Multiple Nodal Solutions for Some Singularly Perturbed Neumann Problems

Chan Sik Kin

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Abstract

We consider the following singularly perturbed elliptic problem

\[
\begin{cases}
\varepsilon^2 \Delta u - u + |u|^{p-1}u = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( \varepsilon > 0 \) is a small parameter, \( 1 < p < +\infty \) for \( N = 1, 2 \), \( 1 < p < \frac{N+2}{N-2} \) for \( N \geq 3 \). It is known that this equation has a least-energy nodal solution concentrating, as \( \varepsilon \) approaches zero, at the global maximum point of the mean curvature function \( H(P), P \in \partial \Omega \).

In this thesis, we prove that for \( \varepsilon \) sufficiently small there exists a nodal solution concentrating at any local maximum point of \( H(P) \). This implies that if \( H(P) \) has \( K \) local maximum points there exist \( K \)-nodal solutions.

We first use the Liapunov-Schmidt method to reduce the problem to finite dimensions. Then we use a minimizing procedure to obtain multiple nodal solutions.
摘要

我們考慮以下奇異攝動橢圓問題

\[
\begin{aligned}
\varepsilon^2 \Delta u - u + |u|^{p-1} u &= 0 & \text{在 } \Omega \\
\frac{\partial u}{\partial \nu} &= 0 & \text{在 } \partial \Omega
\end{aligned}
\]

其中 \( \Omega \) 是 \( \mathbb{R}^N \) 中的有界光滑區域， \( \varepsilon \) 是一個充分小的正實數，對於 \( N=1,2 \) 而言，

\[ 1 < p < \infty \]

對於 \( N \geq 3 \) 而言，

\[ 1 < p < \frac{N+2}{N-2} \]

已知當 \( \varepsilon \) 趨於零時，這個方程式有一個最小能量變號解，而且集中在平均曲率函數 \( H(P) \) 的總體最大值，\( P \) 屬於 \( \Omega \) 的邊界。

在這篇論文中，我們將會證明對於充分小的 \( \varepsilon \)，會存在一個變號解集中在 \( H(P) \) 的局部最大值中。這說明如果 \( H(P) \) 擁有 \( K \) 個局部最大值，會存在 \( K \) 個變號解。

我們首先會用 Liapunov-Schmidt 方法去把這個問題簡化至有限空間，然後用最小化過程取得多個變號解。
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Chapter 1

Introduction

The aim of this thesis is to construct a family of multiple nodal solutions to the following singularly perturbed elliptic problem

\[
\begin{aligned}
\varepsilon^2 \Delta u - u + |u|^{p-1} u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $\varepsilon > 0$ is a constant, the exponent $p$ satisfies $1 < p < \frac{N+2}{N-2}$ for $N \geq 3$ and $1 < p < \infty$ for $N = 1, 2$ and $\nu(x)$ denotes the normal derivative at $x \in \partial \Omega$.

Problem (1.1) arises in many applications, such as Keller-Segel model in chemotaxis [23], population genetics, chemical reactor theory, and Gierer-Meinhardt system [14] etc. The existence and shape of positive solutions of (1.1) have been studied extensively in recent years.

In [25], [26], Ni and Takagi showed that, as $\varepsilon \to 0$, the least energy (positive) solution for (1.1) has a unique maximum point, say $P_\varepsilon$, on $\partial \Omega$. Moreover, $H(P_\varepsilon) \to \max_{P \in \partial \Omega} H(P)$, where $H(P)$ is the mean curvature function on $\partial \Omega$.

Since then, many papers further investigated the higher energy solutions for (1.1) with either Neumann boundary condition or Dirichlet boundary condition. These solutions are called spike layer solutions. A general principle is that the interior spike
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Layer solutions are generated by distance functions. We refer the reader to the articles [4], [10], [11], [12], [7], [8], [17], [18], [16], [31], [34], [35] and the references therein. On the other hand, the boundary peaked solutions are related to the boundary mean curvature function. This aspect is discussed in the papers [3], [9], [19], [22], [32], [36], [37], and the references therein. A good review of the subject is to be found in [24].

All the above results are concerned with positive solutions. Concerning the existence and asymptotic behavior of nodal solutions, the first result was due to Noussair and Wei [27]. In [27], Noussair and Wei extended Ni-Takagi’s result to nodal solutions. They showed that for \( \varepsilon \) sufficiently small, (1.1) has a least energy nodal solution, which has two peaks—a positive maximum and a negative minimum. Furthermore, these two peaks must approach the global maximum points of the mean curvature. There are very few results on nodal solutions. In [28], the corresponding result for the Dirichlet problem is established. Wei and Winter [38] proved the odd symmetry of nodal solutions when the domain is a ball.

In this thesis, we prove a converse result of [27] and study the existence of multiple nodal solutions at a local maximum point of \( H(P) \). In particular, we show that for local maximum point of the mean curvature function \( H(P) \), there exist nodal solutions with one positive local maximum point and one negative local minimum point. This implies that if \( H(P) \) has \( K \) local maximum points, then there are \( 2^K - 1 \) nodal solutions.

Before we introduce our main results, we need some notations.

Associated with (1.1) is the energy functional \( J_{\varepsilon} : W^{1,2}(\Omega) \to \mathbb{R} \) defined by

\[
J_{\varepsilon}(v) = \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} v^2 - \frac{1}{p + 1} \int_{\Omega} |v|^{p+1},
\]

where \( |v| = v_+ + v_- \), where \( v_+ = \max(0, v) \), \( v_- = \max(0, -v) \) and \( v = v_+ - v_- \).

We next introduce the following function. Let \( w \) be the unique solution of

\[
\begin{align*}
\Delta w - w + w^p &= 0 \text{ in } \mathbb{R}^N, \\
w &> 0, \quad w(0) = \max_{z \in \mathbb{R}^N} w(z), \\
w(z) &\to 0, \text{ as } |z| \to \infty.
\end{align*}
\]
We denote the energy of $w$ as
\[ I[w] = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{1}{2} \int_{\mathbb{R}^N} w^2 - \frac{1}{p+1} \int_{\mathbb{R}^N} w^{p+1}. \]

Now we state the main result in this thesis.

**Theorem 1.1** Let $\Gamma \subset \partial \Omega$ be such that
\[ \max_{P \in \partial \Omega} H(P) < \max_{P \in \Gamma} H(P). \]
Then for $\varepsilon$ sufficiently small problem (1.1) has a solution $u_\varepsilon$ which possesses exactly one local maximum point $P^\varepsilon \in \Gamma$ and exactly one local minimum point $Q^\varepsilon \in \Gamma$. Moreover
\[ H(P^\varepsilon) \to \max_{P \in \Gamma} H(P), \quad H(Q^\varepsilon) \to \max_{P \in \Gamma} H(P), \quad w\left(\frac{|P^\varepsilon - Q^\varepsilon|}{\varepsilon}\right) \to 0 \]
as $\varepsilon \to 0$.

Theorem 1.1 can be derived from a more general theorem as follows.

**Theorem 1.2** Let $\Gamma_i, i = 1, \ldots, K$ be $K$ disjoint open sets in $\partial \Omega$ such that
\[ \max_{P \in \partial \Omega_i} H(P) < \max_{P \in \Gamma_i} H(P), i = 1, \ldots, K. \]
Then for $\varepsilon$ sufficiently small problem (1.1) has a solution $u_\varepsilon$ which possesses exactly $K$ local maximum points $P^\varepsilon_1, \ldots, P^\varepsilon_K$ with $P^\varepsilon = (P^\varepsilon_1, \ldots, P^\varepsilon_K) \in \Gamma_1 \times \cdots \times \Gamma_K$ and exactly $K$ local minimum points $Q^\varepsilon_1, \ldots, Q^\varepsilon_K$ with $Q^\varepsilon = (Q^\varepsilon_1, \ldots, Q^\varepsilon_K) \in \Gamma_1 \times \cdots \times \Gamma_K$. Moreover
\[ H(P^\varepsilon_i) \to \max_{P \in \Gamma_i} H(P), \quad H(Q^\varepsilon_i) \to \max_{P \in \Gamma_i} H(P), \quad w\left(\frac{|P^\varepsilon_i - Q^\varepsilon_i|}{\varepsilon}\right) \to 0, \]
i = 1, \ldots, K, as $\varepsilon \to 0$.

As a Corollary of Theorem 1.2, we have

**Corollary 1.3** Suppose the mean curvature function $H(P)$ has $K$ local maximum points. Then problem (1.1) has at least $2^K - 1$ nodal solutions.
In this thesis, we shall only prove Theorem 1.2. Theorem 1.1 is a special case of Theorem 1.2.

Theorem 1.2 can also be generalized to the following singularly perturbed Neumann problem with more general nonlinearities

\[
\begin{aligned}
\left\{ \begin{array}{l}
\varepsilon^2 \Delta u - u + f(u) = 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega,
\end{array} \right.
\end{aligned}
\]

(1.3)

where \( f(u) = f_1(u_+) - f_2(u_-) \) and both \( f_1 \) and \( f_2 \) satisfy the following conditions:

(f1) \( f_1, f_2 \in C^{1+\sigma}(\mathbb{R}) \cap C^2_{\text{loc}}(0, +\infty) \) with \( 0 < \sigma \leq 1 \), \( f_1(0) = f_2(0) = 0 \), \( f'_1(0) = f'_2(0) = 0 \) and \( f_1(t) = f_2(t) \) for \( t \leq 0 \).

(f2) For \( i = 1, 2 \), the problem in the whole space

\[
\begin{aligned}
\left\{ \begin{array}{l}
\Delta w_i - w_i + f_i(w_i) = 0, w_i > 0 \text{ in } \mathbb{R}^N, \\
w_i(0) = \max_{y \in \mathbb{R}^N} w_i(y), \lim_{|y| \to +\infty} w(y) = 0,
\end{array} \right.
\end{aligned}
\]

(1.4)

has a unique solution \( w_i \) which is nondegenerate, i.e.

\[
\text{Kernel} (\Delta - 1 + f'_i(w_i)) = \text{span} \left\{ \frac{\partial w_i}{\partial y_1}, \ldots, \frac{\partial w_i}{\partial y_N} \right\}.
\]

(1.5)

Note that \( f_1(u) = f_2(u) = u^p \) is a special example. Note also that we can allow different nonlinearity for positive and negative parts of \( f \). Since the proofs for more general \( f \) are similar to that of Theorem 1.2, we shall concentrate on the special case \( f_1(u) = f_2(u) = u^p \) throughout this thesis.

