Hilbert C*-modules

by

NG Yin Fun

Thesis
Submitted to the Faculty of the Graduate School of
The Chinese University of Hong Kong
(Division of Mathematics)

In partial fulfillment of the requirements
for the Degree of
Master of Philosophy

July, 2000
ACKNOWLEDGMENTS

I am deeply indebted to my supervisor, Prof. Chi-wai Leung, not only for his immeasurable guidance and valuable advice but also for his kind encouragement and industrious supervision in the course of this research programme. I also wish to acknowledge my classmate Ms. Yin-king Law.

Moreover, I would like to express my gratitude to Prof. M. Frank for sending copies of his papers.
Abstract

Hilbert C*-module is a very interesting structure. It first appeared in the work of Kaplansky, who used it in the theory of AW*-algebra. Besides this, Hilbert C*-module is very useful in many fields, such as operator K-theory, group representation theory and etc.

Broadly speaking, Hilbert C*-module generalizes Hilbert space by allowing the inner product to take values in a C*-algebra rather than the field of complex numbers. Unfortunately, in general, it neither has Riesz representation theorem of Hilbert space, nor has the property that “every inner product inducing equivalent norm to the given one is isomorphic to it”.

In this thesis, we explore some of the works done by M. Frank and W. L. Paschke.
摘要

Hilbert C*-模是一個很有趣的結構，它最先由 Kaplansky 在研究 AW*-代數時引入。此外，Hilbert C*-模亦於多個範圍，如算子 K-理論、群表示理論等有著很大的用處。

概括來說，Hilbert C*-模為 Hilbert 空間的一般化，它容許內積在一給定的 C*-代數而不是複數域取值。不幸的是，一般來說它沒有 Hilbert 空間的 Riesz 表示定理，亦沒有“任一導出等價範數的內積都同構於給定的內積”的特性。

本論文將會展示 M.Frank 及 W.L.Paschke 的一部份工作。
# Contents

Acknowledgments:i

Abstract:ii

Introduction:3

1 Preliminaries:4

1.1 Hilbert C*-modules:4

2 Self-dual Hilbert C*-modules:14

2.1 Self-duality:14

2.2 Self-duality and some related concepts:22

2.3 A criterion of self-duality of $H_A$:23

3 Hilbert W*-modules:25

3.1 Extension of the inner product to $E^\#$:25

3.2 Extension of operators to $E^\#$:33

3.3 Self-dual Hilbert W*-modules:36
3.4 Some equivalent conditions for a Hilbert W*-module
to be self-dual ................. 43

Bibliography

50
Introduction

This thesis is a survey on Hilbert C*-modules, basing on the works of M. Frank and W. L. Paschke. It presupposes a familiarity with the elementary theory of C*-algebras which may be found in [9] and [13].

This article is divided into three chapters.

In chapter 1, we give some background knowledge of Hilbert C*-modules.

In chapter 2, we give the definition of self-dual Hilbert C*-module which is an analogue of the Riesz Representation Theorem in the theory of Hilbert space. We then study some properties and give some examples of self-dual Hilbert C*-module.

In chapter 3, we study Hilbert W*-modules. Following [10], we show that an inner product on a Hilbert W*-module E can be extended to E#, making E# into a self-dual Hilbert C*-module, and every bounded module operator on E can be extended to E#. Then we study the properties of self-dual Hilbert W*-modules. Finally, we give some equivalent conditions for a Hilbert W*-module to be self-dual.
Chapter 1

Preliminaries

Throughout this chapter, let $A$ be an arbitrary $C^*$-algebra.

1.1 Hilbert $C^*$-modules

Definition 1.1.1 A pre-Hilbert module over $A$ is a right $A$-module $E$ (with compatible scalar multiplication: $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for $x$ in $E$, $a$ in $A$ and $\lambda$ in $\mathbb{C}$), equipped with an $A$-valued "inner product" $\langle \cdot, \cdot \rangle : E \times E \to A$, with the following properties:

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$,

(ii) $\langle x, ya \rangle = \langle x, y \rangle a$,

(iii) $\langle y, x \rangle = \langle x, y \rangle^*$,

(iv) $\langle x, x \rangle \geq 0$; if $\langle x, x \rangle = 0$, then $x = 0$,

where $x, y, z$ in $E$, $a$ in $A$, and $\alpha, \beta$ in $\mathbb{C}$.

Remark 1.1.2 From (i) and (iii), $\langle \cdot, \cdot \rangle$ is conjugate-linear in its first variable.

As in the scalar case, we also have the Cauchy-Schwarz inequality in this setting:
**Proposition 1.1.3** If $E$ is a pre-Hilbert module over $A$ and $x, y$ in $E$, then
\[
\langle y, x \rangle \langle x, y \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle.
\]

**Definition 1.1.4** Let $E$ be a pre-Hilbert module over $A$. For $x$ in $E$, we write
\[
\|x\| = \|\langle x, x \rangle\|^\frac{1}{2}. \tag{1.1.4}
\]
Then $\| \cdot \|$ is a norm on $E$ making $E$ into a normed $A$-module. If $E$ is complete with respect to $\| \cdot \|$, $E$ is called a Hilbert $A$-module or a Hilbert module over $C^*$-algebra $A$.

**Lemma 1.1.5** Let $E$ be a Hilbert $A$-module and $(u_i)_{i \in \Lambda}$ be an approximate identity for $A$, then

(i) $\lim \|x - xu_i\| = 0$, for all $x$ in $E$. In particular, $x1 = x$, for all $x$ in $E$, if $A$ is unital.

(ii) $\overline{EA} = E$.

**Remark 1.1.6** If $A$ is non-unital and $\tilde{A}$ denotes the $C^*$-algebra obtained by adjoining an identity $1$ to $A$, then $E$ becomes a Hilbert $\tilde{A}$-module if we define $x1 = x$, for all $x$ in $E$.

**Example 1.1.7**

(a) $A$ itself is a Hilbert $A$-module with $\langle a, b \rangle = a^*b$ ($a, b \in A$). Moreover, any closed right ideal in $A$ is a Hilbert $A$-module.

(b) Let $\{E_i, \langle \cdot, \cdot \rangle_i\}_{i \in I}$ be a family of Hilbert $A$-modules, define the set
\[
\bigoplus_i E_i = \{x = (x_i) \mid x_i \in E_i \text{ and } \sum_i \langle x_i, x_i \rangle_i \text{ exists in } A\}.
\]
Then $\bigoplus_i E_i$ is a Hilbert $A$-module if we define
\[
\langle x, y \rangle = \sum_i \langle x_i, y_i \rangle_i,
\]
where $x = (x_i), y = (y_i)$ in $\bigoplus_i E_i$.

If $I$ is a finite set, and each $E_i = E$, denote $\bigoplus_i E_i$ by $E^{[f]}$.

If $I$ is infinitely countable, and each $E_i = A$, denote $\bigoplus_i E_i$ by $H_A$. 
We now introduce some interesting operators between (pre-)Hilbert $A$-modules.

**Definition 1.1.8** Let $E$ and $F$ be (pre-)Hilbert $A$-modules. We let $\mathcal{B}(E,F)$ or $\mathcal{B}_A(E,F)$ be the set of all bounded $A$-linear maps from $E$ to $F$.

**Theorem 1.1.9** Let $B$ be a $C^*$-subalgebra of $A$, $E$ be a pre-Hilbert $B$-module, and $F$ be a pre-Hilbert $A$-module. Let $t : E \to F$ be a linear map. Then the following conditions are equivalent:

(i) $t$ is bounded and $t(xb) = t(x)b$ for all $x$ in $E$ and $b$ in $B$.

(ii) there exists $K \geq 0$ such that $\langle tx, tx \rangle_A \leq K \langle x, x \rangle_B$ for all $x$ in $E$.

**Proof:** Without loss of generality, we may assume that $A$ and $B$ have the same unit, otherwise, we may consider $E$ as a pre-Hilbert $B$-module and $F$ as a pre-Hilbert $B$-module by Remark 1.1.6. Recall that for any $\tau : B \to A$ which is a linear map with a non-negative real number $K$ such that $\tau(x)^*\tau(x) \leq K x^*x$ for all $x$ in $B$, we have $\tau(x) = \tau(1_A)x$. (c.f.: [10, p.448, 2.7])

We first prove the implication (i)$\Rightarrow$(ii). Assume that $t$ is in $\mathcal{B}(E,F)$. We let $x$ be in $E$, then for any $n$ in $\mathbb{N}$, we set $b_n = ((x,x)_B + n^{-1})^{-\frac{1}{2}}$ and $x_n = xb_n$. Notice that $\langle x_n, x_n \rangle_B = \langle x, x \rangle_B ((x,x)_B + n^{-1})^{-1} \leq 1$, so we have $\|x_n\|_E \leq 1$, and so $\|tx_n\|_F \leq \|t\|$, that is, $\frac{1}{\|t\|^2} \langle tx_n, tx_n \rangle_A \leq 1$. Thus we have $\frac{1}{\|t\|^2} \langle tx_n, tx_n \rangle_A \leq 1_A$. Hence, $\langle tx_n, tx_n \rangle_A \leq \|t\|^2 1_A$. However, $\langle tx_n, tx_n \rangle_A = b_n \langle tx, tx \rangle_A b_n$, and so $\langle tx, tx \rangle_A \leq \|t\|^2 b_n^{-2} = \|t\|^2 ((x,x)_B + n^{-1})$, for all $n$ in $\mathbb{N}$. It follows that $\langle tx, tx \rangle_A \leq \|t\|^2 (x,x)_B$.

To prove the other direction, assume there exists $K \geq 0$ such that $\langle tx, tx \rangle_A \leq K \langle x, x \rangle_B$ for all $x$ in $E$. Then $\|tx\|^2_F = \|\langle tx, tx \rangle_A\| \leq K \|x\|^2_E$ for all $x$ in $E$, and so $\|t\| \leq K^\frac{1}{2}$. Now let $x$ be in $E$ and $y$ be in $F$. Consider the linear map $\tau : B \to A$ defined by

$$\tau(b) = \langle y, t(xb) \rangle_A \quad (b \in B).$$
Using Proposition 1.1.3, we see that

\[
\tau(b)^* \tau(b) = \langle t(xb), y \rangle_A \langle y, t(xb) \rangle_A \\
\leq \|y\|^2_F \langle t(xb), t(xb) \rangle_A \\
\leq \|y\|^2_F K \langle xb, xb \rangle_B \\
= \|y\|^2_F K b^* \langle x, x \rangle_B b \\
\leq \|y\|^2_F K \|x\|^2_E b^* b
\]

for any \( b \) in \( B \). Therefore we have, \( \tau(b) = \tau(1_B)b \) for any \( b \) in \( B \), that is, \( \langle y, t(xb) \rangle_A = \langle y, t(x) \rangle_A b = \langle y, t(x)b \rangle_A \) for any \( b \) in \( B \), \( x \) in \( E \) and \( y \) in \( F \). Hence, \( t(xb) = t(x)b \) for any \( x \) in \( E \) and \( b \) in \( B \). \( \square \)

**Remark 1.1.10** As in the case of Hilbert spaces, for any \( t \in B(E, F) \), we also have

\[
\|t\| = \inf \{K^{\frac{1}{2}} \mid \langle tx, tx \rangle_A \leq K \langle x, x \rangle_B, \forall x \in E \}.
\]

**Definition 1.1.11** Let \( E, F \) be Hilbert \( A \)-modules. \( \mathcal{L}(E, F) \) or \( \mathcal{L}_A(E, F) \) denotes the set of all maps \( t : E \to F \) for which there is a map \( t^* : F \to E \) such that

\[
\langle t(x), y \rangle = \langle x, t^*(y) \rangle,
\]

for all \( x \) in \( E \) and \( y \) in \( F \). We call \( \mathcal{L}(E, F) \) the set of adjointable maps from \( E \) to \( F \).

**Lemma 1.1.12** Let \( E, F \) and \( G \) be Hilbert \( A \)-modules. Let \( t \in \mathcal{L}(E, F) \) and \( s \in \mathcal{L}(F, G) \). Then

(i) Every element of \( \mathcal{L}(E, F) \) is a bounded \( A \)-linear map,

(ii) \( t^* \) is in \( \mathcal{L}(F, E) \),

(iii) $st$ is in $\mathcal{L}(E,G)$,

(iv) $\mathcal{L}(E) := \mathcal{L}(E,E)$ is a $C^*$-algebra.

From now on, for any Hilbert $A$-modules $E$, $F$, $x \in E$ and $y \in F$, we define $\theta_{x,y} : F \to E$ by

$$\theta_{x,y}(z) = x(y,z) \quad (z \in F).$$

We denote by $\mathcal{F}(F,E)$ (or $\mathcal{F}_A(F,E)$) := span\{\theta_{x,y} \mid x \in E, y \in F\} and is called the set of $A$-finite rank operators from $F$ to $E$. One can directly check that:

**Lemma 1.1.13** Let $E$, $F$ and $G$ be Hilbert $A$-modules. Then

(i) $\theta_{x,y}$ is in $\mathcal{L}(F,E)$ with $(\theta_{x,y})^* = \theta_{y,x}$,

(ii) $\theta_{x,y} \theta_{u,v} = \theta_{x(y,u),v} = \theta_{x,v(u,y)}$,

(iii) $t \theta_{x,y} = \theta_{t(x),y}$,

(iv) $\theta_{x,y} s = \theta_{x,s(y)}$,

where $x$ in $E$, $y$, $u$ in $F$, $v$ in $G$, $t$ in $\mathcal{L}(E,G)$ and $s$ in $\mathcal{L}(G,F)$.

