Pseudo-complements and Anti-filters in semigroups

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Pseudo-complements and Anti-filters in semigroups

By an ordered semigroup we shall mean a set S of elements on which is defined a closed binary associative multiplication and a partial ordering with respect to which the multiplication is isotone (i.e. $x \leq y$ implies $zx \leq zy$ and $xz \leq yz$ for all $z \in S$). A subsemigroup F of an ordered semigroup is called an anti-filter of S if $x \in F$, $y \leq x$ implies $y \in F$. If S is an ordered abelian semigroup with zero which is also the minimum element of S under the partial ordering, then an element a of S is said to be Pseudo-complemented if there exists an element a^* of S such that $aa^* = 0$ and $ax = 0$ implies $x \leq a^*$. Pseudo-complements and anti-filters in posets were studied by P.V. Venkatanarasimhan in [9]. He showed that the set S_μ of all anti-filters of a poset P with 0 forms a complete distributive lattice closed for pseudo-complements under the set inclusion as ordering relation. The purpose of this paper is to generalise the concepts of pseudo-complements and anti-filters of posets to ordered semigroups. We shall show that, under certain conditions, the set of all anti-filters of ordered semigroups forms a Noether multiplicative lattice. Some interesting results on pseudo-complements in ordered semigroups are also obtained.

In [8], Mr H. Stone showed that any anti-filter of a poset is the intersection of all prime anti-filters containing it. Guided by this result, Venkatanarasimhan [9] introduced a topology for the set of all prime anti-filters in posets, and obtained extensions of some results of M.H. Stone [8]. But if the poset is an ordered semigroup as well, then the anti-filters cannot always be expressed as the intersection of multiplicative prime anti-filters containing it. However, we shall show that in a special type of ordered semigroups, namely, the Stone semigroups, the topology established by Venkatanarasimkan for anti-filters in posets can be transported to Stone semigroups. The space of anti-filters of a Stone semigroups will be called Stone space. In this paper, we shall prove that: At most 10 distinct sets can be obtained from any subset of a Stone space by taking the operations of closure and complementation successively in any order. This result is a special case of the well-known Kuratowski problem in point set topology.

Anti-filters are usually called semi-ideals in Lattice Theory. But in ordered semigroups, this term would be easily mixed up with the usual definition of ideals. Thus, throughout this paper, we shall abandon the term "semi-ideals". The reader is referred to [9] for definitions and terminology not defined in this paper.

§1. Elementary properties of pseudo-complements.

Let S be an abelian ordered semigroup. S is said to be <u>residuated</u> if, given $a, b \in S$, the set X of elements $x \in S$ satisfying $ax = xa \leqslant b$ is not empty and has a maximum element, denoted by $b : a$ and called the <u>residual</u> of b by a.

By the <u>zero element</u> of a semigroup S we shall mean an element 0 with the property that $0x = x0 = 0$ for all $x \in S$. It is trivial to show that such an element is unique whenever it exists and, if S is residuated, is necessarily the minimum element of S .

We shall say that the ordered semigroup S is <u>pseudo-complemented</u> if S contains a zero element 0 and the residuals $0: a$, as defined above exist for all $a \in S$. The element $0: a$ is usually denoted by a^* and is called the <u>pseudo-complement</u> of a . Throughout this section, we shall assume that all ordered semigroups are pseudo-complemented.

It is well-known that in a pseudo-complemented poset P, the pseudo-complements have the following properties:

(i) $a \leqslant a^{**}$ for every $a \in P$

(ii) $a \leqslant b$ implies that $a^* \geqslant b^*$ for $a, b \in P$

(iii) $a^{***} = a^*$ for every $a \in P$

(iv) P has the greatest element 1 such that $1 = 0^*$

(v) $0 \leqslant a^*$

(vi) $y \leqslant 0^{**} \Leftrightarrow y^* = 0^*$

See Venkatanarasimhan [9] .

Similar results also hold in residuated semigroups.

<u>Theorem 1.1</u> Let S be a pseudo-complemented semigroup with zero which is the least element in S, then

(i) $a \leqslant a^{**}$, $\forall a \in S$.

(ii) $a \leqslant b \Rightarrow a^* \geqslant b^*$ for $a, b \in S$.

(iii) $a^{***} = a^*$ for every $a \in S$.

(iv) S has a maximum element, namely 0^*.

(v) $0 \leq a^*$.

(vi) $y \leq 0^{**} \Leftrightarrow y^* = 0^* \Leftrightarrow xy = 0, \ \forall x \in S$.

(vii) $ab^* \leq b^*$ for all $a, b \in S$.

Proof: The proofs of (i) - (vi) can be found in [9]. We only prove (vii).

As $b^* = 0: b$, then $b^*b \leq 0$ and so, by the isotone property of ordered semigroups, we have $ab^*b \leq a.0 = 0$, whence $ab^* \leq 0: b = b^*$.

We state some special properties of the pseudo-complemented semigroups whose elements are idempotent.

Lemma 1.2 If every element of the pseudo-complemented semigroup is idempotent, then $0^{**} = 0$.

Proof: $0^{**} = 0^{**} . 0^{**} = 0$ by Theorem 1.1 (vi).

Lemma 1.3 With the same hypothesis as above, then $(a_1 \ldots a_n)^{**} = a_1^{**} \ldots a_n^{**}$ and $(a_1 \ldots a_n)^* = (a_1^{**} \ldots a_n^{**})^*$.

Proof: Since $a_i^* .(a_1 \ldots a_n) = 0$, we have $a_1 \ldots a_n \leq a_i^{**}$, whence $(a_1 \ldots a_n)^{**} \leq a_i^{****} = a_i^{**}$ for all $i = 1, \ldots, n$. Then, by the isotone multiplication and the idempotency of elements, we have

$(a_1 \dots a_n)^{**} \leq a_1^{**} \dots a_n^{**}$. Now, let $x = (a_1 \dots a_n)^* . a_1^{**} \dots a_n^{**}$.

If we can show that $x = 0$, then it follows that $a_1^{**} \dots a_n^{**} \leq (a_1 \dots a_n)^{**}$.

The proof will be by induction on n. Assume the result holds for $1, \dots, n-1$. Since $x(a_1 \dots a_n) = 0$, we have $xa_1 \dots \hat{a}_i \dots a_n \leq a_i^*$ (1) Moreover, for any $z, y \in S$, $z^*(zy) = 0$ and so $yz \leq z^{**}$, therefore $xa_1 \dots \hat{a}_i \dots a_n \leq x^{**} \leq a_i^{****} = a_i^{**}$ (2)

The inequalities (1) and (2) together imply that $xa_1 \dots \hat{a}_i \dots a_n \leq a_i^* . a_i^{**} = 0$, whence $xa_1 \dots \hat{a}_i \dots a_n = 0$, and so $x \leq (a_1 \dots \hat{a}_i \dots a_n)^*$.

On the other hand $x \leq a_k^{****} = a_k^{**}$ and therefore $x \leq a_1^{**} \dots \hat{a}_i^{**} \dots a_n^{**}$.

By inducation hypothesis, we have $a_1^{**} \dots \hat{a}_i^{**} \dots a_n^{**} = (a_1 \dots \hat{a}_i \dots a_n)^{**}$.

Consequently $x \leq (a_1 \dots \hat{a}_i \dots a_n)^* . (a_1 \dots \hat{a}_i \dots a_n)^{**} = 0$.

