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The author is deeply indebted to his supervisor, Dr. K. F. Lee, for his valuable criticisms and suggestions. His constant guidance has alleviated much of the difficulties encountered in the research. Without his stimulus, the author would not have aroused his interest in plasma physics.
In this thesis, three types of collision-induced instabilities in plasmas are studied using the cold plasma fluid equations, namely the resistive electromagnetic ion cyclotron instability, the resistive lower hybrid instabilities and the resistive ion-ion hybrid instabilities. These instabilities are found to exist in collisional plasmas carrying a field-aligned current. They all owe their existence to electron-ion collisions which transfer the kinetic energy of the streaming electrons to the low frequency waves so that the growths of the waves are supported. The growth rates of the three instabilities are all proportional to the electron-ion collision frequency. The instability criterion for these instabilities are exactly the same, i.e. they all require that the electron streaming velocities be greater than the phase velocities of the waves along the static magnetic field.

The resistive electromagnetic ion cyclotron instability was studied in a recent paper by Lee and Luhmann. Our investigation here is concerned with verifying the validity of their simple analytical results using parameters representative of the UCLA arcjet plasma. We also study the effect of electron inertia on the instability and find that it is generally negligible.

The other two types of instabilities studied in this thesis are new. They resemble each other in many ways. Firstly, the equations for the finding of the real frequencies and the growth rates of these two instabilities are similar in structure. Secondly, the instability criteria are the same. Thirdly, both types are found to have instabilities occurring concurrently in duality, with different frequencies and growth rates. The frequencies of these two types of instabilities are greater than but
in the order of the lower hybrid frequency and the Buchabaum ion-ion hybrid frequency respectively. Numerical examples are given to illustrate the characteristics of these instabilities.

As the purpose of this thesis is concerned with establishing the existence and the essential features of the instabilities, the cold plasma equations are used throughout the analysis. Thus the results must be regarded as preliminary, the refinements of which (e.g. thermal effects) await future investigation.
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1-1. Current-driven Instabilities in a Collisional Plasma

Drift waves and drift instabilities have been extensively studied by a number of authors. Their importance lies in the fact that under laboratory conditions, gradients in temperature, density, static magnetic field and impurity concentration are bound to exist. Whenever a gradient appears, a plasma current or particle drift happens; drift waves and instabilities are then supported by these drifts. In addition to drifts resulting from gradients in various relevant plasma elements, we also have drifts due to curvature of the confining magnetic field, gravitational force, etc., which are difficult to get rid of. Sometimes, for some special purposes, e.g., heating, drifts in plasmas are deliberately introduced to produce ohmic heating or turbulent heating. In the topside of the ionosphere, particle motions are confined between the ionosphere and the magnetosphere by the magnetic mirror effect of the earth's magnetic field resulting in currents of different types of particles. If the electromagnetic feature of the ionosphere which affects much the radio communications, is to be displayed clearly, drift waves and instabilities in a magnetospheric plasma must first be investigated. Current-driven instabilities are therefore also the important subjects in a variety of laboratory and astrophysical situations.

In the low frequency regime, instabilities driven by current flow parallel and perpendicular to the static confining magnetic field have been treated by many authors. Electrostatic Ion Acoustic (ESIA), Electrostatic Ion Cyclotron (ESIC) and Electromagnetic Ion Cyclotron (EMIC) instabilities have been investigated extensively and are the most
common low frequency current driven instabilities. These instabilities, under certain suitable conditions, can be taken advantage of to heat up the ion or electron for fusion purpose. While many of the earlier work dealt with instabilities in collisionless plasmas, there has been some recent interest in collisional plasmas. As collision rate is proportional to the densities of the colliding particles and inversely proportional to temperature, collisions become an important factor and cannot be ignored in situations where the density is high and the temperature is low. For instance, in the ionosphere, at low and intermediate altitudes above the earth where particle densities are high and the temperature is relatively low, particle collisions must be considered; but in the topside the particle densities are so low and the temperature so high that particle encounters are rare and the plasma there is effectively collision-free. In controlled fusion studies, prior to and at the early stage of heating, the plasma may be treated as collisional since the temperature is then still quite low.

1-2. Resistive-type Instabilities

It has been shown by many authors that a relative streaming between electrons and ions in a plasma can lead to various different instabilities depending on conditions of the systems. When collisions are included, the behaviours of some of the instabilities may be altered, while on the other hand, some more new instabilities may be generated. These resistive-type instabilities occurring in collisional plasmas therefore usually fall into either of the two categories:

(i) Instabilities of which the growth rates and the critical currents may be modified by collisions, but their existence does not depend on the presence of collisions, and

(ii) Instabilities that owe their existence to collisions.
For the first type, good examples are given by Kulsrud and Shen (1966) and Stefant (1971). They found that in collisional plasmas, the ion-ion and electron-electron collisions contribute to the damping of the ion waves due to their tendency to establish their Maxwellian distributions while the electron-ion collisions facilitate the growth rates by diffusing in velocity space the slow particles which originally damp out the waves. The critical currents are found to be lowered by the electron-ion collisions but enhanced by the ion viscosity. In the absence of these collisions instabilities still occur but with a larger current threshold for instabilities.

For the second type, resistive electrostatic and electromagnetic ion cyclotron instabilities have been found to owe their existence to electron-ion collisions. The growth rates for these waves are directly proportional to electron-ion collisions; but the critical currents for instability, i.e. the particle streaming velocity necessary for supporting the instabilities, is not affected by the frequency of collisions as long as electron-ion collisions are present. This is a marked distinction between these two types of instabilities. In the absence of collisions, the waves neither grow nor damp as the imaginary part of the complex angular frequency is then zero. Therefore, instabilities do not exist if the plasma is collision-free.

In this thesis, it is intended to study the second type of instabilities which have only recently been explored by some authors.

1-3. Instabilities Studied in this Thesis

In this thesis, cold plasma models are used to investigate the frequencies and growth rates of three types of instabilities, namely, the resistive electromagnetic ion cyclotron instability, the resistive lower hybrid instability and the resistive ion-ion hybrid instability.
The first instability was recently studied by Lee and Luhmann, and our investigation here is mainly a numerical study verifying their simple analytical results. The second two instabilities are new, to the author's knowledge. The following describes briefly these instabilities.

(i) Resistive electromagnetic ion cyclotron instability

This instability occurs in finite beta plasmas ($\beta = \frac{8\pi n_o KT_o}{B_o^2}$) where the plasma pressure $n_o KT_o$ is an appreciable fraction of the confining magnetic field energy density $B_o^2/8\pi$. In finite beta plasmas, the perturbed magnetic fields are as important as the perturbed electric fields, and, therefore, must be included in the instability calculations. This is why it is called the ELECTROMAGNETIC ion cyclotron instability. The instability criterion requires that the electron streaming velocity be greater than the parallel phase velocity of the wave, i.e. $v_d > x/k_z$, the same condition imposed on the resistive ELECTROSTATIC ion cyclotron instability. However, since the frequency of the electromagnetic ion cyclotron instability is smaller than the ion cyclotron frequency and is lower than the electrostatic one, it has a lower threshold current for instability.

(ii) Resistive lower hybrid instabilities

As pointed out by many authors, in addition to some transverse waves which should be treated in electromagnetic formulation, there are some waves which are essentially longitudinal and can be treated in the electrostatic approximation. As emphasized by Rosenbluth (1964), longitudinal waves are usually more important and grow more rapidly in low beta plasmas. Since small beta simply means that it is rather difficult for the plasma to perturb the imposed magnetic field, the perturbed magnetic field and hence the curl of the perturbed electric field remain small and can be neglected. The elimination of the perturbed magnetic field greatly simplifies the mathematical solution. Therefore, in view
of the fact that lower hybrid resonance is electrostatic in nature, and as a starting point, the electrostatic approximation is made use of to find the instability, leaving the electromagnetic instability for further studies. The wave propagation, as in the case of resonance, is confined to a small region nearly perpendicular to the static magnetic field, i.e. \( \frac{k_z}{k_x} \lesssim (m_e/m_i)^{1/2} \). Instabilities are found to exist in duality in most cases, the one with a frequency closer to the lower hybrid frequency having a larger growth rate and a lower threshold current.

(iii) Resistive ion-ion hybrid instabilities

Just as for the resistive lower hybrid instabilities, electrostatic waves are assumed for the first study of this type of instability. The electromagnetic case is left for further studies. The plasma possesses streaming electrons and two stationary ion species of different charge-to-mass ratios. It is found that waves become unstable when the electron streaming velocity exceeds the phase velocity of the wave along the static magnetic field. Just like the lower hybrid instability, unstable waves exist in duality. The one with a frequency closer to the ion-ion hybrid frequency has a larger growth rate and a lower threshold current.

1-4. Relevant Equations

Throughout this thesis, the fluid model is used to find the dispersion relations of the above-mentioned instabilities. The adoption of the fluid equations is justified as we are dealing with high density, low temperature plasmas. The high density manifests the collective behaviour of the plasma and the low temperature reduces the thermal spread of the particles in the velocity space so that they move as a single fluid element. Hence, we can use the two- or many-component fluid equations to describe the plasma systems.
Let us neglect for simplicity electron viscosity owing to the low frequencies involved. The linearized equations of continuity and momentum transfer for electrons are then

\[ \frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{u}_e + n_{oe} \vec{v}_e) = 0 \quad \text{(1)} \]

\[ \frac{\partial \vec{v}_e}{\partial t} + \frac{\partial}{\partial x_e} (n_e \vec{u}_e \cdot \nabla) \vec{v}_e = -e(\vec{E} + \frac{\vec{v}_e \times \vec{B}_0}{c}) + \frac{\vec{u}_e \times \vec{B}}{c} - \frac{m_e}{e} \vec{v}_e \times \vec{B} - \frac{1}{n_{oe}} K T \frac{\nabla n_e}{n_e} \quad \text{(2)} \]

where \( \nu_{ei} \) is the electron-ion collision frequency,

\( K \) is the Boltzmann constant,

\( \vec{E}, \vec{B} \) are the perturbed electric and magnetic fields,

\( \vec{v}_e, n_e \) are the perturbed velocity and density of the electrons respectively,

\( e \) is the magnitude of the electron charge,

\( T_e, m_e \) are the temperature and mass of the electron respectively,

\( c \) is the velocity of light,

\( \vec{B}_0 \) is the confining static magnetic field density,

\( \vec{u}_e \) is the zero-order drift of the electrons in the equilibrium state, and

\( n_{oe} \) is the equilibrium density of the electrons.

For the ions, as the ion Larmor radius to perpendicular wavelength is appreciable, we include the effect of ion viscosity. The linearized equations of continuity and momentum transfer for the ions are therefore:

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (n_{oi} \vec{v}_i) = 0 \quad \text{(3)} \]
where \( n_i, \vec{v}_i \) are the perturbed density and velocity of the ion respectively,
\( n_{oi}, T_i \) are the equilibrium density and temperature of the ion respectively,
\( \tau_i \) is the collisional part of the stress tensor which is a function of the ion temperature,
\( \mu \) is the viscosity given by the equation
\[
\mu = \frac{4}{3} n_o K_i \nu_{ii}
\]
\( \nu_{ii} \) is the ion-ion collision frequency
\( \Omega_{ci} \) is the ion cyclotron frequency

When two ion species are present in the plasma, four equations are obtained with one equation of continuity and one equation of motion for each ion species.

In the case of a cold plasma, the electron and ion temperatures are all equal to zero and Eqs. (2) and (4) become

\[
\frac{m_i}{\beta t} \frac{\partial \vec{v}_i}{\partial t} = e(\vec{E} + \frac{\vec{v}_i \times \vec{B}}{c}) \quad \text{------- (5)}
\]
\[
\frac{m_e}{\beta t} \frac{\partial \vec{v}_e}{\partial t} + m_e (\vec{u}_e \cdot \vec{\nabla}) \vec{v}_e = -e(\vec{E} + \frac{\vec{v}_e \times \vec{B}}{c} + \frac{\vec{d}_e \cdot \vec{B}}{c}) - m_e \nu_{ei} \vec{v}_e \quad \text{------- (6)}
\]

Eqs. (1) and (3) are unchanged.

An inspection of the equations will find that there are more unknown variables than given equations. Hence, in order to close the system, some more equations must be used. For the electromagnetic instabilities, Maxwell's two curl equations can be taken advantage of to
serve the purpose. These equations are:

\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \text{(7)} \]

\[ \nabla \times \vec{B} = \frac{4\pi}{c} \sum_j q_j (n_{oj} \vec{v}_j + n_{ij} \vec{u}_j), \quad j = e,i \quad \text{(8)} \]

where in equation (8), the displacement current term has been dropped because of the low frequencies concerned.

For the electrostatic instabilities, the perturbed magnetic field is set to zero. To close the system, we may use the scalar potential function and the Poisson's equation:

\[ \vec{E} = -\nabla \phi \quad \text{(9)} \]

\[ \nabla^2 \phi = -\frac{4\pi}{c} \sum_j q_j n_j, \quad j = e,i \quad \text{(10)} \]

where \( \phi \) is the perturbed scalar potential function.

With the above equations, we can now find the solutions in the time and geometric domains. Since we are, however, interested in the location of the instabilities in the frequency spectrum and the wave number space, Fourier and Laplace transformations must be used to convert the variables from time and geometric domains into frequency and wave number domains. After these transformations, we have, for perturbations of the form \( \exp i(\vec{k} \cdot \vec{r} - \omega t) \), the following equivalences:

\[ \nabla \times \vec{A}(\vec{r},t) \longrightarrow i\vec{k} \times \vec{A}(\vec{k},\omega) \quad \text{(11a)} \]

\[ \nabla \cdot \vec{A}(\vec{r},t) \longrightarrow i\vec{k} \cdot \vec{A}(\vec{k},\omega) \quad \text{(11b)} \]

\[ \frac{\partial \vec{A}(\vec{r},t)}{\partial t} \longrightarrow -i\omega \vec{A}(\vec{k},\omega) \quad \text{(11c)} \]

From the given equations and the three conversions above, it
is a straightforward procedure to obtain the dispersion relation relating $k$ and $\omega$, i.e.

$$D(k, \omega) = 0 \quad \text{(12)}$$

As the plasma systems studied in this thesis are assumed to be homogeneous and infinite, we are interested only in the absolute or non-convective instabilities which propagate in all spatial directions and grow at any particular location without limit for all time $t \rightarrow \infty$. The convective instabilities which propagate in a distinct direction in space as they grow with time are not considered here. Hence, in order to find the existence of any absolute instabilities, we let $\omega = \chi + iy$ and assume $|y| \ll \chi$ for small perturbations so that linear theory can be adopted. Substituting this into the dispersion relation, it is now possible to obtain equations for the real frequency, $\chi$, and the growth rate, $y$, of any instabilities that can be found.

1-5. Outline of Thesis

As mentioned in section 1-3, three types of instabilities are to be discussed in this thesis. Chapter 2 concerns the electromagnetic ion cyclotron instability. The main objective is to investigate whether the assumptions made in a paper by Lee and Luhmann to obtain simple analytical results are valid; and if they really are, what are the ranges of validity in the $k$ space. The relative magnitudes of the three components of the perturbed electric field are also calculated. It is found that the $y$ component is indeed small compared to the other two components except for a small range near the boundary of occurrence of instabilities. Besides, the effect of the electron inertia term which was originally neglected, is also studied. It is found that it has in general negligible effect on the ranges of validity and results.

In chapter three, the resistive lower hybrid instabilitiesfor
a fully-ionized, collisional magnetoplasma are studied. The dispersion relation for a quasi-transversely propagating lower hybrid wave is first verified. When relative streaming and electron-ion collisions are included, current-driven instabilities near the lower hybrid frequency are found to exist. It is interesting to note that the criterion for the occurrence of instabilities is just the same as that for the electrostatic and electromagnetic ion cyclotron instabilities. For the lower hybrid instabilities to exist, it requires that the relative streaming velocities be greater than the parallel phase velocities of the waves, i.e.

\[ v_d > \frac{x}{k_z} \]

The frequencies of the unstable waves are discussed qualitatively. Numerical determinations of the waves using typical plasma and magnetic field parameters are given for illustration.

