On Numerical Studies of
Explosion and Implosion in Air

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Master of Philosophy
in
Mathematics

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Explosion and implosion in air are rigorous fluid dynamics phenomena and hence, those studies are highly restricted in a laboratory. Computational fluid dynamics becomes an important tool to investigate such problems. Numerical calculations of hyperbolic conservation laws play an important role in such problems. The resolution of such numerical schemes is highly improved, since the introduction of a class of non-oscillatory explicit second-order accurate finite difference schemes by using the concept of total variation diminishing (TVD). Recently, the development of a class of numerical schemes so called relaxation schemes contributed the field on its simplicity, without using either Riemann solvers spatially or a nonlinear system of algebraic equation solvers temporally. The leading order approximation of the relaxation schemes, so called the relaxed scheme when applying to the one-dimensional scalar conservation law was proved to be TVD.

In this research, the problem of explosion and implosion in air with spherically or cylindrically symmetric flow is studied numerically by using the relaxed scheme. The governing equations are basically the Euler equations with a geometric source term. A splitting method is employed to deal with the inhomogeneous equation. The numerical dissipation of the relaxation scheme is to a large extent determined by a matrix $A$ which is constructed when applying the scheme to the problem. The appropriate choice of matrix $A$ is critical in
the accuracy and practicality of the scheme. An algorithm to choose the matrix and technique in applying the relaxed scheme is developed in addition to maintaining the good features of the relaxed scheme. Some numerical tests are carried out to test the reliability of the algorithm to the Euler system. Numerical simulation of four cases namely spherical explosion, cylindrical explosion, spherical implosion and cylindrical implosion are discussed. The numerical results are presented and compared with some previous numerical, analytical and experimental results.
摘要

空中向外爆炸及向内爆炸是很剧烈的流体动力学现象，所以，在实验室内外有关的研究是有一定程度的困难。于是，计算流体动力学便成为研究有关课题的一个重要的工具，而双曲守恒律的数值计算在此类问题中担任了重要的角色。自从引进了全变差下降(TVD)的概念而发展的一类非振荡、显式、二阶精度差分格式后，数值计算格式的分解能力大大地提升。最近，一类名称为松弛格式(Relaxation Scheme)，它免去运用空间上的黎曼解算子或是时间上的非线性代数方程组解算子，这简单性给予数值计算这范畴上很大的意义。当应用在一维单守恒律时，松弛格式的首阶近似，又名放松格式(Relaxed Scheme)，已证明是TVD的。

在本研究中，放松格式将用来探讨球面对称及圆柱对称的空中向外及向内爆炸的问题。问题的控制方程为欧拉方程带有一个几何有关的源项，这非齐次方程将用分裂法来处理。松弛格式的数值耗散很大程度基于运用这格式时所构造的一个矩阵‘A’，格式的精确度及实用性取决于能否适当地构造它。在保持放松格式的优点的前提下，本论文提供了构造‘A’的算法及应用放松格式的技巧，并使用实际的数值试验来证实了本算法的可靠性，并作了四个数值模拟：球面对称的向外爆炸、圆柱对称的向外爆炸、球面对称的向内爆炸及圆柱对称的向内爆炸。计算结果将会跟以前数值计算的、分析的及实验的结果作一比较。
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"The heavens declare the glory of God; 
the skies proclaim the work of his hands."

Psalms 19.1 (NIV)
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Chapter 1

Introduction

1.1 Background of Explosion and Implosion Problems

Studies of spherical blast waves have long been interested by many researchers ([1], [2], [3], [4], [5]). An experimental investigation of the explosions of a glass spheres under high internal pressure was carried out by Boyer [2] in 1960. A moving outward shock wave and a moving inward rarefaction wave generates at the time the glass sphere broken instantaneously. The former is called the main shock. Due to the dimensionality effect, a shock, also called the second shock, follows from the tail of the rarefaction wave and increases with strength. The second shock reflects from the origin of the sphere and starts propagating outward. Theoretical investigation predicts that once the second shock interacts with the contact discontinuity, similar to the formation of the second shock, a third shock generates and moves inward. Similarly, fourth, fifth, sixth shocks would be formed provided that the time is long enough and the strength of the explosion is large enough. However, hardly can the third shock be observed in experiments carried out in a laboratory ([2], [5]). Some analytical results were studied in order to find the explicit formulae for the location of the shocks and the contact discontinuity ([1], [3]).

For the implosion problem, since it is very difficult to break a spherical diaphragm instantly ([2], [5]). The resulting flow of the experiment was com-
pletely nonuniform and the experimental results were not satisfactory. Because of the difficulty in carrying out an experiment in the laboratory, computational fluid dynamics becomes important to investigate the problem. Sod [6] carried out a numerical experiment on a converging cylindrical shock by using Glimm’s method and operator splitting. Sod successfully captured the second shock produced by the interaction between the main shock and the contact discontinuity.

Due to the limitation of the experimental study, numerical simulations provide a new approach to investigate the fluid dynamics problem. For example, Liu et al. [7] simulated the explosion and implosion problems by using a modified Harten’s scheme. They successfully captured the main features of the problems. Recently, high resolution and robust numerical methods and numerical technique are developed quickly. It would be valuable to carry out some numerical simulations by using different classes of numerical methods to test the practical use of the recently developed methods. Also, the physics of the explosion and implosion problems would understand more thoroughly by the simulations of different numerical schemes.

1.2 Background of the Development of Numerical Schemes

Cylindrical and spherically symmetric wave motion arises naturally in the theory of explosion in air. In these situations, the multidimensional equations may be reduced to essentially one-dimensional equations with a geometric source term to account for the second and third spatial dimensions. The governing equations become a nonlinear system of hyperbolic conservation laws with source terms. One of the approach to tackle such inhomogeneous system is to split for a time step, into the ‘advection problem’ which is a homogeneous hyperbolic problem, and the ‘source problem’ which is a system of ordinary differential equations (ODEs). The solvers for the two split problems can be chosen independently and the numerical methods for both fields are highly
developed. There is a vast literature on ODEs and its numerical methods (see [8] for reference). For the pure advection (homogeneous) hyperbolic problem, two requirements on numerical methods, namely high-order (at least second-order) of accuracy and absence of spurious oscillations have been competing in the history of computational fluid dynamics. A prominent class of nonlinear method, total variation diminishing (TVD) method, is one of the most significant achievements in the development of numerical methods for partial differential equations in the last two decades.

A class of explicit second-order finite difference schemes for computation of weak solutions of hyperbolic conservation laws was developed by Harten [9] in 1983. These highly nonlinear schemes, which were proved to be TVD, are obtained by applying a non-oscillatory first-order accurate scheme to an appropriately modified flux function. Following his work, other robust numerical methods or techniques were developed, e.g., essentially non-oscillatory scheme, subcell resolution, etc ([10], [11], [12], [13]). In simulating the explosion and implosion problems, Liu et al. [7] improved the Harten’s TVD scheme to obtain a high resolution on a contact discontinuity by modifying the scheme with the technique of the artificial compression method (ACM) ([14], [15]). The above schemes are one-dimensional in nature and they can be used to calculate a higher dimensional problem with the help of a dimensional splitting method. However, they are not easy to extend to the multi-dimensional version. So, it would be useful if a class of numerical schemes can be developed, for which can generalize to a higher dimensional version easily and keep its simplicity and explicit nature.

Later, a class of numerical schemes called relaxation schemes for systems of conservation laws in several space dimensions was introduced by Jin & Xin [16] in 1995. The second-order schemes were shown to be TVD in the zero relaxation limit for scalar equations. The advantage of the relaxation scheme is that neither spatially Riemann solvers nor temporally nonlinear system of algebraic equation solvers are required. Also, it keeps its simplicity for general-
ization to a higher dimensional space. Such advantages are interested by many researchers to further investigate and develop the scheme recently ([17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28]). The idea of the relaxation scheme is to use a local approximation by constructing a linear hyperbolic system with a stiff lower order term that approximates the original system with a small dissipative correction. The leading order approximation of the relaxation scheme in the small dissipative correction limit is called the relaxed scheme. During the construction process, a constant matrix 'A' is to be constructed. In order to ensure the dissipative nature of the leading order approximation of the relaxation system, a dissipative condition is required. Although the dissipative condition can always be satisfied by choosing a sufficiently large A. However, larger A introduces more numerical viscosity, so, it is desirable to obtain the smallest A to meet the numerical stability condition. Evje and Fjelde [24] showed that the relaxation scheme produces a poor approximation for a typical mass transport example which involves transition from two-phase flow to single-phase flow. This is due to the 'over-estimate' in choosing the matrix A causing the excessive 'smearing out'. They solved the problem by a flux splitting method so that the choice of the matrix A was possible to become 'reasonably' small. So, the numerical dissipation of the relaxation scheme is to a large extent determined by the matrix A and the choice of A becomes critical in the application of the scheme.

In applying the relaxation scheme to the explosion and implosion problems, we encounter a similar difficulty in choosing the matrix A. As suggested by Jin & Xin [16], the matrix A can be roughly estimated by the characteristics of the original Euler equations such that the characteristics of the relaxation system interlace with those of the Euler equations, or simply take the largest eigenvalue of the Euler equations over the whole space-time domain. However, when the shocks reflect at the origin, the physical quantities will inflate dramatically and causing a 'sudden inflate' of the eigenvalue at this specific time. Surely, the matrix A estimated by the above method will be too large for most of
the calculation time. It would be a main contribution for this research to find an algorithm or at least, some guidelines to estimate the matrix $A$ for the explosion and implosion problems.