Denoted by $\mathcal{K}(F,E)$ (or $\mathcal{K}_A(F,E)$) := $\overline{\mathcal{F}(F,E)} \subseteq \mathcal{L}(F,E)$ and $\mathcal{K}(E) := \mathcal{K}(E,E) \subseteq \mathcal{L}(E)$. Then $\mathcal{K}(E)$ is a closed two-sided ideal of $\mathcal{L}(E)$. We call $\mathcal{K}(F,E)$ the set of "compact" operators from $F$ to $E$.

**Remark 1.1.14**

(i) $\mathcal{K}(A) \cong A$ by identifying $\theta_{a,b}$ with the operation of left multiplication by $ab^*$.

(ii) If $A$ is unital, then $\mathcal{K}(A) \cong \mathcal{L}(A)$.

**Definition 1.1.15** Let $E$ be a (pre-)Hilbert $A$-module. Let $E^\# = B(E,A)$ which is endowed with the following operations:

(i) $(\lambda \tau)(x) = \bar{\lambda} \tau(x)$,
Hilbert C*-modules

(ii) $(\tau_1 + \tau_2)(x) = \tau_1(x) + \tau_2(x)$,

(iii) $(\tau a)(x) = a^* \tau(x)$,

where $\tau_i$ in $E^*$, $x$ in $E$, $a$ in $A$ and $\lambda$ in $\mathbb{C}$. Then $E^*$ becomes a Banach right $A$-module with the operator norm.

**Definition 1.1.16** Let $E$ be a (pre-)Hilbert $A$-module. For each $x$ in $E$, we set

(i) $\hat{x} : E \to A, \hat{x}(y) = \langle x, y \rangle$ ($y \in E$) and

(ii) $L_x : A \to E, L_x(a) = xa$ ($a \in A$).

Then, $\hat{x} \in E^*$. Moreover, the calculation $\langle \hat{x}(y), a \rangle = \langle x, y \rangle^* a = \langle y, x \rangle a = \langle y, xa \rangle = \langle y, L_x a \rangle$ shows that $\hat{x}$ and $L_x$ are adjoints of each other, and hence belong to $\mathcal{L}(E, A)$ and $\mathcal{L}(A, E)$, respectively.

**Lemma 1.1.17** Let $E$ be a Hilbert $A$-module. Then

(i) the map $\hat{\cdot} : x \mapsto \hat{x}$ is an isometric $A$-linear map from $E$ onto $\mathcal{K}(E, A)$, where $\mathcal{K}(E, A)$ is viewed as a Banach $A$-submodule of $E^*$,

(ii) the map $L : x \mapsto L_x$ is an isomorphism from $E$ onto $\mathcal{K}(A, E)$ as Banach space.

**Proof:** See [14, p.21, 2.32].

**Lemma 1.1.18** Let $E$ be a Hilbert $A$-module. For $t$ in $\mathcal{K}(E, A) \subseteq E^*$, $s$ in $\mathcal{K}(A, E)$, $r$ in $\mathcal{K}(E)$ and $a$ in $A \cong \mathcal{K}(A)$, define \[ \begin{pmatrix} a & t \\ s & r \end{pmatrix} : A \oplus E \to A \oplus E \] by

\[ \begin{pmatrix} a & t \\ s & r \end{pmatrix} \begin{pmatrix} b \\ x \end{pmatrix} = \begin{pmatrix} ab + t(x) \\ s(b) + r(x) \end{pmatrix}. \]

Then \[ \begin{pmatrix} a & t \\ s & r \end{pmatrix} \] is in $\mathcal{K}(A \oplus E)$.

Conversely, every elements in $\mathcal{K}(A \oplus E)$ are of this form.
Proof: See [14, p.21-22]. □

Proposition 1.1.19 Let $E$ be a Hilbert $A$-module. For each $x$ in $E$, there exists a unique $y$ in $E$ such that $x = y(y, y)$.

Proof: Let $x$ be in $E$. Notice that $\begin{pmatrix} 0 & \hat{x} \\ L_x & 0 \end{pmatrix}$ is a self-adjoint element in $\mathcal{K}(A \oplus E)$. By Lemma 1.1.18, we know that every self-adjoint element in $\mathcal{K}(A \oplus E)$ which anti-commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is of this form. Now, consider the function $f(t) = t^3$, then $f\left( \begin{pmatrix} 0 & \hat{x} \\ L_x & 0 \end{pmatrix} \right)$ is a self-adjoint element in $\mathcal{K}(A \oplus E)$ which anti-commutes with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, that is, $f\left( \begin{pmatrix} 0 & \hat{x} \\ L_x & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \hat{y} \\ L_y & 0 \end{pmatrix}$, for some $y$ in $E$, and hence $\begin{pmatrix} 0 & \hat{x} \\ L_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{y} \\ L_y & 0 \end{pmatrix}^3$. By computing the bottom left-hand corner, we have $L_x = L_y \hat{y} L_y$, and so $L_x(a) = L_y \hat{y} L_y(a)$, for all $a$ in $A$, that is, $xa = y(y, y)a$, for all $a$ in $A$, and hence $x = y(y, y)$. Now, if there exists another $y_1$ in $E$ such that $x = y_1(y_1, y_1)$, then $\begin{pmatrix} 0 & \hat{y} \\ L_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{x} \\ L_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \hat{y}_1 \\ L_{y_1} & 0 \end{pmatrix}^3$ and so $y = y_1$ since the map $^\sim$ is one-to-one. □

Corollary 1.1.20 Let $E$ be a Hilbert $A$-module. Then $E(E, E) : = \text{span}\{x(y, z) \mid x, y, z \in E\}$ is equal to $E$.

Remark 1.1.21 Let $E$ be a Hilbert $A$-module. Then

$$\overline{\langle E, E \rangle} : = \text{span}\{ \langle x, y \rangle \mid x, y \in E\}$$

is the smallest closed two-sided ideal $I$ of $A$ such that

$$EI : = \text{span}\{xi \mid x \in E, i \in I\} = E.$$
Definition 1.1.22 Let $E$ be a Hilbert $A$-module. We call $\langle E, E \rangle$ the support of $E$. Moreover, $E$ is said to be full if $\langle E, E \rangle = A$.

Remark 1.1.23 Every Hilbert $A$-module can be made into a full Hilbert $\langle E, E \rangle$-module.

Definition 1.1.24 Let $E$ be a Hilbert $A$-module. A closed submodule $F$ of $E$ is said to be complemented if $E = F \oplus F^\perp$ where $F^\perp = \{ y \in E \mid \langle x, y \rangle = 0, \forall x \in F \}$.

Lemma 1.1.25 Let $E$ be a Hilbert $A$-module, $t^* = t$ be in $\mathcal{L}(E)$ and

$$\|tx\| \geq k\|x\| \ (x \in E)$$

for some constant $k > 0$. Then $t$ is invertible in $\mathcal{L}(E)$.

Proof: See [6, p.22, 3.1].

Theorem 1.1.26 Let $E, F$ be Hilbert $A$-modules and $t$ in $\mathcal{L}(E, F)$ has closed range. Then

(i) $\ker(t)$ is a complemented submodule of $E$,

(ii) $\text{ran}(t)$ is a complemented submodule of $F$,

(iii) the mapping $t^*$ in $\mathcal{L}(F, E)$ also has closed range.

Definition 1.1.27 Let $E, F$ be Hilbert $A$-modules. An operator $u$ in $\mathcal{L}(E, F)$ is said to be unitary if $u^*u = 1_E$, $uu^* = 1_F$.

Theorem 1.1.28 Let $E, F$ be Hilbert $A$-modules, $u$ be a linear map from $E$ to $F$. Then the following conditions are equivalent:

(i) $u$ is an isometric, surjective $A$-linear map.
(ii) $u$ is a unitary element of $\mathcal{L}(E, F)$.

**Definition 1.1.29** Let $E$, $F$ be Hilbert $A$-modules. If there exists a unitary element of $\mathcal{L}(E, F)$, or equivalently, there exists an invertible element $t$ in $\mathcal{B}(E, F)$ such that $(tx, ty)_F = (x, y)_E$ for all $x$, $y$ in $E$, then we say that $E$ and $F$ are unitarily isomorphic Hilbert $A$-modules, and we write $E \cong F$ as Hilbert $A$-module.

**Lemma 1.1.30** Let $E$ be a Hilbert $A$-module and $t$ be in $\mathcal{B}(E)$. Then the following conditions are equivalent:

(i) $t$ is a positive element of $\mathcal{L}(E)$.

(ii) $(x, tx) \geq 0$ for all $x$ in $E$.

Recall that a positive element $a$ in $A$ is called a strictly positive element if $\rho(a) > 0$ for any state $\rho$ of $A$. $A$ has a strictly positive element if and only if it has a countable approximate identity. If $A$ has a strictly positive element, then it is said to be $\sigma$-unital.

**Definition 1.1.31** Let $E$ be a Hilbert $A$-module. $E$ is said to be countably generated if there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in $E$ such that

$$E = \text{span}\{x_ia \mid i \in \mathbb{N}, a \in A\}.$$ 

$E$ is said to be algebraically finitely generated if there exists a finite sequence $(x_i)_{i=1}^n$ in $E$ such that

$$E = \text{span}\{x_ia \mid i = 1, 2, \ldots, n, a \in A\}.$$ 

**Remark 1.1.32** If $A$ is $\sigma$-unital, then $H_A$ is countably generated. Indeed, if $a$ is a strictly positive element in $A$, then the set $\{(\delta_i a)_{j \in \mathbb{N}}\}_{i \in \mathbb{N}}$ is a countable generating set for $H_A$. 

**Hilbert C*-modules**
Theorem 1.1.33 Let $E$ be a countably generated Hilbert $A$-module, then $E \oplus H_A \cong H_A$ as Hilbert $A$-module. Thus, $E$ is unitarily isomorphic to a direct summand of $H_A$.

Proof: See [6, p.60, 6.2].

Proposition 1.1.34 Let $E$ be a Hilbert $A$-module. $E$ is countably generated if and only if $\mathcal{K}(E)$ is $\sigma$-unital.

Proof: See [6, p.66, 6.7].
Chapter 2

Self-dual Hilbert C*-modules

Throughout this chapter, let $A$ be an arbitrary C*-algebra.

2.1 Self-duality

In this section, we will see that every algebraically finitely generated Hilbert C*-module over a unital C*-algebra and every Hilbert C*-module over a finite dimensional C*-algebra are self-dual ([2], [8], [15], [16]). Moreover, M. Frank showed that every self-dual Hilbert C*-module has the property that "every inner product inducing equivalent norm to the given one is unitarily isomorphic to it" ([3]).

Definition 2.1.1 Let $E$ be a (pre-)Hilbert $A$-module. $E$ is said to be self-dual if $E = E^*$. (c.f. Definition 1.1.16)

Proposition 2.1.2 If $E$ is a self-dual (pre-)Hilbert $A$-module, then $E$ is complete.

Proof: Suppose that $E$ is not complete. Let $x$ be an element in the completion of $E$ but not in $E$, and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $E$ converges in norm to $x$.
Then the function \( \tau : E \to A \) defined by
\[
\tau(y) = \lim_n \langle x_n, y \rangle \quad (y \in E)
\]
begins to \( E^\# \) but not belongs to \( \bar{E} \), that is \( E \) is not self-dual. Contradiction occurs!

\[ \square \]

**Proposition 2.1.3** Let \( E \) be a self-dual Hilbert \( A \)-module, and \( F \) be a (pre-)Hilbert \( A \)-module, then \( \mathcal{L}(E, F) = \mathcal{B}(E, F) \).

**Proof:** We only need to prove that \( \mathcal{B}(E, F) \subseteq \mathcal{L}(E, F) \). Let \( t \) be in \( \mathcal{B}(E, F) \), then for each \( y \) in \( F \), consider the map \( \tau_y : E \to A \) defined by
\[
\tau_y(x) = \langle y, tx \rangle_F
\]
we see that \( \tau_y \) is in \( E^\# = \bar{E} \), by assumption. Thus, there exists an \( z_y \) in \( E \) such that \( \tau_y = z_y \), that is,
\[
\langle y, tx \rangle_F = \langle z_y, x \rangle_E
\]
for all \( x \) in \( E \). Now, we define the map \( s : F \to E \) by
\[
s(y) = z_y,
\]
then we have \( \langle y, tx \rangle_F = \langle sy, x \rangle_E \) for all \( x \) in \( E \) and \( y \) in \( F \). Hence \( t \) is in \( \mathcal{L}(E, F) \). \[ \square \]

**Proposition 2.1.4** Let \( \{E_i, \langle \cdot, \cdot \rangle_i\}_{i \in I} \) be a family of Hilbert \( A \)-modules. Then \( \{\bigoplus E_i, \langle \cdot, \cdot \rangle\} \) defined in Example 1.1.7(b) is self-dual if and only if \( \{E_i, \langle \cdot, \cdot \rangle_i\}_{i \in I} \) are self-dual for all \( i \) in \( I \).