Let us now sketch the proof of Theorem 1.2. We first introduce a general framework. This framework is a combination of the Liapunov-Schmidt reduction method and the variational principle. The Liapunov-Schmidt reduction method has been introduced and used in a lot of papers. See [1], [2], [3], [4], [5], [13], [17], [18], [20], [29], [30], [36], [37] and the references therein. A combination of the Liapunov-Schmidt reduction method and the variational principle was used in [3], [6], [7], [8], [17] and [18]. We shall follow the procedure in [17] which consists of the following steps:

Step 1. Choose good approximate functions.
We need to project the ground state solution $w$ to $H^1(\Omega)$ with homogeneous Neumann boundary condition.

For any smooth bounded domain $U \subset \mathbb{R}^N$ such that $0 \in U$, we set $P_U w$ to be the unique solution of
\[
\begin{cases}
\Delta u - u + f(u) = 0 \text{ in } U, \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U.
\end{cases}
\]
Let $Q \in \partial \Omega$. Without loss of generality, we may assume that $0 \in \partial \Omega$. We set
\[
\Omega_\varepsilon = \{ z | z \in \Omega \}, \Omega_{\varepsilon,Q} = \{ y | y + Q \in \Omega \}.
\]
Let $\Gamma_i, i = 1, \ldots, K$ be as in Theorem 1.2 and
\[
\eta := \min \left( \frac{1}{10}, \frac{1}{10} \min_{i \neq j}(\Gamma_i, \Gamma_j) \right).
\]
We then define a cut-off function $\chi(z) : \mathbb{R}^N \to \mathbb{R}$ such that
\[
\chi(z) = \begin{cases} 
1 & \text{if } d(x, \cup_{i=1}^{K} \Gamma_i) < \frac{\eta}{2}, \\
0 & \text{if } d(x, \cup_{i=1}^{K} \Gamma_i) \geq \eta.
\end{cases}
\]
Put
\[
\Lambda := \left\{ (P, Q) = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) : P_j, Q_j \in \Gamma_j, w \left( \frac{|P_j - Q_j|}{\varepsilon} \right) < \eta \varepsilon, j = 1, \ldots, K \right\}.
\]
Fix $(P, Q) = (P_1, P_2, \ldots, P_K, Q_1, \ldots, Q_K) \in \Lambda$. We set
\[
w_{P_i}(z) = w \left( z - \frac{P_i}{\varepsilon} \right), w_{Q_i}(z) = w \left( z - \frac{Q_i}{\varepsilon} \right), z \in \Omega_\varepsilon,
\]
\[
w_{\varepsilon,i}(z) = P_{i\Omega} w \left( z - \frac{P_i}{\varepsilon} \right) \chi(\varepsilon z), w_{\varepsilon,Q_i}(z) = P_{i\Omega} w \left( z - \frac{Q_i}{\varepsilon} \right) \chi(\varepsilon z),
\]
Our approximate function is
\[
w_{\varepsilon,P,Q}(z) = \sum_{j=1}^{K} \left[ w_{\varepsilon,P_j}(z) - w_{\varepsilon,Q_j}(z) \right].
\]

**Step 2.** Finite-dimensional reduction.
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We now describe the so-called Liapunov-Schmidt finite dimension reduction procedure. Most of the material is from Sections 3, 4 and 5 in [17]. See also Sections 4, 5 and 6 in [18].

We first introduce some notations.

We observe that solving (1.1) is equivalent to finding a zero of the following nonlinear equation:

\[ S_{\varepsilon}[u] := \Delta u - u + f(u) = 0, \quad u \in H^2(\Omega_{\varepsilon}), \quad (1.10) \]

where \( f(u) = |u|^{p-1}u \) and

\[ H^2(\Omega_{\varepsilon}) = \left\{ u \in H^2(\Omega_{\varepsilon}), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega_{\varepsilon} \right\}. \quad (1.11) \]

We also define the energy functional:

\[ J_{\varepsilon}[u] = \frac{1}{2} \int_{\Omega_{\varepsilon}} |\nabla u|^2 + \frac{1}{2} \int_{\Omega_{\varepsilon}} u^2 - \int_{\Omega_{\varepsilon}} F(u), \quad (1.12) \]

where \( F(u) = \int_0^u f(s) ds = \frac{1}{p+1}|u|^p \).

For any \( u, v \in H^1(\Omega) \), we define the inner product and the norm as follows:

\[ \langle u, v \rangle_{\varepsilon} = \int_{\Omega_{\varepsilon}} (\nabla u \cdot \nabla v + u \cdot v), \quad \|u\|_{\varepsilon} = \langle u, u \rangle^{\frac{1}{2}}. \]

Now we define the approximate kernel and cokernel respectively as follows:

\[ \mathcal{K}_{\varepsilon,P,Q} = \text{span} \left\{ \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_{P_i,j}}, \frac{\partial w_{\varepsilon,Q_i}}{\partial \tau_{Q_i,j}} \right\}_{i=1, \ldots, K, j=1, \ldots, N-1} \subset H^2(\Omega_{\varepsilon}), \]

\[ \mathcal{C}_{\varepsilon,P,Q} = \text{span} \left\{ \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_{P_i,j}}, \frac{\partial w_{\varepsilon,Q_i}}{\partial \tau_{Q_i,j}} \right\}_{i=1, \ldots, K, j=1, \ldots, N-1} \subset L^2(\Omega_{\varepsilon}), \]

where and \( \tau_{P_i,j}, \tau_{Q_i,j} \) are the \((N-1)\) tangential derivatives at \( P_i \) and \( Q_i \) respectively (without loss of generality we assume that the inward normal derivative at \( P_i \) or \( Q_i \) is \( x_N \)-direction and denote \( \tau_{P_i,j}, \tau_{Q_i,j} \) as \( \tau_{P_i,j}, \tau_{Q_i,j} \) respectively in the rest of the paper.)

We construct solutions to (1.1) with the following form:

\[ u = w_{\varepsilon,P,Q} + \Phi_{\varepsilon,P,Q} \in H^2_{\varepsilon}(\Omega_{\varepsilon}). \]
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We first solve (1.10) up to $C_{\varepsilon,P,Q}$:

$$S_\varepsilon[w_\varepsilon,P,Q + \phi_\varepsilon,P,Q] \in C_{\varepsilon,P,Q}, \quad \Phi_\varepsilon,P,Q \in \mathcal{K}_{\varepsilon,P,Q}.$$  \hfill (1.13)

We will show that (1.13) has a unique solution $\Phi_\varepsilon,P,Q$. Moreover, $\Phi_\varepsilon,P,Q$ depends on $P, Q$ smoothly.

Step 3. Solve the finite dimensional problem.

Then we define a reduced energy functional

$$M_\varepsilon(P,Q) = J_\varepsilon[w_\varepsilon,P,Q + \phi_\varepsilon,P,Q] : \bar{\Lambda} \to \mathbb{R}.$$  \hfill (1.14)

It is then shown that a critical point of $M_\varepsilon$ in $\bar{\Lambda}$ gives a critical point of $J_\varepsilon$, and hence a solution to (1.1).

Finally, we minimize the reduced energy functional in $\bar{\Lambda}$ and show that a minimizer exists and must stay in the interior of $\Lambda$.

The thesis is organized as follows. Notation, preliminaries and some useful estimates are explained in Chapter 2. Chapter 3 contains the setup of our problem and we solve (1.1) up to approximate kernel and cokernel, respectively. We set up and solve a minimizing problem in Chapter 4. Finally, in Chapter 5, we show that the solution to the minimizing problem is indeed a solution of (1.1) and satisfies all the properties of Theorem 1.2.

Throughout this thesis, unless otherwise stated, the letter $C$ will always denote various generic constants which are independent of $\varepsilon$, for $\varepsilon$ sufficiently small. The notations $O(a), o(a)$ always mean that $|O(a)| \leq C|a|$, $o(a)/a \to 0$ as $a \to 0$, respectively.
Chapter 2

Preliminary analysis

In this chapter we introduce a projection and derive some useful estimates.

Let \( w \) be the ground state solution of (1.2).

By the well-known result of Gidas, Ni and Nirenberg [15], \( w \) is radially symmetric: \( w(y) = w(|y|) \) and strictly decreasing: \( w'(r) < 0 \) for \( r > 0, r = |y| \). Moreover, we have the following asymptotic behavior of \( w \):

\[
\begin{align*}
  w(r) &= A_N r^{-\frac{N-1}{2}} e^{-r} \left(1 + O \left(\frac{1}{r}\right)\right), \\
  w'(r) &= -A_N r^{-\frac{N-1}{2}} e^{-r} \left(1 + O \left(\frac{1}{r}\right)\right),
\end{align*}
\]

(2.1)

for \( r \) large, where \( A_N > 0 \) is a generic constant. The uniqueness of \( w \) is proved in [21]. From the uniqueness, we derive that

Lemma 2.1

\[
\text{Kernel}(\Delta - 1 + f'(w)) \cap H^2_\nu(R^N_+) = \text{span} \left\{ \frac{\partial w}{\partial y_1}, \ldots, \frac{\partial w}{\partial y_{N-1}} \right\}.
\]

(2.2)

where \( R^N_+ = \{(y', y_N) | y_N > 0\} \).

Proof: See Appendix C of [26]. \( \square \)

Next we introduce a boundary deformation.

Let \( P \in \partial \Omega \). We can define a diffeomorphism straightening the boundary in a neighborhood of \( P \). After rotation of the coordinate system we may assume that the
inward normal to \( \partial \Omega \) at \( P \) is pointing in the direction of the positive \( x_N \)-axis. Denote \( x' = (x_1, \ldots, x_{N-1}) \), \( B'(R_0) = \{ x' \in \mathbb{R}^{N-1} \mid |x'| < R_0 \} \), \( B(P, R_0) = \{ x \in \mathbb{R}^N \mid |x - P| < R_0 \} \), and \( \Omega_0 = \Omega \cap B(P, R_0) = \{ (x', x_N) \in B(P, R_0) \mid x_N - P_N > \rho(x' - P') \} \). Then, since \( \partial \Omega \) is smooth, we can find a constant \( R_0 > 0 \) such that \( \partial \Omega \cap \overline{\Omega_0} \) can be represented by the graph of a smooth function \( \rho_P : B'(R_0) \to \mathbb{R} \) where \( \rho_P(0) = 0, \nabla \rho_P(0) = 0 \).

