A hierarchical fusion of expert opinion in the Transferable Belief Model (TBM)

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The frame of reference: climate sensitivity

Climate sensitivity $\Delta T_{2\times}$ is:
The long term global warming if $[\text{CO}_2]$ in the atmosphere doubles
Uncertain: $1.5^\circ \text{C}$ to $4.5^\circ \text{C}$.

Morgan and Keith (1995) obtained probability density functions by
interviewing 16 leading U.S. climate scientists.

Experts’ uncertainty range subdivided in 7 intervalls to simplify:

$$\Omega = \{\omega_1, \ldots, \omega_7\}$$
$$= \{[-6, 0], [0, 1.5], [1.5, 2.5], [2.5, 3.5], [3.5, 4.5], [4.5, 6], [6, 12]\}$$
Variety of views: everything possible \{2,3\ldots\}, no cooling \{4\ldots\}, reasonable middle \{1\ldots\}, no problem \{5\}
Fusion issues using experts as information sources

- Dependence $\rightarrow$ Avoid unjustified accuracy
- Complete contradiction $\rightarrow$ Need paraconsistency
- Scientific validity $\neq$ popularity $\rightarrow$ No majority rule
- Calibrating experts is not practical $\rightarrow$ don’t!
Categorical beliefs: the indicator function $1_E$

Belief that the state of the world is in the subset $E = \{\omega_2, \omega_3, \omega_4\}$ of the frame of reference $\Omega = \{\omega_1, \ldots, \omega_7\}$ is represented by $m = 1_E$ the indicator function of $E$:

$$\begin{cases} m(\{\omega_2, \omega_3, \omega_4\}) = m(E) = 1 \\ m(A) = 0 \quad \text{for any other } A \subset \Omega, A \neq E \end{cases}$$

(1)
Representing belief with a random subset of $\Omega$

We allocate the unit “mass of belief” among subsets of $\Omega$.

$m : 2^\Omega \rightarrow [0, 1]$ is a Basic Belief Assignment iff:

$$\sum_{A \subseteq \Omega} m(A) = 1$$  \hspace{1cm} (2)
Corner cases included: ignorance and contradiction

Total ignorance, no information  Void beliefs represented by $1_{\Omega}$.

Total confusion  Contradictory beliefs represented by $1_{\emptyset}$.
Discounting and simple beliefs

Discounting is adding a degree of doubt \( r \) to a belief \( m \) by mixing it with the void beliefs:

\[
\text{disc}(m, r) = (1 - r) m + r \mathbf{1}_\Omega
\]  

(3)

Denote \( A^s \) the simple belief that

“The state of the world is in \( A \), with a degree of confidence \( s \)”:

\[
A^s = \text{disc}(1_A, e^{-s})
\]  

(4)

That is:

\[
\begin{align*}
A^s(A) &= 1 - e^{-s} \\
A^s(\Omega) &= e^{-s} \\
A^s(X) &= 0 \quad \text{if } X \neq A \text{ and } X \neq \Omega
\end{align*}
\]
Conjunction $\odot$ and disjunction $\cup$ of beliefs

When two reliable information sources say one $A$ and the other $B$, believe in the intersection of opinions (TBM allows $1_\emptyset$):

$$1_A \odot 1_B = 1_{A \cap B}$$

Generally:

$$(\mu_1 \odot \mu_2)(A) = \sum_{B \cap C = A} \mu_1(B) \mu_2(C) \quad (5)$$

When at least one source is reliable, consider the union of opinions.

$$(\mu_1 \odot \mu_2)(A) = \sum_{B \cup C = A} \mu_1(B) \mu_2(C) \quad (6)$$
Canonical decomposition in simple beliefs

For any \( m \) such that \( m(\Omega) > 0 \), there are weights \( (s(A))_{A \subsetneq \Omega} \) such that:

\[
m = \bigcap_{A \subsetneq \Omega} A^{s(A)}
\]  

(7)

Weights of the \( \cap \) conjunction are the sum of weights:

\[
m_1 \cap m_2 = \bigcap_{A \subsetneq \Omega} A^{s_1(A) + s_2(A)}
\]  

(8)

\( \cap \) Conjunction increases confidence: \( A^s \cap A^s = A^{2s} \).

