Discrete models of dislocations : traveling waves and dynamics of particles

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THÈSE

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Sujet : *Modèles discrets de dislocations : ondes progressives et dynamique de particules*

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À mes parents *Hassan* & *Mariam*,

à ma sœur *Khadija*, son mari *Kamel*
& leurs enfants *Abbass* & *Hussein*,

à mes frères *Mehdi*, *Ali* & *Hussein*,

à mon grand-père *Mohammad*
& toute ma famille.

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Mohammad Al Haj

Résumé : Ce travail se concentre sur l'étude de la dynamique des dislocations dans le réseau cristallin et il est découpé en deux parties : la première partie porte sur les mouvements horizontaux d'une chaîne d'atomes en interaction contenant une dislocation. Bien que, la deuxième partie traite de l'accumulation de dislocations formant ce qu'on appelle des murs de dislocations.

Dans la première partie, nous considérons une généralisation complètement non linéaire des équations de diffusion de réaction discrète également appelée *"modèles de Frenkel-Kontorova complètement amortis"* qui décrivent la dynamique des défauts cristallins (dislocations) dans un réseau. Nous étudions à la fois : les non-linéarités bistable et monostable. Dans des conditions suffisantes, nous montrons l'existence et l'unicité des ondes progressives pour le cas de non-linéarité bistable. Pour le cas monostable, nous étudions l'existence de la branche des solution d'ondes progressives pour une non-linéarité Lipschitz général. Nous montrons également que la vitesse minimale est positive et délimitée ci-dessous. Dans cette partie, nous étudions aussi la généralisation du modèle de Frenkel-Kontorova pour laquelle nous pouvons ajouter un paramètre de force motrice. Nous illustrons également, dans ce cas, la variation de la vitesse de propagation des ondes progressives en fonction du paramètre de force.

Dans la deuxième partie, nous étudions l'accumulation des dislocations dans les murs de dislocations. Nous montrons en fait la convergence de plusieurs dislocations qui interagissent sur les murs de dislocations. Nous présentons aussi les résultats de quelques expériences numériques qui confirment les résultats théoriques que nous obtenons.

Abstract : This work focuses on the study of the dislocation dynamics in the crystal lattice and it is splitted into two parts : the first part is concerned with the horizontal motion of a chain of interacting atoms containing a dislocation. While, the second part deals with the accumulation of dislocations forming what is known as walls of dislocations.

In the first part, we consider a fully nonlinear generalization of the discrete reaction diffusion equations *"fully overdamped Frenkel-Kontorova models"* that describe the dynamics of crystal defects (dislocations) in a lattice. We study both : the bistable and the monostable non-linearities. Under sufficient conditions, we show the existence and uniqueness of traveling wave solution for the bistable non-linearity case. For the monostable case, we study the existence of branch of traveling waves solutions for general Lipschitz non-linearity. We also prove that the minimal velocity is non-negative and bounded below. In this part, we as well study the generalization of Frenkel-Kontorova model for which we can add a driving force parameter. We also illustrate, in this case, the variation of the velocity of propagation of traveling waves in terms of the parameter force.

In the second part, we study the accumulation of dislocations in walls of dislocations. We prove actually the convergence of several interacting dislocations to walls of dislocations. We also present results of some numerical experiments that confirm the theoretical results that we obtain.

Publications issues de la thése

Articles publiés

– (avec N. Forcadel et R. Monneau) *Existence and uniqueness of traveling waves for fully overdamped Frenkel-Kontorova models*, Arch. Ration. Mech. Anal. 210 (1) (2013), 45-99.

Articles soumis et Preprint

– (avec R. Monneau) *Existence of traveling waves for Lipschitz discrete dynamics. Monostable case as a limit of bistable cases.* Soumis.

– (avec Ł. Paszkowski) *Convergence to walls of dislocations in the periodic case.* Soumis.

Proceedings

– (avec N. Forcadel, C. Imbert et R. Monneau) *Modèle de Frenkel-Kontorova : résultats d'homogénéisation et existence de traveling waves*, Equations aux dérivées partielles et leurs applications. Proceedings du colloque Edp-Normandie. Le Havre 2012. 51-60. FNM Fédération Normandie Mathématiques (2013).

Sommaire

Introduction générale

Cette thèse porte sur l'étude mathématique de la *dynamique des dislocations* dans des cristaux.

Nous étudions, dans une première partie (Chapitres 2 et 3), l'existence d'ondes progressives pour la généralisation des modèles de Frenkel-Kontorova complètement amortis qui décrivent le déplacement d'une chaîne d'atomes en interaction et contenant une dislocation. Nous considérons les deux types de non-linéarité bistable et monostable. Nous montrons, sous des hypothèses suffisantes, l'existence et l'unicité des ondes progressives pour le type de non-linéarité bistable et l'existence de la branche de solutions pour le type de non-linéarité monostable. Dans une seconde partie (Chapitre 4), nous étudions l'accumulation des dislocations dans les murs de dislocations. En d'autres termes, nous prouvons la convergence (dynamique) de plusieurs dislocations d'interaction vers ce que nous appelons les murs de dislocations.

Nous présenterons, dans cette introduction, nos résultats pour le cas simplifié et nous renvoyons le lecteur à l'introduction en anglais pour nos résultats dans le cas général et pour plus de détails.

Motivation physique : dislocation

Une dislocation est un type d'imperfection qui se compose de défauts purement géométriques dans le réseau cristallin. Elle peut être définie en spécifiant les atomes qui sont disloqués ou mal connectés, ce qui fausse le réseau cristallin hôte, par rapport au cristal parfait (structure exempte de défauts du cristal hôte). Les dislocations, dont l'ordre de longueur typique dans les matériaux est $10^{-6}m$ et l'épaisseur $10^{-9}m$, ont été introduites dans les années 1930 par Orowan [94], Polanyi [98] et Taylor [108] comme l'une des principales explications à l'échelle microscopique des déformations plastiques macroscopiques des cristaux. Pour une discussion plus complète sur les dislocations, nous nous référons aux textes classiques de Hirth, Lothe [74], Read [99], Hull, Bacon [78] et Bulatov, Cai [27].

Smekal [103] a remarqué que les propriétés des cristaux sont liées à l'absence

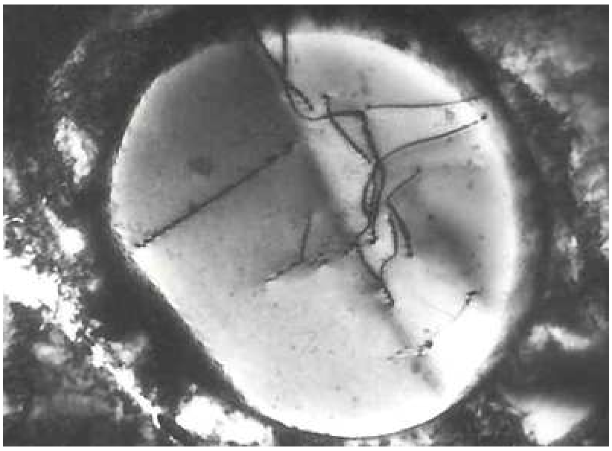

FIGURE 1 – Micrographie de dislocations en acier inoxydable.

(cristal idéal) ou à la présence (cristal réel) de défauts de cristal, par exemple, les propriétés de résistance mécanique, comme l'élasticité, la compressibilité sont très sensibles à la perfection de cristal (indépendant de défauts), tandis que la semi-conductivité et la plasticité dépendent des défauts.

L'omniprésence et l'importance des dislocations pour les plasticités cristallines et d'autres aspects du comportement des matériaux ont été considérés depuis 1950 lorsque les premières observations de dislocations de cristal ont été signalées lors d'expériences en microscopie électronique à transmission (MET) des expériences, voir [75] et [23] (voir Figure 1 pour un exemple d'observation de dislocations).

Chaque dislocation est caractérisée par son vecteur de Burgers et le vecteur de direction de la ligne locale. Nous distinguons les deux types fréquents de dislocations : *edge dislocation,* lorsque le vecteur de Burgers est perpendiculaire au vecteur de direction de ligne et *screw dislocation,* lorsque les deux vecteurs sont parallèles.

1 Enoncés des résultats : ondes progressives

Dans cette thèse, nous nous intéressons à l'étude de la dynamique d'une chaîne d'atomes interagissant ensemble et contenant une *edge dislocation. L'edge dislocation* peut être simplement réalisée par l'insertion d'un demi-plan supplémentaire des atomes dans un cristal parfait par le haut ou par le retrait d'un demi-plan d'atomes de dessous (voir Figure 2).

Les atomes dans un cristal contenant une dislocation sont déplacés de leurs sites

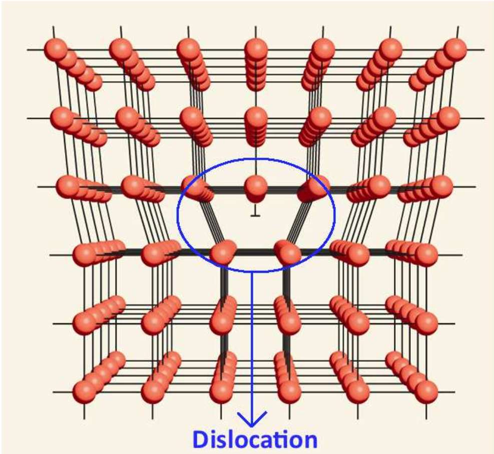

FIGURE 2 – *Edge dislocation* : la ligne de dislocation est marquée par le symbole $\perp$.
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du réseau parfaits, et la déformation qui en résulte produit un champ de contraintes dans le cristal autour de la dislocation. De plus, ces atomes ne sont pas rigidement liés les uns aux autres mais sont couplés élastiquement. Ainsi, en raison de contraintes intérieures qui sont induites par d'autres dislocations ou une variation de température ou quand une contrainte suffisante est appliquée à un cristal (voir [99]), les dislocations peuvent se déplacer sur de petites distances et leur mouvement fournit un mécanisme pour un cristal à se déformer plastiquement par glissement (en cas de *edge dislocations*).

Modèles de Frenkel-Kontorova complètement amortis.
La dynamique de défauts de réseau est décrite par les modèles de Frenkel-Kontorova (FK) complètement amortis (voir par exemple le livre de Braun et Kivshar [25] pour une introduction à ce modèle). Le modèle le plus simple FK complètement amorti est une chaîne d'atomes, où la position $X_i(t) \in \mathbb{R}$ au moment t de la particule $i \in \mathbb{Z}$ résoud

$$\frac{dX_i}{dt} = X_{i+1} + X_{i-1} - 2X_i - \sin(2\pi(X_i - L)) - \sin(2\pi L)$$

où $\frac{dX_i}{dt}$ est la vitesse de la i-ième particule, $-\sin(2\pi L)$ est une force motrice constante qui oblige la chaîne d'atomes à se déplacer et $-\sin(2\pi(X_i - L))$ désigne la force créée par un potentiel périodique reflétant la périodicité du cristal, dont la période est supposée être 1.

Soit f une force générale créée par le potentiel périodique et σ une force constante motrice externe. Nos résultats sur les ondes progressives sont présentés pour l'équation générale suivante :

$$\begin{aligned}\frac{dX_i}{dt} &= X_{i+1} + X_{i-1} - 2X_i + f(X_i) + \sigma \\ &= F(X_{i-1}, X_i, X_{i+1}) + \sigma,\end{aligned} \tag{1}$$

où les propriétés de F sont introduites dans la Sous-section 1.1 et 1.2.

Ondes progressives

Les ondes progressives sont des solutions particulières invariantes par rapport à la translation d'espace et de la forme

$$X_i(t) = \phi(i + ct) \tag{2}$$

où $\phi : \mathbb{R} \to \mathbb{R}$ est l'onde progressive et c est la vitesse de propagation de ϕ (voir Figure 3).

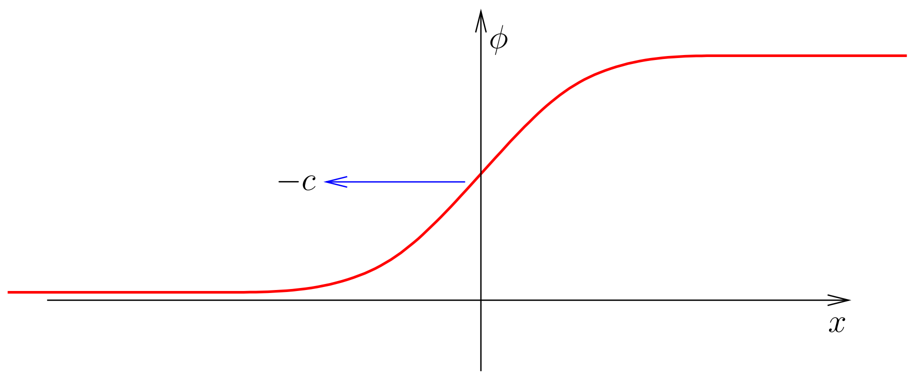

FIGURE 3 – Onde progressive se déplaçer vers la gauche avec une vitesse $-c$.

Dans cet travail, nous recherchons des ondes progressives de la forme (2), pour l'équation de réaction-diffusion discrète (1), et satisfaisant

$$\begin{cases} \phi' \geq 0 \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{3}$$

Nous indiquons que la condition (3) reflète l'existence d'un défaut d'un espace de réseau, appelé dislocation. Par ailleurs, l'expression (2) signifie que les défauts se déplacent avec la vitesse c sous l'impulsion σ. En outre, ϕ est une transition de phase entre $\phi(-\infty)$ et $\phi(+\infty)$, qui sont deux équilibres "stables" du cristal.

Notez que, si nous introduisons l'équation (2) dans l'équation (1), le profil ϕ et la vitesse c doivent satisfaire à l'équation

$$\begin{aligned} c\phi'(z) &= F(\phi(z-1), \phi(z), \phi(z+1)) + \sigma \\ &= \phi(z+1) + \phi(z-1) - 2\phi(z) + f(\phi(z)) + \sigma, \end{aligned} \tag{4}$$

avec $z = i + ct$ et

$$F(X_{i-1}, X_i, X_{i+1}) = X_{i+1} + X_{i-1} - 2X_i + f(X_i) \tag{5}$$

En raison de l'équivalence (quand $c \neq 0$) entre les solutions de (1) et (4), nous allons nous concentrer sur l'équation (4).

1.1 Contrainte σ nulle ($\sigma = 0$)

Dans cette section, nous avons considéré que $\sigma = 0$. Soit $F : [0,1]^3 \to \mathbb{R}$ est définie dans (5), où nous rappelons que $f(v) := F(v, v, v)$ et supposons que f satisfait :

(A_{Lip}) **Régularité :** f est globalement Lipschitz sur $[0, 1]$.

Remarquons que, si ϕ est une solution de

$$c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1))$$

et (3), alors

$$f(\phi(\pm\infty)) = 0.$$

Ici, nous distinguons deux types de non-linéarité f :

1. **Cas bistable.**
Nous disons que la source de la non-linéarité f est de cas bistable si f satisfait les condition suivantes (voir la Figure 4) :

$$\begin{aligned} &f(0) = 0 = f(1) \text{ et il existe } b \in (0,1) \text{ telle que} \\ &f(b) = 0,\ f_{|(0,b)} > 0,\ f_{|(b,1)} < 0 \text{ et } f'(b) > 0. \end{aligned}$$

En d'autres terme, f est bistable puisque les zéros 0 et 1 sont stables (car f est décroissante sur un voisinage de 0 et 1 dans $[0, 1]$).

La non-linéarité bistable se produit plutôt dans la description des réactions chimiques, en particulier pour expliquer les transitions de phases et de la propagation des interfaces. Le prototype de la fonction bistable est donnée par $f(x) = x(b-x)(x-1)$. Pour plus de détails, nous nous référons à des articles de Chen [29], Fife [47, 48], Fife, Mcleod [49] et le livre de Murray [90] et les références qui s'y trouvent.

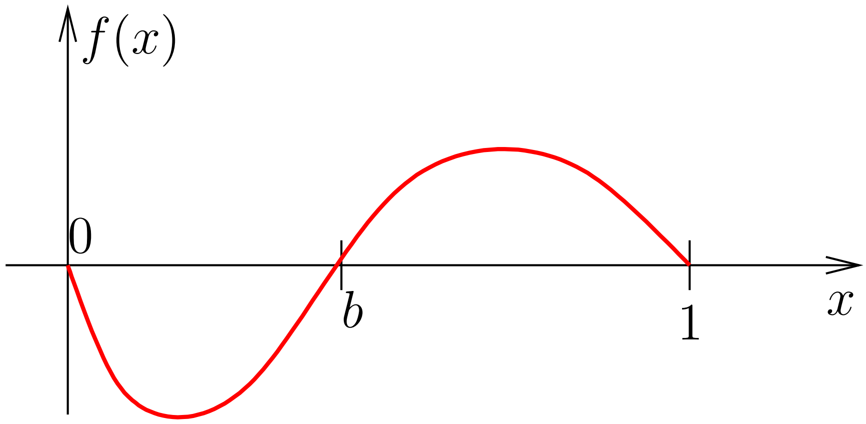

FIGURE 4 – Source de non-linéarité bistable f.

Supposons que :

Hypothèse (B) :

Instabilité : $f(0) = 0 = f(1)$ et il existe $b \in (0,1)$ telle que $f(b) = 0$, $f_{|_{(0,b)}} < 0$, $f_{|_{(b,1)}} > 0$ et $f'(b) > 0$.

Régularité : f est C^1 dans un voisinage de b.

Théorème 1.1. (Existence d'un onde progressive)
Soit F définie dans (5). sous des hypothèses (A_{Lip}), (B), il existe un réel $c \in \mathbb{R}$ et une fonction $\phi : \mathbb{R} \to \mathbb{R}$ qui résoud

$$\begin{cases} c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1)) \quad \textit{sur} \quad \mathbb{R} \\ \phi \textit{ est croissante sur } \mathbb{R} \\ \phi(-\infty) = 0 \quad \textit{et} \quad \phi(+\infty) = 1 \end{cases} \tag{6}$$

dans le sens classique si $c \neq 0$ et presque partout si $c = 0$.

Pour la preuve de ce résultat, nous renvoyons le lecteur à la démonstration du Théorème 2.9, où la preuve utilise le fait que F est croissante relativement à X_i pour tous les $i \neq 0$, ce qui est également garanti par la fonction F définie dans (5).

Afin de prouver l'unicité de la vitesse, nous avons besoin d'introduire l'hypothèse suivante :

Hypothèse (C) : Monotonie inverse près de 0 et 1
Il existe $\beta_0 > 0$ tel que pour $a > 0$, nous avons

$$\begin{cases} f(x+a) < f(x) \quad \text{pour tout} \quad x,\, x+a \in [0, \beta_0] \\ f(x+a) < f(x) \quad \text{pour tout} \quad x,\, x+a \in [1-\beta_0, 1]. \end{cases}$$

Théorème 1.2. (Unicité de la solution)
Soit F définie dans (5) satisfaisant (A_{Lip}) et (c, ϕ) une solution de

$$\begin{cases} c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1)) \quad sur \quad \mathbb{R} \\ \phi(-\infty) = 0 \quad et \quad \phi(+\infty) = 1. \end{cases} \tag{7}$$

Sous l'hypothèse supplémentaire (C), la vitesse c est unique. Par ailleurs, si $c \neq 0$, alors ϕ est unique à une translation en espace près.

A noter que, l'unicité de la vitesse est établie en utilisant un principe de comparaison des deux demi-droites (voir Proposition 2.6 et Corollaire 2.7). Cependant, nous avons eu l'unicité du profil en utilisant un principe du maximum fort (voir le Lemme 2.12 et Proposition 2.15) qui est basé, par exemple, sur le fait que F est strictement croissante par rapport à X_{i+1}.

2. **Cas monostable.**
La non-linéarité source f est dit monostable si nous avons (voir la figure 5) :

Hypothèse (P) :

Monostabilité :
Soit $f(v) = F(v, v, v)$ telle que $f(0) = f(1) = 0, \quad f > 0$ dans $(0, 1)$.

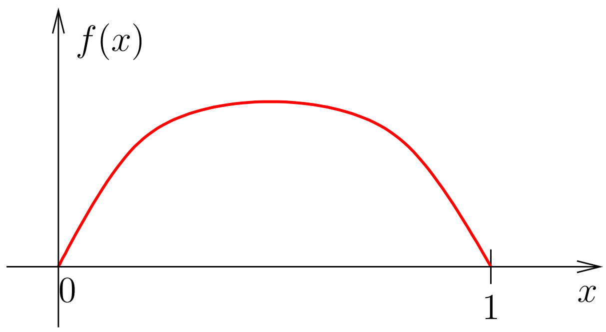

FIGURE 5 – Source de non-linéarité monostable f.

Notons que la non-linéarité f admet un seul zéro stable et l'autre est instable (dans ce cas, 1 est stable et 0 est instable). Ce type de non-linéarité apparaît dans la description de la dynamique des populations ou de la combustion, voir Berestycki, Larrouturou [19]. Un exemple d'une telle non-linéarité est $f(x) = x(1 - x)$.

Théorème 1.3. (Existence des ondes progressives pour une branche de vitesses)
Soit F définie dans (5) *satisfaisant* (A_{Lip}) *et* (P). *Alors il existe une réel c^+ tel que pour tout $c \geq c^+$ il existe une onde progressive $\phi : \mathbb{R} \to \mathbb{R}$ solution (au sens de la viscosité (voir Définition 2.4)) de* (7). *Au contraire pour $c < c^+$, il n'existe pas de solution pour (7).*

Pour la preuve de ce théorème, voir par exemple la preuve du Théorème 2.18. Dans la proposition suivante, nous donnons une minoration de c^+. A cet effet, nous supposons que

Hypothèse (P_{C^1}) :

Monostabilité :
Soit $f(v) = F(v, v, v)$ telle que $f(0) = 0 = f(1)$ et $f > 0$ dans $(0, 1)$.

Régularité près de 0 **:**
f est C^1 dans un voisinage de 0 dans $[0, 1]$ et $f'(0) > 0$.

Proposition 1.4. (Borne inférieure de c^+)
Soit F une fonction définie dans (5) *satisfaisant* (A_{Lip}) *et* (P_{C^1}). *Soit c^+ donné par le Théorème 1.3, alors*

$$c^+ \geq c^* \geq 0,$$

où

$$c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \quad \textit{avec} \quad P(\lambda) := f'(0) + e^{-\lambda} + e^{\lambda} - 2. \tag{8}$$

Nous renvoyons le lecteur à Proposition 2.22 pour le cas plus général.

1.2 Contrainte σ non nulle ($\sigma \neq 0$)

Soit $F\ : \mathbb{R}^3 \to \mathbb{R}$ définie dans (5) et considérons l'équation (4) avec $\sigma \neq 0$. Soit $\theta \in \mathbb{R}$ et supposons que (voir Figure 6)

Hypothèse ($\tilde{A}_{C^1}$) :

Régularité : f est globalement Lipschitz sur $\mathbb{R}$ et C^1 sur un voisinage dans $\mathbb{R}$ des deux intervalles $]0, \theta[$ et $]\theta, 1[$.

Périodicité : $f(v + 1) = f(v)$ pour chaque $v \in \mathbb{R}$.

Hypothèse ($\tilde{B}_{C^1}$) :
Rappelons que $f(v) = F(v, ..., v)$ et supposons que :

Bistabilité : $f(0) = f(1)$ et il esixte $\theta \in (0, 1)$ tel que

$$\begin{cases} f' > 0 & \text{sur} \quad (0, \theta) \\ f' < 0 & \text{sur} \quad (\theta, 1). \end{cases}$$

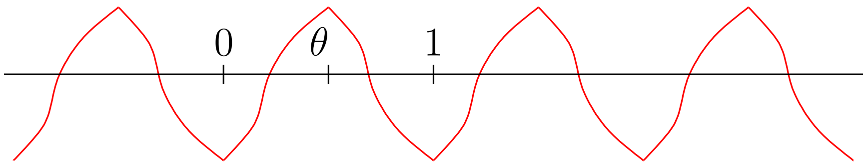

FIGURE 6 – Non-linéarité bistable f

Puisque nous sommes à la recherche des ondes progressives qui sont solutions de (4) et (3), alors nous obtenons

$$f(\phi(\pm\infty)) + \sigma = 0. \tag{9}$$

Définition 1.5. (Plage de σ)
Sous les hypothèses $(\tilde{A}_{C^1})$ et $(\tilde{B}_{C^1})$, définissons $\sigma^{\pm}$ comme

$$\begin{cases} \sigma^+ = -\min f \\ \sigma^- = -\max f. \end{cases} \tag{10}$$

Associons pour chaque $\sigma \in [\sigma^-, \sigma^+]$ les solutions $m_\sigma \in [\theta - 1, 0]$ et $b_\sigma \in [0, \theta]$ de $f(s) + \sigma = 0$.

Notons que si $\sigma \notin [\sigma^-, \sigma^+]$, à cause de (9), alors les équations (4) et (3) n'admettent aucune solution. Nous avons les résultats suivants :

i) Cas bistable : $\sigma \in (\sigma^-, \sigma^+)$
Soit $\sigma \in (\sigma^-, \sigma^+)$. Evidement, à partir de la définition de $\sigma^{\pm}$ dans (10), la fonction $f + \sigma$ obéit à la forme de non-linéarité bistable (voir la Figure 7).

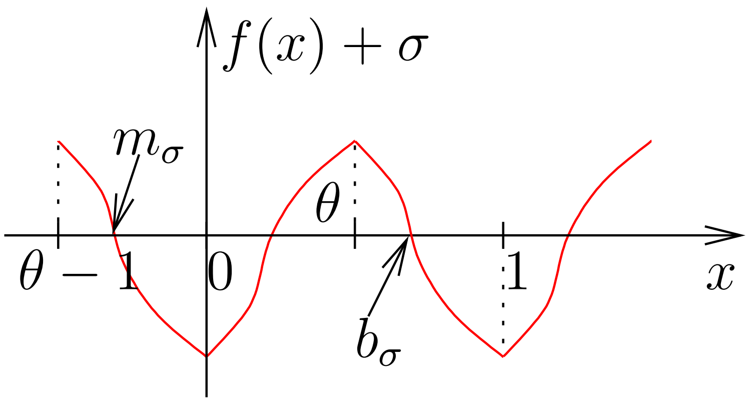

FIGURE 7 – La non-linéarité bistable f

Théorème 1.6. (Existence d'une onde progressive)
Supposons que $(\tilde{A}_{C^1})$ *et* $(\tilde{B}_{C^1})$. *Pour toute* $\sigma \in (\sigma^-, \sigma^+)$, *il existe un réel unique* $c := c(\sigma)$, *telle qu'il existe une fonction* $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ *solution de*

$$\begin{cases} c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1)) + \sigma \quad \textit{sur} \quad \mathbb{R} \\ \phi \textit{ est croissante sur } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \textit{et} \quad \phi(+\infty) = m_\sigma + 1. \end{cases} \tag{11}$$

dans le sens classique si $c(\sigma) \neq 0$ *et presque partout si* $c(\sigma) = 0$.

Voir la preuve du Théorème 2.31 pour la démonstration de ce théorème.