**Proof:** To prove the necessary part, assume that \( \{\bigoplus E_i, \langle \cdot, \cdot \rangle\} \) is self-dual. For each fixed \( j \) in \( I \), let \( \tau \) be an element in \( E^\#_j \). Notice that the map \( \tilde{\tau} : \bigoplus E_i \to A \) defined by
\[
(x_i) \mapsto \tau(x_j)
\]
is an element in \((\bigoplus E_i)^\#\). So, by assumption, there exists an element \(z = (z_i)\) in \(\bigoplus E_i\) such that \(\tau(x_j) = \langle (z_i), (x_i) \rangle = \sum_i \langle z_i, x_i \rangle_i\) for all \((x_i)\) in \(\bigoplus E_i\). Notice also that for each \(i_0\) not equal to \(j\), we have

\[
\langle z_{i_0}, z_{i_0} \rangle_{i_0} = \langle (z_i), (0, \ldots, 0, z_{i_0}, 0, \ldots) \rangle \\
= \tau(0) \\
= 0
\]

which means that \(z_{i_0} = 0\). Hence, \(\tau(x_j) = \langle z_j, x_j \rangle_j\) for all \(x_j\) in \(E_j\). That means \(E_j\) is self-dual.

For proving the sufficient part, we assume that \(\{E_i, \langle \cdot, \cdot \rangle_i\}\) are self-dual for all \(i\) in \(I\). Let \(\tau\) be in \((\bigoplus E_i)^\#\). Notice that for each \(j\) in \(I\), the map \(\tau_j : E_j \to A\) defined by

\[
x_j \mapsto \tau((0, 0, \ldots, 0, x_j, 0, \ldots)j)
\]

is an element in \(E_j^\#\). So, by assumption, there exists an element \(z_j\) in \(E_j\) such that \(\tau((0, 0, \ldots, 0, x_j, 0, \ldots)_j) = \langle z_j, x_j \rangle_j\) for all \(x_j\) in \(E_j\). We now claim that \((z_j)_{j \in I}\) is in \(\bigoplus E_i\). Notice that for each finite subset \(F\) of \(I\),

\[
\| \sum_{j \in F} \langle z_j, z_j \rangle_j \| = \| \sum_{j \in F} (0, 0, \ldots, 0, z_j, 0, \ldots) \| \\
\leq \| \tau \| \sum_{j \in F} (0, 0, \ldots, 0, z_j, 0, \ldots) \|.
\]

Since

\[
\| \sum_{j \in F} (0, 0, \ldots, 0, z_j, 0, \ldots) \| \\
= \| \langle \sum_{j \in F} (0, 0, \ldots, 0, z_j, 0, \ldots)_j, \sum_{k \in F} (0, 0, \ldots, 0, z_k, 0, \ldots)_k \rangle \|^{\frac{1}{2}} \\
= \| \sum_{j \in F} \langle z_j, z_j \rangle_j \|^\frac{1}{2}
\]
we have \( \| \sum_{j \in I} \langle z_j, z_j \rangle \| \leq \| \tau \|^2 \) and hence \( (z_j)_{j \in I} \) is in \( \bigoplus E_i \). Moreover, we have

\[
\tau((x_i)) = \sum_i \tau((0, 0, \ldots, 0, x_i, 0, \ldots)) = \sum_i \langle z_i, x_i \rangle_i = \langle (z_i), (x_i) \rangle
\]

for all \( (x_i) \) in \( \bigoplus E_i \). Hence, \( \bigoplus E_i \) is self-dual. \( \square \)

**Example 2.1.5** Let \( A \) be a \( C^* \)-algebra. Let \( p \) be a projection of \( A \). Then \( pA \) is a self-dual Hilbert \( A \)-module with inner product \( \langle pa, pb \rangle = a^*pb \).

**Proof:** Let \( \tau \) be in \( (pA)^# \). Then for all \( pa \) in \( pA \), we have

\[
\tau(pa) = (\tau(p)p)pa = \langle pr(p)^*, pa \rangle.
\]

So, \( \tau \) is in \( pA \) and hence, \( pA \) is self-dual. \( \square \)

Similar to Theorem 1.1.33 and Proposition 1.1.34, we have the following:

**Proposition 2.1.6** Let \( A \) be a unital \( C^* \)-algebra and \( E \) be a Hilbert \( A \)-module. Then the following conditions are equivalent:

(i) \( E \) is algebraically finitely generated.

(ii) \( 1_E \) is in \( \mathcal{F}(E) \).

(iii) \( E \) is a direct summand of \( A^n \), for some \( n \) in \( \mathbb{N} \).

Moreover, in this case, \( E \) is self-dual.

**Proof:** (i) \( \Rightarrow \) (iii): Assume that \( E \) is algebraically finitely generated. Let \( \{x_1, \ldots, x_n\} \) be a set of generators of \( E \). Consider the \( A \)-linear function \( f : A^n \to E \) defined by

\[
(f_{ij}1)_{j=1}^n \mapsto x_i.
\]
Hilbert C*-modules

We have \( f \in \mathcal{L}(A^n, E) \) and \( f \) has closed range since \( E \) is algebraically finitely generated. So, by Theorem 1.1.26(i), \( \ker(f) \) is a complemented submodule of \( A^n \), and thus \( f|_{\ker(f)^\perp} \) is an isomorphism from \( \ker(f)^\perp \) onto \( E \) as Hilbert module, that is, \( E \) is a direct summand of \( A^n \).

(iii) \( \Rightarrow \) (ii): Assume that \( E \) is a direct summand of \( A^n \). Let \( p \in \mathcal{L}(A^n, E) \) be the projection from \( A^n \) onto \( E \). It is easy to see that \( p \circ \sum_{i=1}^{n} \theta_{(\delta_{ij})_{j=1}^{n}} \circ p^* \) is equal to \( 1_E \) and belongs to \( \mathcal{F}(E) \).

(ii) \( \Rightarrow \) (i): Assume that \( 1_E \) is in \( \mathcal{F}(E) \). Let \( 1_E = \sum_{i=1}^{n} \theta_{y_i, z_i} \) for some finite sequences \( (y_i)_{i=1}^{n}, (z_i)_{i=1}^{n} \) in \( E \). Thus, we have \( x = \sum_{i=1}^{n} y_i \langle z_i, x \rangle \) for any \( x \) in \( E \), that is, \( \{y_i\}_{i=1}^{n} \) is a finite set of generators of \( E \) which generates \( E \) algebraically. Moreover, for any \( \tau \) in \( E^\# \), \( x \) in \( E \), we have

\[
\tau(x) = \tau\left(\sum_{i=1}^{n} y_i \langle z_i, x \rangle\right) = \sum_{i=1}^{n} \tau(y_i) \langle z_i, x \rangle = \langle \sum_{i=1}^{n} z_i \tau(y_i)^*, x \rangle.
\]

Hence, \( \tau \) is in \( \hat{E} \) and so \( E \) is self-dual. \( \square \)

**Proposition 2.1.7** Let \( A \) be a finite dimensional C*-algebra and \( E \) be a Hilbert \( A \)-module. Then \( E \) is self-dual.

**Proof:** Since \( A \) is finite-dimensional, \( A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \), for some \( n_1, \ldots, n_k \) in \( \mathbb{N} \). So, for each \( a \) in \( A \), \( a = (a_1, \ldots, a_k) \) where \( a_i \in M_{n_i}(\mathbb{C}) \), we can define \( r(a) = tr(a_1) + \cdots + tr(a_k) \), where \( tr \) is the trace. Then the function \( r : A \to \mathbb{C}, a \mapsto r(a) \) is a faithful positive linear functional on \( A \) and \( r(ab) = r(ba) \) for all \( a, b \) in \( A \). We can use \( r \) to define a \( \mathbb{C} \)-valued inner product \( r(\langle \cdot, \cdot \rangle) \) on \( E \) to make \( \{E, r(\langle \cdot, \cdot \rangle)\} \) into a Hilbert space. Now, let \( \tau \) be in \( E^\# \), consider the map \( \tau' : E \to \mathbb{C} \) defined by

\[ \tau'(x) = r(\tau(x)). \]
Then \( \tau' \) is a bounded linear functional on \( \{ E, \rho(\langle \cdot, \cdot \rangle) \} \) since \(|\rho(x)|^2 = |\rho(\tau(x))|^2 \leq \| \rho \| \rho(\tau^*(x)\tau(x)) \leq \| \rho \| \| \tau \|^2 \rho(\langle x, x \rangle) \) for all \( x \) in \( E \) by Theorem 1.1.9. So, there exists an element \( y \) in \( E \) such that \( \rho(\tau(x)) = \rho(\langle y, x \rangle) \), for all \( x \) in \( E \). Now, for a fixed \( x \) in \( E \), by polar decomposition, \(|\tau(x) - \langle y, x \rangle| = u^*(\tau(x) - \langle y, x \rangle) \), for some \( u \) in \( A \) (Notice that \( A \) is a von Neumann algebra).

Hence

\[
\rho(|\tau(x) - \langle y, x \rangle|) = \rho(u^*(\tau(x) - \langle y, x \rangle))
\]
\[
= \rho((\tau(x) - \langle y, x \rangle)u^*)
\]
\[
= \rho(\tau(xu^*) - \langle y, xu^* \rangle))
\]
\[
= 0.
\]

Thus, \( \tau(x) = \langle y, x \rangle \) for all \( x \) in \( E \), that is, \( \tau \) is in \( \tilde{E} \).

\[\square\]

**Proposition 2.1.8** Let \( \{ E, \langle \cdot, \cdot \rangle_1 \} \) be a self-dual Hilbert A-module, and let \( \langle \cdot, \cdot \rangle_2 \) be another A-valued inner product on \( E \) such that the norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) induced by them are equivalent. Then \( \{ E, \langle \cdot, \cdot \rangle_2 \} \) is a self-dual Hilbert A-module and there exists a unique \( t \) in \( \mathcal{B}(E) \) such that

(i) \( \langle x, y \rangle_2 = \langle t(x), y \rangle_1 \) for all \( x, y \) in \( E \).

(ii) \( t \) is one-to-one and \( t \) is in \( \mathcal{L}^{(1)}(E)_+ \) and \( \mathcal{L}^{(2)}(E)_+ \), where \( \mathcal{L}^{(i)}(E) \) means the set of all adjointable maps from \( E \) to \( E \) with respect to \( \langle \cdot, \cdot \rangle_i \).

(iii) \( t \) is invertible and \( t^{-1} \) is in \( \mathcal{B}(E) \). Also, \( t^{-1} \) is one-to-one, and \( t^{-1} \) is in \( \mathcal{L}^{(1)}(E)_+ \) and \( \mathcal{L}^{(2)}(E)_+ \), and \( \langle x, y \rangle_1 = \langle t^{-1}(x), y \rangle_2 \) for all \( x, y \) in \( E \).

**Proof:** By assumption, there exist positive real numbers \( k \) and \( l \) such that

\[
 k\|x\|_1 \leq \|x\|_2 \leq l\|x\|_1, \text{ for all } x \in E. \tag{2.19}
\]

Notice that for each \( x \) in \( E \), the map \( y \in E \mapsto \langle x, y \rangle_2 \in A \) is a bounded \( A \)-linear map from \( E \) to \( A \), and since \( \{ E, \langle \cdot, \cdot \rangle_1 \} \)
is self-dual, there exists an element \( t(x) \) in \( E \) such that \( \langle t(x), y \rangle_1 = \langle x, y \rangle_2 \), for all \( y \) in \( E \). We now consider the map \( t : E \to E \) defined by

\[
x \mapsto t(x),
\]

then \( t \) is \( \Lambda \)-linear and \( \langle t(x), y \rangle_1 = \langle x, y \rangle_2 \), for all \( x, y \) in \( E \). Notice that

\[
\|t(x)\|_1^2 = \|\langle t(x), t(x) \rangle_1\| = \|\langle x, t(x) \rangle_2\| \leq \|t(x)\|_2 \|x\|_2 \leq \|t(x)\|_1 \|x\|_1
\]

for all \( x \) in \( E \). We have \( \|t\|_1 \leq \ell^2 \), that is \( t \) is in \( B(E) \).

Notice that for any \( x \) in \( E \), we have

\[
\langle x, t(x) \rangle_1 = \langle t(x), x \rangle_1^* = \langle x, x \rangle_2^* \geq 0
\]

and

\[
\langle x, t(x) \rangle_2 = \langle t(x), t(x) \rangle_1 \geq 0.
\]

Thus, \( t \) is in \( L^{(1)}(E)_+ \) and \( L^{(2)}(E)_+ \) by Lemma 1.1.30.

To show that \( t \) is invertible, notice that

\[
k^2\|x\|_1^2 \leq \|x\|_2^2 = \|\langle x, x \rangle_2\| = \|\langle t(x), x \rangle_1\| \leq \|t(x)\|_1 \|x\|_1
\]

for all \( x \) in \( E \) and so \( k^2\|x\|_1 \leq \|t(x)\|_1 \) for all \( x \) in \( E \). So by Lemma 1.1.25, \( t \) is invertible. The other assertion follows clearly.

It remains to show that \( \{E, \langle \cdot, \cdot \rangle_2\} \) is self-dual. Let \( \tau \) be in \( \{E, \langle \cdot, \cdot \rangle_2\}^\# \). Then \( \tau \) is in \( \{E, \langle \cdot, \cdot \rangle_1\}^\# \). Since \( \{E, \langle \cdot, \cdot \rangle_1\} \) is self-dual, there exists an \( x \) in \( E \) such that
Hilbert C*-modules 21

\[ \tau(y) = \langle x, y \rangle_1 \] for all \( y \) in \( E \). Define \( x' \in E \) by \( x' = t^{-1}(x) \), then \( \tau(y) = \langle x', y \rangle_2 \), for all \( y \) in \( E \). Hence, \( \tau \) is in \( \{ E, \langle \cdot, \cdot \rangle_2 \} \).

We are going to prove the main result in this section:

**Theorem 2.1.9** Let \( E \) be a self-dual Hilbert \( A \)-module. Then \( E \) has the property that every \( A \)-valued inner product inducing equivalent norm to the given one is unitarily isomorphic to the given one.