1.3 Organization of the Thesis

The objective of this research is to understand the physics of the explosion and implosion problems through numerical simulations; meanwhile, the technique of the numerical scheme for the problem is developed. More specifically, an algorithm of using the relaxed scheme on the problems is developed. In chapter 2, the mathematical model are stated. The governing equations of a spherically or cylindrical symmetric flow are basically the Euler equations with a geometric source term. A splitting method is employed to deal with the inhomogeneous equation and the Euler system is mainly solved by the relaxed scheme. The appropriate choice of the matrix ‘$A$’ is critical in the scheme. An algorithm to choose the matrix is also developed in chapter 2. The numerical results are presented in chapter 3. Four cases namely spherical explosion, cylindrical explosion, spherical implosion and cylindrical implosion are discussed. The numerical results are compared with some previous numerical, analytical and experimental results. Also, two-dimensional models are calculated in order to compare with the approximated cylindrical symmetric models. Finally, some concluding remarks are made in the last chapter.
Chapter 2

Governing Equations and Numerical Schemes

The main interest in this research is to simulate the explosion and implosion problems numerically, and to analyze the numerical method involved. In this chapter, we develop the mathematical model for the problems. After choosing a suitable numerical method for the problems, the details and techniques of the method will be discussed.

2.1 Governing Equations

In the theory of explosion or implosion in air, the problem is normally regarded as an inviscid flow (Sod [6], Liu et al. [7]). By dropping the effects of the body force and the thermal conduction, the equations for an unsteady, two-dimensional, compressible inviscid flow, also called the Euler equations are as follows:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0,$$  \hspace{1cm} (2.1)
where

\[ U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E + p)u \end{pmatrix}, \quad G(U) = \begin{pmatrix} \rho v \\ \rho vv \\ \rho v^2 + p \\ (E + pv) \end{pmatrix}. \tag{2.2} \]

Here \( \rho \) is the density, \( p \) is the pressure, \( E \) is the total energy, \( u \) and \( v \) are the velocity components with respect to the \( x \) and \( y \)-direction respectively.

Similarly, the one-dimensional (1D) Euler equations can be written as,

\[ \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \tag{2.3} \]

where

\[ U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \end{pmatrix}. \tag{2.4} \]

The equation of state for a perfect gas is required for the closure of the problem,

\[ E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2, \tag{2.5} \]

where \( \gamma \) is the ratio of specific heats, which is a constant for a calorically perfect gas and dependent on the temperature for a thermally perfect gas. In our model, we assume the gas is calorically perfect and having \( \gamma = 1.4 \) (i.e. for air).

Wave motion in the explosion and implosion problems is normally cylindrical or spherically symmetric in nature. The multi-dimensional Euler equations may be reduced to essentially one-dimensional equations with a geometric source term vector, \( S(U) \), to account for the second and third spatial dimensions. It is usually written in the form

\[ \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = S(U), \tag{2.6} \]
where

\[
U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E + p)u \end{pmatrix}, \quad S(U) = -\frac{\alpha - 1}{x} \begin{pmatrix} \rho u \\ \rho u^2 \\ (E + p)u \end{pmatrix}.
\] (2.7)

Here now \( x \) is the radial distance from the origin and \( u \) is the radial velocity. For \( \alpha = 1 \), (2.6) returns to our one-dimensional Euler equations (2.3) and it represents a plane one-dimensional flow. For \( \alpha = 2 \), (2.6) represents the cylindrical symmetric flow which approximates a two-dimensional flow. For \( \alpha = 3 \), (2.6) represents the spherically symmetric flow, an approximation to a three-dimensional flow.

A difficulty occurs in solving (2.6) due to the arising of the singularity at the origin, \( x = 0 \). In order to remove the singularity, the following form of the conservative equations is adopted to ensure that the resulting finite difference numerical scheme employed is compatible with the original equations at the origin (Liu et al. [7]). At the origin,

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \alpha \frac{\partial \rho u}{\partial x} &= 0, \\
\frac{\partial E}{\partial t} + \alpha \frac{\partial (E + p)u}{\partial x} &= 0, \\
u &= 0.
\end{align*}
\] (2.8)

### 2.2 Numerical Schemes

#### 2.2.1 Splitting Scheme for Partial Differential Equations with Source Terms

The governing equations (2.6) are non-linear system of hyperbolic conservation laws with source terms. Such inhomogeneous system can be calculated numerically by a splitting method. We split the governing equations (2.6) into the pure advection hyperbolic system, which is a homogenous system, same as
(2.3),
\[ \frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \]
and, a system of ordinary differential equations (ODEs),
\[ \frac{d}{dt} U = S(U). \]

By using the data at \( t = n\Delta t \), where \( \Delta t \) is the step size of time, and \( n \) is an integer, as the initial condition and solving the pure advection hyperbolic problem to find the state at \( t = (n+1)\Delta t \), we get an intermediate state. Then, using this intermediate state as the initial condition and solving the ODEs. We get the result at \( t = (n+1)\Delta t \), at last.

By re-written the procedure above as follows:
\[ U(t = (n+1)\Delta t) = S^{(\Delta t)} L^{(\Delta t)} S^{(\Delta t)} U(t = n\Delta t). \] (2.11)

Here, \( S^{(\Delta t)} \) and \( L^{(\Delta t)} \) can be any numerical solvers with the time step \( \Delta t \) for the ODEs and the pure advection hyperbolic system respectively. If \( S^{(\Delta t)} \) and \( L^{(\Delta t)} \) are at least second-order accurate solution operators, Strang [29] showed that the splitting scheme (2.11) is second-order accurate in time.

### 2.2.2 Boundary Conditions

At the origin, the equations (2.8) are discretized as follows:
\[ \begin{align*}
    \rho_0^{n+1} &= \rho_0^n - \alpha \frac{\Delta t \rho_0^n (4u_1^n - u_2^n)}{2\Delta x}, \\
    E_0^{n+1} &= E_0^n - \alpha \frac{\Delta t (E_0^n + p_0^n) (4u_1^n - u_2^n)}{2\Delta x}, \\
    p_0^{n+1} &= (\gamma - 1) E_0^{n+1}, \\
    u_0^{n+1} &= 0.
\end{align*} \] (2.12)

Here and after, the subscript 'j' indicates the spatial positions at \( x = j\Delta x \), where \( \Delta x \) is the step size for space. So subscript '0', '1' and '2' indicate the points at the origin, \( x = \Delta x \) and \( x = 2\Delta x \) respectively. Also, the superscript 'n' indicates the temporal positions at \( t = n\Delta t \).
Reflective boundaries are employed at the origin. This boundary conditions can account for the coming and interaction of waves from the axi-symmetric direction. A fictitious state $U^n_{-j}$ is defined from the known state $U^n_j$ inside the computational domain,

$$
\rho^n_{-j} = \rho^n_j, \quad u^n_{-j} = -u^n_j \quad \text{and} \quad p^n_{-j} = p^n_j. \quad (2.13)
$$

For the end side of the domain, open-end boundary conditions, or so called, transmissive boundaries are used so as to allow the passage of wave without any effect,

$$
\rho^n_{j+1} = \rho^n_j, \quad u^n_{j+1} = u^n_j \quad \text{and} \quad p^n_{j+1} = p^n_j. \quad (2.14)
$$

### 2.2.3 Numerical Solvers for the ODEs - The Second-Order, Two-Stage Runge-Kutta Method

In (2.11), the numerical solvers $S^{(\Delta t)}$ and $L^{(\Delta t)}$ can be chosen independently. We first concentrate on solving the ODEs (2.10). For simplicity, it is preferred to choose an explicit method. However, if the system of ODEs is stiff, numerical difficulties arise especially for explicit methods. Even using implicit method, Leveque [30] observed that large overshoots and non-physical wave speeds occurred for hyperbolic conservation laws with stiff source terms. So checking the stiffness of the ODEs is upmost important. As stated by Lambert [8], a nonlinear system of the form (2.10) is said to be stiff if

(i) \( \Re(\lambda_i) < 0, \quad i = 1, 2, \ldots, m, \) and,

(ii) \( \max_j |\Re(\lambda_i)| \gg \min_j |\Re(\lambda_i)|. \)

Here $\Re(\lambda_i)$ denotes the real part of the complex number $\lambda_i$, which is the eigenvalue of the Jacobian $A(U) = \frac{\partial S}{\partial U}$. In our ODEs, the Jacobian is found to be

$$
A(U) = \begin{pmatrix}
0 & 1 & 0 \\
-u^2 & 2u & 0 \\
-\gamma E u/\rho + (\gamma - 1)u^3 & \gamma E/\rho - 3(\gamma - 1)u^2/2 & \gamma u
\end{pmatrix}. \quad (2.15)
$$
So, the eigenvalues are $u$ and $\gamma u$, and then the stiffness ratio is

$$\max_j |Re(\lambda_j)| : \min_j |Re(\lambda_j)| = \gamma,$$

which is, for air, only 1.4 in our case. Hence, our system of ODEs is not stiff. The second-order, two-stage Runge-Kutta explicit method is chosen to use due to its simplicity and high accuracy. For time step $\Delta t$, when the system evolves from time $t^n$ to $t^{n+1}$, where $t^n = n\Delta t$, the Runge-Kutta method is described as

$$U^{(1)} = U^n + \Delta t S(t^n, U^n),$$

$$U^{(2)} = U^{(1)} + \Delta t S(t^n + \Delta t, U^{(1)}),$$

$$U^{n+1} = \frac{1}{2} [U^n + U^{(2)}].$$

Here and after $U^n$ and $U^{n+1}$ represent the solutions at time $t^n$ and $t^{n+1}$ respectively.

