the algebraic properties of \( R \) will be discussed.

3.1 Basic notions about \( \text{Spec}(R) \)

For the sake of completeness and clarity in the following discussion, some basic notions about \( \text{Spec}(R) \) of a ring \( R \) will be mentioned in this section. As most of them are immediate results from definitions, their proofs will be omitted.

Let \( a \in R \). We write \( V(a) \) instead of \( V(\{a\}) \) to represent the set of prime ideals containing \( a \), the notation \( V_{\text{Spec}(R)}(a) \) will also be used to specify what the underlying ring is. Similar convention is used for \( D(a) \) and \( D_{\text{Spec}(R)}(a) \). It is easy to see that \( \{D(a) | a \in R\} \) form an open base for \( \text{Spec}(R) \). The following propositions conclude some properties about \( D(a), V(A) \) and \( \text{Spec}(R) \) (See [2]).

Proposition 3.1.1 Let \( R \) be a ring and \( a, b \in R \). Then the following properties
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hold:

(a) $D(a) \cap D(b) = D(ab)$;

(b) $D(a) = \emptyset$ if and only if $a \in \text{N}(R)$ (where $\text{N}(R)$ is the nil radical of $R$), i.e. $a$ is nilpotent;

(c) $D(a) = \text{Spec}(R)$ if and only if $a$ is a unit;

(d) $D(a) = D(b)$ if and only if $r((a)) = r((b))$.

Let $\phi : R \rightarrow S$ be a ring homomorphism. Let $X = \text{Spec}(R)$ and $Y = \text{Spec}(S)$. If $q \in Y$, then $\phi^{-1}(q) \in X$. Hence $\phi$ induces a mapping $\phi^* : Y \rightarrow X$ defined by $\phi^*(q) = \phi^{-1}(q)$ for all $q \in Y$.

**Proposition 3.1.2** Let $\phi : R \rightarrow S$ be a ring homomorphism. Then

(a) If $a \in R$ then $\phi^{-1}(D_X(a)) = D_Y(\phi(a))$, hence $\phi^*$ is continuous.

(b) If $A$ is an ideal of $R$, then $\phi^{-1}(V_X(A)) = V_Y(A^\circ)$.\(^2\)

(c) If $B$ is an ideal of $S$, then $\phi^*(V_Y(B)) = V_X(B^\circ)$.

(d) If $\phi$ is surjective, then $\phi^*$ is a homeomorphism of $Y$ onto the closed subset $V(\text{ker}(\phi))$ of $X$. In particular, $\text{Spec}(R)$ and $\text{Spec}(R/\text{N}(R))$ are naturally homeomorphic.

(e) If $\phi$ is injective, then $\phi^*(Y)$ is dense in $X$. More precisely, $\phi^*(Y)$ is dense in $X$ if and only if $\text{ker}(\phi) \subseteq \text{N}(R)$.

(f) If $\psi : S \rightarrow T$ is another ring homomorphism. Then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

\(^1\)We use $r(A)$ to represent the radical of an ideal $A$.

\(^2\)When $\phi : R \rightarrow S$ is a ring homomorphism, $A$ and $B$ are ideals of $R, S$ respectively, we use $A^\circ$ and $B^\circ$ to represent the extension of $A$ in $B$ (i.e. the ideal generated by $\phi(A)$ in $S$) and the contraction of $B$ in $R$ (i.e. $\phi^{-1}(B)$) respectively.
3.2 The Zariski topology on $\text{Spec}(R)$

Unless otherwise stated, we always consider prime spectrum of a ring as a topological space under the Zariski topology in this chapter. It was mentioned in chapter 1 that the prime spectrum of a ring is actually a quasi-compact, sober and coherent space. As in the study of the ordering on $\text{Spec}(R)$, it is desirable to know how much we know about a ring from its prime spectrum.

3.2.1 Hausdorffness

If $R$ is a non-reduced ring (a ring is reduced if its nilradical is zero), then the reduced ring $R/N(R)$, denoted by $R^{\text{red}}$, satisfies $\text{Spec}(R) \approx \text{Spec}(R^{\text{red}})$ by proposition 3.1.2(d). Hence the reducedness of a ring cannot be revealed from its prime spectrum. In other words, a ring cannot be completely determined by its prime spectrum. Hence, the attempt to characterize a ring via topological tools on its prime spectrum confined merely to reduced rings. The followings are some basic results of reduced rings (See [28]).

**Proposition 3.2.1** Let $R$ be a reduced ring, and let $\{P_i|i \in I\}$ be the set of all minimal prime ideals of $R$. Then the following facts hold:

1. $R_{P_i}$ is the quotient field of $R/P_i$.

2. $\bigcup_{i \in I} P_i$ is the set of all zero divisors of $R$.

**Proof.**

1. Let $O_i = \{r \in R|ur = 0 \text{ for some } u \in R \setminus P_i\}$. Then $O_i$ is an ideal of $R$ and $O_i \subseteq P_i$. Since $P_iR_{P_i}$ is the only prime ideal of $R_{P_i}$, every element of $P_iR_{P_i}$ is nilpotent. Thus if $a \in P_i$, there exists $u \in R \setminus P_i$ and $n \in \mathbb{N}$ such that $ua^n = 0$. Hence
\( (ua)^n = 0 \), and since \( R \) is reduced, \( ua = 0 \). Thus \( O_i = P_i \), and hence \( P_i R_{P_i} = 0 \). Therefore \( R_{P_i} \) is the quotient field of \( R/P_i \).

(2) It follows from (1) that every element of \( \bigcup_{i \in I} P_i \) is a zero-divisor in \( R \). Conversely, let \( x \in R \setminus \{0\} \) be a zero-divisor in \( R \). Then there exists \( y \in R \setminus \{0\} \) such that \( xy = 0 \). Since \( \cap_{i \in I} P_i = 0 \), there exists \( j \in I \) such that \( y \not\in P_j \), and hence \( x \in P_j \).

**Definition 3.2.2** Let \( A \) be a subset of a ring \( R \). Then the annihilator \( \text{Ann}_R(A) \) of \( A \) in \( R \) is the set \( \{ r \in R | ra = 0 \text{ for all } a \in A \} \). When \( A = \{a\} \), we write \( \text{Ann}_R(a) \) instead. Subscript will be omitted when it is clear from the context.

**Proposition 3.2.3** Let \( R \) be a reduced ring.

(1) A prime ideal \( P \) of \( R \) is a minimal prime ideal of \( R \) if and only if \( \text{Ann}(x) \not\subset P \) for all \( x \in P \).

(2) Let \( J \) be a finitely generated ideal of \( R \). Then \( J \) is contained in a minimal prime ideal of \( R \) if and only if \( \text{Ann}(J) \neq 0 \).

(3) If \( x \in R \) and \( y \in \text{Ann}(x) \), then \( \text{Ann}(Rx + Ry) = 0 \) if and only if \( x - y \) is not a zero-divisor in \( R \).

**Proof.**

(1) If \( P \) is a minimal prime ideal of \( R \) and \( x \in P \), then by proposition 3.2.1(1), there exists \( u \in R \setminus P \) such that \( ux = 0 \). Hence \( \text{Ann}(x) \not\subset P \). Conversely, suppose that for all \( x \in P \), \( \text{Ann}(x) \not\subset P \). Let \( P_0 \in \text{Spec}(R) \) such that \( P_0 \subset P \). Then there exists \( x \in P \setminus P_0 \) and hence \( \text{Ann}(x) \subset P_0 \subset P \). This contradiction shows that \( P \) is a minimal prime ideal of \( R \).
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(2) Let \( J = Ra_1 + \ldots + Ra_n \), and let \( I = \text{Ann}(J) \). Suppose that \( J \) is contained in a minimal prime ideal \( P \) of \( R \). Then by proposition 3.2.1(1), there exist elements \( u_i \in R \setminus P \) such that \( u_i a_i = 0 \) for all \( i = 1, \ldots, n \). Let \( u = u_1 \ldots u_n \), then \( u \notin P \) and \( u \in I \). Conversely, suppose that \( I \neq 0 \). Then there is a minimal prime ideal \( P \) of \( R \) such that \( I \subseteq P \), and hence \( J \subseteq P \).

(3) Assume \( \text{Ann}(Rx + Ry) = 0 \), and suppose that \( t \in R \) and \( t(x - y) = 0 \). Then \( tx = ty \) and hence \( (tx)^2 = 0 \). Therefore, \( tx = 0 = ty \) and hence \( t \in \text{Ann}(Rx + Ry) = 0 \). Thus \( x - y \) is not a zero-divisor in \( R \). The converse assertion is trivial. □

Though a complete characterization of a ring cannot be obtained from its prime spectrum, several algebraic properties of the ring \( R \) can still be reflected from some topological properties of \( \text{Spec}(R) \). We begin by studying the \( T_2 \) condition on \( \text{Spec}(R) \).

**Definition 3.2.4** A ring \( R \) is Von-Neumann regular (or absolutely flat) if for any \( x \in R \), there exists \( y \in R \) such that \( xyx = x \).

**Proposition 3.2.5** Let \( R \) be a ring, the following statements are equivalent:

(a) \( R \) is Von-Neumann regular.

(b) Every principal ideal is generated by an idempotent.

(c) Every principal ideal is a direct summand in \( R \).

**Proof.**

(a) \( \Rightarrow \) (b) Let \( a \in R \) and \( b \in R \) with \( aba = a \). Then \( e = ba \) is an idempotent and \( Re \subseteq Ra \). Since \( a = aba \in Re \), we have \( Re = Ra \).
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(b) $\Rightarrow$ (c) If $a \in R$ and $e \in R$ is an idempotent with $Re = Ra$, then $1 \in Re + R(1-e)$ implies $R = Ra + R(1-e)$. Let $x \in Ra \cap R(1-e)$, $x = se = t(1-e)$ for some $s, t \in R$, then $x = se = (se)e = t(1-e)e = t(e - e^2) = 0$. Therefore $Ra \oplus R(1-e) = R$, $Ra$ is a direct summand in $R$.

(c) $\Rightarrow$ (a) Let $a \in R$, $Ra$ is a direct summand in $R$ implies there exists ideal $I$ of $R$ such that $Ra \oplus I = R$. Therefore $1 = ra + x$ for some $r \in R$, $x \in I$. Then $a = ra^2 + xa$ and $xa \in Ra \cap I = 0$ implies $a = ra^2$, whence $R$ is Von-Neumann regular.

The following establishes the equivalent conditions for the prime spectrum $Spec(R)$ of a ring $R$ to be Hausdorff, which is also found to be equivalent to $Spec(R)$ being a $T_1$ space (See [2]).

Proposition 3.2.6 Let $R$ be a ring, the following are equivalent:

(1) $R^{red}$ is Von-Neumann regular.

(2) Every prime ideal of $R$ is maximal, i.e. $\dim(R) = 0$.

(3) $Spec(R)$ is $T_1$.

(4) $Spec(R)$ is $T_2$.

If these conditions are satisfied, then $Spec(R)$ is totally disconnected.

Proof.

(4) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (2) Since $Max(R)$ is the set of all closed points in $Spec(R)$, it follows directly from the fact that a space is $T_1$ if and only if every subset containing only one point is closed.
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(2) $\Rightarrow$ (1) As $\dim(R^{\text{red}}) = \dim(R) = 0$, every prime ideal of $R^{\text{red}}$ is a minimal prime ideal. Let $x \in R^{\text{red}} \setminus \{0\}$ and $I = \text{Ann}_{R^{\text{red}}}(x)$. Since $R^{\text{red}}$ is reduced, $R^{\text{red}}x \cap I = 0$. By proposition 3.2.3(1), $R^{\text{red}}x + I$ is not contained in any minimal prime ideal of $R^{\text{red}}$. Therefore $R^{\text{red}}x \oplus I = R^{\text{red}}$ and $R^{\text{red}}x$ is a direct summand of $R^{\text{red}}$. Hence $R^{\text{red}}$ is Von-Neumann regular by proposition 3.2.5.

(1) $\Rightarrow$ (4) Let $P_1, P_2$ be two distinct points in $\text{Spec}(R^{\text{red}})$. As $P_1 \neq P_2$, we may assume $P_2 \subsetneq P_1$ without loss of generality, i.e. there exists $f \in R^{\text{red}}$ such that $f \in P_2 \setminus P_1$. By $R^{\text{red}}$ is Von-Neumann regular, we have $f^2g = f$ for some $g \in R^{\text{red}}$. Then $P_1 \in D(f), P_2 \in D(1 - fg)$ and $D(f) \cap D(1 - fg) = D(f - f^2g) = D(0) = \emptyset$. Hence $\text{Spec}(R) \approx \text{Spec}(R^{\text{red}})$ is $T_2$.

In the proof of the part (1) $\Rightarrow$ (4), we have shown that given two distinct prime ideals $P_1, P_2 \in \text{Spec}(R)$, there exists $f, g \in R$ such that $P_1 \in D(f)$ and $P_2 \in D(1 - fg)$ with $f^2g = f$. Moreover, $D(f) \cup D(1 - fg) = D(1) = \text{Spec}(R)$. This shows that $\text{Spec}(R)$ is totally disconnected. \hfill \Box

3.2.2 Irreducibility

A topological space $X$ is said to be irreducible if $X \neq \emptyset$ and if every pair of non-empty open sets in $X$ intersect, or equivalently if every non-empty open set is dense in $X$. In case that $Y$ is a closed subspace of $X$, then $Y$ is irreducible if and only if it is not a union of two proper closed subspaces of $X$, it coincides with what we defined in chapter 1. It is easy to see that if $Y$ is irreducible subspace of $X$, then the closure $\overline{Y}$ of $Y$ in $X$ is also irreducible. Moreover, every irreducible subspace of $X$ is contained in a maximal irreducible subspace. The maximal irreducible subspaces of $X$ are closed and cover $X$. They are called the irreducible components of $X$. 
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The following propositions summarize some properties about irreducible subspaces of \( \text{Spec}(R) \) of a ring \( R \) (See [2]).

**Proposition 3.2.7** Let \( R \) be a ring. Then a closed subspace of \( \text{Spec}(R) \) is irreducible if and only if it is of the form \( V(p) \) for some \( p \in \text{Spec}(R) \).

**Proof.** Since \( \text{Spec}(R) \) is sober, the result follows immediately. \( \square \)

**Corollary 3.2.8** \( \text{Spec}(R) \) is irreducible if and only if the nilradical \( N(R) \) of the ring \( R \) is a prime ideal.

**Proof.** It follows directly from the fact that \( \text{Spec}(R) = V(N(R)) \), since \( N(R) \) is the intersection of all prime ideals of \( R \). \( \square \)

**Proposition 3.2.9** Let \( R \) be a ring. Then the irreducible components of \( \text{Spec}(R) \) are the closed sets \( V(p) \), where \( p \) is a minimal prime ideal of \( R \).

**Proof.** Every irreducible component \( Y \) of \( \text{Spec}(R) \) is closed and irreducible, hence \( Y = V(p) \) for some \( p \in \text{Spec}(R) \). By Zorn’s lemma, \( p \) always contains a minimal prime ideal \( p_0 \) of \( R \). If \( p \neq p_0 \), then \( Y \subset V(p_0) \) but the latter is closed and irreducible. This contradiction shows that \( p = p_0 \), hence \( p \) is a minimal prime ideal. Conversely, if \( p \) is a minimal prime ideal of \( R \), then \( V(p) \) must be an irreducible component. Otherwise there exists a minimal prime ideal \( p_0 \) of \( R \) such that \( V(p) \subset V(p_0) \), which implies that \( p_0 \subset p \), a contradiction. \( \square \)

A topological space \( X \) is said to be Noetherian if the open subsets of \( X \) satisfy the ascending chain condition (or, equivalently, the maximal condition). Since closed subsets are complements of open subsets, it is equivalent to say that the closed subsets of \( X \) satisfy the descending chain condition (or, equivalently, the minimal condition).
Proposition 3.2.10 If $X$ is a Noetherian space, then the following statements hold.

(a) Every subspace of $X$ is Noetherian;

(b) $X$ is quasi-compact.

Proof.