Proposition 1.7. (Continuité et monotonie de la fonction de vitesse)
Sous les hypothèses $(\tilde{A}_{C^1})$ *et* $(\tilde{B}_{C^1})$, *l'application*

$$\sigma \mapsto c(\sigma)$$

est continue sur (σ^-, σ^+) *et il existe une constante* $K > 0$ *telle que la fonction* $c(\sigma)$ *est croissante et satisfait*

$$\frac{dc}{d\sigma} \geq K|c| \quad \textit{sur} \quad (\sigma^-, \sigma^+)$$

au sens de la viscosité. De plus, il existe des nombres réels $c^- \leq c^+$ *de telle sorte que*

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \textit{et} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+.$$

En outre, soit $c^- = 0 = c^+$, *soit* $c^- < c^+$.

Voir Proposition 2.32 où nous avons le même résultant mais pour le cas général F.

ii) Cas monostable : $\sigma = \sigma^\pm$
Soit $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), alors $f + \sigma$ a la forme monostable positive (resp. négative) de non-linéarité (voir Figure 8).

Théorème 1.8. (Branches verticales pour $\sigma = \sigma^\pm$**)**
Supposons que $(\tilde{A}_{C^1})$ *et* $(\tilde{B}_{C^1})$. *Nous avons*

(i) **(Ondes progressives pour** $\sigma = \sigma^+$**)**
Soit $\sigma = \sigma^+$ *et* c^+ *donné par la Proposition 1.7. Alors pour tout* $c \geq c^+$ *il existe une onde progressive* ϕ *solution de*

$$\begin{cases} c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1)) + \sigma^+ \quad \textit{sur} \quad \mathbb{R} \\ \phi \textit{ est croissante sur } \mathbb{R} \\ \phi(-\infty) = 0 = m_{\sigma^+} \quad \textit{et} \quad \phi(+\infty) = 1 = b_{\sigma^+}. \end{cases} \tag{12}$$

En outre, pour tout $c < c^+$, *l'équation (12) n'admet pas de solution.*

(ii) **(Ondes progressives pour $\sigma = \sigma^-$)**
Soit $\sigma = \sigma^-$ et c^- donné par la Proposition 1.7. Alors pour tout $c \leq c^-$, il existe une onde progressive ϕ solution de

$$\begin{cases} c\phi'(z) = F(\phi(z-1), \phi(z), \phi(z+1)) + \sigma^- \quad sur \quad \mathbb{R} \\ \phi \text{ est croissante sur } \mathbb{R} \\ \phi(-\infty) = \theta - 1 = m_{\sigma^-} \quad et \quad \phi(+\infty) = \theta = b_{\sigma^-}. \end{cases} \tag{13}$$

En outre, pour tout $c > c^-$, l'équation (13) n'admet pas de solution.

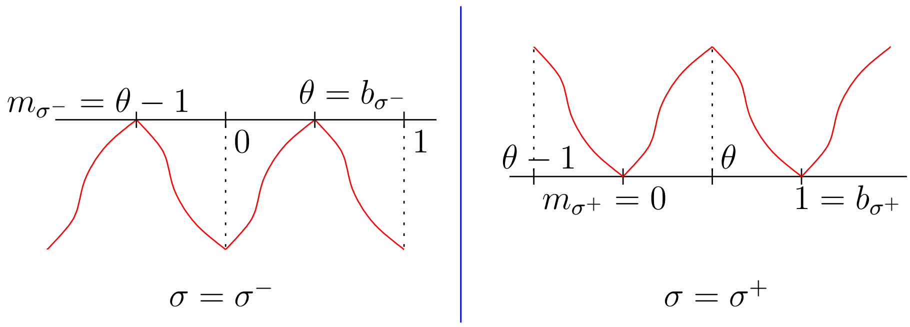

FIGURE 8 – Non-linéarité monostable f

Pour la preuve de ce théorème, nous renvoyons le lecteur à la démonstration du Théorème 2.33 qui est fait pour la non-linéarité F en générale. Les résultats du Théorème 1.6, la Proposition 1.7 et du Théorème 1.8 sont illustrés dans Figure 9.

2 Enoncés des résultats : murs de dislocations

Dans la deuxème partie de la thèse, nous nous intéressons au phénomène d'accumulation de dislocations dans les murs de dislocations qui peu être remarqué dans les matériaux réel qui contiennent des dislocations. Notre objectif est d'étudier la dynamique des dislocations qui interagissent ensemble et forment des murs de dislocations.

Nous considérons plusieurs lignes de dislocation parallèles à l'axe-z et qui se déplaçent horizontalement. Ensuite, nous considérons la section tranversale de ces lignes et nous obtenons des contreparties à deux dimensions où chaque ligne de dislocation est représentée par sa position $(x_i(t), i) \in \mathbb{R}\times\mathbb{Z}$. Le modèle qui caractérise l'évolution horizontale est

$$x_i' = \sum_{j \neq i} f(x_j - x_i, j - i) \quad \text{pour} \quad i \in \mathbb{Z}. \tag{14}$$

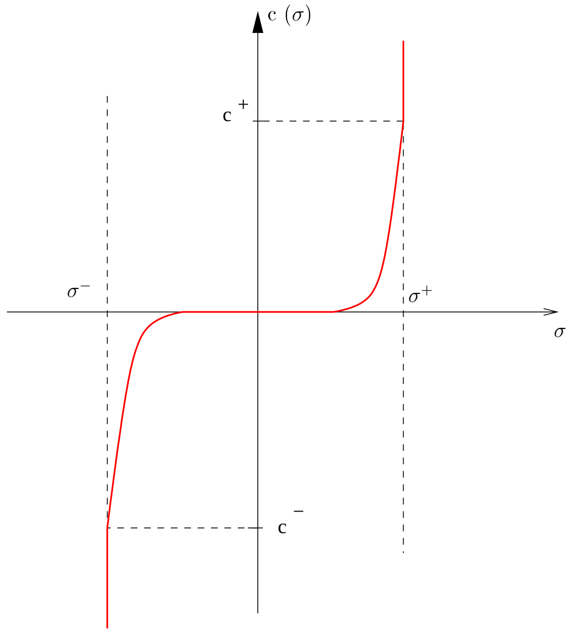

FIGURE 9 – Branches verticales à $\sigma = \sigma^{\pm}$

Ici $f : \mathbb{R} \times \mathbb{Z}\backslash\{0\} \to \mathbb{R}$ est une force anisotrope des interactions à deux corps. Un exemple d'une telle force, selon [41], est

$$f(x, y) = \frac{x(y^2 - x^2)}{(y^2 + x^2)^2}. \tag{15}$$

La force d'interaction donnée par (15) décrit à la fois l'attraction en temps long et la répulsion en temps court entre les atomes. Dans un tel exemple deux particules s'attirent si l'angle vertical entre eux est inférieur à $\frac{\pi}{4}$ et, d'autre part, se repoussent les uns des autres, si l'angle est supérieur à $\frac{\pi}{4}$, voir Figure 10 et Figure 11.

Le système de toutes les particules agissant ensemble sous la force définie ci-dessus peut être réécrit de la manière suivante

$$\begin{cases} \dfrac{d}{dt}X(t) = F(X(t)) \quad t > 0 \\ X(0) = X^0 \in \Omega \cap \ell^\infty, \end{cases} \tag{16}$$

où $X(t) = (x_i(t))_{i\in\mathbb{Z}}, F(X) = (F_i(X))_{i\in\mathbb{Z}}$, $X^0 \in \Omega \cap \ell^\infty$ est une position initiale donnée de dislocations et

$$\Omega = \left\{ X : |x_i - x_j| \leq \sqrt{3 - 2\sqrt{2}}\,|i - j| \right\}. \tag{17}$$

En outre, $F_i(X)$ décrit une force résultante agissant sur la i-ième particule, *i.e.*

$$F_i(X) \overset{def}{=} \sum_{j\neq i} f(x_j - x_i, j-i) \quad \text{pour tout} \quad i \in \mathbb{Z}.$$

Nous avons aussi $\ell^\infty = \ell^\infty(\mathbb{R})$ qui est l'espace de Banach de toutes les suites bornées sur $\mathbb{R}$, doté par la norme $\|\cdot\|_\infty = \sup_n |x_n|$.

Notons que $\arctan\left(\sqrt{3-2\sqrt{2}}\right) = \frac{\pi}{8}$ garantit que la force f limitée à Ω est non seulement attractive mais aussi croissante par rapport à la première variable. Par conséquent, nous sommes en mesure de prouver un principe de comparaison qui nous aide à conclure, par exemple, que les solutions globales de (16) restent dans Ω.

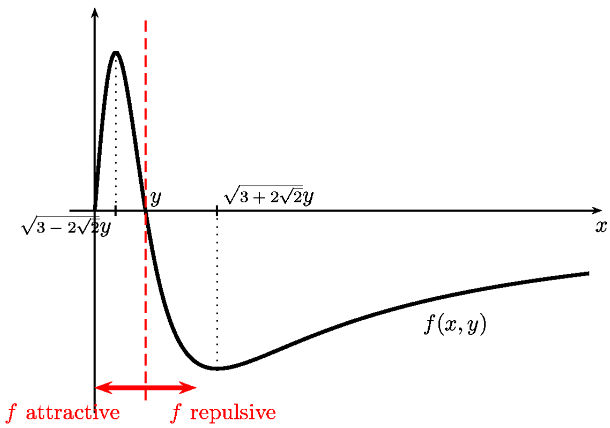

FIGURE 10 – Force d'interaction $f(x,y)$ en fonction de la distance entre deux atomes pour un certain $y \in \mathbb{Z}\backslash\{0\}$ fixé avec la propriété $f(-x,y) = -f(x,y)$. Un angle vertical entre deux particules correspond à $\arctan(\frac{x}{y})$. Ainsi $\frac{\pi}{4}$ se lit comme $x = |y|$.

Nous obtenons les résultats suivants :

Théorème 2.1. (Existence et unicité de la solution)
Soit $X^0 \in \Omega \cap \ell^\infty$. Alors il existe une solution unique $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$ du problème de Cauchy (16)*. En revanche, si la donnèe initiale X^0 est N-périodique (i.e. $x_i^0 = x_{i+N}^0$, pour tout $i \in \mathbb{Z}$), alors la solution reste N-périodique pour tout temps $t > 0$.*

La preuve de l'existence d'une solution globale en temps est basée sur le théorème de Cauchy Lipschitz. Afin de montrer la périodicité de la solution et le fait que $X(t) \in \Omega \cap \ell^\infty$, nous utilisons un résultat de principe de comparaison pour le système (16) (Voir Théorèm 3.1).

Le comportement de la dynamique des particules dans le cas périodique en temps long est donné dans le théorème suivant ce qui prouve que les dislocations s'accumulent en formant ce que nous appelons les murs de dislocations :

Théorème 2.2. (Convergence vers des murs plats)
Soit $X(t)$ la solution N-périodique du problème (16)*. Alors, elle converge vers une solution stationnaire constante du problème* (16) *i.e. pour tout $i \in \mathbb{Z}$, nous avons $\lim_{t\to\infty} x_i(t) = c$, où $c = \frac{1}{N}\sum_{i=1}^{N} x_i^0$ est le barycentre de la donnée initiale.*

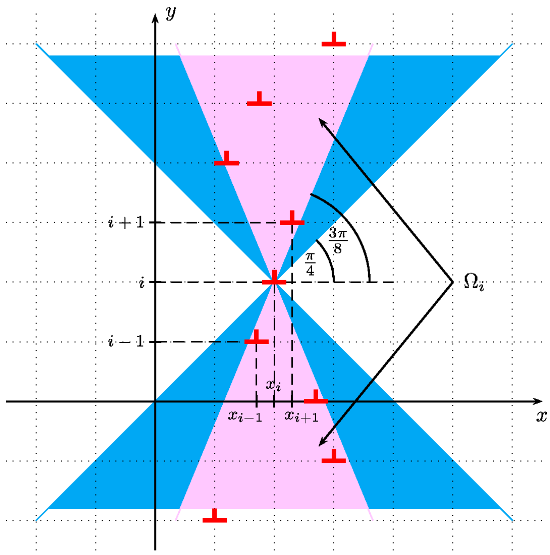

FIGURE 11 – Une particule fixe x_i attire toutes les autres particules si elles sont placées dans une région marquée en bleu et rose. Cependant, la force f est croissante si et seulement si les particules sont situées dans la région repérée en rose. Ce domaine est appelé Ω_i et ainsi nous pouvons représenter Ω, défini dans (17), comme $\Omega = \cap_{i\in\mathbb{Z}}\Omega_i$.

Voir Sections 4 et 6 dans le Chapitre 4 pour la démonstration de ce théorème et des expériences numériques qui montrent la convergence. Nous avons aussi prouvé le résultat de la l^p-contraction de solutions périodiques :

Proposition 2.3. (l^p contraction)
Soit $X(t)$ et $Y(t)$ deux solutions N-périodique du problème (16) *avec les données initiales N-périodiques X^0 et Y^0 respectivement. Alors l'estimation suivante*

$$\|X(t) - Y(t)\|_p \leq \|X^0 - Y^0\|_p, \quad \textit{pour tous } t > 0$$

est vraie pour $p \geq 2$.

Pour la démonstration de cette proposition, nous renvoyons le lecteur à la Section 5 dans le Chapitre 4.

Chapitre 1

General introduction

This thesis focuses on the study of the dislocations dynamics of dislocations in the crystal lattice. Our work was splitted into two parts : the first part is concerned of the horizontal motion of a chain of atoms containing a dislocation. In this part, we study the existence and uniqueness of traveling waves solutions , that illustrate the movement of the dislocation, for different (bistable and monostable) non-linearity types (cf. Chapters 2 and 3). The second parts deals with the accumulation of dislocations forming what walls of dislocation. We prove the existence and uniqueness of the solution for the dynamical system that describes the motion of group of dislocations and we prove that the periodic solution converge to flat walls of dislocation (cf. Chapter 4).

1 Physical motivation

1.1 Historical view about dislocations

Dislocation is a kind of flaw that consists of purely geometrical faults in the crystal lattice. It can be defined by specifying which atoms are dislocated or misconnected, distorting the host crystal lattice, with respect to the perfect crystal (defect-free structure of the host crystal). Dislocation has a typical length of order $10^{-6}m$ and thickness of order $10^{-9}m$ and it has any effect on the lattice at distances greater than few interatomic spacings (see Figure 1.1 for an example of observation of dislocations). For a more comprehensive discussion about dislocation, we refer to the classical texts of Hirth, Lothe [74], Read [99], Hull, Bacon [78] and Bulatov, Cai [27].

The concept of such dislocation arises naturally as a result of the crystallographic [1] nature of plastic flow and it corresponds to a discontinuity in the organization

1. Crystalline solids are materials in which the constituent atoms are arranged in a pattern

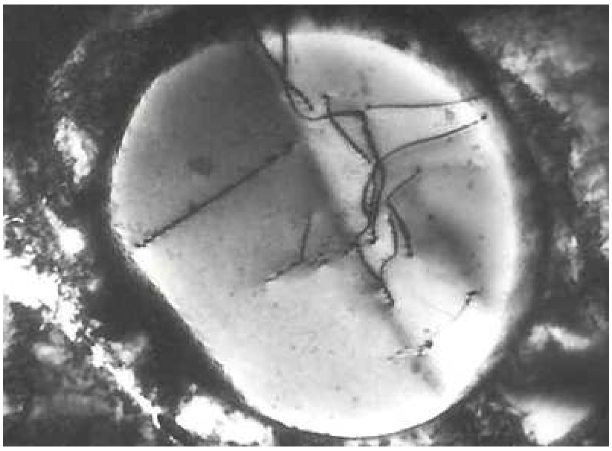

FIGURE 1.1 – Micrograph of dislocations in stainless steel.

of a crystalline structure. Smekal [103] first pointed out that properties of crystal are related to the absence (ideal crystal) and presence (real crystal) of defects in crystal ; for example, mechanical strength properties like elasticity, compressibility are highly sensitive to the crystal perfection (independent of defects) while and semiconductivity and plasticity depend on the defects.

In the late 19th century, Vito Volterra [114] examined mathematical properties of singularities produced by cutting and shifting matter in a continuous solid body. In 1934, Taylor [108], Polanyi [98] and Orowan [94] introduced dislocations into physics interested in understanding what the atoms do when crystal deform. They independently proposed that dislocations may be responsible for a crystal's ability to deform plastically.

The ubiquity and the importance of dislocations for the crystal plasticity other aspects of material behavior have been regarded since 1950s when the first sightings of crystal dislocations were reported in transmission electron microscopy (TEM) experiments (see [75] and [23]). Other evidence which contributed appreciably to the universal acceptance of the existence of dislocations in crystals, was the reconciliation of the classical theory of crystal growth with the experimental observations of growth rates (see [57] and [110]).

Each Dislocation is characterized by it's Burgers vector and the local line direction vector. Here we distinguish the two prevalent types of dislocations : *edge dislocation,* when the Burgers vector is perpendicular to the line direction vector and *screw dislocation,* when the two vectors are parallel.

---

that repeats itself periodically in three dimensions forming the crystal structure of the crystalline.

In this thesis, we are interested in studying the dynamics of a chain of atoms interacting together and containing an *edge dislocations*. To this end, we give in the next subsection some additional information about the formation of *edge dislocation* and explain how the dislocation move (relocating phenomena).

1.2 Dynamics of edge dislocation

Formation of edge dislocation : In real crystals, a number of physical processes produce dislocations. For example, dislocations can appear by shearing along crystal planes, or by condensation of interstitials (extra atoms in the lattice) or vacancies (empty atomic sites) (see [27] and [78]). In other words, an edge dislocation can be simply created by inserting an extra half plane of atoms into a perfect crystal from above or by removing a half-plane of atoms from below (see Figure 1.2).

Movement of dislocation : The atoms in a crystal containing a dislocation are displaced from their perfect lattice sites, and the resulting distortion produces a stress field in the crystal around the dislocation. The dislocation is therefore a source of

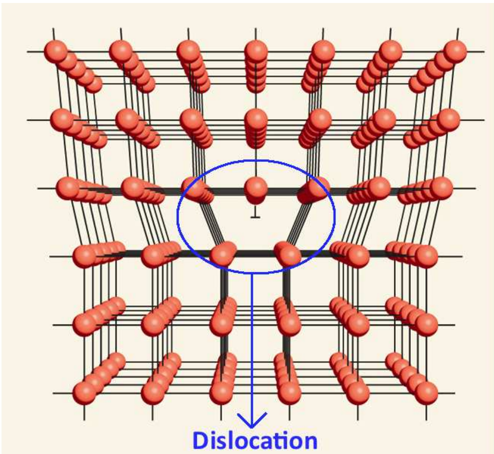

FIGURE 1.2 – An edge dislocation : dislocation line is marked by the symbol $\perp$.

internal stress in the crystal. In addition, these atoms are not rigidly bound to each other but are elastically coupled. Thus due to internal tresses which is induced by other dislocations or any other strain-producing defects, temperature or when a sufficiently high stress [2] is applied to a crystal (containing dislocations) (see [99]), the dislocations can move over small distances and their motion provides a mechanism

2. The applied stress required to overcome the lattice resistance to the movement of the dislocation is the Peierls-Nabarro stress (see [97] and [91])

for a crystal to deform plastically by slip (in case of edge dislocation).
This movement, of *edge dislocation*, takes place along the *glide plane* or *slip plane*, that contains both the Burgers and the dislocation line vectors, and the slipping phenomena occurs in the direction of closest atomic packing in the slipping plane and not in the direction of maximum resolved shear stress (we refer to Hirth and [74] for further mathematical discussions about the motion of dislocations).

In Figure 1.3, we see how, under sufficient stress, the atomic bond joining atoms 1 and 3 is broken while a new atomic bond is built between atoms 1 and 2. We also see how the *edge dislocation* moves one lattice space along the *slip plane*. Such moving phenomena perpetuates until the dislocation reaches a stable state and this will cause a new rearrangement of the crystal structure.

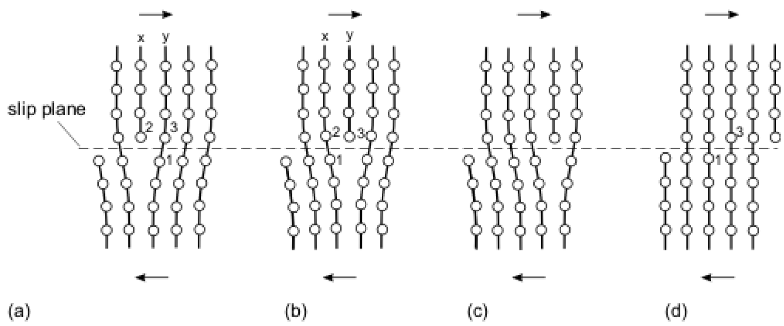

FIGURE 1.3 – Movement of an edge dislocation : the arrows indicate the applied shear stress (taken for [78]).

2 Announcing our results : traveling waves

The atoms belonging to the interface layer are subjected to an external periodic potential produced by the surrounding atoms of the lattice ; this idea gives a birth to the Frenkel-Kontorova model, which is one of the models that describes the dynamics of plane defects. We start this section by a historical view about the Frenkel-Kontorova models and traveling waves then we present our results.

2.1 Frenkel-Kontorova model

Frenkel-Kontorova (FK) model was firstly analytically treated in 1929 by Dehlinger [38] for early work on imperfection in crystal. later in 1938, this model was introduced, as a dynamical discrete model, by Frenkel and Kontorova who invented this simple one-dimensional model for describing the structure and dynamics of a crystal lattice in the vicinity of a dislocation core [3].

Meanwhile, the Frenkel-Kontorova model has become also a model for an adsorbate layer on the surface of a crystal, for ionic conductors, or glassy materials and for sliding friction. The FK model can be also derived for the problem of crowdion in a metal (see Paneth [96] and Frenkel [58, 59]) when one extra atom is inserted into a closely packed row of atoms in a metal with an ideal crystal lattice.

Frenkel-Kontorova model is a simple one dimensional model that describes the dynamics of a chain of particles, presented schematically in Figure 1.4, coupled by a harmonic springs with the nearest-neighbors in the presence of an external periodic potential [4]. For a panoramic view on the general properties and dynamics of solid state models (including the FK model) and summarize the results that involve fundamental physical concepts, we send the reader to the works of Braun and Kivshar [24, 25]. This mechanical model can be derived from the standard Hamiltonian :

$$\mathcal{H} = K + U, \tag{1.1}$$

where K is the kinetic energy and U is the potential one.

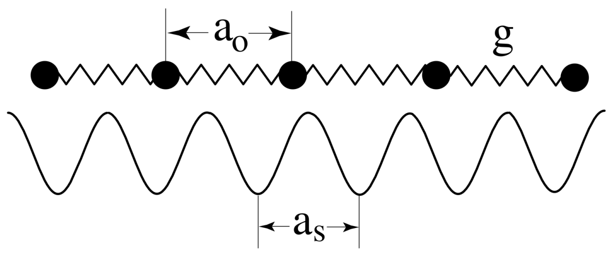

FIGURE 1.4 – A chain of particles interacting via harmonic springs with elastic coupling g is subjected to the action of an external periodic potential with period a_s (taken from [25]).

3. Frenkel and Kontorova denied explicitly in [60] the relation between the analytical solution of a uniformly moving single kink they proved and the dislocation concept developed by Taylor [108], Polanyi [98] and Orowan [94].

4. The periodicity of the potential reflects the periodicity of crystal structure.

The kinetic energy is defined classically by :

$$K = \frac{m}{2} \sum_i \left(\frac{dX_i(t)}{dt} \right)^2, \tag{1.2}$$

where m is the mass of the particle who are supposed to be uniform and $X_i(t)$ is the position of the i-th particle in the chain. The potential energy in decomposed into two parts :

$$U = U_{sub} + U_{int}, \tag{1.3}$$

where

$$U_{sub} = \frac{\varepsilon}{2} \sum_i \left(1 - \cos\left(\frac{2\pi X_i(t)}{a_s} \right) \right) \tag{1.4}$$

characterizes the interaction of the chain of atoms with an external periodic on-site potential of ε potential amplitude and period a_s (see Figure 1.4). The second part of the potential energy (given in (1.3))

$$U_{int} = \frac{g}{2} \sum_i \left(X_{i+1}(t) - X_i(t) - a_0 \right)^2, \tag{1.5}$$

stands for a linear coupling between the nearest neighbors of the chain. In this part, g represents the elastic constant of the harmonic springs and a_0 is the equilibrium distance of the inter-particle potential, in the absence of the on-site potential (see Figure 1.4).

Plugging the Kinetic and the potential energies in (1.1), we get the following Hamiltonian

$$\mathcal{H} = \sum_i \left\{ \frac{m}{2} \left(\frac{dX_i(t)}{dt} \right)^2 + \frac{\varepsilon}{2} \left(1 - \cos\left(\frac{2\pi X_i(t)}{a_s} \right) \right) + \frac{g}{2} \left(X_{i+1}(t) - X_i(t) - a_0 \right)^2 \right\}, \tag{1.6}$$

which can be interpreted under the following simplifying physically relevant assumptions :

(i) The particles of the chain can move along one direction only.

(ii) The general form of the substrate energy is

$$U_{sub} = \sum_i V_{sub}(X_i(t)) \tag{1.7}$$

and we only consider the first term of the Fourier series expansion of the function $V_{sub}(x)$.

(iii) Only the coupling between the nearest-neighbors is included in the inter-particle interaction energy

$$U_{int} = \sum_i V_{int}(X_{i+1}(t) - X_i(t)), \tag{1.8}$$

and, we only realize the harmonic interaction upon expanding $V_{int}(x)$ in a Taylor series, so that $g = V''_{int}(a_0)$.

Introducing now the dimensionless variables, we re-write the Hamiltonian (1.6) in the conventional form, $H = 2\mathcal{H}/\varepsilon$,

$$H = \sum_i \left\{ \frac{1}{2}\left(\frac{dX_i(t)}{dt}\right)^2 + (1 - \cos(X_i(t))) + \frac{g}{2}\left(X_{i+1}(t) - X_i(t) - a_0\right)^2 \right\}, \tag{1.9}$$

where $a_0 \to \frac{2\pi}{a_s}a_0$, $X_i \to \frac{2\pi}{a_s}X_i$, $t \to \frac{2\pi}{a_s}\sqrt{\frac{\varepsilon}{2m}}t$ and the dimensionless coupling constant is changed as $g \to (\frac{a_s}{2\pi})^2 g(\frac{\varepsilon}{2})^{-1}$. Under such a renormalization, the Hamiltonian (1.9) describes a harmonic chain of particles of equal unit mass moving in a sinusoidal external potential with period $a_s = 2\pi$ and amplitude $\varepsilon = 2$.

Remark 2.1. (Corresponding physical values)
In order to obtain all the physical values in the corresponding dimensional form, one should multiply the spatial variables by $\frac{a_s}{2\pi}$*, the frequencies by* $\frac{2\pi}{a_s}\sqrt{\frac{\varepsilon}{2m}}$*, the masses by* m *and the energies by* $\frac{\varepsilon}{2}$*.*

Now, from the Hamiltonian (1.9), we obtain the relevant equation [5] of motion of a discrete chain

$$\frac{d^2X_i(t)}{dt^2} + \sin(X_i(t)) - g(X_{i+1}(t) - 2X_i(t) + X_{i-1}(t)) = 0, \tag{1.10}$$

where the equilibrium lattice spacing a_0 is replaced by the value $X_i - X_{i-1}$.

For more information about the Frenkel-Kontorova model, we suggest for the reader to have a look on the good book of Braun and Kivshar [25].