**Proof:** Let \( \langle \cdot, \cdot \rangle_1 \) be another \( A \)-valued inner product on \( E \). By Proposition 2.1.8, there exists a \( t \) in \( \mathcal{L}(E)_+ \), invertible such that \( \langle x, y \rangle_1 = \langle t(x), y \rangle \) for all \( x, y \) in \( E \). We set \( s = t^\frac{1}{2} \), then \( s \) is in \( \mathcal{L}(E)_+ \), invertible and

\[ \langle x, y \rangle_1 = \langle s(x), s(y) \rangle \]

for all \( x, y \) in \( E \). Hence, \( \{ E, \langle \cdot, \cdot \rangle_1 \} \) is unitarily isomorphic to \( \{ E, \langle \cdot, \cdot \rangle \} \). \( \square \)

**Proposition 2.1.10** Let \( E_1, E_2 \) be Hilbert \( A \)-modules in which \( E_1 \) is self-dual. Suppose \( E_1 \) and \( E_2 \) are unitarily isomorphic, then \( E_2 \) is also self-dual.

**Proof:** By assumption, there exists \( S \in \mathcal{B}(E_1, E_2) \) which is invertible such that \( \langle Sx, Sy \rangle_2 = \langle x, y \rangle_1 \) for all \( x, y \) in \( E_1 \). Now let \( \tau \) be in \( E_1^\# \). Then \( \tau \circ S \) is in \( E_2^\# \) and so there exists an \( x \) in \( E_1 \) such that \( \tau \circ S(y) = \langle x, y \rangle_1 \) for all \( y \) in \( E_1 \). Take \( z = S(x) \), then for all \( \omega \) in \( E_2 \),

\[ \tau(\omega) = \tau(S(S^{-1}(\omega))) = \langle x, S^{-1}(\omega) \rangle_1 = \langle S(x), \omega \rangle_2 = \langle z, \omega \rangle_2, \]

thus \( \tau \) is in \( E_2 \). Hence \( E_2 \) is self-dual. \( \square \)
2.2 Self-duality and some related concepts

In this section, we introduce the concept of orthogonally $A$-complementary and $C^*$-reflexivity and show that a self-dual Hilbert $C^*$-module has these two properties ([3]).

**Definition 2.2.1** Let $E$ be a Hilbert $A$-module. $E$ is said to be orthogonally $A$-complementary if each Hilbert $A$-module $F$ containing $E$ as a Banach $A$-submodule is orthogonally decomposable as $F = E \oplus E^\perp$.

**Theorem 2.2.2** Let $E$ be a self-dual Hilbert $A$-module. Then $E$ is orthogonally $A$-complementary.

**Proof:** Let \( \{F, \langle \cdot, \cdot \rangle_F\} \) be a Hilbert $A$-module containing $E$ as a Banach $A$-submodule. Notice that $\langle \cdot, \cdot \rangle_F|_{E \times E}$ is an $A$-valued inner product on $E$ inducing equivalent norm to the given one. Let $\langle \cdot, \cdot \rangle_E = \langle \cdot, \cdot \rangle_F|_{E \times E}$. Then $\{E, \langle \cdot, \cdot \rangle_E\}$ is also a self-dual Hilbert $A$-module by Proposition 2.1.8. Now consider the inclusion map $\iota : E \to F$. Notice that $\iota \in B(E, F) = \mathcal{L}(E, F)$, since $E$ is self-dual and by Proposition 2.1.3. Hence, there exists $\iota^* : F \to E$ such that

\[
\langle \iota(x), y \rangle_E = \langle x, \iota^*(y) \rangle_E = \langle \iota(x), \iota(\iota^*(y)) \rangle_F
\]

for all $x$ in $E$ and $y$ in $F$. Thus, we have $\langle \iota(x), y - \iota(\iota^*(y)) \rangle_F = 0$ for all $x$ in $E$ and $y$ in $F$, that is, $y - \iota(\iota^*(y))$ is in $E^\perp$ for all $y$ in $F$. Hence, for all $y$ in $F$, $y = [\iota(\iota^*(y))] + [y - \iota(\iota^*(y))] \in E + E^\perp$, that is, $F = E \oplus E^\perp$. \qed

**Definition 2.2.3** Let $E$ be a Hilbert $A$-module. Define $q : E \to E^{\#\#}$ by

\[
(q(x))(\tau) = \tau(x)^*
\]

for $x$ in $E$, $\tau$ in $E^\#$. 
Remark 2.2.4 The map \( q \) defined in Definition 2.2.3 is an \( A \)-linear isometry.

Proof: Clearly, \( q \) is \( A \)-linear. Now, notice that \( \|(q(x))(\tau)\| = \|\tau(x)\| \leq \|\tau\|\|x\| \) for all \( x \) in \( E \), \( \tau \) in \( E^\# \), which implies that \( \|q(x)\| \leq \|x\| \) for all \( x \) in \( E \).

Also, fixed a non-zero element \( x \) in \( E \). Considering \( \tau = \frac{1}{\|x\|} \hat{x} \) in \( E^\# \), we have \( \|q(x)\| = \|(q(x))(\tau)\| = \|\tau(x)\| = \|\frac{1}{\|x\|} (x,x)\| = \|x\| \). Hence, \( \|q(x)\| = \|x\| \).

Definition 2.2.5 A Hilbert \( A \)-module \( E \) is called \( C^* \)-reflexive (or \( A \)-reflexive) if the map \( q \) defined in Definition 2.2.3 is onto.

Proposition 2.2.6 Let \( E \) be a self-dual Hilbert \( A \)-module. Then \( E \) is \( A \)-reflexive.

Proof: Let \( f \) be in \( E^{\#\#} \). Consider the map \( \tau_1 : E \to A \) defined by

\[
\tau_1(x) = f(\hat{x}) \quad (x \in E).
\]

Notice that \( \tau_1 \) is in \( E^\# \) and so in \( \hat{E} \), by assumption. So, there exists an element \( y \) in \( E \) such that \( \tau_1(x) = \langle y, x \rangle \) for all \( x \) in \( E \). Hence,

\[
f(\hat{x}) = \langle y, x \rangle = (\hat{x}(y))^* = (q(y))(\hat{x})
\]

for all \( x \) in \( E \). Since \( E \) is self-dual, we have \( f(\tau) = (q(y))(\tau) \) for all \( \tau \) in \( E^\# \).

Hence \( f = q(y) \), that is, \( E \) is \( A \)-reflexive. \( \square \)

### 2.3 A criterion of self-duality of \( H_A \)

Given any \( C^* \)-algebra \( A \), we recall that \( H_A = \bigoplus_{i \in N} E_i \), where each \( E_i = A \), which is a Hilbert \( A \)-module. It is interesting to see whether \( H_A \) is self-dual. M. Frank gave a criterion of self-duality of \( H_A \) ([3, p.173, 4.3]). He claimed that the following two conditions are equivalent:
(i) $A$ is finite dimensional.

(ii) $H_A$ is self-dual.

We know that (i) $\Rightarrow$ (ii) is always true by Proposition 2.1.7, but there is a gap in the proof of (ii) $\Rightarrow$ (i): We consider an infinite dimensional unital C*-algebra $A$. With Example 2.1.5, we know that $A$ is self-dual. Then, $H_A$ is self-dual by Proposition 2.1.4. It leads to a contradiction.
Chapter 3

Hilbert W* -modules

3.1 Extension of the inner product to $E^\#$

This section is based on the works of W.L. Paschke [10].

Definition 3.1.1 A (pre-)Hilbert W*-module is a (pre-)Hilbert C*-module over a von Neumann algebra.

Throughout this chapter, $A$ denotes a von Neumann algebra and we use the following notations:

- $A_*$ = the predual of $A$.
- $P = \text{the space of all normal positive linear functionals on } A$.
- $\mathbb{M} = \text{the space of all } \sigma(A, A_*)\text{-continuous positive linear functionals on } A$.

and we regard $A_*$ as a subspace of $A^*$ which is the space of all bounded linear functionals on $A$. Also, we regard $P$ as a subset of $A_*$, then we have $\text{span}P = A_*$. The following construction is similar to the GNS construction for a C*-algebra case: Let $E$ be a pre-Hilbert $A$-module. Let $f$ be a positive linear functional on $A$. Notice that $f(\langle \cdot, \cdot \rangle)$ is a semi-inner product on $E$ and if we let $N_f = \{ x \in E \mid f(\langle x, x \rangle) = 0 \}$, then $\frac{E}{N_f}$ is a pre-Hilbert space with $(x + N_f, y + N_f)_f = \frac{f(\langle x, y \rangle) - f(\langle y, y \rangle) + f(\langle x, x \rangle)}{f(\langle x, x \rangle) + f(\langle y, y \rangle)}$.
Let $\tau$ be an element in $E^\#$, then by Theorem 1.1.9, we have $\tau(x)^*\tau(x) \leq ||\tau||^2 \langle x, x \rangle$, for all $x$ in $E$. Hence, for all $x$ in $N_f$, $f(\langle x, x \rangle) = 0$ implies $f(\tau(x)^*\tau(x)) = 0$. Then we have $|f(\tau(x)^*)| \leq ||f||^{\frac{1}{2}} f(\tau(x)^*\tau(x))^{\frac{1}{2}} = 0$, that is, $f(\tau(x)) = \overline{f(\tau(x)^*)} = 0$. So, the map $\frac{\mathcal{E}}{N_f} \to \mathbb{C}$ defined by

$$x + N_f \mapsto f(\tau(x))$$

is a well-defined linear functional. Also, notice that for all $x$ in $E$,

$$|f(\tau(x))| \leq ||f||^{\frac{1}{2}} f(\tau(x)^*\tau(x))^{\frac{1}{2}}$$

$$\leq ||f||^{\frac{1}{2}} ||\tau|| f(\langle x, x \rangle)^{\frac{1}{2}}$$

$$= ||f||^{\frac{1}{2}} ||\tau|| ||x + N_f||_f,$$

thus this map is bounded by $||f||^{\frac{1}{2}} ||\tau||$. Hence, this map can be extended to a bounded linear functional on $H_f$ with norm not larger than $||f||^{\frac{1}{2}} ||\tau||$. And so, by Riesz Representation Theorem, there exists a unique element $\tau_f$ in $H_f$ with $||\tau_f|| \leq ||f||^{\frac{1}{2}} ||\tau||$ such that

$$(\tau_f, x + N_f)_f = f(\tau(x))$$

for all $x$ in $E$. In particular, if $\tau = \hat{y}$ for some $y$ in $E$, then $f(\hat{y}(x)) = f(\langle y, x \rangle) = (y + N_f, x + N_f)_f$ for all $x$ in $E$. Hence, $\hat{y}_f = y + N_f$.

Moreover, let $g$ be another positive linear functional on $A$ with $g \leq f$, then clearly $N_f \subseteq N_g$ and so the map $\frac{\mathcal{E}}{N_f} \to \frac{\mathcal{E}}{N_g}$ defined by

$$x + N_f \mapsto x + N_g \ (x \in E)$$

is a well-defined contractive linear map. And so, this map can be extended to a contractive linear map $V_{f,g} : H_f \to H_g$. Notice that for all $x$ in $E$,

$$V_{f,g}(\hat{\tau}_f) = V_{f,g}(x + N_f)$$

$$= x + N_g$$

$$= \hat{\tau}_g.$$
Proposition 3.1.2 Let $E$ be a pre-Hilbert $A$-module, and let $f$, $g$ be positive linear functionals on $A$ with $g \leq f$. Then $V_{f,g}(\tau_f) = \tau_g$, for all $\tau$ in $E^\#$.

**Proof:** Let $\tau$ be in $E^\#$. Since $\frac{f}{N_f}$ is dense in $H_f$, there exists a sequence $(y_n + N_f)_{n \in \mathbb{N}}$ in $\frac{f}{N_f}$ such that $\|y_n + N_f - \tau_f\|_f \to 0$. And by the contractivity of $V_{f,g}$, we have $V_{f,g}(\tau_f) = \lim_n V_{f,g}(y_n + N_f) = \lim_n (y_n + N_g)$. We claim that $\lim_n (y_n + N_g) = \tau_g$ and then the result follows. To do this, it suffices to show that $(y_n + N_g - \tau_g, x + N_g)_g \to 0$, for all $x$ in $E$, that is, we only need to prove that $g((y_n, x)) \to g(\tau(x))$, for all $x$ in $E$. Now, let $x \in E$. For all $n$ in $\mathbb{N}$, we have

$$
\left| g((y_n, x) - \tau(x)) \right|^2 \\
\leq \| g \| g((y_n, x)(x, y_n) - (y_n, x)\tau(x)^* - \tau(x)(x, y_n) + \tau(x)\tau(x)^*) \\
\leq \| f \| f((y_n, x)(x, y_n) - (y_n, x)\tau(x)^* - \tau(x)(x, y_n) + \tau(x)\tau(x)^*).
$$

Notice that

$$
f((y_n, x)\tau(x)^*) = f((y_n, x\tau(x)^*)) \\
= (y_n + N_f, x\tau(x)^* + N_f)_f \\
\to (\tau_f, x\tau(x)^* + N_f)_f \\
= f(\tau(x\tau(x)^*)) \\
= f(\tau(x)\tau(x)^*).
$$

Hence, it suffices to show that $f((y_n, x)(x, y_n) - \tau(x)(x, y_n)) \to 0$, for all $x$ in $E$.

