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Abstract

The aim of this thesis is to show the maximum principles and comparison theorems of a class of second order of quasilinear elliptic operators, $-\text{div}A(x, Du)$ in an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $A$ satisfies some properties.

Then we use these results together with the moving planes method to get the symmetry properties of the following equation:

$$
\begin{cases}
-\Delta_p u = f(u) & \text{in } B \\
u \geq 0 & \text{in } B \\
u = 0 & \text{on } \partial B,
\end{cases}
$$

where $f$ is locally Lipschitz continuous and $B$ is the unit ball in $\mathbb{R}^N$.

For $f(u) = u^{p-1} - u^q$, where $0 < p - 1 < q$, and $q < \frac{Np-N+1}{N-1}$ if $N \geq 3$, $q < +\infty$ otherwise, we obtain the asymptotic behavior of the energy of the above equation.
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# Contents

1. Introduction and Statement of the Results 3

2. Maximum Principles and Comparison Theorems 12

3. Pohozaev Identity and Symmetry for $p$-Laplacian when $1 < p < 2$ 18

4. Singularly Perturbed $p$-Laplacian Equation 23

5. Appendix 31

Bibliography 39
Chapter 1

Introduction and Statement of the Results

In recent years several researches were devoted to the study of properties of solutions to elliptic equations involving the $p$-Laplacian operator (see [10], [5], [24], [25], [9], [26], [6] and the references therein). The difficulties in extending properties of solutions of strictly elliptic equations to solutions of $p$-Laplace equations are mainly due to the degeneracy of the $p$-Laplacian operator. In particular, comparison theorems widely used for strictly elliptic operators are not available when we consider degenerate operators.

In this paper, we consider a class of second order of quasilinear elliptic operators with a “growth of degree $p - 1$”, $1 < p < \infty$, which includes the $p$-Laplacian operator and prove for them some comparison results. Then we use these results together with the moving planes method to get the symmetry properties. More precisely we consider the operator $-\text{div}A(x, Du)$ in an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and we make the following assumptions on $A$:

\begin{align}
A & \in C^0(\tilde{\Omega} \times \mathbb{R}^N; \mathbb{R}^N) \cap C^1(\bar{\Omega} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R}^N) \\
A(x, 0) &= 0 \quad \forall x \in \Omega \\
\sum_{i,j=1}^N \left| \frac{\partial A_{ij}}{\partial \eta_i}(x, \eta) \right| &\leq \Gamma |\eta|^{p-2} \quad \forall x \in \Omega, \eta \in \mathbb{R}^N \setminus \{0\}
\end{align}
\[
\sum_{i,j=1}^{N} \frac{\partial A_{j}(x, \eta)}{\partial \eta_{i}} \zeta_{i} \zeta_{j} \geq \gamma|\eta|^{p-2}|\zeta|^{2} \quad \forall x \in \Omega, \eta \in \mathbb{R}^{N}\setminus\{0\}, \zeta \in \mathbb{R}^{N}, \tag{1.4}
\]
with \(1 < p < \infty\) and for suitable constants \(\gamma, \Gamma \geq 0\).

In the case of the \(p\)-Laplacian operator \(A = A(\eta) = |\eta|^{p-2}\eta\).

In chapter 2 we prove different forms of weak and strong maximum principles and comparison theorems. The proofs are based on simple estimates contained in Lemma 2.1 below, that explains why maximum principles hold without special hypotheses about the degeneracies, while comparison theorems are not in general available if \(p \neq 2\) in their full generality (see [1], Remark 2-1).

In what follows \(\Omega\) will be an open set in \(\mathbb{R}^{N}, N \geq 2\) and \(A\) a function satisfying (1.1) \(\sim\) (1.4) for \(p \in (1, \infty)\). Moreover all inequalities are meant to be satisfied in a weak sense.

**Theorem 1.1 (Weak Maximum Principle)** Suppose \(\Omega\) is bounded and \(u \in W^{1,p}(\Omega) \cap L^\infty(\Omega), 1 < p < \infty\), satisfies

\[-\text{div} A(x, Du) + w(x)u^{p-1} \leq 0, \quad \text{in } \Omega,
\]

where \(w(x) \geq \Lambda\) for all \(x \in \Omega\) and \(\Lambda \in \mathbb{R}\). Let \(\Omega' \subseteq \Omega\) be open and suppose \(u \leq 0\) on \(\partial \Omega'\). If \(\Lambda \geq 0\) then \(u \leq 0\) in \(\Omega'\). If \(\Lambda < 0\) then there exists a constant \(c > 0\), depending on \(p\) and on \(\gamma, \Gamma\) in (1.3), (1.4) such that if

\[-\Lambda \left(\frac{|\Omega'|}{\omega_{N}}\right)^{\frac{1}{N}} < c,
\]

then \(u \leq 0\) in \(\Omega'\). (where \(|\cdot|\) stands for the Lebesgue measure and \(\omega_{N}\) is the measure of the unit ball in \(\mathbb{R}^{N}\)).

The Strong maximum principle can be obtained via Hopf Lemma (see [26] and [27] for particular cases) or as a consequence of a Harnack type inequality (see section 2). We shall follow the second approach to derive a strong maximum principle.
Theorem 1.2 (Strong Maximum Principle) Suppose that $\Omega$ is connected and $u \in W^{1,p}_0(\Omega) \cap C^0(\Omega)$ satisfies

$$-\text{div} A(x, Du) + w(x)u^{p-1} \geq 0, \quad u \geq 0 \quad \text{in} \quad \Omega,$$

where $w(x) \leq \Lambda$ for some $\Lambda \in \mathbb{R}$.

Then either $u(x) \equiv 0$ in $\Omega$ or $u > 0$ in $\Omega$.

Let us put, if $u$, $v$ are functions in $W^{1,\infty}(\Omega)$ and $A \subseteq \Omega$,

$$M_A = M_A(u, v) = \sup_A (|Du| + |Dv|), \quad m_A = m_A(u, v) = \inf_A (|Du| + |Dv|).$$

Theorem 1.3 (Weak Comparison Theorem) Let $\Omega$ be bounded and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfy

$$-\text{div} A(x, Du) + g(x, u) - \Lambda u^{p-1} \leq -\text{div} A(x, Dv) + g(x, v) - \Lambda v^{p-1} \quad \text{in} \quad \Omega, \quad (1.5)$$

where $\Lambda \in \mathbb{R}$; $u, v > 0$ in $\Omega$ and $g \in C(\bar{\Omega} \times \mathbb{R})$ is such that for each $x \in \Omega$, $g(x, s)$ is nondecreasing in $s$ for $|s| \leq \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$. Let $\Omega' \subseteq \Omega$ be open and suppose $u \leq v$ on $\partial\Omega'$.

(a) If $\Lambda \leq 0$, then $u \leq v$ in $\Omega'$, $\forall p > 1$.

(b) If $p = 2$ there exists $\delta > 0$, depending on $\Lambda, \gamma$ and $\Gamma$ such that if $|\Omega'| < \delta$ then $u \leq v$ in $\Omega'$.

(c) If $1 < p < 2$, $\Lambda > 0$ and $M_\Omega > 0$, there exist $\delta, M > 0$, depending on $p, \Lambda, \gamma, \Gamma, |\Omega|$ and $M_\Omega$, such that the following holds: if $\Omega' = A_1 \cup A_2$ with $|A_1 \cap A_2| = 0$, $|A_1| < \delta$ and $M_{A_2} < M$, then $u \leq v$ in $\Omega'$.

(d) If $p > 2$, $\Lambda > 0$ and $m_\Omega > 0$, there exist $\delta, m > 0$, depending on $p, \Lambda, \gamma, \Gamma, |\Omega|$ and $m_\Omega$, such that the following holds: if $\Omega' = A_1 \cup A_2$ with $|A_1 \cap A_2| = 0$, $|A_1| < \delta$ and $m_{A_2} > m$, then $u \leq v$ in $\Omega'$.
Remark 1.4 The same results hold for

\[-\text{div}A(x, Du) + g(x, u) - \Lambda u \leq -\text{div}A(x, Dv) + g(x, v) - \Lambda v,\]

which have been proved in [1], Theorem 1-2.

Remark 1.5 If \(g(x, u) = -u^q\), where \(q > p - 1\), and \(\Lambda = -1\). Suppose \(u, v < < 1\), if \(u \leq v\) on \(\partial \Omega\), then \(u \leq v\) in \(\Omega\). The proof is similar to Theorem 1.3.

Next we deal with a form of the strong comparison theorem.

Theorem 1.6 (Strong Comparison Theorem) Let \(u, v \in C^1(\Omega)\) satisfy

\[-\text{div}A(x, Du) + \Lambda u^{p-1} \leq -\text{div}A(x, Dv) + \Lambda v^{p-1}, \quad u \leq v \quad \text{in} \quad \Omega, \tag{1.6}\]

and define \(Z_{u,v} = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}\) if \(p \neq 2\), \(Z_{u,v} = \emptyset\) if \(p = 2\). If \(x_0 \in \Omega \setminus Z_{u,v}\) and \(u(x_0) = v(x_0)\) then \(u = v\) in the connected component of \(\Omega \setminus Z_{u,v}\) containing \(x_0\).

Remark 1.7 The same result holds for

\[-\text{div}A(x, Du) + \Lambda u \leq -\text{div}A(x, Dv) + \Lambda v,\]

which has been proved in [1], Theorem 1-4.

In chapter 3, firstly we will prove the Pohozaev identity [25] of the following quasilinear boundary value problem:

\[
\begin{cases}
-\Delta_p u = g(u) \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases} \tag{1.7}
\]
Theorem 1.8 Suppose \( u \) is the solution of (1.7) and \( g \) is a locally Lipschitz continuous function, then we have the following Pohozaev identity:

\[
N \int_{\Omega} G(u) \, dx + \left(1 - \frac{N}{p}\right) \int_{\Omega} ug(u) \, dx = \frac{1}{q} \int_{\partial \Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^q \, dS,
\]

where \( G \) is the primitive of \( g \) such that \( G(0) = 0 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \nu \) stands for the unit outer normal on \( \partial \Omega \).

After that we apply the previous comparison theorems to the study of symmetry and monotonicity properties of solutions to quasilinear elliptic equations. For simplicity we consider here the case of the \( p \)-Laplacian operator that we denote by \( \Delta_p \), so that

\[-\Delta_p u \text{ stands for } -\text{div}(|Du|^{p-2}Du).\]

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N, N \geq 2 \), which is convex and symmetric in the \( x_1 \)-direction and consider the problem

\[
\begin{cases}
-\Delta_p u &= f(u) \quad \text{in } \Omega \\
u &\geq 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{cases}
\]

(1.8)

where \( 1 < p < 2 \).

We are interested in whether the solution of (1.8) is positive or not, then we can further get the monotonicity and symmetry results. In the case \( p = 2 \), Castro and Shivaji in the papers [20] and [21] proved that when \( N = 1 \) there exist solutions of (1.8) with interior zeros, when \( N > 1 \) and \( \Omega \) is a ball then any solution of (1.8) is positive in \( \Omega \). When \( 1 < p < 2 \), if \( f \) is a locally Lipschitz function in \([0, +\infty)\) with \( f(0) < 0 \), then L.Damascelli, F.Pacella and M.Ramaswamy in [22] showed that any solution of (1.8) is positive in \( \Omega \). Also when \( p > 2 \), with additional conditions on \( f \), we can also get the similar result.