### 2.2.4 Numerical Solvers for the Pure Advection Hyperbolic Problem - The Second-Order Relaxed Scheme

A relaxation scheme based on the corresponding relaxation systems was developed by Jin & Xin [16] a decade ago. The advantage of the relaxation scheme is that neither spatially Riemann solvers nor temporally nonlinear system of algebraic equation solvers are required. Also, it keeps its simplicity for generalization to a higher dimensional space. Higher dimensional systems can be treated in the same way as in the one dimension because of the constant linear characteristic fields. Such advantages are interested by many researchers to further investigate and develop the scheme recently.

Our pure advection hyperbolic problem (2.9) is a system of conservation laws in one space variable. A corresponding relaxation system can be introduced,

$$\frac{\partial U}{\partial t} + \frac{\partial V}{\partial x} = 0, \quad V \in \mathbb{R}^2,$$

$$\frac{\partial V}{\partial t} + A \frac{\partial U}{\partial x} = -\frac{1}{\epsilon} (V - F(U)), \quad \epsilon > 0.$$
where $\epsilon$ is a positive constant, calling the relaxation rate. The $3 \times 3$ matrix $A,$

$$A = \text{diag}(a_1, a_2, a_3), \quad a_1, a_2, a_3 > 0,$$

(2.18)

will be determined later.

The idea of the relaxation scheme is to solve the relaxation system (2.17) instead of the original hyperbolic system (2.9). The special feature of the linear characteristic fields and localized lower order terms of the relaxation system results in the advantages of the scheme mentioned before. The discretization of the relaxation system is called the relaxing scheme. The leading order approximation of the relaxing scheme in the small $\epsilon$ limit is called the relaxed scheme.

For simplicity, we may choose

$$A = aI, \quad a > 0,$$

(2.19)

where $I$ is the identity matrix. The reduced system, i.e. the zero $\epsilon$ limit of the relaxation system is dissipative provided that

$$\frac{\lambda^2}{a} \leq 1.$$

(2.20)

Here $\lambda = \max_i |\lambda_i(U)|$ where $\lambda_i$ are the genuine eigenvalues of $F'(U)$. For 1D scalar conservation laws,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u \in \mathbb{R},$$

(2.21)

where $f$ is a scalar valued function. We have a corresponding relaxation system,

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad u, v \in \mathbb{R},$$

$$\frac{\partial v}{\partial t} + a \frac{\partial v}{\partial x} = -\frac{1}{\epsilon} (v - f(u)), \quad \epsilon > 0.$$

(2.22)

The dissipative condition now is referred to as the subcharacteristic condition by Liu [31],

$$-\sqrt{a} \leq f'(u) \leq \sqrt{a} \quad \text{for all} \ u.$$

(2.23)
It will be shown later that this condition plays an important role in the numerical stability. Also, the dissipative structure of the first-order correction to the original system implies that the numerical solution to the relaxation system converges to the entropy solution of the original system [16].

As observed in the numerical tests by Jin & Xin, for the case of compressible Euler equations, the relaxing scheme and the relaxed scheme produce essentially the same result. So, in our problem, we employ the second-order relaxed scheme, instead of the relaxing one, for the pure advection hyperbolic part.

We describe below the one-dimensional version of the relaxed scheme we are going to employ. The following second-order explicit Runge-Kutta is applied in the time discretization.

\[
\begin{align*}
U^{(1)} &= U^n - \frac{\Delta t}{\Delta x} \left( F_{j+\frac{1}{2}}(U^n) - F_{j-\frac{1}{2}}(U^n) \right), \\
U^{(2)} &= U^{(1)} - \frac{\Delta t}{\Delta x} \left( F_{j+\frac{1}{2}}(U^{(1)}) - F_{j-\frac{1}{2}}(U^{(1)}) \right), \\
U^{n+1} &= \frac{1}{2} \left[ U^n + U^{(2)} \right].
\end{align*}
\]

Here \( F_{j+\frac{1}{2}} \) is the numerical flux detailed below, and we use uniform grids for both spatial and temporal space with step sizes \( \Delta x \) and \( \Delta t \) respectively.

In order to obtain a second-order accuracy in spatial discretization, van Leer’s MUSCL scheme [32] are used. The MUSCL scheme uses the piecewise linear interpolation to achieve the second-order accuracy. For the \( p \)-component of the flux, it is written as

\[
F_{j+\frac{1}{2}}^{(p)}(U^n) = \frac{1}{2} \left( F^{(p)}(U^n_j) + F^{(p)}(U^n_{j+1}) \right) - \frac{1}{2} \sqrt{a} (U_j^{(p),n} - U_{j+1}^{(p),n}) \]

\[
+ \frac{1}{4} (\sigma_j^+ - \sigma_{j+1}^-),
\]

where

\[
\sigma_j^+ = \left( F^{(p)}(U_{j+1}^n) \pm \sqrt{a} U_{j+1}^{(p),n} - F^{(p)}(U_j^n) \mp \sqrt{a} U_j^{(p),n} \right) \phi(\theta_j^+), \]

\[
\theta_j^+ = \frac{F^{(p)}(U_j^n) \pm \sqrt{a} U_j^{(p),n} - F^{(p)}(U_{j-1}^n) \mp \sqrt{a} U_{j-1}^{(p),n}}{F^{(p)}(U_{j+1}^n) \pm \sqrt{a} U_{j+1}^{(p),n} - F^{(p)}(U_j^n) \mp \sqrt{a} U_j^{(p),n}}.
\]
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\( \phi(\theta^\pm) \) is called the slope limiter function. Sweby [33] gave out a general condition of the slope-limiter function for the scheme to be TVD,

\[
0 \leq \frac{\phi(\theta)}{\theta} \leq 2, \quad \text{and} \quad 0 \leq \phi(\theta) \leq 2 \tag{2.28}
\]

In particular, we use the one introduced by van Leer [32] in this research,

\[
\phi(\theta) = \frac{\|\theta\| + \theta}{1 + \|\theta\|} \tag{2.29}
\]

We want to remark here that due to the explicit and the special structure of the linear characteristic field of the relaxation scheme, the components of the solution vector can be obtained in parallel computing without any modification of the scheme.

**Theorem:**

For the slope-limiters satisfying (2.28), by choosing the matrix \( A \) as (2.19), the corresponding second-order relaxed schemes (2.24) to (2.27) for the 1D scalar conservation law (2.21) are TVD provided that the subcharacteristic conditions (2.23) and the following CFL condition are satisfied,

\[
\sqrt{a} \frac{\Delta t}{\Delta x} \leq \frac{1}{2}. \tag{2.30}
\]

**Proof of the Second-Order Relaxed Scheme to be TVD for 1D scalar Conservation Law**

Our proof essentially follows the same as Jin & Xin’s [16] analysis with only a slight modification.

A scheme is called TVD if its total variation \( TV(u) := \sum_{j=-\infty}^{\infty} |u_{j+1} - u_j| \) decreases in time,

\[
TV(u^{n+1}) \leq TV(u^n)
\]

The relaxed scheme with the method of line takes the form

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = L(u)
\]
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\[ := - \frac{1}{2\Delta x} \left( f(u_{j+1}^n) - f(u_{j-1}^n) \right) + \frac{\sqrt{a}}{2\Delta x} \left( u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) + \frac{1}{4} \left( \sigma_{j+1}^--\sigma_j^- + \sigma_j^+ - \sigma_{j-1}^+ \right). \] (2.31)

By the subcharacteristic condition (2.23), we have

\[ C_j := \frac{\Delta t}{2\Delta x} \left( \sqrt{a} + \frac{f(u_{j+1}^n) - f(u_j^n)}{u_{j+1}^n - u_j^n} \right) \geq 0, \] (2.32)
\[ D_j := \frac{\Delta t}{2\Delta x} \left( \sqrt{a} - \frac{f(u_{j+1}^n) - f(u_j^n)}{u_{j+1}^n - u_j^n} \right) \geq 0. \] (2.33)

By substituting \( C_j \) and \( D_j \), the relaxed scheme (2.31) becomes

\[ u_j^{n+1} = u_j^n - C_{j-1}(u_j^n - u_{j-1}^n) + D_j(u_{j+1}^n - u_j^n) + \frac{1}{4} \left( \sigma_{j+1}^- - \sigma_j^- + \sigma_j^+ - \sigma_{j-1}^+ \right). \] (2.34)

Then we have

\[ u_{j+1}^{n+1} - u_j^{n+1} = (1 - C_j - D_j)(u_{j+1}^n - u_j^n) + C_{j-1}(u_j^n - u_{j-1}^n) + D_{j+1}(u_{j+2}^n - u_{j+1}^n) + E_{j+\frac{1}{2}}, \] (2.35)

where

\[ E_{j+\frac{1}{2}} = \frac{1}{4} \left[ (\sigma_{j+2}^--\sigma_{j+1}^-) - (\sigma_{j+1}^- - \sigma_j^-) \right. \]
\[ - (\sigma_{j+1}^+ - \sigma_j^+) + (\sigma_j^+ - \sigma_{j-1}^+) \]. (2.36)

By rewriting \( \sigma_j^\pm \) using \( C_j \) and \( D_j \),

\[ E_{j+\frac{1}{2}} = \frac{1}{2} \left[ -\phi(\theta_{j+2})D_{j+2}(u_{j+3}^n - u_{j+2}^n) \right. \]
\[ + (2\phi(\theta_{j+1})D_{j+1} - \phi(\theta_{j+1})C_{j+1})(u_{j+2}^n - u_{j+1}^n) \]
\[ + (2\phi(\theta_j^+)C_j - \phi(\theta_j^-)D_j)(u_{j+1}^n - u_j^n) \]
\[ - \phi(\theta_{j-1})C_{j-1}(u_j^n - u_{j-1}^n) \], (2.37)

and observing that

\[ \theta_j^-C_j(u_{j+1}^n - u_j^n) = C_{j-1}(u_j^n - u_{j-1}^n), \] (2.38)
\[ \theta_j^-D_j(u_{j+1}^n - u_j^n) = D_{j-1}(u_j^n - u_{j-1}^n), \] (2.39)
we get
\[ u_{j+1}^{n+1} - u_j^{n+1} = \left[ 1 - \left( 1 - \phi(\theta_j^+) + \frac{\phi(\theta_j^{+1})}{2\theta_j^{+1}} \right) C_j - \left( 1 + \frac{\phi(\theta_j^-)}{2} \right) D_j \right] (u_{j+1}^n - u_j^n) \]
\[ + \left( 1 - \frac{\phi(\theta_j^{-1})}{2} \right) C_{j-1} (u_j^n - u_{j-1}^n) \]
\[ + \left( 1 + \phi(\theta_j^{-1}) - \frac{\phi(\theta_j^{+2})}{2\theta_j^{+2}} \right) D_{j+1} (u_{j+2}^n - u_{j+1}^n). \]
(2.40)