From now on we omit the use of \( P \) in \( \rho_P \) and write \( \rho \) instead if this can be done without causing confusion. The mean curvatures of \( \partial \Omega \) at \( P \) is

\[
H(P) = \frac{1}{n-1} \sum_{i=1}^{N-1} \rho_i(0),
\]

where

\[
\rho_i = \frac{\partial \rho}{\partial x_i}, \quad i = 1, \ldots, N - 1
\]

and higher derivatives are defined in the same way. By Taylor expansion we have

\[
\rho(x' - P') = \frac{1}{2} \sum_{i,j=1}^{N-1} \rho_{ij}(0)(x_i - P_i)(x_j - P_j)
\]

\[ + \frac{1}{6} \sum_{i,j,k=1}^{N-1} \rho_{ijk}(0)(x_i - P_i)(x_j - P_j)(x_k - P_k) + O(|x' - P'|^4). \]

Recall that \( P_{\Omega, \rho} w \left( \frac{x - P}{\varepsilon} \right) \) is the unique solution of

\[
\begin{align*}
\varepsilon^2 \Delta u - u + w^p \left( \frac{x - P}{\varepsilon} \right) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

By the Maximum Principle, \( P_{\Omega, \rho} w \left( \frac{x - P}{\varepsilon} \right) > 0 \).

Let \( h_{\varepsilon, \rho}(x) = w \left( \frac{x - P}{\varepsilon} \right) - P_{\Omega, \rho} w \left( \frac{x - P}{\varepsilon} \right), x \in \Omega \).

Then \( h_{\varepsilon, \rho} \) satisfies

\[
\begin{align*}
\varepsilon^2 \Delta v - v &= 0 \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= \frac{\partial}{\partial \nu} w \left( \frac{x - P}{\varepsilon} \right) \quad \text{on } \partial \Omega.
\end{align*}
\]

We denote

\[
||v||^2_2 = \varepsilon^{-N} \int_{\Omega} [\varepsilon^2 |\nabla v|^2 + v^2].
\]
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For $x \in \Omega_0$ set now

\[
\begin{cases}
\varepsilon y' = x' - P', \\
\varepsilon y_N = x_N - P_N - \rho(x' - P').
\end{cases}
\]

Furthermore, for $x \in \Omega_0$ we introduce the transformation $T$ by

\[
\begin{cases}
T_i(x') = x_i, & i = 1, \ldots, N - 1, \\
T_N(x') = x_N - P_N - \rho(x' - P').
\end{cases}
\]

Note that then

\[y = \frac{1}{\varepsilon} T(x).\]

We can also define a similar transformation, $\overline{T}$, for $Q$.

Let $v_1$ be the unique solution of

\[
\begin{cases}
\Delta v - v = 0 & \text{in } \mathbb{R}^N_+, \\
\frac{\partial v}{\partial y_N} = -\frac{w'}{|y|} \sum_{i,j=1}^{N-1} \delta_{ij} (0) y_i y_j & \text{on } \partial \mathbb{R}^N_+,
\end{cases}
\]

where $w'$ is the radial derivative of $w$, i.e. $w' = w_r(r)$, and $r = |x - P'|/\varepsilon$.

Note that $v_1$ is an even functions in $y' = (y_1, \ldots, y_{N-1})$. Moreover, it is easy to see that $|v_1| \leq Ce^{-\mu |y|}$ for some $0 < \mu < 1$.

Let $\chi(x)$ be a smooth cutoff function such that $\chi(x) = 1$, $x \in B(0, R_0)$ and $\chi(x) = 0$ for $x \in B(0, R_0)^c$.

Set

\[h_{\varepsilon, P}(x) = \varepsilon v_1(y) \chi(x - P') + \varepsilon^2 \Psi_{\varepsilon, P}(x), x \in \Omega.\]

Then we have

Proposition 2.2

\[\|\Psi_{\varepsilon, P}\|_\varepsilon \leq C.\]

Proof. Proposition 2.2 was proved in [36] by Taylor expansion and a rigorous estimate for the remainder using estimates for elliptic partial differential equations. □

Our next lemma estimates the interactions:
Lemma 2.3 As $\frac{|P-Q|}{\varepsilon} \to +\infty$, we have
\[
\left| \frac{|P-Q|}{\varepsilon} \right| + \left| \frac{y+P-Q}{\varepsilon} \right| = \left\langle y, -\frac{P-Q}{|P-Q|} \right\rangle.
\] (2.7)

As a consequence, we have
\[
\left| w(y + \frac{P-Q}{\varepsilon} \right)+1 (\frac{P-Q}{\varepsilon}) \right| \leq Ce^{|y|}.
\] (2.8)

Proof: Let $\alpha = \frac{P-Q}{\varepsilon}$. By (2.1) and Taylor expansion, as $\varepsilon \to 0$, we have,
\[
|y + \alpha| = |\alpha| + \sum_{i=1}^{N} \frac{\partial |y+\alpha|}{\partial y_i} \bigg|_{y=\alpha} y_i + O(\varepsilon |y|^2)
\]
\[
= |\alpha| + \sum_{i=1}^{N} \frac{y_i + \alpha_i}{|y+\alpha|} \bigg|_{y=\alpha} y_i + O(\varepsilon |y|^2)
\]
\[
= |\alpha| + \sum_{i=1}^{N} \frac{\alpha_i}{|\alpha|} y_i + O(\varepsilon |y|^2)
\]
\[
= |\alpha| + \left\langle y, \frac{\alpha}{|\alpha|} \right\rangle + O(\varepsilon |y|^2),
\]
which proves (2.7). Inequality (2.8) follows from the decay estimate of $w$ at (2.1) and the estimate (2.7). □

Lemma 2.4 Let $P, Q \in \partial \Omega$ be such that $\frac{|P-Q|}{\varepsilon} \to +\infty$. Let $\alpha > \beta > 0$. Then we have
\[
\int_{\Omega} w^{\alpha} \left( \frac{x-P}{\varepsilon} \right) w^{\beta} \left( \frac{x-Q}{\varepsilon} \right) = O \left( \varepsilon^{N} w^{\beta} \left( \frac{P-Q}{\varepsilon} \right) \right)
\] (2.9)

and
\[
\int_{\Omega} w^{p} \left( \frac{x-P}{\varepsilon} \right) w \left( \frac{x-Q}{\varepsilon} \right) = \varepsilon^{N} (\gamma_2 + o(1)) \left( \frac{P-Q}{\varepsilon} \right)
\] (2.10)

where $\gamma_2 = \int_{\mathbb{R}^N^{+}} w^{p}(y)e^{\left\langle y, -\frac{P-Q}{|P-Q|} \right\rangle}$. 

Proof: We first prove (2.10), using Lemma 2.3: Here we let \( y = \frac{x - P}{\varepsilon} \)

\[
\int_\Omega w^p \left( \frac{x - P}{\varepsilon} \right) w \left( \frac{x - Q}{\varepsilon} \right) = \varepsilon^N \int_{\Omega_{\varepsilon,P}} w^p(y) w \left( y + \frac{P - Q}{\varepsilon} \right)
\]

\[
= \varepsilon^N \int_{\mathbb{R}^N_{+}} w^p(y) w \left( y + \frac{P - Q}{\varepsilon} \right) (1 + o(1))
\]

\[
= \varepsilon^N w \left( \frac{P - Q}{\varepsilon} \right) (1 + o(1)) \int_{\mathbb{R}^N_{+}} w^p(y) w \left( y + \frac{P - Q}{\varepsilon} \right) w^{-1} \left( \frac{P - Q}{\varepsilon} \right)
\]

\[
= \varepsilon^N w \left( \frac{P - Q}{\varepsilon} \right) (1 + o(1)) \int_{\mathbb{R}^N_{+}} w^p(y) e^{\langle y, -\frac{P - Q}{\varepsilon} \rangle}
\]

\[
= \varepsilon^N (\gamma_2 + o(1)) w \left( \frac{P - Q}{\varepsilon} \right)
\]

by Lebesgue's Dominated Convergence Theorem.

Thus 2.10 is proven.

We now prove (2.9), similarly, let \( y = \frac{x - P}{\varepsilon} \). Since \( \int_{\mathbb{R}^N_{+}} w^\alpha(y) e^{-\beta |y|} < \infty \), we have

\[
\varepsilon^{-N} w^{-\beta} \left( \frac{P - Q}{\varepsilon} \right) \int_\Omega w^\alpha \left( \frac{x - P}{\varepsilon} \right) w^\beta \left( \frac{x - Q}{\varepsilon} \right) = \int_{\Omega_{\varepsilon,P}} w^\alpha(y) w^\beta \left( y + \frac{P - Q}{\varepsilon} \right) w^{-\beta} \left( \frac{P - Q}{\varepsilon} \right)
\]

\[
= \int_{\mathbb{R}^N_{+}} w^\alpha(y) w^\beta \left( y + \frac{P - Q}{\varepsilon} \right) w^{-\beta} \left( \frac{P - Q}{\varepsilon} \right) (1 + o(1))
\]

\[
= \int_{\mathbb{R}^N_{+}} w^\alpha(y) e^{-\beta |y|} (1 + o(1))
\]

\[
= C < \infty
\]

by Lebesgue's Dominated Convergence Theorem.