In the famous paper [2] Gidas, Ni and Nirenberg, based on [11, 12, 13], used the moving planes method to prove that if \( p = 2 \) every classical positive solution to (1.8) is symmetric with respect to the hyperplane \( T_0 = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\} \) and strictly increasing in \( x_1 \) for \( x_1 < 0 \), provided \( \Omega \) is smooth and \( f \) is locally Lipschitz continuous. As a corollary if \( \Omega \) is a ball then \( u \) is radial and radially decreasing. After
that Berestycki and Nirenberg in [8] improved the method by using a form of the maximum principle in domains with small measure, and also in [14, 15, 16] by using the monotonicity properties of the nonlinearity.

When \( p \neq 2 \), the solutions of (1.8) can only be considered in weak sense since, generally, they belong to the space \( C^{1,\alpha}(\Omega) \) (see [5],[6]). In [9] it is proved, using symmetrization methods, that if \( \Omega \) is a ball, \( p = N \) and \( f \) is continuous with \( f(s) > 0 \) if \( s > 0 \), then \( u \) is radial and radially decreasing. In [10], under the crucial hypothesis that the gradient of the solution vanishes only at the center of the ball, then we can obtain the symmetry result.

Here, using the method of moving planes as in [2], the regularity properties [5, 17, 18, 19], the above comparison results and maximum principles, we can obtain the symmetry of the positive solutions when \( 1 < p < 2 \).

We will focus on \( \Omega = B(0,1) \),

\[
\begin{cases}
-\Delta_p u = f(u) \text{ in } B \\
u \geq 0 \text{ in } B \\
u = 0 \text{ on } \partial B
\end{cases}
\]

where \( B = B(0,1) \) and \( 1 < p < 2 \). To state more precisely the symmetry results we need some notations.

For \(-1 < \lambda < 1\) we define

\[
T_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}, \quad B_\lambda = \{x \in B : x_1 < \lambda\}, \quad B^\lambda = \{x \in B : x_1 > \lambda\}.
\]

If \( x = (x_1, x') \) let \( x_\lambda = (2\lambda - x_1, x') \) be the point corresponding to \( x \) in the reflection through \( T_\lambda \) and if \( u \) is a real function in \( B \) let us put \( u_\lambda(x) = u(x_\lambda) \) where \( x, x_\lambda \in B \).

Finally if \( u \in C^1(\bar{B}) \) we put \( Z = \{x \in B : Du(x) = 0\} \) and

\[
Z_\lambda = \{x \in B_\lambda : Du(x) = Du_\lambda(x) = 0\} \quad \text{for} \quad -1 < \lambda \leq 0,
\]

\[
Z^\lambda = \{x \in B^\lambda : Du(x) = Du^\lambda(x) = 0\} \quad \text{for} \quad 0 \leq \lambda < 1.
\]

We have the following theorem [1]:
**Theorem 1.9** Let $1 < p < 2$ and $u \in C^1(\bar{B})$ a weak solution of (1.9) with $f$ locally Lipschitz continuous. Suppose that the following condition holds:

if $\lambda < 0$ then $B \setminus Z_\lambda$ is connected, with the analogous condition satisfied by $B \setminus Z^\lambda$ for $\lambda > 0$.

Then $u$ is positive, radial and radially decreasing. Further if $Z$ is discrete, then $u$ is radially strictly decreasing.

But the symmetry result relies on the assumption that the set of the critical points of $u$ does not disconnect the caps which are constructed by the moving plane method. After that L.Damascelli and F.Pacella improved it in [28]:

**Theorem 1.10** Let $1 < p < 2$ and $u \in C^1(\bar{B})$ a weak solution of (1.9) with $f$ locally Lipschitz continuous. Then $u$ is radially symmetric and $\partial u / \partial r < 0$ for $0 < r < 1$ in $B \setminus Z$.

In chapter 4, we will deal with the study of solutions to a nonlinear singularly perturbed elliptic problems of the form

$$\varepsilon^p \Delta_p u - u^{p-1} + u^q = 0 \quad \text{in } \Omega,$$

where $0 < p - 1 < q$, and $q < \frac{Np-N+p}{N-p}$ if $N \geq 3$, $q < +\infty$ otherwise. Here $\varepsilon > 0$ is a small parameter and we are interested in positive solutions to this equation satisfying zero Dirichlet boundary condition on $\partial \Omega$.

We observe that if $u_\varepsilon$ solves equation (1.10) in $\Omega$ and $x_\varepsilon$ is a point in $\bar{\Omega}$ where $u_\varepsilon$ maximizes, then the function $v_\varepsilon(y) = u_\varepsilon(x_\varepsilon + \varepsilon y)$ maximizes at the origin and satisfies

$$\Delta_p v - v^{p-1} + v^q = 0 \quad \text{in } \varepsilon^{-1}\{\Omega - x_\varepsilon\},$$

in the expanding domain $\varepsilon^{-1}\{\Omega - x_\varepsilon\}$, which as $\varepsilon \to 0$ becomes the entire space $\mathbb{R}^N$. Now equation (1.11) possesses a least energy solution $w$ (ground state) in entire $\mathbb{R}^N$, maximizing at zero and vanishing exponentially at infinity. The energy functional defined as

$$I_p(v) = \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + v^p) - \frac{1}{q+1} \int_{\mathbb{R}^N} v^{q+1}, \quad v \in W^{1,p}(\mathbb{R}^N),$$

satisfies
has a least positive critical value \( c_* \) characterized as

\[
c_* = \inf_{v \neq 0} \sup_{t > 0} I_p(tv). 
\] (1.13)

An associated critical point \( w \) actually solves equation (1.11) and is called a least energy solution. It also decays exponentially at infinity.

If it happened that the scaled solution \( v_\varepsilon \) converged to one of these \( w \)'s as \( \varepsilon \to 0 \) in, say, the \( W^{1,p} \)-sense, then the actual look \( u_\varepsilon \) would be that of a very sharp like, centered at the point \( x_\varepsilon \), while approximating zero at an exponential rate in \( 1/\varepsilon \) away from it.

Now we consider the positive solutions of equation (1.10) with zero Dirichlet boundary condition

\[
\begin{align*}
\varepsilon^p \Delta_p u - u^{p-1} + u^q &= 0 \quad \text{in } \Omega \\
\ u &> 0 \quad \text{in } \Omega \\
\ u &= 0 \quad \text{on } \partial \Omega.
\end{align*} 
\] (1.14)

The least-energy solutions to the problem are characterized by the means of the mountain pass value of the associated energy functional. In more precise terms, associated to (1.14) we have the "energy" functional

\[
I_\varepsilon(u) = \frac{1}{p} \int_\Omega (\varepsilon^p |\nabla u|^p + u^p) - \frac{1}{q + 1} \int_\Omega u^{q+1}, 
\] (1.15)

whose nontrivial critical points in the space \( W_0^{1,p}(\Omega) \) represent solutions of (1.14).

Clearly the conditions on \( p \) and \( q \) guarantee the validity of the P.S. condition for this functional, so that the mountain pass theorem applies providing a positive critical value characterized as

\[
c_\varepsilon = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_\varepsilon(\gamma(t)) 
\] (1.16)

where \( \gamma \in \Gamma \) if and only if \( \gamma \in C([0,1], W_0^{1,p}(\Omega)) \) and \( \gamma(0) = 0 \), \( I_\varepsilon(\gamma(1)) \leq 0 \). It can be observed that this number can be further characterized as

\[
c_\varepsilon = \inf_{u \neq 0} \sup_{t > 0} I_\varepsilon(tu), 
\] (1.17)

which can be shown to be the least among all nonzero critical values of \( I_\varepsilon \).
Let \( w \) be the solution of the limiting problem

\[
\begin{aligned}
\Delta_p w - w^{p-1} + w^q &= 0 \quad \text{in} \quad \mathbb{R}^N \\
w > 0, \quad w(0) &= \max w, \\
\lim_{|x| \to \infty} w(x) &= 0.
\end{aligned}
\]  

(1.18)

First we obtain the asymptotic behavior of the energy \( c_\varepsilon \) when \( \Omega = B \).

**Theorem 1.11**

\[
c_\varepsilon = \varepsilon^N \left\{ c_* + \exp \left[ - \frac{p}{\varepsilon} \left( \frac{1}{p - 1} \right)^{\frac{1}{p}} + o(1) \right] \right\},
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

As a consequence we obtain the following upper bound of \( c_\varepsilon \) for general \( \Omega \),

**Corollary 1.12**

\[
c_\varepsilon \leq \varepsilon^N \left\{ c_* + \exp \left[ - p \left( \frac{d_0}{\varepsilon} \right)^{\left( \frac{1}{p - 1} \right)^{\frac{1}{p}}} + o(1) \right] \right\},
\]

where \( d_0 = \max_{x \in \Omega} \text{dist}(x, \partial \Omega) \) and \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

The behavior of a least-energy solution to the problems has been well understood. The solutions are characterized by means of mountain pass value of the associated energy functional. Ni and Wei show that a least-energy solution of (0.1) necessarily concentrates around a "most centered point" of the domain, namely around a point of maximum distance to the boundary.
Chapter 2

Maximum Principles and Comparison Theorems

In this chapter we prove the maximum principles and comparison theorems stated in chapter 1. We begin with two simple lemmas, which have been proved by Damascelli (see [1], Lemma 2-1 and 2-2).

Lemma 2.1 There exist constants $c_1, c_2$, depending on $p$ and on constants $\gamma, \Gamma$ in (1.3) and (1.4), such that $\forall \eta, \eta' \in \mathbb{R}^n$ with $|\eta| + |\eta'| > 0$, $\forall x \in \Omega$:

$$|A(x, \eta) - A(x, \eta')| \leq c_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|$$  \hspace{1cm} (2.1)

$$[A(x, \eta) - A(x, \eta')] \cdot [\eta - \eta'] \geq c_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2,$$  \hspace{1cm} (2.2)

where the dot stands for the scalar product in $\mathbb{R}^n$. In particular, since (1.2) holds, we have for any $x \in \Omega, \eta \in \mathbb{R}^N$:

$$|A(x, \eta)| \leq c_1|\eta|^{p-1}$$  \hspace{1cm} (2.3)

$$A(x, \eta) \cdot \eta \geq c_2|\eta|^p.$$  \hspace{1cm} (2.4)

Moreover for each $x \in \Omega, \eta, \eta' \in \mathbb{R}^N$ we have:

$$|A(x, \eta) - A(x, \eta')| \leq c_1|\eta - \eta'|^{p-1} \hspace{1cm} \text{if} \hspace{1cm} 1 < p \leq 2$$  \hspace{1cm} (2.5)

$$[A(x, \eta) - A(x, \eta')] \cdot [\eta - \eta'] \geq c_2|\eta - \eta'|^p \hspace{1cm} \text{if} \hspace{1cm} p \geq 2.$$  \hspace{1cm} (2.6)
Lemma 2.2 (Poincare's Inequality) Let $\Omega$ be a bounded open set and suppose $\Omega = A \cup B$, with $A, B$ measurable subsets of $\Omega$. If $u \in W^{1,p}_0(\Omega)$, $1 < p < \infty$, then
\[
\|u\|_{L^p(\Omega)} \leq \omega_N^{\frac{1}{p'}} \|A\|^{\frac{1}{p'}} \|Du\|_{L^p(A)} + \|B\|^{\frac{1}{p'}} \|Du\|_{L^p(B)},
\]
where $\frac{1}{p} + \frac{1}{p'} = 1$.