Good for independent information sources, but for experts we want to avoid unjustified accuracy.
T. Denœux’s cautious combination operator

Whenever...
Expert 1 has confidence \(s_1(A)\) that state of the world is in \(A\)
Expert 2 has confidence \(s_2(A)\)
...follow the most confident:

\[
m_1 \kappa m_2 = \bigcap_{A \subseteq \Omega} A_{\max(s_1(A), s_2(A))}
\]

Distributivity: \((m_1 \kappa m_3) \kappa (m_2 \kappa m_3) = (m_1 \kappa m_2) \kappa m_3\)

Interpretation:
Expert 1 has beliefs \(m_1 \kappa m_3\)
Expert 2 has beliefs \(m_2 \kappa m_3\)
\(\kappa\) cautious combination of experts counts evidence \(m_1\) only once.
Historical operators: Averaging and Dempster’s rule

Averaging is \[ \frac{m_1(X) + m_2(X)}{2} \]

Renormalizing \( m \) means replacing it with \( m^* \) such that \( m^*(\emptyset) = 0 \) and
\[ m^*(X) = \frac{m(X)}{1 - m(\emptyset)} \]

Dempster’s rule is renormalized conjunction:
\[ m_1 \oplus m_2 = (m_1 \odot m_2)^* \quad (10) \]
There is no satisfying fusion operator

<table>
<thead>
<tr>
<th></th>
<th>Average</th>
<th>⊕, ⨁, ⨀</th>
<th>▲</th>
<th>△</th>
</tr>
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<tbody>
<tr>
<td>Majority rule</td>
<td>☹ 🔴</td>
<td>🔵 🔵</td>
<td>🔵</td>
<td>🔵</td>
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<tr>
<td>Contradiction</td>
<td>🔵 😞</td>
<td>😞 😞</td>
<td>🔵</td>
<td>🔵</td>
</tr>
<tr>
<td>Unjust. accuracy</td>
<td>🔵 😞</td>
<td>😞 🔵</td>
<td>😞</td>
<td>😞</td>
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</tbody>
</table>

Discounting decreases contradiction issues, but calibrating experts is not practical.
A hierarchical approach

1. Partition experts in schools of thought
   (adaptative or sociological methods)

2. Within groups, $\bigwedge$ cautious combination

3. Across theories, $\bigcup$ disjonoction

Using the climate experts dataset:

\[
\begin{align*}
  m_A &= m_2 \otimes m_3 \otimes m_6 & \text{Everything possible} \\
  m_B &= m_4 \otimes m_7 \otimes m_8 \otimes m_9 & \text{No cooling} \\
  m_C &= m_1 \otimes m_{10} \otimes \cdots \otimes m_{16} & \text{Reasonable middle} \\
  m_D &= m_5 & \text{No problem} \\
  m &= m_A \ominus m_B \ominus m_C \ominus m_D
\end{align*}
\]
Probability and plausibility used to present results

Any $m$ defines a probability $p^m$ by:

$$p^m(\omega_i) = \sum_{X \ni \omega_i} \frac{m^*(X)}{|X|} \quad (11)$$

Any $m$ defines a plausibility function $pl$, which is given on singletons by:

$$pl(\{\omega_i\}) = \sum_{X \ni \omega_i} m(X) \quad (12)$$

Levels of probability are generally smaller than levels of plausibility.
Results: fusion of 16 experts on $\Delta T_{2x}$, MK 1995 survey

Noninteractive disjunction of the four groups.

Simple distributions associated with the result BBA:

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_i$</td>
<td>-6.0</td>
<td>0.15</td>
<td>1.5,2.5</td>
<td>2.5,3.5</td>
<td>3.5,4.5</td>
<td>4.5,6.0</td>
<td>6.0,12</td>
</tr>
<tr>
<td>$pl$</td>
<td>0.48</td>
<td>1</td>
<td>1</td>
<td>0.99</td>
<td>0.74</td>
<td>0.59</td>
<td>0.31</td>
</tr>
<tr>
<td>$p^m$</td>
<td>0.08</td>
<td>0.21</td>
<td>0.21</td>
<td>0.21</td>
<td>0.14</td>
<td>0.10</td>
<td>0.05</td>
</tr>
</tbody>
</table>
Symmetric fusions operators vs. Hierarchical approaches

disc. Dempster

disc. Cautious conj.

disc. niConj.

niDisjunction

Averaging

Hierarchical

Hierarchical 3–way

Average within
The likelihood of $\Delta T_{2x} < 1.5^\circ$C has decreased since 1995

IPCC 2001: Climate sensitivity is likely to be in the 1.5 to 4.5$^\circ$C range (unchanged from 1979).