Thus,

\[
\int_\Omega w^\alpha \left( \frac{x - P}{\varepsilon} \right) w^\beta \left( \frac{x - Q}{\varepsilon} \right) = O \left( \varepsilon^N w^\beta \left( \frac{P - Q}{\varepsilon} \right) \right)
\]

(2.9) is proven. □
Lemma 2.5 For any \((P, Q) \in \overline{\Lambda}\) and \(\varepsilon\) sufficiently small

\[
e^{-N} J_\varepsilon \left[ \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right] = \frac{K}{2} I[w] - \varepsilon \gamma \sum_{i=1}^{K} (H(P_i) + H(Q_i)) +
\]

\[
(\gamma_2 + o(1)) \sum_{i=1}^{K} w \left( \frac{|P_i - Q_i|}{\varepsilon} \right) + o(\varepsilon),
\]

where \(\gamma = \frac{1}{N + 1} \int_{\mathbb{R}^{N-1}} |
\nabla w|^2 |y'|^2 dy',\ y' \in \mathbb{R}^{N-1},\ \gamma_2\) is defined in Lemma 2.4.

Proof:

Since \(\Gamma_i \cap \Gamma_j = \emptyset\), it is easy to see that

\[
J_\varepsilon \left[ \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right] = \sum_{i=1}^{K} J_\varepsilon [w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}] + O(e^{-\eta \varepsilon}). \tag{2.11}
\]

So it is enough to prove the case when \(K = 1\). We may omit the index \(i\).

It is well-known that (see [19], [36])

\[
J_\varepsilon [w_{\varepsilon,P}] = \frac{1}{2} I[w] - \gamma \varepsilon H(P) + o(\varepsilon),\ J_\varepsilon [-w_{\varepsilon,Q}] = \frac{1}{2} I[w] - \gamma \varepsilon H(Q) + o(\varepsilon). \tag{2.12}
\]

It remains to consider the interaction terms. First, we consider the two-bump case:

\[
\int_{\Omega} \left[ w_{\varepsilon,P} - w_{\varepsilon,Q} \right]^{p+1} - w_{\varepsilon,P}^{p+1} \cdot w_{\varepsilon,Q}^{p+1} \right].
\]

We divide the domain into three parts:

\[
\Omega_1 = \left\{ |x - P| \leq \frac{1}{2} |P - Q| \right\},
\]

\[
\Omega_2 = \left\{ |x - Q| \leq \frac{1}{2} |P - Q| \right\},
\]

\[
\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2).
\]
On $\Omega_3$, we have
\[
\varepsilon^{-N} \int_{\Omega_3} \left[ |w_{\varepsilon,P} - w_{\varepsilon,Q}|^{p+1} - w_{\varepsilon,P}^{p+1} - w_{\varepsilon,Q}^{p+1} \right] 
\leq C \varepsilon^{-N} \int_{\Omega_3} \left[ w_{\varepsilon,P}^{p+1} + w_{\varepsilon,Q}^{p+1} \right] 
\leq C \int_{\Omega_3} \left( e^{-|x-Q|/(\varepsilon+1)} + e^{-|x-Q|/(\varepsilon+1)} \right) 
\leq C \int_{\Omega_3} \left( e^{-|Q-P|/(\varepsilon+1)} + e^{-|P-Q|/(\varepsilon+1)} \right) 
\leq C e^{-|P-Q|/(\varepsilon+1)} 
\leq C e^{-|P-Q|/\varepsilon} 
= o \left( w \left( \frac{P-Q}{\varepsilon} \right) \right) 
\]
since $p > 1$.

On $\Omega_1$, we have $w_{\varepsilon,Q} \leq w_{\varepsilon,P}$, by Lemma 2.4 and so
\[
\varepsilon^{-N} \int_{\Omega_1} \left[ |w_{\varepsilon,P} - w_{\varepsilon,Q}|^{p+1} - w_{\varepsilon,P}^{p+1} - w_{\varepsilon,Q}^{p+1} \right] 
= \varepsilon^{-N} \int_{\Omega_1} \left[ (w_{\varepsilon,P} - w_{\varepsilon,Q})^{p+1} - w_{\varepsilon,P}^{p+1} \right] + O \left( \int_{\Omega_1} |w_{\varepsilon,Q}^{p+1}| \right) 
= -(p+1)\varepsilon^{-N} \int_{\Omega_1} w_{\varepsilon,P}^{p+1} w_{\varepsilon,Q} + \varepsilon^{-N} \int_{\Omega_1} O \left( |w_{\varepsilon,P}^{p+1}| |w_{\varepsilon,Q}|^{1+\sigma} \right) 
= -(p+1)\gamma_2 + o(1) w \left( \frac{P-Q}{\varepsilon} \right) + o \left( w \left( \frac{P-Q}{\varepsilon} \right) \right), 
\]
where we choose $\sigma$ such that $p - \sigma > 1$.

Similarly, on $\Omega_2$, we have
\[
\varepsilon^{-N} \int_{\Omega_2} \left[ |w_{\varepsilon,P} - w_{\varepsilon,Q}|^{p+1} - w_{\varepsilon,P}^{p+1} - w_{\varepsilon,Q}^{p+1} \right] 
= -(p+1)\gamma_2 + o(1) w \left( \frac{P-Q}{\varepsilon} \right) + o \left( w \left( \frac{P-Q}{\varepsilon} \right) \right). 
\]
Combing the estimates on $\Omega_1, \Omega_2$ and $\Omega_3$ together, we have

$$\varepsilon^{-N} \int_{\Omega} \left[ |w_{e,P} - w_{e,Q}|^{p+1} - w_{e,P}^{p+1} - w_{e,Q}^{p+1} \right]$$

$$= \left( (p+1)\gamma_2 + o(1) \right) w \left( \frac{P - Q}{\varepsilon} \right) + o \left( w \left( \frac{P - Q}{\varepsilon} \right) \right). \tag{2.13}$$

Now we can compute:

$$J_\varepsilon[w_{e,P} - w_{e,Q}]$$

$$= J_\varepsilon[w_{e,P}] + J_\varepsilon[w_{e,Q}] - \int_{\Omega} \nabla w_{e,P} \nabla w_{e,Q} + w_{e,P} w_{e,Q}$$

$$- \frac{1}{p+1} \int_{\Omega} \left[ |w_{e,P} - w_{e,Q}|^{p+1} - w_{e,P}^{p+1} - w_{e,Q}^{p+1} \right]$$

$$= J_\varepsilon[w_{e,P}] + J_\varepsilon[w_{e,Q}] - \int_{\Omega} \nabla w \left( \frac{x - P}{\varepsilon} \right) \nabla w \left( \frac{x - Q}{\varepsilon} \right) + w \left( \frac{x - P}{\varepsilon} \right) w \left( \frac{x - Q}{\varepsilon} \right)$$

$$- \frac{1}{p+1} \int_{\Omega} \left[ |w_{e,P} - w_{e,Q}|^{p+1} - w_{e,P}^{p+1} - w_{e,Q}^{p+1} \right] + O \left( \varepsilon^{-\frac{2}{p+1}} \right)$$

Using (2.13), we obtain

$$J_\varepsilon[w_{e,P} - w_{e,Q}]$$

$$= \varepsilon^{-N} \left[ I[w] - \gamma \varepsilon \left( H(P) + H(Q) \right) + (\gamma_2 + o(1)) w \left( \frac{P - Q}{\varepsilon} \right) + o(\varepsilon) \right]$$

which proves the lemma when $K = 1$. 
Chapter 3

Liapunov-Schmidt Reduction

In this chapter, we reduce problem (1.1) to finite dimensions by the Liapunov-Schmidt method. We first introduce some notations.

Let \( H^2_\varepsilon(\Omega_\varepsilon) \) be the Hilbert space defined by

\[
H^2_\varepsilon(\Omega_\varepsilon) = \left\{ u \in H^2(\Omega_\varepsilon) \left| \frac{\partial u}{\partial \nu_\varepsilon} = 0 \text{ on } \partial \Omega_\varepsilon \right. \right\}.
\]

Define

\[
S_\varepsilon(u) = \Delta u - u + f(u),
\]

where \( f(u) = |u|^{p-1}u \) for \( u \in H^1(\Omega_\varepsilon) \). Then solving equation (1.1) is equivalent to

\[
S_\varepsilon(u) = 0, u \in H^2_\varepsilon(\Omega_\varepsilon).
\]

Fix \( (P, Q) = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \bar{\Lambda} \). To study (1.1) we first consider the linearized operator

\[
\tilde{L}_\varepsilon : u \mapsto \Delta u - u + f' \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) u,
\]

\[
H^2_\varepsilon(\Omega_\varepsilon) \rightarrow L^2(\Omega_\varepsilon).
\]

It is easy to see (integration by parts) that the cokernel of \( \tilde{L}_\varepsilon \) coincides with its kernel. Choose approximate cokernel and kernel as

\[
C_{\varepsilon,P,Q} = K_{\varepsilon,P,Q}
\]

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Let \( \pi_{\varepsilon,P,Q} \) denote the projection from \( L^2(\Omega_\varepsilon) \) onto \( C_{\varepsilon,P,Q} \). Our goal in this chapter is to show that the equation

\[
L_{\varepsilon,P,Q} = 0
\]

has a unique solution \( \Phi_{\varepsilon,P,Q} \in K_{\varepsilon,P,Q} \) provided that \( \varepsilon \) is small enough and \( P, Q = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \overline{\Lambda} \).

As a preparation in the following two propositions we show the invertibility of the corresponding linearized operator.

**Proposition 3.1** Let \( L_{\varepsilon,P,Q} = \pi_{\varepsilon,P,Q} \circ \tilde{L}_\varepsilon \). There exist positive constants \( \varepsilon, \overline{\Lambda} \) such that for all \( \varepsilon \in (0, \varepsilon) \) and \( P, Q = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \overline{\Lambda} \)

\[
\|L_{\varepsilon,P,Q}\Phi\|_{L^2(\Omega_\varepsilon)} \geq \lambda \|\Phi\|_{H^2(\Omega_\varepsilon)}
\]

(3.1)

for all \( \Phi \in K_{\varepsilon,P,Q} \).

**Proposition 3.2** For any \( \varepsilon \in (0, \varepsilon) \) and \( P, Q = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \overline{\Lambda} \) the map

\[
L_{\varepsilon,P,Q} = \pi_{\varepsilon,P,Q} \circ \tilde{L}_\varepsilon : K_{\varepsilon,P,Q} \rightarrow C_{\varepsilon,P,Q}
\]

is surjective.