If $u, v \in W^{1,p}_0(\Omega) \cap L^\infty_0(\Omega)$ we say that (in a weak sense)
\[
\text{div} A(x, Du) + \Lambda u^{p-1} \leq \text{div} A(x, Dv) + \Lambda v^{p-1} \quad \text{in } \Omega \hspace{1cm} (2.8)
\]
\[
\text{div} A(x, Du) + \Lambda u^{p-1} \leq 0 \quad \text{in } \Omega, \hspace{1cm} (2.9)
\]
if for each nonnegative $\varphi \in C_0^\infty(\Omega)$ we have
\[
\int_\Omega [A(x, Du) \cdot D\varphi + \Lambda u^{p-1}\varphi] dx \leq \int_\Omega [A(x, Dv) \cdot D\varphi + \Lambda v^{p-1}\varphi] dx \hspace{1cm} (2.10)
\]
\[
\int_\Omega [A(x, Du) \cdot D\varphi + \Lambda u^{p-1}\varphi] dx \leq 0. \hspace{1cm} (2.11)
\]
If $\Omega$ is bounded and $u, v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ since (2.3) holds, by a density argument (2.10) and (2.11) hold for any nonnegative $\varphi \in W^{1,p}_0(\Omega)$.

Similarly by $u \leq v$ on $\partial \Omega$ (in the weak sense) we mean $(u - v)^+ \in W^{1,p}_0(\Omega)$. Of course if $u$ and $v$ are continuous in $\bar{\Omega}$ and satisfy $u \leq v$ pointwisely on $\partial \Omega$, then they satisfy the inequality also weakly.

Proof of Theorem 1.1: By hypothesis $u^+ \in W^{1,p}_0(\Omega')$ and can be used as a test function in (2.11) yielding
\[
\int_{[u \geq 0]} A(x, Du) \cdot Du dx + \int_{[u \geq 0]} \Lambda u^p dx \leq \int_{[u \geq 0]} A(x, Du) \cdot Du dx + \int_{[u \geq 0]} w(x) u^p dx \leq 0,
\]
where $[u \geq 0] = \{x \in \Omega : u(x) \geq 0\}$.

Since (2.4) holds we get
\[
c_2 \int_{\Omega'} |Du^+|^p dx = c_2 \int_{[u \geq 0]} |Du|^p dx \leq -\Lambda \int_{[u \geq 0]} |u|^p dx = -\Lambda \int_{\Omega'} |u^+|^p dx,
\]
where \(c_2\) is the constant in (2.4).

If \(\Lambda \geq 0\) then
\[
c_2 \int_{\Omega'} |Du^+|^p dx \leq 0,
\]
so that \(u^+ = 0\) in \(\Omega'\).

If \(\Lambda < 0\) then by Lemma 2.2 with \(B = \phi\) we get
\[
c_2 \int_{\Omega'} |Du^+|^p dx \leq -\Lambda \left( \frac{|\Omega'|}{\omega_N} \right) \frac{p}{N} \int_{\Omega'} |Du|^p dx.
\]
So if
\[
-\Lambda \left( \frac{|\Omega'|}{\omega_N} \right) \frac{p}{N} < c_2
\]
\[
0 = \int_{\Omega'} |Du^+|^p dx = \|u\|_{W_0^{1,p}(\Omega')}^p,
\]
therefore \(u^+ = 0\) in \(\Omega'\).

\(\square\)

The following proposition is a particular case of a more general result proved by Trudinger (see, [3], Theorem 1.2]. We can follow his proof, based on Moser's iterative technique, closely.

**Proposition 2.3 (Harnack Type Inequality)** Suppose that \(u \in W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)\) satisfies
\[
-\text{div} A(x, Du) + w(x)u^{p-1} \geq 0, \quad u \geq 0 \quad \text{in} \quad \Omega,
\]
where \(w(x) \leq \Lambda\) for all \(x \in \Omega, \Lambda \in \mathbb{R}\). Let \(x_0 \in \Omega, \delta > 0\) with \(B(x_0, 5\delta) \subseteq \Omega\), and \(s > 0\) with \(s < \frac{N(p-1)}{N-p}\) if \(p \leq N\), and \(s \leq \infty\) if \(p > N\). Then there exists a constant \(c > 0\) depending on \(N, p, s, \Lambda, \gamma, \Gamma\) such that
\[
\|u\|_{L^p(B(x_0, 2\delta))} \leq c \delta^s \inf_{B(x_0, \delta)} u.
\]

Theorem 1.2 follows immediately from the Harnack comparison inequality (Proposition 2.3) whose proof is deferred to the Appendix.
**Proof of Theorem 1.2:** Suppose there is a point \( x_0 \in \Omega \) such that \( u(x_0) = 0 \), and let \( U = \{ x \in \Omega : u(x) = 0 \} \). Clearly \( U \) is nonempty since \( x_0 \in U \). Since \( u \) is continuous, \( U \) is closed relatively to \( \Omega \). Also there is \( \delta > 0 \) such that \( B(x_0, 5\delta) \subseteq \Omega \), then

\[
\inf_{B(x_0, \delta)} u(x) = 0.
\]

By Proposition 2.3 we get

\[
\|u\|_{L^s(B(x_0, 2\delta))} \leq \int_{B(x_0, 2\delta)} u^s dx \leq 0,
\]

for some \( s > 0 \). Since \( u \) is continuous and nonnegative, \( u \equiv 0 \) in \( B(x_0, 2\delta) \). Therefore \( U \) is open, but \( \Omega \) is connected so \( U = \Omega \).

\[\square\]

**Proof of Theorem 1.3:** Using \((u - v)^+ \in W_0^{1,p}(\Omega')\) as a test function in (1.5) we get

\[
\int_{[u \geq v]} [A(x, Du) - A(x, Dv)] \cdot (Du - Dv) dx + \int_{[u \geq v]} [g(x, u) - g(x, v)](u - v) dx
\]

\[
- \Lambda \int_{[u \geq v]} (u^{p-1} - v^{p-1})(u - v) dx \leq 0,
\]

where \([u \geq v] = \{ x \in \Omega' : u(x) \geq v(x) \} \). Since \( g(x, u) \geq g(x, v) \) if \( u \geq v \) we get

\[
\int_{[u \geq v]} [A(x, Du) - A(x, Dv)] \cdot (Du - Dv) dx \leq \Lambda \int_{[u \geq v]} (u^{p-1} - v^{p-1})(u - v) dx. \tag{2.12}
\]

It is easy to see that \((u^{p-1} - v^{p-1})(u - v) \geq 0\) in \([u \geq v], \forall p > 1\).

(a) if \( \Lambda \leq 0, 1 < p < 2 \), by (2.2) we get

\[
c_2 M_\Omega^{p-2} \int_{\Omega'} |D(u - v)^+|^2 dx \leq 0 \quad \Rightarrow \quad (u - v)^+ = 0 \quad \text{in} \quad \Omega',
\]

if \( \Lambda \leq 0, p \geq 2 \), by (2.6) we get

\[
c_2 \int_{\Omega'} |D(u - v)^+|^p dx \leq 0,
\]

Therefore \((u - v)^+ = 0\) in \( \Omega' \).
(b) if \( p = 2 \), by Lemma 2.2 with \( B = \phi \),
\[
c_2 \int_{\Omega'} |D(u - v)^+|^2 \, dx \leq \Lambda \left( \frac{|\Omega'|}{\omega_N} \right)^{\frac{2}{p}} \int_{\Omega'} |D(u - v)^+|^2 \, dx,
\]
so if \( c_2 > \Lambda \left( \frac{|\Omega'|}{\omega_N} \right)^{\frac{2}{p}} \) then we get \( \|(u - v)^+\|_{W^{1,2}_0(\Omega')} = 0 \), Therefore if \( |\Omega'| < \delta \) where \( \delta = \delta(\Lambda, \Gamma, N) \) we get \( (u - v)^+ = 0 \) in \( \Omega' \).

(c) if \( 1 < p < 2 \) and \( \Lambda > 0 \), by (2.2), (2.12) becomes
\[
c_2 M_{\Omega'}^{p-2} \int_{\Omega'} |D(u - v)^+|^2 \, dx \leq \Lambda \int_{\Omega'} C(u - v)^2 \, dx,
\]
where \( C = C(p) \). Now \( \Omega' = A_1 \cup A_2 \) with \( |A_1 \cap A_2| = 0 \), by (2.7) with \( p = 2 \),
\[
c_2 M_{\Omega'}^{p-2} \int_{A_1 \cup [u \geq v]} |D(u - v)|^2 \, dx + c_2 M_{A_2}^{p-2} \int_{A_2 \cup [u \geq v]} |D(u - v)|^2 \, dx \leq
2\Lambda C \omega_N^{\frac{2}{p}} |\Omega'|^{\frac{1}{p}} \left[ |A_1|^{\frac{1}{p}} \int_{A_1 \cup [u \geq v]} |D(u - v)|^2 \, dx + |\Omega|^{\frac{1}{p}} \int_{A_2 \cup [u \geq v]} |D(u - v)|^2 \, dx \right],
\]
if \( |A_1| \) and \( M_{A_2} \) are small, for \( i = 1, 2 \), \( \int_{A_i \cup [u \geq v]} |D(u - v)|^2 = 0 \) so that \( (u - v)^+ = 0 \) in \( \Omega' \).

(d) if \( p > 2 \) and \( \Lambda > 0 \), by (2.2), (2.12) becomes
\[
c_2 m_{\Omega'}^{p-2} \int_{\Omega'} |D(u - v)^+|^2 \, dx \leq \Lambda \int_{\Omega'} C(u - v)^2 \, dx,
\]
where \( C = C(p) \). Now \( \Omega' = A_1 \cup A_2 \) with \( |A_1 \cap A_2| = 0 \), by (2.7) with \( p = 2 \),
\[
c_2 m_{\Omega'}^{p-2} \int_{A_1 \cup [u \geq v]} |D(u - v)|^2 \, dx + c_2 m_{A_2}^{p-2} \int_{A_2 \cup [u \geq v]} |D(u - v)|^2 \, dx \leq
2\Lambda C \omega_N^{\frac{2}{p}} |\Omega'|^{\frac{1}{p}} \left[ |A_1|^{\frac{1}{p}} \int_{A_1 \cup [u \geq v]} |D(u - v)|^2 \, dx + |\Omega|^{\frac{1}{p}} \int_{A_2 \cup [u \geq v]} |D(u - v)|^2 \, dx \right],
\]
if \( |A_1| \) is small and \( m_{A_2} \) is large, for \( i = 1, 2 \), \( \int_{A_i \cup [u \geq v]} |D(u - v)|^2 = 0 \) so that \( (u - v)^+ = 0 \) in \( \Omega' \).
Before proving Theorem 1.6, we need to prove the following Harnack type comparison inequality.