The proof is completed.

Any two distinct element a, b of the ordered semigroup S is said to be equivalent if $a^* = b^*$, that is, $a \sim b$ if and only if $a^* = b^*$. Clearly $\sim$ is an equivalence relation.

<u>Theorem 1.4</u> With the equivalence relation defined above, $\frac{S}{\sim}$ is a semigroup. If every element of S is idempotent, then $\frac{S}{\sim}$ is an ordered semigroup which preserves the partial ordering of S under the natural homomorphism. The kernel of the homomorphism is the set $\{y \mid y \leq 0^{**}\}$.

Proof: Suppose that $a \sim a_1$. Then for any given element $x \in S$, we shall show that $ax \sim a_1x$. Consider $(ax)^*$. Let $p = (ax)^* = 0: ax$, then $axp \leqslant 0$ and so $xp \leqslant 0: a = a^* = a_1{}^* = 0: a_1$. Whence $a_1xp \leqslant 0$ and hence $p \leqslant 0: a_1x$. Similarly we can obtain $0: a_1x \leqslant p$ by interchanging a and a_1. Thus $p = 0: a_1x$, that is, $(ax)^* = (a_1x)^*$. In other words, we have showed that $ax \sim a_1x$, for all $x \in S$. Hence, the equivalence relation $\sim$ is compatible with respect to the semigroup multiplication. $\frac{S}{\sim}$ is a semigroup.

If $\{y\}$, $\{z\}$ are two equivalent classes of S with respect to the equivalence relation $\sim$, then we define $\{y\} \geqslant \{z\}$, if and only if $y^* \leqslant z^*$. We shall show that the above defined relations for elements in $\frac{S}{\sim}$ is a partial ordering. Suppose $\{y\} \geqslant \{z\}$, and $\{w\}$ is an arbitrary element in $\frac{S}{\sim}$. Then $y^{**} \geqslant z^{**}$ and so by lemma 1.3 $\underline{(yw)^{**} = y^{**}w^{**} \geqslant z^{**}w^{**} = (zw)^{**}}$. This shows that $\{yw\} \geqslant \{zw\}$, that is, $\{y\}.\{w\} \geqslant \{z\}.\{w\}$. Hence $\frac{S}{\sim}$ is indeed an ordered semigroup.

By theorem 1.1 (ii), $y \leqslant z$ implies that $y^* \geqslant z^*$, and so $\{y\} \leqslant \{z\}$. Consequently the homomorphism from S onto $\frac{S}{\sim}$ is an order preserving mapping. The set $\{y \mid y \leqslant 0^{**}\}$ is clearly the kernel of the homomorphism. The proof is completed.

Remark: A generalized result in topological semigroups was obtained by K.P. SHUM [7]. The reader is referred to [7]. We now define a lattice structure on the ordered semigroup $\frac{S}{\sim}$.

Lemma 1.5 If the pseudo-complemented semigroup S is idempotent, (that is, each element of S is an idempotent), then $\underset{\sim}{S}$ is pseudo-complemented, and each element of $\underset{\sim}{S}$ is an idempotent.

Proof: Let x be any element of S. Then $\{x\}.\{x^*\} = \{0, \ldots\}$. If $\{x\}.\{y\} = \{xy, \ldots\} = \{0, \ldots\}$, then $(xy)^* = 0^*$ and so by lemma 1.3 $(xy)^{**} = x^{**}y^{**} = 0^{**}$. Apply Theorem 1.1 (vi), we have $(x^{**}y^{**})y^{**} = 0 = x^{**}y^{**}$, and consequently, $y^{**} \leq x^{***} = x^*$. This implies that $y^* \geq x^{**}$ and so $\{y\} \leq \{x^*\}$. Thus $\{x^*\}$ is the pseudo-complement of $\{x\}$ in $\underset{\sim}{S}$.

Hereafter, we shall call the idempotent pseudo-complemented ordered semigroup the Boolean semigroup.

Lemma 1.6 Let S be a Boolean semigroup. For any $\{a\}, \{b\} \in \underset{\sim}{S}$, define $\{a\} \vee \{b\} = \{(a^* b^*)^*\}$. Then $\{a\} \vee \{b\}$ is the least upper bound for $\{a\}$ and $\{b\}$ under the partial ordering $\leq$.

Proof: By lemma 1.5, each element of $\underset{\sim}{S}$ has a pseudo-complement in $\underset{\sim}{S}$. Therefore by Theorem 1.1 (vii), we have $\{a^*\}\{b^*\} \leq \{b^*\}$ so $\{a^* b^*\}^* \geq \{b^{**}\} \geq \{b\}$. Similarly, $\{(a^*b^*)^*\} \geq \{a\}$. To prove that $\{(a^*b^*)^*\}$ is the least upper bound for $\{a\}$ and $\{b\}$, we let $\{x\} \in \underset{\sim}{S}$ such that $\{x\} \geq \{a\}$ and $\{x\} \geq \{b\}$. Thus $\{x^*\} \leq \{a^*\}$, $\{x^*\} \leq \{b^*\}$. By the isotone multiplication, we have $\{x^*\}^2 \leq \{a^*\}\{b^*\}$. Since $\{x^*\}^2 = \{x^*\}$, so $\{x^*\} \leq \{a^*\}\{b^*\}$ which implies $\{x^*\}^* \geq \{a^* b^*\}^*$. But $\{x^*\}^* = \{x\}^{**} = \{x\}$ for $\{x^{**}\}^* = x^*$. Therefore $\{x\} \geq \{a^* b^*\}^*$. Our proof is completed.

<u>Lemma 1.7</u> Let $\underset{\sim}{S}$ be a Boolean semigroup. For any $\{a\}, \{b\} \in \underset{\sim}{S}$, define $\{a\} \wedge \{b\} = \{ab\}$, then $\{a\} \wedge \{b\}$ is the greatest lower bound for $\{a\}$ and $\{b\}$.

Proof: Because $\{a\}^{**} = \{a\}$ for all $\{a\} \in \underset{\sim}{S}$, then by virtue of Theorem 1.1 (vii) we have

$$\{a\} . \{b\} = \{a\}^{**} . \{b\} \leqslant \{a\}^{**} = \{a\} .$$

and similarly $\{a\}\{b\} = \{a\}\{b\}^{**} \leqslant \{b\}^{**} = \{b\}$.

To prove that $\{ab\}$ is the greatest lower bound for $\{a\}$ and $\{b\}$, we let $\{x\} \in \underset{\sim}{S}$ such that $\{x\} \leqslant \{a\}$ and $\{b\}$. By the isotone multiplication and idempotency, we have $\{x\} = \{x\} . \{x\} \leqslant \{a\} . \{b\} = \{ab\}$, which proves that $\{ab\}$ is indeed the greatest lower bound for $\{a\}$ and $\{b\}$.

<u>Theorem 1.8</u> Let S be a pseudo-complemented semigroup whose elements are idempotent, then $\underset{\sim}{S}$ is a complemented lattice with $\{a\} \wedge \{b\} = \{ab\}$ and $\{a\} \vee \{b\} = \{(a^*b^*)^*\}$.

Proof: We need only to show that $\underset{\sim}{S}$ has the greatest element and least element, and for every $\{a\} \in \underset{\sim}{S}$, there exists a complement for $\{a\}$.