In chapter four, the ion-ion current-driven instabilities in a plasma consisting of two ion species are discussed. The dispersion relation is verified by setting some of the parameters to zero to get results already proved by other authors. It is found that the resistive ion-ion instabilities have many similar characteristics as the resistive lower hybrid instabilities, e.g. two unstable waves usually occur simultaneously, the instability criterion requires that the electron streaming velocities be greater than the parallel phase velocities of the waves, the growth rates are directly proportional to the electron-ion collision frequency so that in the absence of collisions, no instability generates, etc. Numerical examples for the finding of the frequencies and the growth rates of the waves are given at the end.

In chapter five, the thesis ends with a brief summary and some discussions.
CHAPTER TWO

THE RESISTIVE ELECTROMAGNETIC ION CYCLOTRON INSTABILITY

2-1. Review of the Electromagnetic Ion Cyclotron Instability

In a recent paper, Lee and Luhmann discussed the stability of obliquely propagating low frequency electromagnetic waves in a fully-ionized collisional plasma carrying a field-aligned current. They pointed out that if electron inertia and displacement current were neglected when compared to the electron momentum change during collisions with the ion and the conduction current respectively, the electric fields of small amplitude perturbations of the form \( \exp(i(k_x x + k_y y - \omega t)) \) satisfied the wave equations

\[
\begin{align*}
R_{xx} E_x + R_{xy} E_y + R_{xz} E_z &= 0 \quad (1a) \\
R_{yx} E_x + R_{yy} E_y + R_{yz} E_z &= 0 \quad (1b) \\
R_{zx} E_x + R_{zy} E_y + R_{zz} E_z &= 0 \quad (1c)
\end{align*}
\]

where the coefficients \( R_{xx} \) etc. are given by the following equations:

\[
\begin{align*}
R_{xx} &= k_z^2 - \frac{\omega \omega_{pi}^2}{c^2 \Delta_i} - \frac{i \nu_{pi} \frac{\Lambda_e}{\Delta_i}}{c^2 \Delta_e} \quad (2a) \\
R_{xy} &= \frac{-i \omega^2 \nu_{pi} \Omega_{ci} (1 - \frac{b}{2}) \Delta_i}{c^2 \Delta_e} + \frac{i \omega^2 \nu_{pe} \Omega_{ce} \Lambda_e}{c^2 \Delta_i} \quad (2b) \\
R_{xz} &= -k_x k_z \frac{i \nu_{pi} \nu_{pe} \Lambda_i}{c^2 \Delta_e} - 2 k_x \nu_{pe} \left\{ k_z \Omega_i + k_y \Omega_{ci} \right\} \quad (2c) \\
R_{yx} &= \frac{i \omega^2 \nu_{pi} \Omega_{ci} (1 - \frac{b}{2}) \Delta_i}{c^2 \Delta_e} - \frac{i \omega^2 \nu_{pe} \Omega_{ce} \Lambda_e}{c^2 \Delta_i} \quad (2d) \\
R_{yy} &= k^2 + \frac{\omega \omega_{pi}^2}{c^2 \Delta_i} (-\omega + \frac{A_{bi}}{\omega} \frac{b}{4}) + \frac{i \nu_{pi} \Omega_{bi}}{c^2 \Lambda_i \Delta_e} \quad (2e)
\end{align*}
\]
In Eqs (2) the following definitions have been used:

\[ \omega_{pe}^2 = \frac{4 \pi n_0 e^2}{m_e} \quad ; \quad \omega_{pi}^2 = \frac{4 \pi n_0 e^2}{m_i} \quad (3a) \]

\[ \Delta_i = (\Omega_{ci}^2 - \omega^2) - \frac{1}{2} i \omega b_i \Omega_i \quad (3b) \]

\[ \Delta_i' = \Delta_i (1 + \frac{ib \Omega_i}{4 \omega}) \quad (3c) \]

\[ \Delta_e = \frac{\Omega_{ce}^2}{\bar{\omega}} + \frac{i b \Omega_{ci} \Omega_i}{4 \omega} \quad (3d) \]

\[ \tilde{\omega} = \omega - k_z v_d \quad ; \quad \omega' = \omega (1 + \frac{ib \Omega_i}{4 \omega}) \quad (3e) \]

\[ v_{te}^2 = \frac{K T_e}{m_e} \quad ; \quad v_{ti}^2 = \frac{K T_i}{m_i} \quad (3f) \]

\[ \Omega_{te} = \frac{e B_o}{m_e c} \quad ; \quad \Omega_{ti} = \frac{e B_o}{m_i c} \quad (3g) \]
In the cold plasma limit, we have

\[ v_{te} = v_{ti} = r_L = b = 0 \]

and Eqs. (2) become

\[
R_{xx} = k_x^2 - \frac{\omega_{pi}^2}{c^2(\Omega_{ci}^2 - \omega^2)} - \frac{i\nu_{ei}^2\omega_{pe}^2}{c^2\Omega_{ce}^2} \quad \text{----- (4a)}
\]

\[
R_{xy} = -R_{yx} = \frac{-i\omega_{pi}\Omega_{ci}\omega}{c^2(\Omega_{ci}^2 - \omega^2)} + \frac{i\omega_{pe}^2}{c^2\Omega_{ce}^2} \quad \text{----- (4b)}
\]

\[
R_{xz} = R_{zx} = -k_xk_z - \frac{i\nu_{ei}k_x\nu_{pe}^2}{c^2\Omega_{ce}^2} \quad \text{----- (4c)}
\]

\[
R_{yy} = k_y^2 - \frac{\omega_{pi}^2}{c^2(\Omega_{ci}^2 - \omega^2)} \quad \text{----- (4d)}
\]

\[
R_{yz} = -R_{zy} = -\frac{ik_x\nu_{pe}^2}{c^2\Omega_{ce}^2} \quad \text{----- (4e)}
\]

\[
R_{zz} = k_z^2 - \frac{i\omega_{pe}^2}{c^2\nu_{ei}^2} - \frac{i\nu_{ei}k_x^2\nu_{pe}^2}{c^2\Omega_{ce}^2} \quad \text{----- (4f)}
\]

The dispersion relation giving non-trivial solutions is obtained from Eqs. (1) as

\[
\begin{vmatrix}
    R_{xx} & R_{xy} & R_{xz} \\
    R_{yx} & R_{yy} & R_{yz} \\
    R_{zx} & R_{zy} & R_{zz}
\end{vmatrix} = 0 \quad \text{----- (5)}
\]

Guided by experimental measurement on the UCLA arcjet plasma, which indicated that waves excited around the ion cyclotron frequency
had a small $E_y$, Lee and Luhmann simplified Eq (5) to the following form

$$\begin{vmatrix} R_{xx} & R_{xz} \\ R_{zx} & R_{zz} \end{vmatrix} = 0$$  \quad (6)$$

The approximation that $E_y$ is negligible for waves propagating almost perpendicular to the magnetic field has been employed by Perkins, when he addressed himself to the problem of ion counterstreaming instabilities in collisionless plasmas. It enables simple analytical solutions to the complicated dispersion relation.

Using the reduced dispersion relation and some simplifying assumptions, the real and imaginary parts of the most fundamental solution corresponding to the case of a cold plasma, were found to be

$$x = \text{Re} \omega = \frac{ckzq_i}{(c^2k_z^2 + \omega_p^2)^{1/2}}$$  \quad (7a)$$

$$y = \text{Im} \omega = -\frac{\nu_e^2c^2k_x^2m_e}{2m_i(c^2k_z^2 + \omega_p^2)} \left(1 - \frac{k_zv_d}{x}\right)$$  \quad (7b)$$

Instability occurs for waves whose parallel phase velocity is smaller than the electron drift velocity, i.e. $x/k_z < v_d$, as $y$ then becomes positive. The frequency of the instability given by Eq (7a) is less than but of the order of the ion cyclotron frequency while the growth rate given by Eq (7b) is directly proportional to the electron-ion collision frequency.

The simple results (7a) and (7b) were obtained from the reduced dispersion relation (6), and not from the full dispersion relation (5). In Ref. 1, there was no discussion on the validity of (6), nor was it shown that the simple analytical results obtained were
consistent with the various assumptions upon which they were based. The main purpose of this chapter is to obtain the conditions of validity for equation (6) and to present some numerical calculations to determine the region of validity in wave number space for results (7a) and (7b) using plasma and magnetic field parameters representative of the UCLA arcjet plasma. The effect of electron inertia is discussed at the end of this chapter.

2-2. Conditions for Validity of the Analytical Results

In order to obtain the reduced dispersion relation (6), we have to eliminate $E_y$ from Eqs. (1) to get

\begin{align}
(R_{xx} - \frac{R_{xy} R_{yx}}{R_{yy}})E_x + (R_{xz} - \frac{R_{xy} R_{yz}}{R_{yy}})E_z &= 0 \quad (8a) \\
(R_{zx} - \frac{R_{yz} R_{zy}}{R_{yy}})E_x + (R_{zz} - \frac{R_{zy} R_{yz}}{R_{yy}})E_z &= 0 \quad (8b)
\end{align}

and assume that

\begin{align}
(i) \quad R_1 &= \left| \frac{R_{xy} R_{yx}}{R_{xx} R_{yy}} \right| \ll 1 \quad (9a) \\
(ii) \quad R_2 &= \left| \frac{R_{xy} R_{yz}}{R_{xz} R_{yy}} \right| = \left| \frac{R_{yx} R_{zy}}{R_{xz} R_{yy}} \right| \ll 1 \quad (9b) \\
(iii) \quad R_3 &= \left| \frac{R_{zy} R_{yz}}{R_{zz} R_{yy}} \right| \ll 1 \quad (9c)
\end{align}

Eqs. (8) then reduce to

\begin{align}
R_{xx} E_x + R_{xz} E_z &= 0 \quad (10a) \\
R_{zx} E_x + R_{zz} E_z &= 0 \quad (10b)
\end{align}
Non-trivial solutions are obtained by setting the determinant of the coefficients of Eqs. (10) to zero which gives the reduced dispersion relation (6). Hence, for (6) to be a good approximation to (5), the results obtained must be consistent with inequalities (9a)-(9c). As the element $R_{yy}$, which is a function of $k_x$, appears in the denominations of all three ratios $R_1$, $R_2$ and $R_3$, we can surely increase $k_x$ to make $R_{yy}$ as large as we please. Thus, for a given $k_z$, the three inequalities (9a)-(9c) will be satisfied for waves with large $k_x$, but may not hold for small values of $k_x$. The value of $k_x$ separating the region of validity and the region of invalidity can be determined numerically for a given set of plasma and magnetic field parameters.

In addition to (9a)-(9c), the fundamental solution given by (7a) and (7b) is based on some further assumptions. For a cold plasma, when $R_{xx}$, $R_{xz}$, $R_{zx}$ and $R_{zz}$ are substituted into (6), it gives a 5th order algebraic equation in $\omega$ which is still very complicated. Lee and Luhmann therefore further simplified the problem by assuming

\begin{align*}
R_4 &= \left| \frac{k_z^2 - \frac{\omega^2 \omega_{pi}}{c^2 (\Omega_{ci}^2 - \omega^2)}}{i \nu \omega \frac{\omega_{pe}}{c \Omega_{ce}}} \right| \ll 1 \quad \text{----- (11a)} \\
R_5 &= \left| \frac{k_x k_z}{\frac{i \nu \omega_{pe}}{c \Omega_{ce}}} \right| \ll 1 \quad \text{----- (11b)} \\
R_6 &= \left| \frac{k_x^2 - \frac{i \omega \omega_{pe}}{\nu_{ei} \nu_{e}}} \right| \ll 1 \quad \text{----- (11c)}
\end{align*}

Then the elements of Eq. (6) were given by

\begin{align*}
R_{xx} &= k_x^2 - \frac{\omega^2 \omega_{pi}}{c^2 (\Omega_{ci}^2 - \omega^2)} \quad \text{----- (12a)} \\
R_{xz} &= R_{zx} = - k_x k_z \quad \text{----- (12b)}
\end{align*}
Now the dispersion relation (6) becomes a quadratic equation in \( \omega \), which, on letting \( \omega = x + iy \) and assuming \( |y| \ll x \), (7a) and (7b) are obtained. Hence, for (7a) and (7b) to be valid, the six inequalities (9a)-(9c), (11a)-(11c) must be satisfied simultaneously.

In fact, similar results can be obtained without the help of (11c). If we substitute (12a), (12b) and (4f) into (6), we obtain

\[
\omega^4 \left( c^2 k_z \omega_p^2 \omega_e^2 \Omega_{ce}^2 c_{ci}^2 \right) + i c^2 k_z^2 \omega_e^2 \Omega_{ci}^2 \nu_{ei} \omega^2 + \omega^2 \left( c^2 k_z^2 + \omega_p^2 \right) \nu_{ei}^2 \omega_e^2 \omega_p^2 k_v^2 v_d^2 - c^2 k_z^2 \omega_e^2 \Omega_{ce}^2 - ic^2 k_z^2 \omega_e^2 \nu_{ci} \omega^2 k_v^2 v_d^2 - c^2 k_z^2 \omega_e^2 \nu_{ei}^2 \Omega_{ci}^2 x_v^2 = 0 \quad \text{(13)}
\]

If we let \( \omega = x + iy \) and assume \( |y| \ll x \), we obtain, on letting the real and imaginary parts separately to zero, the following solutions

\[
y = - \frac{c^2 k_z^2 \nu_{ei}^2}{4m_i (c^2 k_z^2 + \omega_p^2)} (1 - \frac{k_z v_d}{x}) \quad \text{(14a)}
\]

\[
x_1 = \frac{c k_z \Omega_{ci}}{(c^2 k_z^2 + \omega_p^2)^{1/2}} \quad \text{(14b)}
\]

\[
x_2 = - \frac{c k_z \Omega_{ci}}{(c^2 k_z^2 + \omega_p^2)^{1/2}} \quad \text{(14c)}
\]

\[
x_3 = i \frac{\nu_{ei} k_x v_d}{\Omega_{ce}} \quad \text{(14d)}
\]

and

\[
x_4 = -i \frac{\nu_{ei} k_x v_d}{\Omega_{ce}} \quad \text{(14e)}
\]

As the real frequency, \( x \), should be a positive real number, \( x_1 \) is the
correct solution. Note that Eq. (14b) is identical to (7a) while (14a) differs from (7b) by only a factor of 1/2. Since (14a) is obtained without assuming (11c), it is more accurate than (7b). The solutions given by (14a) and (14b) are valid for $k_x$ and $k_z$ satisfying the five inequalities (9a)-(9c) and (11a)-(11b).

2-3. Numerical Determination of Regions of Validity

For numerical calculations determining the regions of validity in $k$ space of solutions (14a) and (14b), we take plasma and magnetic field parameters representative of the UCLA arcjet plasma:

$$n_0 : \quad 3 \times 10^{13} \quad - \quad 3 \times 10^{14} \quad \text{cm}^{-3}$$

$$B_0 : \quad 1 \quad - \quad 4 \quad \text{kG}$$

$$v_d : \quad 2.8 \times 10^6 \quad - \quad 2.8 \times 10^7 \quad \text{cm/sec}$$

$$m_e/m_i = 1/7344 \quad \text{(helium plasma)}$$

$$kT_e = 3 \quad \text{eV}$$

Using the formulas given by Eqs. (3a) and (3g) and the following

$$\nu_{ei} = \frac{n_0 \ln \Lambda}{0.35 T_e^{3/2}} \quad \text{-----} \quad (15a)$$

$$\ln \Lambda = 9.42 + 1.5 \ln T_e - 0.51 \ln n_0 \quad \text{-----} \quad (15b)$$

and constants

$$e = 4.8 \times 10^{-10} \quad \text{e.s.u.}$$

$$c = 3 \times 10^{10} \quad \text{cm/sec}$$

$$m_e = 9.11 \times 10^{-28} \quad \text{gm}$$

$$m_i = 6.69 \times 10^{-24} \quad \text{gm}$$
we get the ranges of electron and ion plasma frequencies, electron and ion cyclotron frequencies, and the electron-ion collision frequency as follows:

$$\begin{align*}
\omega_{pe} & : \quad 3.09 \times 10^{11} \quad - \quad 9.76 \times 10^{11} \quad \text{sec}^{-1} \\
\omega_{pi} & : \quad 3.61 \times 10^{9} \quad - \quad 1.14 \times 10^{10} \quad \text{sec}^{-1} \\
\Omega_{ce} & : \quad 1.75 \times 10^{10} \quad - \quad 6.99 \times 10^{10} \quad \text{sec}^{-1} \\
\Omega_{ci} & : \quad 2.38 \times 10^{6} \quad - \quad 9.52 \times 10^{6} \quad \text{sec}^{-1} \\
\nu_{ei} & : \quad 1.18 \times 10^{8} \quad - \quad 1.18 \times 10^{9} \quad \text{sec}^{-1}
\end{align*}$$

To study the resistive electromagnetic ion cyclotron instability, we compute, for each combination of $n_o$, $B_o$, and $v_d$, the real and imaginary parts of frequency according to (14a) and (14b) for $k_z$ from 0.01 to 1.0 cm$^{-1}$ and $k_x$ from 1.0 to 12.0 cm$^{-1}$. The results obtained are then used to determine the ratios $R_1 - R_5$. For the region in k space in which these five quantities are all less than 0.1, inequalities (9a)-(9c) and (11a)-(11b) are satisfied and hence solutions (14a) and (14b) are valid. It is found that the four ratios $R_2 - R_5$ are all less than 0.1 in the range of $k_z$ and $k_x$ given above. The ratio $R_1$ on the other hand, can be larger than 0.1 for certain values of $k_x$ and $k_z$ within this range. The curve $R_1 = 0.1$ in k space can therefore be taken as the boundary separating the region of validity from the region of invalidity for solutions (14a) and (14b).