**Proposition 2.4 (Harnack Type Comparison Inequality)** Suppose $u, v$ satisfy (1.6), where $\Lambda \in \mathbb{R}$ and $u, v \in W^{1,\infty}_\text{loc}(\Omega)$ if $p \neq 2$; $u, v \in W^{1,2}_\text{loc}(\Omega)$ if $p = 2$. Suppose $B(x_0, 5\delta) \subseteq \Omega$ and, if $p \neq 2$, $\inf_{B(x_0,5\delta)}(|Du| + |Dv|) > 0$. Then for any positive $s < \frac{N(p-1)}{N-p}$ we have

$$
\|u - v\|_{L^s(B(x_0,2\delta))} \leq c\delta^\frac{s}{p} \inf_{B(x_0,\delta)} (u - v),
$$

where $c$ is a constant depending on $N, p, s, \Lambda, \gamma, \Gamma$, and if $p \neq 2$ also on $m$ and $M$, where

$$
m = \inf_{B(x_0,5\delta)} (|Du| + |Dv|), \quad \text{and} \quad M = \sup_{B(x_0,5\delta)} (|Du| + |Dv|).
$$

The proof of Proposition 2.4 is similar to the proof of Proposition 2.3, as shown in the appendix.

**Proof of Theorem 1.6:** We can suppose that $\Omega \setminus Z_{u,v}$ is connected and, as in the proof of Theorem 1.2, we have to prove that $U = \{x \in \Omega \setminus Z_{u,v} : u(x) = v(x)\}$ is open. If $x \in \Omega$ we have $|Du| + |Dv| > 0$ and by continuity there exists $\delta > 0$ and $m > 0$ such that $B(x, 5\delta)$. Since $0 = v(x) - u(x) = \inf_{B(x,\delta)} (v - u)$, by Proposition 2.4 we have $\int_{B(x,2\delta)} (u - v)dx = 0$, so that $u \equiv v$ in $B(x, 2\delta)$ and $U$ is open.

$\square$
Chapter 3

Pohozaev Identity and Symmetry for $p$-Laplacian when $1 < p < 2$

Proof of Theorem 1.8: Let $\{g_\varepsilon\}$ be a sequence of $C^2(\Omega)$ functions converging to $g(u)$ as $\varepsilon \to 0$, and we consider the following problem:

$$
\begin{cases}
-\text{div} \left( (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon \right) = g_\varepsilon \quad \text{in} \quad \Omega \\
\quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega 
\end{cases}
$$

(3.1)

Let $u_\varepsilon$ be the classical solution in $C^3(\overline{\Omega})$ of (3.1), and $P = (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} (Du_\varepsilon \cdot x) Du_\varepsilon$, then

$$
\text{div} P = (Du_\varepsilon \cdot x) \text{div} \left[ (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} Du_\varepsilon \right] + (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} [Du_\varepsilon \cdot D(Du_\varepsilon \cdot x)],
$$

but we know that

$$
Du_\varepsilon \cdot D(Du_\varepsilon \cdot x) = |Du_\varepsilon|^2 + \frac{1}{2} (x \cdot D(|Du_\varepsilon|^2)),
$$

therefore we have

$$
\text{div} P = -(Du_\varepsilon \cdot x) g_\varepsilon + (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \left[ |Du_\varepsilon|^2 + \frac{1}{2} (x \cdot D(|Du_\varepsilon|^2)) \right].
$$

(3.2)
\[
\int_{\Omega} \left( \varepsilon + |Du_\varepsilon|^2 \right)^{\frac{p-2}{2}} |Du_\varepsilon|^2 dx = \int_{\Omega} \left( \varepsilon + |Du_\varepsilon|^2 \right)^{\frac{p-2}{2}} Du_\varepsilon \cdot Du_\varepsilon dx \\
= - \int_{\Omega} \text{div} \left( \left( \varepsilon + |Du_\varepsilon|^2 \right)^{\frac{p-2}{2}} Du_\varepsilon \right) u_\varepsilon dx \\
= \int_{\Omega} u_\varepsilon g_\varepsilon dx, \tag{3.3}
\]

and
\[
\frac{1}{2} \left( x \cdot D(|Du_\varepsilon|^2) \right) (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} = \frac{1}{p} \left( x \cdot D(\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} \right)
\]
\[
\frac{1}{2} \int_{\Omega} \left( x \cdot D(|Du_\varepsilon|^2) \right) (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} dx = \frac{1}{p} \int_{\Omega} x \cdot D(\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dx
\]
\[
= - \frac{N}{p} \int_{\Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dx \\
+ \frac{1}{p} \int_{\partial \Omega} (x \cdot \nu)(\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dS. \tag{3.4}
\]

Therefore, together with (3.2), (3.3) and (3.4), we have
\[
\int_{\Omega} \text{div} P dx = \int_{\Omega} \left\{ - (Du_\varepsilon \cdot x) g_\varepsilon + (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} \left[ |Du_\varepsilon|^2 + \frac{1}{2} \left( x \cdot D(|Du_\varepsilon|^2) \right) \right] \right\} dx
\]
\[
= - \int_{\Omega} (Du_\varepsilon \cdot x) g_\varepsilon dx + \int_{\Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} |Du_\varepsilon|^2 dx \\
+ \frac{1}{2} \int_{\Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} x \cdot D(|Du_\varepsilon|^2) dx
\]
\[
= - \int_{\Omega} (Du_\varepsilon \cdot x) g_\varepsilon dx + \int_{\Omega} u_\varepsilon g_\varepsilon dx - \frac{N}{p} \int_{\Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dx \\
+ \frac{1}{p} \int_{\partial \Omega} (x \cdot \nu)(\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dS. \tag{3.5}
\]

By Divergence theorem, we get
\[
\int_{\Omega} \text{div} P dx = \int_{\partial \Omega} P \cdot \nu dS
\]
\[
= \int_{\partial \Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} (Du_\varepsilon \cdot x)(Du_\varepsilon \cdot \nu) dS. \tag{3.6}
\]
But \( u_\varepsilon = 0 \) and \( Du_\varepsilon = u_\nu \) on \( \partial \Omega \), combining (3.5) and (3.6), we get
\[
- \int_\Omega (Du_\varepsilon \cdot x) g_\varepsilon dx + \int_\Omega u_\varepsilon g_\varepsilon dx - \frac{N}{p} \int_\Omega (\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dx
= \int_{\partial \Omega} (\varepsilon + |Du_\varepsilon|^2)^{\frac{p-2}{2}} (Du_\varepsilon \cdot x)(Du_\varepsilon \cdot \nu) dS - \frac{1}{p} \int_{\partial \Omega} (x \cdot \nu)(\varepsilon + |Du_\varepsilon|^2)^{\frac{p}{2}} dS
= \int_{\partial \Omega} (\varepsilon + |u_\nu|^2)^{\frac{p-2}{2}} (u_\nu \cdot x) dS - \frac{1}{p} \int_{\partial \Omega} (x \cdot \nu)(\varepsilon + |u_\nu|^2)^{\frac{p}{2}} dS
= \int_{\partial \Omega} (x \cdot \nu) \left[ u_\nu^2 \left( 1 - \frac{1}{p} \right) - \frac{\varepsilon}{p} \right] (\varepsilon + |u_\nu|^2)^{\frac{p-2}{2}} dS.
\]
Letting \( \varepsilon \to 0 \),
\[
- \int_\Omega (Du \cdot x) g(u) dx + \int_\Omega u g(u) dx - \frac{N}{p} \int_\Omega |Du|^p dx = \int_{\partial \Omega} (x \cdot \nu)|u_\nu|^2 \left( 1 - \frac{1}{p} \right) |u_\nu|^{p-2} dS
- \int_\Omega (Du \cdot x) g(u) dx + \left( 1 - \frac{N}{p} \right) \int_\Omega u g(u) dx = \left( 1 - \frac{1}{p} \right) \int_{\partial \Omega} (x \cdot \nu) \left| \frac{\partial u}{\partial \nu} \right|^p dS, \tag{3.7}
\]
as
\[
\int_\Omega |Du|^p dx = \int_\Omega u g(u) dx. \tag{3.8}
\]
Clearly \( (Du \cdot x) g(u) = D(G(u)) \cdot x \), then the result follows by (3.7) and (3.8) immediately.

\( \Box \)

**Proof of Theorem 1.9:** Suppose \( u \) is a solution of (1.9). When \( f(0) > 0 \), then the usual strong maximum principle for the Laplace operator gives immediately the positivity of the solutions of (1.9). If \( f(0) = 0 \), since \( f \) is locally Lipschitz continuous in \([0, +\infty)\), from (1.9) we have
\[
-\Delta_p u - (f(u) - f(0)) = 0,
\]
and hence \( u \) satisfies a differential inequality of the type
\[
-\Delta_p u + c(x) u \geq 0, \quad c \in L^\infty(B).
\]
Then, by the strong maximum principle, \( u \) is positive in \( B \). If \( f(0) < 0 \), then by the Theorem 2.1 in [22], \( u > 0 \) in \( B \). Hence, if \( 1 < p < 2 \) and \( f \) is locally Lipschitz continuous, \( u \) is positive in \( B \).
If $-1 < \lambda \leq 0$ the functions $u$ and $u_\lambda$ satisfy the equation

$$-\Delta_p z = f(z) \quad \text{in} \quad B_\lambda,$$

with $f$ is locally Lipschitz continuous. There exist $\delta, M > 0$ such that if $-1 < \lambda \leq 0$, $B'$ is an open subset of $B_\lambda$ with $B' = A_1 \cup A_2$, $|A_1| < \delta$, $M_A = \sup_{A_2} (|Du| + |Du_\lambda|) < M$ and $u \leq u_\lambda$ on $\partial B'$, then $u \leq u_\lambda$ in $B'$.

If $\lambda + 1$ is small, then there exists a $\delta$ such that $|B_\lambda| < \delta$. Moreover if $x \in \partial B_\lambda \cap B$ then $0 = u(x) \leq u_\lambda(x)$; if $x \in \partial B_\lambda \cap T_\lambda$ then $x = x_\lambda$ and $u(x) = u_\lambda(x)$. So $u \leq u_\lambda$ on $\partial B_\lambda$, and by Theorem 1.3 (with $A_2 = \phi$) we get $u \leq u_\lambda$ in $B_\lambda$ for $\lambda$ closes to $-1$.

Define $\lambda_0$ as the supremum of those $\lambda \in (-1, 0)$ such that for each $\mu \in (-1, \lambda)$, we have $u \leq u_\mu$ in $B_\mu$. If we show that $\lambda_0 = 0$, then by continuity $u \leq u_0$ in $B_0$ with $u(x_1, x')$ nondecreasing for $x_1 < 0$, similarly the same procedure in $B^0$ we get $u \equiv u_0$.

We shall prove this by contradiction.

Suppose $\lambda_0 < 0$. Then by continuity $u \leq u_{\lambda_0}$ in $B_{\lambda_0}$. Since $u \leq u_{\lambda_0}$ in $B_{\lambda_0}$ by Theorem 1.6 we have $u < u_{\lambda_0}$ in each connected component of $B_{\lambda_0} \setminus Z_{\lambda_0}$ unless $u$ and $u_{\lambda_0}$ coincide. Suppose $u \equiv u_{\lambda_0}$ in $B_{\lambda_0} \setminus Z_{\lambda_0}$ then by continuity $u \equiv u_{\lambda_0}$ on $\partial Z_{\lambda_0}$. If $Z_{\lambda_0}$ is discrete, then $Z_{\lambda_0} = \partial Z_{\lambda_0}$, so $u \equiv u_{\lambda_0}$ in $B_{\lambda_0}$. Otherwise since $D(u - u_{\lambda_0}) = 0$ in $Z_{\lambda_0}$, it follows that $u - u_{\lambda_0}$ is constant in $(Z_{\lambda_0})^c$. But this constant must be zero because $B_{\lambda_0} \cap \partial Z_{\lambda_0} \neq \emptyset$. So $u \equiv u_{\lambda_0}$ in $B_{\lambda_0}$ and this show that if $u \neq u_{\lambda_0}$ in $B_{\lambda_0}$ then $u \neq u_{\lambda_0}$ in $B_{\lambda_0} \setminus Z_{\lambda_0}$. Since $u \neq u_{\lambda_0}$ in $B_{\lambda_0} \setminus Z_{\lambda_0}$ which is connected, we have $u < u_{\lambda_0}$ in $B_{\lambda_0} \setminus Z_{\lambda_0}$.