The element $\{0\}$ is the least element in $\underset{\sim}{S}$, because 0^* is the greatest element in S. $\{0^*\}$ is the greatest element in $\underset{\sim}{S}$, since $(0^*)^* = 0^{**} = 0$ by lemma 1.2, and 0 is the least element in S. For any $\{a\} \in \underset{\sim}{S}$ we have

$$\{a\} \vee \{a^*\} = \{(a^* a^{**})^*\} = \{0^*\}$$

$$\{a\} \wedge \{a^*\} = \{aa^*\} = \{0\} .$$

Therefore $\{a^*\}$ is the complement of $\{a\}$ in $\underset{\sim}{S}$.

Remark: The above theorem is inspired by the work of V. Glivenko (see G. Birkhoff [1] p. 148) .

Recall the following well-known result of G. Birkhoff ([1] p. 171). If every a in a lattice L has a unique complement a', and if $a \to a'$ is a dual automorphism, then L is a Boolean algebra.

By using the above result we obtain the following theorem.

Theorem 1.9 If S is a pseudo-complemented semigroup whose elements are idempotent, then $\underset{\sim}{S}$ forms a Boolean algebra in which $\{a\} \vee \{b\} = \{(a^*b^*)^*\}$, $\{a\} \wedge \{b\} = \{ab\}$.

Proof: Suppose $\{b\}$ is a complement $\{a\}$, we show that $\{b\} = \{a^*\}$. Since $\{a\} \wedge \{b\} = \{ab\} = \{0\}$, we have $(ab)^{**} = 0^{**} = 0$. But $(ab) \leq (ab)^{**} = 0$ by Theorem 1.1 (i), and so $ab = 0$. That is, $b \leq a^*$, and hence $\{b\} \leq \{a^*\}$. On the other hand, $\{a\} \vee \{b\} = \{(a^*b^*)^*\} = \{0^*\}$. That means $(a^*b^*)^{**} = a^{***}\, b^{***} = a^*\, b^* = 0^{**} = 0$ and so $b^* \leq a^{**}$. Hence $\{b\} \geq \{a^*\}$. Consequently $\{b\} = \{a^*\}$.

To prove that the map $\{a\} \to \{a^*\}$ is a dual automorphism of $\underset{\sim}{S}$, we need to show that $\{a^{**}\} = \{a\}$, $(\{a\} \wedge \{b\})^* = \{a\}^* \vee \{b\}^*$, $(\{a\} \vee \{b\})^* = \{a\}^* \wedge \{b\}^*$. The first equality is trivial. The second equality is obtained as follows:

$$(\{a\} \wedge \{b\})^* = \{ab\}^* = \{a^{**}\, b^{**}\}^* = \{(a^{**}\, b^{**})^*\} = \{a\}^* \vee \{b\}^* .$$

The third equality is obtained dually from the second equality.

Thus $\frac{S}{\sim}$ is a Boolean algebra. In closing this section, we give an example illustrates that the zero element of a pseudo-complemented semigroup S need not be the minimum element of S under the partial ordering $\leqslant$.

Example 1.10 Let S be the set $\{\frac{1}{n} \mid n = 1,2,3,\ldots\} \cup \{0\} \cup \{-n \mid n = 1,2\ldots\}$. For all elements in the interval $[0, 1]$, the multiplication $\circ$ is defined to be the usual multiplication. For elements x and y in the interval $[-\infty, 0]$, the multiplication $\circ$ is defined to be $x \circ y = -xy$. If $x \in [0, 1]$, $y \in [-\infty, 0]$, define $x \circ y = 0$. With this multiplication $\circ$, $(S, \circ)$ is a semigroup. The partial ordering of S is the usual partial ordering in the number system. It can be verified that the multiplication is isotone and non-idempotent. Moreover, the ordered semigroup S is pseudo-complemented. In fact, for each $x \in [0, 1]$, we have $x^* = 0$, and for each $x \in (-\infty, 0)$, $x^* = 1$. Obviously, the zero element of S is not the minimum element in S.

In the above example, the minimum element of S does not exist.

Throughout this paper, the ordered semigroups S to be studied will be semigroups with zero element which is also the minimum element of S.

§2. Anti-filters in ordered semigroups

Anti-filters of posets were studied by R.P. Dilworth [3], P.V. Venkatanarasimhan [9], P.J. McCarthy [6], K.P. Bogart [2] and M.F. Janowitz [4]. In this section, we shall study anti-filters on ordered semigroups. Several results obtained by Venkatanarasimhan and Janowitz on posets are extended to ordered semigroups.

Definition 2.1 A non-empty subset A of a poset P is called an anti-filter of P if $a \in A$, $b \leq a$ $(b \in P)$ implies $b \in A$.

The principal anti-filter generated by an element $a \in P$ is denoted by $(a]$ which is the set $\{x \in P \mid x \leq a\}$. The principal anti-filter generated by a subset A of P is similarly defined and is denoted by $(A]$.

The following theorem on the anti-filters of posets is well-known.

Theorem 2.2 (See Venkatanarasimhan [9] p.339 Theorem A). The set S_μ of all anti-filters of a poset S with 0, forms a complete distributive lattice closed for pseudo-complements under set inclusion as ordering relation. The lattice join and lattice meet in S_μ coincide with the set union and set intersection. This theorem is stated in the paper of Venkatanarasimhan [9] without proof. For the sake of completeness, we produce a proof here.

Proof: Let A_i be anti-filters of the poset S. Suppose $a \in \cup A_i$ such that $b \leq a$, then $a \in A_{io}$ for some i_o. Since A_o is an anti-filter, so $b \in A_{io}$ which implies that $b \in \cup A_i$. Therefore $\cup A_i$ is an anti-filter of S. To prove that $\cap A_i$ is an anti-filter of S, we let $a \in \cap A_i$ such that $b \leq a$, then $a \in A_i$ and $b \leq a$ for all i. Since A_i is an anti-filter, so $b \in A_i$ for all i. Thus $b \in \cap A_i$ and $\cap A_i$ is an anti-filter of S.

To show that S_μ is closed for pseudo-complements. Let A be an anti-filter of S and let $\tilde{A}$ be the set $\{x \in S \mid \text{either } x \geq a \text{ or } x \leq a$ for some $a \neq 0 \in A\}$. Write $A^* = (S \setminus \tilde{A}) \cup (0]$. Clearly A^* is an anti-filter of S and $(A] \cap (A^*] = (0]$. Moreover, if B is another anti-filter of S such that $(A] \cap (B] = (0]$, then we claim that $B \subseteq A^*$. For otherwise, we can find an element $b \neq 0 \in B \setminus A^*$ such that either $b \leq a$ or $a \leq b$ for some $a \neq 0 \in A$. Suppose that $b \leq a$, then $b \in A$ and so $b \in A \cap B \in (A] \cap (B] = (0]$, which contradicts to $b \neq 0$. Likewise, we can prove that $a \leq b$ is impossible. Thus A^* is indeed the pseudo-complement of A. Hence, the set S_μ of all anti-filters of S is closed for pseudo-complements.

<u>Corollary</u> Let S_μ be the set of all anti-filters of a poset S with 0, then $A = A^{**}$ for some $A \in S_\mu$.

Proof: Since S_μ is closed for pseudo-complement, $A \subseteq A^{**}$ for every $A \in S_\mu$. Clearly $A \to A^{**}$ is an isotone function from S_μ into S_μ. Apply the well-known fixed point theorem of lattices (See G. Birkhoff [1] p.54), the result follows.