Notice that for all $x$ in $E$, for all $n$ in $\mathbb{N}$,

$$
f((y_n, x)(x, y_n) - \tau(x)(x, y_n)) \\
= f((y_n, x(x, y_n)) - \tau(x)(x, y_n)) \\
= (y_n + N_f, x(x, y_n) + N_f)_f - (\tau_f, x(x, y_n) + N_f)_f \\
= (y_n + N_f - \tau_f, x(x, y_n) + N_f)_f.
$$
Also,

\[ \|x(x, y_n) + N_f\|_f^2 = f((x(x, y_n), x(x, y_n))) \]
\[ = f((y_n, x(x, y_n))) \]
\[ \leq \|x\|_E^2 f((y_n, x(x, y_n))) \]
\[ \leq \|x\|_E^2 f((y_n, y_n)) \]
\[ = \|x\|_E^2 \|y_n + N_f\|_f^2 \]

and so the sequence \((x(x, y_n) + N_f)_{n \in \mathbb{N}}\) is \(\| \cdot \|_f\)-bounded by the fact that the sequence \((y_n + N_f)_{n \in \mathbb{N}}\) converges. Since \(\|y_n + N_f - \tau_f\|_f \to 0\), we have \(f((y_n, x(x, y_n) - \tau(x(x, y_n))) \to 0\) for all \(x \in E\). The proof is complete. \(\square\)

**Theorem 3.1.3** Let \(E\) be a pre-Hilbert \(A\)-module. The \(A\)-valued inner product \(\langle \cdot, \cdot \rangle\) extends to \(E^* \times E^*\) such that \(E^*\) is a self-dual Hilbert \(A\)-module. In particular, the extended inner product satisfies \(\langle \tau, x \rangle_{E^*} = \tau(x)\), for all \(x \in E\) and \(\tau \) in \(E^*\).

**Proof:** Let \(\tau, \psi\) be in \(E^*\), define \(\Gamma : P \to \mathbb{C}\) by

\[ \Gamma(f) = (\tau_f, \psi_f)_f \quad (f \in P). \]

Notice that for all \(f\) in \(A_*\), by Jordan decomposition, it can be uniquely expressed as \(f = f_1 - f_2 + if_3 - if_4\) where \(f_j\) are in \(P\). So, \(\Gamma\) can be extended to a linear functional on \(A_*\).

We are now going to prove that \(\Gamma\) is bounded. Let \(g \in A_*\), then by Jordan decomposition, we have \(g = f_1 - f_2 + if_3 - if_4\) where \(f_j\) are in \(P\) and \(\sum_{j=1}^{4} \|f_j\| \leq \)
2\|g\|. So,

\[
|\Gamma(g)| \leq \sum_{j=1}^{4} |(\tau_{f_j}, \psi_{f_j})_{f_j}|
\leq \sum_{j=1}^{4} \|\tau_{f_j}\| \|f_j\| \|\psi_{f_j}\|_{f_j}
\leq \sum_{j=1}^{4} \|\tau\| \|f_j\| \|\frac{1}{2}\|\psi\|_{f_j}\|^\frac{1}{2}
= \sum_{j=1}^{4} \|\tau\| \|\psi\|_{f_j}\|
\leq 2\|g\| \|\tau\| \|\psi\|.
\]

Hence, \(\Gamma\) is in \((A_*)^*\).

Since \((A_*)^* = A\), there exists a unique \(\langle \tau, \psi \rangle_{E^*}\) in \(A\) such that

\[
\Gamma(g) = g(\langle \tau, \psi \rangle_{E^*}) \quad (g \in A_*).
\]

In particular, for any \(f\) in \(P\),

\[
(\tau_f, \psi_f)_f = f(\langle \tau, \psi \rangle_{E^*}).
\]

Notice that the map \(\langle \cdot, \cdot \rangle_{E^*} : E^* \times E^* \to A\) is conjugate linear in the first variable and is linear in the second variable by the linearity of the map : \(\tau \mapsto \tau_f\) for any \(f\) in \(P\). Moreover, \(\langle \cdot, \cdot \rangle_{E^*}\) is an \(A\)-valued inner product on \(E^*\). To show this, we need to prove \(\langle \cdot, \cdot \rangle_{E^*}\) satisfies the last three conditions in Definition 1.1.1:

For (iv), notice that for all \(\tau\) in \(E^*\), \(f(\langle \tau, \tau \rangle_{E^*}) = (\tau_f, \tau_f)_f \geq 0\) for all \(f\) in \(P\), which implies that \(\langle \tau, \tau \rangle_{E^*} \geq 0\).

Moreover, let \(\tau\) be in \(E^*\) with \(\langle \tau, \tau \rangle_{E^*} = 0\), we have \((\tau_f, \tau_f)_f = f(\langle \tau, \tau \rangle_{E^*}) = 0\) for all \(f\) in \(P\). Thus \(\tau_f = 0\) for all \(f\) in \(P\) which means that \(f(\tau(x)) = 0\) for all \(f\) in \(P\), \(x\) in \(E\). Hence \(\tau(x) = 0\) for all \(x\) in \(E\).

For (iii), let \(\tau, \psi\) be in \(E^*\), then for all \(f\) in \(P\), we have \(f(\langle \tau, \psi \rangle_{E^*}) = (\tau_f, \psi_f)_f = (\psi_f, \tau_f)_f = f(\langle \psi, \tau \rangle_{E^*}) = f(\langle \psi, \tau \rangle_{E^*}^*).\) Hence, \(\langle \tau, \psi \rangle_{E^*} = \langle \psi, \tau \rangle_{E^*}^*\).
For (ii), let $\tau, \psi$ be in $E^\#$, $b$ in $A$, $f$ in $P$, we define a functional $f_b : A \to \mathbb{C}$ by

$$f_b(a) = f(ab) \quad (a \in A),$$

then $f_b$ is in $A_*$. And so $f_b = \sum_{j=1}^4 \lambda_j f_j$ for some $f_j$ in $P$, $\lambda_j$ in $\mathbb{C}$. Let $g = f + \sum_{j=1}^4 f_j$, then $g$ is in $P$ and $g \geq f$, $f_j$ for $j = 1, 2, 3, 4$. We then have

$$f(\langle \tau, \psi \rangle_{E^\#} b) = \sum_{j=1}^4 \lambda_j f_j (\langle \tau, \psi \rangle_{E^\#})$$

$$= \sum_{j=1}^4 \lambda_j (\tau f_j, \psi f_j) f_j$$

$$= \sum_{j=1}^4 \lambda_j (V_{g,f_j} \tau, \psi f_j) f_j.$$  

Notice that for all $x$ in $E$,

$$\sum_{j=1}^4 \lambda_j (V_{g,f_j} (x + N_g), \psi f_j) f_j = \sum_{j=1}^4 \lambda_j (x + N_{f_j}, \psi f_j) f_j$$

$$= \sum_{j=1}^4 \lambda_j f_j (\psi(x)^*)$$

$$= f_b (\psi(x)^*)$$

$$= f (\psi(x)^* b)$$

$$= f (b^* \psi(x))$$

$$= f ((\psi b)(x))$$

$$= (\psi b)_f (x + N_f) f$$

$$= (x + N_f, (\psi b)_f) f$$

$$= (V_{g,f} (x + N_g), (\psi b)_f) f.$$  

Since $E / N_g$ is dense in $H_g$ with respect to $\| \cdot \|_g$, we have

$$f(\langle \tau, \psi \rangle_{E^\#} b) = \sum_{j=1}^4 \lambda_j (V_{g,f_j} (\tau g), \psi f_j) f_j$$

$$= (V_{g,f} (\tau g), (\psi b)_f) f$$

$$= (\tau f, (\psi b)_f) f$$

$$= f (\langle \tau, \psi \rangle_{E^\#})$$.
for any \( f \) in \( P \). Hence, \( \langle \tau, \psi \rangle_{E^*}^b = \langle \tau, \psi b \rangle_{E^*} \). Therefore, \( \langle \cdot, \cdot \rangle_{E^*} \) is an A-valued inner product.

We are going to show that \( \langle \cdot, \cdot \rangle_{E^*} \) is an extension of \( \langle \cdot, \cdot \rangle \). Let \( x, y \in E \), \( f \in P \). Then \( f(\langle x, y \rangle_{E^*}) = \langle \tilde{x}, \tilde{y} \rangle_f = (x + N_f, y + N_f)_f = f(\langle x, y \rangle) \). Hence, we have \( \langle \tilde{x}, \tilde{y} \rangle_{E^*} = \langle x, y \rangle \).

Moreover, for \( x \) in \( E \), \( \tau \) in \( E^f \), \( f \) in \( P \), we have \( f(\langle \tau, \tilde{x} \rangle_{E^*}) = \langle \tau_f, \tilde{x}_f \rangle_f = (\tau_f, x + N_f)_f = f(\tau(x)) \). Hence, \( \langle \tau, \tilde{x} \rangle_{E^*} = \tau(x) \).

It remains to show that \( \{E^f, \langle \cdot, \cdot \rangle_{E^*} \} \) is self-dual. Let \( \phi \) be in \( (E^f)^f \). Notice that \( \phi|_E \) is in \( E^f \), that is there exists an element \( \tau \) in \( E^f \) such that \( \phi(\tilde{x}) = \tau(x) \) for all \( x \) in \( E \). Define \( \phi_0 \) be in \( (E^f)^f \) by

\[
\phi_0(\psi) = \phi(\psi) - \langle \tau, \psi \rangle_{E^*} \quad (\psi \in E^f).
\]

Then we have \( \phi_0(E) = \{0\} \). If we can prove that \( \phi_0 = 0 \), then the result follows. For this purpose, let \( \psi \in E^f \) and \( f \in P \), then there exists a sequence \( (y_n + N_f)_{n \in \mathbb{N}} \) in \( \frac{E}{N_f} \) which converges to \( \psi_f \). Now, since \( \phi_0 \) is in \( (E^f)^f \), by Theorem 1.1.9, there exists \( K \geq 0 \) such that \( \phi_0(\sigma)^* \phi_0(\sigma) \leq K \langle \sigma, \sigma \rangle_{E^*} \) for all \( \sigma \) in \( E^f \). Therefore, for all \( n \) in \( \mathbb{N} \), we have

\[
f(\phi_0(\psi)^* \phi_0(\psi))
= f(\phi_0(\psi - \tilde{y}_n)^* \phi_0(\psi - \tilde{y}_n))
\leq K f((\psi - \tilde{y}_n, \psi - \tilde{y}_n)_{E^*})
= K[(\psi_f, \psi_f)_f - (y_n + N_f, y_n + N_f)_f - (y_n + N_f, y_n + N_f)_f]
= K\|\psi_f - (y_n + N_f)\|_f^2
\to 0.
\]

Therefore, we have \( f(\phi_0(\psi)^* \phi_0(\psi)) = 0 \) for all \( f \) in \( P \). This implies that \( |f(\phi_0(\psi))| \leq \|f\| \frac{1}{2} f(\phi_0(\psi)^* \phi_0(\psi))^{\frac{1}{2}} = 0 \), that is, \( f(\phi_0(\psi)) = 0 \) for all \( f \) in \( P \). Hence \( \phi_0(\psi) = 0 \), and thus \( \phi_0 = 0 \). □
Remarks 3.1.4

(i) The operator norm $\| \cdot \|$ and the norm $\| \cdot \|_{E^*}$ induced by the $A$-valued inner product $\langle \cdot, \cdot \rangle_{E^*}$ coincide on $E^*$. In fact, let $\tau$ be in $E^*$ and $x$ in $E$,

$$
\tau(x)^* \tau(x) = \langle \hat{x}, \tau \rangle_{E^*} \langle \tau, \hat{x} \rangle_{E^*} \\
\leq \|\tau\|^2_{E^*} \|\hat{x}, \hat{x}\rangle_{E^*} \\
= \|\tau\|^2_{E^*} \langle x, x \rangle
$$

which implies that $\|\tau\| \leq \|\tau\|_{E^*}$ by Remark 1.1.10. On the other hand, notice that $\|\tau_f\| \leq \|\tau\| \|f\|^{\frac{1}{2}}$ for all $f$ in $P$. Hence, we have

$$
\|\tau\|^2_{E^*} = \|\langle \tau, \tau \rangle_{E^*}\| \\
= \sup \{ f(\langle \tau, \tau \rangle_{E^*}) \mid f \in P, \|f\| = 1 \} \\
= \sup \{ (\tau_f, \tau_f) \mid f \in P, \|f\| = 1 \} \\
= \sup \{ \|\tau_f\|^2 \mid f \in P, \|f\| = 1 \} \\
\leq \|\tau\|^2
$$

which implies $\|\tau\|_{E^*} \leq \|\tau\|$.

(ii) Let $A$ be a $C^*$-algebra. Recall that a $C^*$-algebra $A$ is said to be monotone complete if each bounded increasing net in $A_{sa}$ has a least upper bound in $A_{sa}$, where $A_{sa}$ is the set of all self-adjoint elements in $A$. For every Hilbert $A$-module $E$, the $A$-valued inner product $\langle \cdot, \cdot \rangle$ on $E$ can be extended to an $A$-valued inner product $\langle \cdot, \cdot \rangle_{E^*}$ on $E^*$ turning $\{E^*, \langle \cdot, \cdot \rangle_{E^*} \}$ into a self-dual Hilbert $A$-module if and only if $A$ is monotone complete. Moreover, the equalities

$$
\langle \hat{x}, \hat{y} \rangle_{E^*} = \langle x, y \rangle, \quad \langle \tau, \hat{x} \rangle_{E^*} = \tau(x)
$$

are satisfied for every $x, y$ in $E$ and $\tau$ in $E^*$.