Let $C = \{x \in B_{\lambda_0} : u(x) = u(x_{\lambda_0})\} \subseteq Z_{\lambda_0}$. Since $Du = Du_{\lambda_0} = 0$ in $C$ there exists an open set $A$ with $C \subseteq A \subseteq B_{\lambda_0}$ such that $M_{A, \lambda_0} = \sup_A (|Du| + |Du_{\lambda_0}|) < \frac{M}{2}$. Let $K \subseteq B_{\lambda_0}$ be compact with $|B_{\lambda_0} \setminus K| < \frac{\delta}{2}$. In the compact $K \setminus A \subseteq B_{\lambda_0} \setminus C$, $u_{\lambda_0} - u$ is positive and $u_{\lambda_0} - u = m$ where $m > 0$. There exists $\epsilon > 0$ such that, by continuity $\lambda_0 + \epsilon < 0$ and for $\lambda_0 < \lambda < \lambda_0 + \epsilon$ we have $|B_{\lambda_0} \setminus K| < \delta$, $M_{A, \lambda} = \sup_A (|Du| + |Du_\lambda|) < M$ and $u_\lambda - u > 0$ in $K \setminus A$ in particular on $\partial (K \setminus A)$. Moreover $u \leq u_\lambda$ on $\partial (B_\lambda \setminus (K \setminus A))$ (if $x_0$ is a point on $\partial (B_\lambda \setminus (K \setminus A))$ then either $x_0 \in T_\lambda$ where $u = u_\lambda$ or $x_0 \in \partial B$ where $0 = u \leq u_\lambda$ or else $x_0 \in \partial (K \setminus A)$ where $u < u_\lambda$).
Since $B_\lambda \setminus (K \setminus A)$ is the disjoint union of $A_1 = B_\lambda \setminus K$ and $A_2 = K \cap A$, by Theorem 1.3 we get $u \leq u_\lambda$ in $B_\lambda \setminus (K \setminus A)$ so that $u \leq u_\lambda$ in $B_\lambda$ for $\lambda_0 < \lambda < \lambda_0 + \epsilon < 0$, which contradicts the definition of $\lambda_0$. Therefore $\lambda_0 = 0$ and $u$ is symmetric and decreasing in the $x_1-$direction. Repeat the proof for any direction we get $u$ is radial and radially decreasing.

If $Z$ is discrete so is $Z_\lambda$ for each $\lambda \leq 0$. From the above we deduce that for each $\lambda \in (-1, 0)$ we have $u < u_\lambda$ in $B_\lambda \setminus Z_\lambda$. If $(x_1, x')$, $(y_1, x') \in B$ with $x_1 < y_1 < 0$, $\lambda = \frac{x_1 + y_1}{2}$ and $(x_1, x') \notin Z_\lambda$ then $u(x_1, x') < u(y_1, x')$. If $Du(x_1, x') = Du(y_1, x') = 0$, there exist $z_1 \in (x_1, y_1)$ with $Du(z_1, x') \neq 0$ since $Z$ is discrete. By the previous argument we get $u(x_1, x') < u(z_1, x') < u(y_1, x')$, so $u(x_1, x')$ is strictly increasing for $x_1 < 0$. Repeat the proof for any direction we get $u$ is radially strictly decreasing.

\[\square\]

**Proof of Theorem 1.10:** It was also proved by the moving plane method, but with hyperplane

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\},$$

where $\nu \in \mathbb{R}^N$ and $|\nu| = 1$. Therefore if $x \in \mathbb{R}^N$ then $x_\lambda = x + 2(\lambda - x \cdot \nu)\nu$ is the point corresponding to $x$ in the reflection through $T_\lambda^\nu$. We need to define other notation

$$B_\lambda^\nu = \{x \in B : x \cdot \nu < \lambda\},$$

$$\lambda_0(\nu) = \sup\{\lambda \in (-1, 0] : u(x) \leq u(x_\mu^\nu)\} \text{ in } B_\mu^\nu \text{ for any } \mu \in (-1, \lambda]\}$$

The proof is technically quite complicated, the idea is to show that, for any direction $\nu \in \mathbb{R}^N$, if $\lambda_0(\nu) < 0$ then there exists a small set $\Gamma$ of critical points of $u$ in the cap $B_{\lambda_0(\nu)}^\nu$ on which $u$ is constant and whose projection on the hyperplane $T_{\lambda_0(\nu)}^\nu$ contains an open subset of $T_{\lambda_0(\nu)}^\nu$, which reach a contradiction. Then $\lambda_0(\nu) = 0$ and $u$ is symmetric and decreasing in $\nu$-direction, then repeat the proof in any direction we get $u$ is radially decreasing.

\[\square\]
In this chapter we study the behavior of least energy solutions of the nonlinear quasi-linear elliptic equation (1.10) and prove Theorem 1.11.

The least critical value $c_\varepsilon$ associated to $I_\varepsilon$ in $W^{1,p}_0(\Omega)$ can be characterized as

$$c_\varepsilon = \inf_{u \in M_\varepsilon} I_\varepsilon(u)$$

(4.1)

where $M_\varepsilon = \{ u \in W^{1,p}_0(\Omega) : u \geq 0, u \not= 0, \int_{\Omega} (\varepsilon^p |\nabla u|^p + u^p) = \int_{\Omega} u^{q+1} \}$. We observe that given $u \in W^{1,p}_0(\Omega)$, $u \geq 0$ and $u \not= 0$, there exists exactly one $t > 0$ such that $tu \in M_\varepsilon$. In fact given $u \in W^{1,p}_0(\Omega)$, $u \geq 0$ and $u \not= 0$, consider $E_\varepsilon(t) = \int_{\Omega} (\varepsilon^p |\nabla(tu)|^p + (tu)^p) - \int_{\Omega} (tu)^{q+1}$. Then $E_\varepsilon(t) \rightarrow -\infty$ as $t \rightarrow \infty$, since $q > p-1$, and $E_\varepsilon(t) > 0$ when $t$ near 0. Therefore there exist at least one $t$ such that $tu \in M_\varepsilon$.

Suppose $tu$ also in $M_\varepsilon$, i.e.

$$\int_{\Omega} (\varepsilon^p |\nabla(tu)|^p + (tu)^p) = \int_{\Omega} (tu)^{q+1}$$

and

$$\int_{\Omega} (\varepsilon^p |\nabla u|^p + u^p) - \int_{\Omega} t^{q-p+1} u^{q+1} = 0,$$

Similarly, we have

$$\int_{\Omega} (\varepsilon^p |\nabla u|^p + u^p) - \int_{\Omega} t^{q-p+1} u^{q+1} = 0,$$
Hence $i = t$.

Also at this $t$ one has $I_\epsilon(\tau u) = \max_{\tau > 0} I_\epsilon(\tau u)$.

Our proof is based on energy comparisons with the corresponding problem in a ball $B_\rho = \{ x \in \mathbb{R}^N : |x| \leq \rho \}$, where $\rho > 0$. Thus we consider the following equation

$$\begin{cases}
\Delta_p u - u^{p-1} + u^q = 0 \text{ in } B_\rho \\
u > 0 \text{ in } B_\rho \\
u = 0 \text{ on } \partial B_\rho.
\end{cases} \quad (4.2)$$

and its associated functional

$$J_\rho(u) = \frac{1}{p} \int_{B_\rho} (|\nabla u|^p + u^p) - \frac{1}{q+1} \int_{B_\rho} u^{q+1}. \quad (4.3)$$

This functional has a least positive critical value, denoted by $c_\rho$, which can be characterized similarly to (4.1). Using Schwarz’s symmetrization, we find at least one radially symmetric least energy solution of (4.2).

**Lemma 4.1** If $w$ and $w_\rho$ are the solutions of equations (1.18) and (4.2) respectively, then for any large $R < r$, the following estimates hold

$$w(r) = w(R) \exp \left\{ - \left( \frac{1}{p-1} \right)^{\frac{1}{p}} + o(1) \right\} (r - R),$$

$$w_\rho(r) = w_\rho(R) \exp \left\{ - \left( \frac{1}{p-1} \right)^{\frac{1}{p}} + o(1) \right\} (r - R),$$

where $o(1) \to 0$ as $r \to \infty$.

**Proof of Lemma 4.1:** Since $w = w(r)$ is the solution of equation (1.18), then we have

$$\left( |w'|^{p-2} w' \right)' + \frac{N - 1}{r} (|w'|^{p-2} w') - w^{p-1} + w^q = 0.$$

Let $\varphi(r) = -\frac{|w'|^{p-2} w'}{w^{p-1}}$. Yi Li and C. Zhao in [29] proved that

$$\lim_{r \to \infty} \varphi(r) = \left( \frac{1}{p-1} \right)^{\frac{p-1}{p}}.$$
Therefore we have
\[
\lim_{r \to \infty} \frac{w'(r)}{w(r)} = \lim_{r \to \infty} (-\varphi)^{\frac{1}{p-1}} = -\left(\frac{1}{p-1}\right)^{\frac{1}{p}},
\]
if \(r\) is large enough,
\[
\frac{d}{dr} \left(\log(w(r))\right) = \frac{w'(r)}{w(r)} = -\left(\frac{1}{p-1}\right)^{\frac{1}{p}} + o(1),
\]
where \(o(1) \to 0\) as \(r \to \infty\). Then integrate both sides we get
\[
\log(w(r)) - \log(w(R)) = -\left(\frac{1}{p-1}\right)^{\frac{1}{p}} (r - R) + o(1)(r - R)
\]
\[
\log \left(\frac{w(r)}{w(R)}\right) = \left[-\left(\frac{1}{p-1}\right)^{\frac{1}{p}} + o(1)\right] (r - R)
\]
\[
w(r) = w(R) \exp \left\{ \left[-\left(\frac{1}{p-1}\right)^{\frac{1}{p}} + o(1)\right] (r - R) \right\}.
\]

Given \(\varepsilon > 0\), we consider the solution of the equation
\[
(|u'|^{p-2}u')' - (1 - \varepsilon)u^{p-1} = 0 \quad \text{in} \quad (R, \rho) \tag{4.4}
\]
with boundary conditions \(u(R) = w_\rho(R)\) and \(u(\rho) \geq 0\). Then the solution of (4.4) is
\[
u(r) = w_\rho(R) \exp \left\{ -\left(\frac{1 - \varepsilon}{p-1}\right)^{\frac{1}{p}} (r - R) \right\}.
\]

If \(R\) is large enough, we obtain that \(u\) is a supersolution of (4.2) in \([R, \rho]\). Therefore \(w_\rho(r) \leq u(r)\) in \([R, \rho]\), therefore
\[
w_\rho(r) \leq w_\rho(R) \exp \left\{ -\left(\frac{1 - \varepsilon}{p-1}\right)^{\frac{1}{p}} (r - R) \right\}. \tag{4.5}
\]

For the lower estimate, given \(\varepsilon > 0\), we consider the solution of the equation
\[
(|u'|^{p-2}u')' - (1 + \varepsilon)u^{p-1} \geq 0 \quad \text{in} \quad (R, \rho) \tag{4.6}
\]
with the boundary conditions \(u(R) = w_\rho(R)\) and \(u(\rho) = 0\). Then
\[
u(r) = \frac{w_\rho(R)(e^{-\alpha r} - e^{-\alpha \rho})}{e^{-\alpha R} - e^{-\alpha \rho}} \quad \text{where} \quad \alpha = \left(\frac{1 + \varepsilon}{p-1}\right)^{\frac{1}{p}}
\]
is the solution of (4.6). We observe that if $R$ is large enough, $u$ is a subsolution of (4.2) in $[R, \rho)$. Then we have

$$w_\rho(r) \geq \frac{w_\rho(R)(e^{-\alpha r} - e^{-\alpha R})}{e^{-\alpha R} - e^{-\alpha \rho}} \geq w_\rho(R) \exp\left\{ -\left(\frac{1}{p-1}\right)^{\frac{1}{p}} + o(1) \right\}(r - R)$$

(4.7)

Together with (4.5) and (4.7) the estimate immediately follows.