If the poset S is also a ordered semigroup, then the set S_μ of all anti-filters of S is not only a pseudo-complemented complete distributive lattice but also a multiplicative lattice. Moreover, if S has an identity element 1 which is also the greatest element of S, then S_μ is an integral multiplicative lattice.

<u>Definition 2.3</u> (G. Birkhoff [1] p.200) By a multiplicative lattice,

we mean a lattice L with a binary multiplication satisfying

$a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in L$.

A zero of a multiplicative lattice L is an element 0 satisfying

$0 \wedge x = 0 . x = x . 0 = 0$ for all $x \in L$.

A unity of L is an element e satisfying

$ex = xe = x$ for all $x \in L$.

<u>Theorem 2.4</u> Let S be an ordered semigroup with 0 . Then the set S_μ of all anti-filters of S is a pseudo-completed complete multiplicative lattice.

Proof: In Theorem 2.1, we have already showed that the set S_μ is a pseudo-complemented complete lattice under set-inclusion as ordering relation. The lattice join and lattice meet in S_μ are the set union and set intersection. To prove that S_μ is a multiplicative lattice, we only need to show that there is a multiplicative defined on S_μ which is distributive over the union of anti-filters of S . For this purpose, we define the product AB of two anti-filters as the smallest anti-filter containing all products ab where $a \in A$, $b \in B$. Then for any $A, B, C \in S_\mu$, we immediately have $AB \cup AC \subseteq A(B \cup C)$. On the other hand, let $x \in A(B \cup C)$, then $x \leq ab$ or ac for some $a \in A$, $b \in B$, $c \in C$. Thus $x \in AB \cup AC$. Hence the result.

<u>Definition 2.5</u> A lattice L is called an <u>M-lattice</u> if it satisfies the following condition: if $a, b \in L$ and $a \leq b$ then there exists an

element $c \in L$ such that $a = bc$ (See P.J. McCarthy [6])

Follow from the above definition, we call an ordered semigroup S an M-semigroup if for all $a, b \in S$, $a \leq b$ there is an element $c \in S$ such that $a = bc$.

Clearly, if the lattice of anti-filters of an ordered semigroup S is an M-lattice under set union and intersection, then S must be an M-semigroup. However, the converse is not true.

The following example illustrates that the lattice S_μ of an M-semigroup is not necessarily an M-lattice.

Example 2.6 Let Z^+ be the semigroup of positive integers and zero with usual multiplication. For all elements $a, b \in Z^+$, define $a \leq b$ if and only if $b \mid a$ (that is, b divides a). Then $(Z^+, \leq)$ is an M-ordered semigroup. The principal anti-filter $(m]$ is the set which consists of all multiples of m . The sets $(6] \cup (35]$, $(3] \cup (7]$ are obviously anti-filters of S . Clearly, $(6] \cup (35] \subset (3] \cup (7]$. If S_μ is an M-lattice, then there exists an anti-filters A of S such that $(6] \cup (35] = A . [(3] \cup (7]] = A . (3] \cup A . (7]$. Thus $6 \leq a . 3$ for some $a \in A$. This implies that a must equal to 2. For if $a = 1$, then $A . [(3] \cup (7]] \supseteq (3] \cup (7] \neq (6] \cup (35]$. But $a . 7 = 2 . 7 = 14 \in A . (7]$ and $14 \notin (6] \cup (35]$. This contradicts to $(6] \cup (35] = A . (3] \cup A . (7]$. Thus such anti-filter A of S does not exist. S_μ is not an M-lattice.

Definition 2.6 Let L be a lattice. An element M of L is said to be Dilworth meet Principal if $(A \wedge B : M)M = AM \wedge B$, for all elements $A, B \in L$.

An element M of L is said to be Dilworth join Principal if

$(A \vee BM): M = (A: M) \vee B$, for all elements $A, B \in L$.

An element M of L is said to be Dilworth Principal if it is both meet and join principal.

(See R.P. Dilworth [3], p.481).

Dilworth principal elements in lattice theory were studied by Janowitz [4] and McCarthy [6]. We shall here amplify their results to ordered semigroups. We discover that the Dilworth principal elements are closely related to the properties of ordered semigroups.

Definition 2.7 An element a of an ordered semigroup S is called a negative element if $ax \leq a$ and $xa \leq a$ for all elements $x \in S$. An ordered semigroup S is called a negatively ordered semigroup if all of its elements are negative.

The multiplicative semigroup Z^+ of all non-negative elements is trivially a negatively ordered semigroup if $a \leq b$ means $b \mid a$.

The following is another example of negatively ordered semigroup.

Example 2.8 Let S be the set of all order-preserving mappings of a poset T with 0 into itself. Define $f \leq g$ $(f, g \in S)$ if and only if $f(x) \leq g(x)$ for all $x \in T$. Let H be the subset of S which consists of all "retract" mappings h such that $h(x) \leq x$ for all $x \in T$. The multiplication is the composition of mappings.

(i) If $f, g \in H$, then $gf(x) = g[f(x)] \leq f(x) \leq x$ for all $x \in T$. Hence $gf \in H$ and H is a semigroup.

(ii) Since g is an order-preserving mapping and $f(x) \leq x$, we have $g(f(x)) \leq g(x)$ for all $x \in T$. Therefore $gf \leq g$. Similarly $gf \leq f$. Hence H is negatively ordered.

Theorem 2.9 Let S be a negatively ordered semigroup and S_μ be the lattice of anti-filters of S. Then the following conditions are equivalent.

(i) S is an M-semigroup

(ii) For every principal anti-filter $(a]$ of S, $(a]$ satisfies the equality $(B: (a]). (a] = B \cap (a]$, where B is an arbitrary element of S_μ.

Proof: (i) $\Rightarrow$ (ii). Since S is a negatively ordered semigroup, so $(B: (a])(a] \subseteq B$. Thus $(B: (a](a] \subseteq B \cap (a]$. For the converse containment, let $x \in B \cap (a]$. Then $x \leq b \leq a$ for some element $b \in B$. As S is an M-semigroup, we have $x \leq b = ca \leq a$ for some element $c \in S$. We shall show that $c \in B: (a]$. For this purpose, let $y \in (a]$, then $cy \leq ca = b \in B$. This implies that $c. (a] \subseteq B$. As $B: (a]$ is the largest element X satisfying $X(a) \subseteq B$, thus $[c] \subseteq B: (a]$. Now, since $x \leq ca$, so $x \in (B: (a])(a]$ for $(B: (a])(a]$ is an anti-filter of S. Therefore $B \cap (a] \subseteq (B: (a])(a]$.

(ii) $\Rightarrow$ (i) Let a, b be elements of the ordered semigroup S. Assume that $b \leq a$. Then $(b] \subseteq (a]$ and so $(b] = (b] \cap (a] = ((b]: (a])(a]$. Thus $b \leq xt$ for some elements $x \in (b]: (a]$ and $t \in (a]$. It then follows that $b \leq xt \leq xa \leq b$. Hence $b = xa$ for some $x \in S$. This proves that S is an M-semigroup.

Remark The reader should note that the lattice S_μ of anti-filters of an M-semigroup is not necessarily an M-lattice. Thus, Theorem 2.9 amplifies the result of M.F. Janowitz [4; Theorem 1, p.653].