(a) Suppose $Y$ is a subspace of $X$ and $\{E_i\}_{i \in I}$ a non-empty collection of closed subsets of $Y$. Now $E_i$ is closed in $Y$ implies $E_i = F_i \cap Y$ for some closed subset $F_i$ of $X$. As $X$ is Noetherian, $\{F_i\}_{i \in I}$ has a minimal element $F_{i_0}$. Hence, $E_{i_0}$ is a minimal element in $\{E_i\}_{i \in I}$ and $Y$ is Noetherian.

(b) Suppose $X = \bigcup_{i \in I} U_i$, where $U_i$ is open in $X$. Consider the set $\Sigma$ of all finite unions of $U_i$'s. Clearly $\Sigma \neq \emptyset$. Choose a maximal element $V \in \Sigma$ because $X$ is Noetherian. Suppose $U_j \not\subseteq V$ for some $j \in I$. Then we have $V \subset (V \cup U_j)$ and $(V \cup U_j) \in \Sigma$, this contradicts to the maximality of $V$. Hence, $U_i \subseteq V$ for all $i \in I$. Therefore $X = V$ and $\{U_i\}_{i \in I}$ has a finite subcover for $X$. This shows that $X$ is quasi-compact. \hfill \Box

Proposition 3.2.11 Let $X$ be a topological space. Then the following conditions are equivalent:

(a) $X$ is Noetherian.

(b) Every open subspace of $X$ is quasi-compact.

(c) Every subspace of $X$ is quasi-compact.
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Proof.

(a) $\Rightarrow$ (c) By proposition 3.2.10(a), we know that $X$ is Noetherian implies that every subspace of $X$ is Noetherian. Hence every subspace of $X$ is quasi-compact, by proposition 3.2.10(b).

(c) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (a) Suppose $U_1 \subseteq U_2 \subseteq \ldots$ is an ascending chain of open subsets in $X$. Then $Y = \bigcup_{n \in \mathbb{N}} U_n$ is an open subspace of $X$ implies $Y = \bigcup_{k=1}^{r} U_{n_k}$ for some $n_1, \ldots, n_r \in \mathbb{N}$ by (b). Since $\{U_n\}_{n \in \mathbb{N}}$ is an ascending chain, $Y = U_m$ where $m = \max\{n_k | k = 1, \ldots, r\}$, whence $U_k \subseteq Y = U_m$, or $U_k = U_m$ for all $k \geq m$. Therefore $X$ is Noetherian.

Proposition 3.2.12 A Noetherian space is a finite union of irreducible closed subspaces.

Proof. Let $X$ be a Noetherian space and $\Sigma$ be the set of closed subsets of $X$ which are not finite unions of irreducible closed subspaces. Suppose $\Sigma$ is non-empty, then there exists a minimal element $Y$ in $\Sigma$ since $X$ is Noetherian. Now the non-irreducibility of $Y$ implies $Y = E_1 \cup E_2$ for some proper closed subsets $E_1, E_2$ of $Y$. By the minimality of $Y$, we have $E_i \notin \Sigma$, $i = 1, 2$. However, this means that $E_i$ is a finite union of irreducible closed subspaces of $X$, hence so for $Y$, which is a contradiction. Thus, $\Sigma$ is empty and $X$ is a finite union of irreducible closed subspaces.

Proposition 3.2.13 The prime spectrum of a Noetherian ring is a Noetherian space.

Proof. Let $R$ be a Noetherian ring and $X = \text{Spec}(R)$. Suppose $C = \{V(A_i)\}_{i \in I}$ is
a non-empty collection of closed subsets of $X$, then $S = \{\tau(A_i)\}_{i \in I}$ is a non-empty collection of ideals of $R$. Since $R$ is Noetherian, there exists a maximal element $\tau(A_{i_0})$ in $S$. Hence $V(A_{i_0})$ is a minimal element in $C$. Therefore $X$ is Noetherian. \[ \square \]

**Corollary 3.2.14** The set of all minimal prime ideals in a Noetherian ring is finite.

**Proof.** By proposition 3.2.9, the cardinality of the set of minimal prime ideals of a ring is the same as that of the set of irreducible components. The result follows easily from proposition 3.2.12 and the preceding proposition. \[ \square \]

### 3.2.3 Connectedness

**Proposition 3.2.15** If $R = \prod_{i=1}^{n} R_i$ is the direct product of rings $R_i$, then $\text{Spec}(R)$ is the disjoint union of open and closed subspaces $X_i$, where $X_i$ is canonically homeomorphic with $\text{Spec}(R_i)$.

**Proof.** By considering the canonical projection $\pi_j : R \rightarrow R_j$ for all $j \in I$. Then $X_j = \pi_j^*(\text{Spec}(R_j))$ is the set of prime ideals of $R$ containing $\{(x_i) \in \prod_{i=1}^{n} R_i | x_j = 0\}$. The results follow easily from the fact that $\text{Spec}(\prod_{i=1}^{n} R_i) = \bigcup_{i=1}^{n} \{\prod_{i=1}^{n} I_i | I_i = R_i$ for all $i \neq j$ and $I_j \in \text{Spec}(R_j)\}$. \[ \square \]

The preceding proposition shadows something about connectedness on $\text{Spec}(R)$, since it shows that a ring with connected prime spectrum cannot be a finite direct product of non-zero rings. In fact, we have the following characterization of connected prime spectrums (See [31]).

**Proposition 3.2.16** If $R$ is a ring, then the following conditions are equivalent:

1. $R = R_1 \times R_2$ where neither the ring $R_1$ nor the ring $R_2$ is a zero ring.
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(2) $R$ contains an idempotent $\neq 0, 1$.

(3) $X = \text{Spec}(R)$ is disconnected.

Proof.

(1) $\Rightarrow$ (2) Suppose $R = R_1 \times R_2$ with $R_i \neq 0$, $i = 1, 2$. Let $e = (1, 0) \in R$. Clearly, $e^2 = e$. Hence $R$ contains an idempotent $e \neq 0, 1$.

(2) $\Rightarrow$ (3) Let $e$ be an idempotent of $R$ different from 0 and 1. Then, we have $e(1 - e) = 0$. If $P \in \text{Spec}(R)$, then either $e \in P$ or $1 - e \in P$, but not both. This shows that $\text{Spec}(R) = D(Re) \sqcup D(R(1 - e))$. As $e \neq 0, 1$, $Re$ and $R(1 - e)$ are different from $R$, $D(Re)$ and $D(R(1 - e))$ are non-empty. Therefore $\text{Spec}(R)$ is disconnected.

(3) $\Rightarrow$ (1) Let $I, J$ be ideals of $R$ with $\text{Spec}(R) = D(I) \sqcup D(J)$. Then $D(I + J) = \text{Spec}(R)$ and $I + J = R$, and also $D(I \cap J) = \emptyset$. Hence, we have $IJ \subseteq I \cap J \subseteq P$ for all $P \in \text{Spec}(R)$, or $IJ \subseteq I \cap J \subseteq N(R)$. Let $1 = a_1 + a_2$, with $a_1 \in I$, $a_2 \in J$. So $a_1(1 - a_1) = a_1a_2 \in IJ \subseteq N(R)$, and consequently $a_1^n(1 - a_1)^n = 0$ for some $n \in \mathbb{N}$. Now, working in the polynomial ring $\mathbb{Z}[x]$, we obtain $1 = (x + (1 - x))^{2n} = \sum_{r=0}^{2n} C_r^{2n}x^{2n-r}(1-x)^r$. Put $f(x) = \sum_{r=0}^{n} C_r^{2n}x^{2n-r}(1-x)^r$: so $f(x) \equiv 0 \mod x^n$, and $f(x) \equiv 1 \mod (1 - x)^n$. Hence $f(x)^2 \equiv f(x) \mod x^n(1 - x)^n$; also note $f(x) \equiv x^{2n} \mod x(1 - x)$. Going back to $R$, put $e_1 = f(a_1)$, and $e_2 = 1 - e_1$. So $e_1^2 \equiv e_1$ mod $a_1^n(1 - a_1)^n$, that is, mod 0, so $e_1^2 = e_1$. It follows that $e_2^2 = e_2$ and $e_1e_2 = 0$, and $R = R_1 \times R_2$, where $R_1 = Re_1$, $R_2 = Re_2$. Also, since $a_1(1 - a_1) \in IJ$, we have $e_1 \equiv a_1^{2n} \mod IJ$. But $1 = a_1 + a_2$, so $a_1^r = a_1^{r+1} + a_1^r a_2$, or $a_1^r \equiv a_1^{r+1} \mod IJ$ for all $r$. Therefore $e_1 \equiv a_1 \mod IJ$, and also $e_2 \equiv a_2 \mod IJ$ (since $a_1 + a_2 = e_1 + e_2$). So $e_1 \in I$, $e_2 \in J$. Therefore, $e_1, e_2$ are different from 0,1 and hence $Re_1, Re_2$ are non-zero rings.

\footnote{We use $\sqcup$ to denote disjoint union.}
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Remark
(1) The last part of the proof can be greatly simplified if we impose the additional assumption that $R$ is reduced — a condition that is independent of the Zariski topology on Spec($R$).

(2) In fact, we can say a little bit more about the last part in the proof of the preceding proposition. We have shown in the proof that the existence of a disconnection $\text{Spec}(R) = D(I) \sqcup D(J)$ implies that $R = R_1 \times R_2$ where $R_i = R e_i$ for some idempotent $e_i \in R$, $i = 1, 2$. Moreover, $e_1 \in I$, $e_2 \in J$. It follows that $R_1 \subseteq I$ and $R_2 \subseteq J$. Thus $D(R_1) \subseteq D(I)$, $D(R_2) \subseteq D(J)$; but $\text{Spec}(R) = D(I) \sqcup D(J) = D(R_1) \sqcup D(R_2)$, so we conclude $D(R_1) = D(I)$ and $D(R_2) = D(J)$. Thus, we have shown that the given disconnection of $\text{Spec}(R)$ corresponds to the decomposition $R = R_1 \times R_2$.

In fact, we can show that those open subsets closed in Spec($R$) correspond to the idempotents in a ring $R$.

Proposition 3.2.17 Every open and closed subset of Spec($R$) is of the form $D(e)$ for some unique idempotent $e$ in a ring $R$.

Proof. Suppose $D(I)$ is an open and closed subset of Spec($R$), that is, Spec($R$) = $D(I) \sqcup D(J)$ for some ideal $J$ of $R$. Then, there is an idempotent $e$ such that $D(I) = D(Re) = D(e)$ by the proof of proposition 3.2.16. Suppose $e'$ is another idempotent in $R$ such that $D(e) = D(e')$. Taking complements, we have $V(e) = V(e')$ and $V(1 - e) = V(1 - e')$. So, for each $P \in \text{Spec}(R)$, we have either $e, e' \in P$ or $1 - e, 1 - e' \in P$. In either case, $e - e' \in P$. Since $P$ was arbitrary chosen, $e - e' \in N(R)$. Suppose $(e - e')^n = 0$, and without loss of generality we may assume $n$ is odd. Expanding, and using the fact that $e, e'$ are idempotents, we eventually obtain $e = e'$. \qed
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By the preceding proposition, \( \text{Spec}(R) \) being connected is the same as saying that the idempotents in \( R \) are only 0 and 1. For instance, if \( R \) is an integral domain, and \( e \) is an idempotent of \( R \), then \( e(1-e) = 0 \). This implies \( e = 0 \) or \( 1-e = 0 \). Again, if \( R \) is a local ring and \( e \) is an idempotent, then \( e(1-e) = 0 \); if \( e \in U(R) \) (the set of units of \( R \)), then \( 1-e = 0 \), and if \( e \notin U(R) \), then \( 1-e \in U(R) \), so \( e = 0 \). In either case, we have \( \text{Spec}(R) \) is connected. Here is an alternative, somewhat whimsical argument that avoids the mention of idempotents. Consider the unit real closed interval \([0,1]\). If \( P_1, P_2 \in \text{Spec}(R) \) with \( P_1 \subseteq P_2 \), then define \( f : [0,1] \rightarrow \text{Spec}(R) \) by \( f(x) = P_1 \) for \( x < 1 \) and \( f(1) = P_2 \). Thus \( f \) is continuous, so it is a path from \( P_1 \) to \( P_2 \), in the ordinary sense of topology. Therefore, if \( R \) is an integral domain, then \( 0 \in \text{Spec}(R) \), and \( 0 \subseteq P \) for all \( P \in \text{Spec}(R) \). So \( \text{Spec}(R) \) is indeed path-connected. Similarly, if \( R \) is a local ring, \( R\setminus U(R) \in \text{Spec}(R) \), and \( P \subseteq R\setminus U(R) \) for all \( P \in \text{Spec}(R) \), and \( \text{Spec}(R) \) is again path-connected. In either case, we deduce that \( \text{Spec}(R) \) is connected.

**Proposition 3.2.18** If \( R \) is Noetherian and \( \text{Spec}(R) \) is connected, then \( \text{Spec}(R) \) is path-connected.

**Proof.** By corollary 3.2.14, it is known that \( R \) has finite number of minimal prime ideals. Hence, the irreducible components of \( \text{Spec}(R) \) are closed and open. Since \( \text{Spec}(R) \) is connected, \( R \) can only have a unique minimal prime ideal \( P \). Therefore \( \text{Spec}(R) \) is path-connected, by repeating the argument above. \( \square \)

**Definition 3.2.19** A Baer ring is a ring \( R \) such that the annihilator of each element in \( R \) is the principal ideal generated by an idempotent.

**Remark** A Baer ring is necessarily a reduced ring: Suppose \( x \) is nilpotent, and let \( n \) be the smallest positive integer such that \( x^n = 0 \). If \( n \neq 1 \), then \( n \geq 2 \) and...
\(x^{n-1} \neq 0\). Since \(R\) is Baer, \(\text{Ann}(x^{n-1}) = (e)\) for some idempotent \(e\) in \(R\). As \(x \in \text{Ann}(x^{n-1})\), \(x = re\) for some \(r \in R\). Hence \(xe = (re)e = re = x\), which implies \(x^{n-1} = x^{n-1}e = 0\), a contradiction. Therefore \(n = 1\) and \(x = 0\).

**Proposition 3.2.20** A ring \(R\) is an integral domain if and only if it is a Baer ring with connected prime spectrum.

**Proof.** If \(R\) is an integral domain, then \(\text{Spec}(R)\) is connected by the discussion above. Besides, we have \(\text{Ann}(a) = 0\) for all \(a \in R \setminus \{0\}\) since \(R\) is an integral domain. It follows that \(R\) is a Baer ring. Conversely, suppose \(R\) is a Baer ring with connected prime spectrum. For any \(a \in R \setminus \{0\}\), \(\text{Ann}(a) = (e)\) for some idempotent \(e\) in \(R\). By proposition 3.2.17, \(D(\text{Ann}(a)) = D(e)\) is a closed and open subset of \(\text{Spec}(R)\). Since \(\text{Spec}(R)\) is connected, \(D(\text{Ann}(a)) = \text{Spec}(R)\) or \(D(\text{Ann}(a)) = \emptyset\). If \(D(\text{Ann}(a)) = \text{Spec}(R)\), then \(\text{Ann}(a) = R\). This implies that \(a = 0\), a contradiction. Thus, \(D(\text{Ann}(a)) = \emptyset\), whence \(\text{Ann}(a) \subseteq N(R) = 0\), as \(R\) is reduced by the remark above. Hence, \(R\) is an integral domain. \(\square\)

### 3.2.4 Normality

In section 3.2.1, we have seen that a ring \(R\) with its prime spectrum being Hausdorff equivalent to \(R^{\text{red}}\) is Von-Neumann regular. In this section, we will show that a ring \(R\) with its prime spectrum being normal is equivalent to a condition about the ordering on \(\text{Spec}(R)\).

We first give characterizations of closed subsets in \(\text{Spec}(R)\) and \(\text{Max}(R)\).

**Proposition 3.2.21** Let \(R\) be a ring and \(E\) a subset of \(\text{Spec}(R)\). Then \(E\) is closed in \(\text{Spec}(R)\) if and only if \(E = V(\cap E)\).
Proof. If $E$ is closed in $\text{Spec}(R)$, then $E = V(I)$ for some ideal $I$ of $R$. The result follows from the facts that $V(I) = V(r(I))$ and $r(I) = \cap E$. The converse part is trivial. \hfill \Box

The following corollary is a direct consequence of proposition 3.2.21.