The program for the numerical determination of region of validity is given in Appendix A, and the results are depicted in Figs. 2-1 to 2-3. Figure 2-1 shows the $R_1 = 0.1$ curves for $n_o = 1.1 \times 10^{14}$ cm$^{-3}$, $T_e = 3$ eV, $v_d = 1.15 \times 10^7$ cm/sec, and several values of $B_o$. The area above each curve corresponds to $R_1 < 0.1$ while the area below corresponds
to $R_1 > 0.1$. It is seen that, for each value of $B_0$, there is a large region in $k$ space for which (14a) and (14b) are valid. It is interesting to examine the particular set of parameters quoted in Refs. 2 and 3 for the experimental excitation of the EMIC waves: $n_0 = 1.1 \times 10^{14}$ cm$^{-3}$, $T_e = 3$ eV, $B_0 = 1.4$ KG, $v_d = 1.11 \times 10^7$ cm/sec ($0.76v_A$ where $v_A =$ Alfven velocity), and $k_x = 2$ cm$^{-1}$. From the $B_0 = 1.4$ KG curve shown in Fig. 2-1, we see that, for $k_x = 2$ cm$^{-1}$, $k_z$ must be smaller than 0.13 cm$^{-1}$ and larger than 0.29 cm$^{-1}$ for (14a) and (14b) to be valid. Figure 2-2 illustrates the effect of varying $n_0$ on the $R_1 = 0.1$ curve while the effect of varying $v_d$ is shown in Fig. 2-3. As in Fig. 2-1, the area above each curve signifies $R_1 < 0.1$.

From these figures, we see that the analytical results obtained in Ref. 1 are accurate when $k_x$ is large, i.e. when the wave propagates nearly perpendicular to the static magnetic field.

2-4. Verification of the Smallness of the Y Component of the Electric Field

As mentioned in section 2, Lee and Luhmann deduced the reduced dispersion relation by eliminating the $y$ component of the electric field, which was found experimentally small compared to the other two components from the full dispersion relation. While the relative magnitudes of the three components of the electric field do not affect the validity of the analytical results, it is of interest to calculate the ratios of the components of the electric field. Eliminating $E_y$ from Eqs. (1a) and (1b), we get

$$\frac{E_x}{E_z} = \frac{R_{xy} R_{yz} - R_{xz} R_{yy}}{R_{xx} R_{yy} - R_{yx} R_{yx}}$$  \hspace{1cm} (16a)
Figure 2-1. The boundary line $R_1=0.1$ for a helium plasma with $n_0=1.1 \times 10^{14}$ cm$^{-3}$, $T_e=3$ ev, $v_d=1.15 \times 10^7$ cm/sec and several values of $B_0$. In each case, all of k space above the line corresponds to $R_1 < 0.1$ and that below the line corresponds to $R_1 > 0.1$. 

$B_0 = 4$ KG 

$B_0 = 1.4$ KG 

$B_0 = 2$ KG 

$B_0 = 1$ KG
Fig. 2-2 The boundary line $R_1=0.1$ for a helium plasma with $B_0=1.4$KG, $T_e=3$eV, $v_d=1.5	imes10^7$cm/sec (0.8$v_A$) and several values of $n_0$ between $3\times10^{13}$ and $3\times10^{14}$cm$^{-3}$. The area above each line signifies $R_1 < 0.1$. 

INCREASING $n_0$
Fig. 2-3 The boundary line $R_1=0.1$ for a helium plasma with $B_0=1.4$ KG, $T_e=3$ eV, $n_o=1.1 \times 10^{14} \text{cm}^{-3}$, and several values of $v_d$ between $2.88 \times 10^6$ and $2.88 \times 10^7$ cm/sec. The area above each line signifies $R_1 = 0.1$. 

$v_d = 1.44 \times 10^7$ cm/sec ($v_A$) 

$8.64 \times 10^6$ cm/sec ($0.6v_A$) 

$2.02 \times 10^7$ ($1.4v_A$) 

$2.88 \times 10^6$ ($0.2v_A$) 

$2.88 \times 10^7$ ($2v_A$)
The other two ratios can be obtained from Eqs. (1b) and (1c) as given below

\[ \frac{E_y}{E_x} = \frac{R_{yx}}{R_{yy}} - \frac{R_{yz}}{R_{yy}} \cdot \frac{E_z}{E_x} \]  \hspace{1cm} (16b)

\[ \frac{E_y}{E_z} = \frac{R_{yx}}{R_{yy}} \cdot \frac{E_x}{E_z} - \frac{R_{yz}}{R_{yy}} \]  \hspace{1cm} (16c)

The program for the finding of the electric field ratios is given in Appendix B. The results for a plasma with \( B_0 = 1.4 \) KG, \( n_0 = 1.1 \times 10^{14} \) cm\(^{-3} \), \( T_e = 3 \) eV and \( v_d = 1.15 \times 10^7 \) cm/sec (0.8VA) are shown in Fig. 2-4, wherein the three electric field ratios are plotted as a function of \( k_z \) parametric in \( k_x \). It is seen that \( E_x \) is the dominant component. The ratio \( \left| \frac{E_y}{E_x} \right| \) is always very small. The ratio \( \left| \frac{E_y}{E_z} \right| \) is also smaller than 0.1 except for a narrow range of \( k_z \) centered around 0.17 cm\(^{-1} \). As \( k_x \) increases, both \( \left| \frac{E_y}{E_z} \right| \) and \( \left| \frac{E_y}{E_x} \right| \) decrease while \( \left| \frac{E_x}{E_z} \right| \) increases. Referring to Fig. 2-1, we see that, for \( B_0 = 1.4 \) KG, the \( R_1 = 0.1 \) curve, which separates the region of validity and the region of invalidity of the reduced dispersion relation (6), also exhibits a peak around 0.17 cm\(^{-1} \). For values of \( k_z \) lying above the \( R_1 = 0.1 \) curve, \( E_y \) is small compared to both \( E_z \) and \( E_x \). These results appear to lend support to the statement made in Ref. 21 that waves satisfying the reduced dispersion relation (6) have a negligible \( E_y \). The electric field ratios for other combinations of \( n_0, B_0, \) and \( v_d \) within the ranges given in section 2-3 have also been computed. In all cases, it is found that

(i) \( \left| \frac{E_x}{E_z} \right| > 1 \)

(ii) \( \left| \frac{E_y}{E_x} \right| \ll 1 \)

(iii) \( \left| \frac{E_y}{E_z} \right| < 1 \), except for a narrow range of \( k_z \).
Fig. 2-4  The electric field ratios as a function of $k_z$ and two values of $k_x$ for a plasma with $B_o=1.4$ kG, $n_o=1.1 \times 10^{14}$ cm$^{-3}$, $T_e=3$ eV & $v_d=1.15 \times 10^7$ cm/s
Fig. 2-5 The electric field ratio $|E_y|/|E_z|$ as a function of $k_z$ for $k_x=5\text{cm}^{-1}$ and several values of $B_0$ in a plasma with $n_0=1.1\times 10^{14}\text{ cm}^{-3}$, $T=3\text{eV}$ and $v_d=1.15\times 10^7\text{cm/sec (0.8}v_A)$. 
Fig. 2-5 shows the curves of \( \frac{E_y}{E_z} \) versus \( k_z \) for \( k_x = 5 \text{ cm}^{-1} \) and several values of \( B_0 \) in a plasma with the same \( n_o, T_e \) and \( v_d \) as Fig. 2-4. Their similarity to the \( R_1 = 0.1 \) curves of Fig. 2-1 is striking.

In conclusion, \( E_y \) is indeed very small compared to \( E_x \) and \( E_z \); the differences become larger when \( k_x \) increases.

2-5. Effect of Electron Inertia

In Ref. 21, electron inertia was dropped due to the low frequency involved. To see whether it is really unimportant, the term for the electron inertia is included in the force equation which is then solved for instability. The force equation now becomes

\[
-i\omega m_e \vec{v}_e + ik_z v_d \vec{v}_e = -e \left( \frac{\vec{E}}{c} + \frac{\vec{v}_e \times \vec{B}_0}{c} + \frac{\vec{v}_d \times \vec{B}}{c} \right) - m_e \nu_{ei} \vec{v}_e \quad - (17)
\]

which can be rewritten as follows:

\[
0 = -e \left( \frac{\vec{E}}{c} + \frac{\vec{v}_e \times \vec{B}_0}{c} + \frac{\vec{v}_d \times \vec{B}}{c} \right) - m_e \nu'_{ei} \vec{v}_e \quad ----- (18)
\]

where \( \nu'_{ei} = \nu_{ei} - i\omega + ik_z v_d \) \quad ----- (19)

Eq. (18) has the same form as (1b) in Ref. 21. The only difference is that \( \nu'_{ei} \) given by (19) now replaces \( \nu_{ei} \). The other equations in Ref. 21 are unaffected. Using the six assumptions, the dispersion relation therefore turns out to be

\[
\left\{ k_z^2 - \frac{\omega^2 \omega_{pe}^2}{c^2(\alpha_{ci}^2 - \omega^2)} \right\} \left\{ k_x^2 - \frac{i\omega \omega_{pe}^2}{c^2\nu_{ei} - i\omega + ik_z v_d} \right\} - k_x k_z^2 = 0 \quad - (20)
\]

After expanding and rearranging, it gives

\[
i(c^2 \omega_{pe}^2 k_z^2 + c^2 \omega_{pi}^2 k_x^2 + \omega_{pe}^2 \omega^2 k_x^2 \nu_{ei} - (c^2 \omega_{pi}^2 k_x k_z v_d) \omega) +
\]}
\[ c^2 \omega^2_{\text{pi}} k_x^2 \nu_{ei} k_z v_d + i(c^2 \omega^2_{\text{pe}} k_x^2 k_z^2 - c^2 \omega^2_{\text{pe}} \Omega_{\text{ci}} k_z^2) = 0 \]  \hspace{1cm} (21)

Using the same technique for small perturbation, we get two equations for the growth rate and the real frequency of the instability as below:

\[-(c^2 \omega^2_{\text{pe}} k_z^2 + c^2 \omega^2_{\text{pi}} k_x^2 + \omega^2_{\text{pe}} \omega^2_{\text{pi}}) 2xy - c^2 \omega^2_{\text{pi}} k_x x^0 \]

\[ c^2 \omega^2_{\text{pi}} k_x^2 \nu_{ei} k_z v_d = 0 \]  \hspace{1cm} (22)

which when solving for \( y \) gives

\[
y = \frac{\nu_{ei} c^2 \omega^2_{\text{pi}} k_x^2}{2(c^2 \omega^2_{\text{pe}} k_z^2 + c^2 \omega^2_{\text{pi}} k_x^2 + \omega^2_{\text{pe}} \omega^2_{\text{pi}})} \left( \frac{k_z v_d}{x} - 1 \right)
\]

\[
= -\frac{\nu_{ei} c^2 k^2_{m e}}{2m_i (c^2 k_z^2 + c^2 k_x^2 m_e / m_i + \omega^2_{\text{pe}})} \left( 1 - \frac{k_z v_d}{x} \right) \hspace{1cm} (23)
\]

and

\[
(c^2 \omega^2_{\text{pe}} k_z^2 + c^2 \omega^2_{\text{pi}} k_x^2 + \omega^2_{\text{pe}} \omega^2_{\text{pi}}) x^2 - 2c^2 \omega^2_{\text{pi}} k_x k_z v_d x + c^2 \omega^2_{\text{pi}} k_x^2 k_z^2 v_d^2
\]

\[- c^2 \omega^2_{\text{pe}} \Omega_{\text{ci}}^2 k_z^2 = 0 \]  \hspace{1cm} (24)

which can be solved to get

\[
x = \frac{c^2 \omega^2_{\text{pi}} k^2_{x} k_z^2 v_d + c\omega_{\text{pe}} k_z (\omega^2_{\text{pe}} \omega^2_{\text{pi}} \Omega_{\text{ci}}^2 + c^2 \omega^2_{\text{pi}} k_x^2 \omega_{\text{pi}} + c^2 \omega^2_{\text{pe}} k_z^2 \Omega_{\text{ci}} - \omega^2_{\text{pi}} k_z^2 v_d - c^2 \omega^2_{\text{pi}} k_x^2 k_z^2 v_d^2)^{1/2}}{c^2 \omega^2_{\text{pe}} k_z^2 + c^2 \omega^2_{\text{pi}} k_x^2 + \omega^2_{\text{pe}} \omega^2_{\text{pi}}}
\]  \hspace{1cm} (25)

Note that if \( \nu_{ei} \gg 2k_z v_d \) were assumed, the expression for \( y \) would be unchanged, but the expression for \( x \) would become

\[
x = \frac{ck_z \Omega_{\text{ci}}}{(c^2 k_z^2 + c^2 k_x^2 m_e / m_i + \omega^2_{\text{pe}})^{1/2}} \hspace{1cm} (26)
\]
Eq. (26) is much simpler and resembles (13a) in Ref. 21. The condition \( \nu_{ei} \gg 2k_z v_d \) can be satisfied quite easily for collisional plasmas. The present numerical example also meets this requirement.

Comparing (23) and (26) with (13b) and (13a) in Ref. 21 respectively, it is found that they have similar forms but an additional term is included in the denominators of (23) and (26). This term places an upper limit on the propagation angle with respect to the static magnetic field, i.e. \( k_x/k_z \) if electron inertia is to be neglected. When this ratio becomes large, the importance of this new term is enhanced and electron inertia must be considered. From the equations, it is seen that the main effects of electron inertia include stabilizing the wave, i.e. lowering the growth rate and decreasing the real frequency of the instability.

In order to check numerically whether the reduced dispersion relation still holds when electron inertia is included, the author used a program which is listed in Appendix C for reference and the same sets of parameters as before to calculate the six ratios \( R_1 - R_6 \) with \( \nu_{ei} \) now replaced by \( (\nu_{ei} - i\omega + ik_z v_d) \). It is rather encouraging to find that the ratios \( R_2 - R_6 \) are as before all less than 0.1 for the ranges of \( k_z \) and \( k_x \), and the \( R_4 = 0.1 \) curves also have similar shapes. For brevity, only those for the variations of static magnetic field are drawn for comparison (Fig. 2-6). With reference to Fig. 2-1, it is found that the curves are practically unchanged.

2-6. Summary

In this thesis, some of the outstanding problems in Ref. 21 are solved. The six inequalities (9a)-(9c) and (11a)-(11c) adopted to obtain the simple cold plasma results (7a) and (7b) are numerically checked using typical plasma and magnetic field parameters representative of the
Fig. 2-6 The boundary line $R_i=0.1$ for a helium plasma with $n_o=1.1 \times 10^{14} \text{ cm}^{-3}$, $T_e=3 \text{ eV}$, $v_d=1.15 \times 10^7 \text{ cm/sec}$ and several values of $B_0$ when electron inertia is included. In each case, all of $k$ space above the line corresponds to $R_i<0.1$ and that below the line corresponds to $R_i>0.1$. 
UCLA arcjet plasma, in which the ion cyclotron instabilities have recently been found. It is seen that the five inequalities (9b)-(9c) and (11a)-(11c) are easily met while inequality (9a) can only be satisfied with a larger $k_x$. Hence, the simple analytical results are valid when the wave propagates nearly perpendicular to the static magnetic field. More accurate results are also derived by giving up the inequality (11c). We find that the expression for the real frequency of the instability is unchanged while the imaginary frequency, i.e. the growth rate, differs from the original expression by only a factor of 1/2. Similar numerical outcome are obtained for these revised results.

