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Representation for Multiple Right-Hand Sides

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Representation for Multiple Right-Hand Sides

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We are given finitely many polyhedra defined by linear constraints, using the same constraint matrix and different right-hand sides. We consider a simple constraint system and give necessary and sufficient conditions for this system to define the union of the polyhedra.

Key words: formulation, linear inequalities, representation

Running title: Multiple Right-hand Sides

Let $A$ be a $m \times n$ matrix of rank $n$. For any $b \in \mathbb{R}^n$, $\{x | Ax \geq b \}$ is a polyhedron. Suppose we have several right-hand-sides $b^{(1)}, \ldots, b^{(t)}$. These give $t$ polyhedra:

$$P^{(h)} = \{x | Ax \geq b^{(h)} \} \quad 1 \leq h \leq t$$

Define:

$$Q = \text{conv} \left( \bigcup_{h=1}^{t} P^{(h)} \right)$$

$$T = \{x | Ax \geq \sum \lambda_h b^{(h)} \text{, for some } \lambda \text{ with } \sum \lambda_h = 1, \lambda_h \geq 0 \}$$

Jeroslow [1] raises the question of when $Q = T$. The motivation is that $T$ is defined using linear constraints with the auxiliary variables $\lambda_h$. Thus, when $Q = T$, the problem of maximizing a linear objective over the union of $P_h$ can be done by solving a linear program of modest size. In particular, it is not necessary to make one copy of $A$ for each $h$.

[1] gives a sufficient condition (Theorem 1 below) for $Q = T$. In this note we give a modification which is simpler and includes more cases (Theorem 2). Then we give a weaker sufficient condition (Theorem 3). If we make a nondegeneracy assumption, this condition is necessary (Theorem 4). The condition of Theorems 3–4 is not easy to verify. We show (Theorem 5) that the problem of deciding whether $Q = T$ for given $A, b^{(h)}$ is NP-Hard, which suggests that no easily verifiable necessary and sufficient condition exists.

Definitions. For $I$ a subset of the rows of $A$, $1 \leq h \leq t$, we define

$$E_{I,h} = \{x | (Ax)_i = b^{(h)}_i \text{ for all } i \in I \}$$

$$F_{I,h} = \{x | (Ax)_i \geq b^{(h)}_i \text{ for all } i \in I \}$$

When $E_{I,h}$ consists of a single vector, we define $x_{I,h}$ to be that vector. For $h$ fixed, those $x_{I,h}$ which are in $P^{(h)}$ are the extreme points of $P^{(h)}$. 
**Theorem 1** [1, Theorem 2.2]. \( Q = T \) if, for all \( x_{I,h}, x_{I,h} \notin P^{(h)} \) implies that for some \( 1 \leq j \leq m, (Ax_{I,k})_j < b^{(k)}_j \) for all \( 1 \leq k \leq t \).

**Example 1.** We let \( n = t = 2, m = 4 \) and take

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 4 \\ 4 \\ 10 \\ 5 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} 4 \\ 4 \\ 10 \end{pmatrix}
\]

It is easy to see that for both right-hand sides, the inequalities from the last two rows are redundant and that \( Q = T = \{ x | x_i \geq 4 \} \). However, Theorem 1 cannot be used. When \( I \) consists of the bottom two rows of \( A \), \( x_{I,1} = (5,0) \notin P^{(1)} \) because row 2 of \( A \) is violated, and this is the only violated row. However, \( x_{I,2} = (0,5) \) satisfies row 2 so the conditions of Theorem 1 are not satisfied.

This example suggests that the important thing is that when \( x_{I,h} \notin P^{(h)} \) for some \( h \), there must be a reason why \( x_{I,k} \notin P^{(k)} \) for all \( k \), but the reason (i.e., the violated row) may be different for different \( k \). In our example, \( x_{I,2} \) violates row 1 instead of row 2.

**Theorem 2.** \( Q = T \) if, for all \( x_{I,h}, x_{I,h} \notin P^{(h)} \) implies \( x_{I,k} \notin P^{(k)} \) for all \( 1 \leq k \leq t \).

Another way to interpret Theorem 2 is that for all \( h \), the set of \( I \) which give extreme points of \( P^{(h)} \) must be the same— the \( P^{(h)} \) must all have the same shape. Since Theorem 2 follows easily from Theorem 3, we do not give a separate proof.

To develop a necessary and sufficient condition for \( Q = T \), it helps to consider two examples in which the condition of Theorem 2 does not hold, with \( Q = T \) in one case, \( Q \neq T \) in the other.

**Example 2.** We let \( n = t = 2, m = 3 \) and take

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 14 \\ 10 \\ 2 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} 18 \\ 11 \\ 6 \end{pmatrix}
\]

**Example 3.** Same as Example 2, except

\[
b^{(2)} = \begin{pmatrix} 15 \\ 7 \\ 3 \end{pmatrix}
\]

In both examples, when we let \( I \) be the first and third rows of \( A \), \( x_{I,1} = (6,8) \notin P^{(1)} \), but \( x_{I,2} \in P^{(2)} \), so Theorem 2 cannot be used. However, it is easy to show in Example 2 that \( Q = T = P^{(1)} \), but that in Example 3, \( (6,8.5) \in T \setminus Q \) (to see that \( (6,8.5) \in T \), let \( \lambda = (.5,.5) \))

The crucial distinction between the two examples is that in Example 3, the “problem vector” \( x_{I,2} = (6,9) \) was an extreme point of \( Q \), while in Example 2 it was not.
Theorem 3. If $Q \neq T$, there is $c \in \mathbb{R}^n$, $I$, $h$, $j$ with (i) $x_{I,h} \in P^{(h)}$, (ii) $cx_{I,h} = M$, (iii) $cx_{I,j} > M$, where $M = \max \{ cx | x \in Q \}$.