<table>
<thead>
<tr>
<th>$\Delta T_{2x}$ ∈ ...</th>
<th>[0°C, 1.5°C]</th>
<th>[1.5°C, 4.5°C]</th>
<th>[4.5°C, 10°C]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Published PDFs</td>
<td>[0, 0.07]</td>
<td>[0.31, 0.98]</td>
<td>[0.02, 0.62]</td>
</tr>
<tr>
<td>Kriegler (2005)</td>
<td>[0, 0.00]</td>
<td>[0.53, 0.99]</td>
<td>[0.01, 0.47]</td>
</tr>
</tbody>
</table>

IPCC 2007: [2, 4.5°C] is likely, below 1.5°C is very unlikely.

Note:
Likely means $0.66 \leq p \leq 0.90$,
very unlikely means $p \leq 0.1$. 

Conclusions

A hierarchical approach to fusion expert opinions:

- Imprecise
- Deals with dependencies and contradiction
- Avoid majority rule and calibration
- Requires a sociological study of experts groups

About climate sensitivity:

- Above 4.5°C was already plausible in 1995
- Below 1.5°C is less plausible today
Expert 1: bayesian $m$ (top), consonnant $m$ (bottom)
Hierarchical better than symmetric fusion for expert aggregation

<table>
<thead>
<tr>
<th>Fusion method</th>
<th>$m(\Omega)$</th>
<th>$\leq 1.5^\circ C$</th>
<th>In range</th>
<th>$\geq 4.5^\circ C$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>bel–pl</td>
<td>bel–pl</td>
<td>bel–pl</td>
<td>bel–pl</td>
</tr>
<tr>
<td>Hierarchical</td>
<td>0.18</td>
<td>0.–1.</td>
<td>0.–1.</td>
<td>0.–0.61</td>
</tr>
<tr>
<td>Average</td>
<td>0.08</td>
<td>0.07–0.69</td>
<td>0.27–0.93</td>
<td>0.–0.45</td>
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<tr>
<td>disc. Dempster</td>
<td>0.</td>
<td>0.02–0.03</td>
<td>0.97–0.98</td>
<td>0.–0.</td>
</tr>
<tr>
<td>Disjunction</td>
<td>0.99</td>
<td>0.–1.</td>
<td>0.–1.</td>
<td>0.–1.</td>
</tr>
</tbody>
</table>
Sensitivity analysis. Bayesian left, consonnant right.
Cautious combination within groups

Cautious combination of implicit possibilities

Experts groups:

- 2, 3, 6
- 4, 7, 8, 9
- 1, 10-16
- 5
Result of the hierarchical fusion: the belief function

<table>
<thead>
<tr>
<th>subset $A$</th>
<th>$m^*(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2}</td>
<td>0.0001</td>
</tr>
<tr>
<td>{3, 2}</td>
<td>0.0074</td>
</tr>
<tr>
<td>{4, 2}</td>
<td>0.0033</td>
</tr>
<tr>
<td>{4, 3, 2}</td>
<td>0.1587</td>
</tr>
<tr>
<td>{4, 3, 2, 1}</td>
<td>0.0064</td>
</tr>
<tr>
<td>{5, 4, 2}</td>
<td>0.0011</td>
</tr>
<tr>
<td>{5, 4, 3, 2}</td>
<td>0.1321</td>
</tr>
<tr>
<td>{5, 4, 3, 2, 1}</td>
<td>0.0709</td>
</tr>
<tr>
<td>{6, 4, 3, 2}</td>
<td>0.0267</td>
</tr>
<tr>
<td>{6, 4, 3, 2, 1}</td>
<td>0.0129</td>
</tr>
<tr>
<td>{6, 5, 4, 3, 2}</td>
<td>0.0888</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>subset $A$ (cont.)</th>
<th>$m^*(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{6, 5, 4, 3, 2, 1}</td>
<td>0.1811</td>
</tr>
<tr>
<td>{7, 4, 3, 2}</td>
<td>0.0211</td>
</tr>
<tr>
<td>{7, 5, 4, 3, 2}</td>
<td>0.0063</td>
</tr>
<tr>
<td>{7, 6, 4, 3, 2}</td>
<td>0.0135</td>
</tr>
<tr>
<td>{7, 6, 4, 3, 2, 1}</td>
<td>0.0105</td>
</tr>
<tr>
<td>{7, 6, 5, 4, 3, 2}</td>
<td>0.0632</td>
</tr>
<tr>
<td>{7, 6, 5, 4, 3, 2, 1}</td>
<td>0.1956</td>
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</tbody>
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