By using the CFL condition (2.30) and the condition for slope limiter (2.28), we can prove that the three coefficients on the right-hand side of (2.40) are non-negative,
\[ 1 - \left( 1 - \phi(\theta_j^+) + \frac{\phi(\theta_j^{+1})}{2\theta_j^{+1}} \right) C_j - \left( 1 + \frac{\phi(\theta_j^-)}{2} \right) D_j \geq 0, \]
\[ 1 - \frac{\phi(\theta_j^{-1})}{2} \geq 0, \]
\[ 1 + \phi(\theta_j^{-1}) - \frac{\phi(\theta_j^{+2})}{2\theta_j^{+2}} \geq 0. \]
(2.41)

We then obtain
\[ \left| u_{j+1}^{n+1} - u_j^{n+1} \right| \leq \left| 1 - \left( 1 - \frac{\phi(\theta_j^+)}{2\theta_j^+} \right) C_j - \left( 1 + \frac{\phi(\theta_j^-)}{2} \right) D_j \right| |u_{j+1}^n - u_j^n| \]
\[ + \left( 1 - \frac{\phi(\theta_j^{-1})}{2\theta_j^{+1}} \right) C_{j-1} |u_j^n - u_{j-1}^n| \]
\[ + \left( 1 + \phi(\theta_j^{-1}) - \frac{\phi(\theta_j^{+2})}{2\theta_j^{+2}} \right) D_{j+1} |u_{j+2}^n - u_{j+1}^n|. \]
(2.42)

By (2.39), the slope limiter condition (2.28) and the fact that
\[ \phi(\theta) = 0, \ \forall \theta \leq 0, \]
we have
\[ \frac{\phi(\theta_j^+)}{\theta_j^+} C_{j-1} |u_j^n - u_{j-1}^n| = \phi(\theta_j^+) C_j |u_{j+1}^n - u_j^n|, \]
\[ \frac{\phi(\theta_j^-)}{\theta_j^-} D_{j-1} |u_j^n - u_{j-1}^n| = \phi(\theta_j^-) D_j |u_{j+1}^n - u_j^n|. \]
(2.43)
Now we obtain

\[
[1 - \left(1 - \frac{1}{2} \phi(\theta_j^+)\right) C_j - \left(1 + \frac{1}{2} \phi(\theta_j^-)\right) D_j] [u_{j+1}^n - u_j^n] \\
+ \left(1 - \frac{1}{2} \phi(\theta_{j-1}^+)\right) C_{j-1} [u_{j-1}^n - u_j^n] \\
+ \left(1 + \frac{1}{2} \phi(\theta_{j+1}^-)\right) D_{j+1} [u_{j+2}^n - u_{j+1}^n] \\
+ \frac{1}{2} \phi(\theta_j^+) C_j [u_{j+1}^n - u_j^n] \\
- \frac{1}{2} \phi(\theta_{j-1}^+) C_{j-1} [u_{j-2}^n - u_{j-1}^n] \\
+ \frac{1}{2} \phi(\theta_{j+1}^-) D_{j+1} [u_{j+2}^n - u_{j+1}^n] \\
- \frac{1}{2} \phi(\theta_{j-2}) D_{j-2} [u_{j-3}^n - u_{j-2}^n].
\] (2.44)

By summing up over all \(j\) yields,

\[
TV(u^{n+1}) = \sum_{j=-\infty}^{\infty} |u_{j+1}^{n+1} - u_j^{n+1}| \leq \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n| = TV(u^n) \tag{2.45}
\]

Up to now, we have proved that the MUSCL relaxed scheme with one-step time discretization is TVD.

In order to get second-order accuracy in time, the second-order Runge-Kutta splitting scheme is used:

\[
\frac{u_j^{(1)} - u_j^n}{\Delta t} = L(u^n), \quad \frac{u_j^{(2)} - u_j^{(1)}}{\Delta t} = L(u^{(1)}), \\
u^{n+1} = \frac{1}{2} (u^n + u^{(2)}).
\]

We have already proved that

\[
TV(u^{(1)}) = \sum_{j=-\infty}^{\infty} |u_{j+1}^{(1)} - u_j^{(1)}| \\
= \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n + \Delta t (L(u_{j+1}^n) - L(u_j^n))| \\
\leq \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n| = TV(u^n)
\]

By using the result of Shu & Osher [12], we can show that the second-order Runge-Kutta splitting scheme is also TVD,

\[
TV(u^{n+1}) = \sum_{j=-\infty}^{\infty} |u_{j+1}^{n+1} - u_j^{n+1}|.
\]
\[= \sum_{j=-\infty}^{\infty} \frac{1}{2} |u_{j+1}^n - u_j^n| + \frac{1}{2} |u_{j+1}^{(1)} - u_j^{(1)}| + \Delta t \left( L(u_{j+1}^{(1)}) - L(u_j^{(1)}) \right) \]

\[\leq \frac{1}{2} TV(u^n) + \frac{1}{2} TV(u^{(1)}) \]

\[\leq \frac{1}{2} TV(u^n) + \frac{1}{2} TV(u^n)\]

\[= TV(u^n),\]

and we complete the proof of the theorem.

We remark that no rigorous proofs are available for the TVD property for either the multi-dimensional scalar case or 1D system conservation laws. However, we are going to develop an algorithm to choose the matrix \( A \) and setup some criteria for somewhat similar to the CFL condition (2.30) in the next section. Then, a number of numerical test will be carried out to give confidence to our algorithm, before applying to the explosion and implosion problems in the next chapter.

**Algorithm for the Relaxed Scheme**

We want to solve the 1D Euler system (2.9) by using the relaxed scheme mentioned before. For simplicity, we choose the matrix \( A \) as in (2.19) for the system. In order to ensure the relaxed system to be dissipative, the condition (2.20) must be satisfied. In fact, this can always be satisfied by choosing a sufficiently large \( A \). However, larger \( A \) introduces more numerical viscosity, so, it is desirable to obtain the smallest \( A \) to meet the numerical stability condition.

Although we cannot obtain any CFL condition other than the case of 1D scalar for the scheme to be TVD, we use the condition (2.30) as a reference. We define the CFL number as

\[\text{CFL number} := \sqrt{a \frac{\Delta t}{\Delta x}}. \quad (2.46)\]

For returning to the scalar case, the CFL condition (2.30) is expressed as

\[\text{CFL number} \leq \frac{1}{2}. \quad (2.47)\]
Since the eigenvalues of the 1D Euler equations are \( u \) and \( u \pm c \) where

\[
c = \sqrt{\gamma p/\rho},
\]

is the sound speed. In order to obtain the minimum \( A \), and fulfill the dissipative condition (2.20), one can take

\[
a = \max\{\sup |u - c|^2, \sup |u|^2, \sup |u + c|^2\},
\]

(2.49)

here the supremum is taken over all \( t \) and \( x \) in the whole space-time domain. However, as evidenced by the previous researches ([2], [6], [7]), we know that the quantities of the eigenvalues would have a large variation along time, especially when the shocks reflect at the origin and the physical quantities would inflate dramatically. If we simply take the supremum over all time and space, the resulting \( A \) would become too large relative to some time intervals. Excessive numerical viscosity would be injected during such time intervals and the accuracy of the calculation would be lower in general. So, we propose to choose a 'dynamic' matrix \( A \) such that it is tailor-made for each time step interval.

Let us consider a system evolving from time \( t^n \) to \( t^{n+1} \). Instead of taking the supremum of eigenvalues over all time, the idea is to take the supremum over this time step interval only. Now, we want to estimate the maximum eigenvalues inside the time interval \([t^n, t^{n+1}]\). The estimate follows the spirit of the Godunov scheme. We assume the state at \( t^n \) as a piecewise constant. The state keeps constant along every spatial interval \([x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]\). Evoluting the system to the time \( t^{n+1} \) is equivalent to solve a Riemann problem with the initial condition of the state at \( t^n \). The state at the position \( x_j \) will remain unchanged inside the time interval \( \Delta t \) provided that \( \max_i |\lambda_i| \leq \frac{A}{2\Delta t} \) where \( \lambda_i \) is the eigenvalue of the system. Then the constant 'a' is chosen as the same manner as (2.49), but now, the supremum is taken over \( t \in [t^n, t^{n+1}] \) and for all \( x \) in the spatial domain. By using this method of estimate, the constant 'a' will be different for each time step. For each time step interval, we suggest...
the CFL number to be,

\[ \text{CFL number} = \sqrt{a} \frac{\Delta t}{\Delta x} = \max | \lambda_i | \frac{\Delta t}{\Delta x} \leq \frac{1}{2}, \]  

(2.50)

which matches with the CFL condition (2.47) when returning to the scalar case. By using the CFL condition in the scalar case as a reference, we impose no additional constraint on the CFL number for the method proposed in this section.