**Corollary 3.2.22** For any ring $R$, $\text{Max}(R)$ is closed in $\text{Spec}(R)$ if and only if $\text{Max}(R) = V(J(R))$.

**Proposition 3.2.23** Let $R$ be a ring and $F$ a subset of $\text{Max}(R)$. Then $F$ is closed in $\text{Max}(R)$ if and only if $F = V(\cap F) \cap \text{Max}(R)$.

**Proof.** If $F$ is closed in $\text{Max}(R)$, then clearly $F \subseteq V(\cap F) \cap \text{Max}(R)$. Now $F = V(I) \cap \text{Max}(R)$ for some ideal $I$ of $R$ implies that $\cap F \supseteq I$ or $V(\cap F) \subseteq V(I)$. Then $V(\cap F) \cap \text{Max}(R) \subseteq V(I) \cap \text{Max}(R) = F$. The converse part is trivial. \hfill \Box

For each $M \in \text{Max}(R)$, the generization $G(M)$ of $M$ is $\{P \in \text{Spec}(R) | P \subseteq M\}$, let $O_M = \cap G(M)$. We note in passing that:

$$O_M = \{f \in R | \exists g \notin M \text{ such that } \text{Spec}(R) \setminus V(g) \subseteq V(f)\}.$$  

For $\text{Spec}(R) \setminus V(g) \subseteq V(f)$ is equivalent to $fg \in N(R)$, so that the first member contains the second. Next, for $f \in R$, put $S = \{f^n g | n \in \{0\} \cup \mathbb{N}, g \in R \setminus M\}$; $S$ is a multiplicatively closed set, hence if $S \cap N(R) = \emptyset$, there is $P \in \text{Spec}(R)$ contained in $R \setminus S$; then $f \notin P$ and $P \subseteq M$ i.e. $f \notin O_M$.

**Definition 3.2.24** A pm-ring is a ring in which every prime ideal is contained in a unique maximal ideal.

If $R$ is a pm-ring, denote by $\mu$ the map $\text{Spec}(R) \rightarrow \text{Max}(R)$ which sends every prime ideal into the unique maximal ideal containing it.
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**Proposition 3.2.25** Let $R$ be a pm-ring. Then $\mu$ is continuous and the space $\text{Max}(R)$ is a $T_2$ space.

**Proof.** Let $\mathcal{F}$ be a closed subset of $\text{Max}(R)$. Put $F = \cap \mathcal{F}$ and $I = \cap \mu^{-1}(\mathcal{F}) = \cap \{P \in \text{Spec}(R) | \mu(P) \in \mathcal{F}\}$. To show that $\mu^{-1}(\mathcal{F})$ is closed, it suffices to show that $V(I) \subseteq \mu^{-1}(\mathcal{F})$ since the reverse inclusion always holds. We first observe that if $Q \in \text{Spec}(R)$ and $Q \subseteq B = \cup \{M | M \in \mathcal{F}\}$, then $\mu(Q) \in \mathcal{F}:$ As $Q \subseteq B$ implies that $Q + F \subseteq B$, there exists $M \in \text{Max}(R)$ such that $Q + F \subseteq M$. Also, since $F \subseteq M$ and $\mathcal{F}$ is closed in $\text{Max}(R)$, $M \in \mathcal{F}$ by proposition 3.2.23. Moreover, $\mu(Q) \in \mathcal{F}$ since $Q \subseteq M$. Now let $P \in V(I)$, we proceed to show that $P$ contains a prime ideal $Q$ contained in $B$, and this fact leads to $\mu(P) = \mu(Q) \in \mathcal{F}$. In this purpose, let $S = R \setminus B$, $T = R \setminus P$ and pick $s \in S$, $t \in T$. Since $I \subseteq P$, there exists $P' \in \mu^{-1}(\mathcal{F})$ such that $t \notin P'$, and since $s \notin P'$, $st \notin P'$, whence $st \notin I$. Thus, the multiplicatively closed subset $ST = \{st : s \in S, t \in T\}$ does not meet $I$, consequently, there exists a prime ideal $Q$ (containing $I$) which is disjoint from $ST$. Obviously, $Q \subseteq B$ and $Q \subseteq P$. Thus, the continuity of $\mu$ is proved.

Now we show that $\text{Max}(R)$ is $T_2$. For this purpose, pick $M, M' \in \text{Max}(R)$ such that $M \neq M'$. Clearly, the multiplicatively closed subset $S = (R \setminus M)(R \setminus M')$ must contain $0$ for otherwise there is $P \in \text{Spec}(R)$ such that $P \cap S = \emptyset$. This implies that $P \subseteq M \cap M'$, contradicting to $R$ is a pm-ring. So there exist $f \notin M$ and $g \notin M'$ such that $fg = 0$, whence $M \subseteq D(f)$, $M' \subseteq D(g)$ and $D(f) \cap D(g) = D(fg) = D(0) = \emptyset$. Therefore $M, M'$ belong to disjoint open sets in $\text{Spec}(R)$ and so in $\text{Max}(R)$. Thus, $\text{Max}(R)$ is a $T_2$ space.

The following characterization of pm-ring was given by G. DeMarco and A. Orsatti [6].
Theorem 3.2.26 Let $R$ be a ring, then the following statements are equivalent:

(a) $R$ is a pm-ring.
(b) $\text{Max}(R)$ is a retract of $\text{Spec}(R)$.
(c) For each $M \in \text{Max}(R)$, $M$ is the unique maximal ideal containing $O_M$ (i.e. $G(M)$ is closed in $\text{Spec}(R)$).
(d) $\text{Spec}(R)$ is a normal space (in general not $T_2$).

Furthermore, if (a) holds, then the map $\mu$ is the unique retraction of $\text{Spec}(R)$ onto $\text{Max}(R)$.

Proof.

(a) $\Rightarrow$ (b) By proposition 3.2.25, the map $\mu$ is continuous. Hence $\mu$ is a retraction from $\text{Spec}(R)$ onto $\text{Max}(R)$.

(b) $\Rightarrow$ (a) Let $\tau$ be a retraction of $\text{Spec}(R)$ which maps $\text{Spec}(R)$ onto $\text{Max}(R)$. Take $P \in \text{Spec}(R)$ and write $M = \tau(P)$. Then $P \in \tau^{-1}({M})$. Since $\text{Max}(R)$ is always $T_1$, the $\tau^{-1}({M})$ is closed, whence $V(P) \subseteq \tau^{-1}({M})$. Therefore, if $M' \in \text{Max}(R)$ and $P \subseteq M'$, then $M' \in V(P) \subseteq \tau^{-1}({M})$. Consequently, $M' = \tau(M') = M$ i.e. $R$ is a pm-ring.

(c) $\Rightarrow$ (a) For each $P \in \text{Spec}(R)$, $P \subseteq M$ for some $M \in \text{Max}(R)$. Hence $P \in G(M) = V(O_M)$ and $M$ is the unique maximal element of $V(O_M)$ by (c). Therefore $R$ is a pm-ring.

(a) $\Rightarrow$ (d) By proposition 3.2.25, $\text{Max}(R)$ is $T_2$ and $\mu$ is continuous.
Max(R) is always quasi-compact, Max(R) is normal. The result then follows from the fact that \( \mu \) maps the disjoint closed subsets of Spec(R) into the disjoint closed subsets of Max(R).

\[(d) \Rightarrow (a) \] Let \( M, M' \in Max(R) \) with \( M \neq M' \). Then \( \{M\}, \{M'\} \) are disjoint closed subsets of Spec(R). Hence there exist \( a \notin M \) and \( a' \notin M' \) such that \( aa' \in N(R) \), whence \( M \cap M' \) cannot contain any prime ideals. \[\square\]

**Remark** Notice that Max(R) is Hausdorff does not imply that the ring R is a pm-ring since \( D(a) \cap D(a') = \emptyset \) in Max(R) implies only that \( aa' \in J(R) \). But, if \( J(R) = N(R) \) then R is a pm-ring.

**Proposition 3.2.27** Let R be a ring. Then Max(R) is dense in Spec(R) if and only if \( J(R) = N(R) \).

**Proof.** The result follows from the facts that Spec(R) = \( V(N(R)) \) and \( Max(R) = V(J(R)) \) \[\square\]

In view of the preceding proposition, we reformulate the above remark as follows:

**Proposition 3.2.28** If Max(R) is Hausdorff and is dense in Spec(R), then the ring R is a pm-ring.

For each \( M \in Max(R) \), let \( P^J_M = G(M) \cap V(J(R)) = \{ P \in Spec(R) | J(R) \subseteq P \subseteq M \} \) and let \( O^J_M = \cap P^J_M \). Then we have the following theorem cited in [6].

**Theorem 3.2.29** Let R be a ring. Then the following statements are equivalent:

\( (a) \) Every prime ideal of R containing \( J(R) \) is contained in a unique maximal ideal of R.
(b) $\text{Max}(R)$ is a retract of $V(J(R))$.

(c) For each $M \in \text{Max}(R)$, $M$ is the unique maximal ideal containing $O^M$ (i.e. $\mathcal{P}^M$ is closed in $\text{Spec}(R)$).

(d) $V(J(R))$ is a normal space (in general not $T_2$).

(e) $\text{Max}(R)$ is $T_2$.

Furthermore, if (a) holds, then the map $\mu^J : V(J(R)) \rightarrow \text{Max}(R)$ sending every prime ideal of $V(J(R))$ into the unique maximal ideal containing itself is the unique retraction of $V(J(R))$ onto $\text{Max}(R)$.

**Proof.** Just applying theorem 3.2.26 to $R/J(R)$, and using the remark of theorem 3.2.26.

**Remark** It should be noted that Hausdorffness and Normality are equivalent concepts on $\text{Max}(R)$, due to its nature of being a $T_1$ quasi-compact space.

Now we turn to show that the connectedness of $\text{Max}(R)$ precipitates the connectedness of $\text{Spec}(R)$.

**Proposition 3.2.30** If the maximal spectrum of a ring $R$ is connected, then $\text{Spec}(R)$ is connected.

**Proof.** Suppose $E$ is a non-empty closed and open subset of $\text{Spec}(R)$, then $E = D(e)$ for some idempotent $e$ of $R$. As $\text{Max}(R)$ is connected, $E \cap \text{Max}(R) \neq \emptyset$ implies that $E \cap \text{Max}(R) = \text{Max}(R)$, whence $E \supseteq \text{Max}(R)$. Therefore $e$ is a unit of $R$ and $E = \text{Spec}(R)$. Hence, $\text{Spec}(R)$ is connected. $\square$

In case if $R$ is a pm-ring, then the converse statement still holds.
Proposition 3.2.31  If $R$ is a pm-ring with connected prime spectrum, then $\text{Max}(R)$ is connected.

Proof. By proposition 3.2.25, the map $\mu : \text{Spec}(R) \to \text{Max}(R)$ is continuous, the result then follows.

3.3 Topology on $\text{Min}(R)$ and Baer rings

The minimal prime spectrum $\text{Min}(R)$ of a ring $R$ is the set of all minimal prime ideals of $R$, partially ordered by inclusion ordering. As $\text{Min}(R) \subseteq \text{Spec}(R)$, we always consider $\text{Min}(R)$ as a topological space under the subspace topology. A topological space $X$ is called minspectral if it is homeomorphic to $\text{Min}(R)$ for some ring $R$. In this section, a topological characterization of minspectral space given by M. Hochster [17] is mentioned and some of its relation with the underlying ring will be discussed.

We begin by showing that the minspectral space is necessarily a Hausdorff space with a base consisting of closed and open sets (clopen sets, for short).

Proposition 3.3.1  For any element $x$ in a reduced ring $R$, $\text{Min}(R) = (V(\text{Ann}(x)) \cap \text{Min}(R)) \sqcup (V(x) \cap \text{Min}(R))$. Thus, besides being closed by definition, the sets $V(x) \cap \text{Min}(R)$ and $V(\text{Ann}(x)) \cap \text{Min}(R)$ are open.

Proof.  It is easy to see that $V(x) \cup V(\text{Ann}(x)) = \text{Spec}(R)$. The results then follow from proposition 3.2.3(1) that any minimal prime ideal $P$ cannot contain both $x$ and $\text{Ann}(x)$.

The following corollaries are direct consequences of proposition 3.3.1.
Corollary 3.3.2 A minispectral space is necessarily a Hausdorff space with a clopen base (i.e. a base consists of clopen sets), and hence is totally disconnected and completely regular.

Corollary 3.3.3 An element in a reduced ring belongs to some minimal prime ideal if and only if it is a zero-divisor.

Remark The preceding corollary is in fact a particular case of proposition 3.2.3(2).

By an $m$-subbase (resp. $m$-base) $B$ for a Hausdorff space $X$ we mean an open subbase (resp. base) such that each subset of $B$ with the finite intersection property intersects. Thus $B$ is an $m$-base if and only if $B$ is an open subbase and at the same time a subbase for the closed sets of a (usually different) quasi-compact topology on $X$. Besides, we call an open base $B$ full if $\emptyset, X \in B$ and $B$ is closed under finite union and intersection.

Let $R$ be a ring. Denote the set of all quasi-compact open subsets of $\text{Spec}(R)$ by $\mathcal{Q}(R)$. Let $C = \{Q \cap \text{Min}(R) | Q \in \mathcal{Q}(R)\}$. Then we have the following proposition obtained by Hochster [17].

Proposition 3.3.4 The following conditions on an open base $B$ for a Hausdorff space $X$ are equivalent:

1. $B$ is a full $m$-base.

2. There is a ring $R$ and a homeomorphism $h : X \rightarrow \text{Min}(R)$ such that $h$ induces a bijection of $B$ onto $C$.

Proof.
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(2) $\Rightarrow$ (1) Let $R, h$ be given. It suffices to show that $C$ is a full $m$-base for $\text{Min}(R)$. As it is well-known that $Q(R)$ is a full open base for $\text{Spec}(R)$, $C$ is a full open base for $\text{Min}(R)$. Now suppose $\{Q_i\}_{i \in I}$ is a family of sets in $Q(R)$ such that $\{Q_i \cap \text{Min}(R)\}_{i \in I}$ has the FIP. Then $\{Q_i\}_{i \in I}$ has the FIP as well. So $\bigcap_{i \in I} Q_i \neq \emptyset$ because $\text{Spec}(R)$ is quasi-compact. Let $P \in \bigcap_{i \in I} Q_i$. Then $P$ contains a minimal $P'$, by Zorn's lemma. Since $Q_i$ is open, $P \in Q_i$ implies $P' \in Q_i$, and so $P' \in Q_i \cap \text{Min}(R)$. Thus, $P' \in \bigcap_{i \in I} (Q_i \cap \text{Min}(R))$. This means that each subset of $C$ with the FIP intersects. Hence, $B$ is a full $m$-base as required.

(1) $\Rightarrow$ (2) Suppose $X$ is a Hausdorff space with a given full $m$-base $B$. Let $W = \{0, 1\}$ endowed with the topology $T = \{\emptyset, \{0\}, \{0, 1\}\}$. Assume that one copy $W_B$ of $W$ is given for every $B \in B$. Then the map $f_B : X \rightarrow W$ defined by $f_B(x) = 0$ if $x \in B$, $f_B(x) = 1$ if $x \notin B$ is continuous. We then obtain a continuous map $f = \prod_{B \in B} f_B$ from $X$ to $P = \prod_{B \in B} W_B$. Assign $P$ with two topologies: the product topology coming from the topology $T$ specified before, which we call the \textit{weak topology}, denoted it by $W$; and the product topology obtained by letting each $W_B$ have the discrete topology, which we call the \textit{strong topology}, denoted it by $S$. If no specification is made, then we assign $P$ with the weak topology $W$.