The effect of the electron inertia which is neglected in the original paper is also investigated. It is found that it slightly reduces the frequency and the growth rate of the wave as can be seen by comparing Eqs. (7a) and (26), and (7b) and (23) respectively. The region of validity numerically determined by the same sets of parameters used previously are found to remain practically unchanged.
CHAPTER THREE

RESISTIVE LOWER HYBRID INSTABILITIES

3-1. Quasi-perpendicularly Propagating Lower Hybrid Waves

Owing to the rapid development in controlled fusion research, plasma heating has become one of the most attractive topics. Among various heating methods, resonant heating using waves with frequencies matched to a natural mode of oscillation in the plasma and turbulent heating taking advantage of instabilities have been proven efficient. For these two methods, lower hybrid modes were found to be the important ones under practical considerations. Experimental and theoretical results were reported in various papers. It is thus worthwhile to study the stability of lower hybrid waves in current-carrying collisional plasmas. Before we proceed to discuss the stability, let us first verify in the cold plasma limit the lower hybrid resonant frequencies for perpendicular and quasi-perpendicular propagations.

As proved by Stix, the lower hybrid resonant frequency for perpendicular propagation is given by

\[ \omega_{LH} = \left( \frac{\Omega_{ce}^2 \left( \frac{\omega_{pi}^2}{\omega_{pe}^2} + \frac{\Omega_{ci}^2}{\omega_{pe}^2} \right)}{\omega_{pe}^2 + \Omega_{ce}^2} \right)^{1/2} \]  

which for the strong magnetic field cases where \( \Omega_{ci} \ll \omega_{pi} \) but \( \Omega_{ce}^2 \) is of the order of \( \omega_{pe}^2 \), turns out to be

\[ \omega_{LH} = \omega_{pi} \left( 1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2} \right)^{-1/2} \]  

and for the dense plasmas where \( \Omega_{ce} \ll \omega_{pe}^2 \), becomes

\[ \omega_{LH} = \left( \Omega_{ce} \Omega_{ci} \right)^{1/2} \]
The name, hybrid resonance, is due to the fact that electrons and ions resonate differently with the wave. For a frequency $\omega$ such that $\Omega_{ci} \ll \omega \ll \Omega_{ce}$, the wave effectively sees unmagnetized ions and magnetized electrons. The ions oscillate with the electric field of the wave while the electrons move in response to the oscillating $E \times B_0$ drift. Both types of particles oscillate in phase with the electric field. The motions of electrons and ions can be depicted as shown in Fig. 3-1.

Since it is difficult to hold the propagation angle with respect to the static magnetic field at exactly $\pi/2$ to the degree of accuracy required, the lower hybrid frequency for perpendicular propagation is not easy to observe. Fig. 3-1 Lower hybrid resonance

As pointed out by Chen, a critical angle, $\chi_{crit}$ exists for oblique propagations. The critical value is given by the square root of the mass ratio of the electrons and the ions, i.e.

$$\chi_{crit} = \left( \frac{m_e}{m_i} \right)^{1/2}$$

(4)

When the propagation angle exceeds this critical value, i.e. $\chi = k_z/k_x > \chi_{crit}$, the electrons will have enough time to move along the static magnetic field because of the finiteness of $\chi$ to preserve charge neutrality. For these cases, electrostatic ion cyclotron waves result.

On the other hand, for propagation angles smaller than the critical angle, the electrons are not allowed to flow along the lines of force to preserve charge neutrality and hence lower hybrid waves happen. For the convenience of measurement in experiments, it is a normal practice to establish propagation angles a little bit smaller than the critical
angle, i.e. \( \frac{k_z^2}{k_x^2} \lesssim \frac{m_e}{m_i} \). For these propagation angles, the lower hybrid resonant frequency for highly magnetized plasmas is modified as follows:

\[
\omega = \omega_{LH}(1 + \frac{k_z^2}{k_x^2} \cdot \frac{m_i}{m_e})^{1/2}
\]

where \( \omega_{LH} \) is the lower hybrid frequency for perpendicular propagation given by equation (2).

In the following, derivation of the lower hybrid frequency given by Eq. (5) for nearly perpendicular propagation is attempted using the two-fluid cold plasma model. Before the mathematical process begins, let us make some reasonable assumptions to facilitate the derivations. The useful assumptions are listed below:

1/. For the electrostatic lower hybrid wave, we can let \( \vec{E} = -\vec{\nabla}\phi \).

2/. Let \( \vec{B}_o = B_o \hat{z} \), and assume a uniform static magnetic field.

3/. Perturbations are of the form \( \exp(i(k_x x + k_z z - \omega t)) \).

4/. \( \frac{k_z^2}{k_x^2} \lesssim \frac{m_e}{m_i} \) is satisfied for lower hybrid resonance.

5/. Assume the highly-magnetized cases where \( \omega_{pi} \gg \Omega_{ci} \).

6/. Assume \( \Omega_{ce}^2 \gg \omega^2 \gg \Omega_{ci}^2 \), which is appropriate for lower hybrid waves.

7/. Assume a homogeneous and uniform, magnetoplasma which is cold and collisionless.

The wave is depicted in Fig. 3-2.

From chapter one, the following equations are obtained.

\[
-i\omega_{e}\vec{v}_e = iek\vec{\phi} - \frac{e}{c} \vec{v}_e \times \vec{B}_o
\]  \hspace{1cm} (6)

\[
-i\omega_{e} + i\omega_{0}\vec{k} \cdot \vec{v}_e = 0
\]  \hspace{1cm} (7)

\[
-i\omega_{i}\vec{v}_i = -iek\vec{\phi} + \frac{e}{c} \vec{v}_i \times \vec{B}_o
\]  \hspace{1cm} (8)

Fig. 3-2. Geometry of a lower hybrid wave propagating nearly perpendicular to \( \vec{B}_o \).
In the above equations, we have assumed singly charged ions and quasi-neutrality so that \( n_{oe} = n_{oi} = n_0 \).

From (7) and (9), the perturbed electron and ion densities are

\[
n_e = \frac{n_0}{\omega}(k_x v_{ex} + k_z v_{ez}) \quad \text{(11)}
\]

\[
n_i = \frac{n_0}{\omega}(k_x v_{ix} + k_z v_{iz}) \quad \text{(12)}
\]

The \( x \) and \( z \) components of the electron and ion perturbed velocities can be obtained from (6) and (8) respectively. After decomposing the equations into component forms and solving for the \( x \) and \( z \) components, we get

\[
v_{ex} = -\frac{ek_x \phi}{m_e \omega}(1 - \frac{\Omega_{ce}^2}{\omega^2})^{-1} \quad \text{(13a)}
\]

\[
v_{ez} = -\frac{ek_z \phi}{m_e \omega} \quad \text{(13b)}
\]

\[
v_{ix} = \frac{ek_x \phi}{m_i \omega}(1 - \frac{\Omega_{ci}^2}{\omega^2})^{-1} \quad \text{(14a)}
\]

\[
v_{iz} = \frac{ek_z \phi}{m_i \omega} \quad \text{(14b)}
\]

where

\[
\Omega_{ce} = \frac{eB_0}{m_e c} \quad \text{is the electron cyclotron frequency} \quad \text{(15)}
\]

\[
\Omega_{ci} = \frac{eB_0}{m_i c} \quad \text{is the ion cyclotron frequency} \quad \text{(16)}
\]

Substituting Eqs. (11)-(14) into (10), the dispersion relation is found to be

\[
k^2 = \frac{k_x^2 \omega_{pi}^2}{\omega^2 - \Omega_{ci}^2} + \frac{k_x^2 \omega_{pe}^2}{\omega^2 - \Omega_{ce}^2} + \frac{k_z^2 \omega_{pi}^2}{\omega^2} + \frac{k_z^2 \omega_{pe}^2}{\omega^2} \quad \text{(17)}
\]
where $\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e}$ is the electron plasma frequency --- (18)

$\omega_{pi}^2 = \frac{4\pi n_i e^2}{m_i}$ is the ion plasma frequency ---- (19)

Using the assumption 6, Eq. (17) can be simplified as follows:

$$k^2 = \frac{\omega_{pi}^2 k_z^2 + \omega_{pe}^2 k_z^2}{\omega^2} - \frac{\omega_{pe}^2 k_x^2}{\Omega_{ce}^2}$$ ---- (20)

Solving for the complex angular frequency gives

$$\omega = \left( \frac{\omega_{pi}^2 k_z^2 \Omega_{ce}^2 + \omega_{pe}^2 k_z^2 \Omega_{ce}^2}{k^2 \Omega_{ce}^2 + \omega_{pe}^2 k_x^2} \right)^{1/2}$$ ---- (21)

$$= \omega_{pi}(1 + \frac{\omega_{pe}^2}{\Omega_{ce}^2})^{1/2} (1 + \frac{k_z^2}{k^2} \frac{m_i}{m_e})^{1/2}$$ ---- (21)

which is the lower hybrid resonant frequency for propagation angles nearly at right angle but not exactly perpendicular to the static magnetic field.

3-2. Lower Hybrid Current-driven Instabilities

As suggested by the ion cyclotron current-driven instability that a current exceeding a threshold value together with electron-ion collisions effecting energy transfer between electrons and ions, generate instabilities with growth rates proportional to the collision frequency, it is interesting to see whether similar instabilities exist for lower hybrid mode.

Consider an electrostatic wave propagating nearly perpendicular to the uniform and straight static magnetic field. The propagation angle is so arranged that the ratio of the parallel to perpendicular component of the wave number is smaller than the square root of the
mass ratio of the electron to the ion, i.e. \( k_z/k_x \leq (m_e/m_i)^{1/2} \), which is a necessary condition for the existence of lower hybrid waves. In the direction of the static magnetic field, electrons stream with a velocity \( \mathbf{v}_d \) relative to the stationary ions. The presence of the electron-ion collisions prevents the electrons from establishing the equilibrium Maxwellian distribution and electron motion is thus governed by the usual force equation. Assume that the static magnetic field points in the positive z direction and the wave propagates in the xz plane nearly at right angle to the static magnetic field, so that \( k_z^2/k_x^2 \leq m_e/m_i \). The electrons and the ions are then described by the force equation and the continuity equation

\[
\frac{m_e}{2} \frac{\partial^2 \mathbf{v}_e}{\partial t^2} + m_e (\mathbf{v}_d \cdot \nabla) \mathbf{v}_e = -e (\mathbf{\nabla} \phi + \frac{1}{c} \mathbf{v}_e \times \mathbf{B}_0) - m_e \mathbf{v}_i \mathbf{v}_e \quad \text{(22)}
\]

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e + n_e \mathbf{v}_d) = 0 \quad \text{(23)}
\]

\[
m_i \frac{\partial \mathbf{v}_i}{\partial t} = e (\mathbf{\nabla} \phi + \frac{1}{c} \mathbf{v}_i \times \mathbf{B}_0) \quad \text{(24)}
\]

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{v}_i) = 0 \quad \text{(25)}
\]

Let the perturbations be of the form \( \exp i (k_x x + k_z z - \omega t) \). When written in its component forms, Eq. (22) transforms into the following three equations:

\[
\mathbf{v}_{ei}^v_{ex} = \frac{ie k_x \phi}{m_e} - \Omega_{ce} \mathbf{v}_{ey} \quad \text{(26a)}
\]

\[
\mathbf{v}_{ei}^v_{ey} = \Omega_{ce} \mathbf{v}_{ex} \quad \text{(26b)}
\]

\[
\mathbf{v}_{ei}^v_{ez} = \frac{ie k_z \phi}{m_e} \quad \text{(26c)}
\]
where \( \nu_e' = \nu_e - i \omega + i k_z v_d \)  

From Eqs. (26), we can obtain

\[
\nu_{ex} = \frac{ie k_{x} \nu_{e1}' \phi}{m_e (\nu_{e1}'^2 + \Omega_{ce}^2)}
\]

\[
= \frac{ie k_{x} \nu_{e1}' \phi}{m_e \Omega_{ce}^2}
\]

\[
\nu_{ez} = \frac{ie k_{z} \phi}{m_e \nu_{e1}'}
\]

In the above, we have assumed \( \Omega_{ce}^2 \gg \nu_{e1}'^2 \). This can be easily satisfied in most practical situations.

Similarly, we have for the ions

\[
\nu_{ix} = \frac{-ie k_{x} \omega \phi}{m_i (\Omega_{ci}^2 - \omega^2)}
\]

\[
= \frac{-ie k_{x} \phi}{m_i \omega}
\]

\[
\nu_{iz} = \frac{ie k_{z} \phi}{m_i \omega}
\]

since the lower hybrid frequency is much larger than the ion cyclotron frequency. From Eqs. (23) and (25), the electron and ion perturbed densities are

\[
n_e = \frac{n_o k_x v_{ex} + n_o k_z v_{ez}}{\omega - k_z v_d}
\]

\[
n_i = \frac{n_o k_x v_{ix} + n_o k_z v_{iz}}{\omega}
\]

Substituting Eqs. (27)-(30) into Poisson's equation gives
In order to solve for the complex angular frequency, the value of $\nu_{ei}^\prime$ given by (26d) is substituted back into (31) which is then expanded to get

$$
-k^2_{ce} + \omega^2_{pe} x^2 \nu^\prime - \omega^2_{pe} x^2 \nu_{ei}^\prime + 2\omega^2_{pe} k^2 x \nu_{ei}^\prime + i2k^2_{ce} k_v v_d + 
$$

$$
i2\omega^2_{pe} k_v v_d \nu^\prime - (k^2_{ce} k_v v_d \nu_{ei}^\prime + 2\omega^2_{pe} k_v v_d \nu_{ei}^\prime + ik^2_{ce} k_v v_d - 
$$

$$
i\omega^2_{pi} k^2_{ce} \nu^\prime - i\omega^2_{pe} x^2 \nu_{ei}^\prime + i\omega^2_{pe} k_v v_d \nu^\prime - i\omega^2_{pi} k^2_{ce} k_v v_d + 
$$

$$
i2\omega^2_{pi} k^2_{ce} k_v v_d \nu^\prime + \omega^2_{pi} k^2_{ce} k_v v_d \nu_{ei}^\prime + i\omega^2_{pi} k^2_{ce} k_v v_d = 0 \quad \cdots (32)
$$

Letting $\omega = x + iy$ and assuming $|y|<<x$ for linear approximation, we get, on equating the real and imaginary parts separately to zero, the following two equations:

**Re:**

$$
4x^3 (k^2_{ce} + \omega^2_{pe} x^2) y + (k^2_{ce} \nu_{ei}^\prime + 2\omega^2_{pe} k^2 x \nu_{ei}^\prime) x^3 - (k^2_{ce} k_v v_d \nu_{ei}^\prime + 
$$

$$
2\omega^2_{pe} k_v v_d \nu_{ei}^\prime) x^2 - \omega^2_{pi} k^2_{ce} \nu_{ei}^\prime \nu_x + \omega^2_{pi} k^2_{ce} k_v v_d \nu_{ei}^\prime = 0 \quad \cdots (33)
$$

**Im:**

$$
-(k^2_{ce} + \omega^2_{pe} x^2) x^4 + (2k^2_{ce} k_v v_d + 2\omega^2_{pe} k^2 x^2 \nu_{ei}^\prime) x^3 - (k^2_{ce} k_v v_d - 
$$

$$
\omega^2_{pi} k^2_{ce} - \omega^2_{pe} x^2 \nu_{ei}^\prime + \omega^2_{pe} k_v v_d \nu_{ei}^\prime - \omega^2_{pe} k^2_{ce} \nu_x x^2 - \omega^2_{pi} k^2_{ce} k_v v_d x + 
$$

$$
\omega^2_{pi} k^2_{ce} k^2_{ce} k_v v_d = 0 \quad \cdots \cdots (34)
$$

The growth rate can be obtained from (33) as given below

$$
y = \frac{\nu_{ei}(k_v v_d/x - 1) \left[(k^2_{ce} + \omega^2_{pe} x^2) x^2 - \omega^2_{pi} k^2_{ce} \right]}{4x^2 (k^2_{ce} + \omega^2_{pe} x^2)} \quad \cdots (35)
$$
The real frequencies at which the lower hybrid current-driven instabilities occur are given by (34) which, after rearranging, reads

\[
(\omega_{pi}^2 - k_{ce}^2 x^2 - \omega_{pe}^2 k_{x}^2) (x - k_{z} v_{d})^2 + \omega_{pe}^2 k_{ce}^2 x^2 + \omega_{pe}^2 k_{x}^2 y^2 = 0 \quad \text{(36)}
\]

3-3. The Instability Criterion

If we let \( v_{d} \) go to infinity, the above equation will give

\[
x^2 = \frac{\omega_{pi}^2 k_{ce}^2}{k_{ce}^2 + \omega_{pe}^2 k_{x}^2} \quad \text{(37)}
\]

which deduces the following inequality

\[
x^2 > \frac{\omega_{pi}^2 k_{ce}^2}{k_{ce}^2 + 2 \omega_{pe}^2 k_{x}^2} \quad \text{(38)}
\]

For this case, therefore, \( y \) is positive and instability with real frequency given by (37) occurs.