Simple Frenkel-Kontorova model

The simplest Frenkel-Kontorova model that describes the evolution of a chain of atoms of uniform mass m is given by

$$m\frac{d^2X_i}{dt^2} + \gamma\frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} - \sin(2\pi(X_i - L)) - \sin(2\pi L), \tag{1.11}$$

5. The condition for a solution with no forces on the atoms is that $\partial H/\partial X_i = 0$ for all i.

where again $X_i(t) \in \mathbb{R}$ denotes the position of $i \in \mathbb{Z}$ particle at time t, $\frac{dX_i}{dt}$ and $\frac{d^2X_i}{dt^2}$ are respectively the velocity and the acceleration of the ith particle, γ denotes the friction constant. Here, $-\sin(2\pi L)$ is the constant driving force that will cause the movement of the chain of atoms and $-\sin(2\pi(X_i - L)$ denotes the force created by the periodic potential whose period is assumed to be 1.

Fully overdamped FK model

In order to get the fully overdamped FK model, we assume that the mass is negligible in comparison with the friction term ; i.e.

$$m << \gamma.$$

For simplicity, we set one the friction constant ($\gamma = 1$). Thus we obtain the following one dimensional discrete reaction-diffusion equation

$$\frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} - \sin(2\pi(X_i - L) - \sin(2\pi L). \tag{1.12}$$

2.2 Traveling waves

Traveling waves are particular solutions invariant with respect to space translation and have the form

$$X_i(t) = \phi(i + ct), \tag{1.13}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the traveling wave and c is the velocity of propagation of ϕ (see Figure 1.5).

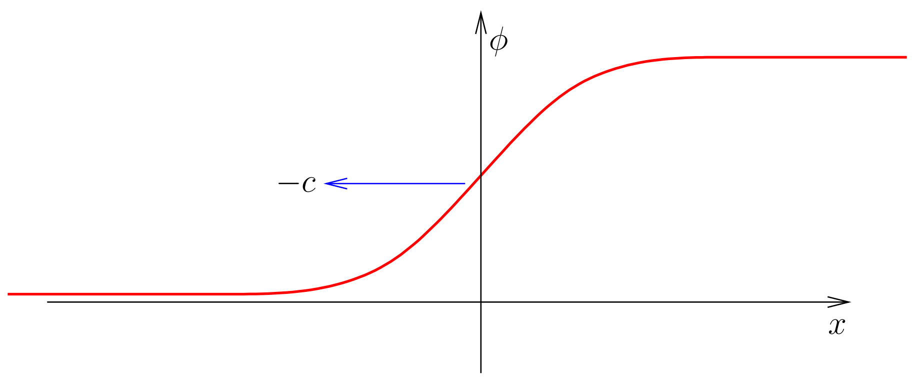

FIGURE 1.5 – Traveling wave moving to the left with velocity c when $c > 0$.

In the past decades, traveling wave solutions have been extensively and intensively studied. Thus more and more evidence indicates that the traveling wave solutions play an important role in the study of lattice dynamical systems. More precisely, the traveling wave solutions can determine the long term behavior of the corresponding initial value problems of lattice dynamical systems, which partly arise from the stability of traveling wave solutions ; e.g. we can refer to [13, 30, 66, 87, 88].

For nonlinear reaction-diffusion equations models describing a variety of physical, chemical and biological phenomena, traveling wave solutions are important since in many situations they determine the long term behavior of other solutions, and account for phase transitions between different states of physical systems, propagation of patterns, and domain invasion of species in population biology. Moreover, the existence of traveling waves appears to be very common in nonlinear equations and the importance of the traveling waves solutions is the possibility to use them to determine the behavior of the solutions of general Cauchy-type problems. It is proved, for some cases, that solutions of the Cauchy problem converge to traveling waves in some sense (by speed or by profile)(see [12, 79, 82, 81, 92]).

The study of traveling waves in reaction-diffusion equations is of independent interest and has a substantial history. It can be traced back to the pioneering works of Fisher [52] and Kolmogorov, Petrovsky, and Piskunov [83] in 1937, in order to describe the propagation of mutant genes that are advantageous to the survival of populations distributed in linear habitats. Since then, this field has gone through enormous continuous growth and development.

After the celebrated paper [83] in 1937, the problem of studying traveling wave solutions for parabolic equations attracted much attention. This is a very rich subject of a great relevance in genetic theory (see, for instance, Aronson and Weinberger [6, 7], Barenblatt and Zel´dovich [10], Fife [48]. See also Freidlin [56], Rothe [100] or Stokes [105] and Murray [90] for a derivation of reaction-diffusion equation in models for population dynamics (like models for the spread of advantegeous genetic traits in a population).

The theory of traveling waves was also developed in chemical physics, like the work of Zel´dovich and Frank-Kamenetskii [125, 126] and [55] in the combustion theory, the work of Semenov [102, 115] on cold flames.

Many physical, chemical and biological phenomena which are observed experimentally can be modeled, according to the type of non-linearity, by traveling wave solutions of parabolic systems see ([8, 9, 12, 13, 45, 46, 109]). We also refer the reader to Fife [48], Hadeler and Rothe [69] ; and to the important contributions by Kanel´ [79, 80, 82] and the celebrated papers of Fife and McLeod [49, 50] which settled most issues in great generality.

We also mention here the related works on traveling wave solutions of spatially discrete reaction-diffusion type equations of Zinner and his coworkers [128, 129, 130], Fath [44], Erneux and Nicolis [43], Gao [62], Mallet-Paret[88], Carpio et al. [28] and the seminal work of Weinberger [116].

Study of traveling waves arises in nonlinear nonlocal differential equations in domains with nonlocal interactions, such as on a spatial lattice, Hankerson and Zinner [73]. They have been also studied, since the classical work of Fife and Mcleod [49], for nonlocal evolution equations [12, 30, 39, 51], for spatially varying systems [4], and in the context of numerical discretizations [21].

Traveling waves were also studied for reaction-diffusion-convection equations in periodic media, see [119, 120, 121, 122] ; and for various heterogeneous media Xin [123], Weinberger [117], Berestycki and Hamel [14], Berestycki, Hamel and Nadirachvili [16], Berestycki, Hamel and Roques [17, 18].

The reader may also consult the excellent survey by A.I. Volpert [113] written as some comments to the famous paper of Kolmogorov, Petrovsky and Piskunov [83] ; and the list [63, 64, 89, 93, 95, 101, 107, 111, 112, 127] and the references therein.

In our work, we look for traveling waves, of the form (1.13), for the discrete reaction-diffusion equation (1.12), and satisfying

$$\begin{cases} \phi' \geq 0 \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{1.14}$$

Here, we point out that condition (1.14) reflects the existence of a defect of one lattice space, called dislocation. Moreover, expression (1.13) means that the defect moves with velocity c under the driving force $sin(2L)$. In addition, ϕ is a phase transition between $\phi(-\infty)$ and $\phi(+\infty)$, which are two “stable” equilibriums of the crystal.

Clearly, if we plug (1.13) into (1.12), the profile ϕ and the velocity c have to satisfy the equation

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)), \tag{1.15}$$

with $z = i + ct$ and f_L defined as

$$f_L(x) := -\sin(2\pi(x-L) - \sin(2\pi L). \tag{1.16}$$

In this thesis, due to the equivalence (when $c \neq 0$) between the solutions of (1.12) and (1.15), we will focus on Equation (1.15) and it's generalizations.

Theorem 2.2. (Existence and uniqueness of traveling waves for the (FK) model ([1, Theorem 1.1]))
There exists a unique real c and a function $\phi : \mathbb{R} \to \mathbb{R}$ solution of

$$\begin{cases} c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1, \end{cases} \tag{1.17}$$

in the classical sense, if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, if $c \neq 0$, the profile ϕ if unique (up to space translation) and $\phi' > 0$ on $\mathbb{R}$.

Let us, in this thesis, mention that our results about existence and uniqueness of traveling wave still true even for less regular non-linearity in comparison with the Frenkel-Kontorova non-linearity. To make this point clear, consider the function

$$G(X_{i-1}, X_i, X_{i+1}) := \max\left(\frac{1}{2}, X_{i-1}\right) + \min\left(\frac{1}{2}, X_{i+1}\right) - X_i - \frac{1}{2} + f_L(X_i), \tag{1.18}$$

where always f_L defined in (1.16). Then consider the system

$$\dot{X}_i = G(X_{i-1}, X_i, X_{i+1}) \quad \text{for } i \in \mathbb{Z}. \tag{1.19}$$

Theorem 2.3. (Existence and uniqueness of traveling waves for (1.18) **model ([1, Theorem 1.2]))**
For any $L \in \left(\frac{-1}{4}, \frac{1}{4}\right) \setminus \{0\}$, the results of Theorem 2.2 hold true for the system

$$\begin{cases} c\phi'(z) = G(\phi(z-1), \phi(z), \phi(z+1)) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1, \end{cases} \tag{1.20}$$

Here, we mention that Theorem 2.2 has been proved in several works (see for instance, the pioneering works [130] and [73], and [88] in full generality). However, the result of Theorem 2.3 is new. Notice that this result is for instance not included in Mallet-Paret's work [88], since G does not satisfy $\frac{\partial G}{\partial X_{i-1}} > 0$ and $\frac{\partial G}{\partial X_{i+1}} > 0$. Such a condition is important in [88] to construct the traveling waves for bistable nonlinearities using deformation (continuation) method. They have used such continuation argument to connect the discrete dynamical system that he studied and a PDE model for which the existence and uniqueness are known.

2.3 Results for the Generalized model

In this thesis, we got similar results in a framework general than (1.15). Assume that we have a chain of $N+1$ interacting atoms and let F be a real function (whose properties will be specified in Subsection 2.3.1 and 2.3.2), then consider the following generalized equation with $\sigma \in \mathbb{R}$:

$$c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma, \tag{1.21}$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$ such that

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \;\; \text{if} \;\; i \neq j, \tag{1.22}$$

which does not restrict the generality. The goal is to find both the profile ϕ and the velocity c for different non-linearity types of F.

Remark that if we take $N = 2$, $r_0 = 0$, $r_1 = -1$, $r_2 = 1$ and we set $\sigma = 0$ and

$$F = F_0(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + f_L(X_0), \tag{1.23}$$

with f_L defined in (1.16), the we get (1.15).

Let

$$r^* = \max_{i=0,...,N} |r_i| \tag{1.24}$$

and, for simplicity, set for a general function h

$$F((h(z+r_i))_{i=0,...,N}) = F(h(z+r_0), h(z+r_1), ..., h(z+r_N)).$$

We now introduce the viscosity notion of solutions that we will use throughout the whole thesis. To this end, let

$$u^*(y) = \limsup_{x\to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x\to y} u(x)$$

be the upper and the lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u.

Definition 2.4. (Viscosity solution)
Let $I = I' = \mathbb{R}$ (or $I = (-r^*, +\infty)$ and $I' = (0, +\infty)$) and $u : I \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and a function F defined on $\mathbb{R}^{N+1}$.

- *The function u is a sub-solution (resp. a super-solution) of*

$$cu'(x) = F((u(x+r_i))_{i=0,...,N}) + \sigma \quad on \quad \mathbb{R}, \tag{1.25}$$

if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(\mathbb{R})$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at x^*, we have

$$c\psi'(x^*) \leq F((u(x^*+r_i))_{i=0,...,N}) + \sigma \quad \Big(resp.\ c\psi'(x^*) \geq F((u(x^*+r_i))_{i=0,...,N}) + \sigma\Big).$$

- *A function u is a viscosity solution of (1.25) if u^* is a sub-solution and u_* is a super-solution.*

2.3.1 Stress σ is null ($\sigma = 0$)

We assume, in this subsection, that the external stress $\sigma = 0$ and we consider a function $F : [0,1]^{N+1} \to \mathbb{R}$. Our aim is to construct the traveling waves solution of

$$c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)), \tag{1.26}$$

and (1.14). Because of (1.14), we can normalize the limits of the profile ϕ at infinity as follows

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \tag{1.27}$$

Note that the function F_0 defined in (1.23) is compatible with the normalization condition (1.27). Therefore, the system that we will study is illustrated as

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{1.28}$$

Define now the restriction f of F over the diagonal as

$$f(v) = F(v, ..., v) \quad \text{for every } v \in [0,1]. \tag{1.29}$$

For example, when $F = F_0$ in (1.23), we have that $f = f_L$.

Clearly, if ϕ is a solution of the system (1.28), then the profile ϕ is bounded and monotone over $\mathbb{R}$. This means that the derivative ϕ' tends to zero as $x \to \pm\infty$, and then passing to the limit in the first equation of (1.28), we obtain (always if ϕ exists)

$$\left\| \begin{array}{l} F(0, ..., 0) = f(0) = 0 \\ F(1, ..., 1) = f(1) = 0. \end{array} \right.$$

Therefore, in order to prove the existence of monotone and bounded traveling waves solution of (1.28), then it is obviously necessary that $F(0, ..., 0) = 0 = F(1, ..., 1)$ and this will be clear in our assumptions.

Let $F : [0,1]^{N+1}$ and assume that :

Assumption $(A_{\mathbf{Lip}})$ **:**

Regularity : F is globally Lipschitz continuous over $[0,1]^{N+1}$.

Monotonicity : $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Existence of hull function. This result is adapted to our problem proceeding the joint work of Forcadel, Imbert and Monneau [53].

Lemma 2.5. (Existence of a hull function ([1, Lemma 2.6]))
Let F be a given function satisfying assumption (A_{Lip}) and $p > 0$. There exists a unique λ_p such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense) :

$$\begin{cases} \lambda_p h_p' = F((h_p(y + pr_i))_{i=0,\dots,N}) \quad on \quad \mathbb{R} \\ h_p(y+1) = h_p(y) + 1 \\ h_p'(y) \geq 0 \\ |h_p(y+y') - h_p(y) - y'| \leq 1 \text{ for all } y' \in \mathbb{R}. \end{cases} \tag{1.30}$$

Such a function h_p is called a hull function. Moreover, there exists a constant $K > 0$, independent on p, such that

$$|\lambda_p| \leq K(1+p).$$

Here, we mention that this Lemma is proved in [53] only for $r_i \in \mathbb{Z}$, however, the proof for the general case $r_i \in \mathbb{R}$ is exactly the same.

Comparison principle on both half-lines. Under assumption (A_{Lip}), we get the following comparison principle results on both half-lines :

Proposition 2.6. (Comparison principle on $(-\infty, r^*]$ ([1, Theorem 4.1]))
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) and assume that

$$\left| \begin{array}{l} \text{there exists } \beta_0 > 0 \text{ such that if} \\ Y = (Y_0, ..., Y_N),\ Y + (a, ..., a) \in [0, \beta_0]^{N+1} \\ \text{then } F(Y + (a, ..., a)) < F(Y) \text{ if } a > 0. \end{array} \right. \tag{1.31}$$

Let u, $v : (-\infty, r^*] \to [0,1]$ be respectively a sub and a super-solution of

$$cu'(x) = F((u(x + r_i))_{i=0,\dots,N}) \quad on \quad (-\infty, 0) \tag{1.32}$$

in the sense of Definition 2.4. Assume moreover that

$$u \leq \beta_0 \quad on \quad (-\infty, r^*]$$

and

$$u \leq v \quad on \quad [0, r^*].$$

Then

$$u \leq v \text{ on } (-\infty, r^*].$$

Using the transformation

$$\left\| \begin{array}{l} \widehat{u}(x) := 1 - u(-x),\ \widehat{v}(x) := 1 - v(-x) \\ \widehat{F}(X) := F(1 - X_0, ..., 1 - X_N) \\ \widehat{c} := -c \text{ and } \widehat{r_i} := -r_i, \end{array} \right. \tag{1.33}$$

we can simply get the following comparison principle on $[-r^*, +\infty)$:

Corollary 2.7. (Comparison principle on $[-r^*, +\infty)$ ([1, Corollary 4.2]))
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) and assume that :

$$\left| \begin{array}{l} \text{there exists } \beta_0 > 0 \text{ such that if} \\ X = (X_0, ..., X_N),\ X + (a, ..., a) \in [1 - \beta_0, 1]^{N+1} \\ \text{then } F(X + (a, ..., a)) < F(X) \text{ if } a > 0. \end{array} \right. \tag{1.34}$$

Let $u,\ v : [-r^*, +\infty) \to [0,1]$ be respectively a sub and a super-solution of (1.32) on $(0, +\infty)$ in sense of Definition 2.4. Moreover, assume that

$$v \geq 1 - \eta_0 \quad on \quad [-r^*, +\infty),$$

and that

$$u \leq v \quad on \quad [-r^*, 0].$$

Then

$$u \leq v \text{ on } [-r^*, +\infty).$$

Remark 2.8. (Inverse monotonicity)
Notice that assumptions (1.31) and (1.34) are satisfied if F is C^1 on a neighborhood of $\{0\}^{N+1}$ and $\{1\}^{N+1}$ in $[0,1]^{N+1}$ and $f'(0) < 0,\ f'(1) < 0$. This condition means that 0 and 1 are stable equilibria.

Studying the existence and uniqueness of traveling waves, is in fact strongly correlated to the state of zeros of F ($\{0\}^{N+1}$ and $\{1\}^{N+1}$) and then to the source type non-linearity f. Hereafter, our study of traveling waves is made taking into account the type of nonlinear source. We distinguish two nonlinear source types : bistable and monostable non-linearities.

1. **Bistable case.**

We say that the non-linearity source is of bistable case if f (defined in (1.29)) satisfies (see Figure 1.6) :

$$\begin{array}{l} f(0) = 0 = f(1) \text{ and there exists } b \in (0,1) \text{ such that} \\ f(b) = 0,\ f_{|_{(0,b)}} > 0,\ f_{|_{(b,1)}} < 0 \text{ and } f'(b) > 0. \end{array}$$

In other words, f is bistable since the zeros 0 and 1 are stable [6] (because f is non-decreasing over a neighborhood of 0 and 1 in $[0,1]$).

Bistable nonlinearities occurs rather in the description of chemical reactions, in particular to explain the phases transitions and the propagation of interfaces. Indeed, the two states $x = 0$ and $x = 1$ represents the stable steady states of the system as we said before and the traveling wave ϕ describes the transitions from one stable

6. A zero of F is stable if it is stable for the dynamics $\dot{X} = F(X)$.

state to another with constant speed c. The prototype of bistable function is given by $f(x) = x(b-x)(x-1)$. For more details, we refer to the articles of Chen [29], Fife [47, 48], Fife, Mcleod [49] and to the book of Murray [90] and the references therein.

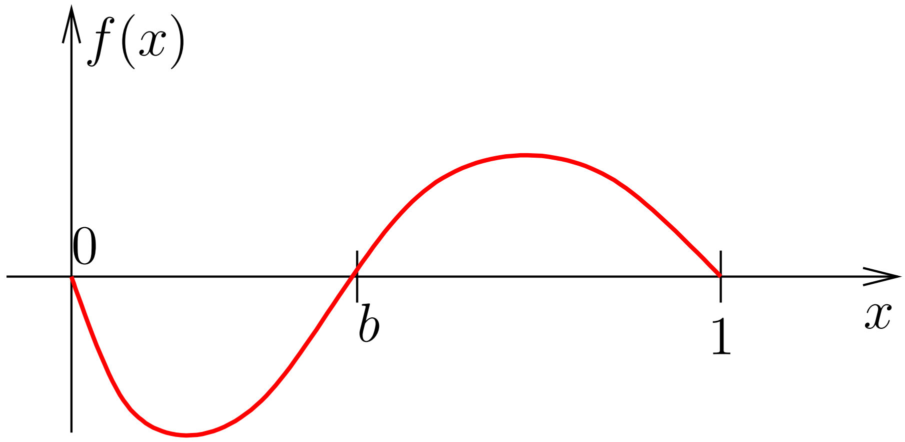

FIGURE 1.6 – Bistable non-linearity source f.

For the bistable case, we study the existence of traveling waves and the uniqueness of the velocity and the profile.

Existence of traveling waves. Here, we resume our main result about the existence of traveling waves. Assume that

Assumption (B) :

Instability : $f(0) = 0 = f(1)$ and there exists $b \in (0,1)$ such that $f(b) = 0$, $f_{|_{(0,b)}} < 0$, $f_{|_{(b,1)}} > 0$ and $f'(b) > 0$.

Smoothness : F is C^1 in a neighborhood of $\{b\}^{N+1}$.

Theorem 2.9. (Existence of a traveling wave ([1, Theorem 1.4]))
Under assumptions (A_{Lip}), (B), there exist a real $c \in \mathbb{R}$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ that solves

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1 \end{cases} \tag{1.35}$$

in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$.

Remark 2.10. *The point b is supposed to be unstable and this is the meaning of the condition $f'(b) > 0$. Moreover, to avoid the unstability at infinity, we assume that F is smooth over a neighborhood of $\{b\}^{N+1}$.*

Our method to prove the existence is completely new. In our approach, the existence of traveling waves relies on the construction of hull functions of slope p (like correctors) for an associated homogenization problem. Passing to the limit $p \to 0$, one major difficulty is to identify a traveling wave joining two stable states. In particular, we have avoided this traveling wave to degenerate to the intermediate unstable state.

In the case of bistable non-linearity f, the existence and uniqueness of traveling waves are well known for the model equation

$$u_t = u_{xx} + f(u). \tag{1.36}$$

Starting from equation (1.36), and using a continuation method, Bates et al. [12] proved in particular the existence of traveling waves for the convolution model

$$u_t = J * u - u + f(u) \tag{1.37}$$

where J is a kernel.

In [88], Mallet-Paret (see also Carpio et al. [28] for semi-linear case) used also a global continuation method (i.e. a homotopy method) to get existence of traveling waves for bistable non-linearities and information about the uniqueness and the dependence of solutions on parameters. This continuation argument was applied to connect the discrete dynamical system that he studied and a PDE model (similar to (1.36)) for which the existence and uniqueness are known. He proved the continuation between the solutions of the two systems using a general Fredholm alternative method [87] for the linearized traveling waves equations.

Traveling waves were also studied by Chow et al. [35] for lattice dynamical systems (lattice ODE's) and for coupled maps lattices (CML's) that arise as time-discretizations of lattice ODE's. Using a geometric approach, the authors studied the stability of traveling waves for lattice ODE's and proved existence of traveling waves of their time discretized CML's. More precisely, they constructed a local coordinate system in a tubular neighborhood of the traveling wave solution in the phase space of their system.

Zinner [128] proved the existence of traveling waves for the discrete Nagumo equation

$$\dot{x}_i = d(x_{i+1} - 2x_i + x_{i-1}) + f(x_i) \quad i \in \mathbb{Z}. \tag{1.38}$$

The construction is done by introducing first a simplified problem (using a projection to 0 or 1 for $|i| \geq N$) for which the existence is attained by Brouwer's fixed point theorem. Hankerson and Zinner [73] also proved existence of traveling waves (for an equation more general than (1.38)) obtained as the long time limit of the solution with Heaviside initial data, using an interesting lap number argument.

In [34], Chen, Guo and Wu constructed traveling waves for a lattice ODE's with bistable non-linearity. They rephrase the solution ϕ of (1.26) as a fixed point of an integral formulation. First, they considered a simplified problem (using a projection on 0 or 1 for large indices $|i| \geq N$) and they show, for any $c \neq 0$, the existence of a solution $\phi^{N,\,c}$ using the monotone iteration method. Finally, they recover the existence of a solution in the limit $N \to +\infty$ for a suitable choice $c = c(N)$ converging to a limit velocity.

Uniqueness of the velocity. In order to prove the uniqueness of the velocity, we need to introduce the following assumption :

Assumption (C) : Inverse monotonicity close to $\{0\}^{N+1}$ and $E = \{1\}^{N+1}$
There exists $\beta_0 > 0$ such that for $a > 0$, we have

$$\begin{cases} F(X + (a, ..., a)) < F(X) & \text{for all} \quad X,\, X + (a, ..., a) \in [0, \beta_0]^{N+1} \\ F(X + (a, ..., a)) < F(X) & \text{for all} \quad X,\, X + (a, ..., a) \in [1 - \beta_0, 1]^{N+1}. \end{cases}$$

Theorem 2.11. (Uniqueness of the velocity ([1, Theorem 1.5 (a)]))
Assume (A_{Lip}) and let (c, ϕ) be a solution of

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) & on \quad \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1. \end{cases} \tag{1.39}$$

Under the additional assumption (C), the velocity c is unique.

The proof of uniqueness of the velocity is based on the comparison principle results on both half-lines ; that we get under the assumption (C).

Uniqueness of the profile for $c \neq 0$. The uniqueness of the profile is either proved by a Strong Maximum Principle or using the weak asymptotics of the profile. Assumptions are splitted into two similar categories : category $^+$ contains the assumptions superscript by $^+$; and we use such assumptions to prove the uniqueness of the profile when $c > 0$. The category $^-$ consists of assumptions superscript by $^-$ and are used to prove of uniqueness when $c < 0$.

Strong Maximum Principle. We start with the following half Strong Maximum Principle result that we base on to get the full Strong Maximum Principle results

Lemma 2.12. (Half Strong Maximum Principle ([1, Lemma 6.1]))
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying assumption (A_{Lip}) and let ϕ_1, $\phi_2 : \mathbb{R} \to [0,1]$ be respectively a viscosity sub and a super-solution of (1.25). Assume that

$$\begin{cases} \phi_2 \geq \phi_1 \quad on \quad \mathbb{R} \\ \phi_2(0) = \phi_1(0). \end{cases}$$

If $c > 0$ (resp. $c < 0$), then

$$\phi_1 = \phi_2 \quad for\ all \quad x \leq 0 \quad (resp.\ \ x \geq 0).$$

In order to announce the complete Strong Maximum Principle results ; we assume that

Assumption (D+) :
i) All the r_i's "Shifts" have the same sign : Assume that $r_i \leq 0$ for all $i \in \{0, ..., N\}$.

ii) Strict monotonicity : F is increasing in X_{i^+} with $r_{i^+} > 0$.

Assumption (D−) :
i) All the r_i's "Shifts" have the same sign : Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

ii) Strict monotonicity : F is increasing in X_{i^-} with $r_{i^-} < 0$.