Let $Y$ be the $S$-closure of $f(X)$ in $P$. Topologize $Y$ by the topology inherited from $(P, W)$. By proposition 1.4.2, $Y$ is spectral, i.e. $Y \approx \text{Spec}(R)$ for some ring $R$. Let $p_y$ be the prime ideal of $R$ corresponding to a given $y \in Y$. Then, given $y, y' \in Y$, the following conditions are equivalent:

(a) $y' \in \overline{\{y\}}_{Y, W|_Y}$.

(b) For every open (resp., quasi-compact open) neighbourhood $U$ of $y'$, $y \in U$.

(c) $p_y \subseteq p_{y'}$. 

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The subspace \( \text{Min}(R) \) clearly corresponds to the following set:

\[
Y_0 = \{ y \in Y | y' \in \overline{\{y\}}_{Y,W|_V} \text{ implies } y' = y \}.
\]

Now, since \( \mathcal{B} \) is an open base for the Hausdorff space \( X \), the maps \( f_B \) separate points and closed sets, and hence \( f \) is an embedding which maps \( X \) into the topological space \( (\mathcal{P}, \mathcal{W}) \). We now wish to show that \( f(X) = Y_0 \). First, we prove that for each \( y \in Y \), there is an \( x \in X \) such that \( y \in \overline{\{f(x)\}}_{Y,W|_V} \). Let \( y \in Y \) and \( B_y = \{ B \in \mathcal{B} | f_B(y) = 0 \} = \{ B \in \mathcal{B} | y \in B \} \). An open base for the \( \mathcal{W} \)-neighbourhoods of \( y \) in \( P \) is given by the sets of the form \( U(\mathcal{D}) = \{ z \in \mathcal{P} | f_D(z) = 0 \text{ for each } D \in \mathcal{D} \} \), where \( \mathcal{D} \) runs through the finite subsets of \( B_y \).

To show that there is an \( x \in X \) such that \( y \in \overline{\{f(x)\}}_{Y,W|_V} \), it suffices to show that \( \bigcap_{\mathcal{D}} (U(\mathcal{D}) \cap f(X)) \neq \emptyset \); for if \( f(x) \) is in the intersection, then \( y \in \overline{\{f(x)\}}_{Y,W|_V} \). This is equivalent to show that \( \bigcap_{\mathcal{D}} f^{-1}(U(\mathcal{D})) \neq \emptyset \). Trivially, \( f^{-1}(U(\mathcal{D})) = \bigcap_{D \in \mathcal{D}} D \). Since each \( D \in \mathcal{B} \) and a family of sets in \( \mathcal{B} \) with the FIP intersects, we only need to show that if \( \mathcal{D}_1, \ldots, \mathcal{D}_k \subseteq \mathcal{B}_y \), then \( \bigcap_{i=1}^k (\bigcap_{D \in \mathcal{D}_i} D) \neq \emptyset \). Now \( y \) is in the \( S \)-closure of \( f(X) \) in \( P \) and \( \bigcap_{i=1}^k (\bigcap_{D \in \mathcal{D}_i} D) = \bigcap_{D \in \bigcup_{i=1}^k \mathcal{D}_i} D \) is a \( S \)-open neighbourhood of \( y \) in \( P \); hence it meets \( f(X) \). But this says precisely that the intersection of the sets in \( \bigcup_{i=1}^k \mathcal{D}_i \) is non-empty.

We now proceed to show that \( f(X) = Y_0 \). Let \( y \in Y_0 \). Choose \( x \in X \) such that \( y \in \overline{\{f(x)\}}_{Y,W|_V} \). By definition of \( Y_0 \), \( y = f(x) \). Thus \( Y_0 \subseteq f(X) \). Now suppose \( x \in X \), then \( f(x) \in \overline{\{y\}}_{Y,W|_V} \) for some \( y \in Y_0 \), since \( Y \) is spectral. As \( Y_0 \subseteq f(X) \), \( y = f(x') \) for some \( x' \in X \). Because \( f \) induces a homeomorphism of \( X \) onto \( f(X) \) and \( f(x) \in \overline{\{f(x')\}}_{Y,W|_V} \), we must have \( x \in \overline{\{x'\}} \). Since \( X \) is Hausdorff, \( x = x' \) and \( f(x) = y \in Y_0 \).

Thus, \( f \) induces a homeomorphism of \( X \) onto \( Y_0 \). It remains to show that this map induces a bijection between \( \mathcal{B} \) and \( \{ Q \cap Y_0 | Q \text{ is quasi-compact open subset of } Y \} \).
To see this, we first notice that for a given $B \in \mathcal{B}$, we have $f(B) = \{y \in Y_0 | f_B(y) = 0\} = (Q_B \cap Y) \cap Y_0$, where $Q_B = \{z \in P | f_B(z) = 0\}$ is quasi-compact open in $P$, and hence $Q_B \cap Y$ is quasi-compact open in $Y$. As the sets $Q_{B_1} \cap \ldots \cap Q_{B_k} \cap Y$ form an open base for $Y$, every quasi-compact open subset $Q$ of $Y$ is a finite union of sets of this form. The inverse image of $Q \cap Y_0$ is a finite union of finite intersections of sets $B_i \in \mathcal{B}$, and is thus in $\mathcal{B}$, since $\mathcal{B}$ is full.

If $\mathcal{B}$ is an open subbase for $X$, then the topology having $\mathcal{B}$ as a subbase for its closed sets is called the dual topology on $X$ determined by $\mathcal{B}$. It can be observed that any open subbase $\mathcal{B}$ generates a least full open base containing it, consisting of $\emptyset$, $X$, and the finite unions of finite intersections of sets $B_i \in \mathcal{B}$. This full open base and $\mathcal{B}$ obviously determine the same dual topology.

**Proposition 3.3.5** Let $X$ be a Hausdorff space. Then we have the following properties:

1. If $\mathcal{B}$ is an $m$-subbase, then the full open base generated by $\mathcal{B}$ is an $m$-base.

2. If $\mathcal{B}$ is an $m$-subbase, then any subset of $\mathcal{B}$ which is a subbase is an $m$-subbase.

3. If $\mathcal{B}$ is an $m$-subbase (resp. $m$-base, full $m$-base) for $X$, and $Y \subseteq X$ is closed in the dual topology determined by $\mathcal{B}$, then $\{B \cap Y | B \in \mathcal{B}\}$ is an $m$-subbase (resp. $m$-base, full $m$-base) for $Y$.

4. If $\mathcal{B}$ is an $m$-base for $X$ and $U \subseteq X$ is open, then $\{B \in \mathcal{B} | B \subseteq U\}$ is an $m$-base for $U$.

5. If $\mathcal{B}$ is an $m$-base for $X$, then each set in $\mathcal{B}$ is clopen.

**Proof.**

1. As the dual topologies involved are actually equal, they are quasi-compact.
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(2) Trivial.

(3) This part follows from the quasi-compactness of $Y$ in the inherited dual topology determined by $\{B \subseteq Y \mid B \in \mathcal{B}\}$.

(4) Trivial.

(5) Let $B \in \mathcal{B}$ and $p \in \overline{B}$. Let $\mathcal{B}_p = \{C \in \mathcal{B} \mid p \in C\}$. Then, for each finite subset $\{C_1, \ldots, C_k\}$ of $\mathcal{B}_p$, $(C_1 \cap \ldots \cap C_k) \cap B \neq \emptyset$ since it contains $p$. Hence, $(\cap_{C \in \mathcal{B}_p} C) \cap B \neq \emptyset$. But $\cap_{C \in \mathcal{B}_p} C = \{p\}$, since $X$ is Hausdorff. Therefore, we have $p \in B$. □

Hochster [17] obtained the following characterization of a Hausdorff space to be minspectral.

Theorem 3.3.6 The following conditions on a Hausdorff space $X$ are equivalent:

(1) $X$ is minspectral.

(2) $X$ has an $m$-subbase.

(3) $X$ has a full $m$-base.

Proof. $(1) \Leftrightarrow (3)$ and $(3) \Leftrightarrow (2)$ are obvious from proposition 3.3.4 and proposition 3.3.5. □

Corollary 3.3.7 Open subspaces of minspectral spaces are minspectral.

Proof. This is an immediate result of proposition 3.3.5(4). □

In view of theorem 3.2.26, J. Kist obtained a similar result on $\text{Min}(R)$ in [23]. First, we need some preliminary results.
For $P \in \text{Spec}(R)$, define $O_P = \cap G(P)$. Then we have the following lemma.

**Lemma 3.3.8** Let $R$ be a reduced ring. Then $O_P = \{a \in R | \text{Ann}(a) \not\subset P\}$ for each $P \in \text{Spec}(R)$.

**Proof.** Suppose $a \in R$ and $\text{Ann}(a) \not\subset P$. Then $\text{Ann}(a) \not\subset Q$ for each $Q \in G(P)$, whence $a \in \cap G(P) = O_P$. Conversely, suppose $a \in O_P$. Let $S = \{a^n b | n \in \{0\} \cup \mathbb{N}, b \in R \setminus P\}$. Then $S$ is a multiplicatively closed subset of $R$. If $0 \not\in S$, then there exists $Q \in \text{Spec}(R)$ such that $Q \cap S = \emptyset$. Hence, we have $Q \in G(P)$ and $a \not\in Q$. This contradiction implies that $a^n b = 0$ for some $n \in \{0\} \cup \mathbb{N}$ and $b \in R \setminus P$, whence $ab = 0$ since $R$ is reduced. Therefore $b \in \text{Ann}(a) \setminus P$, and $\text{Ann}(a) \not\subset P$. \qed

The following theorem is Kist's characterization [23] of reduced Baer ring by its minimal prime spectrum.

**Theorem 3.3.9** A reduced ring $R$ is a Baer ring if and only if the minimal prime spectrum of $R$ is a retract of the prime spectrum of $R$.

**Proof.**

$\Rightarrow$) If $a \in R$, then $V(\text{Ann}(a) \cap \text{Ann}(\text{Ann}(a))) = \text{Spec}(R)$. It follows that $O_P \subseteq \{a \in R | \text{Ann}(\text{Ann}(a)) \subseteq P\}$ for all $P \in \text{Spec}(R)$ by lemma 3.3.8. Since $R$ is a Baer ring, $\text{Ann}(a) + \text{Ann}(\text{Ann}(a)) = R$ for all $a \in R$; so $O_P = \{a \in R | \text{Ann}(\text{Ann}(a)) \subseteq P\}$. As $R$ is a reduced ring, $\text{Ann}(\text{Ann}(ab)) = \text{Ann}(\text{Ann}(a)) \cap \text{Ann}(\text{Ann}(b))$ for each pair $a, b \in R$. Therefore, $O_P$ is a prime ideal for each prime ideal $P$ in a Baer ring.

Now $\{P \in \text{Spec}(R) | a \not\in O_P\} = V(\text{Ann}(a))$, and $V(\text{Ann}(a))$ is clopen in the Baer ring $R$ by proposition 3.2.17, so $\nu : \text{Spec}(R) \rightarrow \text{Min}(R)$ defined by $\nu(P) = O_P$ is a continuous mapping.

In any ring, clearly $a \in \text{Ann}(\text{Ann}(a))$, and thus $O_P \subseteq P$. Hence, in a Baer ring,
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$O_P = P$ for each minimal prime ideal $P$. Therefore $\nu$ is a retraction of Spec$(R)$ onto Min$(R)$.

$\Leftarrow$ Suppose $\tau$ is a retraction of Spec$(R)$ onto Min$(R)$. If $Q \in \text{Min}(R)$, then $Q = \tau(Q)$, i.e. $Q \in \tau^{-1}(Q)$. Since $\tau$ is a continuous mapping and Min$(R)$ is Hausdorff, $\tau^{-1}(Q)$ is closed in Spec$(R)$, so $V(Q) \subseteq \tau^{-1}(Q)$. Thus, if $P \in \text{Spec}(R)$ and $Q \subseteq P$, then $\tau(P) = Q$. Consequently, each prime ideal in $R$ contains a unique minimal prime ideal, and so $\tau(P) = O_P$ for each $P \in \text{Spec}(R)$.

Since $R$ is reduced, lemma 3.3.8 ensures that $\{P \in \text{Spec}(R)|a \notin O_P\} = V(\text{Ann}(a))$ for each $a \in R$. Since $\tau$ is continuous, $V(\text{Ann}(a)) = \tau^{-1}(D(a) \cap \text{Min}(R))$ is clopen for each $a \in R$. Then there exists an ideal $I$ of $R$ such that $V(\text{Ann}(a)) \sqcup V(I) = \text{Spec}(R)$. This fact implies that $\text{Ann}(a) \oplus I = R$. As $1 \in R$, $1 = e + x$ for some $e \in \text{Ann}(a)$, $x \in I$. Now $e = e^2 + ex$ but $ex \in \text{Ann}(a) \cap I = 0$, it follows that $e = e^2$. Clearly $Re \subseteq \text{Ann}(a)$. Let $y \in \text{Ann}(a)$. Then $y = ey + xy$, but $xy \in \text{Ann}(a) \cap I = 0$. Hence, $y \in Re$ and $\text{Ann}(a) = Re$. This shows that $R$ is a Baer ring.

The following theorem is an application of theorem 3.3.9 provided by J. Kist himself [23].

**Theorem 3.3.10** Let $R$ be a pm-ring. For each $P \in \text{Spec}(R)$, let $\mu(P)$ be the unique maximal ideal containing $P$, and $\tau$ the restriction of $\mu$ on Min$(R)$. Then the following statements hold:

(a) $\tau$ is a continuous mapping of Min$(R)$ onto Max$(R)$.

(b) If $J(R) = 0$, then $\tau$ maps no proper closed subset of Min$(R)$ onto Max$(R)$.

(c) $\tau$ is injective if and only if each prime ideal in $R$ contains a unique minimal prime ideal.
(d) Let $R$ be a reduced ring. Then $\tau$ is a homeomorphism if and only if $R$ is a Baer ring.

(e) Assume each prime ideal of the reduced ring $R$ contains a unique minimal prime ideal. Then $\text{Min}(R)$ is quasi-compact if and only if $R$ is a Baer ring.

Proof.

(a) Immediate consequence of proposition 3.2.25.

(b) Clearly, every proper closed set in $\text{Min}(R)$ is contained in a set of the form $V(a)$ for some $a \in R \setminus \{0\}$, since such sets form a base for the closed sets. Thus, if $M \in \text{Max}(R)$ with $a \not\in M$, then $M \not\in \tau(V(a))$.

(c) It is easy to see that the following three statements are equivalent:

(i) $\tau$ is one-to-one.

(ii) Each maximal ideal contains a unique minimal prime ideal.

(iii) $O_M$ is a minimal prime ideal for each maximal ideal $M$.

For each $P \in \text{Spec}(R)$, we have $O_{\mu(P)} \subseteq P \subseteq \mu(P)$. Thus, if (iii) holds, then each prime ideal contains a unique minimal prime ideal, in particular, $O_M \in \text{Min}(R)$ for each $M \in \text{Max}(R)$. So, $\tau$ is one-to-one with inverse $M \mapsto O_M$.

(d) If $R$ is a Baer ring, then by theorem 3.3.9, the mapping $\nu : P \mapsto O_P$ is a retraction of $\text{Spec}(R)$ onto $\text{Min}(R)$. In particular, the restriction of $\nu$ on $\text{Max}(R)$ is the inverse of $\tau$. So, $\tau$ is a homeomorphism. Conversely, if $\tau$ is a homeomorphism, then $\tau^{-1} \circ \mu : P \mapsto \mu(P) \mapsto O_{\mu(P)} = O_P$ is a retraction of $\text{Spec}(R)$ onto $\text{Min}(R)$. This shows that the reduced ring $R$ is a Baer ring by theorem 3.3.9.
(e) Suppose \( \text{Min}(R) \) is quasi-compact. Observe that \( \tau \) is then a bijective continuous mapping which maps \( \text{Min}(R) \) onto \( \text{Max}(R) \), whence \( \tau \) is a homeomorphism, since \( \text{Max}(R) \) is Hausdorff by proposition 3.2.25. Therefore, \( R \) is a Baer ring by (d). Conversely, if \( R \) is a Baer ring, then by (d), we have \( \text{Min}(R) \approx \text{Max}(R) \) which is quasi-compact. 