□

Lemma 4.2 Let $u$ and $v$ be the solutions of the following equations

$$(|u'|^{p-2}u')' - \frac{N-1}{r}(|u'|^{p-2}u') - u^{p-1} = 0 \quad \text{in} \quad (\rho - 1, \infty)$$

(4.8)

$$(|v'|^{p-2}v')' - \frac{N-1}{r}(|v'|^{p-2}v') - e(\rho)v^{p-1} = 0 \quad \text{in} \quad (\rho - 1, \rho)$$

(4.9)

with the boundary conditions $u(\rho - 1) = v(\rho - 1) = 1$ and $u(\infty) = 0$, $v(\rho) = 0$, where $e(\rho) \to 1$ as $\rho \to \infty$ uniformly. Then for some $\lambda_0 > 0$, we have

$$|u'(\rho - 1)|^{p-2}u'(\rho - 1) - |v'(\rho - 1)|^{p-2}v'(\rho - 1) \geq \lambda_0$$

for large $\rho$.

Proof of Lemma 4.2: Given $\epsilon > 0$, we consider the following equation

$$(|u'|^{p-2}u')' - (1 + \epsilon)u^{p-1} = 0 \quad \text{in} \quad (\rho - 1, \infty)$$

(4.10)

with the boundary conditions $u(\rho - 1) = 1$ and $u(\infty) = 0$. We know that

$$\bar{u}(r) = \exp\left\{ -\left(\frac{1+\epsilon}{p-1}\right)^{\frac{1}{p}} (r - \rho + 1) \right\}$$

is the solution of (4.10). Also if $\rho$ is large $\bar{u}$ is a subsolution of (4.8). Then we have

$$u'(\rho - 1) \geq \lim_{r \to \rho - 1} \frac{\bar{u}(r) - \bar{u}(\rho - 1)}{r - (\rho - 1)}$$

$$= -\left(\frac{1+\epsilon}{p-1}\right)^{\frac{1}{p}}.$$
Next consider

\[ (|v'|^{p-2}v')' - (1 - \varepsilon)e(\rho)v^{p-1} \leq 0 \quad \text{in} \quad (\rho - 1, \rho) \quad (4.11) \]

with the boundary conditions \( v(\rho - 1) = 1 \) and \( v(\rho) = 0 \). Then

\[ \tilde{v}(r) = \frac{e^{\alpha(r-r)} - e^{-\alpha(r-r)}}{e^\alpha - e^{-\alpha}} \quad \text{where} \quad \alpha = \left( \frac{(1 - \varepsilon)e(\rho)}{p - 1} \right)^{\frac{1}{p}} \]

is the solution of (4.11). Also if \( \rho \) is large \( \tilde{v} \) is a supersolution of (4.9). Similarly we have

\[ v' (\rho - 1) \leq \lim_{r \to \rho - 1} \frac{\tilde{v}(r) - \tilde{v}(\rho - 1)}{r - (\rho - 1)} = \frac{\alpha e^{\alpha} + \alpha e^{-\alpha}}{e^\alpha - e^{-\alpha}}. \]

Hence

\[ |u'(\rho - 1)|^{p-2}u'(\rho - 1) - |u'(\rho - 1) - p - 1|^{p-2}u'(\rho - 1) \geq \left( \frac{(1 - \varepsilon)e(\rho)}{p - 1} \frac{e^{\alpha} + e^{-\alpha}}{e^\alpha - e^{-\alpha}} \right)^{p-1} - \left( \frac{1 + \varepsilon}{p - 1} \right)^{p-1} \geq \lambda_0 > 0 \]

if \( \varepsilon \) is small.

\[ \square \]

**Lemma 4.3**

\[ c_\rho = c_* + \exp \left\{ - p \left[ \left( \frac{1}{p - 1} \right)^{\frac{1}{p}} + o(1) \right] \rho \right\} \]

where \( c_* \) is given by (1.13), and \( o(1) \to 0 \) as \( \rho \to \infty \).

**Proof of Lemma 4.3:** First we find an upper estimate for \( c_\rho \) with the required form. Let \( v_\rho(r) \) be the solution of the equation

\[ \Delta_\rho u - u^{p-1} = 0 \quad (4.12) \]

in \( B_\rho \setminus \overline{B}_{\rho - 1} \), with boundary conditions \( v_\rho(\rho - 1) = w(\rho - 1) \), and \( v_\rho(\rho) = 0 \). We define \( \tilde{w}_\rho(r) = w(r) \) if \( 0 \leq r \leq \rho - 1 \), and \( \tilde{w}_\rho(r) = v_\rho(r) \) if \( \rho - 1 \leq r \leq \rho \). Then we have

\[ c_\rho = J_\rho(w_\rho) = \max_{t \geq 0} J_\rho(tw_\rho) \leq \max_{t \geq 0} J_\rho(t\tilde{w}_\rho) = J_\rho(t\tilde{w}_\rho). \]
Since $\bar{w}_\rho$ converges in the $W^{1,p}$ sense to $w$, it is easy to see that $t_\rho \to 1$ as $\rho \to \infty$.

Next we see that

$$J_\rho(t_\rho \bar{w}_\rho) \leq \frac{1}{p} \int_{B_\rho} |\nabla (t_\rho \bar{w}_\rho)|^p + \frac{1}{p} \int_{B_\rho} (t_\rho \bar{w}_\rho)^p - \frac{1}{q + 1} \int_{B_\rho} (t_\rho \bar{w}_\rho)^q + 1$$

$$\leq \frac{t_\rho^p}{p} \int_{B_{\rho-1}} |\nabla w_\rho|^p + \frac{t_\rho^p}{p} \int_{B_{\rho-1}} w_\rho^p - \frac{t_\rho^{q+1}}{q + 1} \int_{B_\rho} w_\rho^q + 1 + \frac{t_\rho^p}{p} \int_{\partial B_{\rho-1}} (|\nabla v_\rho|^p + v_\rho^p)$$

$$\leq I(t_\rho w_\rho) + \frac{t_\rho^p}{p} \int_{\partial B_{\rho-1}} v_\rho |Dv_\rho| p^{-2} \partial v_\rho \partial$$

$$\leq c_* + \frac{t_\rho^p}{p} \omega_{N} r^{N-1} |v_\rho | p^{-1} \right|_r$$

$$\leq c_* + \frac{1}{p} \left( \frac{1}{p-1} \right)^{\frac{p-1}{p}} \right|_{r-\rho} \exp \left\{-p \left( \frac{1}{p-1} \right)^{1} + o(1) \right\}. \right.$$ 

Therefore the upper estimate follows directly from the above inequality.

As for the lower estimate, let $\bar{v}_\rho$ be the solution of equation (4.12) in $\mathbb{R}^N \setminus B_{\rho-1}$, with boundary conditions $\bar{v}_\rho(\rho-1) = w_\rho(\rho-1)$ and $\bar{v}_\rho(\infty) = 0$. Let us define $\bar{w}_\rho(r) = w_\rho(r)$ if $0 \leq r \leq \rho - 1$ and $\bar{w}_\rho(r) = \bar{v}_\rho(r)$ if $\rho - 1 \leq r < \infty$. Then we have, for all $t > 0$ that

$$J_\rho(w_\rho) - I_\rho(t \bar{w}_\rho) \geq J_\rho(t w_\rho) - I_\rho(t \bar{w}_\rho)$$

$$= \frac{t_\rho^p}{p} \int_{B_\rho} (|\nabla w_\rho|^p + w_\rho^p) - \frac{t_\rho^{q+1}}{q + 1} \int_{B_\rho} w_\rho^{q+1} + \frac{t_\rho^p}{p} \int_{\mathbb{R}^N} (|\nabla \bar{w}_\rho|^p + \bar{w}_\rho^p)$$

$$\geq \left[ \frac{t_\rho^p}{p} \int_{B_\rho} (|\nabla w_\rho|^p + w_\rho^p) - \frac{t_\rho^{q+1}}{q + 1} \int_{B_\rho} w_\rho^{q+1} - \frac{t_\rho^p}{p} \int_{\mathbb{R}^N} (|\nabla \bar{w}_\rho|^p + \bar{w}_\rho^p) \right]$$

$$\geq \left[ \frac{t_\rho^p}{p} \int_{B_\rho} (|\nabla w_\rho|^p + w_\rho^p) - \frac{t_\rho^{q+1}}{q + 1} \int_{B_\rho} w_\rho^{q+1} - \frac{t_\rho^p}{p} \int_{B_\rho} \left( |\nabla \bar{w}_\rho|^p + \bar{w}_\rho^p \right) \right]$$

$$= \frac{t_\rho^p}{p} \int_{B_\rho} (|\nabla w_\rho|^p + w_\rho^p) - \frac{p}{q + 1} \left( (t w_\rho)^{q+1-p} \cdot w_\rho^p \right) - \frac{t_\rho^p}{p} \int_{B_{\rho-1}} \left( |\nabla \bar{w}_\rho|^p + \bar{w}_\rho^p \right)$$

$$= \frac{t_\rho^p}{p} \left\{ \int_{B_\rho} (|\nabla w_\rho|^p + (1 - \frac{p}{q + 1} (t w_\rho)^{q+1-p}) \cdot w_\rho^p) - \int_{\mathbb{R}^N \setminus B_{\rho-1}} \left( |\nabla \bar{w}_\rho|^p + \bar{w}_\rho^p \right) \right\}. \right.$$
taking \( e(\rho, t) = \max\{1 - \frac{p}{q+1}(tw_\rho)^{q+1-p} : \rho - 1 \leq r \leq \rho\} \), we get

\[
J_\rho(w_\rho) \geq I_p(t\tilde{w}_\rho) + \frac{t^p}{p} \left\{ \int_{B_\rho} \left( |\nabla w_\rho|^p + e(\rho, t)w_\rho^p \right) - \int_{\mathbb{R}^N \setminus B_{\rho-1}} (|\nabla \tilde{v}_\rho|^p + \tilde{v}_\rho^p) \right\}. \tag{4.13}
\]