It is seen from Eq. (35) that the criterion for instability is exactly the same as that for ion cyclotron current-driven instability, if the inequality (38) holds for any value of \( v_{d} \). In order to check whether the inequality (38) holds for every streaming velocity, Eq. (36) is rewritten as follows:

\[
(x - k_{z} v_{d})^2 = \frac{(\omega_{pe}^2 k_{ce}^2 + \omega_{pe}^2 k_{x}^2 y_{i}^2)}{(k_{ce}^2 + k_{x}^2 \omega_{pe}^2)^2 - k_{ce}^2 \omega_{pi}^2} \quad \text{(39)}
\]

and assign \( F(x) = (x - k_{z} v_{d})^2 \) \quad \text{(40)}
The function \( G(x) \) exhibits a pole at

\[
x_p = \frac{\omega_{pi} k \Omega_{ce}}{(k^2 \Omega_{ce}^2 + k^2 \omega_{pe}^2)^{1/2}}
\]

which is the lower hybrid frequency for waves propagating at right angle to the static magnetic field in strongly magnetized plasmas, and is the same frequency as that obtained through the mathematical calculation for the infinite streaming velocity. It gives in linear scales a curve as shown in Fig. 3-3. For the other function \( F(x) \), a zero exists at

\[
x_z = k_z v_d
\]

and the function is always positive (Fig. 3-4).

Mathematical argument indicates that for infinite electron streaming velocity, the two functions intersect at the pole of \( G(x) \). For smaller drift velocities, they will have solutions at higher real frequencies. The real frequency, \( x_p \), producing an infinite value of \( G(x) \) is apparently the minimum frequency of any instability that can exist, as it is the minimum frequency which starts a posi-
itive excursion of the function $G(x)$ which only cuts with the always positive function $F(x)$ in the upper half plane. Hence, the real frequency of any possible instabilities must satisfy the following inequality

$$x > x_p \quad ------ \quad (44)$$

i.e. the real frequency of the lower hybrid current-driven instability is in general greater than the lower hybrid resonant frequency which is the limiting frequency for infinite streaming velocity.

Since the following relation

$$x_p^2 = \frac{k^2 \Omega^2_{ce} \omega^2_{pi}}{k^2 \Omega^2_{ce} + k^2 \omega^2_{pe}} > \frac{k^2 \Omega^2_{ce} \omega^2_{di}}{k^2 \Omega^2_{ce} + 2k^2 \omega^2_{pe}}$$

holds, we always have

$$(k^2 \Omega^2_{ce} + 2\omega^2_{pe} k^2 x^2 - k^2 \Omega^2_{ce} \omega^2_{pi}) > 0 \quad ------ \quad (45)$$

With other terms positive, the sign of $y$ is determined solely by the sign of the factor $(k_z v_d / x - 1)$. Hence, the instability criterion of the lower hybrid instability is

$$v_d > x / k_z \quad ------ \quad (46)$$

which is exactly the same as that for the resistive ion cyclotron instability.

3-4. The Real Frequencies of the Instabilities

The real frequencies of the instabilities are given by the solutions of Eq. (36) which is a $4^{th}$ order polynomial in $x$. Owing to the high order of the equation, no general simple analytical results are obtainable. Although it can only be solved numerically for specific values of parameters, its order of magnitude can be obtained by considering the cases $v_d \rightarrow 0$ and $v_d \rightarrow \infty$. Denoting these limiting values of
By substituting these values of $v_d$ into Eq. (36) the following solutions:

$$x^2_\infty = \frac{\omega^2_{pi}}{(1 + \omega^2_{pe}/\Omega^2_{ce})} = \omega_{LH} \quad \text{(47)}$$

$$x^2_o = \frac{\omega^2_{pi} k^2 \Omega^2_{ce} + \omega^2_{pe} k^2 \Omega^2_{ce} + \omega^2_{pe} k^2 \nu^2_{ei}}{k^2 \Omega^2_{ce} + k^2 \omega^2_{pe}}$$

$$= \omega^2_{LH} (1 + \frac{k^2 m_i^2}{k^2 x^2 e} + \frac{m_j^2 \nu^2_{ei}}{m_j \Omega^2_{ce}}) \quad \text{(48)}$$

Eqs. (47) and (48) show that both $x_o$ and $x_\infty$ are of order of the lower-hybrid frequency. It is therefore reasonable to expect the real frequencies of the instabilities to be of the same order when $v_d$ is finite.

In addition to the above argument, the method used in Section 3-3 may be adopted to discuss qualitatively the possibility of occurrence of instability. As noted before, Eq. (36) is rewritten as Eq. (39) and two function names, $F(x)$ and $G(x)$, are assigned respectively to the functions on both sides as given by Eqs. (40) and (41). The graphs of these two functions are depicted in Figs. 3-3 and 3-4 respectively. The solutions of the original equation are now obtainable from the intersection frequencies of these two new equations. As $F(x)$ is a non-negative function, it can only intersect the other function, $G(x)$, in the upper half plane. From the graphs of these two functions, it is seen that, depending on system parameters, the conditions of intersection can be divided into three different cases, namely, intersection at only one point, at two points and three points.

3-4-1. Intersection at only one point

For system parameters such that the two function cut at only
one frequency (Figs. 3-5 a and b), the solution of Eq. (36) which is the real frequency of the possible instability, is given by that cutting frequency. Here, we see that the cutting frequency is always greater than $x_z$ and $x_p$. This makes the imaginary frequency of the complex angular frequency negative as

$$x_1 > x_z = k_Z v_d$$

Hence, the wave is damped and no instability occurs.

![Graphs](image)

Fig. 3-5 F(x) and G(x) cut at only one point

3-4-2. Intersection at two points

When values of parameters are adjusted so that the two functions cut at exactly two points as shown in Fig. 3-6, the two curves are tangent to each other at $x_1$. Of the two solutions, $x_1$ and $x_2$, the higher frequency $x_2$ is larger than both $x_z$ and $x_p$, and as discussed in section 3-4-1, this renders only a damped wave.

For the smaller solution, it satisfies the instability criterion as

$$x_1 < x_z = k_Z v_d \text{ or } v_d > x_1/k_Z$$

![Graphs](image)

Fig. 3-6 F(x) and G(x) cut at exactly two points.
Therefore, waves with real frequency at $x_1$ will grow with time and become unstable, while waves with real frequency at $x_2$ are damped. However, the requirement that $F(x)$ and $G(x)$ pass tangent to each other at $x_1$ is difficult to obtain.

3-4-3. Intersection at three points

As discussed in the last section, waves at $x_3$ (Fig. 3-7) are stable while waves at $x_1$ and $x_2$ will grow with time. From the figure, we see that one of the unstable waves has a frequency very close to the lower hybrid frequency, and the other has a higher frequency.

Let us now study two numerical cases as illustration. For the following plasma and magnetic field parameters:

- $n_0 = 3 \times 10^{11} \text{ cm}^{-3}$
- $B_0 = 750 \text{ Gauss}$
- $m_e/m_i = 1/7344$ (helium plasma)
- $T_e = 3 \text{ eV}$
- $v_d = 1.15 \times 10^7$ and $2 \times 10^9 \text{ cm/sec}$

the electron and ion plasma frequencies, electron cyclotron frequency and the electron-ion collision frequency are

$$\omega_{pe} = 3.09 \times 10^{10} \text{ sec}^{-1}$$
$$\omega_{pi} = 3.61 \times 10^8 \text{ sec}^{-1}$$
\[ \nu_{ei} = 1.18 \times 10^6 \text{ sec}^{-1} \]
\[ \Omega_{ce} = 1.31 \times 10^{10} \text{ sec}^{-1} \]

In order to find the real frequencies of the waves for the above parameters, we assign each time a different value to \( x \) and have the functions, \( F(x) \) and \( G(x) \), computed for these \( x \)'s. When the values of \( F(x) \) and \( G(x) \) become equal, the corresponding \( x \) gives the real frequency of the wave. The program for this calculation is given in Appendix D. Using this method, we obtain the real frequencies of the waves for the above parameters with \( k_z \) fixed at \( 0.25 \text{ cm}^{-1} \) and \( k_x \) varied from \( 10 \text{ cm}^{-1} \) to \( 45 \text{ cm}^{-1} \). Two graphs corresponding to the two different values of \( v_d \) are constructed (Figs. 3-8 and 3-9). Here we find that for small \( v_d \), \( F(x) \) and \( G(x) \) only cut at one point and the real frequency of the wave is such that \( x/k_z > v_d \). The wave is therefore stable. For large \( v_d \), however, \( F(x) \) and \( G(x) \) cut at more than one point when \( k^2_z/k_x < m_e/m_i \) is satisfied. The wave with the highest frequency is stable and the other two waves are unstable as determined by the instability criterion. The frequency of one of the two unstable waves approaches the lower hybrid frequency as \( k_x \) is increased (Instability A). The frequency of the other unstable wave deviates further from the lower hybrid frequency as \( k_x \) is increased, but the slope of deviation becomes smaller and smaller so that it stays in the range of the lower hybrid frequency even when \( k_x \) is increased to a very large value (Instability B).

3-5. The Growth Rates

Since there are two instabilities occurring simultaneously, it is interesting to compare their growth rates. From Eq. (35), it is seen that the growth rate of the instability is directly proportional to the absolute value of the factor \( (x - k_z v_d) \) and inversely proportional to the
Fig. 3-8 Waves are stable when \( F(x) \) and \( G(x) \) cut at only one point.
Fig. 3-9 Two unstable and one stable waves result when $F(x)$ and $G(x)$ cut at three points.
frequency. Hence, instability A in Fig. 9 should have a larger growth rate than instability B since it is further from the boundary separating the stable region and the unstable region and also has a lower frequency. This can be seen by plotting the graphs of the growth rates in Fig. 3-10, where the growth rate of instability A is shown to be much larger and remains fairly constant as $k_x$ is varied. The growth rate of instability B is smaller and decreased exponentially as $k_x$ is increased. We can, therefore, conclude that instability A grows much faster than instability B at a frequency very close to the lower hybrid frequency.

3-6. **Summary and Discussions**

In this chapter, we have discussed the possibility of current-driven instability at a frequency equal to or greater than the lower hybrid frequency. The model adopted is an infinite homogeneous and uniform magnetoplasma so that gradients and boundary conditions can be got rid of to simplify the problem. In order to dispense with viscous terms and pressure gradients, the system is further simplified by assuming a cold or low-temperature plasma with $v_d > v_{the}$, where $v_{the}$ is the electron thermal velocity. As a first study, electrostatic perturbations are considered for this kind of instability. Joule heating effect is discarded as the real frequencies of the waves are only weakly temperature-dependent. Two other assumptions used are

(i) the frequency of the instability is much larger than the ion cyclotron frequency, and

(ii) the electron cyclotron frequency is much greater than the electron-ion collision frequency and the frequencies of the instabilities.

These two assumptions can easily be met for cold lower hybrid waves.

After some lengthy mathematical development, we eventually come to the result that the instability criterion is exactly the same as that
Fig. 3-10 The growth rates of the two unstable waves.
for the ion-cyclotron instability which says that the wave becomes unstable only when the streaming velocity of the electrons exceeds the phase velocity along the static magnetic field. The real frequencies of the instabilities are in general equal to or greater than the lower hybrid frequency. Their exact location depend on system parameters. In most cases, two unstable waves exist concurrently with frequencies both greater than the lower hybrid frequency. One of the two instabilities with a frequency close to the lower hybrid frequency has a fairly constant growth rate in the order of the ion cyclotron frequency. The growth rate of the other unstable wave is smaller and decreases exponentially as \( k_x \) is increased. Its frequency is higher and approaches the instability boundary for large \( k_x \).
4-1. The Dispersion Relation

In chapter 2, it was shown that in a fully-ionized collisional magnetoplasma carrying a field-aligned current, a resistive instability occurs for obliquely propagating ion cyclotron waves when the parallel phase velocity is smaller than the electron drift velocity, the growth rate being directly proportional to the electron-ion collisional frequency $\nu_d$ and the difference of the parallel phase velocity and the electron drift velocity. In chapter 3, similar instabilities were shown to exist for waves with frequencies around the lower hybrid frequency. These instabilities occur in plasmas having one type of ion species. A question naturally arises as to whether such resistive-type instability occurs for plasmas having two ion species. Plasmas having more than one ion species are frequently found in controlled-thermal nuclear fusion research, ionospheric studies, etc. In this chapter, it is shown that in a current-carrying plasma composed of two ion species, both singly charged and one much heavier than the other, resistive type instabilities do occur for electrostatic waves with frequencies around the Buchsbaum ion-ion hybrid frequency.

Let $N_a$ and $N_b$ be the equilibrium densities of ion species a and b and $N_e$ the electron density. Since the ions are assumed singly charged, charge neutrality requires $N_e = N_a + N_b$. The infinite and uniform plasma is immersed in a static magnetic field $\vec{B}_o = B_o \hat{z}$ and the electrons have a zero-order drift velocity $\vec{v}_d = v_d \hat{z}$ with respect to the ions. For simplicity, the cold plasma model is used. Assuming perturbations of the form $\exp(i(k_x x + k_z z - \omega t))$ and considering electrostatic waves with $\vec{E} = -\vec{\nabla}\phi$, ...
the relevant equations for this system are

\begin{align*}
(-i\omega + i k_v d + \nu_{ei}) m_e \vec{v}_e &= -e(-i k \phi + \frac{\vec{v}_e \times \vec{B}_0}{c}) \quad \cdots \quad (1a) \\
-i\omega n_e + i k \cdot (N_e \vec{v}_e + n_e \vec{v}_d) &= 0 \quad \cdots \quad (1b) \\
-i\omega m_a \vec{v}_a &= e(-i k \phi + \frac{\vec{v}_a \times \vec{B}_0}{c}) \quad \cdots \quad (1c) \\
-i\omega n_a + N_a k \cdot \vec{v}_a &= 0 \quad \cdots \quad (1d) \\
-i\omega m_b \vec{v}_b &= e(-i k \phi + \frac{\vec{v}_b \times \vec{B}_0}{c}) \quad \cdots \quad (1e) \\
-i\omega n_b + N_b k \cdot \vec{v}_b &= 0 \quad \cdots \quad (1f) \\
\vec{k}^2 \phi &= 4\pi e(n_a + n_b - n_e) \quad \cdots \quad (1g)
\end{align*}

Solving (1a)-(1f) for the perturbed densities, \(n_a, n_b\) and \(n_e\), and substituting the results in (1g) yields the dispersion relation

\begin{align*}
\vec{k}^2 &= \frac{\omega_{pa} k_x^2}{\omega^2 - \Omega_a^2} + \frac{\omega_{pb} k_x^2}{\omega^2 - \Omega_b^2} + \frac{(\omega_{pa}^2 + \omega_{pb}^2) k_z^2}{\omega^2} + \frac{\omega_{pe}^2 k_x^2}{(\omega^2 - \Omega_{pe}^2)(\omega - k_z v_d)} + \frac{\omega_{pe}^2 k_z^2}{\omega(\omega - k_z v_d)} \quad \cdots \quad (2)
\end{align*}

where \(\omega' = \omega - k_z v_d + i \nu_{ei}\)

4-2. Review of Ion-ion Hybrid Wave Theory

In order to check the correctness of this dispersion relation, we set some of the relevant parameters to zero and see if it reduces to the usual forms for the ion-ion hybrid waves.