In view of proposition 3.3.10(e), the quasi-compactness of \( \text{Min}(R) \) is related to whether the ring \( R \) is Baer or not. In fact, G. Artico and U. Marconi [1] explored more details along this direction. They studied first the situation that \( \text{Min}(R) \) is quasi-compact.

**Theorem 3.3.11** Let \( R \) be a reduced ring. Then the following statements are equivalent:

1. The family of sets \( \{ V(a) \cap \text{Min}(R) | a \in R \} \) is a subbase for the topology of \( \text{Min}(R) \).
2. The space \( \text{Min}(R) \) is quasi-compact.
3. For each \( a \in R \), there exists a finite number of elements \( a_1, \ldots, a_n \in R \) such that \( aa_i = 0 \) for each \( i = 1, \ldots, n \) and \( \text{Ann}(a_1, \ldots, a_n, a) = 0 \).

**Proof.**

(1) \( \Rightarrow \) (2) Suppose \( \cap_{a \in B}(D(a) \cap \text{Min}(R)) = \emptyset \) for some \( B \subseteq A \). Since \( \cap_{a \in B}(D(a) \cap \text{Min}(R)) \) coincides with the set of minimal prime ideals disjoint from \( B \). Let \( S \) be the multiplicatively closed subset generated by \( B \). Then, it is known that a prime ideal is disjoint from \( S \) if and only if it is disjoint from \( B \). Now \( 0 \in S \), for if otherwise, there exists a prime ideal, and then a minimal prime one, disjoint from \( B \), which leads to a contradiction. But \( 0 \in S \) implies there exists \( a_1, \ldots, a_n \in B \) such that \( a_1 \ldots a_n = 0 \), and so \( \cap_{i=1}^n(D(a_i) \cap \text{Min}(R)) = D(a_1 \ldots a_n) \cap \text{Min}(R) = \)
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\[ D(0) \cap \text{Min}(R) = \emptyset. \] Therefore \( \text{Min}(R) \) is indeed quasi-compact, by Alexander's subbase theorem.

(2) => (3) If \( \text{Min}(R) \) is quasi-compact, then \( \text{V}(a) \cap \text{Min}(R) \) is a quasi-compact open set, and therefore it is a union of basic open sets, i.e.

\[ \text{V}(a) \cap \text{Min}(R) = \bigcup_{i=1}^{n} (\text{D}(a_i) \cap \text{Min}(R)) \]

for some \( a_i \in R \) \( i = 1, \ldots, n \). Since \( D(a_i) \cap \text{Min}(R) \subseteq V(a) \cap \text{Min}(R), \text{Min}(R) \subseteq V(aa_i) \), for each \( i = 1, \ldots, n \), and so \( aa_i = 0 \) for each \( i \). Moreover, the above relation implies that \( V(a_1, \ldots, a_n, a) \cap \text{Min}(R) = \emptyset \); by proposition 3.2.3(2), \( \text{Ann}(a_1, \ldots, a_n, a) = 0 \).

(3) => (1) Choose a basic open set \( D(a) \cap \text{Min}(R) \). Let \( a_1, \ldots, a_n \) be the elements given by (3). By proposition 3.2.3(2), the ideal \( I = (a_1, \ldots, a_n, a) \) is contained in no minimal prime; this implies that \( D(a) \cap \text{Min}(R) \supseteq V(a_1) \cap \ldots \cap V(a_n) \cap \text{Min}(R) \). But since \( aa_i = 0 \) for all \( i \), equality actually holds. \( \square \)

Theorem 3.3.9 implies that, in a Baer ring \( R \), every prime ideal contains a unique minimal prime ideal and that \( \text{Min}(R) \) is quasi-compact. In fact, these two conditions do characterize the Baer rings (See [1]).

**Theorem 3.3.12** Let \( R \) be a reduced ring. Then the following statements are equivalent:

(1) \( R \) is a Baer ring.

(2) Every prime ideal contains a unique minimal prime ideal and \( \text{Min}(R) \) is quasi-compact.

(3) \( \text{Min}(R) \) is a retract of \( \text{Spec}(R) \).

**Proof.**
Chapter 3. The topology on \( \text{Spec}(R) \)

(1) \( \Rightarrow \) (2) By theorem 3.3.9, \( \text{Min}(R) \) is a retract of \( \text{Spec}(R) \). Hence, \( \text{Min}(R) \) is quasi-compact. Again by theorem 3.3.9, \( O_P \) is the unique minimal prime ideal contained in \( P \) for each \( P \in \text{Spec}(R) \). Therefore every prime ideal contains a unique prime ideal.

(2) \( \Rightarrow \) (3) Let \( \tau : \text{Spec}(R) \to \text{Min}(R) \) be defined by \( \tau(P) = O_P \) for each \( P \in \text{Spec}(R) \). For any \( a \in R \), by proposition 3.3.1, \( \tau^{-1}(D(a) \cap \text{Min}(R)) = \tau^{-1}(V(\text{Ann}(a)) \cap \text{Min}(R)) = V(\text{Ann}(a)) \), is closed in \( \text{Spec}(R) \). By theorem 3.3.11, \( \tau \) is continuous. Therefore \( \text{Min}(R) \) is a retract of \( \text{Spec}(R) \).

(3) \( \Rightarrow \) (1) Follows immediately from theorem 3.3.9.

Remark

(1) Let \( K \) be a field and \( x, y \) be indeterminates over \( K \). Then, \( R = K[x, y]/(xy) \) is a ring with quasi-compact minimal prime spectrum, but it is not a Baer ring. Since \( R \) is Noetherian, \( \text{Min}(R) \) is finite and hence quasi-compact. It is easy to see that \( R \) is a reduced ring with no idempotent differs from 0 and 1. Using the fact that \( K[x, y] \) is a unique factorization domain, it can be shown that \( \text{Ann}(x + (xy)) \) is generated by \( y + (xy) \). So, \( R \) is not a Baer ring.

(2) If \( X \) is a topological space, let \( C(X) \) be the ring of all real-valued continuous functions on \( X \). In [11, page 208], \( X \) is said to be an F-space if every prime ideal of \( C(X) \) contains a unique minimal prime ideal; \( X \) is called basically disconnected if the closure of every cozero-set (i.e. a subset of \( X \) in the form \( \{x \in X : |f(x)| > 0\} \) for some \( f \in C(X) \)) is an open set. One can easily prove that \( C(X) \) is a Baer ring if and only if \( X \) is basically disconnected. Since there exist F-spaces \( X \) that are not basically disconnected, for instance \( \beta \mathbb{R}\setminus\mathbb{R} \) (where \( \beta \mathbb{R} \) is the Stone-Čech compactification of \( \mathbb{R} \)), there are rings in which every prime ideal contains a unique minimal prime ideal, without being Baer rings.
Now we turn to the notion of connectedness on \( \text{Min}(R) \), indeed it is just the case that \( \text{Min}(R) \) is a singleton, due to its nature of being totally disconnected.

**Proposition 3.3.13** For a ring \( R \), \( \text{Min}(R) \) is connected if and only if it is a singleton.

**Proof.** This result follows from the fact that \( \text{Min}(R) \) is totally disconnected. \( \square \)

**Corollary 3.3.14** For a ring \( R \), \( \text{Min}(R) \) is connected if and only if \( \text{Spec}(R) \) is irreducible.

Hence \( \text{Min}(R) \) is connected implies \( \text{Spec}(R) \) is connected.

Similar to proposition 3.2.31, we also have the following proposition:

**Proposition 3.3.15** If \( R \) is a Baer ring with connected prime spectrum, then \( \text{Min}(R) \) is connected.

**Proof.** The result is immediate as \( \text{Min}(R) \) is a retract of \( \text{Spec}(R) \). \( \square \)

**Remark** In fact, by proposition 3.2.20, we know that \( R \) is Baer with connected \( \text{Spec}(R) \) means that \( R \) is an integral domain. Thus, \( \text{Min}(R) \) is connected.
Chapter 4

Study algebraic properties from \( \text{Spec}(R) \)

Having discussed the order and topology on \( \text{Spec}(R) \) in chapters 2 and 3, we are now going to put these two things together in this chapter. We begin by studying the works of Thomas S. Fischer about the prime spectrums [8] and maximal spectrums [7] of Bézout rings. D. Lazard's characterization [25] of an algebraic property (known as property A(0)) of a ring in terms of the D-closed subsets of its prime spectrum will be presented, with detailed proof. W. J. Lewis and J. Ohm [27] pointed out by an example that the property A(0) cannot be characterized in terms of the ordering of \( \text{Spec}(R) \) alone. Evidently, the ordering on \( \text{Spec}(R) \) provides less information than the topology on \( \text{Spec}(R) \). Moreover, the notion of \( C(m) \) topology suggested by W. J. Lewis and J. Ohm will also be mentioned, it provides a path to generalize the theorem of W. J. Lewis that every finite poset is spectral. In section 4, evaporation of the difference between the ordering and topology on prime spectrum of Noetherian ring is mentioned, other related results will also be described. To unify what has been discussed in the preceding chapters, applications of some results mentioned before will be provided in the remaining sections, with emphasis on how to recognize an
algebraic property of a ring either from the ordering or the topology on its prime spectrum.

4.1 Prime spectrums of Bézout rings

Thomas S. Fischer in [8] suggested the general problem of characterizing the prime spectrum of a Bézout ring, which is motivated by answering the following question:

Let \( R \) be a reduced coherent ring (see definition 4.1.2 below) and \( I \) a finitely generated ideal of \( R \). Is \( \text{Min}(R/I) \) a compact space?

This question is an interesting one since a reduced coherent ring always has compact minimal prime spectrum (See [9, Corollary 4.2.16]). Fischer first answered this question negatively by using proposition 2.4.2 and the proposition below. His basic idea is to construct a Bézout domain without the desired property.

**Proposition 4.1.1** Let \( R \) be a Bézout domain and \( G \) its group of divisibility. Then \( \text{Spec}(R) \) is homeomorphic to \( \text{Spec}(G) \).

**Proof.** Let \( K \) be the field of quotients of \( R \) and \( \mu : K^* \rightarrow G \) the canonical map. Then, from the definition of \( G \), we have for any \( a, b \in R \),

\[
\mu(a) \leq \mu(b) \text{ if and only if } b \in Ra.
\]

Let \( P \in \text{Spec}(R) \), \( P^* = P \setminus \{0\} \) and \( a \in R \setminus \{0\} \). Then we have

\[
\mu(r) \leq \mu(P^*) \text{ if and only if } r \in P^*.
\]

Since \( P \) is a prime ideal, it follows easily that \( \mu(P^*) \in \text{Spec}(G) \). Define \( \psi : \text{Spec}(R) \rightarrow \text{Spec}(G) \) by \( \psi(P) = \mu(P^*) \). Then, by (2), it can be shown that \( \psi \) is a one-to-one, continuous open mapping. Let \( Q \in \text{Spec}(G) \) and \( P = \mu^{-1}(Q) \cup \{0\} \).
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Since the ring $R$ is Bézout, it follows that $P \in \text{Spec}(R)$ by (1). Hence $\psi(P^*) = Q$, and $\psi$ is surjective. □

**Remark** In fact, this theorem is a topological analogue of theorem 2.4.3.

Proposition 4.1.1, together with theorem 2.4.2, provided a method for the investigation of prime spectrum of a Bézout domain. In particular, we can show that a topological space $X$ having a unique minimal element is homeomorphic to the prime spectrum of a Bézout domain if and only if there exists a lattice-ordered group $G$ with $\text{Spec}(G)$ which is homeomorphic to $X$.

In general, every finitely presented ideal of a ring is finitely generated, however, the converse is not always true.

**Definition 4.1.2** A coherent ring is a ring in which every finitely generated ideal is finitely presented.

Now we can mention the example given by Fischer in [8], showing that there exists a coherent ring such that its prime spectrum contains a closed subspace $Y$ with $\text{Min}(Y)$ being non-compact. In this example, the method of constructing a reduced coherent ring is demonstrated.

**Example 4.1** Let $G$ be the group of sequence $(a_n)$ of integers that are eventually in arithmetic progression, and let $G^+$ be the set of sequences in $G$ such that all of whose terms are non-negative. Let $(a_n), (b_n) \in G$ and define $c_n = \min(a_n, b_n)$. Then either $c_n = a_n$ for all large $n$ or $c_n = b_n$ for all large $n$. So, $(a_n) \wedge (b_n) = (c_n) \in G$ and $G$ is a lattice-ordered group.

Let $P_i = \{(a_n) \in G^+|a_i > 0 \}$, $P' = \{(a_n) \in G^+|a_n > 0 \text{ for large } n \}$ and $P'' =$
\{(a_n) \in G^+ | (a_n) \text{ is eventually strictly increasing} \}. In order to see that these sets are all the non-empty prime \( V \)-segments of \( G \), we let \( Q \in \text{Spec}(G) \setminus \{\emptyset\} \). Consider the following cases:

**Case (1).** There exists \((a_n) \in Q\) such that \(a_n = 0\) for all sufficiently large \(n\). Then \(Q \subseteq P_k\) for some integer \(k\) since \(Q\) is closed under taking infimums. Also, \((a_n) \in Q \subseteq P_k\) implies \(a_n = b_n + c_n\), where \(b_n = a_n\) if \(n \neq k\) and \(b_n = 0\) if \(n = k\); and \(c_n = a_n - b_n\). But then we have \((c_n) \in Q\). This implies that \((d_n) \in Q\) where \(d_n = 1\) if \(n = k\) and \(d_n = 0\) if \(n \neq k\). It follows that \(Q = P_k\).

**Case (2).** There does not exist \((a_n) \in Q\) such that \(a_n = 0\) for all sufficiently large \(n\). Let \(m\) be the least integer \(d\) such that there exists \((a_n) \in Q\) with \(a_{n+1} - a_n = d\) for all sufficiently large \(n\). Since \(Q\) is prime, \(m\) is either 0 or 1. If \(m = 0\), then the constant sequence (1) is obviously in \(Q\). This implies that \(Q = P'\). If \(m = 1\), then there exists \((a_n) \in Q\) such that \(a_{n+1} = a_n + 1\), whence \(Q = P''\). Therefore, \(\text{Spec}(G) = \{\emptyset, P', P'', P_1, P_2, P_3, \ldots\}\). As a poset, \(\text{Spec}(G)\) can be expressed by the following diagram:

Let \(m\) be a fixed positive integer. Define \(a_n = n\) if \(n \neq m\) and \(a_n = 0\) if \(n = m\).
Let $b_n = 1$ and $c_n = n$ for all $n$. Then, $D((a_n)) = \{0, P_m\}$, $D((b_n)) = \{0, P''\}$, and $D((c_n)) = \{0\}$. Notice that $\text{Min}(V((c_n))) = \{P'', P_1, P_2, P_3, \ldots\}$ is an infinite discrete space in $\text{Spec}(G) \setminus \{0\}$. By theorem 2.4.2 and proposition 4.1.1, there exists a Bézout domain $R$ with $G$ as its group of divisibility. Since $\{0\}$ is open in $\text{Spec}(G)$, there exists $r \in R$ such that $V((c_n))$ and $\text{Spec}(R/(r))$ are homeomorphic. As a Bézout domain is a reduced coherent ring, $R$ is thus a reduced coherent ring with $\text{Min}(R/(r))$ not being quasi-compact.

Let $X$ be a spectral tree. We define $X^*$ to be the spectral tree obtained by adjoining a unique minimal element to $X$. That is, $X^* = X \cup \{\omega\}$ where $\omega \notin X$, and $X^*$ is topologized by declaring a non-empty set $A \subseteq X^*$ is open if and only if $\omega \in A$ and $A \setminus \{\omega\}$ is open in $X$.

Let $X$ be a spectral space and $A(X)$ the Boolean algebra of sets generated by the quasi-compact open subsets of $X$. Members of $A(X)$ are called constructible sets. Notice that the constructible sets are precisely those sets that are both open and closed in the patch topology. Since every patch is itself a spectral space, every chain in a patch has an infimum. In other words, every patch (in particular, every constructible set) has minimal elements. This fact was used repeatedly without explicit reference, in the proof of the following theorem (See [8, Theorem 2.1]).