**Proof of Proposition 3.1:** We will follow the method used in [17], [18], [19], [36] and [37]. Suppose that (3.1) is false. Then there exist sequences \( \{\varepsilon_k\}, \{P_k, Q_k\} = \{(P_{1,k}, \ldots, P_{K,k}, Q_{1,k}, \ldots, Q_{K,k})\} \), and \( \{\Phi_k\} (i = 1, 2, \ldots, K, k = 1, 2, \ldots) \) with \( \varepsilon_k > 0 \),
$P_k, Q_k \in \overline{\Lambda}, \Phi_k \in \mathcal{K}_{n_k, P_k, Q_k}^\perp$ such that

$$\varepsilon_k \to 0,$$

$$P_k, Q_k \to P, Q \in \overline{\Lambda},$$

$$\|L_{n_k, P_k, Q_k} \Phi_k\|_{L^2(\Omega_{n_k})} \to 0,$$

$$\|\Phi_k\|_{H^2(\Omega_{n_k})} = 1, \quad k = 1, 2, \ldots .$$

For $j = 1, 2, \ldots, N - 1$ denote

$$e_{ij,k} = \frac{\partial}{\partial \tau(P_{i,k})} w_{e, P_{i,k}} \left/ \left\| \frac{\partial}{\partial \tau(P_{i,k})} w_{e, P_{i,k}} \right\|_{L^2(\Omega_{n_k})} \right.,$$

$$\bar{e}_{ij,k} = \frac{\partial}{\partial \tau(Q_{i,k})} w_{e, Q_{i,k}} \left/ \left\| \frac{\partial}{\partial \tau(Q_{i,k})} w_{e, Q_{i,k}} \right\|_{L^2(\Omega_{n_k})} \right.,$$

where

$$w_{e, P_{i,k}}(y) = P_{i,k} w \left( y - \frac{P_{i,k}}{\varepsilon_k} \right) \chi(\varepsilon y), \quad y \in \Omega_{n_k},$$

$$w_{e, Q_{i,k}}(y) = P_{i,k} w \left( y - \frac{Q_{i,k}}{\varepsilon_k} \right) \chi(\varepsilon y), \quad y \in \Omega_{n_k}.$$

Note that since

$$\frac{\partial w_{P_{i,k}}}{\partial \tau_{P_{i,j}}} = \frac{\partial w_{Q_{i,k}}}{\partial \tau_{Q_{i,j}}} \quad \text{and} \quad \frac{\partial w_{Q_{i,j}}}{\partial \tau_{Q_{i,j}}} = \frac{\partial w_{P_{i,j}}}{\partial \tau_{P_{i,j}}}$$

are exponentially small,

$$< e_{ij,k}, e_{ij,j,k} > = \delta_{ij} \delta_{j_1 j_2} + O(\varepsilon_k) \quad \text{as} \quad k \to \infty,$$

$$< \bar{e}_{ij,k}, \bar{e}_{ij,j,k} > = \delta_{ij} \delta_{j_1 j_2} + O(\varepsilon_k) \quad \text{as} \quad k \to \infty,$$

by the symmetry of the function $w$ and the fact that $P, Q \in \overline{\Lambda}$ (recall that $\Phi \left( \frac{|P_k - Q_k|}{\varepsilon} \right) \leq \eta \varepsilon$). Here $\delta_{ij}$ is the Kronecker symbol. Furthermore, because of (3.4),

$$\|\tilde{L}_{n_k} \Phi_k\|_{L^2}^2 - \sum_{i=1}^{K} \sum_{j=1}^{N-1} \left( \int_{\Omega_{n_k}} \tilde{L}_{n_k} \Phi_k e_{ij,k} \right)^2 - \sum_{i=1}^{K} \sum_{j=1}^{N-1} \left( \int_{\Omega_{n_k}} \tilde{L}_{n_k} \Phi_k \bar{e}_{ij,k} \right)^2 \to 0 \quad (3.6)$$

as $k \to \infty$. Let $\Omega_0, \chi, \rho$ and $T$ be as defined in chapter 2. (Note that we allow $R_0 \to 0$ but $\frac{R_0}{\varepsilon} \to \infty$). Then $T$ has an inverse $T^{-1}$ such that

$$T^{-1} : T \left( B(P, R_0) \cap \overline{\Omega} \right) \to B(P, R_0) \cap \overline{\Omega}.$$
Recall that $\varepsilon y = T(x)$. We use the notation $T^{(i)}$ if $P$ is replaced by $P_i$. We introduce new sequences \( \{\varphi_{i,k}\} \) by

$$
\varphi_{i,k}(y) = \chi \left( \frac{1}{\varepsilon_k} (T^{(i)})^{-1}(\varepsilon_k y) \right) \Phi_k \left( (T^{(i)})^{-1}(\varepsilon_k y) \right)
$$

(3.7)

for $y \in \mathbb{R}^N_+$. Since $T^{(i)}$ and $(T^{(i)})^{-1}$ have bounded derivatives it follows from (3.5) and the smoothness of $\chi$ that

$$
\|\varphi_{i,k}\|_{H^2(\mathbb{R}^N_+)} \leq C
$$

for all $k$ sufficiently large. Since also

$$
\|\varphi_{i,k}\|_{H^2(\mathbb{R}^N_+ \setminus B(0,R))} \to 0 \quad \text{as } R \to \infty
$$

uniformly in $k$ for all $k$ large enough there exists a subsequence, again denoted by \( \{\varphi_{i,k}\} \) which converges weakly in $H^2(\mathbb{R}^N_+)$ to a limit $\varphi_{i,\infty}$ as $k \to \infty$. We are now going to show that $\varphi_{i,\infty} \equiv 0$. As a first step we deduce

$$
\int_{\mathbb{R}^N_+} \varphi_{i,\infty} \frac{\partial w}{\partial y_j} = 0, \quad j = 1, \ldots, N - 1.
$$

(3.8)
This statement is shown as follows (note that $\det DT = \det DT^{-1} = 1$)

$$
\int_{R^N} \varphi_{i,k}(y) \left[ \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{(T^{(i)})^{-1}(e_k y)}{e_k} \right) \right] dy \\
= \varepsilon_k^{-N} \int_{\Omega_0} \chi(x - P_{i,k}) \Phi_k \left( \frac{x}{\varepsilon_k} \right) \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) dx \\
= \varepsilon_k^{-N} \int_{\Omega} \Phi_k \left( \frac{x}{\varepsilon_k} \right) \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \\
- \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \\
- \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} [1 - \chi(x - P_{i,k})] \Phi_k \left( \frac{x}{\varepsilon_k} \right) \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \\
= 0 - \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} \Phi_k \left( \frac{x}{\varepsilon_k} \right) \left[ \frac{\partial w}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \right] \\
- \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} [1 - \chi(x - P_{i,k})] \Phi_k \left( \frac{x}{\varepsilon_k} \right) \left[ \frac{\partial w}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \right] \\
- \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} \Phi_k \left( \frac{x}{\varepsilon_k} \right) \left[ \frac{\partial w}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right) \right] \\
- \varepsilon_k^{-N} \int_{\Omega \setminus \Omega_0} [1 - \chi(x - P_{i,k})] \Phi_k \left( \frac{x}{\varepsilon_k} \right) \frac{\partial w}{\partial \tau_{(P_i,k)}} \left( \frac{x - P_{i,k}}{e_k} \right),
$$

where $\Omega_0$ is as defined in chapter 2. In the last expression the first two terms tend to zero as $k \to \infty$ since $\varepsilon_k^{-N} \Phi_k$ is bounded in $L^2(\Omega)$ and the term in the square bracket converges to 0 strongly in $L^2(\Omega)$. The last two terms tend to zero as $k \to \infty$ because of the exponential decay of $\partial w / \partial \tau_{(P_i,k)}$ at infinity.

We conclude

$$
\limsup_{k \to \infty} \int_{R^N} \varphi_{i,k}(y) \left[ \frac{\partial w_{e,P_i,k}}{\partial \tau_{(P_i,k)}} \left( \frac{(T^{(i)})^{-1}(e_k y)}{e_k} \right) \right] = 0
$$

$$
i = 1, \ldots, K, \quad j = 1, \ldots, N - 1.
$$

(3.9)

Similarly, we have

$$
\limsup_{k \to \infty} \int_{R^N} \varphi_{i,k}(y) \left[ \frac{\partial w_{e,Q_i,k}}{\partial \tau_{(Q_i,k)}} \left( \frac{(T^{(i)})^{-1}(e_k y)}{e_k} \right) \right] = 0
$$
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\[ i = 1, \ldots, K, \quad j = 1, \ldots, N - 1. \quad (3.10) \]

These imply (3.8).

Let \( K_0 \) and \( C_0 \) be the kernel and cokernel, respectively, of the linear operator \( S_0'(w) \) which is the Fréchet derivative at \( w \) of

\[
S_0(v) = \Delta v - v + f(v),
\]

\[
S_0 : H^2_N(\mathbb{R}^N_+) \to L^2(\mathbb{R}^N_+),
\]

where

\[
H^2_N(\mathbb{R}^N_+) = \left\{ u \in H^2_N(\mathbb{R}^N_+) \mid \frac{\partial u}{\partial y_N} = 0 \right\}.
\]

Note that

\[
S_0'(w)v = \Delta v - v + f'(w)v,
\]

\[
K_0 = C_0 = \text{span} \left\{ \frac{\partial w}{\partial y_j} \mid j = 1, \ldots, N - 1 \right\}.
\]

Equation (3.8) implies that \( \varphi_{i,\infty} \in K_0^\perp \). By the exponential decay of \( w \) and by (3.4) we have after possibly taking a further subsequence that

\[
\Delta \varphi_{i,\infty} - \varphi_{i,\infty} + f'(w)\varphi_{i,\infty} = 0,
\]

i.e. \( \varphi_{i,\infty} \in K_0 \). Therefore \( \varphi_{i,\infty} = 0 \).