Strong Maximum Principle under $(D\pm)$ $i)$ **:** we have

Proposition 2.13. (Strong Maximum principle under $(D\pm)$ $i)$ **([1, Lemma 6.4]))**
Assume $c > 0$ (resp. $c < 0$) and let F satisfying (A_{Lip}) and $(D+)$ $i)$ (resp. $(D-)$ $i)$). Let ϕ_1, ϕ_2 be two solutions of (1.25) such that

$$\phi_1(0) = \phi_2(0) \quad and \quad \phi_1 \leq \phi_2 \quad on \quad \mathbb{R}.$$

Then

$$\phi_1(x) = \phi_2(x) \quad for\ all \quad x \in \mathbb{R}.$$

The Proof of Proposition 2.13 is based on Lemma 2.12 and the following comparison principle result :

Lemma 2.14. (Comparison principle, under $(D\pm)$ $i)$ **([1, Lemma 6.3]))**
Assume that $c > 0$ (resp. $c < 0$) and let F satisfying (A_{Lip}) and $(D+)$ $i)$ (resp. $(D-)$ $i)$). Let ϕ_1, ϕ_2 be respectively a viscosity sub and a viscosity super-solution of (1.25). Assume that $\phi_1(0) = \phi_2(0)$ and

$$\phi_1 \leq \phi_2 \quad on \quad [-r^*, 0] \quad \big(resp.\ on\ [0, r^*]\big),$$

then

$$\phi_1(x) \leq \phi_2(x) \quad for\ all \quad x \geq -r^* \quad \big(resp.\ x \leq r^*\big).$$

Strong Maximum Principle under $(D\pm)$ $i)$**) :** we have

Proposition 2.15. (Strong Maximum Principle under $(D\pm)$ $ii)$ **([1, Lemma 6.2]))**
Let $F : [0,1]^{N+1} \to \mathbb{R}$ *satisfying* (A_{Lip})*. Let* $\phi_1,\ \phi_2 : \mathbb{R} \to [0,1]$ *be respectively a viscosity sub and super-solution of (1.25) such that*

$$\phi_2 \geq \phi_1 \quad on \quad \mathbb{R} \quad \text{and} \quad \phi_2(0) = \phi_1(0)$$

a) If F *is increasing w.r.t.* X_{i_0} *for certain* $i_0 \neq 0$ *then*

$$\phi_2(kr_{i_0}) = \phi_1(kr_{i_0}) \quad for\ all \quad k \in \mathbb{N}.$$

b) If we assume moreover that F *satisfies* $(D+)$ $ii)$ *if* $c > 0$*, or* $(D-)$ $ii)$ *if* $c < 0$*, then*

$$\phi_1(x) = \phi_2(x) \quad for\ all\ x \in \mathbb{R}.$$

Here, we also note that the proof of item $b)$ in Proposition 2.15 is established using Lemma 2.12 and item $a)$.

Asymptotics of the profile. Let ϕ be a solution of (1.25) and assume that

Assumption (E+) :

i) Strict monotonicity close to 0 **:** Assume that $\dfrac{\partial F}{\partial X_{i^+}}(0) > 0$ with $r_{i^+} > 0$.

ii) Smoothness close to $\{0\}^{N+1}$ **:**
There exists $\nabla F(0)$, with $f'(0) < 0$, and there exists $\alpha \in (0,1)$ and $C_0 > 0$ such that for all $X \in [0,1]^{N+1}$

$$|F(X) - F(0) - X.\nabla F(0)| \leq C_0 |X|^{1+\alpha}.$$

Assumption (E−) :
i) Strict monotonicity close to 1 **:** Assume, for $E = (1, ..., 1) \in \mathbb{R}^{N+1}$, that $\dfrac{\partial F}{\partial X_{i^-}}(E) > 0$ with $r_{i^-} < 0$.

ii) Smoothness close to $\{1\}^{N+1}$ **:**
There exists $\nabla F(E)$ with $f'(1) < 0$ and there exists $\alpha \in (0,1)$ and $C_0 > 0$ such that for all $X \in [0,1]^{N+1}$

$$|F(X) - F(E) - (X - E).\nabla F(E)| \leq C_0 |X - E|^{1+\alpha},$$

with $E = (1, ..., 1) \in \mathbb{R}^{N+1}$.

We have the following asymptotics of the profile ϕ near $\pm\infty$:

Proposition 2.16. (Asymptotics near $\pm\infty$ ([1, Proposition 5.1]))
Consider a function F defined on $[0,1]^{N+1}$ satisfying (A_{Lip}) and (C), and assume that $c \neq 0$. Then

i) **asymptotics near $-\infty$**
Let $\phi : \mathbb{R} \to [0,1]$ be a solution of (1.25), satisfying

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi \geq \delta > 0 \quad \text{on} \quad [0, r^*]$$

for some $\delta > 0$ and assume $(E+)$ $ii)$. If there exists a unique $\lambda^+ > 0$ solution of

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) e^{\lambda r_i} \tag{1.40}$$

then for any sequence $(x_n)_n$, $x_n \to -\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{\phi(x + x_{n'})}{e^{\lambda^+ x_{n'}}} \longrightarrow Ae^{\lambda^+ x} \quad \text{locally uniformly on } \mathbb{R} \text{ as } n' \to +\infty.$$

ii) **asymptotics near $+\infty$**
Let $\phi : \mathbb{R} \to [0,1]$ be a solution of (1.25), satisfying

$$\phi(+\infty) = 1 \quad \text{and} \quad \phi \leq 1 - \delta < 1 \quad \text{on} \quad [0, r^*]$$

for some $\delta > 0$ and assume $(E-)$ $ii)$. If there exists a unique $\lambda^- < 0$ solution of

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(1, ..., 1) e^{\lambda r_i}, \tag{1.41}$$

then for any sequence $(x_n)_n$, $x_n \to +\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{1 - \phi(x + x_{n'})}{e^{\lambda^- x_{n'}}} \longrightarrow Ae^{\lambda^- x} \quad \text{locally uniformly on } \mathbb{R} \text{ as } n' \to +\infty.$$

Now, we resume our main result about the uniqueness of the profile in both cases $c > 0$ or $c < 0$:

Theorem 2.17. (Uniqueness of the profile ([1, Theorem 1.5 (b)]))
Assume (A_{Lip}) and let (c, ϕ) be a solution of (1.39). If $c \neq 0$, then under the additional assumptions (C) and $(D+)$ $i)$ or $ii)$ or $(E+)$ if $c > 0$ (resp. $(D-)$ $i)$ or $ii)$ or $(E-)$ if $c < 0$), the profile ϕ is unique (up to space translation) and $\phi' > 0$ on $\mathbb{R}$.

Notice that, using weak asymptotics (in comparison with those of Mallet-Paret [88]) allow us to have weaker assumptions. Here, we re-emphasize that we get the existence of solution under very weak assumptions in comparison with similar results in previous works ; moreover, our method is still effective in higher dimensional problems. Consider, for instance, the model

$$\frac{d}{dt}X_I(t) = f(X_I) + \sum_{|J|=1}\Big(X_{I+J} - X_I\Big) \qquad (1.42)$$

that describes the interaction of an atom $I \in \mathbb{Z}^n$ with its nearest neighbors ($X_I \in \mathbb{R}$ denotes the position of atom I). We can look for traveling waves $X_I(t) = \phi(ct+\nu\cdot I)$ that propagates in a direction $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. That is for $z = ct + \nu \cdot I$, we look for ϕ solution of

$$c\phi'(z) = f(\phi(z)) + \sum_{|J|=1}\Big(\phi(z+\nu\cdot J) - \phi(z)\Big),$$

where f denotes a bistable non-linearity. Setting $r_j := \nu \cdot J$, we recover an equation of type (1.26) for $N = 2n$. Therefore, the results of higher dimensional problems follow from our one dimensional results (Theorems 2.9 and 2.17) as far as they hold for general shifts r_j's.

Finally, we remark that Theorems 2.2 and 2.3 are particular cases of Theorems 2.9, 2.11 and Theorem 2.17. Indeed, existence of the solution in Theorem 2.3 follows from Theorem 2.9 and the fact that $b \neq \frac{1}{2}$ in assumption (B), when $L \in \left(-\frac{1}{4}, \frac{1}{4}\right) \setminus \{0\}$. Uniqueness of the profile in Theorem 2.3 follows from Theorem 2.17 and the fact that the function G defined in 1.18 verifies assumptions $(E^{\pm})$.

2. Monostable.

The non-linearity source f is said to be monostable if f satisfies (see Figure 1.7) :

Assumption (P_{Lip}) :

Positive monostability :

Let $f(v) = F(v, ..., v)$ such that $f(0) = f(1) = 0, \quad f > 0 \;$ in $\; (0, 1)$.

In other words, the non-linearity f is monostable if it has only one stable zero and the other is unstable (in this case, 1 is stable and 0 is unstable). This type of non-linearity appears in describing population dynamics or combustion, see Berestycki, Larrouturou [19], Kanel´[80, 81] and Zel´dovich, Frank-Kamenetskii [125]. An example of such non-linearity is $f(x) = x(1-x)$ or $f(x) = x^2(1-x)^2$.

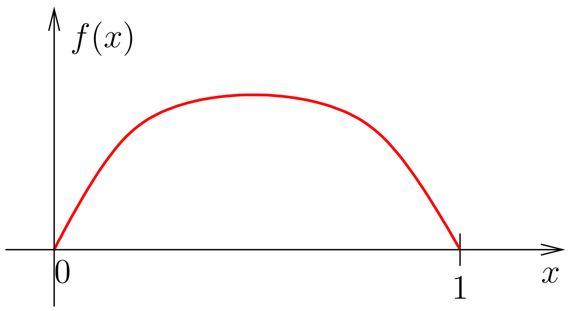

FIGURE 1.7 – Positive monostable non-linearity source f.

Under the assumption (P_{Lip}), Kolmogorov, Petrovsky and Piskunov [83] proved, for the nonlinear diffusion equation

$$u_t = u_{xx} + f(u) \quad \text{in} \quad \mathbb{R}, \tag{1.43}$$

the existence of traveling waves connecting zeroes of f for a branch of possible velocities $c \geq c^*$, where c^* is the minimal velocity [69]. They also assert that if, furthermore, f satisfies

$$f'(1) < 0 \quad \text{and} \quad 0 < f(s) \leq f'(0)s \text{ for all } s \in (0,1),$$

then $c^* = 2\sqrt{f'(0)}$, (see [16]).

For the discrete reaction diffusion equation (1.26), we prove under the assumption (P_{Lip}) the existence of a branch of traveling waves solutions for $c \geq c^+$ for some critical velocity c^+, with no existence of solution for $c < c^+$. We also give certain sufficient conditions to insure that $c^+ \geq 0$ and we give an example when $c^+ < 0$. We as well prove a lower bound of c^+, precisely we show that $c^+ \geq c^*$, where c^* is associated to a linearized problem at infinity. On the other hand, under a KPP condition we show that $c^+ \leq c^*$. we also give an example where $c^+ > c^*$.

Many works have been devoted for such equation that appears in biological models for developments of genes or populations dynamics and in combustion theory (see for instance, Aronson, Weinberger [6, 7] and Hadeler, Rothe [69]). For more developments and applications in biology of reaction diffusion equations, the reader may refer to [111, 70, 71, 72, 76, 131, 14, 15, 92] and to the references cited therein.

We also distinguish [77] (for nonlocal non-linearities with integer shifts) and [36, 86, 116, 124] (for problems with linear nonlocal part and with integer shifts also). See also [67] for particular monostable non-linearities with irrational shifts. We also refer to [65, 31, 68, 32, 33, 71, 130] for different positive monostable non-linearities. We have to underline the work of Hudson and Zinner [77] (see also [130]), where they proved the existence of a branch of solutions $c \geq c^*$ for general Lipschitz non-linearities (with possibly an infinite number of neighbors $N = +\infty$, and possibly p types of different particles, while $p = 1$ in our study) but with integer shifts $r_i \in \mathbb{Z}$. However, they do not state the nonexistence of solutions for $c < c^*$. Their method of proof relies on an approximation of the equation on a bounded domain (applying Brouwer's fixed point theorem) and an homotopy argument starting from a known solution. The full result is then obtained as the size of the domain goes to infinity. Here we underline that our results hold for the fully nonlinear case with real shifts $r_i \in \mathbb{R}$.

In [32], Chen and Guo proved the existence of a solution starting from an approximated problem. They constructed a fixed point solution of an integral reformulation (approximated on a bounded domain) using the monotone iteration method (with sub and supersolutions). This approach was also used to get the existence of a solution in [61, 33, 67, 68]. Another approach based on recursive method for monotone discrete in time dynamical systems was used by Wienberger et al. [86, 116]. See also [124], where this method is used to solve problems with a linear nonlocal part. In a third approach [65], Guo and Hamel used global space-time sub and supersolutions to prove the existence of a solution for periodic monostable equations.

There is also a wide literature about the uniqueness and the asymptotics at infinity of a solution for a monostable non-linearities, see for instance [31, 76] (for a degenerate case), [32, 33] and the references therein. Let us also mention that certain delayed reaction diffusion equations with some KPP-Fisher non-linearities do not admit traveling waves (see for example [61, 130]).

Finally, we mention that our method opens new possibilities to be adapted to more general problems. For example, we can think to adapt our approach to a case with possibly p types of different particles similar to [54]. The case with an infinite number of neighbors $N = +\infty$ could be also studied. We can also think to study fully nonlinear parabolic equations.

Our main result is :

Theorem 2.18. (Existence of traveling waves for a branch of velocities ([2, Theorem 1.1]))
Assume (A_{Lip}) *and* (P_{Lip}). *Then there exists a real* c^+ *such that for all* $c \geq c^+$ *there exists a traveling wave* $\phi : \mathbb{R} \to \mathbb{R}$ *solution (in the viscosity sense (see Definition 2.4)) of*

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1. \end{cases} \tag{1.44}$$

On the contrary for $c < c^+$, *there is no solution of (1.44).*

We believe that the critical velocity c^+ contains information about $f'(0)$; similar to classical result in [83] which asserts that the critical velocity of reaction diffusion equation (1.43) is $c^+ = 2\sqrt{f'(0)}$. This shows that when F is only Lipschitz, it becomes very difficult to capture c_F^+ and to show Theorem 2.18. To prove Theorem 2.18, we first construct traveling waves solutions of (1.44) for every $c >> 1$. Then, we show the existence of the critical velocity c^+ by perturbing F. Finally, we fill the gap by proving the result for every $c \geq c^+$.

Up to our knowledge, Theorem 2.18 is the first result for discrete dynamics with real shifts $r_i \in \mathbb{R}$ in the fully nonlinear case. Even when $r_i \in \mathbb{Z}$, the only result that we know for fully nonlinear dynamics is the one of Hudson and Zinner [77]. However, the nonexistence of solutions for $c < c^+$ is not addressed in [77].

See Figure 1.8 for an explicit Lipschitz non-linearity example for which our result (Theorem 2.18) is still true, even if $f'(0)$ is not defined.

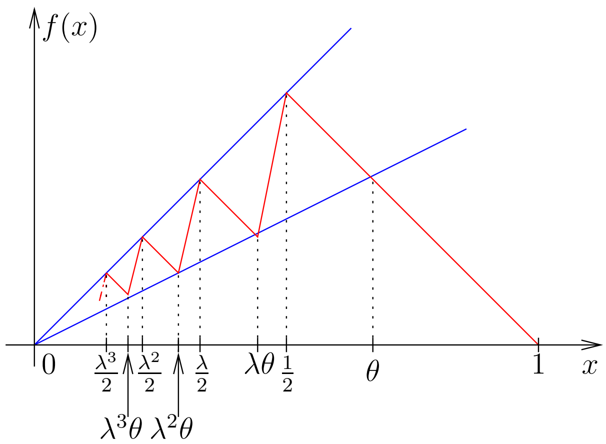

FIGURE 1.8 – Lipschitz positive degenerate monostable non-linearity ; the rest of the figure over $[0, \frac{\lambda^3}{2}]$ is completed by dilation of center 0 and ratio λ.

If F is smooth and strictly monotone near $\{0\}^{N+1}$ (F satisfies (P_{C^1}) below which is stronger then (P_{Lip})), then we can show the following result about the sign of the critical velocity c^+ (given in Theorem 2.18) :

Assumption (P_{C^1}) :

Positive degenerate monostability :

Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$:

F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$.

Proposition 2.19. (Non-negative c^+ for particular F ([2, Proposition 1.3])) *Consider a function F satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 2.18. Then we have $c^+ \geq 0$, if one of the three following conditions i), ii) or iii) holds true :*

i) **Reflection symmetry of F**
Let $X = (X_i)_{i\in\{0,...,N\}} \in [0,1]^{N+1}$. Assume that for all $i \in \{0,...,N\}$ there exists $\bar{i} \in \{0,...,N\}$ such that $r_{\bar{i}} = -r_i$; and

$$F(\overline{X}) = F(X) \quad \textit{for all} \;\; X \in [0,1]^{N+1},$$

where

$$\overline{X}_i = X_{\bar{i}} \quad \textit{for} \quad i \in \{0,...,N\}.$$

ii) **All the r_i's "shifts" are non-negative**
Assume that $r_i \geq 0$ for all $i \in \{0,...,N\}$.

iii) **Strict monotonicity**
Let

$$I = \Big\{ i \in \{1,...,N\} \textit{ such that there exists } \bar{i} \in \{1,...,N\} \;\; \textit{with} \;\; r_{\bar{i}} = -r_i \Big\} \tag{1.45}$$

and assume that

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i\in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) > 0. \tag{1.46}$$

Notice that because of the monotonicity of F in X_j for $j \neq 0$, condition (3.12) is satisfied if

$$\frac{\partial F}{\partial X_0}(0) > 0.$$

Moreover, if

$$I = \{1,...,N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \;\; \text{for all} \;\; i \in I, \tag{1.47}$$

then condition (1.46) is equivalent to $f'(0) > 0$. In particular, under condition *i)* property (1.47) holds true. This shows that condition *iii)* is more general than condition *i)*.

Remark that if we replace (P_{C^1}) by (P_{Lip}) assuming *i)*, *ii)* or *iii)*, we do not know if $c^+ \geq 0$. However, if we don't assume any of the three conditions *i)*, *ii)* or *iii)*, then c^+ could be negative even if F satisfies (P_{C^1}) or more regular. Here is a counter example with $c^+ < 0$:

Proposition 2.20. (Counter example with $c^+ < 0$ ([2, Proposition 1.4]))
There exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity (given in Theorem 2.18) is negative, i.e. $c^+ < 0$.

One of the reasoning behind that could be the following instability result of c^+, which holds under the assumption (P_{Lip}) :

Proposition 2.21. (Instability of the minimal velocity c_F^+ ([2, Proposition 1.2]))
There exists a function F satisfying (A_{Lip}) and (P_{Lip}) with a minimal velocity c_F^+ such that there exists a sequence of functions F_δ (satisfying (A_{Lip}) and (P_{Lip})) with associated critical velocity $c_{F_\delta}^+$ satisfying

$$F_\delta \to F \quad in \quad L^\infty$$

when $\delta \to 0$, but

$$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

Lower bound of c^+. In the next result, we give a lower bound of the critical velocity c^+.

Proposition 2.22. (Lower bound for c^+ ([2, Proposition 1.5]))
Let F be a function satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 2.18 and assume

$$\exists\ i_0 \in \{0, ..., N\} \quad such\ that \quad r_{i_0} > 0 \quad and \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0, \tag{1.48}$$

then

$$c^+ \geq c^*,$$

where

$$c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \quad with \quad P(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) e^{\lambda r_i}. \tag{1.49}$$

The proof of Proposition 2.22 is based on the following Harnack type inequality :

Proposition 2.23. (Harnack inequality ([2, Proposition 9.14]))
Let F be a function satisfying (A_{Lip}) and (P_{Lip}). Assume in addition that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and satisfying (1.48). Let (c, u) with $c \neq 0$ be a solution of

$$\begin{cases} cu'(x) = F((u(x + r_i))_{i=0,\ldots,N}) \quad on \quad \mathbb{R} \\ u' \geq 0 \\ u(-\infty) = 0 \quad and \quad u(+\infty) = 1. \end{cases} \tag{1.50}$$

Then for every $\rho > 0$ there exists a constant $\overline{\kappa}_1 = \overline{\kappa}_1(\rho) > 1$ such that for every $x \in \mathbb{R}$, we have

$$\sup_{B_\rho(x)} u \leq \overline{\kappa}_1 \inf_{B_\rho(x)} u. \tag{1.51}$$

Moreover, there exists $\overline{\kappa}_0 > 1$ such that

$$u(x + r^*) \leq \overline{\kappa}_0 u(x), \tag{1.52}$$

where $r^* = \max_{i=0,...,N} |r_i|$.

Note that, establishing the Harnack inequality (Proposition 2.23) uses the strong maximum principle and the lower bound on a solution results (given below) for an associated evolution problem :

Proposition 2.24. (Strong maximum principle for a linear evolution problem ([2, Proposition 9.11]))
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$. Assume also that F satisfies (1.48). Let $T > 0$ and $u : \mathbb{R} \times [0,T) \to [0,+\infty)$ be a lower semi-continuous function which is a supersolution of

$$u_t(x,t) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0) u(x + r_i, t) \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,T). \tag{1.53}$$

If $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \mathbb{R} \times (0,T)$, then

$$u(x_0 + kr_{i_0}, t) = 0 \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad 0 \leq t \leq t_0.$$

Proposition 2.25. (Lower bound on a solution of the evolution nonlinear problem ([2, Proposition 9.13]))
Consider a function F satisfying $(\tilde{A}_{Lip})$ and (P_{Lip}). Assume moreover that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and that (1.48) holds true. Then there exists $\varepsilon_0 \in (0,1]$ and $T_0 > 0$ such that for all $\delta \in (0, T_0)$ and $R > 0$, there exists $\kappa = \kappa(\delta, R) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, the solution $\psi = \psi_\varepsilon$ of

$$\psi_t(x,t) = F((\psi(x + r_i, t))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \times (0, +\infty) \tag{1.54}$$

with initial condition

$$\psi^*(\cdot, 0) = \varepsilon H^* \quad \text{and} \quad \psi_*(\cdot, 0) = \varepsilon H_* \quad (H = 1_{[0,+\infty)} \text{ is the Heaviside function}), \tag{1.55}$$

satisfies

$$\psi_\varepsilon(x,t) \geq \kappa\varepsilon \quad \text{for all} \quad (x,t) \in [-R, R] \times [\delta, T_0]. \tag{1.56}$$

Remark that the solution of the nonlinear evolution problem (1.54) with initial condition (1.55) is given by the following lemma :

Lemma 2.26. (Existence of a solution for the nonlinear problem ([2, Lemma 9.12]))
Consider a function F satisfying $(\tilde{A}_{Lip})$, (P_{Lip}) and let $\varepsilon \in (0,1]$. Then there exists $\psi : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ a viscosity solution of the nonlinear evolution problem (1.54) with initial condition satisfying (1.55).

Here, it is natural to ask if we may have $c^+ = c^*$ in general or not. Here below, we give an example of a non-linearity where we have $c^+ > c^*$ which answers the question.

Lemma 2.27. (Example with $c^+ > c^*$ ([2, Lemma 8.4]))
Consider the function $F^0 : [0,1]^3 \to \mathbb{R}$ defined as

$$F^0(X_0, X_{-1}, X_1) := g(X_1) + g(X_{-1}) - 2g(X_0) + f(X_0),$$

with $r_0 = 0$, $r_{\pm 1} = \pm 1$ and $f,\ g : [0,1] \to \mathbb{R}$ are C^1 over a neighborhood of 0, Lipschitz on $[0,1]$ and satisfying

$$\begin{cases} f(0) = f(1) = 0 \\ f > 0 \quad \text{on} \quad (0,1) \\ f'(0) > 0 \end{cases} \quad \text{and} \quad \begin{cases} g'(0) = 0 \\ g(1) = 1 + g(0) \\ g' \geq 0. \end{cases}$$

Let c^+ given by Theorem 2.18 (with F replaced by F^0), then

$$c^+ > c^*,$$

where c^* is defined in (1.49).

An example of such g is $g(x) = x - \frac{1}{2\pi}\sin(2\pi x)$. On the other hand, we can find a KPP type condition to insure the inequality $c^+ \leq c^*$, as show the following result :

Proposition 2.28. (KPP condition for $c^+ \leq c^*$ ([2, Proposition 1.6]))
Let F be a function satisfying (A_{Lip}) and (P_{Lip}). Let c^+ given by Theorem 2.18 and assume that F is differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. If moreover F satisfies the KPP condition :

$$F(X) \leq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) X_i \quad \text{for every} \quad X \in [0,1]^{N+1}, \tag{1.57}$$

then $c^+ \leq c^*$ with c^* defined in (1.49).

Remark that the result of Proposition 2.19 under conditions $i)$ and $ii)$ can be deduced directly from Proposition 2.22. Indeed, If F satisfies (P_{C^1}), then we get $P(0) = f'(0) > 0$, where P is defined 1.49. Moreover, if we assume condition $i)$ or $ii)$ in Proposition 2.19, then we obtain that $P'(0) \geq 0$. This implies that $c^* = \inf \dfrac{P(\lambda)}{\lambda} \geq 0$. Using Proposition 2.22, we deduce that $c^+ \geq c^* \geq 0$. This consequence does not hold true for the result of Proposition 2.19 under condition $iii)$. We refer the reader to Chapter 3 (Section 8.1), where we prove Proposition 2.19 using a different approach without passing through this consequence.

2.3.2 Stress σ is non-zero ($\sigma \neq 0$)

Let $\sigma \in \mathbb{R}$ be any constant and define a real function $F : \mathbb{R}^{N+1} \to \mathbb{R}$, the consider, for $\sigma \in \mathbb{R}$, the equation

$$c\phi'(z) = F(\phi(z + r_0), ..., \phi(z + r_N)) + \sigma \quad \text{on } \mathbb{R}. \tag{1.58}$$

We study the existence of traveling waves ϕ solutions of (1.58) and satisfying

$$\begin{cases} \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{1.59}$$

Again, if ϕ is a solution of (1.58) and (1.59), then the profile ϕ is bounded and monotone. Thus, upon passing to the limit in (1.58), we obtain

$$F(\phi(\pm\infty), ..., \phi(\pm\infty)) + \sigma = 0.$$

Therefore, in order to prove that (1.58) and (1.59) admit a solution, then it is necessary that the equation

$$F(v, ..., v) + \sigma := f(v) + \sigma = 0 \quad \text{admits at least two solutions.} \tag{1.60}$$

Before resuming the results that we obtained, let us introduce the following assumptions on F. Let $E = (1, ..., 1)$ and $\Theta = (\theta, ..., \theta) \in \mathbb{R}^{N+1}$ and assume that

Assumption $(\tilde{A}_{C^1})$:

- **Regularity :** F is globally Lipschitz continuous over $\mathbb{R}^{N+1}$ and C^1 over a neighborhood in $\mathbb{R}^{N+1}$ of the two intervals $]0, \Theta[$ and $]\Theta, E[$.
- **Monotonicity :** $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.
- **Periodicity :** $F(X_0+1, ..., X_N+1) = F(X_0, ..., X_N)$ for every $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.

Notice that, since F is periodic in E direction, then F is C^1 over a neighborhood of $\mathbb{R}E\backslash(\mathbb{Z}E \cup \mathbb{Z}\Theta)$. We also have that $f(v+1) = f(v)$.

Assumption $(\tilde{B}_{C^1})$:

Bistability : $f(0) = f(1)$ and there exists $\theta \in (0,1)$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0,\theta) \\ f' < 0 & \text{on} \quad (\theta,1). \end{cases}$$

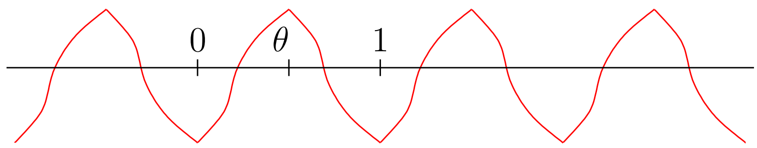

FIGURE 1.9 – Bistable non-linearity f

See Figure 1.9 for an example of f satisfying $(\tilde{B}_{C^1})$. Remark that assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ holds true in particular for the Frenkel-Kontorova model for $\beta > 0$

$$\frac{d}{dt}X_i = X_{i+1} + X_{i-1} - 2X_i - \beta \sin\left(2\pi\left(X_i + \frac{1}{4}\right)\right) + \sigma. \tag{1.61}$$

Definition 2.29. (Range of σ)
Under assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$, define $\sigma^\pm$ as

$$\begin{cases} \sigma^+ = -\min f \\ \sigma^- = -\max f. \end{cases} \tag{1.62}$$

Associate for each $\sigma \in [\sigma^-,\sigma^+]$ the solutions $m_\sigma \in [\theta-1,0]$ and $b_\sigma \in [0,\theta]$ of $f(s) + \sigma = 0$.