We remark that the above method provides only a rough estimate since the Godunov scheme is only first-order accurate. So, this method in no way guarantees that the estimate always satisfies the dissipative condition, however, a close estimate to the minimum \( A \) is expected to obtain. In practical use, we set the time step in the computer program by

\[ \Delta t = \frac{(\text{CFL number}) \times \Delta x}{\sqrt{\beta \times \max(|u + c|, |u|, |u - c|)}}. \]  

(2.51)

By changing the adjusting factor \( \beta \) as long as the numerical stability is still maintained, we believe that the solution is reliable.

**Numerical Test for the Algorithm**

As mentioned in the previous section that we can only prove the TVD property of the 1D scalar conservation law. However, in this research, what we want to calculate instead, is the system of equations. In order to apply the relaxed scheme on the Euler system with the algorithm developed in the previous section, we have done a series of numerical tests to support it.

It should be noted that we are not going to test the accuracy of the numerical scheme in this section, instead, we want to give evidence that the scheme do converge to the 'correct' solution and it is numerically stable. So in the following parts, instead of comparing with other numerical methods, we compare our numerical solution with the exact solution. Once the result shows that the scheme converge to the exact solution, we conclude that the scheme is numerically stable and hence reliable.
CHAPTER 2. GOVERNING EQUATIONS AND NUMERICAL SCHEMES

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<th>$\rho_L$</th>
<th>$u_L$</th>
<th>$\rho_L$</th>
<th>$u_R$</th>
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</tbody>
</table>

Table 2.1: Data for Five Riemann Problem Tests.

Five tests are selected to test the performance of the numerical scheme. They have been used by Toro [34]. Relaxed scheme with the algorithm developed in the previous section are tested. In all the tests, the spatial domain is $x \in [0, 1]$. The ratio of specific heats is $\gamma = 1.4$. The step size is $\Delta x = 0.005$ and the CFL number $= 0.3$. For each test, different step size, CFL number and adjusting factor $\beta$ in (2.51) have already been tested repeatedly to ensure that the result is numerically stable. The tests are Riemann problems with the initial data summarized in table 2.1, in terms of the primitive variables. All quantities are dimensionless. Subscript 'L' and 'R' represents the left and right constant values of the initial data discontinuous at the position $x = x_0$.

The first test is similar to the ‘Sod test problem’ [35]. It is, in fact, a modification of the Sod test and the solution consists of a right shock wave, a right traveling contact wave and a left sonic rarefaction wave. This test is very useful in assessing the entropy satisfaction property of numerical methods.

The second test is called ‘123 problem’ and its solution consists of two strong rarefactions and a trivial stationary contact discontinuity. The solution region between the non-linear waves is close to vacuum, so this test is useful in assessing the performance of numerical methods for low density flows [36]. The Richtmyer (or two-step Lax-Wendroff) method fails to give a solution to this problem.
Figure 2.1: Test 1: Exact solution (solid line) and numerical solution (square symbol) for pressure, density, velocity and temperature at time $t = 0.2$. 
Figure 2.2: Test 2: Exact solution (solid line) and numerical solution (square symbol) for pressure, density, velocity and temperature at time $t = 0.15$. 
Figure 2.3: Test 3: Exact solution (solid line) and numerical solution (square symbol) for pressure, density, velocity and temperature at time $t = 0.012$.

Test 3 is the left half of the ‘blast wave problem’ by Woodward and Colella [37], which is a very severe test problem. The solution consists of a left rarefaction, a contact discontinuity and a strong right shock with Mach number $= 198$. The Richtmyer (or two-step Lax-Wendroff) method fails to give a solution to this problem.

Test 4 is a very severe test, too. Its solution represents the collision of two strong shocks produced by the above test (Test 3) and consists of a left facing shock traveling very slowly to the right, a right traveling contact discontinuity and a right traveling shock wave. So, all three discontinuities are traveling to
Figure 2.4: Test 4: Exact solution (solid line) and numerical solution (square symbol) for pressure, density, velocity and temperature at time $t = 0.035$.

The fifth test simulates the collision of two uniform streams. Its solution consists of two strong symmetric shock waves and a trivial contact discontinuity. In this test, the adjusting factor $\beta$ should be at least 6.5 for stable solution, while $\beta \approx 1$ is large enough for all other tests.

In figures 2.1 to 2.5, it is shown that the relaxed scheme with the algorithm in the previous section passes all the five tests above. Although we cannot prove the TVD property of the scheme when applying to the Euler system, the numerical experiments above show the applicability of our scheme to the
Figure 2.5: Test 5: Exact solution (solid line) and numerical solution (square symbol) for pressure, density, velocity and temperature at time $t = 0.8$. 
Euler system.

Two-Dimensional Case

One of the advantages of the relaxation scheme is its simplicity for generalization to a higher dimensional space. Since the governing equation (2.6) is an approximation to the multi-dimensional case. So, it is worthwhile and convenient to calculate the original multi-dimensional system by the relaxation scheme in order to analyze the effectiveness of the approximated equation. We choose to calculate the two-dimensional case for comparison with the cylindrical symmetric approximated model.

Considering the two-dimensional Euler System (2.1) and (2.2), similar to the 1D case, a corresponding relaxation system can be introduced,

\[
\begin{align*}
\frac{\partial U}{\partial t} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} &= 0, \quad V, W \in \mathbb{R}^3, \\
\frac{\partial V}{\partial t} + A \frac{\partial U}{\partial x} &= -\frac{1}{\epsilon} (V - F(U)), \\
\frac{\partial W}{\partial t} + B \frac{\partial U}{\partial y} &= -\frac{1}{\epsilon} (W - G(U)), \quad \epsilon > 0.
\end{align*}
\]

(2.52)

For simplicity, as in (2.19), we consider the $3 \times 3$ matrices $A = a I$ and $B = b I$, where $a, b > 0$ are to be chosen. Jin & Xin [16] showed that the reduced system (2.52) is dissipative provided that

\[
\lambda^2 \geq \frac{a^2}{b} \leq 1.
\]

(2.53)

Here $\lambda = \max_i |\lambda_i(U)|$ and $\mu = \max_i |\mu_i(U)|$ where $\lambda_i$ and $\mu_i$ are the genuine eigenvalues of $F'(U)$ and $G'(U)$ respectively. The relaxed scheme can be expressed as follows:

\[
\begin{align*}
U^{(1)} &= U^n - \frac{\Delta t}{\Delta x} \left( F_{j+\frac{1}{2}}(U^n) - F_{j-\frac{1}{2}}(U^n) \right) \\
&\quad - \frac{\Delta t}{\Delta y} \left( G_{j+\frac{1}{2}}(U^n) - G_{j-\frac{1}{2}}(U^n) \right), \\
U^{(2)} &= U^{(1)} - \frac{\Delta t}{\Delta x} \left( F_{j+\frac{1}{2}}(U^{(1)}) - F_{j-\frac{1}{2}}(U^{(1)}) \right) \\
&\quad - \frac{\Delta t}{\Delta y} \left( G_{j+\frac{1}{2}}(U^{(1)}) - G_{j-\frac{1}{2}}(U^{(1)}) \right),
\end{align*}
\]
\[ U^{n+1} = \frac{1}{2} \left[ U^n + U^{(2)} \right]. \] (2.54)

where \( \Delta t, \Delta x \) and \( \Delta y \) are the step size for time, for spatial \( x \)-direction and spatial \( y \)-direction respectively. Indices ‘n’, ‘i’ and ‘j’ are corresponded to ‘t’, ‘x’ and ‘y’ respectively.

For the \( p \)-component of the flux, it is written as

\[ F_{j+\frac{1}{2}}^{(p)} (U^n) = \frac{1}{2} \left( F^{(p)}(U_{i,j+1}^n) + F^{(p)}(U_{i,j+1}^n) \right) - \frac{1}{4} \sqrt{a} (U_{i+1,j}^{(p),n} - U_{i,j}^{(p),n}) + \frac{1}{4} (\sigma_{i,j}^{x,\pm} - \sigma_{i+1,j}^{x,\pm}) \] (2.55)

\[ G_{j+\frac{1}{2}}^{(p)} (U^n) = \frac{1}{2} \left( G^{(p)}(U_{i,j+1}^n) + G^{(p)}(U_{i,j+1}^n) \right) - \frac{1}{4} \sqrt{b} (U_{i+1,j}^{(p),n} - U_{i,j}^{(p),n}) + \frac{1}{4} (\sigma_{i,j}^{y,\pm} - \sigma_{i,j+1}^{y,\pm}) \] (2.56)

where the slope limiters are

\[ \sigma_{i,j}^{x,\pm} = \left( F^{(p)}(U_{i+1,j}^n) \pm \sqrt{a} U_{i+1,j}^{(p),n} - F^{(p)}(U_{i,j}^n) \right) \phi(\theta_{i,j}^{x,\pm}) \] (2.57)

\[ \theta_{i,j}^{x,\pm} = \frac{F^{(p)}(U_{i,j}^n) \pm \sqrt{a} U_{i,j}^{(p),n} - F^{(p)}(U_{i-1,j}^n) \pm \sqrt{a} U_{i-1,j}^{(p),n}}{F^{(p)}(U_{i,j}^n) \pm \sqrt{a} U_{i,j}^{(p),n} - F^{(p)}(U_{i,j}^n) \pm \sqrt{a} U_{i,j}^{(p),n}} \] (2.58)

\[ \sigma_{i,j}^{y,\pm} = \left( G^{(p)}(U_{i,j+1}^n) \pm \sqrt{b} U_{i,j+1}^{(p),n} - G^{(p)}(U_{i,j}^n) \right) \phi(\theta_{i,j}^{y,\pm}) \] (2.59)

\[ \theta_{i,j}^{y,\pm} = \frac{G^{(p)}(U_{i,j}^n) \pm \sqrt{b} U_{i,j}^{(p),n} - G^{(p)}(U_{i,j-1}^n) \pm \sqrt{b} U_{i,j-1}^{(p),n}}{G^{(p)}(U_{i,j}^n) \pm \sqrt{b} U_{i,j}^{(p),n} - G^{(p)}(U_{i,j}^n) \pm \sqrt{b} U_{i,j}^{(p),n}} \] (2.60)

We define the CFL number as

\[ \text{CFL number} := \max \left( \sqrt{a} \frac{\Delta t}{\Delta x}, \sqrt{b} \frac{\Delta t}{\Delta y} \right), \] (2.61)

so that it would naturally return to (2.47) for degenerating to the case of 1D scalar conservation law.