This result is due to M. Frank ([4]).
3.2 Extension of operators to $E^\#$

Throughout this section, $A$ is still a von Neumann algebra.

**Theorem 3.2.1** Let $E$ be a Hilbert $A$-module, $E_0$ be a Hilbert submodule of $E$. For each $\varphi \in E_0^\#$, there exists an extension $\tilde{\varphi}$ in $E^\#$ of $\varphi$ such that $\|\tilde{\varphi}\| = \|\varphi\|.$

**Proof:** Define $\varphi' : E_0^\# \to A$ by

$$\varphi'(\tau) = \langle \varphi, \tau \rangle_{E_0^\#} \quad (\tau \in E_0^\#).$$

Then $\varphi'$ is an $A$-linear map and $\|\varphi'\| = \|\varphi\|$ by Remark 3.1.4(i). Next define $P : E \to E_0^\#$ by

$$P_x(h) = \langle x, h \rangle \quad (x \in E, h \in E_0),$$

then $P$ is an $A$-linear map.

We set $\tilde{\varphi} = \varphi' \circ P : E \to A$, then $\tilde{\varphi}$ is an $A$-linear map. Notice that for $h$ in $E_0$,

$$\tilde{\varphi}(h) = \varphi' \circ P(h) = \langle \varphi, P(h) \rangle_{E_0^\#} = \varphi(h).$$

Thus $\tilde{\varphi}$ is an extension of $\varphi$.

To show that $\|\tilde{\varphi}\| = \|\varphi\|$, we first notice that for all $x$ in $E$,

$$\|\tilde{\varphi}(x)\| = \|\varphi' \circ P(x)\|$$

$$= \|\langle \varphi, P(x) \rangle_{E_0^\#}\|$$

$$\leq \|\varphi\| \|P(x)\|$$

$$= \|\varphi\| \sup\{\|P(x)(h)\| \mid h \in E_0, \|h\| = 1\}$$

$$= \|\varphi\| \sup\{\|\langle x, h \rangle\| \mid h \in E_0, \|h\| = 1\}$$

$$\leq \|\varphi\| \|x\|. $$
This implies that \( \|\tilde{\varphi}\| \leq \|\varphi\| \). On the other hand, since \( \tilde{\varphi} \) is an extension of \( \varphi \), we have

\[
\|\tilde{\varphi}\| = \sup\{\|\tilde{\varphi}(x)\| \mid x \in E, \|x\| = 1\} \\
\geq \sup\{\|\tilde{\varphi}(x)\| \mid x \in E_0, \|x\| = 1\} \\
= \|\varphi\|.
\]

Hence, \( \|\tilde{\varphi}\| = \|\varphi\| \).

\[\Box\]

**Proposition 3.2.2** Let \( E, F \) be pre-Hilbert \( A \)-modules and \( t \in \mathcal{B}(E, F) \). Then there exists a unique extension \( \tilde{t} \) in \( \mathcal{B}(E^#, F^#) \) of \( t \). Moreover, \( \|\tilde{t}\| = \|t\| \).

**Proof:** First, we define \( t' : F \to E^# \) by

\[
(t'y)(x) = \langle y, tx \rangle_F \quad (x \in E, y \in F).
\]

Notice that \( \|(t'y)(x)\| \leq \|t\| \|x\|_E \|y\|_F \) for all \( x \in E, y \in F \). By Remark 3.1.4(i), we have \( \|t'y\|_{E^#} \leq \|t\| \|y\|_F \) for all \( y \in F \). Thus, \( t' \) is bounded with \( \|t'\| \leq \|t\| \).

Notice also that for all \( a \in A, x \in E, y \in F \), we have

\[
(t'(ya))(x) = \langle ya, tx \rangle_F \\
= a^* \langle y, tx \rangle_F \\
= a^*(t'(y)(x)) \\
= (t'y)a(x).
\]

Hence, \( t' \) is in \( \mathcal{B}(F, E^#) \).

Now, similarly, we define \( \tilde{t} : E^# \to F^# \) by

\[
(\tilde{t}\tau)(y) = \langle \tau, t'y \rangle_{E^#} \quad (\tau \in E^#, y \in F).
\]
In fact, \( \tilde{t} = (t')' \), so we have \( \tilde{t} \) in \( \mathcal{B}(E^*, F^*) \) with \( ||\tilde{t}|| \leq ||t'|| \). For all \( x \) in \( E \), \( y \) in \( F \), we have

\[
(\tilde{t}\hat{x})(y) = \langle \hat{x}, t'y \rangle_{E^*} = (t'y)(x)^* = \langle y, tx \rangle_F = \langle tx, y \rangle_F = \tilde{t}x(y).
\]

Hence, we have \( \tilde{t}(\hat{x}) = \tilde{t}x \) for all \( x \) in \( E \), that is, \( \tilde{t} \) is an extension of \( t \).

Notice that, by above, \( ||\tilde{t}|| \leq ||t'|| \leq ||t|| \), and on the other hand, \( \tilde{t} \) is an extension of \( t \) and so \( ||\tilde{t}|| \geq ||t|| \). Hence, we have \( ||\tilde{t}|| = ||t|| \).

It remains to show that this extension is unique. It suffices to show that if \( V \) is in \( \mathcal{B}(E^#, F^#) \) with \( V(\tilde{E}) = \{0\} \), then we have \( V = 0 \). Now since \( \{E^#, \langle \cdot, \cdot \rangle_{E^#}\} \) is self-dual, by Proposition 2.1.3, \( V \) is in \( \mathcal{L}(E^#, F^#) \), that is \( V \) is adjoinable. Therefore for \( \psi \) in \( F^# \), \( x \) in \( E \), we have

\[
(V^*\psi)(x) = \langle V^*\psi, \hat{x} \rangle_{E^*} = \langle \psi, V\hat{x} \rangle_{E^*} = 0.
\]

Thus, \( V^* = 0 \), that is \( V = 0 \). \( \Box \)

**Corollary 3.2.3** Let \( E \) be a pre-Hilbert \( \mathcal{A} \)-module. Each element \( t \) in \( \mathcal{L}(E) \) can be extended to a unique \( \tilde{t} \) in \( \mathcal{L}(E^#) \). Moreover, the map \( \mathcal{L}(E) \rightarrow \mathcal{L}(E^#) \) defined by

\[
t \mapsto \tilde{t}
\]

is an 1-1 *-homomorphism.
Proof: Let \( t \) be in \( \mathcal{L}(E) \), by Proposition 3.2.2, there exists a unique extension \( \tilde{t} \) in \( \mathcal{B}(E^\#) \) and \( \|\tilde{t}\| = \|t\| \). Since \( \{E^\#, \langle \cdot, \cdot \rangle_{E^\#}\} \) is self-dual, we have \( \tilde{t} \) is in \( \mathcal{L}(E^\#) \) by Proposition 2.1.3.

We now consider the map : \( \mathcal{L}(E) \to \mathcal{L}(E^\#) \) defined by

\[
 t \mapsto \tilde{t}.
\]

Clearly, this map is linear. Moreover, for \( t, s \) in \( \mathcal{L}(E) \), \( \tilde{ts} \) and \( \tilde{(t^*)} \) are extensions of \( ts \) and \( t^* \) respectively. So, we have \( \tilde{ts} = \tilde{t}s \) and \( \tilde{(t^*)} = \tilde{t^*} \) by the uniqueness of the extension. Thus this map is a *-homomorphism. Finally, notice that this map is an isometry. Hence, this map is one-to-one. \( \square \)

### 3.3 Self-dual Hilbert W*-modules

It is well known that every von Neumann algebra has a predual. It is natural to ask whether every Hilbert W*-module has a predual. In this section, we prove that a self-dual Hilbert W*-module \( E \) is a conjugate space, that is, it has a predual ([10]). Throughout this section, \( A \) denotes a von Neumann algebra.

**Proposition 3.3.1** Let \( E \) be a self-dual Hilbert \( A \)-module. Then \( E \) is a conjugate space.

**Proof:** Let \( Y \) be the linear space \( E \) with "twisted" scalar multiplication, that is, \( \lambda \cdot x = \bar{\lambda}x \) for \( \lambda \) in \( \mathbb{C} \) and \( x \) in \( Y \). Consider \( A \otimes Y \) the tensor product with the greatest cross-norm. For each \( x \) in \( E \), define the map \( \tilde{x} : A \otimes Y \to \mathbb{C} \) by

\[
\tilde{x}(\sum_{j=1}^{n} f_j \otimes y_j) = \sum_{j=1}^{n} f_j(\langle y_j, x \rangle)
\]

for \( f_1, \cdots, f_n \) in \( A \), \( y_1, \cdots, y_n \) in \( Y \).
Notice that \( \bar{x} \) is a well-defined bounded linear functional with \( \| \bar{x} \| = \| x \|_E \).

The reason is that

\[
|\bar{x}(\sum_{j=1}^{n} f_j \otimes y_j)| \leq \|x\|_E \sum_{j=1}^{n} \|f_j\| \|y_j\|_E
\]

for all \( f_1, \ldots, f_n \) in \( A_* \), \( y_1, \ldots, y_n \) in \( X \) and thus \( \| \bar{x} \| \leq \| x \|_E \). Also, let \( \{g_n\} \) be a sequence of bounded linear functionals of norm 1 in \( A_* \) such that \( |g_n(\langle x, x \rangle)| \to \|x\|_E^2 \). Notice that \( g_n \otimes x \) is in \( A_* \otimes X \), \( \|g_n \otimes x\| = \|x\|_E \) and \( \bar{x}(g_n \otimes x) \to \|x\|_E^2 \).

Hence we have \( \|x\|_E \leq \| \bar{x} \| \).

Thus, the map \( \bar{x} : E \to (A_* \otimes X)^* \) defined by

\[
x \mapsto \bar{x}
\]

is a linear isometry.

It remains to show that \( \tilde{E} \) is weak*-closed in \( (A_* \otimes X)^* \), since then \( \tilde{E} \) is isometric to the dual space of a quotient space of \( A_* \otimes X \). Let \( \{x_\alpha\} \) be a net in \( \tilde{E} \) weak* convergent to some \( F \) in \( (A_* \otimes X)^* \). We claim that \( F \) is in \( \tilde{E} \) and then the result follows. To do this, for each \( y \) in \( E \), define \( \psi_y : A_* \to \mathbb{C} \) by

\[
\psi_y(g) = F(g \otimes y) \quad (g \in A_*).
\]

Notice that \( \psi_y \) is a bounded linear functional on \( A_* \) with norm not exceeding \( \|F\| \|y\|_E \). Since \( (A_*)^* = A \), there exists an element \( \tau(y) \) in \( A \) such that

\[
\|\tau(y)\| \leq \|F\| \|y\|_E
\]

and

\[
F(g \otimes y) = \psi_y(g) = g(\tau(y)^*)
\]

for all \( g \) in \( A_* \). Consider the map \( \tau : E \to A \) defined by

\[
y \mapsto \tau(y).
\]

Notice that \( \tau \) is bounded linear. Also, \( \tau \) is \( A \)-linear. The reason is that if we let \( y \) be in \( E \), \( a \) in \( A \), \( f \) in \( A_* \), define \( g : A \to \mathbb{C} \) by

\[
g(b) = f(a^* b) \quad (b \in A),
\]
Hilbert $C^*\text{-modules}$

then we have

$$f(\tau(ya)^*) = F(f \otimes ya)$$

$$= \lim_{a} x_{a}^{-1}(f \otimes ya)$$

$$= \lim_{a} f((ya, x_{a}))$$

$$= \lim_{a} g((y, x_{a}))$$

$$= F(g \otimes y)$$

$$= g(\tau(y)^*)$$

$$= f(a^* \tau(y)^*).$$

Thus, $\tau(ya) = \tau(y)a$. Hence, $\tau$ is in $E^\#$. Since $E$ is self-dual, there exists an $x_{0}$ in $E$ such that

$$\tau(y) = \langle x_{0}, y \rangle \quad (y \in E),$$

then $F = x_{0}$ is in $\bar{E}$.

Remark 3.3.2 Let $\mathcal{T}_{E}$ denote the weak*-topology on $E$ defined in Proposition 3.3.1. Notice that a bounded net $\{x_{a}\}$ in $E$ converges to $x$ in $E$ with respect to $\mathcal{T}_{E}$ if and only if $f((y, x_{a})) \to f((y, x))$ for all $f$ in $A_{*}$ and $y$ in $E$.

Proposition 3.3.3 Let $E$ be a self-dual Hilbert $A$-module. Then $\mathcal{L}(E)$ is a von Neumann algebra.

Proof: By Lemma 1.1.12 (iv), $\mathcal{L}(E)$ is a $C^*$-algebra. So, it suffices to show that $\mathcal{L}(E)$ is a conjugate space. We let $Y$ be the linear space $E$ with "twisted" scalar multiplication (that is $\lambda \cdot y = \bar{\lambda}y$ for $\lambda$ in $\mathbb{C}$ and $y$ in $Y$). Consider $E \otimes Y \otimes A_{*}$ the tensor product with the greatest cross-norm. For each $t$ in $\mathcal{L}(E)$, define a map $\hat{t} : E \otimes Y \otimes A_{*} \to \mathbb{C}$ by

$$\hat{t}(\sum_{j=1}^{n} x_{j} \otimes y_{j} \otimes g_{j}) = \sum_{j=1}^{n} g_{j}((y_{j}, tx_{j}))$$
for $x_1, \ldots, x_n$ in $E$, $y_1, \ldots, y_n$ in $Y$, and $g_1, \ldots, g_n$ in $A$.