A/. Perpendicular propagation of collisionless plasmas having no relative particle drift
In this case, the values of \( v_{ei} \), \( v_d \) and \( k_z \) in Eq. (2) are all set to zero giving

\[
1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_{ce}^2} - \frac{\omega_{pa}^2}{\omega^2 - \Omega_a^2} - \frac{\omega_{pb}^2}{\omega^2 - \Omega_b^2} = 0 \quad \text{(3)}
\]

which is consistent with that developed by Stix in his book.

If we now further simplify Eq. (3) with the help of assumptions applicable to ion-ion hybrid waves, i.e.

\[
\omega_{pe}^2 \gg \Omega_{ce}^2 \gg \omega^2 \quad \text{(4a)}
\]

\[
\omega = 0(\Omega_a, \Omega_b) \quad \text{(4b)}
\]

the equation becomes

\[
\frac{\omega_{pe}^2}{\Omega_{ce}^2} = \frac{\omega_{pa}^2}{\omega^2 - \Omega_a^2} + \frac{\omega_{pb}^2}{\omega^2 - \Omega_b^2} \quad \text{(5)}
\]

Solution of Eq. (5) yields the Buchsbaum ion-ion hybrid frequency

\[
\omega_b^2 = \frac{\Omega_a \omega_{pb}^2 + \Omega_b \omega_{pa}^2}{\omega_{pi}^2} \quad \text{(6a)}
\]

where

\[
\omega_{pi}^2 = \omega_{pa}^2 + \omega_{pb}^2 \quad \text{(6b)}
\]

It can easily be shown that \( \omega_b \) lies between the two ion cyclotron frequencies. When the ion species in the plasma satisfy the following inequality

\[
\frac{m_a}{m_b} \ll \frac{N_b}{N_a} \ll \frac{m_b}{m_a} \quad \text{(7)}
\]

the ion-ion frequency takes the form
An investigation of Eqs. (7) and (8) reveals that (7) is actually the equivalence of

\[ \Omega_a^2 \gg \omega^2 \gg \Omega_b^2 \]  \hspace{1cm} (9)

B/. Oblique propagation of collisionless plasma having no relative particle drift

Here, the z-component of the wave number is non-zero and Eq. (2) becomes

\[ k^2 = \frac{\omega_{pa}^2 k_x^2}{\omega^2 - \Omega_a^2} + \frac{\omega_{pb}^2 k_x^2}{\omega^2 - \Omega_b^2} + \frac{\omega_{pe}^2 k_z^2}{\omega^2 - \Omega_{ce}^2} \]  \hspace{1cm} (10)

Using the usual assumptions for ion-ion hybrid waves (4), Eq. (10) is simplified as below

\[ 0 = \frac{k_z^2 \omega_{pe}^2}{k^2 \omega^2} + \frac{k_x^2}{k^2} \left( \frac{\omega_{pa}}{\omega^2 - \Omega_a^2} + \frac{\omega_{pb}}{\omega^2 - \Omega_b^2} \right) \]  \hspace{1cm} (11)

which is exactly the same as that obtained by Bujarbarua et al. For plasmas satisfying the following inequalities

\[ \Omega_a^2 \gg \omega^2 \gg \Omega_b^2 \]  \hspace{1cm} (12a)

\[ k_z^2/k^2 \ll 1 \]  \hspace{1cm} (12b)

and

\[ \frac{m_a \alpha}{m_a (1 - \alpha)} \gg \frac{k_z^2}{k^2} \beta \gg \frac{\alpha m_a}{(1 - \alpha) m_b} \]  \hspace{1cm} (12c)

where

\[ \alpha = \frac{N_a}{N_e}, \quad \beta = \frac{m_b}{(1 - \alpha) m_e} = \frac{\omega_{pe}^2}{\omega_{pb}^2} \]  \hspace{1cm} (12d)

Bujarbarua et al further showed that the Buchsbaum frequency was modified
by the z-component of the wave vector as follows:

\[ \omega = \omega_b (1 + \frac{k_z^2}{k^2})^{1/2} \]  \hspace{1cm} (13)

Note that the third inequality in (12) is actually the same as

\[ \frac{\omega_{pa}^2}{\omega_{pb}^2} \gg \frac{k_z^2 \omega_{pe}^2}{k^2 \omega_{pb}^2} \gg \frac{\omega_{pe}^2 \nu_{ei}^2}{\omega_{pb}^2 \Omega_a^2} \]  \hspace{1cm} (14)

4-3. The Instability Criterion

Let us now come back to our plasma system where electron-ion collisions and relative particle drift are present. For plasmas satisfying inequality (7), i.e., plasmas with ion species b much heavier than species a, it can be verified a posteriori that the ion-ion hybrid frequency meets inequality (9). With the additional assumption \( \omega^2 \ll \Omega_{ce}^2 \) which requires a easily met condition, \( \nu_{ei}^2 \ll \Omega_{ce}^2 \), Eq. (2) turns out to be

\[ k^2 = \frac{\omega_{pa}^2 A_x^2}{\Omega_a^2} + \frac{\omega_{pa}^2 k_z^2 + \omega_{pb}^2 k^2}{\omega^2} \left( \frac{\omega_{pe}^2 k_z^2}{\Omega_{ce}^2 (\omega - k_z v_d)} + \frac{\omega_{pe}^2 k_z^2}{\omega (\omega - k_z v_d)} \right) \]  \hspace{1cm} (15)

After expanding and rearranging, Eq. (15) becomes

\[ i \omega^4 (k_x^2 \Omega_{ce}^2 + k_x^2 \omega_{ce}^2 + k_x^2 \omega_{a}^2) - \omega^2 (k_x^2 \Omega_{ce}^2 \nu_{ei} + k_x^2 \omega_{ce}^2 \nu_{ei} + 2k_x^2 \omega_{a}^2 \nu_{ei}) + i 2k_x^2 \omega_{ce}^2 k_z v_d + i 2k_x^2 \omega_{a}^2 k_z v_d + i 2k_x^2 \omega_{ce}^2 k_z v_d + i 2k_x^2 \omega_{a}^2 k_z v_d + i k_x^2 \Omega_{ce}^2 k_z v_d \]

\[ + 2k_x^2 \omega_{ce}^2 \nu_{ei} k_z v_d + 2k_x^2 \omega_{a}^2 \nu_{ei} k_z v_d + k_x^2 \Omega_{ce}^2 \nu_{ei} k_z v_d + k_x^2 \omega_{ce}^2 \nu_{ei} k_z v_d + i k_x^2 \Omega_{ce}^2 k_z v_d \]

\[ - i k_x^2 \Omega_{ce}^2 - i k_x^2 \omega_{ce}^2 + i k_x^2 \omega_{a}^2 k_z v_d - i k_x^2 \omega_{ce}^2 \nu_{ei} + i k_x^2 \omega_{a}^2 \nu_{ei} + i k_x^2 \Omega_{ce}^2 k_z v_d \]
If we let \( \omega = x + iy \) in (16) and assume \( |y| \ll |x| \) for linear approximation, we obtain, on equating the real and imaginary parts separately to zero, the following equations for \( x \) and \( y \):

\[
(x - k_z v_d)^2 = \frac{C x^2}{A x^2 - B} \quad ----- \quad (17)
\]

\[
y = \frac{\nu_{ei}}{4A x^2} \left( \frac{k_z v_d}{x} - 1 \right) \left[ (A + \omega_{pa}^2 k_x^2) x^2 - B \right] \quad ----- \quad (18)
\]

where

\[
A = k_x^2 (\omega_{pa}^2 + \omega_{pe}^2) + k_{a ce}^2 \quad ----- \quad (19a)
\]

\[
B = \omega_{pa}^2 (\omega_{pa}^2 k_z^2 + \omega_{pb}^2 k^2) \quad ----- \quad (19b)
\]

\[
C = \omega_{pe}^2 (k_{x e i}^2 + k_{z ce}^2) \quad ----- \quad (19c)
\]

In arriving at (19a), we have made use of the assumption

\[
\omega_{pa}^2 \gg \Omega_a^2 \quad ----- \quad (19d)
\]

which is easily fulfilled in normal situations. Eq. (18) shows that \( y \) becomes positive and instability results if the two inequalities

\[
k_z v_d > x \quad ----- \quad (20a)
\]
are both satisfied or failed at the same time.

Using the method mentioned in the last chapter, i.e. for a specific set of parameters, plotting the left and right hand sides of Eq. (17) as a function of \( x \) and locating the point(s) of intersection, it is immediately found that for a finite value of \( v_d \), the real frequency of the instability always satisfies the following inequality

\[
(A + \frac{\omega_{pe}^2 \omega_a^2 k_x^2}{k_x^2}) x^2 > B \quad \text{(20b)}
\]

\[
x > \frac{B}{A}
\]

From this, we conclude that inequality (20b) is always satisfied and instability occurs only when (20a) is also satisfied. Inequality (20a) is therefore the instability criterion.

4-4. Numerical Illustrations

4-4-1. The Real frequency of the instability

In order to obtain an idea of the order of magnitude of the real frequency of the resistive ion-ion hybrid instability, we calculate the frequency for two extreme cases, i.e. for \( v_d \rightarrow 0 \) and \( v_d \rightarrow \infty \).

Substituting these values of \( v_d \) into Eq. (17), we obtain the respective results:

\[
x_0^2 = \frac{B + C}{A} = \frac{\omega_{rb}^2 \omega_a^2}{\omega_{pa}^2} + \frac{m_e \nu_{ei}}{m_i} + \frac{k_x^2 \omega_{pe}^2 \omega_a^2}{k_x^2 \omega_{pa}^2}
\]

\[
x_\infty^2 = \frac{B}{A} = \frac{\omega_{rb}^2}{\omega_{pa}^2} + \frac{k_x^2 \omega_a^2}{k_x^2}
\]

In deriving the above equations, \( k_x^2 = k^2 \) has been assumed.

For Eqs. (22) and (23), it is seen that both \( x_0 \) and \( x_\infty \) are
approximately of the order of $\Omega_{\omega_{pb}}/\omega_{pe}$, which is the Buchsbaum ion-ion hybrid frequency for a plasma satisfying (7). It is therefore, reasonable to conclude that the real frequency of the instability should lie in the order of the ion-ion hybrid frequency.

The actual solutions of (17) which has exactly the same form as Eq. (3-39) depend on the values of the plasma and magnetic field parameters. As discussed in the last chapter, Eq. (17) may have one, two or three solutions, the second case being a critical case and is difficult to obtain. The procedures will not be repeated here; but two numerical examples are given for illustration. The plasma parameters and magnetic field strength selected are

$$m_e / m_n = 1/7344$$
(Helium)

$$m_e / m_b = 1/73440$$
(Argon)

$$N_e = 3 \times 10^{13} \text{ cm}^{-3}$$

$$N_n = N_b = N_e / 2$$

$$B_o = 2 \text{ mG}$$

$$T_e = 3 \text{ eV}$$

These correspond to plasma frequencies, cyclotron frequencies and electron-ion collision frequency as given below:

$$\omega_{pe} = 3.09 \times 10^{11} \text{ sec}^{-1}$$

$$\omega_{pa} = 2.35 \times 10^{9} \text{ sec}^{-1}$$

$$\omega_{pb} = 8.58 \times 10^{8} \text{ sec}^{-1}$$

$$\Omega_m = 4.76 \times 10^{6} \text{ sec}^{-1}$$

$$\Omega_0 = 4.76 \times 10^{5} \text{ sec}^{-1}$$
In order to investigate the effect of the electron streaming velocity on the frequency of the wave, we calculate the frequency using two values of $v_d$, namely, $1.15 \times 10^7$ cm/sec and $1 \times 10^9$ cm/sec. The parallel component of the wave vector is fixed at $0.3$ cm$^{-1}$ while the perpendicular component is varied from $20$ to $45$ cm$^{-1}$. The program for this propose is attached in Appendix E and the results are depicted in Figs. 4-1 and 4-2. Here we find that for the smaller streaming velocity, only one wave exists and the wave is always stable. For the larger $v_d$, $F(x)$ and $G(x)$ cut at three points, two of which corresponding to unstable waves. One of the unstable waves has a frequency very close to the Buchsbaum ion-ion hybrid frequency. The frequency of the other instability is a bit away from the Buchsbaum frequency but approaches the instability boundary, i.e. $x = k_z v_d$ as a limit as $k_x$ is increased.

It is worthwhile to note that Figs. 4-1 and 4-2 look very much like Figs. 3-8 and 3-9. This is as expected since both kinds of instabilities result in equations of the same form for the finding of the real frequencies. It may have been noticed that owing to the high frequency and the necessary condition for such a wave, i.e. $k_z/k_x \ll (m_e/m_i)^{1/2}$, the resistive lower hybrid instability is difficult to achieve except for very large $v_d$ and small magnetic field, the latter being used to lower the real frequency of the wave. On the other hand, because of the lower frequency of the wave and the less stringent condition for the existence of the instabilities, i.e. $k_z/k_x \gg (m_e/m_i)^{1/2}$, the resistive ion-ion hybrid instabilities can exist for much smaller electron streaming velocity and also for larger confining magnetic field. This means that the ion-ion hybrid instability is more easy to happen.

\[ \Omega_e = 3.59 \times 10^{10} \text{ sec}^{-1} \]

\[ \nu_{ei} = 1.18 \times 10^{8} \text{ sec}^{-1} \]
\[ v_d = 1.15 \times 10^7 \text{ cm/sec} \]
\[ k_z = 0.3 \text{ cm}^{-1} \]
\[ \omega_B = 1.58 \times 10^6 \text{ sec}^{-1} \]

Fig. 4-1  \( F(x) \) and \( G(x) \) cut at only one point.
Fig. 4-2 $F(x)$ and $G(x)$ cut at three points.
4-5. The Growth Rates

Since there are two possible instabilities for this type of instability, it is worthwhile to construct from Fig. 4-2 the graphs of the growth rates of the two unstable waves and see which wave grows faster. These graphs are shown in Figs. 4-3 and 4-4. It is seen that the growth rate of instability A which has a frequency very close to the Buchsbaum ion-ion hybrid frequency, is much greater than that of instability B. This in fact can be obtained from Eq. (18) which says that the growth rate of any possible instability is directly proportional to the difference of the electron streaming velocity and the parallel phase velocity, and inversely proportional to the frequency of the wave. In Figs. 4-3 and 4-4, both growth rates decrease exponentially as $k_x$ is increased and approach the ion cyclotron frequency of the lighter ion species as a limit for large $k_x$. The current threshold for this type of instability requires that the electron streaming velocity be greater than the phase velocity of the wave along the static magnetic field. Since instability A has a much smaller frequency, it has a much lower current threshold than instability B.

4-6. Summary and Discussions

For obliquely propagating electrostatic waves in a current-carrying collisional plasma consisting of two stationary ion species of different charge-to-mass ratios, we have shown that instabilities occur when the electron streaming velocity exceeds the phase velocity of the wave along the static magnetic field. Under the following assumptions:

(i) The plasma pressure is small compared to the magnetic field energy density so that electrostatic approximation is applicable.

(ii) The plasma is infinite in extent, uniform, homogeneous and cold.

This eliminates considerations of boundary conditions, density
Fig. 4-3 The normalized growth rate of instability B.
Fig. 4.4 The growth rate of instability A.

Since the conditions involved with this type of instability are quite stringent, e.g., the interaction with the unstable component becomes negligible for large distances.
gradient, viscosity, pressure gradient, etc.

(iii) The static magnetic field is uniform and straight so that gradient and curvature drifts can be discarded.

(iv) Electron streaming is along the static magnetic field.

(v) One of the two ion species is much heavier than the other.

(vi) Cyclotron frequency of the electron is much higher than the electron-ion collision frequency and the frequency of the wave.

we found that instabilities usually occur in duality, one of them having a frequency very close to the ion-ion hybrid frequency. The growth rate of each unstable wave is directly proportional to the electron-ion collision frequency which is a must for the instability to exist, the difference of the electron streaming velocity and the parallel phase velocity, and inversely proportional to the frequency of the wave. Hence, the unstable wave with a frequency which is lower and further from the boundary separating the stable region and the unstable region, has a much greater growth rate than the other unstable wave (see Figs. 4-3 and 4-4). The instability criterion requires the threshold streaming velocity of the electron

\[ v_{\text{threshold}} = \frac{x}{k_z} \]

be exceeded. Since the instability with a frequency close to the ion-ion hybrid frequency has a much smaller frequency, it has also a much lower threshold velocity, i.e. it grows much faster. The growth rates of both instabilities decrease exponentially with increasing \( k_x \) and remain in the order of the ion cyclotron frequency of the lighter ion species for large \( k_x \).