Similar algorithm as the previous section in order to obtain a rough estimation of matrix \( A \) and \( B \) are used. However, numerical stability should be taken more carefully by the adjusting factor \( \beta \) because of the more uncertainty in this 2D case now.

For the next chapter, we would use the numerical scheme stated and the algorithm developed in this chapter to calculate the explosion and implosion problems.
In this chapter, results of numerical simulations of explosion and implosion problems are presented. The numerical method developed in the previous chapter is used. The results are shown to be reliable by comparing with the data calculated by the modified Harten's scheme [7]. The modified Harten's scheme attempts to have a high resolution for contact discontinuities in addition to maintaining the good features of Harten's TVD scheme. A technique in the same spirit of the artificial compression method (ACM) is used in the modified Harten's scheme. The original ACM [14] is capable to maximize the resolution of a steady progressing profile. In addition to the five tests in the previous chapter, in order to test the smearing problem of our numerical scheme at contact discontinuities, we carry out the sixth test as shown in table 3.1.

This Test 6 corresponds to an isolated contact discontinuity moving slowly to the left. In figure 3.1, the relaxed scheme (square symbol) is compared with the Harten's scheme [9] (dashed line) and the modified Harten's scheme [7]

<table>
<thead>
<tr>
<th>Test</th>
<th>(\rho_L)</th>
<th>(u_L)</th>
<th>(p_L)</th>
<th>(\rho_R)</th>
<th>(u_R)</th>
<th>(p_R)</th>
<th>(x_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1.4</td>
<td>-0.1</td>
<td>1.0</td>
<td>1.0</td>
<td>-0.1</td>
<td>1.0</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 3.1: Data for the 6th Riemann Problem Test.
(dash-dot line). It is seen that the resolution for contact discontinuities of the relaxed scheme is not as high as the modified Harten's but better than the Harten's scheme. In the following presentation of the explosion and implosion problems, the results of the relaxed scheme and the modified Harten's scheme would be shown on the same graph. This comparison is significant because these two numerical schemes are based on completely different ideas.

### 3.1 Spherical Explosion Problem

For the spherical explosion problem, for purpose of comparison, we choose the same model as such experimentally investigated by Boyer [2], analytically analyzed by Brode [1] and numerically discussed by Liu et al. [7]. A glass sphere with radius $x_0 = 1$ in. initially filled with air at pressure $p_H' = 326$ p.s.i. and temperature $T' = 299$ K is exploded in air at the atmospheric pressure and the same temperature inside the sphere. The superscript 'prime' (') denotes the dimensional quantity. The dependent variables are non-dimensionalized via

$$
\rho = \rho' / \rho_0', \quad p = p' / (\rho_0' c_0'^2), \quad u = u' / c_0'.
$$

Here $c_0'$ is the initial sound speed calculated by (2.48), in which the corresponding density is $\rho_0' = \rho_0'$ and the pressure is determined by the ideal gas law. The independent variables are non-dimensionalized via

$$
t = t' / (4x_0' / c_0'), \quad x = x' / (4x_0').
$$

As mentioned in chapter 2, in our model, we assume the gas is calorically perfect and having $\gamma = 1.4$ (i.e. for air). The initial values corresponding to the non-dimensionalized quantities are calculated and summarized in table 3.2.

At time $t = 0$, the glass sphere with radius $x = 0.25$ is broken and it is an analogue to a Riemann problem. In the following numerical results, $\Delta x = 0.05$, CFL number = 0.3, the adjusting factor $\beta = 1$. 
Figure 3.1: Test 6: Comparison of the smearing effect at the contact discontinuity among exact solution (solid line) and numerical solutions (relaxed scheme - square symbol, Harten’s scheme - dashed line, modified Harten’s scheme - dash-dot line) at time $t = 1.0$. 
CHAPTER 3. NUMERICAL RESULTS 32

<table>
<thead>
<tr>
<th>Internal Density</th>
<th>( \rho'_H = 26.1886 \text{ kgm}^{-3} )</th>
<th>( \rho_H = 21.733 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Velocity</td>
<td>( u'_H = 0 \text{ m/s} )</td>
<td>( u_H = 0 )</td>
</tr>
<tr>
<td>Internal Pressure</td>
<td>( p'_H = 326 \text{ p.s.i.} )</td>
<td>( p_H = 15.514 )</td>
</tr>
<tr>
<td>External Density</td>
<td>( \rho'_L = 1.205 \text{ kgm}^{-3} )</td>
<td>( \rho_L = 1.0 )</td>
</tr>
<tr>
<td>External Velocity</td>
<td>( u'_L = 0 \text{ m/s} )</td>
<td>( u_L = 0 )</td>
</tr>
<tr>
<td>External Pressure</td>
<td>( p'_L = 15 \text{ p.s.i.} )</td>
<td>( p_L = 0.715 )</td>
</tr>
<tr>
<td>Unit Time</td>
<td>( t' = 293 \mu s )</td>
<td>( t = 1 )</td>
</tr>
<tr>
<td>Sphere Radius</td>
<td>( x'_0 = 1 \text{ in.} )</td>
<td>( x_0 = 0.25 )</td>
</tr>
</tbody>
</table>

Table 3.2: Initial Data for Explosion Problem.

3.1.1 Physical Description

The present numerical results are in fairly good agreement with the results obtained by Liu et al. [7]. Immediately after the broken of the glass sphere at \( t = 0 \), a spherical shock (main shock) is generated and moves outward in the air. The compressed gas is expanded through a spherical rarefaction wave and a contact discontinuity separates the expanded gas from the air compressed by the shock. Different from an one-dimensional shock tube problem, due to the three-dimensionality of the flow, the region between the tail of the rarefaction wave and the main shock wave is not a steady-state region. The high pressure gas upon passing through a spherical rarefaction wave, due to the increase of volume from the dimensionality, must expand to a lower pressure than that reached through an equivalent one-dimensional expansion. This 'over-expansion' must be compensated by a compression wave or shock. This is the physical explanation of the formation of the second shock in a spherical explosion (see [2], [3] for reference). This second shock is rather weak and propagates outward initially. Then its strength increases with time and the second shock stops moving outward, but starts reversing to the inward direc-
tion at about $t = 0.6$. The contact discontinuity propagates outward slowly from $x = 0.25$ to $0.7$ and starts reversing the direction at about $t = 0.8$ (see figures 3.2 and 3.3). It is seen from the figures that the results calculated from the relaxed scheme is highly agreed with those calculated by the modified Harten's scheme.

The second shock reflects from the origin at about $t = 1.2$ and interacts with the contact discontinuity at about $t = 1.8$. It propagates through the contact surface and continues moving outward. The interaction causes the contact discontinuity propagates slightly outward again and; meanwhile, an inward rarefaction wave is generated during the interaction. Similar to the formation of the second shock, a rather weak, inward moving third shock follows (see figures 3.4 and 3.5).

Similarly, the third shock reflects from the origin at about $t = 2.7$ (see figures 3.6 and 3.7). It can be expected that a similar situation as stated above would repeatedly occur resulting in the formation of the fourth, fifth and sixth shocks, but they are too weak to be observed or formed in our simulation. The third shock seems a bit more sharp when calculated by the modified Harten’s scheme than by the relaxed scheme. Anyway, the main features of the explosion problem are captured by the relaxed scheme successfully.

3.1.2 Comparison with Previous Analytical and Experimental Results

Since the shocks and contact discontinuity especially the second and third shock waves, are rather weak, the positions are obtained based on a rather objective observation. However, for the purpose of comparison of the main features of this problem, this rough observation is enough. In figure 3.8(a), the analytical result predicted by Brode [1] are compared. The relaxed scheme shows a good agreement with Brode’s result. Figure 3.8(b) shows the comparison between the result of the present numerical scheme and the experimental result obtained by Boyer [2]. There are two main deviations for the results.
CHAPTER 3. NUMERICAL RESULTS

(a) Pressure (Dimensionless)

- $t=0.2$ Relaxed Scheme
- $t=0.3$ Relaxed Scheme
- $t=0.4$ Relaxed Scheme

- $t=0.2$ Modified Harten's Scheme
- $t=0.3$ Modified Harten's Scheme
- $t=0.4$ Modified Harten's Scheme

Radial Distance (Dimensionless)

- Rarefaction Wave
- Second Shock
- Main Shock

(b) Velocity (Dimensionless)

- $t=0.2$ Relaxed Scheme
- $t=0.3$ Relaxed Scheme
- $t=0.4$ Relaxed Scheme

- $t=0.2$ Modified Harten's Scheme
- $t=0.3$ Modified Harten's Scheme
- $t=0.4$ Modified Harten's Scheme

Radial Distance (Dimensionless)

- Second Shock
- Main Shock
Figure 3.2: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 0.2$ to $t = 0.4$. 
CHAPTER 3. NUMERICAL RESULTS

(a) Pressure (Dimensionless) vs. Radial Distance (Dimensionless)
- Main Shock
- Second Shock
- $t=0.6$ Relaxed Scheme
- $t=0.8$ Relaxed Scheme
- $t=1.0$ Relaxed Scheme
- $t=0.6$ Modified Marten's Scheme
- $t=0.8$ Modified Haiten's Scheme
- $t=1.0$ Modified Marten's Scheme