Since \( w_\rho \rightarrow w \) in the \( W^{1,p} \) sense, we see that \( t_\rho \rightarrow 1 \) as \( \rho \rightarrow \infty \). We consider the comparison function \( z_\rho \) given as the solution of the equation

\[
\Delta_p u - e(\rho, t_\rho)u^{p-1} = 0 \quad \text{in} \quad B_\rho \setminus B_{\rho-1}
\]

with the boundary conditions \( z_\rho(\rho - 1) = w_\rho(\rho - 1) \) and \( z_\rho(\rho) = 0 \). Then it follows from (4.13) that

\[
c_\rho \geq c_* - \frac{t^p}{p} \int_{\partial B_{\rho-1}} \left( z_\rho |Dz_\rho|^{p-2} \frac{\partial z_\rho}{\partial \nu} - \tilde{v}_\rho |D\tilde{v}_\rho|^{p-2} \frac{\partial \tilde{v}_\rho}{\partial \nu} \right)
\]

\[
= c_* - \frac{t^p}{p} \omega_N N^{p-1} w_\rho(r)^{p-1} \left( |z'_\rho|^{p-2} z'_\rho - |\tilde{v}'_\rho|^{p-2} \tilde{v}'_\rho \right)_{r=\rho-1}.
\]

But one has

\[
|\tilde{v}'_\rho|^{p-2} \tilde{v}'_\rho - |z'_\rho|^{p-2} z'_\rho \geq w_\rho(\rho - 1) \lambda_0 \geq \exp \left\{ - \left[ \left( \frac{1}{p-1} \right)^{\frac{1}{p}} + o(1) \right] \right\}
\]

as it follows directly from Lemmas 4.1 and 4.2 and the lower estimate follows immediately.

\[\square\]

**Proof of Theorem 1.11:** Now we consider when \( \Omega = B \), then after scaling (1.14) becomes (4.2) with \( \rho = 1/\varepsilon \), also we have

\[
I_\varepsilon(u) = \varepsilon^N J_\rho(u),
\]

by Lemma 4.3 we have

\[
c_\varepsilon = \varepsilon^N c_\rho = \varepsilon^N \left\{ c_* + \exp \left\{ - \frac{p}{\varepsilon} \left[ \left( \frac{1}{p-1} \right)^{\frac{1}{p}} + o(1) \right] \right\} \right\}.
\]

\[\square\]
Proof of Corollary 1.12: We consider a ball of maximal radius contained in $\Omega$, then since the least energy values for $I_\varepsilon$ in $\Omega$ and that in the ball are ordered, namely $c_\varepsilon \leq \varepsilon^N c_\rho$ with $\rho = d_0/\varepsilon$, where $d_0 = \max_{x \in \Omega} \text{dist}(x, \partial \Omega)$, we obtain from Lemma 4.3,

$$c_\varepsilon \leq \varepsilon^N \{ c_* + \exp \left[ - p \left( \frac{d_0}{\varepsilon} \right) \left( \frac{1}{p-1} \right)^{\frac{1}{p}} + o(1) \right] \}.$$
Chapter 5

Appendix

In this Appendix we prove Proposition 2.3, we shall use the following theorem, which is a consequence of the John-Nirenberg Lemma, [4].

**Theorem 5.1** Let $u \in W^{1,p}(B)$, where $B$ is a ball in $\mathbb{R}^n$, and suppose that there exists a constant $C'$ such that

$$
\left( \int_{B_R} |Du|^p dx \right)^{\frac{1}{p}} \leq C' R^{\frac{n-p}{p}}, \quad \forall B_R \subset B.
$$

Then there exist constant $r_0 = r_0(C', p, N) > 0$ and $C = C(N)$ such that

$$
(\int_B e^{r_0 u} dx)(\int_B e^{-r_0 u} dx) \leq C |B|^2.
$$

**Proof of Proposition 2.3:** It may be assumed without loss of generality that $u(x) \geq \varepsilon > 0$. For otherwise, we may replace $u$ by $u + \varepsilon$ and let $\varepsilon \to 0$ in the final results, and for simplicity we denote $B = B(x_0, 5\delta)$.

Use $\eta^p(x) u^\beta(x)$ as a test function, where $\beta \neq 0$, $\eta(x) \in C_0^1(B)$ is to be further specified.
if $\beta < 0$,

$$-|\beta| \int_B \eta^p u^{\beta-1} A(x, Du) \cdot Du \, dx + p \int_B \eta^{p-1} u^\beta A(x, Du) \cdot D\eta \, dx + \int_B w(x) \eta^p u^{\beta-1} \, dx \geq 0,$$

(5.1)

by Lemma 2.1,

$$-|\beta| \int_B \eta^p u^{\beta-1} c_2 |Du|^p \, dx + p \int_B \eta^{p-1} u^\beta c_1 |Du|^{p-1} |D\eta| \, dx + \int_B w(x) \eta^p u^{\beta-1} ds \geq 0,$$

where $c_1, c_2$ are the constants in (2.1) and (2.2), depending on $p, \gamma, \Gamma$. Since $w(x) \leq \Lambda$ we have

$$c_2 |\beta| \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq c_1 p \int_B \eta^{p-1} u^\beta |Du|^{p-1} |D\eta| \, dx + \int_B w(x) \eta^p u^{\beta-1} \, dx$$

$$c_2 |\beta| \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq c_1 p \int_B \eta^{p-1} u^\beta |Du|^{p-1} |D\eta| \, dx + \Lambda \int_B \eta^p u^{\beta-1} \, dx$$

$$|\beta| \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq E_1 \left( \int_B \eta^{p-1} u^\beta |Du|^{p-1} |D\eta| \, dx + \int_B \eta^p u^{\beta-1} \, dx \right),$$

where $E_1 = \max \left( \frac{c_1 p \Lambda}{c_2}, \Lambda \right) = F_1 (\Gamma, \gamma, \Lambda, p)$.

By Young’s Inequality

$$ab^{p-1} \leq \frac{1}{p} a^p + \left( 1 - \frac{1}{p} \right) b^p,$$

we choose

$$\epsilon = \left( \frac{|\beta|}{E_1} \right)^{\frac{1-p}{p}}, \quad a = u|D\eta|, \quad b = \eta|Du|,$$

then we get

$$|\beta| \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq E_1 \left[ \frac{|\beta|^{1-p}}{pE_1^{1-p}} \int_B u^{p+\beta-1} |D\eta|^p \, dx + \left( 1 - \frac{1}{p} \right) \frac{|\beta|}{E_1} \int_B \eta^p u^{\beta-1} |Du|^p \, dx \right]$$

$$+ \int_B \eta^p u^{p+\beta-1} \, dx$$

$$\frac{|\beta|}{p} \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq E_1 \left( \frac{|\beta|^{1-p}}{pE_1^{1-p}} \int_B u^{p+\beta-1} |D\eta|^p \, dx + \int_B \eta^p u^{p+\beta-1} \, dx \right)$$

$$|\beta| \int_B \eta^p u^{\beta-1} |Du|^p \, dx \leq E_1^\frac{p}{|\beta|^{1-p}} \int_B u^{p+\beta-1} (|D\eta|^p + \eta^p) \, dx,$$
since \( E_1^p |\beta|^{1-p} \geq E_1^p \) if \( E_1 \) is chosen to be large enough.

\[
\int_B \eta^p u^{\beta-1} |Du|^p dx \leq E_1^p (1 + \frac{1}{|\beta|})^p \int_B u^{\beta+1-1} (|Du|^p + \eta^p) dx.
\]

(5.2)

Let us put, if \( h > 0 \), and \(-\infty < t < +\infty, t \neq 0\),

\[
\Phi(t, h) = \left( \int_{B(x_0, h)} u^t dx \right)^{\frac{1}{t}},
\]

so that

\[
\sup_{B(x_0, h)} u = \Phi(+\infty, h) \quad \text{and} \quad \inf_{B(x_0, h)} u = \Phi(-\infty, h).
\]

Put in (5.2) \( \beta = 1 - p < 0 \). For \( y \in B(x_0, 2\delta) \), \( r < \text{dist}(y, \partial B(x_0, 2\delta)) \), we choose \( \eta \in C^1_0(B) \) with \( \eta(x) = 1 \) in \( B(y, r) \), supp \( \eta \subseteq B(y, 2r) \) and \( w(x) = \log u(x) \),

\[
\int_{B(y, r)} |Du|^p dx = \int_{B(y, r)} u^{-p} |Du|^p dx
\]

\[
\leq E_1^p (1 + \frac{1}{|\beta|})^p \int_{B(y, r)} dx
\]

\[
= E_1^p (1 + \frac{1}{|\beta|})^p |B(y, r)|
\]

\[
= E_1^p (1 + \frac{1}{|\beta|})^p (\omega_N r^N)
\]

\[
\left( \int_{B(y, r)} |Du|^p dx \right)^{\frac{1}{p}} \leq E_2 (1 + \frac{1}{|\beta|}) r^{N-p},
\]

if \( r \) is small enough, where \( E_2 = E_2(\Gamma, \gamma, \Lambda, p, N) \).

Therefore by Theorem 5.1, there exist constant \( r_0 > 0 \) and \( C \) depending on \( N \) such that

\[
\left( \int_{B(x_0, 2\delta)} e^{r_0^2 u} dx \right) \left( \int_{B(x_0, 2\delta)} e^{-r_0 u} dx \right) \leq C |B(x_0, 2\delta)|^2,
\]

which implies

\[
\Phi(r_0, 2\delta) \leq \tilde{C} \delta^{-\frac{2N}{p}} \Phi(-r_0, 2\delta),
\]

(5.3)

where \( \tilde{C} = \bar{C}(N) \) and \( r_0 = r_0(\Gamma, \gamma, \Lambda, p, N) \).
Now consider (5.2) when $\beta \neq 1 - p, \beta < 0$.

By Sobolev Inequality

$$\|\eta w\|_{x^p,h_1} \leq C\|D(\eta w)\|_{p,h_1}$$

(5.4)

where $\chi = \left\{ \begin{array}{ll}
N - p, & \text{if } p < N; \\
\infty, & \text{if } p > N.
\end{array} \right.$

$C = \left\{ \begin{array}{ll}
C(p, N), & \text{if } p < N; \\
h_1^{1-\beta} C(p, N), & \text{if } p > N,
\end{array} \right.$

and

$$\|w\|_{P^r} = \left( \int_{B(x_0,r)} w^p dx \right)^{1/p}.$$

If $p = N$, then we take $\chi$ to be any arbitrary number.