**Theorem 4.1.3** Let $X$ be a spectral tree and assume that the closure of every constructible set is constructible. Then $X$ is homeomorphic to the prime spectrum of a Bézout ring.

**Sketch of proof.** Let $G$ be the group of integer valued functions on $X$ that are continuous in the patch topology. For each $f \in G$, let $M(f)$ denote the minimal
support of $f$.

Let $G^+ = \{ f \in G \mid f(x) > 0 \text{ for all } x \in M(f) \}$. Our goal is to show that $G$ is a lattice-ordered group, and that $\text{Spec}(G)$ is homeomorphic to $X^*$.

We can check by definition that $G$ is an ordered group. To see that $G$ is a lattice-ordered group, let $f, g \in G$; we construct their infimum as follows:

Let $x \in X$. If $f(y) = g(y)$ for all $y \leq x$, define $h(x) = f(x)$. Otherwise, let $y$ be the unique minimal element of the patch $G(x) \cap \text{Supp}(f - g)$ where $\text{Supp}(f - g)$ is the support of $f - g$; if $f(y) < g(y)$ then define $h(x) = f(x)$, and if $f(y) > g(y)$, then define $h(x) = g(x)$. To show that $h \in G$, let $A = \{ x \in X\mid (g - f)(x) > 0 \} \setminus \{ x \in X\mid (g - f) < 0 \}$. As the closure of a constructible set is constructible, we have $\overline{A}$ is constructible by the boundedness of $g - f$. In order to show that $h \in G$, it remains to verify the equality $h^{-1}(n) = (f^{-1}(n) \cap \overline{A}) \cup (g^{-1}(n) \setminus \overline{A})$ holds for every integer $n$. The part that $h = f \wedge g$ can be checked by definition.

For each $x \in X$, let $Q_x = \{ f \in G^+ \mid f(x_0) > 0 \text{ for some } x_0 \leq x \}$. It can be checked that $Q_x \in \text{Spec}(G)$.

Now, define $\phi : X^* \rightarrow \text{Spec}(G)$ by $\phi(\omega) = 0$ and $\phi(x) = Q_x$ for all $x \in X$. We wish to show that $\phi$ is a homeomorphism.

Let $f \in G^+$ and consider $\phi^{-1}(D(f))$. Let $\alpha = \text{Supp}(f)$. Since $\alpha \in A(X)$, $\alpha$ is a patch. Suppose $y \in \phi^{-1}(D(f))$. Then $y_0 \leq y$ implies $y_0 \notin \alpha$, so $y \in X \setminus \overline{\alpha} \subseteq X \setminus \alpha$. Also, if $x \notin \overline{\alpha}$, then $f(x_0) = 0$ for all $x_0 \leq x$, so $x \in \phi^{-1}(D(f))$. Therefore, $y \in X \setminus \overline{\alpha} \subseteq \phi^{-1}(D(f))$, and $\phi^{-1}(D(f))$ is open. Hence $\phi$ is continuous.

Let $U \subseteq X$ be any open set and suppose $Q_x \in \phi(U)$. Let $\alpha$ be a quasi-compact open set with $x \in \alpha \subseteq U$. Then $Q_x \in D(1 - \chi_\alpha) \cap \phi(X) \subseteq \phi(U)$. This shows that $\phi$ is continuous.
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is an open map.

If \( x \neq y \), then may assume \( x \not\in y \), or \( y \not\in \{x\} \). Hence, there exists \( \alpha \in A(X) \) such that \( x \in \alpha \) but \( y \not\in \overline{\alpha} \). Thus, \( \chi_\alpha \), the characteristic function of \( \alpha \), is in \( Q_x \setminus Q_y \). Hence, \( \phi \) is injective.

We now show that \( \phi \) is surjective. Let \( Q \subseteq Spec(G) \setminus \{\emptyset\} \). We can show that \( Q \subseteq Q_{z_0} \) for some \( z_0 \in X \) by using the quasi-compactness of \( X \). Clearly, the set \( A = \{y \in X|Q \subseteq Q_y \subseteq Q_{z_0}\} \) is a chain in \( X \). This implies that \( y_0 = \inf(A) \) exists in \( A \). Let \( f \in Q \) and \( \alpha = \text{Supp}(f) \). Then \( f \in Q_y \) for all \( y \in A \) and so, \( A \subseteq \alpha \). Hence \( y_0 \in \alpha \) since \( A \) is a chain in the patch (which is a spectral space) \( \alpha \). This means that \( f \in Q_{y_0} \). Thus, \( Q \subseteq Q_{y_0} \). Let \( B = \{x \in X|Q_x \subseteq Q\} \). By the definition of \( y_0 \), we know that if \( x < y_0 \) then \( Q_x \subseteq Q \). So we have either \( B \neq \emptyset \) or \( y_0 \) is a minimal in \( X \). In either case, we can show that \( Q_{y_0} \subseteq Q \), whence \( Q = Q_{y_0} \). Therefore, \( \phi \) is surjective and hence a homeomorphism.

Finally, let \( R \) be the Bézout domain with \( Spec(G) \), then \( Spec(R) \) and \( X^* \) are homeomorphic. Since \( X \) is closed in \( X^* \), there exists an ideal \( I \) of \( R \) such that \( X \) is homeomorphic to \( Spec(R/I) \). Clearly, \( R/I \) is a Bézout ring. \( \square \)

**Remark** In fact, what we have shown is that if \( X \) is a spectral tree such that the closure of every constructible set is constructible, then \( X^* \) is homeomorphic to the prime spectrum of a Bézout domain.

Fischer also proved that if \( R \) is the Bézout ring given by the theorem 4.1.3, then we can have a topological characterization for the case that a maximal ideal \( M \) of \( R \) is principal.

**Proposition 4.1.4** Let \( X \) be a spectral tree such that the closure of every con-
structible set is constructible. Also, let $x$ be a maximal element of $X$. Then $Q_x$ is principal if and only if $x$ is isolated in the patch topology on $X$.

Proof.

$\Rightarrow$ Let $Q_x = \{h + g | g \in G^+\}$. Then $\{Q_x\} = V(h)$. This means that $Q_x$ is isolated in the patch topology on $\text{Spec}(G)$ and $x$ is isolated in the patch topology on $X$.

$\Leftarrow$ Let $\alpha = \{x\}$, where $\alpha$ is open in the patch topology. Then, $\alpha \in A(X)$ and so $h = \chi_\alpha \in Q_x$. Let $f \in Q_x \setminus \{h\}$. If $(f - h)(x) < 0$, then $f(x) < h(x) = 1$, so $f(x) \leq 0$. Since $f \in Q_x$, there exists $x_0 < x$ with $f(x_0) > 0$. Then $(f - h)(x_0) = f(x_0) > 0$. If $(f - h)(y) < 0$ and $y = x$, then $f(y) \leq 0$. Since $f \in Q_x$, there exists $y_0 < y \ (\text{and so } y_0 \neq x \text{ since } x \text{ is a maximal element of } X)$ such that $f(y_0) > 0$. Then $(f - h)(y_0) > 0$. Thus, $f - h \in G^+$. As $f = (f - h) + h$, we have $Q_x = \{h + g | g \in G^+\}$. \qed

Remark Let $\Gamma$ denote the class of spectral trees such that the closure of every constructible set is constructible. Fischer also discussed the following questions: (1) Whether the class $\Gamma$ contains every spectral tree? (2) Whether the class $\Gamma$ contains every spectral tree that is the prime spectrum of a Bézout ring? In fact, Fischer answered both questions negatively by exhibiting a space that is known to be the prime spectrum of a Bézout ring, but this space is not a member of $\Gamma$. The reader is referred to [8].

Let $X$ be a spectral tree. In order to show that $X \in \Gamma$, it suffices to check that $\overline{\alpha \setminus \beta} \in A(X)$ as members of $A(X)$ are finite unions of sets of the form $\alpha \setminus \beta$, where $\alpha$ and $\beta$ are quasi-compact open subsets of $X$. In fact, we only need to check that $\overline{\alpha \setminus \beta} \in A(X)$ as $\alpha$ and $\beta$ range over any QCI-base for $X$, by the lemma below.

Lemma 4.1.5 Let $X$ be a spectral tree. Let $\alpha, \beta_1$ and $\beta_2$ be quasi-compact open
subsets of $X$. Then, $[\alpha \setminus (\beta_1 \cup \beta_2)] = (\alpha \setminus \beta_1) \cap (\alpha \setminus \beta_2)$.

**Proof.** Clearly, $[\alpha \setminus (\beta_1 \cup \beta_2)] \subseteq (\alpha \setminus \beta_1) \cap (\alpha \setminus \beta_2)$. For the converse containment, let $x \in (\alpha \setminus \beta_1) \cap (\alpha \setminus \beta_2)$. By proposition 1.2.2, there exist $x_1 \in G(x) \cap (\alpha \setminus \beta_1)$ and $x_2 \in G(x) \cap (\alpha \setminus \beta_1)$. Without loss of generality, we may assume $x_2 \leq x_1$. Then we have $x_1 \in (\alpha \setminus (\beta_1 \cup \beta_2))$. Consequently, $x \in [\alpha \setminus (\beta_1 \cup \beta_2)]$. Thus, $(\alpha \setminus \beta_1) \cap (\alpha \setminus \beta_2) \subseteq [\alpha \setminus (\beta_1 \cup \beta_2)]$.

It should be noticed theorem 2.5.2 is a corollary of theorem 4.1.3 if we can show that a tree $X$ with properties (K1) and (K2) is a member of $\Gamma$ for the COP-topology (See [8]).

**Definition 4.1.6** A Hilbert ring (or Jacobson ring) is a ring such that each prime ideal is the intersection of all maximal ideals containing it.

A subset of a topological space $X$ is **locally closed** if it is the intersection of an open set and a closed set, or equivalently, if it is open in its closure. A subset $X_0$ of $X$ is said to be **very dense** in $X$ if it satisfies one of the following equivalent conditions:

1. Every non-empty locally closed subset of $X$ meets $X_0$.
2. For every closed set $E$ in $X$, we have $E \cap X_0 = E$.
3. The mapping $U \mapsto U \cap X_0$ of the collection of open sets of $X$ onto the collection of open sets of $X_0$ is bijective.

In fact, we have the following topological characterization of Hilbert ring (See [2, pages 71-72]).

**Proposition 4.1.7** The following conditions are equivalent for a ring $R$:
(a) $R$ is a Hilbert ring.

(b) $\text{Max}(R)$ is very dense in $\text{Spec}(R)$.

(c) Every locally closed subset of $\text{Spec}(R)$ consisting of a single point is closed.

As an application, Fischer used theorem 4.1.3 and proposition 4.1.4 to answer the following question raised by A. V. Geramita:

If $R$ is a Hilbert domain in which every maximal ideal is finitely generated, is $R$ Noetherian?

**Example 4.2** [W. J. Lewis] Let $X$ be the poset pictured below:

With the COP-topology, $X$ is a member of $\Gamma$. By theorem 4.1.3, we then obtain a two-dimensional (hence non-Noetherian) Bézout domain $R$ whose prime spectrum is homeomorphic to $X$. In particular, $R$ is a Hilbert domain, and by proposition 4.1.4, each maximal ideal of $R$ is principal. Therefore a Hilbert domain in which every maximal ideal is finitely generated is not necessarily Noetherian.

Along the related directions, Fischer also studied the maximal spectrum of
Bézout ring in [7]. He gave a condition, in addition to being quasi-compact and $T_1$, that is necessary for a space to be homeomorphic to the maximal spectrum of a Bézout ring. He also showed that these three conditions are also sufficient under certain assumptions.

Let $X$ be a topological space. For any integer $n$, the sequence $(x_1, \ldots, x_n) \in X^n$ is said to be sticky if $\cap_{i=1}^n N_i \neq \emptyset$ for all open neighbourhood $N_i$ of $x_i$ for all $i$. A quasi-compact $T_1$ space is called a $B$-space provided every closed subspace $Y$ has the following property: If $(y_{i-1}, y_i) \in Y^2$ is sticky for $2 \leq i \leq n$, then $(y_1, \ldots, y_n)$ is sticky.

The following proposition is due to Fischer [7].

**Proposition 4.1.8** The maximal spectrum of a Bézout ring is a $B$-space.

**Proof.** Let $R$ be a Bézout ring and $X = \text{Max}(R)$. Let $F$ be closed in $X$ and $(x_1, \ldots, x_n)$ be a sequence in $F$. By passing to the Bézout ring $R/(\cap F)$, we may assume $F = X$ and $X = \text{Spec}(R)$. Suppose $(x_{i-1}, x_i)$ is sticky for all $2 \leq i \leq n$. If $N_i$ is an arbitrary open neighbourhood of $x_i$, then $N_i = U_i \cap X$ where $U_i$ is open in $\text{Spec}(R)$. Now if $U = \cap_{i=1}^n U_i \neq \emptyset$, then since $X$ is dense in $\text{Spec}(R)$, it follows that $U \cap X \neq \emptyset$, as desired. Hence, we suffice to prove that $U \neq \emptyset$, however, it is equivalent to find a point $z \leq x_i$ for all $i$. To find such a point $z$, we may assume by induction that there is a point $x \leq x_i$ for $i = 1, \ldots, n-1$. Let $\mathcal{F}$ be the collection of all subsets in the form of $D(rs)$, where $r \in R \setminus x_{n-1}$, $s \in R \setminus x_n$. This collection has the FIP since $(x_{n-1}, x_n)$ is sticky. So, there is a point $y \in \cap \mathcal{F}$ by the quasi-compactness in the patch topology. Then $x, y \in G(x_{n-1})$ implies $x, y$ are comparable. Since $y \leq x_n$, we have $z = \min(x, y)$, which is the desired one. \hfill $\square$

Let $R$ be a ring. Define the $j$-spectrum $j\text{spec}(R)$ of $R$ to be the subset of $\text{Spec}(R)$
consisting of those prime ideals which are the intersections of maximal ideals. Hence, a ring \( R \) is a Hilbert ring if and only if \( jspec(R) = Spec(R) \). We give \( jspec(R) \) the subspace topology which is inherited from the Zariski topology on \( Spec(R) \). R. G. Swan [32] observed the following two facts about the \( j \)-spectrum of a ring.

**Proposition 4.1.9** The lattice of closed sets of \( jspec(R) \) is isomorphic to the lattice of closed sets of \( Max(R) \).

**Proof.** Let \( J = jspec(R) \) and \( M = Max(R) \). Then \( M \subseteq J \). If \( A \) is a closed set of \( M \), then let \( \overline{A} \) be its closure in \( J \). If \( B \) is a closed set of \( J \), let \( B^* = B \cap M \), which is a closed set of \( M \). Clearly, these operations preserve the order. We claim that they are actually the inverse operations. If \( A \) is closed in \( M \), then obviously \( \overline{\overline{A}} = \overline{A} \cap M = A \). Let \( B \) be closed in \( J \). Then, we have \( B = V(I) \cap J \) with \( I = \cap B \).

Let \( A = B^* = B \cap M \). Since all \( P \in B \) are the intersections of maximal ideals, \( I \) is also an intersection of maximal ideals and so \( I = \cap A \). Therefore, if \( A \) is given, then we can recover \( I \) and thus \( B \). In other words, \( B \) is completely determined by \( A \). Since \( B \cap M = A = \overline{A} \cap M \), it follows that \( B = \overline{A} = B^* \). \( \square \)

**Proposition 4.1.10** Every closed irreducible subset of \( jspec(R) \) contains a unique generic point.

**Proof.** Let \( F \) be a closed irreducible subset of \( jspec(R) \). For the same reason as in the proof of proposition 4.1.9, we have \( F = V(I) \cap jspec(R) \) with \( I = \cap F \).

If \( I \) is not a prime ideal, then there exist ideals \( A, B \) of \( R \) such that \( A, B \not\subseteq I \) but \( AB \subseteq I \). Since \( AB \subseteq P \) implies \( A \subseteq P \) or \( B \subseteq P \) for each \( P \in Spec(R) \), \( F \) is then the union of the proper closed subsets \( V(A) \cap jspec(R) \) and \( V(B) \cap jspec(R) \). This fact contradicts to \( F \) being irreducible, whence \( I \) is a prime ideal in \( jspec(R) \).