Notice that $\tilde{t}$ is a well-defined bounded linear functional on $E \otimes Y \otimes A$, with $||\tilde{t}|| = ||t||_{L(E)}$. The reason is that

$$||\tilde{t}|| \leq ||t||_{L(E)} \sum_{j=1}^{n} ||g_j|| ||y_j||_E ||x_j||_E$$

for all $x_1, \ldots, x_n$ in $E$, $y_1, \ldots, y_n$ in $Y$ and $g_1, \ldots, g_n$ in $A$, which means that $||\tilde{t}|| \leq ||t||_{L(E)}$. On the other hand, fix an element $x$ in $E$. Let $\{g_n\}$ be a sequence of bounded linear functionals of norm 1 in $A$ such that $|g_n(\langle tx, tx \rangle)| \to ||tx||^2_E$.

Notice that $x \otimes tx \otimes g_n$ is in $E \otimes Y \otimes A$, $||x \otimes tx \otimes g_n|| = ||x||_E ||tx||_E$ and $||\tilde{t}(x \otimes tx \otimes g_n)|| = |g_n(\langle tx, tx \rangle)| \to ||tx||^2_E$. Thus we have $||\tilde{t}|| \geq \frac{||tx||_E}{||x||_E}$ for all $x$ in $E$, that is, $||\tilde{t}|| \geq ||t||_{L(E)}$.

Hence the map $\gamma : L(E) \to (E \otimes Y \otimes A)^*$ defined by

$$t \mapsto \tilde{t}$$

is a linear isometry.

To complete the proof, it suffices to show that $(L(E))^\prime$ is weak*-closed in $(E \otimes Y \otimes A)^*$, since then $(L(E))^\prime$ is isometric to the dual space of a quotient space of $E \otimes Y \otimes A$. Let $\{t_\alpha\}$ be a net in $L(E)$ with $\{t_\alpha\}$ weak* convergent to $\phi$ in $(E \otimes Y \otimes A)^*$. We need to claim that $\phi$ is in $(L(E))$. Now, for each $x$, $y$ in $E$, define a map $\tau_{x,y} : A \to \mathbb{C}$ by

$$\tau_{x,y}(g) = \phi(x \otimes y \otimes g) \quad (g \in A).$$

Notice that $\tau_{x,y}$ is a bounded linear functional on $A$ with norm not greater than $||\phi|| ||x||_E ||y||_E$. Since $(A)^* = A$, there exists an $\tau_x(y)$ in $A$ such that

$$||\tau_x(y)|| \leq ||\phi|| ||x||_E ||y||_E$$

and

$$\phi(x \otimes y \otimes g) = g(\tau_x(y))$$
for all $g$ in $A_\ast$.

Now we claim that for $x, y$ in $E$, $a$ in $A$, we have $\tau_{xa}(y) = \tau_x(y)a$ and $\tau_x(ya) = a^*\tau_x(y)$. Take $f$ in $A_\ast$, define $g$ in $A_\ast$ by $g(b) = f(ba)$, for $b$ in $A$. Then

$$f(\tau_{xa}(y)) = \phi(xa \otimes y \otimes f) = \lim_{a} \tilde{t}_a(xa \otimes y \otimes f) = \lim_{a} f(\langle y, t_\alpha(xa) \rangle) = \lim_{a} f(\langle y, t_\alpha x \rangle a) = \lim_{a} g(\langle y, t_\alpha x \rangle) = \lim_{a} \tilde{t}_a(x \otimes y \otimes g) = \phi(x \otimes y \otimes g) = g(\tau_x(y)) = f(\tau_x(y)a).$$

Since this holds for all $f$ in $A_\ast$, we have $\tau_{xa}(y) = \tau_x(y)a$. Similarly, we can prove that $\tau_x(ya) = a^*\tau_x(y)$.

So, if for each $y$ in $E$, we define a map $: E \to A$ by

$$x \mapsto \tau_x(y),$$

then this map is a bounded $A$-linear map. Since $E$ is self-dual, there exists a unique element $Uy$ in $E$ satisfying

$$\tau_x(y) = \langle Uy, x \rangle \quad (x \in E).$$

Now, we consider the map $U : E \to E$ defined by

$$y \mapsto Uy.$$
Notice that $U$ is $A$-linear. In fact, for $x, y$ in $E$, $a$ in $A$,

$$
\langle U(ya), x \rangle = \tau_x(ya)
$$

$$
= a^* \tau_x(y)
$$

$$
= a^* \langle Uy, x \rangle
$$

$$
= \langle (Uy)a, x \rangle
$$

and thus $U(ya) = (Uy)a$.

Moreover, $U$ is bounded. Indeed,

$$
\|Uy\|_E = \|\langle Uy, Uy \rangle\|
$$

$$
= \|\tau_{Uy}(y)\|
$$

$$
\leq \|\phi\| \|Uy\|_E \|y\|_E
$$

for all $y$ in $E$. Hence $U$ is in $\mathcal{B}(E) = \mathcal{L}(E)$, by Proposition 2.1.3. So, $U^*$ exists, and we have $\phi = (U^*)$ which is in $(\mathcal{L}(E))^\prime$. \hfill \Box

The following result is the polar decomposition for Hilbert module setting.

**Proposition 3.3.4** Let $E$ be a self-dual Hilbert $A$-module. For each $x$ in $E$, it can be written as $x = u\langle x, x \rangle^{\frac{1}{2}}$ where $u$ is in $E$ such that $\langle u, u \rangle$ is the range projection of $\langle x, x \rangle^{\frac{1}{2}}$, that is, $\langle u, u \rangle$ is the least projection of all the projections $p$ in $A$ such that $\langle x, x \rangle^{\frac{1}{2}} p = p \langle x, x \rangle^{\frac{1}{2}} = \langle x, x \rangle^{\frac{1}{2}}$. This decomposition is unique in the sense that if $x = vb$ where $b \geq 0$ and $\langle v, v \rangle$ is the range projection of $b$, then $v = u$ and $b = \langle x, x \rangle^{\frac{1}{2}}$.

**Proof:** Let $x$ be in $E$, for each $n$ in $\mathbb{N}$, define

$$
b_n = (\langle x, x \rangle + n^{-1})^{\frac{1}{2}}
$$

and

$$
x_n = xb_n^{-1}.
$$
Notice that $\langle x_n, x_n \rangle = \langle x, x \rangle (\langle x, x \rangle + n^{-1})^{-1}$ and so $\|x_n\|_E \leq 1$. Now, consider the closure of convex hull of $\{x_n : n \in \mathbb{N}\}$ which is closed, bounded convex subset of $E$. By Banach-Alaoglu Theorem, it is $T_E$-compact, where $T_E$ is the weak*-topology which is defined in Remark 3.3.2. Hence $\{x_n : n \in \mathbb{N}\}$ has a $T_E$-accumulation point in $E$, says $y$. Notice that

$$\|b_n - \langle x, x \rangle^{\frac{1}{2}}\| \to 0$$

and

$$x_nb_n = x$$

for all $n$ in $\mathbb{N}$. So, we have $x = y\langle x, x \rangle^{\frac{1}{2}}$.

Now, let $p$ be the range projection of $\langle x, x \rangle^{\frac{1}{2}}$, we have $\langle x, x \rangle^{\frac{1}{2}}p = p\langle x, x \rangle^{\frac{1}{2}} = \langle x, x \rangle^{\frac{1}{2}}$ which implies that $x = yp\langle x, x \rangle^{\frac{1}{2}}$ and $\langle x, x \rangle = \langle x, x \rangle^{\frac{1}{2}}p(y, y)p\langle x, x \rangle^{\frac{1}{2}}$. Hence, $\langle x, x \rangle^{\frac{1}{2}}(p-p(y, y)p)\langle x, x \rangle^{\frac{1}{2}} = 0$. Notice that $\|x_n\|_E \leq 1$ for all $n$ in $\mathbb{N}$ which implies $\|y\|_E \leq 1$ and thus $p-p(y, y)p \geq 0$. Hence if $\langle x, x \rangle^{\frac{1}{2}}(p-p(y, y)p)\langle x, x \rangle^{\frac{1}{2}} = 0$, then $\langle x, x \rangle^{\frac{1}{2}}(p-p(y, y)p)^{\frac{1}{2}} = 0$. From this we have $p(p-p(y, y)p)^{\frac{1}{2}} = 0$ since $p$ is range projection of $\langle x, x \rangle^{\frac{1}{2}}$. This forces $p(p-p(y, y)p)p = 0$, that is $p = p(y, y)p$.

Now let $u = yp$, then $u\langle x, x \rangle^{\frac{1}{2}} = yp\langle x, x \rangle^{\frac{1}{2}} = x$ and $\langle u, u \rangle = p(y, y)p = p$.

Next, we want to prove the uniqueness of the decomposition. Suppose $x = vb$ where $b \geq 0$ and $\langle v, v \rangle$ is the range projection of $b$. We need to claim that $v = u$ and $b = \langle x, x \rangle^{\frac{1}{2}}$. First notice that $\langle x, x \rangle = b\langle v, v \rangle b = b^2$. Then we have $b = \langle x, x \rangle^{\frac{1}{2}}$ and $\langle v, v \rangle = p$. Since $\langle v - vp, v - vp \rangle = p - p - p + p = 0$, we see that $v = vp$. Similarly, we have $u = up$. We note also that

$$\langle x, u \rangle = \langle u\langle x, x \rangle^{\frac{1}{2}}, u \rangle$$

$$= \langle x, x \rangle^{\frac{1}{2}} \langle u, u \rangle$$

$$= \langle x, x \rangle^{\frac{1}{2}} p.$$
On the other hand, we have
\[
\langle x, u \rangle = \langle vb, u \rangle \\
= b\langle v, u \rangle \\
= \langle x, x \rangle^{\frac{1}{2}}\langle v, u \rangle.
\]
From this if \( \langle x, x \rangle^{\frac{1}{2}}(\langle v, u \rangle - p) = 0 \), then \( \langle vp, u \rangle - p = 0 \), that is \( \langle v, u \rangle - p = 0 \).
Hence, \( \langle v - u, v - u \rangle = p - p - p + p = 0 \). This completes the proof. \qed

3.4 Some equivalent conditions for a Hilbert \( \mathcal{W}^* \)-module to be self-dual

Throughout this section, \( A \) still denotes a von Neumann algebra.

Definition 3.4.1 Let \( E \) be a pre-Hilbert \( A \)-module. Let \( P_1 \) be the set of all normal states on \( A \). The topology on \( E \) induced by the semi-norms
\[
f((\cdot, \cdot))^{\frac{1}{2}} \quad (f \in P_1)
\]
is denoted by \( \tau_1 \). Moreover, the topology induced on \( E \) by the linear functionals
\[
f((\langle y, \cdot \rangle)) \quad (f \in P_1, y \in E)
\]
is denoted by \( \tau_2 \).

Remarks 3.4.2

(i) The topology \( \tau_2 \) is same as \( \mathcal{T}_E \) defined in Remark 3.3.2.

(ii) In the case of \( A = \mathbb{C} \) and so \( E \) is a Hilbert space, \( \tau_1 \) is the norm-topology on \( E \) and \( \tau_2 \) is the weak*-topology on \( E \). So, \( \tau_1 \) and \( \tau_2 \) do not coincide, in general.
Lemma 3.4.3 Let $E$ be a Hilbert $A$-module. For each $\tau$ in $E^*$, there exists a net $\{x_\alpha\}$ in $E$ such that $x_\alpha \tau_1$-converges to $\tau$.

Proof: Let $\tau$ be an element in $E^*$. Let $s = \{f_1, f_2, \ldots, f_n\}$ be a finite subset of $\mathcal{P}_1$. We set $f = f_1 + f_2 + \cdots + f_n$. Notice that $f$ is a normal positive linear functional on $A$. Using the same notation as in Section 3.1, there exists a unique element $\tau_f$ in $H_f$ with $\|\tau_f\| \leq \|f\|^{\frac{1}{2}} \|\tau\|$ such that

$$(\tau_f, x + N_f) = f(\tau(x)) \quad (x \in E)$$

and there exists an element $x_s$ in $E$ satisfying

$$(x_s + N_f - \tau_f, x_s + N_f - \tau_f) \leq \frac{1}{n}.$$ 

Notice that in the proof of Theorem 3.1.3, we have

$$(\tau_f, \tau_f) = f((\tau, \tau)_{E^*}).$$

Hence we have $f((\tau - x_s, \tau - x_s)_{E^*}) = (x_s + N_f - \tau_f, x_s + N_f - \tau_f) \leq \frac{1}{n}$. Therefore, $f_i((\tau - x_s, \tau - x_s)_{E^*}) \leq \frac{1}{n}$ for all $i = 1, 2, \cdots, n$. Consequently, the net $\{x_s\}$ which is indexed by the inclusion-directed family of finite subset of $\mathcal{P}_1$ has the required property. \hfill \square

The following theorem is the main result of this section:

Theorem 3.4.4 Let $E$ be a Hilbert $A$-module. Then the following conditions are equivalent:

(i) $E$ is self-dual.

(ii) $E$ is $A$-reflexive.

(iii) The unit ball of $E$ is $\tau_1$-complete.

(iv) The unit ball of $E$ is $\tau_2$-complete.
(v) there exists a collection \( \{ p_\alpha \}_{\alpha \in I} \) of (not necessary distinct) non-zero projections of \( A \) such that \( E \) and \( \bigoplus p_\alpha A \) (c.f.: Example 1.1.7) are unitarily isomorphic, where \( p_\alpha A \) is a Hilbert \( A \)-module with inner product \( \langle p_\alpha a, p_\alpha b \rangle = a^* p_\alpha b \).