Since the conditions imposed on this type of instability are less stringent, e.g. the propagation angle restriction imposed in the resistive lower hybrid instability does not exist for this type of
instability, the parallel phase velocity and hence the threshold velocity is smaller, etc., resistive ion-ion hybrid instabilities are much easier to occur.
CHAPTER FIVE

SUMMARY AND DISCUSSIONS

In this chapter, we present a summary of the results of this thesis and offer some discussions.

5-1. Resistive Electromagnetic Ion Cyclotron Instability

In a recent paper by Lee and Luhmann, they used the two-fluid model to prove theoretically the existence of a resistive electromagnetic ion cyclotron instability which was experimentally found on the UCLA arcjet plasma. Using the continuity equation and the force equation for electron and ion respectively, and Maxwell's two curl equations, they obtained the dispersion relation

\[
\begin{vmatrix}
R_{xx} & R_{xy} & R_{xz} \\
R_{yx} & R_{yy} & R_{yz} \\
R_{zx} & R_{zy} & R_{zz}
\end{vmatrix} = 0 \quad \text{----- (1)}
\]

The elements \(R_{xx}\), etc. are given by Eqs. (2-2) and (2-3).

Even for the cold plasma model, Eq. (1) is very complicated and can not be solved analytically. Guided by experimental measurements on the UCLA arcjet plasma, Lee and Luhmann tried to get rid of the \(y\)-component of the perturbed electric field which was found to be small compared to the \(x\) and \(z\) components, by assuming

\[
R_1 = \left| \frac{R_{xy} R_{yx}}{R_{xx} R_{yy}} \right| \ll 1 \quad \text{----- (2a)}
\]

\[
R_2 = \left| \frac{R_{xy} R_{yz}}{R_{xz} R_{yy}} \right| = \left| \frac{R_{yx} R_{zy}}{R_{zx} R_{yy}} \right| \ll 1 \quad \text{----- (2b)}
\]
When these inequalities are satisfied, the full dispersion relation is reduced to

\[
\begin{vmatrix}
R_{xx} & R_{xz} \\
R_{zx} & R_{zz}
\end{vmatrix} = 0 \quad (3)
\]

which gives a 5th order algebraic equation in \( \omega \).

In order to further simplify the equation, Lee and Luhmann assumed that the last terms of \( R_{xx}, R_{xz}, R_{zx}, \) and \( R_{zz} \) given by Eqs. (2-4) were negligible compared to the other terms of the elements respectively (Eqs. 2-11). With these assumptions satisfied, they finally got the simple analytical results for a cold plasma

\[
x = \text{Re} = \frac{ck_{\Omega ci}}{(c^2k_z^2 + \omega_{pi}^2)^{1/2}} \quad (4a)
\]

\[
y = \text{Im} = -\frac{c^2k_m^2 \omega_i}{2m_i(c^2k_z^2 + \omega_{pi}^2)} \left(1 - \frac{k_zv_d}{x}\right) \quad (4b)
\]

In our research, we find that if inequality (2-11c) is discarded, the real frequency of the wave given by (4a) is unchanged while the growth rate given by (4b) is only modified by a factor of 1/2 as below:

\[
y = -\frac{c^2k_m^2 \omega_i}{4m_i(c^2k_z^2 + \omega_{pi}^2)} \left(1 - \frac{k_zv_d}{x}\right) \quad (5)
\]

Using plasma and magnetic field parameters representative of the UCLA arcjet plasma, we find that the four ratios \( R_2 - R_5 \) are all less
than 0.1 for $k_z$ from 0.01 to 1.0 cm$^{-1}$ and $k_x$ from 1.0 to 12.0 cm$^{-1}$.

However, the ratio $R_1$ can be larger than 0.1 for certain values of $k_x$ and $k_z$ within this range. From the $R_1 = 0.1$ curves shown in Figs. 2-1, 2-2 and 2-3, we see that the regions of validity for various sets of parameters are quite large and the analytical results are accurate for waves propagating nearly perpendicular to the static magnetic field.

The smallness of the $y$ component of the perturbed electric field compared to the other two components is verified using the same parameter sets.

The effect of electron inertia which was discarded by Lee and Luhmann is found to be negligible in most cases. Based on similar inequalities (2-9a)-(2-9c) and (2-11a)-(2-11c), now with $\nu_{ei}$ replaced by $\nu_{ei} = i\omega + ik_z v_d$, the solutions are found to be

$$x = \frac{ck_zN_{ci}}{(c^2k_z^2 + c^2k_x^2m_e/m_i + \omega_{pi}^2)^{1/2}}$$  (6a)

$$y = -\frac{c^2k_x^2m_e\nu_{ei}}{2m_i(c^2k_z^2 + c^2k_x^2m_e/m_i + \omega_{pi}^2)(1 - \frac{k_zv_d}{x})}$$  (6b)

It is seen that the electron inertia term decreases significantly the real frequency and the growth rate of the wave only when the wave propagated nearly perpendicular to the static magnetic field. The consistency of the six inequalities with the solutions the accuracy of which depend on the validity of the six inequalities, is also verified using the afore-mentioned parameters and the $R_1 = 0.1$ curves separating the region of validity and the region of invalidity are practically unchanged.

Since the resistive ion cyclotron instabilities for low and high $\beta$ plasmas have already been investigated by some authors, there are not much future studies left for this kind of instability. Perhaps the
effect of the inhomogeneity of the plasma and the magnetic field can be considered. Boundary conditions may also be included.

5-2. **Resistive Lower Hybrid Instabilities**

As a second topic, we studied the stability of low frequency electrostatic waves propagating nearly perpendicular to the static magnetic field in a fully-ionized collisional plasma carrying a field-aligned current. The lower hybrid frequency given by Eq. (3-21) for quasi-perpendicular propagation obtained by some other authors was also derived using the two fluid cold plasma model.

As a first study in this kind of instability and also guided by characteristics of the lower hybrid wave, we restrict ourself for the time being to study only the electrostatic instability and confine the propagation angle with respect to the static magnetic field within a small region where $k_z^2/k_x^2 \ll m_e/m_i$ is satisfied. Since the lower hybrid resonance frequency is in the order of the ion plasma frequency, we expect that the wave has a frequency much greater than the ion cyclotron frequency. We also take advantage of the common assumption that the square of the electron cyclotron frequency is much greater than the square of the electron-ion collision frequency and the square of the ion plasma frequency. Under these assumptions, the cold plasma two fluid equations are solved using the small perturbation technique, to get the following results for the real frequency and the growth rate of the instability:

\[
(x - k_z v_d)^2 = \frac{C}{A x^2 - B} \quad \quad \quad (7a)
\]

\[
y = -\frac{\nu e_i}{4 A x^2} \left\{(A + k_x^2 \omega_p^2)x^2 - B\right\}(1 - \frac{k_z v_d}{x}) \quad \quad (7b)
\]
where \( A = k^2 \omega^2_{ce} + k^2 \omega^2_{pe} \) \hspace{1cm} (8a)

\( B = \frac{\omega^2}{\pi} k^2 \omega^2_{ce} \) \hspace{1cm} (8b)

\( C = \omega^2_{pe} (k^2 \omega^2_{ce} + k^2 \nu^2_{ei}) \) \hspace{1cm} (8c)

In chapter three, we proved that for any wave with a frequency \( \omega \) satisfying Eq. (7a) the condition

\[
(A + k^2 \omega^2_{pe})x^2 - B > 0 \hspace{1cm} (9)
\]

is always true. Hence, the sign of \( y \) is determined solely by the sign of the factor \( (x - k_z v_d) \) and the instability criterion is found to be

\[
v_d > x/k_z \hspace{1cm} (10)
\]

which is exactly the same as that for the ion cyclotron instabilities.

We have qualitatively shown that solutions of Eq. (7a) can be obtained by finding the intersection points of the two functions on both sides. These intersection points give the real frequencies of the waves. We find that depending on the parameters, there may be one, two or three intersection points occurring simultaneously. In cases where there is only one intersection point, the wave is found to be stable. When two intersection points exist concurrently, we find that one of the two waves which has a higher frequency is stable while the other wave with a frequency close to the lower hybrid frequency is unstable. However, this case is difficult to obtain and instabilities are found to exist in duality for most of the time. This corresponds to cases where three waves exist concurrently with the stable wave having the highest frequency. The unstable wave with a lower frequency close to the lower hybrid frequency is found to have a greater growth rate than the other instability. This is numerically proved using typical plasma and magnetic field parameters.
In order to have an idea of the order of magnitude of the real frequency, two limiting cases corresponding to $v_d \to 0$ and $v_d \to \infty$ were obtained for reference:

\[ x_0^2(v_d=0) = \frac{B + C}{A} = \omega_{\text{LH}}^2 \left( 1 + \frac{k_z^2 m_i}{k_x^2 m_e} \right) + \omega_{\text{LH}}^2 \frac{\nu_{e i m_i}}{\Omega_{e i m_e}} \]  \quad (11)

\[ x_\infty^2(v_d=0) = \omega_{\text{LH}}^2 \]  \quad (12)

Note that the first term of Eq. (11) is the lower hybrid frequency for quasi-perpendicular propagation. Since these two frequencies are in the order of the lower hybrid frequency, it is quite safe to presume that the frequencies of the possible waves are also in the order of the lower hybrid frequency. This is verified numerically using typical plasma and magnetic field parameters.

From the analysis, we see that the frequencies of the waves are in the order of the ion plasma frequency which is much greater than the ion cyclotron frequency. The instability criterion thus requires high threshold streaming velocities. The parallel phase velocities and hence the threshold streaming velocities can be lowered by increasing the wave number; but this makes the assumption $k_z/k_x \lesssim (m_e/m_i)^{1/2}$ more difficult to attain. From Eq. (3-2), we learn that the lower hybrid frequency can also be lowered by reducing the magnitude of the static magnetic field strength. This in turn lowers the threshold streaming velocities. It should, however, be noted that the confining static magnetic field must not be reduced too much or the leakage of the plasma material will increase to an unbearable amount. From the above argument, we see that the threshold streaming velocities for this kind of instabilities are in general much greater than that for the resistive ion cyclotron instability and the resistive ion-ion hybrid instabilities studied in chapters two and four.
respectively and can not be reduced significantly.

Although results of our research are rather encouraging, there are still many outstanding problems, e.g. the frequencies of the waves can not be expressed by a simple equation, the duality of the instabilities, etc. Future studies may therefore include some more appropriate assumptions to further simplify Eq. (7a). Experimental verifications of the existence of this kind of instabilities are of course inviting. Thermal effect seems quite suitable for the next study and the possible electromagnetic lower hybrid instability in a high $\beta$ plasma is another valuable research subject.

5-3. Resistive Ion-ion Hybrid Instabilities

Guided by the resistive ion cyclotron instability and the resistive lower hybrid instabilities, we extended our research to one more similar instability. This time we assume that the collisional plasma consists of two ion species of different charge-to-mass ratios, ion species $b$ being much heavier than ion species $a$. For electrostatic waves propagating perpendicular to the static magnetic field in high density collisionless plasmas consisting two ion species, it is well known that a Buchsbaum ion-ion hybrid resonance exists with a frequency

$$\omega_B^2 = \frac{\Omega_a^2 \omega_{pb}^2 + \Omega_b^2 \omega_{pa}^2}{\omega_{pa}^2 + \omega_{pb}^2}$$

(13)

which can be shown to lie between the two ion cyclotron frequencies of the two ion species, $\Omega_a$ and $\Omega_b$. For a plasma satisfying

$$\frac{m_a}{m_b} \ll \frac{N_b}{N_a} \ll \frac{m_b}{m_a}$$

(14)

Eq. (13) is found to become

$$\omega_B = \frac{\omega_{pb} \Omega_a}{\omega_{pa}}$$

(15)
In our derivation of the instability, we have made use of this fact and assumed that the square of the ion cyclotron frequency of ion species \(a\) is much greater than the square of the complex angular frequency of the wave which in turn is much greater than the square of the ion cyclotron frequency of ion species \(b\). In order to further simplify the problem, the electron cyclotron frequency is assumed to be much greater than the electron-ion collision frequency which can be satisfied quite easily. For a fully ionized collisional plasma meeting the above assumptions, we have shown that the solutions to the linearized force and continuity equations and Poisson's equation gives:

\[
(x - k_z v_d)^2 = \frac{C x^2}{A x^2 - B} \quad \cdash\cdash \quad (16a)
\]

\[
y = -\frac{\nu_{ei}}{4 A x^2} (1 - \frac{k_z v_d}{x}) \{ (A + \omega_{pe}^2 \Omega_a^2 k_z x^2) x^2 - B \} \cdash\cdash \quad (16b)
\]

where

\[
A = k_x^2 \left( \omega_{pa}^2 \Omega_{ce}^2 + \omega_{pe}^2 \Omega_a^2 \right) + k_a^2 \Omega_a^2 \Omega_{ce}^2
\]

\[
B = \Omega_{a ce}^2 \left( \omega_{pa}^2 k_z^2 + \omega_{pb}^2 k_z^2 \right) \cdash\cdash \quad (17a)
\]

\[
B = \Omega_{a ce}^2 \left( \omega_{pe}^2 k_z^2 + \omega_{pe}^2 k_z^2 \right) \cdash\cdash \quad (17b)
\]

\[
C = \Omega_{a ce}^2 \left( k_x^2 \nu_{ei}^2 + k_z^2 \Omega_{ce}^2 \right) \cdash\cdash \quad (17c)
\]

Just as in the study of the resistive lower hybrid instabilities, the instability criterion for this kind of instabilities is found to be

\[
v_d > \frac{x}{k_z}
\]

since the condition

\[
(A + \omega_{pe}^2 \Omega_a^2 k_x^2) x^2 - B > 0 \quad \cdash\cdash \quad (18)
\]
is always satisfied for waves obtained from Eq. (16a). Although the
instability criterion bears the same form as that for the resistive
lower hybrid instabilities, it predicts much smaller threshold streaming
velocities as the frequencies of the instabilities are much lower than
that of the resistive lower hybrid instabilities and there is also
practically no restriction on the propagation angle with respect to the
static magnetic field so that a larger $k_z$ can be chosen to reduce the
threshold streaming velocities.

using the method outlined in chapter three, it is found that
there may be one, two or three solutions to Eq. (16a). If only one single
wave exists in the plasma, the instability criterion predicts a stable
system. For the critical case where two waves happen concurrently, one
of the waves with a higher frequency is found to be stable while the
other wave with a frequency close to the ion-ion hybrid frequency is
unstable. This empirical case is however very difficult to obtain and
to maintain. Actually, we have shown that together with a third wave
which has the highest frequency and is stable, instabilities usually
occur in duality. One of the instabilities has a lower frequency close
to the ion-ion hybrid frequency and a very large growth rate. The other
instability has a frequency a bit higher than the ion-ion hybrid frequency
and its growth rate is smaller. Numerical illustrations of the one-wave
and three-wave cases using typical parameters are depicted in Figs. 4-1
to 4-4.

The order of magnitude of the real frequencies obtainable from
Eq. (16a) has also been investigated analytically by calculating the two
limiting cases corresponding to $v_d \rightarrow 0$ and $v_d \rightarrow \infty$:

$$x_0^2(v_d=0) = \frac{B + C}{A} = \frac{\omega_{pb}^2}{\omega_{pa}^2} \left(1 + \frac{k_z^2 \omega_{pe}^2}{k_x \omega_{pb}^2} \right) + \frac{m_e v_e^2}{m_a e_i} \quad \cdots \quad (19)$$
Note that the first term of Eq. (19) is the ion-ion hybrid frequency for quasi-perpendicularly propagating waves. Eqs. (19) and (20) show that both \( x_0 \) and \( x_\infty \) are in the order of \( \omega_pb \Omega_a/\omega_pa \) which is the Buchsbaum ion-ion hybrid frequency for a plasma satisfying inequality (14).