Remark 2.30. *i)* **Existence of m_σ and b_σ.** *Under assumption $(\tilde{B}_{C^1})$ and from the definition of $\sigma^\pm$ (see (1.62)), the associated $m_\sigma \in [\theta-1,0]$ and $b_\sigma \in [0,\theta]$ exist uniquely for every $\sigma \in [\sigma^-,\sigma^+]$. This implies that the two maps $\sigma \to m_\sigma$, b_σ are well defined.*

ii) **No solution for $\sigma \notin [\sigma^-,\sigma^+]$.** *From the definition of $\sigma^\pm$ (see (1.62) and from* (1.60), *we conclude that (1.58) and* (1.59) *do not admit a bounded and monotone solution if $\sigma \notin [\sigma^-,\sigma^+]$.*

We distinguish our results according to $\sigma \in [\sigma^-,\sigma^+]$:

i) Bistable case : $\sigma \in (\sigma^-, \sigma^+)$
Because of the definition of $\sigma^\pm$ in (1.62), then for every $\sigma \in (\sigma^-, \sigma^+)$, the function $f + \sigma$ obeys the bistable shape, i.e. $f + \sigma$ satisfies (see Figure 1.10) :

$$\left| \begin{array}{l} f(v) + \sigma = 0 \quad \text{for} \quad v = m_\sigma, \; b_\sigma \quad \text{and} \quad m_\sigma + 1 \\ (f+\sigma)_{|_{(m_\sigma, b_\sigma)}} < 0, \quad (f+\sigma)_{|_{(b_\sigma, m_\sigma+1)}} > 0. \end{array} \right.$$

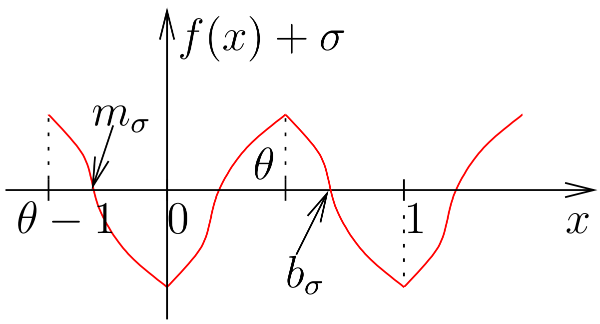

FIGURE 1.10 – Bistable non-linearity f

Theorem 2.31. (Existence of a traveling wave ([2, Theorem 1.7-1 (i)]))
Assume $(\tilde{A}_{C^1})$ *and* $(\tilde{B}_{C^1})$. *For any* $\sigma \in (\sigma^-, \sigma^+)$, *there exists a unique real* $c := c(\sigma)$, *such that there exists a function* $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ *solution (in the viscosity sense) of*

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \text{and} \quad \phi(+\infty) = m_\sigma + 1. \end{cases} \tag{1.63}$$

Remark that in our case, saying that ϕ_σ is a viscosity solution of (1.63) is equivalent to say that ϕ solves (1.63) in the classical sense if $c(\sigma) \neq 0$ and almost everywhere if $c(\sigma) = 0$ (as in Theorem 2.9). See for instance Lemma 2.11 in Chapter 2 about the equivalence between viscosity and almost everywhere monotone solutions.

Proposition 2.32. (Continuity and monotonicity of the velocity function ([2, Theorem 1.7-1 (ii)]))
Under the assumptions $(\tilde{A}_{C^1})$ *and* $(\tilde{B}_{C^1})$, *the map*

$$\sigma \mapsto c(\sigma)$$

is continuous on (σ^-, σ^+) *and there exists a constant* $K > 0$ *such that the function* $c(\sigma)$ *is non-decreasing and satisfies*

$$\frac{dc}{d\sigma} \geq K|c| \quad \text{on} \quad (\sigma^-, \sigma^+)$$

in the viscosity sense. In addition, there exists real numbers $c^- \leq c^+$ such that

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \text{and} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+.$$

Moreover, either $c^- = 0 = c^+$ or $c^- < c^+$.

The continuity of the velocity function $c(\sigma)$ is proved simply by taking a sequence $\sigma_n \in (\sigma^-, \sigma^+)$, then by passing to the limit $n \to +\infty$. Using the comparison principle for an associated evolution problem, we show that the velocity function is non-decreasing over (σ^-, σ^+) and we prove the existence of the critical limits $c^\pm$.

ii) Monostable case : $\sigma = \sigma^\pm$
Since $\sigma^+ = -\min f$, then $f + \sigma^+ \geq 0$ over $\mathbb{R}$, thus the non-linearity $f + \sigma^+$ has a positive monostable shape (see Figure 1.11 for $\sigma = \sigma^+$). Similarly, since $\sigma^- = -\max f$, i.e. $f + \sigma^- \leq 0$, then $f + \sigma^-$ has a negative monostable shape (see Figure 1.11 for $\sigma = \sigma^-$).

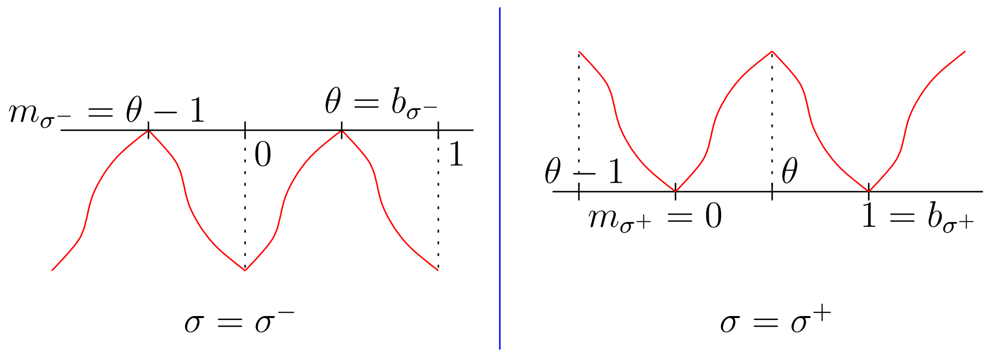

FIGURE 1.11 – Bistable non-linearity f

Theorem 2.33. (Vertical branches for $\sigma = \sigma^\pm$ ([2, Theorem 1.7-2]))
Assume $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. We have

(*i*) **(Existence of traveling waves for $c \geq c^+$ when $\sigma = \sigma^+$ ([2, Theorem 1.7-2 (*i*)]))**
Let $\sigma = \sigma^+$ and c^+ given in Proposition 2.32. Then for every $c \geq c^+$ there exists a traveling wave ϕ solution of

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma^+ & \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 = m_{\sigma^+} \quad \text{and} \quad \phi(+\infty) = 1 = b_{\sigma^+}. \end{cases} \tag{1.64}$$

Moreover, for any $c < c^+$, there is no solution ϕ of (1.64).

(ii) **(Existence of traveling waves for $c \leq c^-$ when $\sigma = \sigma^-$ ([2, Theorem 1.7-2 *(ii)*]))**
Let $\sigma = \sigma^-$ and c^- given in Proposition 2.32. Then for every $c \leq c^-$, there exists a traveling wave ϕ solution of

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^- \quad on \quad \mathbb{R} \\ \phi \textit{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = \theta - 1 = m_{\sigma^-} \quad and \quad \phi(+\infty) = \theta = b_{\sigma^-}. \end{cases} \tag{1.65}$$

Moreover, for any $c > c^-$, there is no solution ϕ of (1.65).

The proof of the existence of vertical branch for $\sigma = \sigma^-$ (Theorem 2.33 *(ii)*) follows from the case $\sigma = \sigma^+$ (Theorem 2.33 *(i)*) using the transformation

$$\left\| \begin{array}{l} \overline{\phi}(z) := \theta - \phi(-z) \\ \overline{F}(X) := F((\theta - X_i)_{i=0,...,N}) \\ \overline{c} := -c, \ \overline{r_i} := -r_i \ \text{ and } \ \overline{\sigma}^+ = -\sigma^-. \end{array} \right. \tag{1.66}$$

Moreover, the variation of the velocity function in terms of the external stress σ. is illustrated in Figure 1.12.

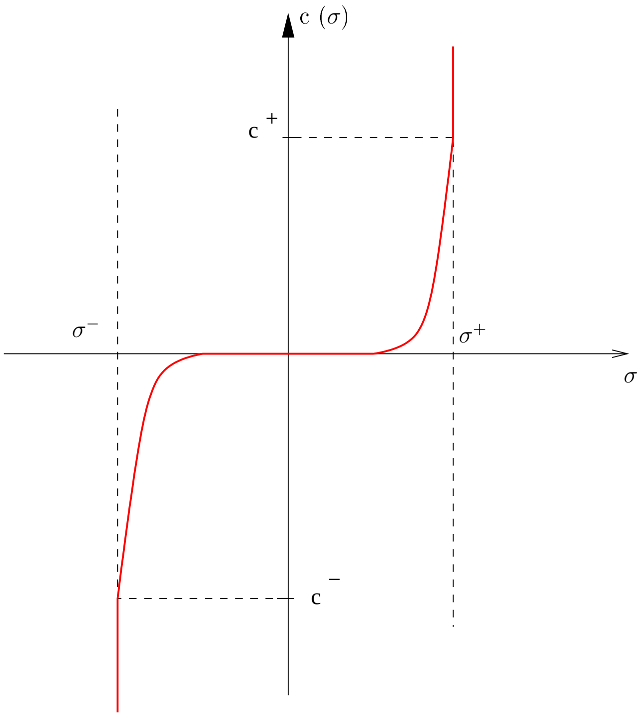

FIGURE 1.12 – Vertical branches at $\sigma = \sigma^{\pm}$

Remark 2.34. (Link between results of Sections 2.3.1 and 2.3.2)
Remark that Theorem 2.18 is a generalization of Theorem 2.33 (*i*) *for* $\sigma = \sigma^+$. *Notice also that the result of Theorem 2.31 is contained in Theorem 2.9.*

Remark 2.35. (sign of c^+ and c^-)
If we can apply Proposition 2.19 for $F + \sigma^+$, *we deduce that* $c^+ \geq 0$. *Similarly, by symmetry (using the transformation* (1.66)*), it is possible to introduce similar assumptions to conclude that* $c^- \leq 0$.

3 Announcing our results : walls of dislocations

We are interested, in this part, in the accumulation phenomenon of disloactions in walls of dislocations which can be seen in real material that contain dislocations. Our aim is to investigate the dynamics of dislocations that interact together and form walls of dislocations.

We consider several dislocation lines parallel to the z-axis and moving horizontaly. Then we consider the cross section of these lines and we get the two-dimensional counterpart where each dislocation line is represented by its position $(x_i(t), i) \in \mathbb{R} \times \mathbb{Z}$. The model that characterize the horizontal evolution is

$$x_i' = \sum_{j \neq i} f(x_j - x_i, j - i) \quad \text{for} \quad i \in \mathbb{Z}, \tag{1.67}$$

where $f \colon \mathbb{R} \times \mathbb{Z} \backslash \{0\} \to \mathbb{R}$ is an anisotropic force of two-body interactions. An example of such a force, according to [41], is

$$f(x, y) = \frac{x(y^2 - x^2)}{(y^2 + x^2)^2}. \tag{1.68}$$

The interaction force given by (1.68) describes both long-range attraction and short-range repulsion between atoms. In such an example two particles attract each other if the vertical angle between them is less then $\frac{\pi}{4}$ and, on the other hand, repel each other if the angle is greater then $\frac{\pi}{4}$, see Figure 1.13 and Figure 1.14.

The system of all particles acting together under the above defined force can be rewritten in the following way

$$\begin{cases} \dfrac{d}{dt} X(t) = F(X(t)) \quad t > 0 \\ X(0) = X^0 \in \Omega \cap \ell^\infty, \end{cases} \tag{1.69}$$

where $X(t) = (x_i(t))_{i \in \mathbb{Z}}, F(X) = (F_i(X))_{i \in \mathbb{Z}}$, $X^0 \in \Omega \cap \ell^\infty$ is some given initial position of dislocations and

$$\Omega = \left\{ X : |x_i - x_j| \leq \sqrt{3 - 2\sqrt{2}}\, |i - j| \right\}. \tag{1.70}$$

Moreover, $F_i(X)$ describes a resultant force acting on an i-th particle, *i.e.*

$$F_i(X) \stackrel{def}{=} \sum_{j\neq i} f(x_j - x_i, j-i) \quad \text{for each} \quad i \in \mathbb{Z}.$$

We also have $\ell^\infty = \ell^\infty(\mathbb{R})$ is the Banach space of all bounded sequences over $\mathbb{R}$ supplemented with the norm $\|\cdot\|_\infty = \sup_n |x_n|$.

Notice here that $\arctan\left(\sqrt{3-2\sqrt{2}}\right) = \frac{\pi}{8}$ which guarantees that the force f restricted to Ω is not only attractive but also nondecreasing with respect to the first variable. Therefore, we are able to prove a comparison principle which helps us to conclude *e.g.* the global-in-time solutions stays in Ω.

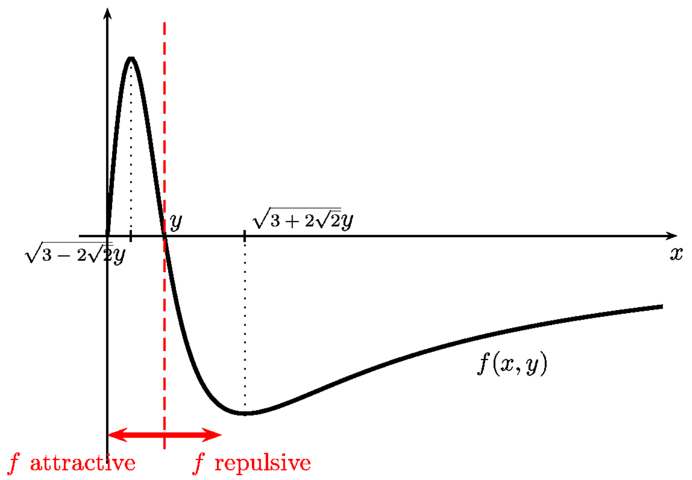

FIGURE 1.13 – Interaction force $f(x,y)$ as a function of the distance between two atoms for some fixed $y \in \mathbb{Z}\backslash\{0\}$ with the property $f(-x,y) = -f(x,y)$. A vertical angle between two particles corresponds to $\arctan(\frac{x}{y})$. Thus $\frac{\pi}{4}$ reads as $x = |y|$.

We have the following results :

Theorem 3.1. (Existence of a unique solution ([3, Theorem 1.2]))
Let $X^0 \in \Omega \cap \ell^\infty$. Then there exists a unique solution $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$ of the Cauchy problem (1.69)*. Moreover, if the initial data X^0 is N-periodic (i.e. $x_i^0 = x_{i+N}^0$, for every $i \in \mathbb{Z}$), then the solution remains N-periodic for every time $t > 0$.*

The proof of existence of global-in-time solution is based on the Cauchy Lipschitz theorem. However, in order to show the periodicity of the solution and the fact that $X(t) \in \Omega \cap \ell^\infty$, we use a comparison principle result for the system (1.69).

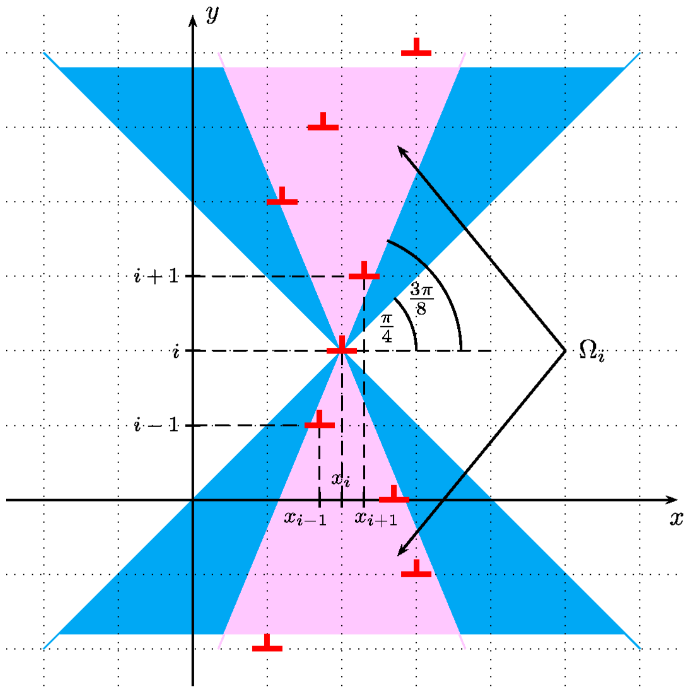

FIGURE 1.14 – A fixed particle x_i attracts all other particles if they are placed in a region marked in blue and pink. However, the force f is non-decreasing only if the particles are located in the region marked in pink. Such domain we call Ω_i and thus we can present Ω, defined in (1.70), as $\Omega = \cap_{i\in\mathbb{Z}}\Omega_i$.

The long time behavior of the dynamics of particles in the periodic case is given in the following theorem which proves that dislocations accumulate forming so-called walls of dislocations :

Theorem 3.2. (Convergence to flat walls ([3, Theorem 1.3]))
Let $X(t)$ be the N-periodic solution of the problem (1.69). Then it converges to a constant stationary solution of the problem (1.69) i.e. for every $i \in \mathbb{Z}$, we have $\lim_{t\to\infty} x_i(t) = c$, where $c = \frac{1}{N}\sum_{i=1}^{N} x_i^0$ is the barycenter of the initial data.

We have also proved the following ℓ^p contraction for periodic solutions :

Proposition 3.3. (l^p contraction ([3, Proposition 1.4]))
Let $X(t)$ and $Y(t)$ be two N-periodic solutions of the problem (1.69) with N-periodic initial data X^0 and Y^0 respectively. Then the following estimate

$$\|X(t) - Y(t)\|_p \leq \|X^0 - Y^0\|_p, \quad \text{for all } t > 0$$

holds true provided $p \geq 2$.

We perform, in this part, some numerical experiments that confirm the convergence to a flat wall result that we obtain in Theorem 3.2. Our adaptive scheme is constructed as follows. Let $N > 0$ denote the total number of interacting particles. Let Δt denote a time-step and let us define an approximate solution of (1.69) by a solution $X^n = (X_1^n, \ldots, X_N^n)$ of the following forward Euler scheme

$$X^{n+1} = X^n + \Delta t F(X^n) \stackrel{def}{=} S(X^n). \tag{1.71}$$

Lemma 3.4. (Monotonicity of the scheme ([3, Lemma 6.1]))
The scheme derived in (4.24) *is monotone if and only if the time-step satisfies* $\Delta t \leq \dfrac{3}{\pi^2}$ *and the initial data* $X^0 \in \Omega$ *defined in* (1.70).

In our numerical experiments we assume the initial data $X^0 \in \Omega \cap \ell^\infty$ which is denoted by "**x**" on the left-upper plot in Figure 1.15. Furthermore, in every picture, by "*" we emphasised what the limit solution is (by Theorem 3.2 the limit solution is at the barycenter of initial data). We observe in Figure 1.15 the evolution of dislocations which eventually converge.

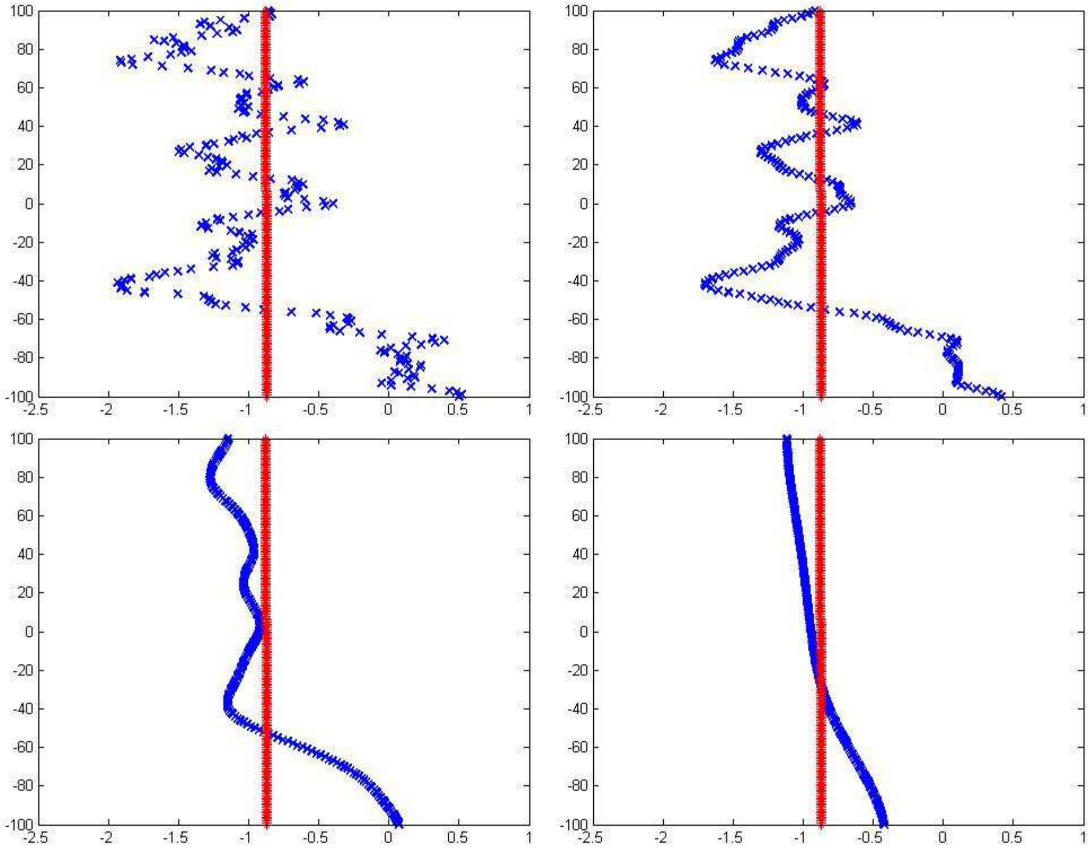

FIGURE 1.15 – Evolution of dislocations of (1.69) with initial data $X^0 \in \Omega$.

Chapitre 2

Existence et unicité d'ondes progressives pour les modèles Frenkel-Kontorova complètement amortis

Ce chapitre est un travail en collaboration avec N. Forcadel et R. Monneau [1]. Dans ce chapitre, nous étudions l'existence et l'unicité d'ondes progressives pour une équation de réaction-diffusion discrète non-linéaire bistable, à savoir une généralisation des modèles de Frenkel-Kontorova complètement amortis. Ces modèles consistent en un système d'EDO qui décrivent la dynamique des défauts cristallins dans une matière solide. Sous des hypothèses très faibles, nous démontrons l'existence d'une solution sous forme d'onde progressive et l'unicité de la vitesse de propagation de cette onde progressive. La question de l'unicité du profil est également étudiée en prouvant un principe du maximum fort ou certains asymptotiques faibles sur le profil à l'infini.

Existence and uniqueness of traveling waves for fully overdamped Frenkel-Kontorova models

M. Al Haj, N. Forcadel, R. Monneau

Abstract

In this article, we study the existence and the uniqueness of traveling waves for a discrete reaction-diffusion equation with bistable non-linearity, namely a generalization of the fully overdamped Frenkel-Kontorova model. This model consists in a system of ODE's which describes the dynamics of crystal defects in a lattice solids. Under very poor assumptions, we prove the existence of a traveling wave solution and the uniqueness of the velocity of propagation of this traveling wave. The question of the uniqueness of the profile is also studied by proving Strong Maximum Principle or some weak asymptotics on the profile at infinity.
Keywords : Frenkel-Kontorova models, traveling waves, viscosity solutions, comparison principle.

1 Introduction

In this work, we are interested in the fully overdamped Frenkel-Kontorova (FK) model which describes the dynamics of crystal defects in a lattice (see for instance the book of Braun and Kivshar [25] for an introduction to this model). This model (and its generalization) is a discrete reaction-diffusion equation with "bistable" non-linearity. For this model, we show the existence and the uniqueness of traveling waves.

1.1 Setting of the problem

We first give an example of the simplest fully overdamped Frenkel Kontorova model, and then we provide a general framework for which we will establish our results.

(i) The simplest Frenkel-Kontorova model

The simplest fully overdamped FK model is a chain of atoms, where the position $X_i(t) \in \mathbb{R}$ at the time t of the particle $i \in \mathbb{Z}$ solves

$$\frac{dX_i}{dt} = X_{i+1} + X_{i-1} - 2X_i - sin(2\pi(X_i - L)) - \sin(2\pi L), \tag{2.1}$$

where $\frac{dX_i}{dt}$ is the velocity of the ith particle, $-\sin(2\pi L)$ is a constant driving force which will cause the movement of the chain of atoms and $sin(2\pi X_i)$ denotes the force created by a periodic potential reflecting the periodicity of the crystal, whose period is assumed to be 1. Set, for simplicity,

$$f_L(x) := -\sin(2\pi(x-L)) - \sin(2\pi L). \tag{2.2}$$

We look for particular *traveling wave* solutions of (2.1), namely solutions of the form

$$X_i(t) = \phi(i + ct) \tag{2.3}$$

with

$$\begin{cases} \phi' \geq 0 \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{2.4}$$

Here c is the velocity of propagation of the traveling wave ϕ, and (2.4) reflects the existence of a defect of one lattice space, called dislocation. Moreover, expression (2.3) means that the defect moves with velocity c under the driving force L. In addition, ϕ is a phase transition between $\phi(-\infty)$ and $\phi(+\infty)$ which are two "stable" equilibriums of the crystal.

Clearly, if we plug (2.3) in (2.1), the profile ϕ and the velocity c have to satisfy

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)), \tag{2.5}$$

with $z = i + ct$ and f_L defined in (2.2).

Due to the equivalence (for $c \neq 0$) between solutions of (2.1) and (2.5), from now on, we will focus on equation (2.5).

Theorem 1.1. (Existence and uniqueness of traveling waves for (FK) model)
There exists a unique real c and a function $\phi : \mathbb{R} \to \mathbb{R}$ solution of

$$\begin{cases} c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f_L(\phi(z)) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{2.6}$$

in classical sense if $c \neq 0$ and almost every where if $c = 0$. Moreover, if $c \neq 0$, then the profile ϕ is unique (up to space translation) and $\phi' > 0$ on $\mathbb{R}$.

This theorem has be proved in several works (see for instance, the pioneering works [128] and [73], and [88] in full generality).

(ii) A simple example not covered by the literature
Define the function G as

$$G(X_{i-1}, X_i, X_{i+1}) := \max\left(\frac{1}{2}, X_{i-1}\right) + \min\left(\frac{1}{2}, X_{i+1}\right) - X_i - \frac{1}{2} + f_L(X_i), \quad (2.7)$$

where f_L defined in (2.2), then consider the following system

$$\dot{X}_i = G(X_{i-1}, X_i, X_{i+1}) \quad \text{for} \quad i \in \mathbb{Z}. \quad (2.8)$$

Theorem 1.2. (Existence and uniqueness of traveling waves for example (2.7))
For any $L \in \left(\frac{-1}{4}, \frac{1}{4}\right) \backslash \{0\}$, the results of Theorem 1.1 hold true for system (2.6) replaced by the following system

$$\begin{cases} c\phi'(z) = G\big(\phi(z-1), \phi(z), \phi(z+1)\big) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \quad (2.9)$$

Up to our knowledge, this result is new. Notice that this result is for instance not included in Mallet-Paret's work [88], since G does not satisfy $\dfrac{\partial G}{\partial X_{i-1}} > 0$ and $\dfrac{\partial G}{\partial X_{i+1}} > 0$. Such a condition is important in [88] to construct the traveling waves using deformation (continuation) method.