Note that (ii) and (iii) imply $x_{I,j} \notin Q$, hence $x_{I,j} \notin P^{(j)}$.

Proof: Let $y \in T \setminus Q$. There is $c \in \mathbb{R}^n$ with $cy > \max \{ cx | x \in Q \} = M$. The maximum of $cx$ over $Q$ is obtained by finding, for each $h$, the maximum of $cx$ over $P^{(h)}$. Standard linear programming results (with the assumption that $A$ is of rank $n$) imply that there is $I$ (consisting of $n$ rows), $h$ such that $cx_{I,h} = M = \max \{ cx | x \in F_{I,h} \}$. For any $j$, $\max \{ cx | x \in P^{(j)} \} \leq \max \{ cx | x \in F_{I,j} \} = cx_{I,j}$. Since $y \in T$, there is $\lambda$ with $Ay \geq \sum \lambda_j b^{(j)}$. For those $\lambda_j > 0$, let $y^{(j)}$ be the solution to:

\[
(Ay^{(j)})_i = b_i^{(j)} + \frac{1}{\lambda_j} \left( Ay - \sum_{j=1}^t \lambda_j b^{(j)} \right)_i \quad \text{for all } i \in I
\]

By considering $(Ay)_i$, it can be shown that $y = \sum \lambda_j y^{(j)}$. Since $cy > M$, $cy^{(j)} > M$ for some $j$. Since $y^{(j)} \in F_{I,j}$, $cx_{I,j} > M$.

Thus the nonexistence of $c$, $I$, $h$, $j$ satisfying (i)-(iii) is a sufficient condition for $Q = T$. It is not necessary in some special cases.

Example 4. We let $n = t = 2$, $m = 3$ and take

\[
A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad b^{(1)} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix}, \quad b^{(2)} = \begin{pmatrix} 4 \\ 6 \\ 4 \end{pmatrix}
\]

It is easy to see that $Q = T = P^{(1)} = P^{(2)}$, but if $I$ is the second and third rows of $A$, $x_{I,1} = (4,4)$, $x_{I,2} = (3,4)$ can be used to satisfy conditions (i)-(iii) of Theorem 3. To avoid this type of pathology we make a nondegeneracy assumption.

Theorem 4. Assume that, whenever $x_{I,h}$, $x_{J,h}$ are both defined, that they are equal only if $I = J$. Then the existence of $I$, $c$, $h$, $j$ satisfying (i)-(iii) of Theorem 3 implies $Q \neq T$.

Proof: Our assumption implies that the system $Ax_{I,h} \geq b^{(h)}$ has all rows other than those corresponding to $I$ as strict inequalities. This implies that the solution to the system

\[
(Ax)_i = \left( (1 - \epsilon)b^{(h)} + \epsilon b^{(j)} \right)_i \quad \text{for all } i \in I
\]

will be a member of $T$ for small positive $\epsilon$. But such solutions will have $cx > M$, hence not be members of $Q$.

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Theorem 5. The problem of deciding whether $Q = T$ is NP-Hard.

Proof: Given natural numbers $n_i$, $N$ we construct $A$, $b^{(h)}$ so that $Q \neq T$ if and only if there is some subset of the $n_i$ which adds up to exactly $N$ (this is the knapsack problem,
which is NP-Hard). Our problem will have \( t = 2 \) and one \( x_i \) for each \( n_i \). The inequalities defining \( P^{(1)} \), \( P^{(2)} \) are:

\[
- \sum n_i x_i \geq -N \quad - \sum n_i x_i \geq -N + \epsilon \\
-x_i \geq -1 + \epsilon^2 \quad -x_i \geq -1 - \epsilon \\
x_i \geq 0 \quad x_i \geq 0
\]

where \( \epsilon > 0 \) is chosen so that \( \epsilon (1 + \sum n_i) < 1 \).

If there is no subset \( S \) which adds up to exactly \( N \) then for any \( S \)

\[
\sum_{i \in S} n_i (1 - \epsilon^2) \leq N \quad \text{iff} \quad \sum_{i \in S} n_i \leq N - 1 \quad \text{iff} \quad \sum_{i \in S} n_i (1 + \epsilon) \leq (N - 1) + (1 - \epsilon)
\]

Thus the extreme points of \( P^{(1)} \), \( P^{(2)} \) are the same and Theorem 2 implies that \( Q = T \).

If there is \( S \) whose members add up to \( N \), then by letting \( \lambda = (1 - .5\epsilon, .5\epsilon) \), we can show that \( y \in T \), where \( y_i = 1 - .5\epsilon^2 \) for all \( i \in S \), all other components 0. Since

\[
\sum_{i \in S} n_i x_i \leq (1 - \epsilon^2) N \text{ for } x \in P^{(1)} \quad \sum_{i \in S} n_i x_i \leq N - \epsilon \text{ for } x \in P^{(2)} \quad \sum_{i \in S} n_i y_i = (1 - .5\epsilon^2) N
\]

\( y \notin Q \), hence \( Q \neq T \). 

Q. E. D.

References