Theorem 2.10 Let S be a negatively ordered M-semigroup with 0 and S_μ be the lattice of anti-filters of S. Then the following conditions are equivalent:

(i) S is a cancellative semigroup

(ii) For every principal anti-filter $(a]$ of S, $(a]$ satisfies the equality $(B.\ (a]):(a] = B \cup (0:(a])$

Proof: (i) $\Rightarrow$ (ii). Trivially, we have $B \subseteq (B.\ (a]):(a]$. Also, since $(0:(a])(a] = (0] \subseteq B.\ (a]$, we have $0:(a] \subseteq (B.\ (a]):(a]$. Therefore $B \cup (0:(a]) \subseteq (B:(a]):(a]$. For the converse containment, let $x \in (B.\ (a]):(a]$. Then $xa \leq by \leq ba$ for some $y \in (a]$, $b \in B$. Because S is an M-semigroup, there exists an element $c \in S$ such that $xa = c(ba)$. If $xa \neq 0$, then by the cancellation property of (i), we have $x = cb \leq b \in B$. If $xa = 0$, then $x.\ (a] = (0]$ and $x \in (0]:(a]$. Therefore, in any case, we have $(B.\ (a]):(a] \subseteq B \cup (0:(a])$.

(ii) $\Rightarrow$ (i). Suppose $xa = ya \neq 0$ holds in S. Then $(xa]:(a) = \{(x].\ (a]:(a]$. By (ii), we have $\{(x].\ (a]\}:(a] = (x] \cup (0:(a])$. Similarly, $(ya]:(a] = (y] \cup (0:(a])$. Therefore, by assumption, we have $(x] \cup (0:(a]) = (y] \cup (0:(a])$. Since $xa \neq 0$ implies that $x \notin 0:(a]$, x is the greatest element in the set $(x] \cup (0:(a])\setminus(0:(a])$. Similarly, y is the greatest element in the set $(y] \cup (0:(a])\setminus(0:(a])$. Therefore $x = y$ and the proof is completed.

By Theorem 2.9, Theorem 2.10 and K.P. Bogart's Theorem 1 [2 ; p.215-216], we have the following theorem:

<u>Theorem 2.11</u> Let S be a negatively ordered semigroup with 0. Then every principal anti-filter of S is a Dilworth principal element in S_μ, if and only if S is a cancellative M-semigroup.

At this point, it is natural to ask when the Dilworth principal element would coincide with the principal anti-filter. In order to answer this question, the following lemma is needed.

<u>Lemma 2.12</u> Let S be an M-semigroup and S_μ be the lattice of anti-filters of S. Then $(\bigcup_\alpha A_\alpha): M = \bigcup_\alpha (A_\alpha : M)$ holds for all $A_\alpha \in S_\mu$ and for every Dilworth principal element M of S_μ.

Proof: The proof can be obtained verbatim from M.F. Janowitz [4 ; p.654, Theorem 3] and hence omitted.

We now answer the question mentioned above.

<u>Theorem 2.13</u> Let S be a negatively ordered M-semigroup with identity. Then every Dilworth principal element in the lattice S_μ, the lattice of anti-filters of S, is the principal anti-filter (a] of S.

Proof: Since S is negatively ordered M-semigroup with identity, then it is trivial to see that S is the identity and the greatest element in the multiplicative lattice S_μ. If $S = \bigcup_\alpha A_\alpha$ for some $A_\alpha \in S_\mu$, then, as $1 \in S = \bigcup_\alpha A_\alpha$, we have $S = (1] \subseteq A_{\alpha_o}$ for some $A_{\alpha_o} \in S_\mu$. That is, $S = A_{\alpha_o}$.

Now, let M be an arbitrary Dilworth principal element of S_μ, then $M = \bigcup_{a \in M} (a]$. By lemma 2.12, we have $S = (\bigcup_{a \in M} (a]): M = \bigcup_{a \in M} ((a]: M)$. By the above argument, we immediately have $S = (a_o]: M$ for some $a_o \in M$. It then follows that $M = (a_0]$.

A commutative ring R is said to be an ordered ring if R is an ordered semigroup under multiplication. Let R be a negatively M-ordered commutative ring with identity, then it is trivial to see that every ideal of R is an anti-filter of R, and every principal ideal (a) of R is a principal anti-filter (a] of R and vice versa.

Thus, when viewed in the content of ring theory, Theorem 2.13 can be translated to the following:

Theorem 2.14 Let R be a negatively M-ordered integral domain with 1 and L(R) be the set of ideals of R, then the followings are equivalent:

(i) Every element of L(R) is Dilworth principal element in S_μ .

(ii) R is an principal ideal domain.

§3. Prime anti-filters

In this section, we shall introduce a topology for the prime anti-filters of Stone semigroups. The topological concepts used in this paper can be found in P.V. Venkatanarasimhan [9].

Definition 3.1 Let S be an negatively ordered semigroup with identity. A proper anti-filter P of S is called a prime anti-filter of S if $(a].(b] \subseteq P$ implies either $(a] \subseteq P$ or $(b] \subseteq P$.

The following is an analogy of prime ideals in semigroups.

Theorem 3.2 Let P be prime anti-filter of an negatively ordered semigroup S with identity. Then P is a prime anti-filter if and only if for any anti-filter A_1 and A_2 of S, $A_1A_2 \subseteq P$ implies that either $A_1 \subseteq P$ or $A_2 \subseteq P$.

<u>Corollary</u> If a prime anti-filter P contains the product of a finite number of anti-filters, then it contains at least one of them.

<u>Definition 3.3</u> Let S be an negatively ordered semigroup with identity. An anti-filter A of S is called a dense anti-filter of S if A^*, the pseudo-complement of A is the zero anti-filter, that is, $A^* = (0]$.

An anti-filter A of S is called a normal anti-filter of S if $A = A^{**}$. (See Venkatanarasimhan [9]).

The following results concerning prime anti-filters in posets can be translated almost verbatim to negatively ordered semigroups with identity.

<u>Theorem 3.4</u> (P.V. Venkatanarasimhan [9]) Let S be an negatively ordered semigroup with identity. Then

(i) $S\setminus\{1\}$ is the maximal anti-filter, and it is a prime anti-filter.

(ii) A prime anti-filter of S is either normal or dense.

(iii) The set-union of any family of prime anti-filters of S is a prime anti-filter.

(iv) The set-intersection of any lower-directed family of the prime anti-filters is a prime anti-filter.

Let $\mathcal{P}$ be the set of all prime anti-filters of an negative ordered semigroup S with identity. The set of all prime anti-filters containing a given anti-filter A of S is denoted by F(a). The set theoretical complement of F(A) in $\mathcal{P}$ is denoted by F'(A). Then, mimic to Venkatanarasimhan [9], we obtain the following theorems for negatively ordered semigroups.

Theorem 3.5 (cf. Venkatanarasimhan [9], Theorem 19, p.344)

(i) $F(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} F(A_i)$

(ii) $F(A_1 \cap A_2 \cap \ldots \cap A_n) = F(A_1) \cup F(A_2) \cup \ldots \cup F(A_n)$

(iii) $F(S) = \phi$

(iv) $F(0] = \mathcal{P}$

Theorem 3.6 (cf. Venkatanarasimkan [9], Theorem 20)

(i) $F'(\bigcup_i A_i) = \bigcup_i F'(A_i)$

(ii) $F'(A_1 \cap \ldots \cap A_n) = F'(A_1) \cap \ldots \cap F'(A_n)$

(iii) $F'(S) = \mathcal{P}$

(iv) $F'((0]) = \phi$

Equipped with theorem 3.5, we can introduce a (unique) topology $\mathcal{F}$ on $\mathcal{P}$ whose closed subsets are precisely the set F(A).