Hence

\[
\varphi_{i,k} \to 0 \quad \text{weakly in } H^2(\mathbb{R}^N_+) \quad \text{as } k \to \infty. \quad (3.11)
\]

By the definition of \( \varphi_{i,k} \) we get \( \Phi_k \to 0 \) in \( H^2 \) and

\[
||\Phi_k||_{L^2(\Omega_k)} \to 0 \quad \text{as } k \to \infty.
\]

Furthermore,

\[
||f'\left(\sum_{i=1}^{K}(w_{\varepsilon,P_i} - w_{\varepsilon,\Omega_i})\right)\Phi_k||_{L^2(\Omega_k)} \to 0
\]

and therefore

\[
|| (\Delta - 1) \Phi_k ||_{L^2(\Omega_k)} \to 0 \quad \text{as } k \to \infty.
\]
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Since
\[ \int_{\Omega_{ek}} |\nabla \Phi_k|^2 + \Phi_k^2 = \int_{\Omega_{ek}} [(1 - \Delta) \Phi_k]\Phi_k \leq C(\Delta - 1) \Phi_k \|L^2(\Omega_{ek})|, \]
we have that
\[ \|\Phi_k\|_{H^1(\Omega_{ek})} \to 0 \quad \text{as } k \to \infty. \]

In summary:
\[ \|\Delta \Phi_k\|_{L^2(\Omega_{ek})} \to 0 \quad \text{and} \quad \|\Phi_k\|_{H^1(\Omega_{ek})} \to 0. \] (3.12)

From (3.12) and the following elliptic regularity estimate (for a proof see Appendix B in [36])
\[ \|\Phi_k\|_{H^2(\Omega_{ek})} \leq C(\|\Delta \Phi_k\|_{L^2(\Omega_{ek})} + \|\Phi_k\|_{H^1(\Omega_{ek})}) \] (3.13)
for \( \Phi_k \in H^2_N(\Omega_{ek}) \), we deduce that
\[ \|\Phi_k\|_{H^2(\Omega_{ek})} \to 0 \quad \text{as } k \to \infty. \]

This contradicts the assumption
\[ \|\Phi_k\|_{H^2(\Omega_{ek})} = 1 \]
and the proof of Proposition 3.1 is completed. \( \square \)

Proof of Proposition 3.2:

We define a linear operator \( T \) from \( L^2(\Omega_e) \) to itself by
\[ T = \pi_{e,P,Q} \circ \tilde{L} \circ \pi_{e,P,Q}. \]

Its domain of definition is \( H^2_N(\Omega_e) \). By the theory of elliptic equations and by integration by parts it is easy to see that \( T \) is a (unbounded) self-adjoint operator on \( L^2(\Omega_e) \) and a closed operator. The \( L^2 \) estimates of elliptic equations imply that the range of \( T \) is closed in \( L^2(\Omega_e) \). Then by the Closed Range Theorem ([39], page 205), we know that the range of \( T \) is the orthogonal complement of its kernel which is, by Proposition 3.1, \( K_{e,P,Q} \). This leads to Proposition 3.2. \( \square \)
We are now in a position to solve the equation

\[ \pi_{\varepsilon,P,Q} \circ S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P,Q} \right) = 0. \] (3.14)

Since \( L_{\varepsilon,P,Q}^{-1}|_{K_{\varepsilon,P,Q}} \) is invertible (call the inverse \( L_{\varepsilon,P,Q}^{-1} \)), we can rewrite

\[ \Phi = - (L_{\varepsilon,P,Q}^{-1} \circ \pi_{\varepsilon,P,Q}) \left( S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right) + (L_{\varepsilon,P,Q}^{-1} \circ \pi_{\varepsilon,P,Q}) N_{\varepsilon,P,Q}(\Phi) \]

\[ \equiv G_{\varepsilon,P,Q}(\Phi), \] (3.15)

where

\[ N_{\varepsilon,P,Q}(\Phi) = S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi \right) - S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) + S'_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \Phi \]

and the operator \( G_{\varepsilon,P,Q} \) is defined by the last equation for \( \Phi \in H^2_N(\Omega_\varepsilon) \). We are going to show that the operator \( G_{\varepsilon,P,Q} \) is a contraction on

\[ B_{\varepsilon,\delta} = \{ \Phi \in H^2(\Omega_\varepsilon) \mid \| \Phi \|_{H^2(\Omega_\varepsilon)} < \delta \} \]

if \( \delta \) is small enough.

In fact we have the following lemma.

**Lemma 3.3** For \( \varepsilon \) sufficiently small, we have

\[ |N_{\varepsilon,P,Q}| \leq C|\Phi_{\varepsilon,P,Q}|^{1+\sigma}, \] (3.16)

\[ \left\| \frac{\partial}{\partial t} S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right\|_{L^2(\Omega_\varepsilon)} \leq C_{\varepsilon}^{1+\frac{\sigma}{2}}. \] (3.17)

**Proof:** (3.16) follows from the mean value theorem.

To prove (3.17), we divide the domain into \((2K + 1)\) parts: let

\[ \Omega = \bigcup_{i=1}^{K} \Omega_{P_i} \cup \bigcup_{i=1}^{K} \Omega_{Q_i} \cup \Omega_{C}, \]
where
\[
\Omega_{P_i} = \left\{ |x - P_i| \leq \frac{1}{2} \min_k |P_k - Q_k| \right\}, \quad i = 1, \ldots, K,
\]
\[
\Omega_{Q_i} = \left\{ |x - Q_i| \leq \frac{1}{2} \min_k |P_k - Q_k| \right\}, \quad i = 1, \ldots, K,
\]
\[
\Omega_C = \Omega \setminus \left( \bigcup_{i=1}^K \Omega_{P_i} \cup \bigcup_{i=1}^K \Omega_{Q_i} \right).
\]

Note that
\[
S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) = f \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) - \sum_{i=1}^K \left( f(w_{P_i}) - f(w_{Q_i}) \right).
\]

We now estimate \( S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \) in each domain.

In \( \Omega_C \), we have
\[
\left| S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right| \leq \left( (w_{P_1} + \cdots + w_{P_K}) - (w_{Q_1} + \cdots + w_{Q_K}) \right)^{1+\sigma}
\leq O \left( \varepsilon^{\frac{1+\sigma}{2}} \right).
\]

Hence, using also the fact that \( w(y) \) decays exponentially in \( |y| \) we obtain
\[
\left\| S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right\|_{L^2((\Omega_C)_\varepsilon)} \leq O \left( \varepsilon^{\frac{1+\sigma}{2}} \right).
\]

In \( \Omega_{P_i}, \ i = 1, \ldots, K \), we have
\[
\left| S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right| \leq \sum_{j \neq i} \left( \left| f'(w_{P_j}) w_{P_j} \right| + \left| f'(w_{P_i}) (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) - w_{P_j} \right| \right)
+ O \left( \sum_{j \neq i} \left( \left| w_{\varepsilon,P_j} - w_{\varepsilon,Q_j} \right|^{1+\sigma} + \left| w_{P_j} \right|^{1+\sigma} \right) \right)
+ O \left( \left| w_{\varepsilon,P_i} - w_{\varepsilon,Q_i} - w_{P_i} \right|^{1+\sigma} \right).
\]

Using Proposition 2.2 and the facts that \( w_{\varepsilon,P}, w_P \) and \( v_1 \) decay exponentially, we obtain
\[
\left\| S_{\varepsilon} \left( \sum_{i=1}^K (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) \right) \right\|_{L^2((\Omega_{P_i})_\varepsilon)} \leq O \left( \varepsilon^{\frac{1+\sigma}{2}} \right).
\]
Similarly, we have
\[
\left\| S_\epsilon \left( \sum_{i=1}^K (w_{\epsilon, P_i} - w_{\epsilon, Q_i}) \right) \right\|_{L^2(\Omega, \epsilon)} \leq O \left( \epsilon^{\frac{1+2}{4}} \right).
\]
Thus
\[
\| G_{\epsilon, P, Q}(\Phi) \|_{H^2(\Omega, \epsilon)}
\leq \lambda^{-1} \left( \| \pi_{\epsilon, P, Q} \circ N_{\epsilon, P, Q}(\Phi) \|_{L^2(\Omega, \epsilon)} + \| \pi_{\epsilon, P, Q} \circ \left( S_\epsilon \left( \sum_{i=1}^K (w_{\epsilon, P_i} - w_{\epsilon, Q_i}) \right) \right) \|_{L^2(\Omega, \epsilon)} \right)
\leq \lambda^{-1} C \left( c(\delta) \epsilon^{\frac{1+2}{4}} + \epsilon^{\frac{1+2}{4}} \right),
\]
where \( \lambda > 0 \) is independent of \( \delta > 0 \) and \( c(\delta) \to 0 \) as \( \delta \to 0 \). Similarly we show
\[
\| G_{\epsilon, P, Q}(\Phi) - G_{\epsilon, P, Q}(\Phi') \|_{H^2(\Omega, \epsilon)} \leq \lambda^{-1} C \| \Phi - \Phi' \|_{H^2(\Omega, \epsilon)},
\]
where \( c(\delta) \to 0 \) as \( \delta \to 0 \). Therefore \( G_{\epsilon, P, Q} \) is a contraction on \( B_\delta \). The existence of a fixed point \( \Phi_{\epsilon, P, Q} \) now follows from the Contraction Mapping Principle and \( \Phi_{\epsilon, P, Q} \) is a solution of (3.15).

Because of
\[
\| \Phi_{\epsilon, P, Q} \|_{H^2(\Omega, \epsilon)}
\leq \lambda^{-1} \left( \| N_{\epsilon, P, Q}(\Phi_{\epsilon, P, Q}) \|_{L^2(\Omega, \epsilon)} + \| S_\epsilon \left( \sum_{i=1}^K (w_{\epsilon, P_i} - w_{\epsilon, Q_i}) \right) \|_{L^2(\Omega, \epsilon)} \right)
\leq \lambda^{-1} C(\epsilon^{\frac{1+2}{4}} + c(\delta) \| \Phi_{\epsilon, P, Q} \|_{H^2(\Omega, \epsilon)}),
\]
we have
\[
\| \Phi_{\epsilon, P, Q} \|_{H^2(\Omega, \epsilon)} \leq C \epsilon^{\frac{1+2}{4}}.
\]

We have proved

**Lemma 3.4** There exists \( \epsilon > 0 \) such that for every \((2N+1)\)-tuple \( \epsilon, P_1, \ldots, P_K, Q_1, \ldots, Q_K \) with \( 0 < \epsilon < \epsilon \) and \( P, Q = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \bar{A} \) there is a unique \( \Phi_{\epsilon, P, Q} \in K_{\epsilon, P, Q} \) satisfying \( S_\epsilon \left( \sum_{i=1}^K (w_{\epsilon, P_i} - w_{\epsilon, Q_i}) + \Phi_{\epsilon, P, Q} \right) \in C_{\epsilon, P, Q} \) and
\[
\| \Phi_{\epsilon, P, Q} \|_{H^2(\Omega, \epsilon)} \leq C \epsilon^{\frac{1+2}{4}}.
\] (3.18)
The next lemma is our main estimate.