Since the closed set \( V(C) \cap jspec(R) \) contains \( I \) if and only if \( C \subseteq I \), the closure of
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$I$ in $j\text{spec}(R)$, denoted by $\{I\}$, is $V(I) \cap j\text{spec}(R)$. This shows that $F = \{I\}$. Also $Q \in \{I\}$ if and only if $Q \supseteq I$, so $\{I\} = \{Q\}$ if and only if $I = Q$. Thus $I$ is unique.

We call a topological space $j$-spectral if it is homeomorphic to the $j$-spectrum of some ring. Then the following corollary is a direct consequence.

**Corollary 4.1.11** A $j$-spectral space is spectral if and only if it has a QCI-base.

If $X$ is a Noetherian space, Fischer [7] showed that the converse of theorem 4.1.8 is also true.

**Proposition 4.1.12** Let $X$ be a Noetherian topological space. Then $X$ is homeomorphic to the maximal spectrum of a Bézout domain if and only if $X$ is a B-space.

**Proof.** Let $X$ be a B-space. Then, since $X$ is quasi-compact and $T_1$, $X$ is homeomorphic to the maximal spectrum of some ring $R$. Let $Y$ be the $j$-spectrum of $R$. By proposition 4.1.9, the lattices of closed subsets of $X$ and $Y$ are isomorphic. An immediate consequence is that $Y$ is Noetherian. Then any open base of $Y$ is a QCI-base. Hence, $Y$ is spectral by corollary 4.1.11.

Suppose $x_1, x_2$ are incomparable elements of $Y$ with $x \in \{x_1\} \cap \{x_2\} \cap X$. Let $U_1 = Y \setminus \{x_2\}$, and $U_2 = Y \setminus \{x_1\}$, and let $y_i \in U_i \cap X$ with $x_i < y_i$. Note that $(y_1, x)$ and $(y_2, x)$ are sticky in $Z = (\{x_1\} \cup \{x_2\}) \cap X$. Since $X$ is a B-space, $(y_1, y_2, x)$ is also sticky in $Z$. However, $U_1 \cap U_2 \cap Z = \emptyset$, which is a contradiction. Thus, $Y$ is a tree.

Because $Y$ is a spectral tree, $Y^*$ is also a spectral tree. Moreover, $X = \text{Max}(Y^*)$. Since $Y^*$ is Noetherian, the closure of every constructible set in $Y^*$ is still constructible. Invoking theorem 4.1.3 to $Y^*$, $X$ is then homeomorphic to the maximal
spectrum of a Bézout domain. The converse part is just the result of proposition 4.1.8.

Let $X$ be a topological space, then the (combinatorial) dimension $\dim(X)$ of $X$ is defined to be the supremum of integers $n$ such that there is a sequence $C_0 \subset \ldots \subset C_n$ of closed irreducible sets in $X$. A compact space $X$ is called a Boolean space if it is totally disconnected. Equivalently, $X$ is Boolean if and only if it is compact and has a clopen base.

**Lemma 4.1.13** A topological space $X$ is a Boolean space if and only if $X$ is a zero-dimensional spectral space.

**Proof.** This result follows immediately from the definition of Boolean space and the fact that a zero-dimensional spectral space has an open base consisting of clopen sets. \qed

The following proposition is a characterization of the maximal spectrum of one-dimensional Bézout domains obtained by Fischer [7].

**Proposition 4.1.14** A topological space $X$ is homeomorphic to the maximal spectrum of a one-dimensional Bézout domain if and only if either (1) $X$ is Boolean, or (2) $X$ is a one-dimensional irreducible quasi-compact $T_1$ space having a QCI-base.

**Proof.**

\( \Rightarrow \) Let $R$ be a one-dimensional Bézout domain and let $X = \text{Max}(R)$. If $X$ is closed in $\text{Spec}(R)$, then $X$ is a zero-dimensional spectral space, hence $X$ is Boolean. On the other hand, suppose $X$ is not closed in $\text{Spec}(R)$. Then $\text{jspec}(R) = \text{Spec}(R)$, so the lattices of closed subsets of $\text{Spec}(R)$ and $X$ are isomorphic by proposition 4.1.9. In particular, $D(r) \cap X$ is quasi-compact for each $r \in R$. Thus,
the family \( \{D(r) \cap X \mid r \in R\} \) is a QCI-base for \( X \). Also, since \( \text{Spec}(R) \) is one-dimensional and irreducible, the same is true for \( X \). Since \( X \) is clearly \( T_1 \) and quasi-compact, the condition (2) holds.

\( \Leftarrow \) Let \( X \) be a Boolean space. Then \( X \) is a zero-dimensional spectral space by lemma 4.1.13, so \( X \in \Gamma \). Clearly, \( X = \text{Max}(X^*) \) and \( X^* \) is homeomorphic to the prime spectrum of a one-dimensional Bézout domain by the remark of theorem 4.1.3.

Let \( X \) be a one-dimensional irreducible quasi-compact \( T_1 \) space having a QCI-base. Let \( Y = X \cup \{\{\omega\} \mid \omega \notin X\} \) where \( \omega \) is a symbol not contained in \( X \). Topologize \( Y \) by declaring a non-empty subset \( U \subseteq Y \) to be open if and only if \( \omega \in U \) and \( U \setminus \{\omega\} \) is open in \( X \). Since the space \( X \) is irreducible, there exists a QCI-base \( B' \) for \( X \) not containing \( \emptyset \). Let \( B = \{V \cup \{\omega\} \mid V \in B'\} \), then \( B \) is a QCI-base for \( Y \). Also, it is easy to see that \( Y \) is \( T_0 \) and quasi-compact. Let \( K \) be a non-empty closed irreducible subset of \( Y \). If \( \omega \in K \), then \( K = \{\omega\} \). If \( \omega \notin K \), then \( \{\omega\} \subseteq K \subseteq X \) for some \( x \in X \). Since \( X \) is not closed in \( Y \), \( K \neq X \), whence \( K = \{x\} \) by \( \text{dim}(X) = 1 \). Therefore \( Y \) is a spectral tree and \( X = \text{Max}(Y) \).

Let \( \alpha, \beta \in B \) and consider \( \overline{\alpha \setminus \beta} \). If \( \omega \in \alpha \setminus \beta \), then \( \overline{\alpha \setminus \beta} = Y \in A(Y) \). If \( \omega \notin \alpha \setminus \beta \), then \( \overline{\alpha \setminus \beta} = \overline{\alpha} \setminus \overline{\beta} \in A(Y) \), by proposition 1.2.2. Therefore \( Y \in \Gamma \), and \( X \) is homeomorphic to the maximal spectrum of a one-dimensional Bézout domain by theorem 4.1.3. \( \square \)

**Remark** In fact, proposition 4.1.14 characterizes the maximal spectrum of any one-dimensional integral domain. In [7], Fischer also gave a partial result about the maximal spectrum of Bézout domain of higher dimensions.

Fischer pointed out in [7] that the only compact spaces which he showed to be homeomorphic to the maximal spectrums of Bézout rings are totally disconnected.
He argued that the possibility of a compact space to be homeomorphic with the maximal spectrum of a Bézout ring should be independent of the connectedness of it, by mentioning an example of compact connected space $X$ (namely $X = \beta \mathbb{R}^+ \setminus \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers) such that the ring of continuous real valued functions on $X$ is a Bézout ring, given by Gillman and Henriksen in [10]. Fischer then suggested the question that whether every compact space is homeomorphic to the maximal spectrum of some Bézout ring, though he suspected this is not the case. As a test case, he proposed the following question:

**Question** Is there a Bézout ring whose maximal spectrum is homeomorphic to the unit interval?

Owing to the fact that every compact space $X$ is homeomorphic to the maximal spectrum of the ring of real-valued continuous functions $C(X)$ (See [11, Chapter 4] or [2, pages 14-15]), it is natural to ask whether the ring $C([0, 1])$ is Bézout or not. The answer is negative due to the following result [11, pages 208-210].

**Theorem 4.1.15** For any space $X$, the following are equivalent.

1. The prime ideals contained in any given maximal ideal form a chain.
2. Given any $f \in C(X)$, there exists a constant function $k \in C(X)$ such that $f = k|f|$.
3. For all $f, g \in C(X)$, $(f, g) = (|f| + |g|)$.
4. $X$ is an F-space, i.e. every finitely generated ideal in $C(X)$ is principal.

Clearly, the continuous function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x - 0.5$ does not satisfy the condition (2) in theorem 4.1.15, therefore $[0, 1]$ is not an F-space. Along the same line, one may wonder if $C[0, 1]$ is an Arithmetical ring, though it
is not Bézout. However, the condition (1) in theorem 4.1.15 does tell us that this is not the case.

Proposition 4.1.16 For any space $X$, $C(X)$ is Bézout if and only if it is Arithmetical.

Proof. If $C(X)$ is Arithmetical, then condition (1) in theorem 4.1.15 holds, by theorem 2.3.5. Hence $X$ is an F-space, or $C(X)$ is Bézout. The converse is well-known. □

4.2 D-closed subsets of Spec($R$)

Let $n$ be a nonnegative integer. Call a ring $R$ a $A(n)$ ring if given any exact sequence $0 \rightarrow M \rightarrow E_1 \rightarrow \ldots \rightarrow E_n$ of finitely generated $R$-modules with $M$ flat and $E_i$ free for each $i$, then $M$ is a projective $R$-module. In particular, a ring $R$ is $A(0)$ if all finitely generated flat $R$-modules are projective. D. Lazard in [25] gave a characterization of $A(0)$ rings in terms of the topology of the prime spectrum. We will describe here Lazard’s work, together with an example of a poset having two spectral topologies, one of which yields an $A(0)$ ring and the other does not, from [27]. This shows that the partial ordering of $Spec(R)$ is not sufficient by itself to determine whether or not $R$ is $A(0)$.

By the D-component of an element $x$ in a poset $X$, we mean the intersection of all sets containing $x$ that are closed under generization and specialization. Thus $y$ is in the D-component of $x$ if and only if there exist elements $x_1, \ldots, x_n \in X$ such that $x \leq x_1 \geq x_2 \leq x_3 \geq \ldots \leq x_n \geq y$. Moreover, if $T$ is an order compatible topology for $X$, a subset of $X$ is defined to be D-closed if it is $T$-closed and is a union of D-components. Hence a subset of $X$ is D-closed if and only if it is $T$-closed.
and closed under generization.

The following is a standard result about finitely generated projective $R$-module for a ring $R$ (For instance, see [14, page 56]):

**Lemma 4.2.1** Let $R$ be a ring, a finitely generated flat $R$-module $M$ is projective if and only if the mapping $f : \text{Spec}(R) \to \mathbb{Z}$ defined by $f(P) = \text{rank}_{R_P}(M_P)$ is continuous.

**Lemma 4.2.2** Let $R$ be a ring and $M$ be a finitely generated flat $R$-module. Then $F_n = \{P \in \text{Spec}(R) | \text{rank}_{R_P}(M_P) \geq n\}$ is a $D$-closed subset of $\text{Spec}(R)$ for any integer $n$.

**Proof.** In fact, $F_n = \text{Supp}(\bigwedge^n M)$ and $\bigwedge^n M$ is a finitely generated flat $R$-module. As the support of a finitely generated $R$-module, $F_n$ is closed in $\text{Spec}(R)$. Let $P \in F_n$ and $Q \subseteq P$. Then $Q \cap (R \setminus P) = \emptyset$ implies $M_Q \simeq (M_P)_{QR_P}$ as $R_Q$-module, hence $\text{rank}_{R_Q}(M_Q) = \text{rank}_{R_Q}((M_P)_{QR_P}) = \text{rank}_{R_P}(M_P) \geq n$, it follows that $Q \in F_n$. Therefore $F_n$ is $D$-closed. \qed

**Proposition 4.2.3** Let $R$ be a ring and $E$ a $D$-closed subset of $\text{Spec}(R)$. Then $E$ is the support of a finitely generated flat $R$-module.

**Proof.** Let $E = V(A)$ be a $D$-closed subset in $\text{Spec}(R)$. Put $S = 1 + A$, a multiplicatively closed subset. Let $\psi : R \to R_S$ be the canonical map, and $B = \ker(\psi) = \{a \in R|sa = 0 \text{ for some } s \in 1 + A\}$. It is easy to see that $E = V(A) = V(B)$. Now if $P \notin V(B)$, then $B_P = R_P$ and $(R/B)_P = 0$. On the other hand, if $P \in V(B)$, then $P \in V(A)$ and $S \cap P = \emptyset$, whence $B_P = 0$ and $(R/B)_P = R_P$. Therefore, $E$ is the support of the finitely generated (cyclic, in fact) flat $R$-module $(R/B)$. \qed
The following theorem is a characterization of A(0) rings [25].

**Theorem 4.2.4** A ring \( R \) is A(0) if and only if the D-closed subsets of Spec\((R)\) are open.

**Proof.** Suppose \( R \) is a A(0) ring and \( E \) is a D-closed subset of Spec\((R)\). By proposition 4.2.3, \( E = \text{Supp}(M) \) for some finitely generated flat \( R \)-module \( M \). Now \( R \) is A(0) implies the \( R \)-module \( M \) is projective, whence \( E = \bigcup_{n \geq 1} f^{-1}(n) \) is open, by lemma 4.2.1. Conversely, assume that D-closed subsets of Spec\((R)\) are open. Let \( M \) be a finitely generated flat \( R \)-module, then for any integer \( n \), \( F_n = \{ P \in \text{Spec}\,(R) \mid \text{rank}_{R_P}(M_P) \geq n \} \) is D-closed by lemma 4.2.2. Hence \( F_n \) is open implies \( f^{-1}(n) = \{ P \in \text{Spec}\,(R) \mid \text{rank}_{R_P}(M_P) = n \} = F_n \setminus F_{n+1} \) is open. It follows that \( f(P) = \text{rank}_{R_P}(M_P) \) is continuous, \( M \) is then projective by lemma 4.2.1. \( \square \)

**Remark** From the proofs of proposition 4.2.3 and theorem 4.2.4, we can see that \( R \) is A(0) if and only if every cyclic flat \( R \)-module is projective. For other characterizations of A(0) rings, the reader is referred to [34, Theorem 2.1].

Since the D-components partition Spec\((R)\), it follows that \( R \) is A(0) whenever Spec\((R)\) has only finitely many D-components each of which is closed. In particular, if \( R \) has only finitely many minimal prime ideals, then \( R \) is A(0). It is also true, but for a different reason, that if \( R \) has only finitely many maximal ideals, then \( R \) is A(0) (See [19]). In either case, it suffices to glance merely at the ordering of Spec\((R)\) in order to conclude that \( R \) is A(0). The following example, however, shows that the A(0) property cannot be characterized by the ordering of Spec\((R)\) alone (See [27, page 824]).

**Example 4.3** Let \( X = \{ m_i | i = 0, 1, 2, \ldots, \infty \} \cup \{ p_j | j = 1, 2, \ldots, \infty \} \), and order \( X \) by defining \( p_j \leq m_{j-1}, m_j \) if \( j < \infty \) and \( p_\infty \leq m_\infty \) (See the figure below).
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$X$ has exactly two $D$-components, namely $D_\infty = \{m_\infty, p_\infty\}$ and $X \setminus D_\infty$.

For any $m \in X$, let $C(m)$ be the topology for $X$ having closed sets

1. finite sets closed under specialization; and

2. sets containing $m$ and closed under specialization.

It is immediate that $C(m)$ is a $T_0$, order compatible and quasi-compact topology for which closed irreducible sets have generic points. Moreover, the cofinite (i.e. finite complement) sets containing $m$ and closed under generization, together with the finite sets not containing $m$ and closed under generization, form a base of quasi-compact open sets which is closed under finite intersections. Thus, $(X, C(m))$ is a spectral space.