The following theorems are true for the topological space $\mathcal{P}$.

Theorem 3.7 (Venkatanarasimkan [9])

(i) If X is a subset of the space $\mathcal{P}$, then the closure of X is the set $F(X_o)$, where X_o is the intersection of all the members of X.

(ii) The space $\mathcal{P}$ is a T_0-space.

(iii) The space $\mathcal{P}$ is a T_1-space if and only if the ordered semigroup S is the chain of two elements.

(iv) The space $\mathcal{P}$ is compact and non-regular.

Proof: The proofs mimic to Venkatanarasimkan [9] and hence omitted.

The topology theory of anti-filters on posets developed by Venkatanarasimkan [9] depends upon the following theorem of M.H. Stone [8].

Theorem 3.8 (M.H. Stone) (i) Given an anti-filter A of a poset P and $b \notin A$ $(b \in P)$, among all the anti-filters containing A and not containing b, there exists a maximal one, and it is prime.

(ii) Any anti-filter of a poset is the intersection of all the prime anti-filters containing it.

By using the above result, P.V. Venkatanarasimkan successively shows that the topological space $\mathcal{P}$ is connected and non-Hausdorff.

However, as multiplication is involved Theorem 3.8 is not generally true for ordered semigroups. The following is a counter-example.

Example 3.9 Let $S = \{0, 1, a, b, c\}$ with multiplication table as follows

*	1	0	a	b	c
1	1	0	a	b	c
0	0	0	0	0	0
a	a	0	0	0	0
b	b	0	0	0	0
c	c	0	0	0	0

The Hasse diagram of S is:

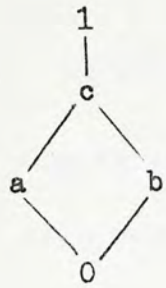

Figure 1

Then it is easy to check that $(S, *, \leqslant)$ is a negatively ordered semigroup. Since $(c] . (b] = (0]$, the prime anti-filters of S must contain either $(c]$ or $(b]$. Similarly $(c] . (a] = (0]$, the prime anti-filters of S must contain either $(c]$ or $(a]$. Thus the prime anti-filters of S are the sets $(c] = \{0, a, b, c\}$ and $(a] \cup (b] = \{0, a, b\}$. However, the anti-filter $(b]$ cannot be expressed as the intersection of prime anti-filters of S. Thus, Stone's Theorem does not hold in ordered semigroups.

In order to develop a topology theory for ordered semigroup parallel to those established by Venkatanarasimkan [9], we need the following definition.

<u>Definition 3.10</u> A negatively ordered semigroup S is called a Stone semigroup, if for any $b \neq 0 \in S$, there exists $x \in S$ such that $x \not\geqslant b^k$ for all positive integers $k \geqslant 1$. In other words, a negatively ordered semigroup S is a Stone semigroup if for any cyclic subsemigroup $\{b, b^2, \ldots\}$ of S, the filter generated by this subsemigroup is not the whole of S.

The semigroup Z^+ of all non-negative integers with usual multiplication is a Stone semigroup if the partial ordering $\leqslant$ in Z^+ is defined as $a \leqslant b$ if and only if $b \mid a$. For let $n \in Z^+$ and let p be a prime number which is not a prime factor of n, then we would have $p \not\geq n^k$ for all integers $k \geq 1$.

Let R be a Boolean ring. We define an order relation on R by the rule $x \leq y \Leftrightarrow xy = x^2$. It is easy to show that this is indeed a negatively partial order relation, and R is a non-cancellative Stone-semigroup.

The Stone theorem (cf. Theorem 3.8) for Stone semigroup is as follows:

<u>Theorem 3.11</u> Let S be a Stone semigroup. Then for any non-zero element b of S, there exists a prime anti-filter P of S which does not contain b.

Proof: Let C be the set consisting of all elements x of S such that $x \not\geq b^k$ for all integers $k \geq 1$.

In order to prove that C is an anti-filter of S, we let $y \leq x$. If $y \geq b^m$ for some integer $m \geq 1$, then $x \geq b^m$, which contradicts to the choice of x. Hence $y \not\geq b^k$ for all $k \geq 1$. This means that $y \in C$ and C is an anti-filter.

Now, let $(x] \cdot (y] \subseteq C$, then we have $xy \in C$. Assume that $x \notin C$ and $y \notin C$. Then $x \geq b^m$ and $y \geq b^n$ for some integers $m \geq 1$, $n \geq 1$. Consequently, $xy \geq b^m b^n = b^{m+n}$. This contradicts to $xy \in C$. Therefore either x or $y \in C$ and so either $(x] \subseteq C$ or $(y] \subseteq C$. Thus, C is a prime anti-filter of S. Clearly, $b \notin C$. Our proof is completed.

By using Theorem 3.11, we can establish the following theorem which enlarges the result of V.K. Balachandran (See Venkatanarasimkan [9]) from posets to ordered semigroups.

Theorem 3.12 (cf. Venkatanarasimkan [9], Theorem 17, p.343). Let S be a Stone semigroup and A be an anti-filter of S . Then the pseudo-complement of A is the intersection of all the prime anti-filters not containing A .

Proof: Let B be the intersection of all the prime anti-filters B_i not containing A . As $A . A^* = (0]$ and B_i does not contain A, we must have $B_i \supseteq A^*$ and consequently $B \supseteq A^*$. Suppose if possible that $B \supsetneq A^*$, then there exists an element $x \in S$ such that $x \in B \setminus A^*$. We claim that there is some element $y \in A$ such that $(x] \cap (y] \neq 0$. For if not, then we shall have $(x] \cap (y] = 0$ for all $y \in A$. This implies that $(x] \cap A = (0]$ and so $(x] \subseteq A^*$, a contradiction. Our claim is therefore established. Now applying Theorem 3.12, we can find a prime anti-filter C of S such that $(x] \cap (y] \not\subseteq C$. Clearly both (x] and (y] are not contained in C, and so both the anti-filters A and B are not contained in C . But B is the intersection of all prime anti-filters not containing A, so B must be contained in C , a contradiction again. Therefore, we have to conclude that $B = A^*$. Our proof is completed.

The following results which were established by P.V. Venkatanarasimkan [9] can now be transported to Stone semigroup.

Theorem 3.13 Let S be a Stone semigroup and A be an anti-filter of S. Let $\mathcal{P}$ denote the stone space of prime anti-filters of S. Then the following statements are valid:

(i) The closure of the $F'(A)$ is the set $F(A^*)$

(ii) The interior of the set $F(A)$ is the set $F'(A^*)$

(iii) A closed subset $F(A)$ of the space $\mathcal{P}$ has empty interior if and only if A is a dense anti-filter of S.

(iv) An open subset $F'(A)$ of the space $\mathcal{P}$ is dense if and only if A is a dense anti-filter of S.

(v) The space $\mathcal{P}$ is connected.

(vi) The isolated point of $\mathcal{P}$ is a normal anti-filter of S. Conversely, a normal and completely meet-irreducible prime anti-filter of S is an isolated point of $\mathcal{P}$. (We call an anti-filter A to be completely meet-irreducible, if A is not the intersection of any family of anti-filters which does not contain A as a member.)