(b) Velocity (Dimensionless) vs. Radial Distance (Dimensionless)
- Main Shock
- Second Shock
- $t=0.6$ Relaxed Scheme
- $t=0.8$ Relaxed Scheme
- $t=1.0$ Relaxed Scheme
- $t=0.6$ Modified Marten's Scheme
- $t=0.8$ Modified Haiten's Scheme
- $t=1.0$ Modified Marten's Scheme
Figure 3.3: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t=0.6$ to $t=1.0$. 
CHAPTER 3. NUMERICAL RESULTS

(a) Pressure (Dimensionless)

- \( t = 1.15 \) Relaxed Scheme
- \( t = 1.20 \) Relaxed Scheme
- \( t = 1.30 \) Relaxed Scheme
- \( t = 1.15 \) Modified Harten’s Scheme
- \( t = 1.20 \) Modified Harten’s Scheme
- \( t = 1.30 \) Modified Harten’s Scheme

Reflected Second Shock from Origin
Main Shock
Second Shock
Radial Distance (Dimensionless)

(b) Velocity (Dimensionless)

- \( t = 1.15 \) Relaxed Scheme
- \( t = 1.20 \) Relaxed Scheme
- \( t = 1.30 \) Relaxed Scheme
- \( t = 1.15 \) Modified Harten’s Scheme
- \( t = 1.20 \) Modified Harten’s Scheme
- \( t = 1.30 \) Modified Harten’s Scheme

Second Shock Reflected from Origin
Main Shock
Second Shock
Radial Distance (Dimensionless)
Figure 3.4: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 1.15$ to $t = 1.30$. 
CHAPTER 3. NUMERICAL RESULTS

(a) Interaction between Second Shock and Contact Discontinuity Forming Third Shock Moving Toward Origin

- Second Shock
- Main Shock

(b) Second Shock Interacting with Contact Discontinuity Forming Third Shock

- Second Shock
- Main Shock
CHAPTER 3. NUMERICAL RESULTS

(c) $t=1.65$ Relaxed Scheme
$\square$ $t=1.80$ Relaxed Scheme
$\Diamond$ $t=1.90$ Relaxed Scheme

$\text{ Modified Harten's Scheme}$

(d) $t=1.65$ Relaxed Scheme
$\square$ $t=1.80$ Relaxed Scheme
$\Diamond$ $t=1.90$ Relaxed Scheme

Figure 3.4: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 1.15$ to $t = 1.30$. 

Figure 3.5: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 1.65$ to $t = 1.90$. 
CHAPTER 3. NUMERICAL RESULTS

(a)  
\[ r(t) = \begin{cases} 
2.0 & \text{Relaxed Scheme} \\
2.2 & \text{Relaxed Scheme} \\
2.4 & \text{Relaxed Scheme} \\
2.0 & \text{Modified Marten's Scheme} \\
2.2 & \text{Modified Marten's Scheme} \\
2.4 & \text{Modified Marten's Scheme} 
\end{cases} \]

(b)  
\[ \frac{\partial v}{\partial r} = \begin{cases} 
0.5 & \text{Relaxed Scheme} \\
0.25 & \text{Relaxed Scheme} \\
0 & \text{Relaxed Scheme} \\
0.25 & \text{Modified Marten's Scheme} \\
0 & \text{Modified Marten's Scheme} \\
0 & \text{Modified Marten's Scheme} 
\end{cases} \]
Figure 3.6: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 2.0$ to $t = 2.4$. 
CHAPTER 3. NUMERICAL RESULTS

(a)

![Graph showing pressure distribution over radial distance for different schemes at various times.

(b)

![Graph showing velocity distribution over radial distance for different schemes at various times.]
CHAPTER 3. NUMERICAL RESULTS

Figure 3.7: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time \( t = 2.5 \) to \( t = 2.9 \).
CHAPTER 3. NUMERICAL RESULTS

First, the time for the second shock to reflect at the origin is later in the experiment. Second, before the interaction with the second shock, the contact discontinuity in the experiment do not indicate any inward motion. Our deviations obtained are the same as those calculated by the modified Harten’s scheme. Liu et al. [7] suggested three reasons for the deviations. First, the glass sphere may not have broken instantaneously and it results in a not completely spherically symmetric flow. Second, the moving fragments of the broken glass diaphragm causes the energy loss in the flow system. The adjustment by increasing the pressure may partly compensate the loss of the kinetic energy residing in each fragment, but the loss of the thermal energy is not considered. Third, the effect of the slower moving fragments, which have much greater inertia due to their higher density than air, in the flow and across the contact discontinuity may greatly contribute to the asymmetry of the flow.

3.2 Cylindrical Explosion Problem

For the cylindrical explosion problem, same numerical settings are used as in the spherical case except that $\alpha = 2$.

3.2.1 Physical Description

The physics of the cylindrical flow is similar to the spherical flow. Figure 3.9 compares the movement of shocks and contact discontinuities for these two flows. The main shock is moving faster in the cylindrical flow than in the spherical flow. Both second shocks propagate outward at first and then reverse backward to the origin. The cylindrical second shock reverses at a later time (about $t = 1$) than the spherical case ($t = 0.6$). So, the reflection of the second shocks from the origin is later for the cylindrical case ($t = 2.05$) than the spherical case($t = 1.2$). Although it is not shown in the figure, it is expected that the third shock of the cylindrical case would be formed at about $t = 3.2$ when the second shock interacts with the contact discontinuity.
Figure 3.8: Comparison between the relaxed scheme and the (a) analytical (Brode), (b) experimental (Boyer) results of the spherical explosion.
Figure 3.9: Comparison between the spherical and the cylindrical explosion.
3.2.2 Two-Dimensional Model

As mentioned before, the convenience of the generalization to a higher dimensional space of the relaxation scheme is one of its advantages. We take this advantage to calculate the two-dimensional model of the explosion problem in order to compare with the cylindrical approximation model (quasi-1D model). The 2D scheme has been described in the last section of the previous chapter. As a 2D 'Riemann problem', the two sets of initial data are separated by a circle with radius 0.25. For the points adjacent to the quadrilateral cells cutting the initial discontinuity, the initial data is modified by assigning the area-weighted value of the two sets of data. This 'smoothing' process avoids the formation of small amplitude waves created at early times by the staircase configuration of the data. In the following numerical results, CFL number= 0.3, the adjusting factor $\beta = 1$, $\Delta x = \Delta y = 0.1$

In figure 3.10(a), the quantities of pressure, density, velocity and temperature between the quasi-1D and 2D models are compared at $t = 1.65$. The radial distance in the 2D model is extracted from the positive x-axis. At this particular time position, the second shock has been reflected from the origin but has not interacted with the contact discontinuity yet. Figure 3.10(b) shows the situation at time $t = 2.2$, where the third shock has just been formed and propagating inward. Agreement is good between the two models except for the position of the contact discontinuity and the position very near the origin. We also observe that there is a strange 'trough' for the velocity profile at about the position of the contact discontinuity. Since the diversities stated above are much worse if we don't carry out the 'smoothing' process of the area-weighted for the initial values, or using a coarser grid. So, it is believed that the rectangular grid of our finite difference scheme which results in a non-circular geometry of the initial data causes those diversities stated above. Although the density and temperature profiles at the position very near the origin are different for the two models, however, the highly agreement of the
Figure 3.10: Comparison between the 2D model and the quasi-1D model of the cylindrical explosion at time (a) $t = 1.65$ and (b) $t = 2.2$. 
shock position away from the origin shows that the quasi-1D model is a good approximation to the cylindrical symmetric problem. Our reflective boundaries in (2.13) and the compatible equations at the origin (2.8) are acceptable for the simulation of the reflection of shock at the origin. Also, negligible or even none dimensional effect are observed in affecting the waves interaction and reflection on the numerical scheme. Figure 3.11 shows the three-dimensional plot of the pressure contour at time $t = 2.5$. The roughly circular shape indicates the cylindrical symmetric of the flow.
CHAPTER 3. NUMERICAL RESULTS

<table>
<thead>
<tr>
<th>Internal Density</th>
<th>( \rho_L = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Internal Velocity</td>
<td>( u_L = 0 )</td>
</tr>
<tr>
<td>Internal Pressure</td>
<td>( p_L = 1.0 )</td>
</tr>
<tr>
<td>External Density</td>
<td>( \rho_H = 4.0 )</td>
</tr>
<tr>
<td>External Velocity</td>
<td>( u_H = 0 )</td>
</tr>
<tr>
<td>External Pressure</td>
<td>( p_H = 4.0 )</td>
</tr>
</tbody>
</table>

Table 3.3: Initial Data for Implosion Problem.

3.3 Spherical Implosion Problem

The initial condition of the implosion problem are summarized in table 3.3. The radius of the sphere is 0.25. In the implosion problem, the numerical settings are \( \Delta x = 0.05 \), CFL number = 0.3, the adjusting factor \( \beta = 1 \).

3.3.1 Physical Description

At the time of the glass sphere broken (\( t = 0 \)), a moving inward shock wave (main shock) and a moving outward rarefaction wave form. A contact discontinuity, which is moving inward separates the shock and the rarefaction waves. Different from the explosion problem, there is no second shock follows at this moment (see figure 3.12). The main shock then reflects from the origin and interacts with the contact discontinuity at about \( t = 0.2 \). Due to the interaction, the contact surface stops moving inward. The contact discontinuity propagates slightly outward for a short time and then stays stationary at the position \( x = 0.175 \). A second shock which is moving inward is generated due to the interaction. The main features of the implosion problem are captured by the relaxed scheme, as effectively as the modified Harten's scheme. Pressure is imploded at about \( t = 0.3 \) that indicates the reflection of the second shock at the origin (figure 3.14). We don't observe the formation of the third shock.
when the second shock encounters the contact discontinuity. It may because it is physically too weak to observe.