Moreover, in this case, the linear span of the completion of the unit ball of \( E \) with respect to \( \tau_1 \) coincides with \( E^\# \).

**Proof:** (i) \( \Rightarrow \) (ii) follows from Proposition 2.2.6. (ii) \( \Rightarrow \) (i): Let \( \tau \) be in \( E^\# \). Consider the map \( f : E^\# \rightarrow A \) defined by

\[
 f(\sigma) = \langle \tau, \sigma \rangle_{E^\#} \quad (\sigma \in E^\#).
\]

Then \( f \) is in \( E^{\#\#} \). By the reflexivity of \( E \), there exists \( y \in E \) such that

\[
 f(\sigma) = (q(y))(\sigma) = (\sigma(y))^* 
\]

for all \( \sigma \) in \( E^\# \), that is, \( \langle \tau, \sigma \rangle_{E^\#} = (\sigma(y))^* \) for all \( \sigma \) in \( E^\# \). In particular, for all \( x \) in \( E \), we have

\[
 \tau(x) = \langle \tau, x \rangle_{E^\#} \\
= (\bar{x}(y))^* \\
= \langle y, x \rangle \\
= \bar{y}(x).
\]

Hence, \( \tau = \bar{y} \). Thus \( E \) is self-dual.

Now, let \( L \) be the linear span of the completion of the unit ball of \( E \) with respect to \( \tau_1 \). (i) \( \Rightarrow \) (iii): Assume that \( E \) is self-dual. We prove by contradiction. Suppose that the unit ball of \( E \) is not complete with respect to \( \tau_1 \). Then there exist an element \( y \) in \( L \backslash E \), and a norm-bounded net \( \{ y_\alpha \}_{\alpha \in I} \) in \( E \) such that \( y_\alpha \rightarrow y \) with respect to \( \tau_1 \). Now, fix an element \( f \) in \( P_1 \) and \( \epsilon > 0 \), there exists an \( \alpha \) in \( I \) such that

\[
f((y - y_\beta, y - y_\beta)) < \epsilon
\]
for all $\beta \geq \alpha$. Let $x \in E$, we have
\[ |f(\langle y_\beta, x \rangle) - f(\langle y_\gamma, x \rangle)| = |f(\langle y_\beta - y_\gamma, x \rangle)| \]
\[ \leq f(\langle y_\beta - y_\gamma, y_\beta - y_\gamma \rangle)^{\frac{1}{2}} f(\langle x, x \rangle)^{\frac{1}{2}} \]
\[ \leq (2e f(\langle x, x \rangle))^{\frac{1}{2}} \]
for all $\beta, \gamma \geq \alpha$. Hence, $w^* - \lim \{\langle y_\alpha, x \rangle \mid \alpha \in I\}$ exists and is denoted by $R(x)$. Similarly, we can easily prove that $y$ is the $\tau_2$-limit of a norm-bounded net $\{y_\alpha\}_{\alpha \in I}$. We consider the map $R : E \to A$ defined by
\[ x \mapsto R(x). \]
Notice that $|f(\langle y_\beta, x \rangle)| \leq \|x\| \sup \{\|y_\alpha\| \mid \alpha \in I\}$ for all $\beta$ in $I$, we have $R$ is bounded. Clearly, $R$ is $A$-linear. Hence $R$ is in $E^\#$. So, by the assumption, there exists an element $z$ in $E$ such that $R(x) = \langle z, x \rangle$ for all $x$ in $E$, that is,
\[ w^* - \lim \{\langle y_\alpha, x \rangle \mid \alpha \in I\} = \langle z, x \rangle \]
for all $x$ in $E$. Hence $z$ is the $\tau_2$-limit of the norm-bounded net $\{y_\alpha\}_{\alpha \in I}$. Thus $y = z$ is in $E$, where the contradiction occurs. (i) $\Rightarrow$ (iii) follows.

Notice that $\{E^\#, \langle \cdot, \cdot \rangle_{E^\#}\}$ is self-dual, so by the implication (i) to (iii), we remark here that the unit ball of $E^\#$ is $\tau_1$-complete and thus $L \subseteq E^\#$. Conversely, by Lemma 3.4.3, we have $E^\# \subseteq L$ and hence $L = E^\#$.

(iii) $\Rightarrow$ (i): Assume that the unit ball of $E$ is $\tau_1$-complete. Then $E = L = E^\#$ and hence $E$ is self-dual.

(iii) $\Rightarrow$ (iv): Assume that the unit ball of $E$ is $\tau_1$-complete. Then by the implication (iii) $\Rightarrow$ (i), $E$ is self-dual. Using Proposition 3.3.1 and Remark 3.3.2, we know that $(E, \tau_2)$ is a conjugate space. Hence, the unit ball of $E$ is $\tau_2$-complete.

(iv) $\Rightarrow$ (iii): Assume that the unit ball of $E$ is $\tau_2$-complete. Let $(x_\alpha)_{\alpha \in I}$ be a norm-bounded net in $E$ which is Cauchy and $x_\alpha \to t$ in $L$ with respect to $\tau_1$. Notice that for all $y$ in $E$, $f$ in $P_1$, $\beta, \gamma$ in $I$, we have
\[ |f(\langle y, x_\beta \rangle) - f(\langle y, x_\gamma \rangle)|^2 \leq f(\langle x_\beta - x_\gamma, x_\beta - x_\gamma \rangle) f(\langle y, y \rangle). \]
So, \((x_\alpha)_{\alpha \in I}\) is \(\tau_2\)-Cauchy. By assumption, there exists an element \(x\) in \(E\) such that \(x_\alpha \tau_2\)-converges to \(x\). Notice that \(x_\alpha \to t\) with respect to \(\tau_1\). Using the same argument, we have \(x_\alpha \to t\) with respect to \(\tau_2\). Hence \(t = x\) is in \(E\). Hence the unit ball of \(E\) is \(\tau_1\)-complete.

\((v) \Rightarrow (i)\): Assume that \(E \cong \bigoplus p_\alpha A\) as Hilbert \(A\)-module. Notice that for each \(\alpha\) in \(I\), \(p_\alpha A\) is self-dual (Example 2.1.5). By Proposition 2.1.4, \(\bigoplus p_\alpha A\) is self-dual. Thus by Proposition 2.1.10, \(E\) is self-dual.

\((i) \Rightarrow (v)\): Assume that \(E\) is self-dual. Then by Proposition 3.3.4 and Zorn's lemma, there exists a subset \(\{e_\alpha : \alpha \in I\}\) of \(E\) which is maximal with respect to the following properties:

\(\langle e_\alpha, e_\alpha \rangle\) is a non-zero projection,

and \(\langle e_\alpha, e_\beta \rangle = 0\) for \(\alpha \neq \beta\).

We set \(p_\alpha = \langle e_\alpha, e_\alpha \rangle\) for each \(\alpha\) in \(I\). Notice that \(\langle e_\alpha - e_\alpha p_\alpha, e_\alpha - e_\alpha p_\alpha \rangle = 0\) which implies \(e_\alpha = e_\alpha p_\alpha\) for all \(\alpha\) in \(I\).

Now, let \(x\) be in \(E\). For each \(\alpha\) in \(I\), we have

\[
\langle e_\alpha, x \rangle = \langle e_\alpha p_\alpha, x \rangle = p_\alpha \langle e_\alpha, x \rangle \in p_\alpha A.
\]

For a finite subset \(\mathcal{F}\) of \(I\), let \(y = \sum_{\alpha \in \mathcal{F}} e_\alpha \langle e_\alpha, x \rangle\), \(z = x - y\), then

\[
\langle y, y \rangle = \langle \sum_{\alpha \in \mathcal{F}} e_\alpha \langle e_\alpha, x \rangle, \sum_{\beta \in \mathcal{F}} e_\beta (\langle e_\beta, x \rangle) \rangle
= \sum_{\alpha \in \mathcal{F}} \langle x, e_\alpha \rangle p_\alpha \langle e_\alpha, x \rangle
= \sum_{\alpha \in \mathcal{F}} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle.
\]

Also, we have

\[
\langle y, z \rangle = \langle y, x - y \rangle
= \langle \sum_{\alpha \in \mathcal{F}} e_\alpha \langle e_\alpha, x \rangle, x \rangle - \sum_{\alpha \in \mathcal{F}} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle
= 0.
\]
Thus

\[ \langle x, x \rangle = \langle y + z, y + z \rangle \]
\[ = \langle y, y \rangle + \langle z, z \rangle \]
\[ \geq \langle y, y \rangle \]

that is, for all finite subset \( \mathcal{F} \) of \( I \), we have

\[ \sum_{\alpha \in \mathcal{F}} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle \leq \langle x, x \rangle. \]

Hence \( \sum_{\alpha \in I} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle \leq \langle x, x \rangle. \)

So, the map \( T : E \to \bigoplus p_\alpha A \) defined by

\[ Tx = (\langle e_\alpha, x \rangle)_{\alpha \in I} \]

is well-defined and is in \( B(E, \bigoplus p_\alpha A) \) by Theorem 1.1.9.

We first show that \( T \) is onto. Let \( (p_\alpha a_\alpha)_{\alpha \in I} \) be in \( \bigoplus p_\alpha A \). Let \( \mathcal{F} \) be a finite subset of \( I \) and set \( y_\mathcal{F} = \sum_{\alpha \in \mathcal{F}} e_\alpha a_\alpha \). Notice that

\[ \langle y_\mathcal{F}, y_\mathcal{F} \rangle = \langle \sum_{\alpha \in \mathcal{F}} e_\alpha a_\alpha, \sum_{\beta \in \mathcal{F}} e_\beta a_\beta \rangle \]
\[ = \sum_{\alpha \in \mathcal{F}} a_\alpha^* p_\alpha a_\alpha \]
\[ \leq \sum_{\alpha \in I} (p_\alpha a_\alpha)^* (p_\alpha a_\alpha). \]

Thus \( \{y_\mathcal{F}\}_{\mathcal{F}} \) is a norm-bounded net in \( E \). Consider the closure of convex hull of \( \{y_\mathcal{F} : \mathcal{F} \} \) which is closed, bounded, convex subset of \( E \), and so by Banach-Alaoglu Theorem, it is \( T_E \)-compact, where \( T_E \) is the weak*-topology which is defined in Remark 3.3.2. Hence \( \{y_\mathcal{F} : \mathcal{F} \} \) has a \( T_E \)-accumulation point in \( E \), says \( y \). So, for all \( f \in A_* \), \( x \in E \), we have

\[ f((x, y_\mathcal{F})) \to f((x, y)), \]

in particular, we have

\[ f((e_\alpha, y_\mathcal{F})) \to f((e_\alpha, y)) \]
for all $\alpha$ in $I$. Notice that for all $\alpha$ in $I$, $\langle e_\alpha, y_f \rangle = p_\alpha a_\alpha$ for sufficiently large $F$ and thus $\langle e_\alpha, y \rangle = p_\alpha a_\alpha$. Hence $T(y) = (p_\alpha a_\alpha)_{\alpha \in I}$. Therefore, $T$ is onto.

Next, we show that $T$ is one-to-one. Let $x \in E$ with $\langle e_\alpha, x \rangle = 0$ for all $\alpha$ in $I$. Using the same notation as in Proposition 3.3.4, we have $\langle e_\alpha, x_n \rangle = \langle e_\alpha, x b_n^{-1} \rangle = 0$ for all $\alpha$ in $I$ and $n$ in $\mathbb{N}$. This implies that $y = T_E - \lim x_n$ has the property that $\langle e_\alpha, y \rangle = 0$ for all $\alpha$ in $I$. Then we have $\langle e_\alpha, u \rangle = \langle e_\alpha, y p \rangle = 0$ for all $\alpha$ in $I$. Since $u \in E$ and $\langle u, u \rangle$ is a projection of $A$, by the maximality of $\{e_\alpha : \alpha \in I\}$, we have $\langle u, u \rangle = 0$. Hence $x = 0$.

It remains to show that $\langle Tx, Tx \rangle = \langle x, x \rangle$ for all $x$ in $E$. Let $x$ be in $E$ and $F$ be a finite subset of $I$. We set $x_F = \sum_{\alpha \in F} e_\alpha \langle e_\alpha, x \rangle$. Then $\langle x_F, x_F \rangle = \sum_{\alpha \in F} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle \leq \langle x, x \rangle$ and so the net $\{x_F : F\}$ is norm-bounded. Similarly, $\{x_F : F\}$ has a $T_E$-accumulation point in $E$, says $y$. Notice that for all $\alpha$ in $I$, $\langle e_\alpha, x_F \rangle = \langle e_\alpha, x \rangle$ for sufficiently large $F$. Thus $\langle e_\alpha, y \rangle = \langle e_\alpha, x \rangle$ for all $\alpha$ in $I$. Thus $x = y$ is a $T_E$-accumulation point of $\{x_F : F\}$ by injectivity of $T$. Now, for $f$ in $A_*$, we have

$$f(\langle x, x \rangle) = \lim_{F} f(\langle x_F, x_F \rangle)$$
$$= \lim_{F} f(\langle x_F, x_F \rangle)$$
$$= \lim_{F} f(\sum_{\alpha \in F} \langle x, e_\alpha \rangle \langle e_\alpha, x \rangle)$$
$$= f(\langle Tx, Tx \rangle).$$

Hence $\langle x, x \rangle = \langle Tx, Tx \rangle$. 

$\square$
Bibliography


Hilbert C*-modules