**Lemma 3.5** Let $\Phi_{\varepsilon, P, Q}$ be defined by Lemma 3.4. Then we have

$$J_{\varepsilon} \left[ \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right]$$

$$= \frac{K}{2} I[w] - \varepsilon \gamma \sum_{i=1}^{K} \left( H(P_i) + H(Q_i) \right) + \left( \gamma_2 + o(1) \right) \sum_{i=1}^{K} w \left( \frac{|P_i - Q_i|}{\varepsilon} \right) + o(\varepsilon),$$

where $\gamma$ and $\gamma_2$ are defined in Lemma 2.5.

**Proof:**

In fact for any $P, Q \in \overline{\Lambda}$, we have

$$J_{\varepsilon} \left[ \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right]$$

$$= J_{\varepsilon} \left[ \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \right] + g_{\varepsilon, P, Q}(\Phi_{\varepsilon, P, Q}) + O\left( \|\Phi_{\varepsilon, P, Q}\|_{H^2(\Omega_\varepsilon)}^2 \right),$$

where

$$g_{\varepsilon, P, Q}(\Phi_{\varepsilon, P, Q})$$

$$= \int_{\Omega_\varepsilon} \sum_{i=1}^{K} \left( \nabla (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \nabla \Phi_{\varepsilon, P, Q} + (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \Phi_{\varepsilon, P, Q} \right)$$

$$- \int_{\Omega_\varepsilon} f \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \right) \Phi_{\varepsilon, P, Q}$$

$$= \int_{\Omega_\varepsilon} \left[ \sum_{i=1}^{K} \left( f(w_{P_i}) - f(w_{Q_i}) \right) - f \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \right) \right] \Phi_{\varepsilon, P, Q}$$

$$\leq \left\| \sum_{i=1}^{K} \left( f(w_{P_i}) - f(w_{Q_i}) \right) - f \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) \right) \right\|_{L^2(\Omega_\varepsilon)} \|\Phi_{\varepsilon, P, Q}\|_{L^2(\Omega_\varepsilon)}$$

$$\leq O(\varepsilon^{1+\sigma})$$

by Lemma 3.3 and Lemma 3.4.

Estimate (3.19) now follows from Lemma 2.5 and Lemma 3.4. □

Finally, we show that $\Phi_{\varepsilon, P, Q}$ is actually smooth in $P, Q$. 
Lemma 3.6 Let $\Phi_{\varepsilon, P, Q}$ be defined by Lemma 3.4. Then $\Phi_{\varepsilon, P, Q} \in C^1$ in $P, Q$.

Proof. Recall that $\Phi_{\varepsilon, P, Q}$ is a solution of the equation

$$
\pi_{\varepsilon, P, Q} \circ S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right) = 0
$$

such that

$$
\Phi_{\varepsilon, P, Q} \in K_{\varepsilon, P, Q}^\perp.
$$

By differentiating equation (3.20) twice we easily conclude that the functions $w_{\varepsilon, P_i}$ and $\partial^2 w_{\varepsilon, P_i}/(\partial \tau_{P_i, j} \partial \tau_{P_i, k})$ are $C^1$ in $P$, $w_{\varepsilon, Q_i}$, and $\partial^2 w_{\varepsilon, Q_i}/(\partial \tau_{Q_i, j} \partial \tau_{Q_i, k})$ are $C^1$ in $Q$. This implies that the projection $\pi_{\varepsilon, P, Q}$ is $C^1$ in $P, Q$. Applying $\partial/\partial \tau_{P_i, j}$ gives

$$
\pi_{\varepsilon, P, Q} \circ DS_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right) \left( \sum_{i=1}^{K} \frac{\partial w_{\varepsilon, P_i}}{\partial \tau_{P_i, j}} + \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)
$$

$$
+ \frac{\partial \pi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \circ S_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right) = 0,
$$

where

$$
DS_\varepsilon \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right) = \Delta - 1 + f \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right).
$$

We decompose $\frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}}$ into two parts:

$$
\frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} = \left( \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)_1 + \left( \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)_2,
$$

where $\left( \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)_1 \in K_{\varepsilon, P, Q}$ and $\left( \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)_2 \in K_{\varepsilon, P, Q}^\perp$.

We can easily show that $\left( \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_i, j}} \right)_1$ is continuous in $P$ since

$$
\int_{\Omega_\varepsilon} \Phi_{\varepsilon, P, Q} \frac{\partial w_{\varepsilon, P_k}}{\partial \tau_{P_{l,i}}} = 0, \quad k = 1, \ldots, K, \quad l = 1, \ldots, N - 1
$$

and

$$
\int_{\Omega_\varepsilon} \frac{\partial \Phi_{\varepsilon, P, Q}}{\partial \tau_{P_{l,i}}} \frac{\partial w_{\varepsilon, P_k}}{\partial \tau_{P_{l,i}}} + \int_{\Omega_\varepsilon} \Phi_{\varepsilon, P, Q} \frac{\partial^2 w_{\varepsilon, P_k}}{\partial \tau_{P_{l,i}} \partial \tau_{P_{l,j}}} = 0
$$

$$
k, i = 1, \ldots, K, \quad l, j = 1, \ldots, N - 1.
$$
We can write equation (3.22) as

\[
\pi_{\varepsilon,P,Q} \circ Ds_e \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P,Q} \right) \left( \left( \frac{\partial \Phi_{\varepsilon,P,Q}}{\partial \tau_{P,i,j}} \right)_{2} \right)
\]

\[+ \pi_{\varepsilon,P,Q} \circ Ds_e \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P,Q} \right) \left( \sum_{i=1}^{K} \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_{P,i,j}} + \left( \frac{\partial \Phi_{\varepsilon,P,Q}}{\partial \tau_{P,i,j}} \right)_{1} \right)
\]

\[+ \frac{\partial \pi_{\varepsilon,P,Q}}{\partial \tau_{P,i,j}} \circ Ds_e \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P,Q} \right) = 0. \quad (3.23)
\]

As in the proof of Propositions 3.1 and 3.2, we can show that the operator

\[
\pi_{\varepsilon,P,Q} \circ Ds_e \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P,Q} \right)
\]

is invertible from \( K_{\varepsilon,P,Q} \) to \( C_{\varepsilon,P,Q} \). Then we can take inverse of \( \pi_{\varepsilon,P,Q} \circ Ds_e \left( \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P} \right) \) in the above equation and the inverse is continuous in \( P \).

Since \( \frac{\partial w_{\varepsilon,P_i}}{\partial \tau_{P,i,j}} \in K_{\varepsilon,P,Q} \) are continuous in \( P, Q \) and so is \( \frac{\partial \Phi_{\varepsilon,P,Q}}{\partial \tau_{P,i,j}} \) we conclude that \( (\partial \Phi_{\varepsilon,P,Q}/\partial \tau_{P,i,j})_{2} \) is also continuous in \( P, Q \). This is the same as the \( C^1 \) dependence of \( \Phi_{\varepsilon,P,Q} \) in \( P \).

Similarly, by applying \( \partial / \partial \tau_{Q,i,j} \), \( \Phi_{\varepsilon,P,Q} \in C^1 \) in \( Q \).

Hence \( \Phi_{\varepsilon,P,Q} \in C^1 \) in \( P, Q \).

The proof is finished. \( \square \)
Chapter 4

The reduced problem: A
Minimizing Procedure

In this chapter, we study a minimizing problem.

Fix $P, Q \in \Lambda$. Let $\Phi_{e,P,Q}$ be the solution given by Lemma 3.4. We define a new functional

$$M_{e}(P, Q) = J_{e} \left[ \sum_{i=1}^{K} (w_{e,P_{i}} - w_{e,Q_{i}}) + \Phi_{e,P,Q} \right] : \Lambda \rightarrow \mathbb{R}. \quad (4.1)$$

We shall prove

**Proposition 4.1** For $\varepsilon$ small, the following minimizing problem

$$\min \{ M_{e}(P, Q) : (P, Q) \in \Lambda \} \quad (4.2)$$

has a solution $(P^e, Q^e) \in \Lambda$.

**Proof:** Since $J_{e} \left[ \sum_{i=1}^{K} (w_{e,P_{i}} - w_{e,Q_{i}}) + \Phi_{e,P,Q} \right]$ is continuous in $P, Q$, the minimizing problem has a solution. Let $M_{e}(P^e, Q^e)$ be the minimum where $P^e, Q^e \in \Lambda$.

We claim that $P^e, Q^e \in \Lambda$.

In fact for any $P, Q \in \Lambda$, by Lemma 3.5, we have

$$M_{e}(P, Q) = \frac{K}{2} \int |w| - \varepsilon \gamma \sum_{i=1}^{K} (H(P_{i}) + H(Q_{i})) + (\gamma_{2} + o(1)) \sum_{i=1}^{K} w \left( \frac{|P_{i} - Q_{i}|}{\varepsilon} \right) + o(\varepsilon).$$
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Since $M_\varepsilon(P^\varepsilon, Q^\varepsilon)$ is the minimum, we have

$$
-\gamma \sum_{i=1}^{K} \left( H(P^\varepsilon_i) + H(Q^\varepsilon_i) \right) + \frac{1}{\varepsilon} \sum_{i=1}^{K} \left( \frac{1}{2} \gamma_2 + o(1) \right) w \left( \frac{|P^\varepsilon_i - Q^\varepsilon_i|}{\varepsilon} \right)
$$

$$
\leq -\gamma \sum_{i=1}^{K} \left( H(P_i) + H(Q_i) \right) + \frac{1}{\varepsilon} \sum_{i=1}^{K} \left( \frac{1}{2} \gamma_2 + o(1) \right) w \left( \frac{|P_i - Q_i|}{\varepsilon} \right) + o(1)
$$

for any $P, Q = (P_1, \ldots, P_K, Q_1, \ldots, Q_K) \in \Lambda$.