Proof: The proofs of (i) - (v) can be found in Venkatanarasimhan [9] and hence omitted. We only prove (vi).

Suppose that A is an isolated point of $\mathcal{P}$. Then $\{A\} = \{A\}^o = \{A\}^{'-'} = F'(B)$, where B is the intersection of all the prime anti-filters of S other than A, by Theorem 3.7(i). Hence $A \supsetneq B$, for otherwise we would have $A \supseteq B$ and so $A \in F(B)$ which contradicts to $A = F'(B)$. By Theorem 3.13(i), we have $F(A) = \{A\}^- = \{F'(B)\}^- = F(B^*)$. Also by Theorem 3.11 and the choice of B, we know that $B \cap A = (0]$ and so $B \subseteq A^*$. If A is dense, then $A^* = (0]$ and $B = (0]$. This is a contradiction with $A \supsetneq B$. So A is not dense. Therefore A is normal.

Conversely, suppose A is a normal and completely meet-irreducible prime anti-filter. Then $A^* \neq (0]$ and so $A \supsetneq A^*$. Let A_1 be the intersection of all the prime anti-filters strictly containing A and let B be the intersection of all the prime anti-filters other than A. Since $A \not\supseteq A^*$ and $A \cup A^*$ must be contained in the maximal anti-filter $S\setminus\{1\}$, so $A \subsetneq S\setminus\{1\}$. As $S\setminus\{1\}$ is a prime anti-filter strictly containing A, the existence of A_1 is assured. Since A is completely meet-irreducible, we have $A \not\supseteq A_1$. Since A is prime, it then follows that $A \not\supseteq A^* . A_1$. By the negative order and Theorem 3.12, we have $A \not\supseteq A^* \cap A_1 = B$ whence $A \in F'(B)$. But this implies that $A = F'(B)$. Therefore A is an isolated point.

§4 Kuratowski problem on Stone space

K. Kuratowski [5] has shown that at most 14 distinct sets can be constructed from a subset of a topological space by successive applications in any order of the closure and the complement operations. Let us call the above sets the relatives of A. In this section we shall consider the Kuratowski problem on Stone space. We shall prove that at most 10 distinct relatives can be obtained from a subset of the Stone space.

We shall use the following concepts:

Let A be a subset of a topological space T. A^- is the topological closure of A in T. A^o is the interior of the set A in T. It is well-known that $A^o = A'^{-'}$ where A' is the set theoretical complement of A in T.

By theorem 3.7 and theorem 3.13, we have the following:

<u>Theorem 4.1</u> Let $\mathcal{P}$ be a Stone space (that is, the space of prime anti-filters of a Stone semigroup). If $\mathcal{A}$ is a subset of the space $\mathcal{P}$, then the following equalities hold:

(i) $\mathcal{A}^{-} = F(\bigcap A_\lambda \mid A_\lambda \in \mathcal{A})$

(ii) $\mathcal{A}^{-o} = F(\bigcap A_\lambda)^{o} = F'[(\bigcap A_\lambda)^*]$

(iii) $\mathcal{A}^{-o-} = F'[(\bigcap A_\lambda)^*]^{-} = F[(\bigcap A_\lambda)^{**}]$

(iv) $\mathcal{A}^{o} = \mathcal{A}^{\prime - \prime} = \mathcal{P} \setminus (\mathcal{P} \setminus A)^{-} = \mathcal{P} \setminus F(\bigcap B_\mu) = F'(B)$

where B is the intersection of all elements of $\mathcal{P} \setminus \mathcal{A}$

(v) $\mathcal{A}^{o-} = F'(B)^{-} = F(B^*)$

(vi) $\mathcal{A}^{o-o} = F(B^*)^{o} = F'(B^{**})$.

Proof: Equality (i) follows from Theorem 3.7(i). Equality (ii) follows from Theorem 3.13(ii). Equality (iii) follows from Theorem 3.13(i). Equality (v) follows from Theorem 3.13(i). Equality (vi) follows from Theorem 3.13(ii).

<u>Theorem 4.2</u> Let $\mathcal{A}$ be a subset of a finite Stone space $\mathcal{P}$. If B is the intersection of all elements of $\mathcal{P} \setminus \mathcal{A}$ and $\bigcap A_\lambda$ is the intersection of all elements of $\mathcal{A}$, then

(i) $B^* = (\bigcap A_\lambda)^{**}$

(ii) $B^{**} = (\bigcap A_\lambda)^*$

Proof: From Theorem 3.12, $B^* = \bigcap C_k$, where C_k are prime anti-filters of S which does not contain B. As B is the intersection of all elements of $\mathcal{P} \setminus \mathcal{A}$, so each C_k cannot belong to $\mathcal{P} \setminus \mathcal{A}$. Hence C_k is an element of the set $\mathcal{A}$. Therefore, $B^* = \bigcap C_k \supseteq \bigcap\{A_\lambda \mid A_\lambda \in \mathcal{A}\}$. By the properties of pseudo-complements, we have $B^{**} \subseteq (\bigcap A_\lambda)^*$ and so $B^* = B^{***} \supseteq (\bigcap A_\lambda)^{**}$.

As $B = \bigcap\{B_\mu \mid B_\mu \in \mathcal{P} \setminus \mathcal{A}\}$, so by lemma 1.3, $B^{**} = \{B_\mu \mid B_\mu \in \mathcal{P} \setminus \mathcal{A}\}^{**} = \bigcap B_\mu^{**}$.

Thus, in order to prove the converse containment, we only need to prove $B_\mu^* \subseteq A_\lambda^{**}$ for all μ and λ. Since then, we would have $B_\mu^* \subseteq \bigcap A_\lambda^{**} = (\bigcap A_\lambda)^{**}$ and so $B_\mu^{**} \supseteq (\bigcap A_\lambda)^{***} = (\bigcap A_\lambda)^*$ for all μ. Hence, it follows that $B^{**} = \bigcap B_\mu^{**} \supseteq (\bigcap A_\lambda)^*$, that is, $B^* = B^{***} \subseteq (\bigcap A_\lambda)^{**}$.

Since A_λ and B_μ are prime anti-filters for all λ and μ. By Theorem 3.4(i), we know that A_λ and B_μ are either dense or normal.

If B_μ is dense, then we have $B_\mu^* = (0] \subseteq A_\lambda^{**}$. If A_λ is dense, then we have $A_\lambda^{**} = (0]^* = S$ and hence $B_\mu^* \subseteq A_\lambda^{**}$.

If both A_λ and B_μ are normal, then from the facts that $A_\lambda^* \cdot A_\lambda = (0] \subseteq B_\mu$ and $B_\mu^* \cdot B_\mu = (0] \subseteq A_\lambda$ we obtain the following four cases: $B_\mu \subseteq A_\lambda$ or $B_\mu^* \subseteq A_\lambda$; at the same time $A_\lambda^* \subseteq B_\mu$ or $A_\lambda \subseteq B_\mu$. Notice that the case for $A_\lambda \subseteq B_\mu$ and $B_\mu \subseteq A_\lambda$ is impossible as $A_\lambda \neq B_\mu$. For the other cases, we have $B_\mu^* \subseteq A_\lambda = A_\lambda^{**}$ as required. The proof is completed.

In view of Theorem 4.1 and Theorem 4.2, we obtain the following version of the Kuratowski's problem on Stone space.