3.4 Cylindrical Implosion Problem

For the cylindrical implosion problem, same numerical settings are used as in the spherical case except that \( \alpha = 2 \). For the initial setting stated above, it is exactly the same model as Sod [6] and Liu et al. [7].

3.4.1 Physical Description

As in the explosion case, the physics of the cylindrical implosion is similar to the spherical implosion. Figure 3.16 compares the movement of shocks and contact discontinuities for these two implosion flows. The main shock interacts with the contact discontinuity at about the same position in both cases. Unlike the case of the explosion problem, the main and second shocks are moving a bit slower in the cylindrical flow than in the spherical flow. No third shock is observed in both cases and it may because the third shock is too weak to be observed.

3.4.2 Two-Dimensional Model

Similar to the analysis in the explosion problem, a comparison between 2D model and the quasi-1D model is made. The numerical settings of the 2D model are, CFL number=0.3, the adjusting factor \( \beta = 1 \) and \( \Delta x = \Delta y = 0.05 \). Good agreement is shown in figure 3.17 between the 2D model and the quasi-1D model except at the position near the origin. Due to the reasons stated before in the explosion section, same conclusion can be drawn in this implosion case. The quasi-1D model as an approximation model to the cylindrical flow is acceptable. Figure 3.18 shows the three-dimensional plot of the pressure contour at time \( t = 0.5 \). The roughly circular shape indicates the cylindrical symmetric of the flow.
CHAPTER 3. NUMERICAL RESULTS

(a) Pressure (Dimensionless) vs Radial Distance (Dimensionless)

(b) Velocity (Dimensionless) vs Radial Distance (Dimensionless)
Figure 3.12: Numerical Simulation of Implosion for (a) Pressure (b) Velocity (c) Density and (d) Temperature from time $t = 0.05$ to $t = 0.15$. 
CHAPTER 3. NUMERICAL RESULTS

(a) Pressure (Dimensionless) vs. Radial Distance (Dimensionless) for different time steps and schemes:
- t=0.18 Relaxed Scheme
- t=0.20 Relaxed Scheme
- t=0.24 Relaxed Scheme
- t=0.27 Relaxed Scheme
- t=0.18 Modified Harten's Scheme
- t=0.20 Modified Harten's Scheme
- t=0.24 Modified Harten's Scheme
- t=0.27 Modified Harten's Scheme

Main Shock
Second Shock Moving Toward Origin

(b) Velocity (Dimensionless) vs. Radial Distance (Dimensionless) for different time steps and schemes:
- t=0.18 Relaxed Scheme
- t=0.20 Relaxed Scheme
- t=0.24 Relaxed Scheme
- t=0.27 Relaxed Scheme
- t=0.18 Modified Harten's Scheme
- t=0.20 Modified Harten's Scheme
- t=0.24 Modified Harten's Scheme
- t=0.27 Modified Harten's Scheme

Main Shock
Second Shock Moving Toward Origin
Figure 3.13: Numerical simulation of the implosion problem for (a) pressure, (b) velocity, (c) density, and (d) temperature from time $t = 0.18$ to $t = 0.27$. 
CHAPTER 3. NUMERICAL RESULTS

(a) Implosion of Pressure due to Reflection of Second Shock at Origin

- Pressure at t=0.28 Relaxed Scheme
- Pressure at t=0.30 Relaxed Scheme
- Pressure at t=0.32 Relaxed Scheme
- Pressure at t=0.35 Relaxed Scheme

- Pressure at t=0.28 Modified Harten's Scheme
- Pressure at t=0.30 Modified Harten's Scheme
- Pressure at t=0.32 Modified Harten's Scheme
- Pressure at t=0.35 Modified Harten's Scheme

(b) Reflection of Second Shock

- Reflection at t=0.28 Relaxed Scheme
- Reflection at t=0.30 Relaxed Scheme
- Reflection at t=0.32 Relaxed Scheme
- Reflection at t=0.35 Relaxed Scheme

- Reflection at t=0.28 Modified Harten's Scheme
- Reflection at t=0.30 Modified Harten's Scheme
- Reflection at t=0.32 Modified Harten's Scheme
- Reflection at t=0.35 Modified Harten's Scheme

Radial Distance (Dimensionless)
Figure 3.14: Numerical simulation of the explosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 0.28$ to $t = 0.35$. 
CHAPTER 3. NUMERICAL RESULTS

(a) 

(b) 

Radial Distance (Dimensionless)
Figure 3.15: Numerical simulation of the implosion problem for (a) pressure (b) velocity (c) density and (d) temperature from time $t = 0.4$ to $t = 1.0$. 
CHAPTER 3. NUMERICAL RESULTS

Figure 3.16: Comparison between the spherical and the cylindrical implosion.
Figure 3.17: Comparison between the 2D model and the quasi-1D model of the cylindrical implosion at time $t = 0.4$. 
Figure 3.18: Three-dimensional plot of the pressure contour of the cylindrical implosion at time $t = 0.5$. 
Chapter 4

Conclusion

The present development provides a demonstration on the numerical simulation of explosion and implosion problems by using the relaxed scheme.

Cylindrical and spherically symmetric explosion and implosion in air are numerically simulated. The governing equation is a nonlinear system of hyperbolic conservation laws with source terms. The inhomogeneous system is split for a time step, into the advection problem which is a homogeneous hyperbolic problem, and the source problem which is a system of ordinary differential equations (ODEs). The ODEs is solved by the second-order, two-stage Runge-Kutta method. The pure advection hyperbolic problem is calculated by the second-order relaxed scheme developed by Jin & Xin [16]. A constant matrix ‘A’ is constructed in the application of the relaxed scheme. The numerical dissipation of the relaxation scheme is to a large extent determined by the matrix A and the choice of A becomes critical in applying the relaxation scheme. An algorithm to choose the matrix is developed. The matrix is chosen ‘dynamically’ such that it is tailor-made for each time step interval. The matrix constructed as $A = aI$ where the constant $a$ defined as the square of the maximum eigenvalue inside the time interval $[t^n, t^{n+1}]$ for all spatial domain. We impose no additional criteria on the CFL number for the algorithm proposed in practical use. Though we cannot prove the TVD property of the scheme in system of conservation laws, a series of numerical tests are carried
out to exam the applicability of the algorithm on the Euler system. The test results are satisfactory.

By applying the relaxed scheme in the explosion and implosion problem, four cases namely spherical explosion, cylindrical explosion, spherical implosion and cylindrical implosion are presented. For the explosion problem, at the time immediately after the broken of the spherical or cylindrical diaphragm, the main shock is generated and moves outward in the air. The compressed gas is expanded through a rarefaction wave and a contact discontinuity separates the expanded gas from the air compressed by the main shock. Different from an one-dimensional shock tube problem, due to the three-dimensionality of the flow, the region between the tail of the rarefaction wave and the main shock wave is not a steady-state region. The high pressure gas upon passing through a spherical rarefaction wave, due to the increase of volume from dimensionality, must expand to a lower pressure than that reached through an equivalent one-dimensional expansion. This 'over-expansion' is compensated by a second shock. This second shock is rather weak and propagates outward initially. Its strength increases with time and the second shock stops moving outward, but starts reversing to the inward direction and reflects from the origin. Also, the contact discontinuity propagates outward slowly and starts reversing the direction until it interacts with the reflected second shock. The contact discontinuity propagates slightly outward again and a third shock generated similarly. The numerical simulation has successfully captured all the main features of the explosion problem including the propagation of the main shock, the reflection of the second shock, the interaction with the contact discontinuity and the formation of the third shock, as well. Comparison is made between previous numerical results [7], analytical results [1] and experimental results [2]. Also, thanks to the advantage of the relaxation scheme to generalize to a higher dimensional space easily, a 2D model with circular initial condition is calculated to check the cylindrical symmetric model. The results of our quasi-1D model is found acceptable.
For the implosion problem, at the time of the spherical or cylindrical diaphragm broken, an inward moving shock wave (main shock) and an outward moving rarefaction wave form. A contact discontinuity, which is moving inward separates the shock and the rarefaction waves. Unlike the explosion problem, there is no second shock follows at this moment. The main shock then reflects from the origin and interacts with the contact discontinuity causing the formation of the second shock (similar to the formation of the third shock in the case of the explosion problem). Due to the interaction, the contact surface stops moving inward but propagates slightly outward for a short time and then stays stationary. The main features of the implosion problem, the reflection of the main shock, the movement of the contact discontinuity and the formation of the second shock are captured by the relaxed scheme, as effectively as the modified Harten's scheme. We don't observe the formation of the third shock when the second one encounters the contact discontinuity. It may because it is physically too weak to observe. A 2D model has been presented so as to compare to our cylindrical symmetric quasi-1D model.

The present research contributes a picture to understand the physics of the explosion and implosion problems through the numerical simulation by the relaxed scheme. Meanwhile, the technique of the application of the relaxed scheme is developed especially on the choosing of the important matrix $A$, which plays the critical role on the accuracy and practicality of the scheme. Although the matrix $A$ chosen by the algorithm developed in this research has shown to be practically capable for most Euler problems by the numerical tests, it is still a rough estimate and in no way guarantee the dissipative condition to be always satisfied. Based on the present finding, it would be valuable for the future to find a procedure for choosing $A$ that will rigorously be proved to fulfill the dissipative condition. Moreover, noted the smearing effect on the contact discontinuity (figure 3.1(b)), equipping the relaxation scheme with ACM technique can also be considered as a future development so as to eliminate the smearing effect.
Bibliography