Since this thesis gives only a preliminary study on this type of instabilities, there are still many problems left unsolved. For examples, experimental verifications of the existence of these instabilities, finding a simpler expression for the real frequencies of the waves, thermal effect on this kind of instabilities, similar electromagnetic instabilities in high \( \beta \) plasmas, etc. are all interesting subjects for further studies.
APPENDIX A  

PROGRAM TO CHECK THE CONSISTENCY OF THE ANALYTICAL RESULTS WITH THE SIX INEQUALITIES R1 TO R6 (ELECTRON INERTIA EXCLUDED)

A = ICN CYCLOTRON FREQUENCY
L = VELOCITY OF LIGHT
E = ELECTRON-IC COLLISION FREQUENCY
G = ICN PLASMA FREQUENCY

B = ELECTRON CYCLOTRON FREQUENCY
D = ELECTRON DRIFT VELOCITY
F = ICN-ELECTRON MASS RATIO
H = ELECTRON PLASMA FREQUENCY

IMPLICIT COMPLEX (C,R,W)
READ (5,4) N

DO, 2 NC = 1,N
READ (5,10) A,E,L,D,E,F,G,H

10 FORMAT (E20.1)

DA = A * A
DB = E * B
DL = L * L
DD = D * D
DE = E * E
DG = G * G
DH = H * H

C = CC/CDL
P = DL/(F+F) * E
G = 1/F
Z = E/F
S = Z * D
T = DH/DB
U = T/DB
V = E * U
BA = C * A
BB = T/B
BC = C * V
BD = EB * D
BE = V * CD
BF = T/E
EG = LIH
BH = DD/DB * CE

WRITE (6,20) L,A,D,E,F,E,G,H


DO 1CC KX = 1,12
DKX = KX * KX
AKX = KX * BG
CAKX = AKX * AKX
AE = -KX * BC
AF = Z * DAKX
CB = CMPLX ( C,AF)
AG = S * DAKX
AH = FD * KX
APPENDIX A CONTINUED FROM LAST PAGE

AI = RE * DKK
AJ = EH * DKK
WRITE (6,30) KK

3C FORMAT (1HO,5X,5HKX = ,I2)
DO 80 I = 1,1CC
ZK = I/1CC
AZK = ZK * ZK
AZK = AZK * AZK
AP = C + DAIZK
CA = CMPLX (AF,0.)
AQ = DA * DAK
AR = AG * ZK
CC = CMPLX((AC,AR)
CD = 2. * CA
CX = -CB/CD
CY = CX * CX + 4. * CA/CD * CC/CD
W = CX + CSCRT(CY)
X = REAL(W)
XDASH = X - ZK * D
Y = AIMAG(W)
YDASH = CMPLX(XDASH,Y)
WSC = W * W
CE = (DA - WSC)/C
CF = WSC/CE
CG = (ZK - CF
CH =-(0.,1.) * V * WDASH
RXX = CG + CH
RXZ = ZKX + (C.,1.) * AE
RYX = -RXY
RYY = DKK + CC
RYZ = (C.,-1.) * AH
RZX = RXZ
RZY = -RYZ
CI = DKK - (0.,1.) * BF/WDASH * WSC
CJ = (0.,-1.) * AI/WDASH
RZZ = CI + CJ
CP = RXY/RYY
T1 = CABS (CP * RYX/RXX)
T2 = CABS (CP * RYZ/RXZ)
T3 = CABS (RZY/RZZ * RYZ/RYY)
T4 = CABS (CH/CC)
T5 = FC/ZK
T6 = CABS((CJ/CI)
WRITE (6,50) ZK,X,Y,T1,T2,T3,T4,T5,T6
50 FORMAT (1H ,F5.2,2E16.5,6E13.4)
80 CONTINUE
100 CONTINUE
200 CONTINUE
STOP
END
APPENDIX B
PROGRAM TO CHECK THE ELECTRIC FIELD RATIOS (ELECTRON INERTIA NAGLICTED)

A = ICN CYCLOTRON FREQUENCY
B = ELECTRON CYCLOTRON FREQUENCY
L = VELOCITY OF LIGHT
D = ELECTRON DRIFT VELOCITY
E = ELECTRON-ICN COLLISION FREQUENCY
F = ICN-ELECTRON MASS RATIO
G = ICN PLASMA FREQUENCY
H = ELECTRON PLASMA FREQUENCY
C L = VELOCITY OF LIGHT
C D = ELECTRON DRIFT VELOCITY
C E = ELECTRON-ICN COLLISION FREQUENCY
C F = ICN-ELECTRON MASS RATIO
C G = ICN PLASMA FREQUENCY
C H = ELECTRON PLASMA FREQUENCY
C I = ELECTRON CYCLOTRON FREQUENCY
C J = ICN PLASMA FREQUENCY

IMPLICIT COMPLEX (C,R,W)
READ (5,4) N
4 FORMAT (I5)
C DO 20 CC NO = 1,N
C READ (5,10) A, B, L, D, E, F, G, H
C 10 FORMAT (E20.1C)
C DA = A * A
C DB = B * B
C DL = L * L
C DD = D * D
C DE = E * E
C DG = G * G
C DH = H * H
C C = DC/DL
C P = DL/(F + F) * E
C C = E/F
C S = C * D
C T = DH/DL
C U = T/DB
C V = E * U
C BA = C * A
C RR = T/B
C BC = D * V
C BD = EB * D
C BE = V * CD
C BF = T/E
C BG = L/H
C BH = DD/DB * CE
C WRITE (6,20) L, A, D, B, E, F, G, H
C DC ICX KX = 1,12
C DKX = KX * KX
C AKX = KX * BG
C DAKX = AKX * AKX
C AE = -KX * BC
C AF = C * DAKX
C CB = CPPLX ( C., AF)
C AG = S * DAKX
C AH = PD * KX
C AI = PE * DKX
APPENDIX B

CONTINUED FROM LAST PAGE

AJ = BH + DKX
WRITE (6, 30) KX

3C FORMAT (1HO, 5X, 5HKX = 12)
DO 60 I = 1, 1CC
ZK = 1/10C.
DZK = ZK * ZK
ZKK = -KX * ZK
AZK = ZK * BG
DAZK = AZK * AZK
AP = 1/F + DAZK
CA = CMPLX (AF, C.)
AQ = DA * DAZK
AR = AG * ZK
CC = CMPLX (AQ, AR)
CD = 2. * CA
CX = -CB/CD
CY = CX * CX + 4. * CA/CC * CC/CD
W = CX + CSCRT (CY)
X = REAL (W)
XDASH = X - ZK * D
Y = AIMAG (W)
WDASH = CMPLX (XDASH, Y)
WSC = W * W
CE = (DA - WSC)/C
CF = WSC/CE
CG = ZK - CF
CH = -(0., 1.) * V * WDASH
RX = CG + CH
RY = (C., 1.) * (BB * WDASH - A/CE * W)
RXZ = ZKK + (C., 1.) * AE
RYX = -RYY
RYY = DKX + CC
RYZ = (0., -1.) * AH
RZX = RXZ
RYZ = -RYZ
CI = CKX - (0., 1.) * BF/WDASH * WSC
CJ = (C., -1.) * AI/WDASH
RZZ = CI + CJ
CP = (RX * RYZ - RXZ * RYY)/(RX * RYY - RXZ * RYX)
EXZ = CARX (CP)
EYX = CARX (-RYX/RYY - RYX/RYX)
EYZ = CARX (-RYX/RYY * CP - RYX/RYX)
WRITE (6, 50) ZK, X, Y, EXZ, EYX, EYZ
5C FORMAT (1H, F5.2, 2E16.5, 3E25.5)
8C CONTINUE
1C CONTINUE
2C CONTINUE
STOP
END
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APPENDIX C

PROGRAM TO CHECK THE CONSISTENCY OF THE ANALYTICAL RESULTS WITH THE SIX INEQUALITIES B1 TO B6 (ELECTRON INERTIA INCLUDED)

A = ICN CYCLOTRON FREQUENCY
C = VELOCITY OF LIGHT
E = ELECTRON-ICN COLLISION FREQUENCY
G = ICN PLASMA FREQUENCY

R = ELECTRON CYCLOTRON FREQUENCY
D = ELECTRON DRIFT VELOCITY
F = ICN-ELECTRON MASS RATIO
H = ELECTRON PLASMA FREQUENCY

IMPLICIT COMPLEX (P, W, R)
READ (5,4) N
4 FORMAT (15)
DO 2CC NC = 1, N
READ (5,10) A, B, C, D, E, F, G, H
10 FORMAT (E20.1C)
DA = A * A
DB = E * B
DC = C * C
DD = E * D
DE = E * E
DG = G * G
DH = F * H
DCG = DC * DG
DCH = DC * DH
DGH = DG * DH
G = DCG * E
S = C * D
T = DC * DD
U = DCH * DA
V = D * DCG * C
B1 = DH/DC
BA = CG/DC
BB = B1/DB
BC = BA * A
BD = EB * D
BE = E1/B * D
BF = EB * DD
WRITE (6,20) C, A, D, B, F, E, G, H
20 FORMAT (1H1,15X,59HNUMERICAL JUSTIFICATION OF THE SIMPLIFIED COLD
1PLASMA MODEL, /*/, 16X, 19HVELOCITY OF LIGHT =, E20.1C, 15X, 25HION CYCL
2CTRON FREQUENCY =, E20.1C, 15X, 10X, 25HELECTRON DRIFT VELOCITY =, E20.10
3, 1CX, 30HELECTRON CYCLOTRON FREQUENCY =, E20.1C, 15X, 1CX,
4 25HICN-ELECTRON MASS RATIO =, E20.10, 6X, 34HICN-ION COLLISION
5FREQUENCY =, E20.10, 13X, 22HICN PLASMA FREQUENCY =, E20.1C, 13X,
6 27HELECTRON PLASMA FREQUENCY =, E20.10, 7X, 9HREAL FREQ,
7 6X, 9HIMAG FREQ, 9X, 2HR1, 11X, 2HR2, 11X, 2HR3, 11X, 2HR4, 11X, 2HR5,
8 11X, 2HR6)
DO 1CC KX = 1, 12
DKX = KX * KX
AA = ECG * DKX
AB = C * DKX
AC = V * DKX
AD = S * DKX
AE = T * DKX
AF = BD * KX
AG = BE * KX
AH = BF * DKX

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C APPENDIX C CONTINUED

C

WRITE (6,30) KX
3C FORMAT (1HO,5X,5HKX = ,I2)
DO 80 I = 1,ICC
ZK = I/100.
DZK = ZK * ZK
ZKX = -KX * ZK
AQ = ECH * DZK + AA + EGH
PA = CMPLX(0.,AQ)
AP = AC * ZK
PB = CMPLX(AB,AP)
AQ = AD * ZK
AR = AE * DZK
AS = L * DZK
PC = CMPLX(AC,AR - AS)
PX = FB/(PA + PA)
PY = CSGRT (PX * PX - PC/PA)
W = PX + PY
X = REAL(W)
XDASH = X - ZK * D
Y = AIMAG(W)
WDASH = CMPLX(XDASH,Y)
WSG = W * W
PD = DA - WSG
PXX2 = WSG/PD * BA
PE = E - (0.,1.) * WDASH
PXX3 = (0.,1.) * BB * PE * WDASH
PXX12 = DZK - PXX2
RXX = PXX12 - PXX3
RXY = (0.,1.) * (BB * WDASH - BC/PC * W)
RYX = - RXY
PXZ2 = (0.,1.) * AF * PE
RXZ = ZKX - PXZ2
RZK = RXZ
RYY = DKX + DZK - PXX2
RYZ = CMPLX(0.,-AC)
RZY = -RZY
PZZ2 = (0.,1.) * B1/WDASH * WSG/PE
PZZ12 = DKX - PZZ2
PZZ3 = (0.,1.) * AH/WDASH * PE
RZZ = PZZ12 - PZZ3
PP = RXY/RYY
T1 = CABS(PP/RXX * RYX)
T2 = CABS(PP/RXZ * RYZ)
T3 = CABS(RZY/RYY * RYZ/RZZ)
T4 = CABS(-PXX3/PXX12)
T5 = CABS(-PXZ2/ZKX)
T6 = CABS(-PZZ3/PZZ12)
WRITE (6,50) ZK,X,Y,T1,T2,T3,T4,T5,T6
5C FORMAT (1H ,F5.2,2E16.5,6E13.4)
8C CONTINUE
1C CONTINUE
2C CONTINUE
STOP
END
APPENDIX D  PROGRAM FOR THE FINDING OF THE REAL FREQUENCY OF
THE RESISTIVE LOWER HYBRID INSTABILITY

B = ELECTRON CYCLOTRON FREQUENCY  D = ELECTRON DRIFT VELOCITY
E = ELECTRON-ION COLLISION FREQUENCY  F = ICN-ELECTRON MASS RATIO
G = ICN PLASMA FREQUENCY  H = ELECTRON PLASMA FREQUENCY

READ (5,4) J
4 FORMAT (15)
DO 200 NO = 1,J
READ (5,30) KX,B,D,E,G,H,ZK
30 FORMAT (15/(E20.10))
DB = B * B
DE = E * E
DG = G * G
DH = H * H
DKX = KX * KX
DZK = ZK * ZK
DK = DKX + DZK
DBH = DB * DH
DHKX = DH * DKX
DBK = DB * DK
DBGK = DG * DBK
DBHZK = DBH * DZK
DEHKX = DE * DHKX
P = DBK + DHKX
C = DBHZK + DEHKX
R = ZK * D
WRITE (6,40) D,B,F,E,G,H,KX,ZK
40 FORMAT (1H1,35X,55HNUMERICAL JUSTIFICATION OF THE LOWER HYBRID INS
1TABILITY,///,10X,25HELECTRON DRIFT VELOCITY = ,E20.10,
2 10X,30HELECTRON CYCLOTRON FREQUENCY = ,E20.10,/,10X,
3 25HICN-ELECTRON MASS RATIO = ,E20.10,6X,34HELECTRON-ION COLLISION
4FREQUENCY = ,E20.10,/,13X,22HION PLASMA FREQUENCY = ,E20.1C,13X,
5 27HELECTRON Plasma FREQUENCY = ,E20.10,/,35X,5HKX = ,15,10X,
65HZK = ,F8.3,///,40X,1HX,29X,1HF,29X,1HG)
DO 100 I = 1,1000
X = 1.E08 + I * 1.E06
XSQ = X * X
G = Q/(P * XSQ - DBGK) * XSQ
BA = X - R
F = BA * BA
WRITE (6,50) X,F,G
50 FORMAT (1H ,20X,3E30.10)
100 CONTINUE
200 CONTINUE
STOP
END
APPENDIX E  
PROGRAM FOR THE FINDING OF THE REAL FREQUENCY OF  
THE RESISTIVE ION-ION HYBRID INSTABILITY  

\[ A = \text{ICN CYCLOTRON FREQUENCY (A)}, \quad B = \text{ION CYCLOTRON FREQUENCY (B)} \]
\[ C = \text{ELECTRON CYCLOTRON FREQUENCY}, \quad D = \text{ELECTRON DRIFT VELOCITY} \]
\[ E = \text{ELECTRON-ICN COLLISION FREQUENCY}, \quad F = \text{ION PLASMA FREQUENCY (A)} \]
\[ G = \text{ICN PLASMA FREQUENCY (B)}, \quad H = \text{ELECTRON PLASMA FREQUENCY} \]
\[ I = \text{ICN(A) TO ELECTRON MASS RATIO}, \quad J = \text{ION (B) TO ELECTRON MASS RATIO} \]

READ (5, 4) N

4 FORMAT (15)
DO 20 C NO = 1, N
READ (5, 20) KX, I, J, ZK, A, C, D, E, F, G, H

20 FORMAT (3 I10/(E20.10))
DA = A * A
DC = C * C
DE = E * E
DF = F * F
DG = G * G
DH = H * H
DKX = KX * KX
DZK = ZK * ZK
DK = DKX + DZK
DAC = DA * DC
CAH = DA * DH
DEKX = DE * DKX
DCZK = DC * DZK
DFZK = DF * DZK
DGK = DG * DK
DCFKX = DC * DF * DKX
P = DEKX + DCZK
Q = DFZK + DGK
R = ZK * D
BA = CAH * P
BB = DAC * Q
WRITE (6, 40) A, C, D, E, F, G, H, I, J, KX, ZK

40 FORMAT (6, 40) A, C, D, E, F, G, H, I, J, KX, ZK

DO 100 N = 1, 1000
X = N * 1.5E05
XSQ = X * X
G = BA/(XSQ * DCFKX - BB) * XSQ
CA = X - R
F = CA * CA
WRITE (6, 60) X, F, G

60 FORMAT (1 H, 20X, 3 E20.10)
100 CONTINUE
200 CONTINUE
STOP
END
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36. Ott, E., Phys. Fluids 18, 566 (1975)