<u>Theorem 4.3</u> Let $\mathcal{P}$ be the finite Stone space. Then at most 10 distinct relatives can be obtained from a subset $\mathcal{A}$ of $\mathcal{P}$ by successive applications in any order of the closure and the complement operation.

Proof: We have in fact proved that $\mathcal{A}^{o-} = \mathcal{A}^{-o-}$, $\mathcal{A}^{-o} = \mathcal{A}^{o-o}$. Dually, let $\mathcal{B}$ be the set theoretical complement of the set $\mathcal{A}$, then $\mathcal{B}^{-o} = \mathcal{B}^{o-o}$, $\mathcal{B}^{o-} = \mathcal{B}^{-o-}$.

The following lattice diagram is drawn to illustrate the inclusion situations

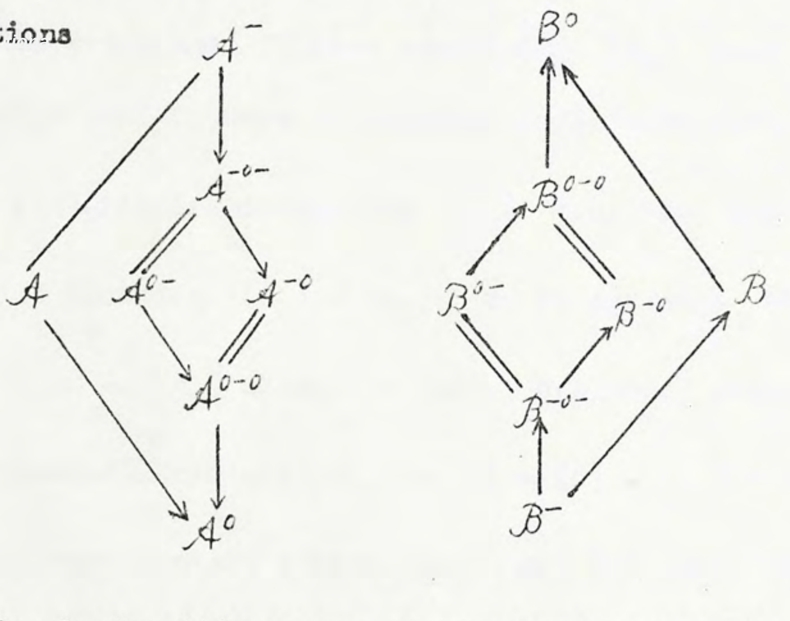

Figure 2

In closing this paper, we construct an example of Stone space $\mathcal{P}$ with a subset $\mathcal{A}$ which possesses exactly 10 distinct relatives.

<u>Example 4.4</u> Let S be a poset with the following Hasse diagram:

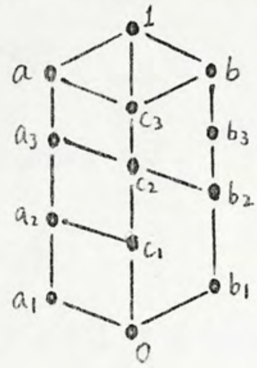

Figure 3

If the multiplication on S is defined to be the greatest lower bound (that is, $a \cdot b = a \wedge b$), then S can be easily verified to be a Stone semigroup with zero and identity.

Observe the following facts:

The prime anti-filters containing $(c_3]$ must contain $(a]$ or $(b]$; the prime anti-filters containing $(c_1]$ must contain (a_2) or $(b]$; the prime anti-filters containing $(c_2]$ has three different form: it contains $(a]$; it contains $(a_3] \cup (b_3]$; or it contains $(b]$, since $(c_2] = (a_3] \cap (b] = \{(c_2] \cup (b_3]\} \cap (a]$. Moreover, since $(0]$ belongs to all prime anti-filters and $(a_1] \cap (b] = (0] = (a_2] \cap (b_3]$, thus all prime anti-filters must contain either $(a_1]$ or $(b]$ and $(a_2]$ or $(b_3]$.

In view of the above facts, we find the set of all prime anti-filters of S as follows:

$$\mathcal{P} = \{(a_2],\ (a_2] \cup (b_1],\ (a_2] \cup (b_3],\ (a_3],\ (a_3] \cup (b_3],\ (a_3] \cup (b],$$
$$(a],\ (a] \cup (b_3],\ (a_1] \cup (b_3],\ (b],\ (b] \cup (a_1],\ (b] \cup (a_2],\ (b] \cup (a]\}$$

Now let $A_1 = (a_2] \cup (\ _1]$, $A_2 = (a_1] \cup (b_3]$, $A_3 = (b]$ and let $\mathcal{A} = \{A_1, A_2, A_3\}$.

Clearly $\mathcal{A}$ is a subset of $\mathcal{P}$. Then

(i) $\mathcal{A} = \{A_1, A_2, A_3\}$

(ii) $\mathcal{A}^- = F(\cap A_i) = F(b_1] = \{A_1, (a_2] \cup (b_3], (a_2] \cup (b], (a_3],$
$(a_3] \cup (b_3], (a_3] \cup (b], (a] \cup (b], (a] \cup (b_3], (a],$
$A_3, (b] \cup (a_1], A_2\}$

(iii) $\mathcal{A}^{o-} = \mathcal{A}^{-o-} = F(b_3] = \{(a_2] \cup (b_3], (a_2] \cup (b], (a_3] \cup (b_3],$
$(a_3] \cup (b], (a] \cup (b_3], (b] \cup (a], (b] \cup (a_1], A_3, A_2\}$

(iv) $\mathcal{A}^{-o} = \mathcal{A}^{o-o} = F'(a_2] = \{(b] \cup (a_1], A_3, A_2\}$

(v) $\mathcal{A}^{o} = F'\{(a_1] \cup (c_1]\} = \{A_3, A_2\}$.

On the other hand, let $\mathcal{B} = \mathcal{A}'$. Then we have

(i)' $\mathcal{B} = \mathcal{A}' = \{(a_2], (a_2] \cup (b_3], (a_3], (a_3] \cup (b], (a], (a] \cup (b_3],$
$(b] \cup (a_3], (b] \cup (a_1], (b] \cup (a_2], (b] \cup (a]\}$

(ii)' $\mathcal{B}^- = F(\cap A_j \mid A_j \in \mathcal{A}') = F[(a_1] \cup (c_1]] = \mathcal{B} \cup \{A_1\}$

(iii)' $\mathcal{B}^{o-} = \mathcal{B}^{-o-} = F[(\cap A_j)^{**}] = F[\{(a_1] \cup (c_1]\}^{**}\} = F(a_2] =$
$\{A_1, (a_2], (a_2] \cup (b_3], (a_3], (a_3] \cup (b_3], (a_3] \cup (b],$
$(a], (a] \cup (b_3], (a_2] \cup (b], (a] \cup (b]\}$

(iv)' $\mathcal{B}^{-o} = \mathcal{B}^{o-o} = F'[(\cap A_j)^*] = F'(b_3] = \{(a_2], (a_2] \cup (b_1], (a_3], (a]\}$

(v)' $\mathcal{B}^{o} = F'[B]$, where B is the intersection of all elements of $\mathcal{A}$. Hence $\mathcal{B}^{o} = F'(b_1] = \{(a_2)\}$.

Thus exactly 10 distinct relatives of $\mathcal{A}$ can be found in $\mathcal{P}$.

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