The $D$-component $D_\infty$ is closed in the $C(m)$ topology for any choice of $m$, and hence $(X, C(m))$ has the property that $D$-closed sets are open if and only if $D_\infty$ is open. But $D_\infty$ is open if and only if $m \in D_0$. Thus, by choosing the rings $R_1, R_2$ with $\text{Spec}(R_1), \text{Spec}(R_2)$ homeomorphic to $(X, C(m_0))$ and $(X, C(m_\infty))$ respectively, we get a ring $R_1$ which has $A(0)$ property and a ring $R_2$ which does not, yet the prime spectrum of both rings are also order isomorphic to $X$. 
4.3 The $C(m)$ topology

In this section, we describe how W. J. Lewis and J. Ohm [27] generalized the topology given in example 4.3 to an arbitrary poset, and how they used it to generalize the result that every finite poset is spectral obtained by Lewis himself.

Given a poset $X$, choose an element $m \in X$. Define a topology, called the $C(m)$ topology, by choosing the following collection of sets as a base for the closed sets of the topology:

(1) finite sets not containing $m$ and closed under specialization (including $\emptyset$), and

(2) cofinite sets containing $m$ and closed under specialization (including $X$).

Lemma 4.3.1 Let $X$ be a poset and let $m \in X$. Then the $C(m)$ topology is compatible with the order of $X$ if and only if the following conditions hold:

(a) if $x \in X$ and $S(x)$ is infinite, then $x \leq m$; and

(b) if $x \in X$ and $G(x)$ is infinite, then $x \geq m$.

Proof. For all $x \nleq m$, $\{x\} = S(x)$ if and only if condition (a) holds. Similarly, for all $x \leq m$, $\{x\} = S(x)$ if and only if condition (b) holds. \qed

Theorem 4.3.2 Let $X$ be a poset with the $C(m)$ topology for some $m \in X$. If the topology is compatible with the ordering of $X$, then $(X, C(m))$ is a spectral space.

Proof. Since the $C(m)$ topology is compatible with the order, $X$ is $T_0$. $X$ is quasi-compact since any open set containing $m$ is cofinite. Corresponding to the closed base for the topology, we have the following open base:

(1) cofinite sets containing $m$ and closed under generization; and
(2) finite sets not containing $m$ and closed under generization.

Clearly these sets are quasi-compact, and they are closed under finite intersections. Now let $F$ be a closed irreducible subset of $X$. If $x \geq m$ for all $x \in F$, then either $F = \{m\}$, or $F$ is finite and has a generic point. We may assume here that there exists an $x \in F$ such that $x \not\geq m$. Since $G(x)$ is finite, we may choose $x$ to be a minimal element of $F$. Then $\{x\}$ and $X \setminus G(x)$ are closed sets whose union contains $F$. Since $x \in F$ and $F$ is irreducible, we get $F = \{x\}$. \qed

As a special case of theorem 4.3.2, W. Lewis and J. Ohm obtained the following generalization of the fact that any finite poset is spectral.

**Corollary 4.3.3** If $X$ is a poset with the property that $S(x) \cup G(x)$ is finite for all $x \in X$, then $X$ is spectral.

### 4.4 Prime spectrum of Noetherian ring

As we have seen in chapter 2, theorem 1.3.1 provides a criterion for the determination of a given poset to be spectral or not. However, this depends on whether we can find an order-compatible spectral topology on the given poset. The problem is that there may be a number of spectral topologies that induce the same partial ordering. (Recall that the partial ordering induced by a $T_0$ topology is: $x \leq y$ if and only if $y \in \overline{\{x\}}$.) For instance, consider the set $X = \{x, y_1, y_2, \ldots\}$ endowed with the ordering defined by specifying $x < y_n$ for all $n \geq 1$. One spectral topology on can be obtained by taking the closed sets to be the finite subsets of $Y = \{y_1, y_2, \ldots\}$. It can be easily seen that it is just the Zariski topology on $Spec(\mathbb{Z})$ of the ring $\mathbb{Z}$. But one can enrich the topology without changing the ordering, by declaring any subset of $Y$ containing $y_1$ to be closed, for instance.
However, the distinction between topology and order evaporates for Noetherian rings. To see this, we note that if $R$ is Noetherian ring then $\text{Spec}(R)$ is a Noetherian space, i.e. it has the descending chain condition on closed sets. It follows that every closed set is a finite union of irreducible closed sets; that is, the closed sets are the finite unions of sets of the form $\{p\} = \{q \in \text{Spec}(R) | q \geq p\}$. Thus the partial ordering on the prime spectrum determines the topology.

In fact, we can see from the discussion above that the equivalence between the topology and order occurs not only for Noetherian rings, but for all rings which have Noetherian prime spectrums. J. Ohm and R. L. Pendleton [29] showed that the class of all rings which possess Noetherian prime spectrum corresponds to those have property (RFG) defined below.

An ideal $A$ of a ring $R$ is said to be an RFG-ideal if the radical of $A$ is the radical of a finitely generated ideal, i.e. there exist $a_1, a_2, \ldots, a_n \in R$ such that $r(A) = r((a_1, \ldots, a_n))$. Clearly we may choose $a_i$ to be in $A$. We say that a ring $R$ has property (RFG) if every ideal of $R$ is an RFG-ideal.

Proposition 4.4.1 The prime spectrum of a ring $R$ is Noetherian if and only if $R$ has property (RFG).

Proof. Let $A$ be an ideal of $R$. Then $r(A) = r((a_1, \ldots, a_n))$ for some $a_i \in R$, it means $V(A) = \cap_{i=1}^n V(a_i)$, or $D(A) = \cup_{i=1}^n D(a_i)$. Since $\{D(a) | a \in R\}$ is an open base consisting of quasi-compact sets in $\text{Spec}(R)$, it follows that $R$ has property (RFG) is equivalent to every open subset of $\text{Spec}(R)$ being quasi-compact. Hence the (RFG) property of $R$ is equivalent to $\text{Spec}(R)$ being Noetherian by proposition 3.2.11. 

The following proposition concludes some properties about RFG-ideals.
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Proposition 4.4.2 Let $A, B$ be ideals of a ring $R$ and $f \in R \setminus \{0\}$. Then the following statements hold:

(a) If $A$ and $B$ are RFG-ideals, then so are $AB$ and $A \cap B$.

(b) If $A + (f)$ and $AR_f$ are RFG-ideals, then so is $A$.

(c) If $A \subseteq B$ and $B$ is an RFG-ideal, then so is $B/A$.

(d) If $A \subseteq B$ and $A, B/A$ are RFG-ideals, then so is $B$.

Proof.

(a) This is immediate from the equality $r(AB) = r(A \cap B) = r(A)r(B)$.

(b) Since $A + (f)$ and $AR_f$ are RFG-ideals, we have $r(A + (f)) = r((a_1 + r_1 f, \ldots, a_n + r_n f))$ and $r(AR_f) = r((b_1/f^k, \ldots, b_m/f^k))$ for some $a_i, b_j \in A$ and $r_i \in R$. Then, $r(A) = r((a_1, \ldots, a_n, b_1, \ldots, b_m))$ for if $P \in V((a_1, \ldots, a_n, b_1, \ldots, b_m))$, then either $f \in P$, in which case $P \supseteq A + (f) \supseteq A$, or $f \notin P$, in this case $PR_f \supseteq AR_f$. So, in both cases, $P \supseteq A$. Therefore, $A$ is an RFG-ideal.

(c) Suppose $r(B) = r((b_1, \ldots, b_m))$ for some $b_j \in B$. Then $r(B/A) = r((\overline{b_1}, \ldots, \overline{b_m}))$, where $\overline{b_j} = b_j + A$ is the residue class of $b_j$ in $R/A$.

(d) Suppose $r(A) = r((a_1, \ldots, a_n))$ and $r(B/A) = r((\overline{b_1}, \ldots, \overline{b_m}))$ for some $a_i \in A$ and $b_j \in B$, then $r(B) = r((a_1, \ldots, a_n, b_1, \ldots, b_m))$, for if $P \in V((a_1, \ldots, a_n, b_1, \ldots, b_m))$, then $P \supseteq (\overline{b_1}, \ldots, \overline{b_m})$ which leads to $P \supseteq B$. \qed

Proposition 4.4.3 Let $\mathcal{F}$ be a family of non-RFG-ideals in a ring $R$. If $\mathcal{F} \neq \emptyset$, then $\mathcal{F}$ contains some maximal elements, and any such maximal element is prime.

By $R_f$, we mean the localization of $R$ with respect to the multiplicatively closed subset $S = \{f^n | n \in \mathbb{N}\} \cup \{1\}$. 
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Proof. Let \( C \) be a chain of ideals in \( \mathcal{F} \), then \( C = \bigcup C \) is a non-RFG-ideal in \( R \), for otherwise \( r(C) = r(c_1, \ldots, c_l) \) for some \( c_i \in C \) implies that there exist \( C_t \in C \) such that \( c_t \in C_t \) for \( t = 1, 2, \ldots, l \), whence \( C' = \bigcup_{t=1}^l C_t \in C \) and \( r(C') = r(c_1, \ldots, c_l) \). This is clearly a contradiction. Thus, \( C \) is an upper bound of \( C \) in \( \mathcal{F} \). By Zorn's lemma, there exists a maximal element \( A \) in \( \mathcal{F} \). If \( A \) is not prime, then there exist ideals \( B, C \) with \( A \subset B, A \subset C \) and \( BC \subset A \subset B \cap C \). Since \( A \) is maximal in \( \mathcal{F} \), \( B, C \) are RFG-ideals, so \( BC \) is an RFG-ideal by proposition 4.4.2(1). But then \( r(BC) \subset r(A) \subset r(B) \cap r(C) = r(BC) \). This fact implies that \( A \) is an RFG-ideal, a contradiction. \( \square \)

The following corollary is the radical ideal analogue of Cohen’s theorem.

Corollary 4.4.4 The prime spectrum of a ring \( R \) is Noetherian if and only if every prime ideal of \( R \) is an RFG-ideal.

Proof. It follows immediately from proposition 4.4.1 and proposition 4.4.3. \( \square \)

At the end of this section, we mention an example of a non-Noetherian ring having Noetherian prime spectrum.

Example 4.4 Let \( R \) be the set of all sequences \( (a_n) \ (n \geq 1) \) of elements of the field \( \mathbb{Z}_2 \), such that for some \( m_0 \), depending on the sequence, \( a_m = a_{m_0} \) for all \( m \geq m_0 \).

If we define operations on \( R \) by

\[
(a_n) + (b_n) = (a_n + b_n) \quad \text{and} \quad (a_n)(b_n) = (a_nb_n),
\]

then \( R \) is a commutative ring with unity. For each \( i > 0 \), let \( P_i = \{(a_n) \in R | a_i = 0 \} \).

Let \( P_0 = \{(a_n) \in R | \text{For some } m_0, \text{depending on the sequence, } a_m = 0 \text{ for } m \geq m_0 \} \).

Then we can show that \( R \) is not Noetherian and \( \text{Spec}(R) = \{P_i | i \geq 0 \} \) (See [24, page 57]). Since any closed subset of \( \text{Spec}(R) \) other than \( \text{Spec}(R) \) itself is finite, \( \text{Spec}(R) \) is thus Noetherian.
Let $R$ be the ring constructed in example 4.4, it is easy to see that $\text{Spec}(R) \approx \text{Spec}(\mathbb{Z})$, hence the property of being a Noetherian ring cannot have a topological characterization on the prime spectrum.

4.5 Reduced Bézout rings that are coherent

It is known that a Bézout domain is always a coherent ring, it is natural to ask whether the statement can be generalized to reduced Bézout ring.

Question. Is a reduced Bézout ring always be coherent?

The answer to this question is negative, due to example 4.1. In the example, Fischer did obtain a quotient ring $S$ of a Bézout domain such that $\text{Min}(S)$ is not quasi-compact. Thus the quotient ring $S$ obtained by Fischer is a reduced Bézout ring but is not coherent.

In fact, we can show that a reduced Bézout ring being coherent is equivalent to its minimal prime spectrum being quasi-compact. The following characterization of Baer rings is crucial to the proof of this statement.

Proposition 4.5.1 A ring $R$ is Baer if and only if every principal $R$-module is projective.

Proof. Suppose $R$ is a Baer ring. Let $a \in R$, then $\text{Ann}(a) = eR$ for some idempotent $e$ of $R$. Since $R = eR \oplus (1-e)R = \text{Ann}(a) \oplus (1-e)R$, $aR$ is isomorphic to $R/\text{Ann}(a)$, or $(1-e)R$, as $R$-modules. Hence, $aR$ is projective. Conversely, suppose every principal $R$-module is projective. Let $a \in R$. $R/\text{Ann}(a)$ is isomorphic to $aR$ as $R$-modules, with the latter being projective. Therefore, $\text{Ann}(a)$ is a direct summand of $R$, we can then repeat the same argument as in the last part of the
proof of theorem 3.3.9 to show that $\text{Ann}(a)$ is generated by an idempotent of $R$. Hence, $R$ is a Baer ring.

**Proposition 4.5.2** A reduced Bézout ring $R$ is coherent if and only if $\text{Min}(R)$ is quasi-compact.

**Proof.** Suppose $R$ is a reduced Bézout ring with $\text{Min}(R)$ being quasi-compact, then $R$ is a Baer ring by proposition 3.3.12. Since $R$ is Bézout, every finitely generated ideal is principal, which is then projective, whence is finitely presented. Therefore $R$ is coherent. The converse is a standard result (For instance, see [9, Corollary 4.2.16]).

### 4.6 Applications

We conclude our discussion on the topic by mentioning some applications of those results that we obtained.

The following example of Lewis and Ohm [27, page 830] shows how the ordering of a spectral poset can influence an algebraic property of any corresponding ring.

**Example 4.5** Let $X = \{x_m | m \in \mathbb{N}\} \cup \{y_n | n \in \mathbb{N}\}$ which is partially ordered by defining $x_m < y_n$ for all $m \geq n$ (See the figure below).
It is easy to see that $X$ is not a spectral poset, since it does not satisfy the condition (H) mentioned in chapter 2. In fact, this is just the example 2.1 with the order reversed.

Let $X' = X \cup \{m\}$ by requiring that, in addition to the ordering of $X$, $m < x$ for all $x \in X$. It is easy to show that the COP-topology for $X'$ is spectral, since in this topology all the closed sets other than $X'$ are finite.

Since $X'$ is spectral and has a unique minimal element, there exists an integral domain $R$ such that $X' \approx \text{Spec}(R)$. However, $X$ itself is not spectral and thus cannot be a closed subset of $X'$; so it follows that the intersection of the non-zero prime ideals must be zero. Thus, in this case the ordering of $\text{Spec}(R)$ implies a very concrete algebraic property of $R$.

Consider what happens when the ordering of $X'$ is reversed. Then any ring $R$ such that $\text{Spec}(R)$ is this new $X'$ has a unique maximal ideal $M$ such that $M$ is the union of the non-maximal prime ideals, because the set of all non-maximal prime ideals cannot be spectral.

This example shows how the ordering of a spectral poset can influence an algebraic
The following propositions are applications of the topological notions on prime spectrum.

**Proposition 4.6.1** An Arithmetical ring with connected prime spectrum which is also a $A(0)$ ring must have unique minimal prime ideal.

**Proof.** Let $P$ be a minimal prime ideal of $R$. Since $R$ is Arithmetical, $V(P)$ is a D-closed subset, so is open, by theorem 4.2.4. As a non-empty closed and open subset of the connected space $Spec(R)$, $V(P) = Spec(R)$. Therefore $P \subseteq Q$ for all $Q \in Spec(R)$. Suppose then $P'$ is any minimal prime ideal of $R$, then $P \subseteq P'$ implies that $P = P'$, whence $P$ is the unique minimal prime ideal of $R$. 

**Corollary 4.6.2** A reduced Arithmetical ring $R$ is an integral domain if and only if it is $A(0)$ and $Spec(R)$ is connected.

In view of proposition 4.5.2, we have a similar result.

**Proposition 4.6.3** A reduced Arithmetical ring $R$ is Baer if and only if $Min(R)$ is quasi-compact.

**Proof.** Suppose $Min(R)$ is quasi-compact. Since $R$ is Arithmetical, every prime ideal contains a unique minimal prime ideal of $R$, whence $R$ is a Baer ring by theorem 3.3.12. Applying the same theorem, the converse part is trivial.
Bibliography