(iii) General framework
We now consider a generalization of equation (2.5). To this end, we introduce a real function (whose properties to be specified later in Subsection 1.2) :

$$F : [0,1]^{N+1} \to \mathbb{R}. \quad (2.10)$$

We then consider the following equation

$$c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)), \quad (2.11)$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$. We also normalize the limits of the profile at infinity as follows :

$$\phi(-\infty) = 0, \quad \phi(+\infty) = 1. \quad (2.12)$$

Note that, for $N = 2$ and $F = F_0(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + f_L(X_0)$, equation (2.5) is a particular case of (2.11). Moreover, F_0 is compatible with (2.12).

Assume, without loss of generality, for the whole work that :

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \text{ if } i \neq j.$$

1.2 Main results

In order to present our results, we have to introduce some assumptions on F defined in (2.10). Note that, for later use, we split these assumptions into assumptions (A) and (B).

Assumption (A) :

Regularity : F is globally Lipschitz continuous over $[0,1]^{N+1}$.

Monotonicity : $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

We set $f(v) = F(v, ..., v)$.

Assumption (B) :

Instability : $f(0) = 0 = f(1)$ and there exists $b \in (0,1)$ such that $f(b) = 0$, $f_{|(0,b)} < 0$, $f_{|(b,1)} > 0$ and $f'(b) > 0$.

Smoothness : F is C^1 in a neighborhood of $\{b\}^{N+1}$.

Remark 1.3.

1. *The point b is supposed to be unstable and this is the meaning of the condition $f'(b) > 0$.*
2. *Notice that the instability part of assumption (B) means in particular that f is of "Bistable" shape (see [88]).*

Theorem 1.4. (Existence of a traveling wave)
Under assumptions (A), (B), there exist a real $c \in \mathbb{R}$ and a function $\phi : \mathbb{R} \to \mathbb{R}$ that solves

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) & \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1 \end{cases} \tag{2.13}$$

in the classical sense if $c \neq 0$ and almost every where if $c = 0$.

Our method to construct a solution relies on the construction of a hull function for an associated homogenization problem (see the work of Forcadel, Imbert, Monneau [53]). In order to prove the uniqueness of the traveling wave, we need the following additional assumptions :

Assumption (C) : Inverse monotonicity close to $\{0\}^{N+1}$ and $E = \{1\}^{N+1}$
There exists $\beta_0 > 0$ such that for $a > 0$, we have

$$\begin{cases} F(X + (a, ..., a)) < F(X) \quad \text{for all} \quad X, X + (a, ..., a) \in [0, \beta_0]^{N+1} \\ F(X + (a, ..., a)) < F(X) \quad \text{for all} \quad X, X + (a, ..., a) \in [1 - \beta_0, 1]^{N+1}. \end{cases}$$

This condition is important to get the comparison principle (see Theorem 4.1).

Assumption (D+) :

i) All the r_i's "Shifts" have the same sign : Assume that $r_i \leq 0$ for all $i \in \{0, ..., N\}$.

ii) Strict monotonicity : F is increasing in X_{i^+} with $r_{i^+} > 0$.

Assumption (D−) :

i) All the r_i's "Shifts" have the same sign : Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

ii) Strict monotonicity : F is increasing in X_{i^-} with $r_{i^-} < 0$.

Assumption (E+) :

i) Strict monotonicity close to 0 **:** Assume that $\dfrac{\partial F}{\partial X_{i^+}}(0) > 0$ with $r_{i^+} > 0$.

ii) Smoothness close to $\{0\}^{N+1}$ **:**

There exists $\nabla F(0)$, with $f'(0) < 0$, and there exists $\alpha \in (0,1)$ and $C_0 > 0$ such that for all $X \in [0,1]^{N+1}$

$$|F(X) - F(0) - X.\nabla F(0)| \leq C_0 |X|^{1+\alpha}.$$

Assumption (E−) :

i) Strict monotonicity close to 1 **:** Assume, for $E = (1, ..., 1) \in \mathbb{R}^{N+1}$, that $\dfrac{\partial F}{\partial X_{i^-}}(E) > 0$ with $r_{i^-} < 0$.

ii) Smoothness close to $\{1\}^{N+1}$ **:**

There exists $\nabla F(E)$ with $f'(1) < 0$ and there exists $\alpha \in (0,1)$ and $C_0 > 0$ such that for all $X \in [0,1]^{N+1}$

$$|F(X) - F(E) - (X - E).\nabla F(E)| \leq C_0 |X - E|^{1+\alpha},$$

with $E = (1, ..., 1) \in \mathbb{R}^{N+1}$.

Theorem 1.5. (Uniqueness of the velocity and of the profile)
Assume (A) and let (c, ϕ) be a solution of

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) \quad on \quad \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1. \end{cases} \tag{2.14}$$

(a) Uniqueness of the velocity : *Under the additional assumption (C), the velocity c is unique.*
(b) Uniqueness of the profile ϕ : *If $c \neq 0$, then under the additional assumptions (C) and (D+) i) or ii) or (E+) if $c > 0$ (resp. (D−) i) or ii) or (E−) if $c < 0$), the profile ϕ is unique (up to space translation) and $\phi' > 0$ on $\mathbb{R}$.*

Remark 1.6. (Interpreting the assumptions)
(1) **If** $\mathbf{c} > \mathbf{0}$ **:** *Assumptions* $(D+)$ *i), ii) and* $(E+)$ *are respectively important to prove a Strong Maximum Principle (cf Lemma 6.2 and 6.4) and the asymptotics of the profile near* $-\infty$ *(cf Lemma 6.6) that we use to prove the uniqueness of the profile of a solution.*

(2) **If** $\mathbf{c} < \mathbf{0}$ **:** *Under* $(D-)$ *i), ii) and* $(E-)$*, we respectively get the same results as for* $c > 0$*, however, the asymptotics are proven near* $+\infty$ *in this case.*

Remark also that Theorems 1.1 and 1.2 are particular cases of Theorem 1.4 and Theorem 1.5. Indeed, existence of the solution in Theorem 1.2 follows from Theorem 1.4 and the fact that $b \neq \frac{1}{2}$ in assumption (B), when $L \in \left(-\frac{1}{4}, \frac{1}{4}\right) \setminus \{0\}$. Uniqueness of the profile in Theorem 1.2 follows from Theorem 1.5 (b), and the fact that the function G defined in (2.7) verifies assumptions $(E\pm)$.

For the whole paper, we define

$$r^* = \max_{i=0,\ldots,N} |r_i| \tag{2.15}$$

and we set, as a notation, for a general function h :

$$F((h(y + r_i))_{i=0,\ldots,N}) := F(h(y + r_0), h(y + r_1), ..., h(y + r_N)).$$

1.3 Brief review of the literature

The study of traveling waves in reaction-diffusion equations has been introduced in pioneering works of Fisher [52] and Kolmogorov, Petrovsky and Piskunov [83]. Existence of traveling waves solutions has been for instance obtained in [6, 20, 81, 49]. More generally, there is a huge literature about existence, uniqueness and stability of traveling waves with various non linearities with applications in particular in biology and combustion and we refer for instance to the references cited in [14, 34]. There are also several works on discrete or nonlocal versions of reactions-diffusion equations (see for instance [13, 28, 30, 39, 42, 66, 118, 130] and [34, 88] and the references cited therein).

As explained above, in the case of bistable non-linearity f, the existence and uniqueness of traveling waves are well known for the model equation

$$u_t = u_{xx} + f(u). \tag{2.16}$$

Starting from equation (2.16), and using a continuation method, Bates et al. [12] proved in particular the existence of traveling waves for the convolution model

$$u_t = J * u - u + f(u) \tag{2.17}$$

where J is a kernel.

In [88], Mallet-Paret (see also Carpio et al. [28] for semi-linear case) used also a global continuation method (i.e. a homotopy method) to get existence of traveling waves for bistable non-linearities and information about the uniqueness and the dependence of solutions on parameters. This continuation argument was applied to connect the discrete dynamical system that he studied and a PDE model (similar to (2.16)) for which the existence and uniqueness are known. He proved the continuation between the solutions of the two systems using a general Fredholm alternative method [87] for the linearized traveling waves equations.

Traveling waves were also studied by Chow et al. [35] for lattice dynamical systems (lattice ODE's) and for coupled maps lattices (CML's) that arise as time-discretizations of lattice ODE's. Using a geometric approach, the authors studied the stability of traveling waves for lattice ODE's and proved existence of traveling waves of their time discretized CML's. More precisely, they constructed a local coordinate system in a tubular neighborhood of the traveling wave solution in the phase space of their system. Such an approach is used to transform lattice ODE's into a nonautonomous time-periodic ODE and traveling waves to periodic solutions of this ODE. In addition, they gain from this transformation the possibility to use the standard tools of dynamical systems and to see traveling waves of CML as certain orbits for a circle diffeomorphism whose rotational number is equal to the wave speed.

Zinner [128] proved the existence of traveling waves for the discrete Nagumo equation

$$\dot{x}_i = d(x_{i+1} - 2x_i + x_{i-1}) + f(x_i) \quad i \in \mathbb{Z}. \tag{2.18}$$

The construction is done introducing first a simplified problem (using a projection to 0 or 1 for $|i| \geq N$) for which the existence is attained by Brouwer's fixed point theorem. Hankerson and Zinner [73] also proved existence of traveling waves (for an equation more general than (2.18)) obtained as the long time limit of the solution with Heaviside initial data, using an interesting lap number argument.

In [34], Chen, Guo and Wu constructed traveling waves for a lattice ODE's with bistable non-linearity. They rephrase the solution ϕ of (2.11) as a fixed point of an integral formulation. First, they considered a simplified problem (using a projection on 0 or 1 for large indices $|i| \geq N$) and they show, for any $c \neq 0$, the existence of a solution $\phi^{N,\, c}$ using the monotone iteration method. Finally, they recover the existence of a solution in the limit $N \to +\infty$ for a suitable choice $c = c(N)$ converging to a limit velocity.

In this paper, we introduce a completely new method at least to prove the existence of traveling waves. In our approach, the existence of traveling waves relies on the construction of hull functions of slope p (like correctors) for an associated homogenization problem. Passing to the limit $p \to 0$, one important difficulty is to identify a traveling wave joining two stable states. In particular, we have avoided this traveling wave to degenerate to the intermediate unstable state. The uniqueness of the profile is proved using either strong maximum principle or weak asymptotics of the profile. Notice that, using weak asymptotics (in comparison with those of Mallet-Paret [88]) allow us to have weaker assumptions.

We also mention that our method is still effective in higher dimensional problems. Consider, for instance, the model

$$\frac{d}{dt}X_I(t) = f(X_I) + \sum_{|J|=1} \Big(X_{I+J} - X_I \Big) \tag{2.19}$$

that describes the interaction of an atom $I \in \mathbb{Z}^n$ with its nearest neighbors ($X_I \in \mathbb{R}$ denotes the position of atom I). We can look for traveling waves $X_I(t) = \phi(ct + \nu \cdot I)$ that propagates in a direction $\nu \in \mathbb{R}^n$ with $|\nu| = 1$. That is for $z = ct + \nu \cdot I$, we look for ϕ solution of

$$c\phi'(z) = f(\phi(z)) + \sum_{|J|=1} \Big(\phi(z + \nu \cdot J) - \phi(z) \Big),$$

where f denotes a bistable non-linearity. Setting $r_j := \nu \cdot J$, we recover an equation of type (2.11) for $N = 2n$. Therefore, the results of higher dimensional problems follow from our one dimensional results (Theorems 1.4 and 1.5) as far as they hold for general shifts r_j's.

We get the existence of solutions under very poor assumptions in comparison with similar results in previous works. Our framework is very flexible, and does not require a setting in any particular functional space. We also think that our method opens new perspectives and could be used to study many models : for example, fully overdamped FK models with time dependent non-linearities, accelerated FK models, FK with multi-particles.

1.4 Organization of the paper

In Section 2, we introduce an extension of F onto $\mathbb{R}^{N+1}$ and we recall, for the extension function, the notion of viscosity solutions, the existence of hull functions for our model and we prove some results about monotone functions. We prove Theorem 1.4 (for the extended function) in Section 3. In Section 4, we prove the uniqueness of the velocity of a profile (Theorem 1.5 part (a) = Proposition 4.5) and a comparison principle result on the half-line. Section 5 is devoted to the asymptotics of a profile near $\pm\infty$ (Proposition 5.1). In Section 6, we prove the uniqueness of the profile (Theorem 1.5 part (b)). Finally, we prove in the Appendices A and B the extension result, namely Lemma 2.1 and some results about monotone function, namely Lemmas 2.10 and 2.11 respectively.

2 Preliminary results

This section is divided into four subsections. In the first subsection, we extend the function F onto $\mathbb{R}^{N+1}$. In the second subsection, we recall the definition of a viscosity solution. We apply a result of existence of hull functions associated to the homogenization of our problem with the extended F in the third subsection. We dedicate the fourth subsection for some results about monotone functions that we will use in Section 3.

2.1 Extension of F

The proof of existence of traveling waves is based on the construction of hull functions (like correctors) associated to a homogenization problem (see [53]). To this end, we first need to extend the function F in $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ and satisfying the following assumption :

Assumption ($\tilde{A}$) :

Regularity : $\tilde{F}$ is globally Lipschitz continuous over $\mathbb{R}^{N+1}$.

Periodicity : $\tilde{F}(X_0+1, ..., X_N+1) = \tilde{F}(X_0, ..., X_N)$ for every $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.

Monotonicity : $\tilde{F}(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

The extension result is the following :

Lemma 2.1. (Extension of F)
Given a function F defined over $Q = [0,1]^{N+1}$ satisfying (A) and $F(1, ..., 1) = F(0, ..., 0)$, there exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ such that

$$\tilde{F}_{|Q} = F \quad \textit{and} \quad \tilde{F} \textit{ satisfies } (\tilde{A}).$$

The proof of this lemma is postponed in Appendix A.

Remark 2.2. *We notice that, if ϕ is a traveling wave constructed for (2.13) with F replaced by $\tilde{F}$, then ϕ is a traveling wave of (2.13). This is a direct consequence of Lemma 2.1 and the fact that*

$$\begin{cases} \phi \textit{ is non-decreasing on } \mathbb{R} \\ \phi(-\infty) = 0 \textit{ and } \phi(+\infty) = 1. \end{cases}$$

By convention, we will say that $\tilde{F}$ satisfies (B) (resp. (C), (D) or (E)) if and only if $F = \tilde{F}_{|Q}$ satisfies (B) (resp. (C), (D) or (E)).

We now give a result corresponding to Theorem 1.4 for $\tilde{F}$, whose proof is given in Section 3.

Proposition 2.3. (Result corresponding to Theorem 1.4 for $\tilde{F}$)
Assume that $\tilde{F}$ satisfies $(\tilde{A})$, (B). Then there exist a real c and a function ϕ solution of

$$\begin{cases} c\phi'(z) = \tilde{F}((\phi(z+r_i))_{i=0,\dots,N}) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing on } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1, \end{cases} \tag{2.20}$$

in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$.

For simplicity, in the rest of this section and in Section 3, we call $\tilde{F}$ as F.

Proof of Theorem 1.4
The proof of Theorem 1.4 is a straightforward consequence of Remark 2.2 and Proposition 2.3. □

2.2 Viscosity solution

In the whole paper, we will use the notion of viscosity solution that we introduce in this subsection. To this end, we recall that the upper and the lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(y) = \limsup_{x \to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \to y} u(x).$$

Definition 2.4. (Viscosity solution)
Let $u : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and F defined on $\mathbb{R}^{N+1}$.

- *The function u is a sub-solution (resp. a super-solution) of*

$$cu'(x) = F((u(x+r_i))_{i=0,\dots,N}) \quad on \quad \mathbb{R}, \tag{2.21}$$

  if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(\mathbb{R})$ such that $u - \psi$ attains a local maximum (resp. a local minimum) at x^*, we have

$$c\psi'(x^*) \leq F((u(x^*+r_i))_{i=0,\dots,N}) \quad \Big(resp.\ c\psi'(x^*) \geq F((u(x^*+r_i))_{i=0,\dots,N})\Big).$$

- *A function u is a viscosity solution of (2.21) if u^* is a sub-solution and u_* is a super-solution.*

We also recall the stability result for viscosity solutions (see [11, Theorem 4.1]).

Proposition 2.5. (Stability of viscosity solutions)
Consider a function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A})$. Assume that $(u_\varepsilon)_\varepsilon$ is a

sequence of sub-solutions (resp. super-solutions) of (2.21). Suppose that the functions $(u_\varepsilon)_\varepsilon$ are uniformly locally bounded on $\mathbb{R}$ and let

$$\overline{u}(x) = \limsup_{\varepsilon\to 0}{}^* u_\varepsilon(x) := \limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{\varepsilon\to 0}{}_* u_\varepsilon(x) := \liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y),$$

be the relaxed upper and lower semi-limits. If $\overline{u}$ (resp. $\underline{u}$) is finite, then $\overline{u}$ is a sub-solution (resp. $\underline{u}$ is a super-solution) of (2.21).

2.3 On the hull function

In this subsection, we first adapt the result of existence of a hull function associated to the homogenization of our problem, then we make the link between the existence of a hull function and the existence of the traveling wave.

Lemma 2.6. (Existence of a hull function ([53, Theorem 1.5]))
Let F be a given function satisfying assumption $(\tilde{A})$ and $p > 0$. There exists a unique λ_p such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense) :

$$\begin{cases} \lambda_p h_p' = F((h_p(y + pr_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ h_p(y + 1) = h_p(y) + 1 \\ h_p'(y) \geq 0 \\ |h_p(y + y') - h_p(y) - y'| \leq 1 \text{ for all } y' \in \mathbb{R}. \end{cases} \tag{2.22}$$

Such a function h_p is called a hull function. Moreover, there exists a constant $K > 0$, independent on p, such that

$$|\lambda_p| \leq K(1 + p).$$

Notice that Lemma 2.6 is proven in [53] only for $r_i \in \mathbb{Z}$. However, the proof for the generalization $r_i \in \mathbb{R}$ is still valid (it is exactly the same).
After this recall, and using the hull function h_p, we define the function ϕ_p as :

$$\phi_p(x) := h_p(px). \tag{2.23}$$

Moreover we set, as a velocity, the ratio

$$c_p := \frac{\lambda_p}{p}. \tag{2.24}$$

Remark 2.7. *It is possible that $c_p = 0$ for all $p > 0$. Our proof of existence of traveling wave is done for the general case. However, we state throw out the proof the different situations for the velocity.*

Notice that the above ϕ_p satisfies the following lemma :

Lemma 2.8. (Properties of ϕ_p)
Let $p > 0$ and assume $(\tilde{A})$. Then the function ϕ_p defined in (2.23) satisfies in the viscosity sense :

$$\begin{cases} c_p \phi_p' = F((\phi_p(z + r_i))_{i=0,\ldots,N}) \quad on \quad \mathbb{R} \\ \phi_p' \geq 0 \\ \phi_p\left(z + \dfrac{1}{p}\right) = \phi_p(z) + 1. \end{cases} \tag{2.25}$$

Moreover, if $c_p \neq 0$ then there exists $M > 0$ independent on p such that

$$|\phi_p'| \leq \frac{M}{|c_p|}, \tag{2.26}$$

for $0 < p \leq \dfrac{1}{r^*}$, with r^* given in (2.15).

Proof of Lemma 2.8.
Let h_p be a viscosity solution given by Lemma 2.6. Then we get (2.25) by the change of variables (2.23)-(2.24). We now show (2.26). We choose $p > 0$ such that

$$\frac{1}{p} \geq r^*.$$

Since ϕ_p is non-decreasing, then we have

$$\begin{cases} |\phi_p(x + r_i) - \phi_p(x)| \leq \left|\phi_p\left(x + \dfrac{1}{p}\right) - \phi_p(x)\right| = 1 \quad \text{if} \quad r_i \geq 0 \\ |\phi_p(x + r_i) - \phi_p(x)| \leq \left|\phi_p\left(x - \dfrac{1}{p}\right) - \phi_p(x)\right| = 1 \quad \text{if} \quad r_i \leq 0 \end{cases}$$

Moreover, since $F \in Lip(\mathbb{R}^{N+1})$, then

$$|F((\phi_p(x + r_i)_{i=0,\ldots,N})) - F((\phi_p(x))_{i=0,\ldots,N})| \leq L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix} =: L^1,$$

where L is the Lipschitz constant of F. On the other hand, f is bounded (because f is Lipschitz and periodic) and $F((\phi_p(x))_{i=0,\ldots,N}) = f(\phi_p(x))$, thus

$$|F((\phi_p(x + r_i)_{i=0,\ldots,N}))| \leq L^1 + |f|_{L^\infty} =: M.$$

This implies that

$$|c_p \phi_p'| \leq M$$

in the viscosity sense. If in addition $c_p \neq 0$, then we get the Lipschitz bound

$$|\phi_p'| \leq \frac{M}{|c_p|}.$$

□

2.4 Useful results about monotone functions

In this subsection, we recall miscellaneous results about monotone functions that we will use later in Section 3 for the proof of Proposition 2.3. We state Helly's Lemma on the one hand, and the equivalence between viscosity and almost everywhere solution on the other hand.

Lemma 2.9. (Helly's Lemma, (see [5], Section 3.3, **page** 70))
Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on $[a,b]$ verifying $|g_n| \leq M$ uniformly in n. Then there exists a subsequence $(g_{n_j})_{j\in\mathbb{N}}$ such that

$$g_{n_j} \to g \quad a.e. \ on \quad [a,b],$$

with g non-decreasing and $|g| \leq M$.

Lemma 2.10. (Complement of Helly's Lemma)
Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of non-decreasing functions on a bounded interval I and suppose that

$$g_n \to g \quad a.e. \ on \quad I.$$

If g is constant on $\mathring{I}$, then for every closed subset interval $I' \subset \mathring{I}$,

$$g_n \to g \quad uniformly \ on \quad I'.$$

The proof of Lemma 2.10 is done in Appendix B.

We introduce now a lemma that shows the equivalence between viscosity and almost everywhere solutions under the monotonicity of the solution.

Lemma 2.11. (Equivalence between viscosity and a.e. solutions)
Let F satisfying assumption $(\tilde{A})$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.Then ϕ is a viscosity solution of

$$0 = F((\phi(x + r_i))_{i=0,\ldots,N}) \quad on \quad \mathbb{R} \tag{2.27}$$

if and only if ϕ is an almost everywhere solution of the same equation.

The proof of Lemma 2.11 is also delayed in Appendix B.

3 Construction of a traveling wave : proof of Proposition 2.3

This section is devoted to the proof of existence of a traveling wave for system (2.20). We control both the velocity of propagation and the finite difference of a solution in the first subsection. Then we prove Proposition 2.3 in the second subsection.

3.1 Preliminary results

We have

Lemma 3.1. (Velocity c_p is bounded)
Under the assumption $(\tilde{A})$, (B), let c_p be the velocity given by (2.24). Then there exists $M_1 > 0$ such that

$$|c_p| \leq M_1$$

for $0 < p \leq \dfrac{1}{r^*}$, with r^* given in (2.15).

Proof of Lemma 3.1.
Consider the function ϕ_p given by (2.23) which satisfies (2.25). Let c_p be the associated velocity given by (2.24) and assume by contradiction that when $p \to p_0 \in [0, \frac{1}{r^*}]$

$$\lim_{p\to p_0} c_p = +\infty, \tag{2.28}$$

(the case $c_p \to -\infty$ being similar). Let $\overline{\phi}_p(x) := \phi_p(c_p x)$ solution of

$$\overline{\phi}_p'(x) = F\left(\left(\overline{\phi}_p\left(x + \frac{r_i}{c_p}\right)\right)_{i=0,\ldots,N}\right).$$

Since $\overline{\phi}_p$ is invariant w.r.t. space translations, we may assume that

$$\overline{\phi}_p(0) = b - \varepsilon$$

for some $\varepsilon > 0$ small enough. Moreover, by (2.26) we have

$$|\overline{\phi}_p'| = |c_p \phi_p'| \leq M$$

for some $M > 0$ independent on p. Thus using Ascoli's Theorem and the diagonal extraction argument, $\overline{\phi}_p$ converges as $p \to p_0$ (up to a subsequence) to some $\overline{\phi}$ locally uniformly on $\mathbb{R}$, and $\overline{\phi}$ satisfies classically

$$\begin{aligned} \overline{\phi}'(x) &= F((\overline{\phi}(x))_{i=0,\ldots,N}) \\ &= f(\overline{\phi}(x)) \end{aligned}$$

and $\overline{\phi}(0) = b - \varepsilon$. But $\overline{\phi}_p' \geq 0$ (because (2.28) implies trivially that $c_p \geq 0$), thus $\overline{\phi}' \geq 0$. Hence $f(\overline{\phi}(x)) \geq 0$ for all x, in particular $f(\overline{\phi}(0)) = f(b-\varepsilon)$, a contradiction since $f(b - \varepsilon) < 0$ (see assumption (B)). □

Next, we introduce an important proposition on the control of the finite difference that will be used in the proof of existence of a traveling wave.

Proposition 3.2. (Control on the finite difference)
Assume that F satisfies $(\tilde{A})$ and let $a > r^*$ with r^* given by (2.15) and $M_0 > 0$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all function ϕ (viscosity) solution of

$$\begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(x + 1) \leq \phi(x) + 1 \\ |c| \leq M_0 \\ |c\phi'| \leq M_0, \end{cases}$$

and for all $x_0 \in \mathbb{R}$ satisfying

$$\phi_*(x_0 + a) - \phi^*(x_0 - a) \leq \delta,$$

we have

$$\text{dist}(\alpha, \{0, b\} + \mathbb{Z}) < \varepsilon \quad \text{for all} \quad \alpha \in [\phi_*(x_0), \phi^*(x_0)].$$

Note that, $\{0, b\} + \mathbb{Z} \equiv \mathbb{Z} \cup (b + \mathbb{Z})$. Roughly speaking, this proposition says that if ϕ is flat enough around x_0, then $\phi(x_0)$ is close to a zero of f.