Choose $P_j^0, Q_j^0$ be such that

$$
H(P_j^0) = \max_{Q \in \Gamma_j} H(Q), \quad |Q_j^0 - P_j^0| = \varepsilon.
$$

(4.3)

By our assumption, $P_j^0 \notin \partial \Gamma_j$. Thus $Q_j^0, P_j^0 \in \Gamma_j$, $w \left( \frac{P_j^0 - Q_j^0}{\varepsilon} \right) = O(\varepsilon^3)$.

Let

$$(P^0, Q^0) = (P_1^0, \ldots, P_K^0, Q_1^0, \ldots, Q_K^0).$$

Then we compute

$$
M_\varepsilon (P^0, Q^0) = -2\gamma \sum_{i=1}^{K} \max_{Q \in \Gamma_i} H(Q) + o(\varepsilon).
$$

(4.4)

This implies that

$$
-\gamma \sum_{i=1}^{K} \left( H(P^\varepsilon_i) + H(Q^\varepsilon_i) \right) + \frac{1}{\varepsilon} \sum_{i=1}^{K} \left( \frac{1}{2} \gamma_2 + o(1) \right) w \left( \frac{|P^\varepsilon_i - Q^\varepsilon_i|}{\varepsilon} \right)
$$

$$
\leq -2\gamma \sum_{i=1}^{K} \max_{P \in \Gamma_i} H(P) + o(\varepsilon).
$$

(4.5)

Note that $\partial \Lambda \subset \{ P_i \in \partial \Gamma_i \text{ or } Q_i \in \partial \Gamma_i \text{ or } w \left( \frac{|P_i - Q_i|}{\varepsilon} \right) = \varepsilon \eta \}$. Hence if $(P, Q) \in \partial \Lambda$

, we have that either

$$
H(P_i) \geq \max_{P \in \Gamma} H(P) - \eta_0
$$

or

$$
H(Q_i) \geq \max_{P \in \Gamma} H(P) - \eta_0
$$

for some $i = 1, \ldots, K$ and $\eta_0 > 0$, or

$$
\frac{1}{\varepsilon} w \left( \frac{|P_i - Q_i|}{\varepsilon} \right) = \eta.
$$
Hence if \((P, Q) \in \partial \Lambda\) we have

\[
- \gamma \sum_{i=1}^{K} (H(P_i^e) + H(Q_i^f)) + \frac{1}{\varepsilon} \sum_{i=1}^{K} \left( \frac{1}{2} \gamma_2 + o(1) \right) w \left( \frac{|P_i^e - Q_i^f|}{\varepsilon} \right) 
\geq -2 \gamma \sum_{P \in \Gamma_i} \max_{P \in \Gamma_i} H(P) + \min_{\gamma \eta_0, \gamma \eta_2}.
\]

Note that

\[
\min_{\gamma \eta, \gamma \eta^2} \gamma_2 \geq \delta > 0
\]

since

\[
\int_{R^N} w^p(y) e^{\langle y, (P^e - Q^f) \rangle} = \frac{1}{2} \int_{R^N} w^p(y) e^{\langle y, (P^e - Q^f) \rangle} > 0.
\]

A contradiction to (4.5).

It follows that \((P_e^e, Q_e^f)\) must stay in the interior of \(\Lambda\).

This completes the proof of Proposition 4.1. \(\Box\)
In this chapter, we apply results in chapter 3 and chapter 4 to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2:

By Lemma 3.4 and Lemma 3.6, there exists $\varepsilon_0$ such that for $\varepsilon < \varepsilon_0$ we have a $C^1$ map which, to any $P, Q \in \Lambda$, associates $\Phi_{\varepsilon, P, Q} \in K_{\varepsilon, P, Q}$ such that

$$S_{\varepsilon} \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P, Q} \right) = \sum_{k=1}^{K} \sum_{l=1}^{N-1} \alpha_{kl} \frac{\partial w_{\varepsilon, P_k}}{\partial P_{k,l}} + \beta_{kl} \frac{\partial w_{\varepsilon, Q_k}}{\partial Q_{k,l}}$$

(5.1)

for some constants $\alpha_{kl}, \beta_{kl} \in \mathbb{R}^{K(N-1)}$.

By Proposition 4.1, we have $P^\varepsilon, Q^\varepsilon \in \Lambda$, achieving the minimum of the minimization problem in Proposition 4.1. Let $\Phi_{\varepsilon} = \Phi_{\varepsilon, P^\varepsilon, Q^\varepsilon}$ and $u_{\varepsilon} = \sum_{i=1}^{K} (w_{\varepsilon, P_i} - w_{\varepsilon, Q_i}) + \Phi_{\varepsilon, P^\varepsilon, Q^\varepsilon}$.

Then we have

$$\left. \frac{\partial}{\partial P_{i,j}} \right|_{P, Q = P^\varepsilon, Q^\varepsilon} M_{\varepsilon}(P, Q) = 0, \ i = 1, \ldots, K, \ j = 1, \ldots, N - 1.$$  

(5.2)

$$\left. \frac{\partial}{\partial Q_{i,j}} \right|_{P, Q = P^\varepsilon, Q^\varepsilon} M_{\varepsilon}(P, Q) = 0, \ i = 1, \ldots, K, \ j = 1, \ldots, N - 1.$$  

(5.3)

Hence we have

$$\int_{\Omega_{\varepsilon}} \left[ \nabla u_{\varepsilon} \nabla \left( \sum_{i=1}^{K} \left( w_{\varepsilon, P_i} + \Phi_{\varepsilon, P, Q} \right) \right) \right]_{P, Q = P^\varepsilon, Q^\varepsilon} + \left. \frac{\partial}{\partial P_{i,j}} \right|_{P, Q = P^\varepsilon, Q^\varepsilon} \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} + \Phi_{\varepsilon, P, Q}) \right)$$

$$= \left. \frac{\partial}{\partial Q_{i,j}} \right|_{P, Q = P^\varepsilon, Q^\varepsilon} \left( \sum_{i=1}^{K} (w_{\varepsilon, P_i} + \Phi_{\varepsilon, P, Q}) \right) \bigg|_{P, Q = P^\varepsilon, Q^\varepsilon}$$

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\[-|u_\varepsilon|^p u_\varepsilon \frac{\partial (\sum_{i=1}^{K} w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q})}{\partial P_{i,j}}|_{P,Q=P^*,Q^*} = 0.\]

Thus

\[
\int_{\Omega_\varepsilon} \left[ \nabla u_\varepsilon \cdot \nabla \left( w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q} \right) \right]_{P,Q=P^*,Q^*} \frac{\partial (w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q})}{\partial P_{i,j}} + u_\varepsilon \frac{\partial (w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q})}{\partial P_{i,j}} - |u_\varepsilon|^{p+1} u_\varepsilon \frac{\partial (w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q})}{\partial P_{i,j}}|_{P,Q=P^*,Q^*} = 0
\]

for \(i = 1, \ldots, K\) and \(j = 1, \ldots, N - 1\).

Therefore we have

\[
\sum_{k=1}^{K} \sum_{l=1}^{N-1} \alpha_{kl} \int_{\Omega_\varepsilon} \frac{\partial w_{\varepsilon,P_k}}{\partial P_{k,l}} \frac{\partial (w_{\varepsilon,P_i} + \Phi_{\varepsilon,P,Q})}{\partial P_{i,j}} = 0. \tag{5.4}
\]

Since \(\Phi_{\varepsilon,P,Q} \in K_{\varepsilon,P,Q}^L\), we have that

\[
\left| \int_{\Omega_\varepsilon} \frac{\partial w_{\varepsilon,P_k}}{\partial P_{k,l}} \frac{\partial \Phi_{\varepsilon,P,Q}}{\partial P_{i,j}} \right| \leq \left| \int_{\Omega_\varepsilon} \frac{\partial^2 w_{\varepsilon,P_k}}{\partial P_{k,l} \partial P_{i,j}} \Phi_{\varepsilon,P,Q} \right| \leq \|\Phi_{\varepsilon,P,Q}\|_{L^2} \frac{\partial^2 w_{\varepsilon,P_k}}{\partial P_{k,l} \partial P_{i,j}} \left( A + o(1) \right).
\]

Note that

\[
\int_{\Omega_\varepsilon} \frac{\partial w_{\varepsilon,P_k}}{\partial P_{k,l}} \frac{\partial w_{\varepsilon,P_i}}{\partial P_{i,j}} = \frac{1}{\varepsilon^2} \delta_k \delta_{ij} \left( A + o(1) \right),
\]

where

\[
A = \int_{\mathbb{R}^N} \left( \frac{\partial w}{\partial y_1} \right)^2 > 0.
\]

Thus equation (5.4) becomes a system of homogeneous equations for \(\alpha_{kl}\) and the matrix of the system is nonsingular since it is diagonally dominant. So \(\alpha_{kl} = 0, k = 1, \ldots, K, l = 1, \ldots, N - 1\).

Similarly, by applying (5.3), \(\beta_{kl} = 0, k = 1, \ldots, K, l = 1, \ldots, N - 1\).

Hence \(u_\varepsilon = \sum_{i=1}^{K} (w_{\varepsilon,P_i} - w_{\varepsilon,Q_i}) + \Phi_{\varepsilon,P_i} + \cdots + \Phi_{\varepsilon,P_K} + \cdots + \Phi_{\varepsilon,Q_K}\) is a solution of (1.1).

By the structure of \(u_\varepsilon\) we see that (up to a permutation) \(Q_1^\varepsilon - P_i^\varepsilon = o(1)\). This proves Theorem 1.2.
Corollary 1.3 follows immediately from Theorem 1.2, since if $H(P)$ has $K$ local maximum points on $\partial \Omega$, there are $K$ subsets $\Gamma_i, i = 1, ..., K$ satisfying the assumptions of Theorem 1.2. By arbitrarily combining these $K$ subsets (there are $2^K - 1$ distinct combinations), we obtain $2^K - 1$ (distinct) nodal solutions.
Bibliography


Multiple Nodal Solutions


Multiple Nodal Solutions


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