For $\delta \leq h_2 < h_1 \leq 5\delta$, we take $\eta \in C_0^1(B)$ with $\eta(x) = 1$ in $B(x_0, h_2)$, $\eta(x) \leq 1$ in $B(x_0, h_1)$, supp $\eta \subseteq B(x_0, h_1)$ and $|D\eta| \leq \frac{2}{h_1 - h_2}$, then by (5.4) we obtain

$$\|w\|_{x^p,h_2} = \|\eta w\|_{x^p,h_2} \leq \|\eta w\|_{x^p,h_1} \leq C\|D(\eta w)\|_{x^p,h_1}.$$  

(5.5)

If $w(x) = u(x)^q$ where $pq = p + \beta - 1$, from (5.2) we get

$$\int_{B(x_0,h_1)} \eta^p |Dw|^p dx = \int_{B(x_0,h_1)} \eta^p q^p u^{q-1} |Du|^p dx$$

$$= \int_{B(x_0,h_1)} \eta^p q^p u^{q-\beta-1} |Du|^p dx$$

$$\left( \int_{B(x_0,h_1)} \eta^p |Dw|^p dx \right)^{1/p} = |q| \left( \int_{B(x_0,h_1)} \eta^p u^{q-\beta-1} |Du|^p dx \right)^{1/p}$$

$$\leq |q| E_1 \left( 1 + \frac{1}{|\beta|} \right) \left[ \int_{B(x_0,h_1)} (\eta^p + |D\eta|^p) u^{q+\beta-1} dx \right]^{1/p}$$

$$= |q| E_1 \left( 1 + \frac{1}{|\beta|} \right) \left[ \int_{B(x_0,h_1)} (\eta^p + |D\eta|^p) u^{\beta} dx \right]^{1/p}$$

$$\leq |q| E_1 \left( 1 + \frac{1}{|\beta|} \right) \left[ \int_{B(x_0,h_1)} (1 + \frac{2p}{(h_1 - h_2)^p}) u^p dx \right]^{1/p}$$

$$\leq E_3 |q| \left( 1 + \frac{1}{|\beta|} \right) \frac{1}{h_1 - h_2} \left( \int_{B(x_0,h_1)} u^p dx \right)^{1/p},$$

where $E_3 = E_3(\Gamma, \gamma, \Lambda, p)$. 
It follows that

\[ \| D(\eta w) \|_{p;h_1} = \| \eta Dw + w D\eta \|_{p;h_1} \]
\[ \leq \| \eta Dw \|_{p;h_1} + \| w D\eta \|_{p;h_1} \]
\[ \leq E_3 |q| \left( 1 + \frac{1}{|\beta|} \right) \frac{1}{h_1 - h_2} \| w \|_{p;h_1} + \frac{2}{h_1 - h_2} \| w \|_{p;h_1} \]
\[ = \left( 2 + E_3 |q| \left( 1 + \frac{1}{|\beta|} \right) \right) \frac{1}{h_1 - h_2} \| w \|_{p;h_1} \]
\[ \leq E_4 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] \frac{1}{h_1 - h_2} \| w \|_{p;h_1}, \quad (5.6) \]

where \( E_4 = E_4(\Gamma, \gamma, \Lambda, p) \).

Set \( r = pq = p + \beta - 1 \),

\[ \left( \int_{B(x_0, h_2)} u^{x_p} dx \right)^{\frac{1}{x_p}} = \left( \int_{B(x_0, h_2)} u^{x_q} dx \right)^{\frac{1}{x_q}} \]
\[ = \left[ \int_{B(x_0, h_2)} u^{r} dx \right]^{\frac{1}{r}} \]
\[ = \Phi(xr, h_2)^{q} \]

\[ \left( \int_{B(x_0, h_1)} u^{p} dx \right)^{\frac{1}{p}} = \left( \int_{B(x_0, h_1)} u^{p} dx \right)^{\frac{1}{p}} \]
\[ = \left[ \int_{B(x_0, h_1)} u^{r} dx \right]^{\frac{1}{r}} \]
\[ = \Phi(r, h_1)^{q} \]

Then, together with (5.5) and (5.6), we get

\[ \| w \|_{x;p,h_2} \leq E_4 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] \frac{1}{h_1 - h_2} \| w \|_{p;h_1} \]
\[ \left( \int_{B(x_0, h_2)} u^{x_p} dx \right)^{\frac{1}{x_p}} \leq E_4 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] \frac{1}{h_1 - h_2} \left( \int_{B(x_0, h_1)} u^{p} dx \right)^{\frac{1}{p}} \]
\[ \left( \int_{B(x_0, h_2)} u^{r} dx \right)^{\frac{2}{r}} \leq E_4 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] \frac{1}{h_1 - h_2} \left( \int_{B(x_0, h_1)} u^{r} dx \right)^{\frac{2}{r}}. \quad (5.7) \]

Now if \( q < 0 \), then (5.7) will become

\[ \Phi(xr, h_2) \geq \left( E_5 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] (h_1 - h_2)^{-1} \right)^{\frac{p}{r}} \Phi(r, h_1), \]
where \( E_5 = E_5(\Gamma, \gamma, \Lambda, p, N) \).

For \( r_0 > 0 \) in (5.2), define \( r_k = -\chi^k r_0 \), \( h_k = \delta[1 + \frac{3}{2}(\frac{1}{2})^k] \).

We have \( r_k \to -\infty \), \( \beta_k = r_k - (p - 1) \to -\infty \), \( h_0 = \frac{5\delta}{2} \), and \( \frac{1}{p} \leq p - 1 \),

\[
\Phi(-\chi^k r_0, h_0) \geq E_5^{-\frac{p}{r_0}} (1 + |r_0|)^{-\frac{p}{r_0}} \left( \frac{3\delta}{4} \right)^{\frac{p}{r_0}} \Phi(-r_0, h_1)
\]

\[
\Phi(-\chi^{k+1} r_0, h_{k+1}) \geq E_5^{-\frac{p}{r_0^k}} (1 + |r_0|\chi^k)^{-\frac{p}{r_0^k}} \left( \frac{3\delta}{4} \right)^{\frac{p}{r_0^k}} \frac{1}{2^{k}} \Phi(-\chi^k r_0, h_k)
\]

\[
\geq E_5^{-\frac{p}{r_0}} \left( \frac{3\delta}{4} \right)^{\frac{p}{r_0}} \sum \frac{k}{x^k} \Phi(-r_0, h_0)
\]

where \( E_6 = E_6(E_5, r_0) = E_6(\Gamma, \gamma, \Lambda, p, N) \).

Since

\[
\sum_{k=1}^{\infty} \frac{1}{x^k} = \frac{N}{p},
\]

\[
\Phi(-\infty, \delta) \geq E_7 \delta^{\frac{N}{p}} \Phi(-r_0, \frac{5\delta}{2}), \tag{5.8}
\]

where \( E_7 = E_7(\Gamma, \gamma, \Lambda, p, N) \).

If \( 0 < s \leq r_0 \), by Hölder inequality,

\[
\Phi(s, 2\delta) = \left( \int_{B(x_0, 2\delta)} u^s dx \right)^{\frac{1}{s}}
\]

\[
= (\int_{B(x_0, 2\delta)} u^s dx)^{\frac{1}{s}}
\]

\[
\leq (\int_{B(x_0, 2\delta)} u^r dx)^{\frac{1}{r}} (\int_{B(x_0, 2\delta)} dx)^{\frac{r_0 - s}{r_0^2}}
\]

\[
= \Phi(r_0, 2\delta) C_N \delta^{N(\frac{1}{s} - \frac{1}{r_0})},
\]

where \( C_N = C_N(N) \).

Therefore, by (5.3) and (5.8), we get

\[
\Phi(s, 2\delta) \leq C_N \delta^{N(\frac{1}{s} - \frac{1}{r_0})} \Phi(r_0, 2\delta)
\]

\[
\leq C_N C' \delta^{N(\frac{1}{s} - \frac{1}{r_0})} \delta^2 \Phi(-r_0, 2\delta)
\]

\[
\leq E_8 \delta^{\frac{N}{p}} \Phi(-\infty, \delta),
\]
where \( E_8 = E_8(\Gamma, \gamma, \Lambda, p, N) \).

If \( r_0 < s < \frac{N(p-1)}{N-p} \), and \( q > 0 \), (5.7) becomes

\[
\Phi(\chi r, h_1) \leq \left\{ E_5 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] (h_1 - h_2)^{-1} \right\} \Phi(\chi, h_2),
\]

then \( \frac{s}{\chi^{k_0+1}} = r_1 \leq r_0 \) for some \( k_0 \in \mathbb{N} \).

Set \( \bar{r}_k = r_1 \chi^k \), \( k = 0, 1, \ldots, k_0 + 1 \) and \( h_0 = \frac{s}{2} > h_1 > h_2 > \ldots > h_{k_0+1} = 2\delta \) with

\[
h_k - h_{k+1} = \frac{1}{k_0 + 1} (\delta + \frac{\delta}{2}).
\]

Therefore (5.9) becomes

\[
\Phi(\chi^{k_0+1} r_1, h_{k_0+1}) \leq \left\{ E_5 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] (h_{k_0} - h_{k_0+1})^{-1} \right\} \Phi(\chi^{k_0+1} r_1, h_{k_0}).
\]

Since \( |\beta_{k_0+1}| \leq |\beta_k| \) for \( k = 0, 1, 2, \ldots, k_0 \) with \( |\beta_{k_0}| = |r_{k_0} - p + 1| = -s + p - 1 > 1 \), we get

\[
\Phi(\chi^{k_0+1} r_1, h_{k_0+1}) \leq \left\{ E_5 \left[ 1 + |q| \left( 1 + \frac{1}{|\beta|} \right) \right] \frac{2(k_0 + 1)}{\delta} \right\} \Phi(\chi^{k_0+1} r_1, h_{k_0})
\leq E_9 \left( \frac{1}{\delta} \right) \sum_{k=0}^{k_0} \frac{1}{\chi^k} \Phi(\chi, h_0),
\]

where \( E_9 \) and \( E_{10} \) depend on \( \Gamma, \gamma, \Lambda, p, N \) and \( s \).

Since

\[
\sum_{k=0}^{k_0} \frac{1}{\chi^k} = \frac{N(s - r_1)}{ps} = \frac{N}{p} \left( 1 - \frac{r_1}{s} \right),
\]

therefore

\[
\Phi(s, 2\delta) \leq E_{10} \delta \frac{N}{r_1} \Phi(r_1, \frac{5\delta}{2}).
\]
Similarly we can deduce
\[ \phi(b,\delta) \leq E_{11} \delta \frac{N}{s} \phi(-\infty, \delta), \]
where \( E_{11} = E_{11}(\Gamma, \gamma, \Lambda, p, N, s) \).

\[ \square \]

**Proof of Proposition 2.4:** The proof is similar to the proof of Proposition 2.3, but with \( \eta^p(x)(v(x) - u(x))^\beta \) as a test function, then (5.1) will become

\[ -|\beta| \int_B \eta^p(v - u)^{\beta - 1}[A(x, Du) - A(x, Dv)] \cdot D(v - u) dx + \]

\[ p \int_B \eta^{p-1}(v - u)\beta[A(x, Du) - A(x, Dv)] \cdot D\eta dx + \Lambda \int_B \eta^p(v - u)^\beta(|u|^{p-2}u - |v|^{p-2}v) dx \leq 0. \]

Using mean value theorem and estimates (2.1) and (2.2), if \( 1 < p < 2 \) we get

\[ c_2|\beta|M^{p-2} \int_B \eta^p(v - u)^{\beta - 1}|Dv - Du|^2 dx \leq pc_1 m^{p-2} \int_B \eta^{p-1}(v - u)^\beta|D(v - u)||D\eta| dx + C\Lambda \int_B \eta^p(v - u)^{\beta + 1} dx, \]

where \( C = C(p) \), and \( c_1, c_2 \) are the constants in (2.1) and (2.2), depending on \( p, \gamma, \Gamma \). If \( p > 2 \) we obtain the same inequality with the roles of \( m, M \) interchanged. Then using the same technique we can get (2.13).

\[ \square \]
Bibliography