Proof of Proposition 3.2.
The proof is done by contradiction.
Step 1 : construction of a sequence, by contradiction
We assume by contradiction that there exists $\varepsilon_0 > 0$ such that for all $\delta_n \to 0$, there exists ϕ^n solution of

$$\begin{cases} c^n(\phi^n)'(x) = F((\phi^n(x + r_i))_{i=0,\dots,N}) \\ (\phi^n)' \geq 0 \\ \phi^n(x + 1) \leq \phi^n(x) + 1 \\ |c^n| \leq M_0 \\ |c^n(\phi^n)'| \leq M_0, \end{cases} \tag{2.29}$$

such that there exists $x_n \in \mathbb{R}$ satisfying

$$\phi^n_*(x_n + a) - (\phi^n)^*(x_n - a) \leq \delta_n \to 0, \tag{2.30}$$

and there exists $\alpha_n \in [\phi^n_*(x_n), (\phi^n)^*(x_n)]$ such that

$$\operatorname{dist}(\alpha_n, \{0, b\} + \mathbb{Z}) \geq \varepsilon_0 > 0. \tag{2.31}$$

Up to replace $\phi^n(x)$ by $\phi^n(x + e_n) + k_n$ with $e_n \in \mathbb{R}$, $k_n \in \mathbb{Z}$, we can assume that

$$\left| \begin{array}{l} x_n \equiv 0 \\ \phi^n(0) \in [0, 1) \quad \text{for all } n. \end{array} \right. \tag{2.32}$$

Step 2 **: passing to limit** $n \to +\infty$
Because $|c^n| \leq M_0$ then, up to extract a subsequence as $n \to +\infty$, we have

$$c^n \to c.$$

Case 1 **:** $c \neq 0$
For n large enough, we have $|c^n| \geq \frac{|c|}{2} \neq 0$. Hence

$$|(\phi^n)'| \leq \frac{2M_0}{|c|} \quad \text{for large } n,$$

thus ϕ^n is uniformly Lipschitz continuous. Using Ascoli's Theorem and the diagonal extraction argument, $\phi^n \to \phi$ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover, ϕ satisfies (in the viscosity sense)

$$\left\{ \begin{array}{l} c\phi'(x) = F((\phi(x + r_i))_{i=0,\ldots,N}) \\ \phi' \geq 0 \end{array} \right. \tag{2.33}$$

Case 2 **:** $c = 0$
Notice that $\phi^n(x + 1) \leq \phi^n(x) + 1$ implies (using (2.32))

$$\left\{ \begin{array}{lll} \phi^n(x) \leq \lceil x \rceil + 1 & \text{for} & x \geq 0 \\ \phi^n(x) \geq -\lceil |x| \rceil & \text{for} & x \leq 0. \end{array} \right. \tag{2.34}$$

Therefore, using Helly's Lemma (Lemma 2.9) and the diagonal extraction argument, ϕ^n converges (up to a subsequence) to ϕ locally a.e. Moreover, we have (using Lemma 2.11 if $c^n = 0$)

$$c^n \int_{b_1}^{b_2} (\phi^n)'(z)dz = \int_{b_1}^{b_2} F((\phi^n(z + r_i))_{i=0,\ldots,N})dz$$

for every $b_1 < b_2$. That is,

$$c^n(\phi^n(b_2) - \phi^n(b_1)) = \int_{b_1}^{b_2} F((\phi^n(z + r_i))_{i=0,\ldots,N})dz.$$

But
$$F((\phi^n(z+r_i))_{i=0,\dots,N}) \to F((\phi(z+r_i))_{i=0,\dots,N}) \quad \text{a.e.}$$
and
$$|F((\phi^n(z+r_i))_{i=0,\dots,N})| \leq m_0(1+|z|)$$
for some constant $m_0 > 0$ (because of (2.34) and the fact that F is globally Lipschitz with f bounded). Thus, using Lebesgue's dominated convergence theorem, we pass to the limit $n \to +\infty$, and we get

$$0 = \int_{b_1}^{b_2} F((\phi(z+r_i))_{i=0,\dots,N})dz$$

which implies (since b_1 and b_2 are arbitrary) that

$$0 = F((\phi(z+r_i))_{i=0,\dots,N}) \quad \text{a.e.}$$

Since $(\phi^n)' \geq 0$ implies $\phi' \geq 0$, then by Lemma 2.11, ϕ verifies

$$\begin{cases} 0 = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \geq 0 \end{cases} \tag{2.35}$$

in the viscosity sense.
Step 3 : getting a contradiction
Passing to the limit in (2.30) with $x_n = 0$ implies that

$$\phi_*(a) \leq \phi^*(-a).$$

But ϕ is non-decreasing, then $\phi = const =: k$ over $(-a, a)$. Since $a > r^*$, then from (2.33) and (2.35), we get for $x = 0$

$$\begin{aligned} 0 &= F((\phi(x+r_i))_{i=0,\dots,N}) \\ &= F((k)_{i=0,\dots,N}) = f(k), \end{aligned}$$

hence $k \in \{0, b\} + \mathbb{Z}$. On the other hand, since $\alpha_n \in [\phi^n_*(0), (\phi^n)^*(0)]$, then (up to a subsequence)
$$\alpha_n \to \alpha \in \{k\} = [\phi_*(0), \phi^*(0)].$$
Moreover, if we pass to limit in (2.31), we get

$$\text{dist}(\alpha = k, \{0, b\} + \mathbb{Z}) \geq \varepsilon_0 > 0,$$

which is a contradiction. □

3.2 Proof of Proposition 2.3

Proof of Proposition 2.3
The proof is done in several steps.
Step 0 **: introduction**
Let $p > 0$ and ϕ_p (given by (2.23)) be a non-decreasing solution of

$$c_p \phi'_p(x) = F((\phi_p(x + r_i))_{i=0,\dots,N})$$

with

$$\phi_p\left(x + \frac{1}{p}\right) = 1 + \phi_p(x)$$

and c_p is given by (2.24). Up to translate ϕ_p, let us suppose that

$$\begin{cases} (\phi_p)_*(0) \le b \\ (\phi_p)^*(0) \ge b. \end{cases} \tag{2.36}$$

Our aim is to pass to limit as p goes to zero.
Step 0.1 **: introduce** z_p **and** y_p
For any $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{2}\min(b, 1-b)$), let z_p, $y_p \in \mathbb{R}$ such that

$$\begin{cases} (\phi_p)^*(z_p) \ge b + \varepsilon \\ (\phi_p)_*(z_p) \le b + \varepsilon, \end{cases} \tag{2.37}$$

and

$$\begin{cases} (\phi_p)^*(y_p) \ge b - \varepsilon \\ (\phi_p)_*(y_p) \le b - \varepsilon. \end{cases} \tag{2.38}$$

From Proposition 3.2, since $(\phi_p)^*(z_p) > b$ and $(\phi_p)_*(y_p) < b$, we deduce that (for $a > r^*$)

$$(\phi_p)_*(z_p + a) - (\phi_p)^*(z_p - a) \ge \delta(\varepsilon) > 0 \tag{2.39}$$

and

$$(\phi_p)_*(y_p + a) - (\phi_p)^*(y_p - a) \ge \delta(\varepsilon) > 0, \tag{2.40}$$

with $\delta(\varepsilon)$ independent of p. Moreover, we notice that

$$y_p \le 0. \tag{2.41}$$

(Otherwise, $b - \varepsilon \ge (\phi_p)_*(y_p) \ge (\phi_p)^*(0) \ge b$, a contradiction).
Step 1 **: viscosity super-solution**
Let

$$\psi_p(x) := (\phi_p)_*(x + a) - (\phi_p)^*(x - a).$$

Notice that ψ_p is lower semi continuous and $\psi_p(x) \geq 0$ for all $x \in \mathbb{R}$ (because $(\phi_p)_*$ is l.s.c, $(\phi_p)^*$ is u.s.c and ϕ_p is non-decreasing). Since (in the viscosity sense)

$$\begin{cases} c_p((\phi_p)_*)'(x+a) \geq F(((\phi_p)_*(x+a+r_i))_{i=0,\dots,N}) \\ c_p((\phi_p)^*)'(x-a) \leq F(((\phi_p)^*(x-a+r_i))_{i=0,\dots,N}), \end{cases}$$

then we can show (using a doubling of variables) the following inequality

$$c_p\,(\psi_p)'_*\,(x) \geq F(((\phi_p)_*\,(x+a+r_i))_{i=0,\dots,N}) - F(((\phi_p)^*\,(x-a+r_i))_{i=0,\dots,N}), \quad (2.42)$$

which holds in the viscosity sense.
Step 2 : passing to the limit $p \to 0$
Since c_p is bounded (see Lemma 3.1), then

$$c_p \to c,$$

up to a subsequence.
Case 1 : $c \neq 0$
For p small enough, we have $|c_p| \geq \frac{|c|}{2} \neq 0$. From (2.26), we deduce that

$$|\phi_p'| \leq \frac{2M}{|c|} \quad \text{for } p \text{ small,}$$

thus ϕ_p is uniformly Lipschitz continuous. Using Ascoli's Theorem and the diagonal extraction argument, $\phi_p \to \phi$ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover, ϕ satisfies, at least in the viscosity sense (using the stability result, Proposition 2.5),

$$\begin{cases} c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \geq 0, \end{cases} \quad (2.43)$$

and

$$\begin{cases} (\phi)_*(0) \leq b \\ (\phi)^*(0) \geq b. \end{cases}$$

Case 2 : $c = 0$
Let $R > 0$ and choose p small enough such that $R < \frac{1}{2p}$. Since

$$\phi_p\left(\frac{1}{2p}\right) = 1 + \phi_p\left(\frac{-1}{2p}\right), \quad (2.44)$$

then for all $x \in [-R, R]$, we have

$$|\phi_p(x) - \phi_p(0)| \leq \left|\phi_p\left(\frac{1}{2p}\right) - \phi_p\left(\frac{-1}{2p}\right)\right| = 1.$$

Notice that (2.36), the monotonicity of ϕ_p and (2.44) implies that

$$b-1 \le \phi_p\left(-\frac{1}{2p}\right) \le (\phi_p)_*(0) \le b \le (\phi_p)^*(0) \le \phi_p\left(\frac{1}{2p}\right) \le b+1,$$

thus

$$b-1 \le \phi_p(0) \le b+1.$$

Hence

$$|\phi_p|_{L^\infty[-R,R]} \le 3.$$

Using Helly's Lemma (Lemma 2.9) and the diagonal extraction argument, ϕ_p converges locally a.e. (up to a subsequence) to non-decreasing function ϕ. Thus, ϕ satisfies

$$\begin{cases} 0 = c\phi'(x) = F((\phi(x+r_i))_{i=0,\dots,N}) \\ \phi' \ge 0 \end{cases} \tag{2.45}$$

almost everywhere. Moreover, from Lemma 2.11, we deduce that ϕ is a viscosity solution of (2.45) with

$$\begin{cases} \phi_*(0) \le b \\ \phi^*(0) \ge b. \end{cases}$$

Step 3 **: first properties of the limit** ϕ
Step 3.1 **: the oscillation of** ϕ **is bounded**
Consider any $R > 0$. Choose p_0 such that $R \le \dfrac{1}{2p_0}$ and let $p \in (0, p_0]$. Then

$$\phi_p(R) - \phi_p(-R) \le \phi_p\left(\frac{1}{2p_0}\right) - \phi_p\left(\frac{-1}{2p_0}\right) = 1.$$

But ϕ_p converges (up to a subsequence and at least almost everywhere) to ϕ, (see Step 2), thus

$$\phi(R) - \phi(-R) \le 1$$

for almost every R. Now let R goes to $+\infty$, we conclude that

$$\phi(+\infty) - \phi(-\infty) \le 1.$$

Step 3.2 **:** $\phi(\pm\infty) \in \mathbb{Z} \cup (\{b\} + \mathbb{Z})$
Since (2.43) is invariant by translation, then

$$\phi^n(x) = \phi(x-n)$$

is a viscosity solution of

$$c(\phi^n)'(x) = F((\phi^n(x+r_i))_{i=0,\dots,N}).$$

Moreover, ϕ is non-decreasing bounded (see Step 3.1), thus $(\phi^n)_n$ is a non-increasing sequence of bounded functions. Therefore, ϕ^n converges pointwise as $n \to +\infty$. Moreover, since

$$\lim_{n\to+\infty} (\phi^n(x) - \phi(-\infty)) = 0,$$

then ϕ^n converges to $\phi(-\infty)$. Now, using the stability for viscosity solutions (see Proposition 2.5), we deduce that $\phi(-\infty)$ is a solution of

$$c(\phi(-\infty))' = F((\phi(-\infty))_{i=0,\dots,N}) = f(\phi(-\infty)).$$

That is

$$f(\phi(-\infty)) = 0.$$

Similarly we get $f(\phi(+\infty)) = 0$. Therefore the assertion of the step follows from (B).
Step 4 **:** $\phi(\pm\infty) \notin \{b\} + \mathbb{Z}$
Since $\phi(+\infty) - \phi(-\infty) \leq 1$ and

$$\begin{cases} \phi_*(0) \leq b \\ \phi^*(0) \geq b, \end{cases}$$

we get that $\phi(-\infty) \in \{b-1, 0, b\}$ and $\phi(+\infty) \in \{b, 1, b+1\}$. We want to exclude the cases $\phi(\pm\infty) = b$, $b \pm 1$. Notice that if $\phi(+\infty) = b + 1$, then $\phi(-\infty) = b$. Similarly, if $\phi(-\infty) = b - 1$, then $\phi(+\infty) = b$. Therefore, it is sufficient to exclude the cases $\phi(\pm\infty) = b$. At the end, this will show that $\phi(+\infty) = 1$ and $\phi(-\infty) = 0$.
Suppose to the contrary that

$$\phi(+\infty) = b,$$

(the case $\phi(-\infty) = b$ being similar). Let $x_0 = 2r^*$, where $r^* = \max_{i=0,\dots,N} |r_i|$. Since

$$b = \phi(+\infty) \geq \phi^*(0) \geq b,$$

then $\phi(x) = b$ for all $x > 0$. Hence

$$\phi(x_0) = \phi(x_0 \pm a) = b,$$

for $r^* < a < 2r^*$. Using the uniform convergence of ϕ_p to ϕ (see Lemma 2.10 if $c = 0$), we deduce that

$$\phi_p(x_0) \to b$$

and

$$\psi_p(x_0) = (\phi_p)_*(x_0 + a) - (\phi_p)^*(x_0 - a) \to 0 \quad \text{as} \quad p \to 0.$$

Step 4.1 : Equation satisfied by ψ_p at its point of minimum
Since (for z_p and y_p defined in (2.37) and (2.38)) we have

$$\begin{cases} z_p \to +\infty \quad \text{as} \quad p \to 0 & (\phi \text{ is non-decreasing and } \phi(+\infty) = b) \\ y_p \leq 0 & (\text{by (2.41)}), \end{cases}$$

then $x_0 \in [y_p, z_p]$ for p small enough. Next, set

$$m_p = \min_{x \in [y_p, z_p]} \psi_p(x) = \psi_p(x_p^*) \geq 0 \quad \text{with} \quad x_p^* \in [y_p, z_p],$$

thus

$$m_p = \psi_p(x_p^*) \leq \psi_p(x_0) \to 0 \quad \text{as} \quad p \to 0. \tag{2.46}$$

In addition, since

$$\begin{cases} \psi_p(y_p) \geq \delta(\varepsilon) > 0 \\ \psi_p(z_p) \geq \delta(\varepsilon) > 0, \end{cases}$$

then

$$x_p^* \in (y_p, z_p). \tag{2.47}$$

Therefore from (2.42), we get

$$0 = c_p((\psi_p)_*)'(x_p^*) \geq F(((\phi_p)_* (x_p^* + a + r_i))_{i=0,\ldots,N}) - F(((\phi_p)^* (x_p^* - a + r_i))_{i=0,\ldots,N}) \tag{2.48}$$

in the viscosity sense (and pointwisely).
Step 4.2 : $\psi_p(x_p^* + r_i) \geq \psi_p(x_p^*) = m_p$ for all i
Because of (2.47), we have

$$b - \varepsilon \leq (\phi_p)^*(y_p) \leq (\phi_p)^*(x_p^*) \leq (\phi_p)_*(z_p) \leq b + \varepsilon. \tag{2.49}$$

Therefore doing a reasoning similar to the one of Step 2, we show that

$$\phi_p(x_p^* + .) \to \phi_0 \quad \text{a.e. on} \quad \mathbb{R},$$

and ϕ_0 is a viscosity solution of (2.43). Since

$$m_p = \psi_p(x_p^*) = (\phi_p)_* (x_p^* + a) - (\phi_p)^* (x_p^* - a) \to 0 \quad \text{as} \quad p \to 0, \tag{2.50}$$

we deduce that

$$\phi_0 = const := k \quad \text{on} \quad (-a, a). \tag{2.51}$$

From Lemma 2.10 and (2.49), we deduce that $k \in [b - \varepsilon, b + \varepsilon]$. Moreover, we have

$$0 = c\phi_0'(0) = F((\phi_0(0 + r_i))_{i=0,\ldots,N}) = f(k),$$

hence $k = b$. Again, using Lemma 2.10 we deduce that

$$\sup_{(x_p^*-a+\delta,x_p^*+a-\delta)} |\phi_p(x) - b| \to 0 \quad \text{for any} \quad \delta > 0.$$

Moreover, because of (2.50), we can even conclude that

$$(\phi_p)_*(x_p^* + a),\ (\phi_p)^*(x_p^* - a) \to b \quad \text{as} \quad p \to 0. \tag{2.52}$$

Now, since

$$\begin{cases} (\phi_p)_*(y_p) \le b - \varepsilon \\ (\phi_p)^*(z_p) \ge b + \varepsilon, \end{cases}$$

then $y_p\,, z_p \notin (x_p^* - a + \delta, x_p^* + a - \delta)$ for every fixed δ. Since $y_p < x_p^* < z_p$, thus choosing $0 < \delta \le a - r^*$ implies that

$$y_p \le x_p^* + r_i \le z_p \quad \text{for all} \quad i.$$

Therefore,

$$\psi_p(x_p^* + r_i) \ge \psi_p(x_p^*) = m_p. \tag{2.53}$$

Step 4.3 **: getting a contradiction**
In this step, we assume that $m_p > 0$ (it will be shown in Step 5) and we want to get a contradiction. Set

$$k_i = \begin{cases} (\phi_p)_* \left(x_p^* + a + r_i\right) & \text{if} \quad r_i \le 0 \\ (\phi_p)^* \left(x_p^* - a + r_i\right) & \text{if} \quad r_i > 0. \end{cases}$$

Hence from (2.53) and using the monotonicity of F together with inequality (2.48), we get

$$0 \ge F((a_i)_{i=0,\ldots,N}) - F((c_i)_{i=0,\ldots,N}),$$

where

$$a_i = \begin{cases} k_i & \text{if} \quad r_i \le 0 \\ k_i + m_p & \text{if} \quad r_i > 0 \end{cases} \quad \text{and} \quad c_i = \begin{cases} k_i - m_p & \text{if} \quad r_i \le 0 \\ k_i & \text{if} \quad r_i > 0. \end{cases}$$

Notice that

$$k_i \in [(\phi_p)^*(x_p^* - a), (\phi_p)_*(x_p^* + a)].$$

Therefore from (2.52) and the fact that $m_p \to 0$, we deduce that

$$a_i \to b \quad \text{and} \quad c_i \to b \quad \text{as} \quad p \to 0.$$

Since F is C^1 near $\{b\}^{N+1}$ and $c_i + t(a_i - c_i) = c_i + tm_p$, then

$$\begin{aligned} 0 &\geq \int_0^1 dt \sum_{i=0}^{N} \left((a_i - c_i) \frac{\partial F}{\partial X_i}((c_j + t(a_j - c_j))_{j=0,\dots,N}) \right) \\ &= \int_0^1 dt \sum_{i=0}^{N} \left(m_p \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) \right). \end{aligned}$$

Since $m_p > 0$, we get

$$\begin{aligned} 0 &\geq \int_0^1 dt \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) \\ &= f'(b) + \int_0^1 dt \left(\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) - \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(b, ..., b) \right). \end{aligned}$$

But F is C^1 near $\{b\}^{N+1}$ and $c_i + tm_p \to b$ for all i, thus

$$\int_0^1 dt \left(\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}((c_j + tm_p)_{j=0,\dots,N}) - \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(b, ..., b) \right) \to 0 \quad \text{as} \quad p \to 0.$$

This implies that

$$0 \geq f'(b) > 0,$$

which is a contradiction because of assumption (B).

Step 5 **:** $m_p > 0$
We split this step into two cases :
Case 1 **:** F **is strongly increasing in some direction**
Assume that F verifies in addition :

$$\frac{\partial F}{\partial X_{i_1}} \geq \delta_0 > 0, \tag{2.54}$$

for certain i_1 with $r_{i_1} > 0$ (assuming $r_{i_1} < 0$ being similar).
Assume to the contrary that $m_p = 0$. Thus

$$\psi_p(x_p^*) = (\phi_p)_*(x_p^* + a) - (\phi_p)^*(x_p^* - a) = 0.$$

Since ϕ_p is non-decreasing, then

$$\phi_p(x_p^*) = \phi_{p_{|(x_p^*-a, x_p^*+a)}} = k = const,$$

where k is a zero of f, i.e

$$f(k) = 0. \tag{2.55}$$

Let $d \geq x_p^* + a$ be the first real number such that

$$\phi_p(d+\eta) > k \quad \text{for every} \quad \eta > 0.$$

Choose $0 < \eta < r_{i_1}$ and set

$$x_1 = d + \eta - r_{i_1}.$$

From the definition of d, we deduce that

$$\phi_p = k \quad \text{on a neighborhood of} \quad x_1,$$

hence $\phi_p'(x_1) = 0$. Moreover, we have

$$\begin{cases} \phi_p(x_1 + r_i) \geq k \text{ for all } i \neq i_1 \\ \phi_p(x_1 + r_{i_1}) = \phi_p(d+\eta) > k \text{ for } i = i_1, \end{cases}$$

therefore

$$\begin{aligned} 0 = c\phi_p'(x_1) &= F((\phi_p(x_1 + r_i))_{i=0,\dots,N}) \\ &\geq F(k, \dots, \overbrace{\phi_p(x_1 + r_{i_1}))}^{i_1}, \dots, k) \\ &\geq f(k) + \delta_0(\phi_p(d+\eta) - k) \\ &= \delta_0(\phi_p(d+\eta) - k) > 0, \end{aligned}$$

where we have used (2.55) for the last line. This is a contradiction.

Case 2 : create the monotonicity
In fact, we can always assume hypothesis (2.54) for a modification F_p of F, where

$$F_p(X_0, X_1, \dots, X_N) = F(X_0, X_1, \dots, X_N) + p(X_{i_1} - X_0).$$

Then the whole construction works for F replaced by F_p with the additional monotonicity property (2.54) with $\delta_0 = p$. Once we pass to the limit $p \to 0$, we still get the same contradiction as in Step 4.3 and we recuperate the construction of traveling wave ϕ of (2.20) for the function F. □

4 Uniqueness of the velocity c

We prove in this section the uniqueness of the velocity of a traveling wave ϕ solution of (2.14) (part (a) of Theorem 1.5). We show in the first subsection a comparison principle on the half-line, and we prove the uniqueness of the velocity in the second subsection.

4.1 Comparison principle on the half-line

In this subsection, we prove a comparison principle on the half-line that is essentially used to prove the uniqueness of the velocity (in the second subsection of this section) and the uniqueness of the profile ϕ that solves (2.14) (in Section 6).

Theorem 4.1. (Comparison principle on $(-\infty, r^*]$)
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A) and assume that

$$\left| \begin{array}{l} \textit{there exists } \beta_0 > 0 \textit{ such that if} \\ Y = (Y_0, ..., Y_N),\ Y + (a, ..., a) \in [0, \beta_0]^{N+1} \\ \textit{then } F(Y + (a, ..., a)) < F(Y) \textit{ if } a > 0. \end{array} \right. \tag{2.56}$$

Let u, $v : (-\infty, r^*] \to [0,1]$ be respectively a sub and a super-solution of

$$cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (-\infty, 0) \tag{2.57}$$

in the sense of Definition 2.4. Assume moreover that

$$u \le \beta_0 \quad on \quad (-\infty, r^*]$$

and

$$u \le v \quad on \quad [0, r^*].$$

Then

$$u \le v \ on \ (-\infty, r^*].$$

Before giving the proof of this result, we give a corollary which is a comparison principle on $[-r^*, +\infty)$.

Corollary 4.2. (Comparison principle on $[-r^*, +\infty)$)
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A) and assume that :

$$\left| \begin{array}{l} \textit{there exists } \beta_0 > 0 \textit{ such that if} \\ X = (X_0, ..., X_N),\ X + (a, ..., a) \in [1 - \beta_0, 1]^{N+1} \\ \textit{then } F(X + (a, ..., a)) < F(X) \textit{ if } a > 0. \end{array} \right. \tag{2.58}$$

Let u, $v : [-r^*, +\infty) \to [0,1]$ be respectively a sub and a super-solution of (2.57) on $(0, +\infty)$ in sense of Definition 2.4. Moreover, assume that

$$v \ge 1 - \eta_0 \quad on \quad [-r^*, +\infty),$$

and that

$$u \le v \quad on \quad [-r^*, 0].$$

Then

$$u \le v \ on \ [-r^*, +\infty).$$

Remark 4.3. (Inverse monotonicity)
Notice that assumptions (2.56) and (2.58) are satisfied if F *is* C^1 *on a neighborhood of* $\{0\}^{N+1}$ *and* $\{1\}^{N+1}$ *in* $[0,1]^{N+1}$ *and* $f'(0) < 0$, $f'(1) < 0$. *This condition means that* 0 *and* 1 *are stable equilibria.*

Lemma 4.4. (Transformation of a solution of (2.57))
Let u, $v : (-\infty, r^*] \to [0,1]$ *be respectively a sub and super-solution of (2.57) in the sense of Definition 2.4. Then*

$$\widehat{u}(x) := 1 - u(-x) \quad \text{and} \quad \widehat{v}(x) := 1 - v(-x)$$

are respectively a super and a sub-solution of (2.57) on $[-r^*, +\infty)$ *with* F, c *and* r_i *(for all* $i \in \{0, ..., N\}$*) replaced by* $\widehat{F}$, $\widehat{c}$ *and* $\widehat{r}_i$, *given by*

$$\begin{cases} \widehat{F}(X_0, ..., X_N) = -F(1 - X_0, ..., 1 - X_N) \\ \widehat{c} := -c \\ \widehat{r}_i := -r_i. \end{cases} \tag{2.59}$$

Moreover,

$$\widehat{F} : [0,1]^{N+1} \to \mathbb{R}$$

satisfies (A), (B) *and* (C), *where* b *and* f *are replaced by*

$$\begin{cases} \widehat{b} := 1 - b \\ \widehat{f}(v) := -f(-v) \end{cases}$$

in (B).

Notice that, Lemma 4.4 is still true even though u, $v : \mathbb{R} \to [0,1]$ are a sub and a super-solution of (2.57) on $\mathbb{R}$.

Proof of Lemma 4.4
Let $u : (-\infty, r^*] \to [0,1]$ be a sub-solution of (2.57) and set $\widehat{u}(x) = 1 - u(-x)$. It is then easy to see that in the viscosity sense

$$\begin{aligned} c\widehat{u}'(x) = cu'(-x) &\leq F((u(-x + r_i))_{i=0,...,N}) \\ &= F((1 - \widehat{u}(x - r_i))_{i=0,...,N}). \end{aligned}$$

Hence $\widehat{u}$ is a super-solution of (2.57) on $[-r^*, +\infty)$ with F, r_i and c replaced by $\widehat{F}$, $\widehat{r}_i := -r_i$ and $\widehat{c} := -c$ given in (2.59). Similarly, we show that $\widehat{v}$ is a sub-solution of the same equation on $[-r^*, +\infty)$. □

Proof of Corollary 4.2.
Let u, $v : [-r^*, +\infty) \to [0,1]$ be a sub and super-solution of (2.57) on $(0, +\infty)$ such

that $v \geq 1 - \beta_0$ on $[-r^*, +\infty)$. We set $\widehat{u}(x) = 1 - u(-x)$ and $\widehat{v}(x) = 1 - v(-x)$. It is then easy to see that $\widehat{u}$, $\widehat{v} \in [0,1]$, $\widehat{v} \leq \beta_0$ on $(-\infty, r^*]$.

Using Lemma 4.4, we show that $\widehat{u}$ and $\widehat{v}$ are respectively a super and a sub-solution of (2.57) with (F, c, r_i) replaced by $(\widehat{F}, \widehat{c}, \widehat{r_i})$ defined in (2.59). Moreover, using the fact that F satisfies (2.58), we deduce that $\widehat{F}$ satisfies (2.56).

We then deduce by Theorem 4.1 that

$$\widehat{v} \leq \widehat{u} \quad \text{on} \quad (-\infty, r^*]$$

i.e.

$$u \leq v \quad \text{on} \quad [-r^*, -\infty).$$

□

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1.
Let u, $v : (-\infty, r^*] \to [0,1]$ be respectively a sub and a super-solution of (2.57) such that

$$u \leq \beta_0 \quad \text{on} \quad (-\infty, r^*],$$

and $u \leq v$ on $[0, r^*]$.

Step 0 : Introduction
Set

$$\overline{v} := \min(v, \beta_0).$$

According to (2.56) we have

$$F(\beta_0, ..., \beta_0) < F(0, ..., 0) = f(0) = 0$$

thus the constant β_0 is a super-solution of (2.57). Hence $\overline{v}$ is a super-solution of (2.57) on $(-\infty, 0)$ with $u \leq \overline{v}$ on $[0, r^*]$. Moreover, since $\overline{v} \leq v$, it is sufficient to prove the comparison principle (Theorem 4.1) between u and $\overline{v}$ which satisfy in addition u, $\overline{v} \in [0, \beta_0]$.
For simplicity, we note $\overline{v}$ as v with u, $v \in [0, \beta_0]$ and $u \leq v$ on $[0, r^*]$.

Step 1 : Doubling the variables
Suppose by contradiction that

$$M = \sup_{x \in (-\infty, r^*]} u(x) - v(x) > 0.$$

Let ε, $\alpha > 0$ and define

$$\begin{aligned} M_{\varepsilon,\alpha} \quad &:= \quad \sup_{x,\, y \in (-\infty, r^*]} \left(u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \alpha |x|^2 \right) \\ &= \quad u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) - \frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} - \alpha |\overline{x}_\varepsilon|^2, \end{aligned}$$

for certain $\overline{x}_\varepsilon$, $\overline{y}_\varepsilon \in (-\infty, -r^*]$. Note that the maximum is reached since the function

$$(x,y) \mapsto \psi(x,y) = u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2$$

is upper semi-continuous and satisfies $\psi(x,y) \to -\infty$ as $|x|, |y| \to +\infty$. Moreover, for all $\delta > 0$, there exists $x_\delta \in (-\infty, r^*]$ such that

$$M \geq u(x_\delta) - v(x_\delta) \geq M - \delta.$$

Hence

$$\begin{aligned} M_{\varepsilon,\alpha} &\geq u(x_\delta) - v(x_\delta) - \alpha|x_\delta|^2 \\ &\geq M - \delta - \alpha|x_\delta|^2 \\ &\geq \frac{M}{2} > 0, \end{aligned}$$

for $\delta = \dfrac{M}{4}$ and α chosen small enough such that $\alpha < \dfrac{M}{4|x_\delta|^2}$. Moreover, since $u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) \leq \beta_0$, we have

$$\frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} + \alpha|\overline{x}_\varepsilon|^2 \leq \beta_0. \tag{2.60}$$

Step 2 **: There exists** α **small enough and** $\varepsilon \to 0$ **such that** $\overline{x}_\varepsilon \in [0, r^*]$ **or** $\overline{y}_\varepsilon \in [0, r^*]$

Assume that $\overline{x}_\varepsilon \in [0, r^*]$ (the case $\overline{y}_\varepsilon \in [0, r^*]$ being similar). Using (2.60), we deduce that $\overline{y}_\varepsilon \in [-\sqrt{2\beta_0\varepsilon}, r^*]$. Then $\overline{x}_\varepsilon$ and $\overline{y}_\varepsilon$ converge (up to a subsequence) to a certain $\overline{x}_0 \in [0, r^*]$ as $\varepsilon \to 0$ (from (2.60), the two limits coincide). We then deduce that

$$\begin{aligned} 0 < \frac{M}{2} &\leq \limsup_{\varepsilon \to 0} (u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon)) \\ &\leq u(\overline{x}_0) - v(\overline{x}_0) \leq 0, \end{aligned}$$

which is a contradiction. The last inequality takes place since $u \leq v$ on $[0, r^*]$.

Step 3 **: For all** α **and** ε **small enough, we have** $\overline{x}_\varepsilon$, $\overline{y}_\varepsilon \in (-\infty, 0)$

Step 3.1 **: Viscosity inequalities**

We have

$$u(x) \leq v(\overline{y}_\varepsilon) + M_{\varepsilon,\alpha} + \frac{|x - \overline{y}_\varepsilon|^2}{2\varepsilon} + \alpha|x|^2 := \phi(x),$$

and $u(\overline{x}_\varepsilon) = \phi(\overline{x}_\varepsilon)$. Thus

$$c\left(\frac{\overline{x}_\varepsilon - \overline{y}_\varepsilon}{\varepsilon} + 2\alpha\overline{x}_\varepsilon\right) = c\phi'(\overline{x}_\varepsilon) \leq F((u(\overline{x}_\varepsilon + r_i))_{i=0,\dots,N}). \tag{2.61}$$

Similarly, we get

$$c\left(\frac{\overline{x}_\varepsilon - \overline{y}_\varepsilon}{\varepsilon}\right) \geq F((v(\overline{y}_\varepsilon + r_i))_{i=0,\ldots,N}). \tag{2.62}$$

Subtracting (2.62) from (2.61) implies that

$$2c\alpha\overline{x}_\varepsilon \leq F((u(\overline{x}_\varepsilon + r_i))_{i=0,\ldots,N}) - F((v(\overline{y}_\varepsilon + r_i))_{i=0,\ldots,N}). \tag{2.63}$$

Note that from (2.60)

$$\alpha|\overline{x}_\varepsilon| \leq \sqrt{\alpha\beta_0}.$$

This implies that for ε fixed, $\alpha\overline{x}_\varepsilon \to 0$ as $\alpha \to 0$.

Step 3.2 **: Passing to the limit** $\alpha \to 0$

We have

$$u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2 \leq u(\overline{x}_\varepsilon) - v(\overline{y}_\varepsilon) - \frac{|\overline{x}_\varepsilon - \overline{y}_\varepsilon|^2}{2\varepsilon} - \alpha|\overline{x}_\varepsilon|^2.$$

Set

$$\left|\begin{array}{l} u_i^\alpha = u(\overline{x}_\varepsilon + r_i) \\ v_i^\alpha = v(\overline{y}_\varepsilon + r_i), \end{array}\right.$$

then

$$\begin{cases} u_i^\alpha \leq v_i^\alpha + m_\alpha + \delta_i^\alpha & \text{if} \quad i \neq 0 \\ u_0^\alpha = v_0^\alpha + m_\alpha & \text{if} \quad i = 0, \end{cases}$$

where $m_\alpha = u_0^\alpha - v_0^\alpha$ and $\delta_i^\alpha = 2\alpha\overline{x}_\varepsilon r_i + \alpha|r_i|^2$. For ε fixed, since u_i^α, $v_i^\alpha \in [0, \beta_0]$ and $\dfrac{M}{2} \leq m_\alpha \leq \beta_0$, we deduce that as $\alpha \to 0$ and up to a subsequence,

$$\begin{cases} u_i^\alpha \to u_i^0 \\ v_i^\alpha \to v_i^0 \\ m_\alpha \to m_0 \\ \delta_i^\alpha \to 0, \end{cases}$$

with u_i^0, $v_i^0 \in [0, \beta_0]$, $0 < \frac{M}{2} \leq m_0 \leq \beta_0$ and

$$\begin{cases} u_i^0 \leq v_i^0 + m_0 & \text{if} \quad i \neq 0 \\ u_0^0 = v_0^0 + m_0 & \text{if} \quad i = 0. \end{cases}$$

Moreover, passing to the limit in (2.63) implies that

$$0 \leq F((u_i^0)_{i=0,\ldots,N}) - F((v_i^0)_{i=0,\ldots,N}). \tag{2.64}$$

Step 4 : Getting a contradiction
We claim that for all i, there exists l_i, $l'_i \geq 0$ such that

$$u_i^0 + l_i = v_i^0 - l'_i + m_0, \tag{2.65}$$

and

$$\begin{cases} \overline{u}_i^0 := u_i^0 + l_i \leq \beta_0 \\ \overline{v}_i^0 := v_i^0 - l'_i \geq 0. \end{cases}$$

Recall that for all $i \in \{0, ..., N\}$, we have

$$\begin{cases} u_i^0,\, v_i^0 \in [0, \beta_0] \\ u_i^0 \leq v_i^0 + m_0 \\ u_0^0 - v_0^0 = m_0 \leq \beta_0. \end{cases}$$

If for some i, $u_i^0 = v_i^0 + m_0$, then it suffices to take $l_i = l'_i = 0$. Assume then that $u_i^0 < v_i^0 + m_0$.

Case 1 : $u_i^0,\, v_i^0 \in (v_0^0, u_0^0)$
Set $l_i = u_0^0 - u_i^0$ and $l'_i = v_i^0 - v_0^0$. Then

$$\begin{cases} \overline{u}_i^0 = u_i^0 + l_i = u_0^0 \leq \beta_0 \\ \overline{v}_i^0 = v_i^0 - l'_i = v_0^0 \geq 0, \end{cases}$$

and $\overline{u}_i^0 = \overline{v}_i^0 + m_0$.

Case 2 : $u_i^0 > u_0^0$ **and** $v_i^0 > v_0^0$
Since $u_i^0 - v_0^0 > m_0$, then there exists $l'_i < v_i^0 - v_0^0$ such that

$$u_i^0 = v_i^0 - l'_i + m_0$$

and $\overline{v}_i^0 = v_i^0 - l'_i > v_0^0 \geq 0$. Thus, it is sufficient to take $l_i = 0$.

Case 3 : $u_i^0 < u_0^0$ **and** $v_i^0 < v_0^0$
This case can be treated as Case 2 by taking $l'_i = 0$ and $l_i < u_0^0 - u_i^0$.

Finally, going back to (2.64), since F is non-decreasing, we deduce that

$$\begin{aligned} 0 &\leq F((u_i^0)_{i=0,...,N}) - F((v_i^0)_{i=0,...,N}) \\ &\leq F((\overline{u}_i^0)_{i=0,...,N}) - F((\overline{v}_i^0)_{i=0,...,N}) \\ &= F((\overline{u}_i^0)_{i=0,...,N}) - F((\overline{u}_i^0 - m_0)_{i=0,...,N}) \\ &< 0. \end{aligned}$$

Last inequality takes place since F verifies (2.56) for $\overline{u}_i^0$, $\overline{u}_i^0 - m_0 \in [0, \beta_0]$ and $m_0 > 0$. Therefore, we get a contradiction. $\square$

4.2 Uniqueness of the velocity

This subsection is devoted to prove the uniqueness of the velocity c of a traveling wave that solves (2.14).

Proposition 4.5. (Uniqueness of c)
Under assumptions (A), consider the function F defined on $[0,1]^{N+1}$. Let (c_j, ϕ_j) be a solution of (2.14) for $j = 1, 2$. If F satisfies in addition (C), then $c_1 = c_2$.

Proof of Proposition 4.5.
Suppose that for $j = 1, 2$, (c_j, ϕ_j) is a solution of (2.14) and assume by contradiction that $c_1 < c_2$. We have,

$$\phi_j(-\infty) = 0 \quad \text{and} \quad \phi_j(+\infty) = 1.$$

We set $\delta = \min(\beta_0, \frac{1}{4})$ where β_0 is given in assumption (C). Up to translate ϕ_1 and ϕ_2, we can assume that

$$\phi_1(x) \geq 1 - \delta \quad \forall x \geq -r^*$$

and

$$\phi_2(x) \leq \delta \quad \forall x \leq r^*.$$

This implies that

$$\phi_2 \leq \phi_1 \text{ over } [-r^*, r^*].$$

Moreover, since $c_1 < c_2$, we have

$$c_1\phi_2'(x) \leq c_2\phi_2'(x) = F((\phi_2(x + r_i))_{i=0,\dots,N}).$$

Hence (c_1, ϕ_2) is a sub-solution of (2.14). Since

$$\phi_1 \geq 1 - \delta \quad \text{on} \quad [-r^*, +\infty),$$

we deduce using Corollary 4.2 that

$$\phi_2 \leq \phi_1 \text{ over } [-r^*, +\infty).$$

Similarly, since

$$\phi_2 \leq \delta \quad \text{on} \quad (-\infty, r^*],$$

we deduce using Theorem 4.1 that

$$\phi_2 \leq \phi_1 \text{ over } (-\infty, r^*].$$

Therefore,

$$\phi_2 \leq \phi_1 \text{ over } \mathbb{R}.$$

Next, set

$$\left| \begin{array}{l} u_1(t,x) = \phi_1(x + c_1 t) \\ u_2(t,x) = \phi_2(x + c_2 t), \end{array} \right.$$

then for $j = 1, 2$, we have

$$\partial_t u_j(t,x) = F((u_j(t, x + r_i))_{i=0,\dots,N}). \tag{2.66}$$

Moreover, at time $t = 0$, we have

$$u_1(0,x) = \phi_1(x) \geq \phi_2(x) = u_2(0,x) \text{ over } \mathbb{R},$$

thus applying the comparison principle for equation (2.66) (see [53]), we get

$$u_1 \geq u_2 \quad \forall\, t \geq 0\, \forall\, x \in \mathbb{R}.$$

Taking $x = y - c_1 t$, we get

$$\phi_1(y) \geq \phi_2(y + (c_2 - c_1)t), \quad \forall\, t \geq 0,\ \forall\, y \in \mathbb{R}.$$

Using that $c_1 < c_2$, and passing to the limit $t \to +\infty$, we get

$$\phi_1(y) \geq \phi_2(+\infty) = 1, \quad \forall\, y \in \mathbb{R}.$$

But $\phi_1(-\infty) = 0$, hence a contradiction. Therefore $c_1 \geq c_2$. Similarly, we show that $c_2 \geq c_1$, hence $c_1 = c_2$. □

5 Asymptotics for the profile

In this section, our main result is the asymptotics near $\pm\infty$ for solutions $\phi : \mathbb{R} \to [0,1]$ of

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R}, \tag{2.67}$$

namely Proposition 5.1.

Proposition 5.1. (Asymptotics near $\pm\infty$)
Consider a function F defined on $[0,1]^{N+1}$ satisfying (A) and (C), and assume that $c \neq 0$. Then

$i)$ **asymptotics near $-\infty$**
Let $\phi : \mathbb{R} \to [0,1]$ be a solution of (2.67), satisfying

$$\phi(-\infty) = 0 \quad \textit{and} \quad \phi \geq \delta > 0 \quad \textit{on} \quad [0, r^*]$$

for some $\delta > 0$ and assume $(E+)$ $ii)$. If there exists a unique $\lambda^+ > 0$ solution of

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i} \tag{2.68}$$

then for any sequence $(x_n)_n$, $x_n \to -\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{\phi(x + x_{n'})}{e^{\lambda^+ x_{n'}}} \longrightarrow Ae^{\lambda^+ x} \quad \textit{locally uniformly on } \mathbb{R} \textit{ as } n' \to +\infty.$$

$ii)$ **asymptotics near $+\infty$**
Let $\phi : \mathbb{R} \to [0, 1]$ be a solution of (2.67), satisfying

$$\phi(+\infty) = 1 \quad \textit{and} \quad \phi \leq 1 - \delta < 1 \quad \textit{on} \quad [0, r^*]$$

for some $\delta > 0$ and assume $(E-)$ $ii)$. If there exists a unique $\lambda^- < 0$ solution of

$$c\lambda = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(1, ..., 1)e^{\lambda r_i}, \tag{2.69}$$

then for any sequence $(x_n)_n$, $x_n \to +\infty$, there exists a subsequence $(x_{n'})_{n'}$ and $A > 0$ such that

$$\frac{1 - \phi(x + x_{n'})}{e^{\lambda^- x_{n'}}} \longrightarrow Ae^{\lambda^- x} \quad \textit{locally uniformly on } \mathbb{R} \textit{ as } n' \to +\infty.$$

5.1 Uniqueness and existence of $\lambda^\pm$

In this subsection, we address the question of the existence and uniqueness of $\lambda^\pm$.

Lemma 5.2. (Uniqueness and existence of λ^+)
Assume (A) and suppose that $\nabla F(0)$ exists with $f'(0) < 0$. Then there is at most one solution $\lambda^+ > 0$ of (2.68). Moreover, if $c < 0$ or if we assume $(E+)$ $i)$, then there exists a (unique) solution $\lambda^+ > 0$ of (2.68).

Proof of Lemma 5.2
Step 1 : Uniqueness
Let

$$g(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)e^{\lambda r_i} - c\lambda. \tag{2.70}$$

Because of assumption (A), the function g is convex and

$$g(0) = f'(0) < 0.$$

Thus, there exists at most one solution $\lambda^+ > 0$ of (2.68) and if λ^+ exists, then we have

$$g < 0 \quad \text{on} \quad (0, \lambda^+) \quad \text{and} \quad g > 0 \quad \text{on} \quad (\lambda^+, +\infty). \tag{2.71}$$

Step 2 : Existence
Assume $c < 0$. We have

$$g(\lambda) \geq \frac{\partial F}{\partial X_0}(0, ..., 0) - c\lambda,$$

which implies that $\lim_{\lambda \to +\infty} g(\lambda) = +\infty$. On the other hand, if we assume $(E+)$ $i)$, then

$$g(\lambda) \geq \frac{\partial F}{\partial X_0}(0, ..., 0) + \frac{\partial F}{\partial X_{i_+}}(0, ..., 0)e^{\lambda r_{i_+}} - c\lambda,$$

which implies that $\lim_{\lambda \to +\infty} g(\lambda) = +\infty$.

Therefore, there exists a unique $\lambda^+ > 0$ such that $g(\lambda^+) = 0$. □

In the same way (or using Lemma 4.4), we can prove the following lemma concerning λ^-

Lemma 5.3. (Uniqueness and existence of λ^-)
Assume (A) and suppose that $\nabla F(1, ..., 1)$ exists with $f'(1) < 0$. Then there is at most one solution $\lambda^- < 0$ of (2.69). Moreover, if $c > 0$ or if we assume $(E-)$ $i)$, then there exists a (unique) solution $\lambda^- < 0$ of (2.69).

5.2 Proof of Proposition 5.1

In this subsection, we prove that any solution of (2.67) is exponentially bounded (from above and below) near $-\infty$. Finally, we prove Proposition 5.1 $i)$.

Lemma 5.4. (Exponential bounds for a solution of (2.67) near $-\infty$)
Assume (A), (C) and $(E+)$ $ii)$. Let $\phi : (-\infty, 0] \to [0, 1]$ be a solution of (2.67) on $(-\infty, -r^*)$ satisfying $\phi(-\infty) = 0$ and assume that there exists $\lambda^+ > 0$ solution of (2.68). Then there exists k_2 such that

$$\phi(x) \leq k_2 e^{\lambda^+ x} \quad \textit{for all} \quad x \leq 0.$$

Moreover, if

$$\phi \geq \delta > 0 \quad \textit{on} \quad [-r^*, 0] \quad \textit{for some} \quad \delta > 0, \tag{2.72}$$

then there exists $k_1 > 0$ such that

$$k_1 e^{\lambda^+ x} \leq \phi(x) \quad \textit{for all} \quad x \leq 0.$$

Remark 5.5. *Notice that the exponential bounds of Lemma 5.4 do not holds if we do not assume (E+) ii). To see this, it suffices to define* $f(u) = -u'$ *with* $u(x) = -xe^x$*. A simple computation then gives that*

$$\begin{cases} f(0) = 0 \\ f'(0) = -1 \\ f'(u) - f'(0) \sim_{u \to 0} \dfrac{-1}{\ln u} \end{cases}$$

and so f *does not satisfies (E+) ii) and* u *is not exponentially bounded.*

Proof of Lemma 5.4
The idea of the proof is to construct a sub and super-solution of

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,...,N}) \quad \text{on} \quad (-\infty, -r^*) \tag{2.73}$$

then, using the comparison principle (Theorem 4.1), we deduce the existence of k_1 and k_2. Let $\lambda^+ > 0$ be the solution of (2.68) and consider the perturbation $\lambda^+ < \lambda' < (1 + \alpha)\lambda^+$ with α given in assumption $(E+)$ *ii*).

Step 1 : existence of k_1
Step 1.1 : construction of a sub-solution of (2.73)
Set

$$\underline{\phi}(x) = A\left(e^{\lambda^+ x} + e^{\lambda' x}\right)$$

defined on $(-\infty, 0]$, where $A > 0$ will be chosen such that $\underline{\phi}$ is a sub-solution of (2.73). Since λ^+ is a solution of (2.68), then for $x \in (-\infty, -r^*)$ we have

$$\begin{aligned} c\underline{\phi}'(x) &= c\lambda^+ A e^{\lambda^+ x} + cA\lambda' e^{\lambda' x} \\ &= \nabla F(0, ..., 0).((Ae^{\lambda^+(x+r_i)})_{i=0,...,N}) + cA\lambda' e^{\lambda' x} \\ &= \nabla F(0, ..., 0).((\underline{\phi}(x + r_i))_{i=0,...,N}) - Ae^{\lambda' x}\big(\nabla F(0, ..., 0).((e^{\lambda' r_i})_{i=0,...,N}) - c\lambda'\big) \\ &\leq F((\underline{\phi}(x + r_i))_{i=0,...,N}) + C_0|\Phi(x)|^{1+\alpha} - Ae^{\lambda' x} g(\lambda'), \end{aligned}$$

where for the last line we have used $(E+)$ *ii*), $\Phi(x) = \big((\underline{\phi}(x + r_i))_{i=0,...,N}\big)$ and g defined in (2.70). Using the fact that for $x \in (-\infty, -r^*)$, we have $\underline{\phi}(x + r_i) \leq 2Ae^{\lambda^+(x+r^*)}$. We get

$$\begin{aligned} c\underline{\phi}'(x) - F((\underline{\phi}(x + r_i))_{i=0,...,N}) &\leq A\left(2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+(x+r^*)}|E|^{1+\alpha} - e^{\lambda' x} g(\lambda')\right) \\ &\leq A\left(2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+ r^*}|E|^{1+\alpha} - g(\lambda')\right) e^{\lambda' x}, \end{aligned}$$

with $E = (1, ..., 1) \in \mathbb{R}^{N+1}$. Since $g(\lambda') > 0$ (see (2.71)),

$$c\underline{\phi}'(x) \leq F((\underline{\phi}(x + r_i))_{i=0,...,N}) \leq 0 \quad \text{for } A \text{ small enough.}$$

This shows that $\underline{\phi}$ is a sub-solution of (2.73) on $(-\infty, -r^*)$.

Step 1.2 **: applying the comparison principle**
Up to decrease $A > 0$, let us assume moreover that $2A \leq \min(\delta, \beta_0)$ with δ given in (2.72) and β_0 given in assumption (C) (this is possible since A can be chosen as small as we want). Thus

$$\phi \geq \delta \geq 2A \geq \underline{\phi} \quad \text{on} \quad [-r^*, 0]$$

and

$$\underline{\phi} \leq 2A \leq \beta_0 \quad \text{on} \quad (-\infty, 0].$$

Hence using the comparison principle (Theorem 4.1 and a shift of the functions), we deduce that

$$\underline{\phi}(x) \leq \phi(x) \quad \text{for all} \quad x \leq 0.$$

This implies that ϕ satisfies

$$k_1 := A \leq \frac{\phi(x)}{e^{\lambda^+ x}} \quad \text{for all} \quad x \leq 0.$$

Step 2 **: existence of** k_2
Step 2.1 **: construction of a super-solution of (2.73)**
Define for $x \in (-\infty, 0]$ the function

$$\overline{\phi}(x) = A\left(2e^{\lambda^+ x} - e^{\lambda' x}\right).$$

Repeating the same proof as in Step 1, we get

$$\begin{aligned} c\overline{\phi}'(x) - F((\overline{\phi}(x + r_i))_{i=0,...,N}) &\geq A\left(-2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+(x+r^*)} |E|^{1+\alpha} + e^{\lambda' x} g(\lambda')\right) \\ &\geq A\left(-2^{1+\alpha} C_0 A^\alpha e^{(1+\alpha)\lambda^+ r^*} |E|^{1+\alpha} + g(\lambda')\right) e^{\lambda' x}, \end{aligned}$$

with $E = (1, ..., 1) \in \mathbb{R}^{N+1}$. Again, since $g(\lambda') > 0$, then $\overline{\phi}$ is a super-solution of (2.73) for $A > 0$ small enough.

Step 2.2 **: applying the comparison principle**
Define, for $a > 0$ large enough, the function $\tilde{\phi}(x) = \phi(x - a)$ such that

$$\tilde{\phi} \leq \min\left(\beta_0, Ae^{-\lambda^+ r^*}\right) \quad \text{on} \quad (-\infty, 0],$$

with β_0 given in assumption (C). This is possible because we assume that $\phi(-\infty) = 0$. Thus

$$\tilde{\phi} \leq Ae^{-\lambda^+ r^*} \leq \overline{\phi} \quad \text{on} \quad [-r^*, 0].$$

Hence, applying the comparison principle result (Theorem 4.1, up to a shift of the functions), we deduce that

$$\tilde{\phi} \leq \overline{\phi} \quad \text{on} \quad (-\infty, 0].$$

This implies that

$$\frac{\phi(x)}{e^{\lambda^+ x}} \leq 2Ae^{\lambda^+ a} \quad \text{for all} \quad x \leq -a.$$

Using the fact that $\phi \leq 1$, we get

$$\frac{\phi(x)}{e^{\lambda^+ x}} \leq k_2 \quad \text{for all} \quad x \leq 0,$$

where $k_2 := \max\left(2Ae^{\lambda^+ a}, \max_{x\in[-a,0]} \frac{\phi(x)}{e^{\lambda^+ x}}\right)$. □

We only prove Proposition 5.1 *i*) (the proof of Proposition 5.1 *ii*) being similar).

Proof of Proposition 5.1 *i*)
Let $\phi : [0,1] \to \mathbb{R}$ be a solution of (2.67) such that

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi \geq \delta \quad \text{for some } \delta > 0.$$

We recall, from Lemma 5.4, that

$$0 < k_1 \leq \frac{\phi(x)}{e^{\lambda^+ x}} \leq k_2 < +\infty \quad \text{for all} \quad x \leq 0, \tag{2.74}$$

where λ^+ is the solution of (2.68).

Step 1 **: Shifting and rescaling** ϕ
For a sequence $x_n \to -\infty$ and for all $x \leq 0$, define the function v_n as

$$v_n(x - x_n) := \frac{\phi(x)}{e^{\lambda^+ x}}.$$

We have

$$c\phi'(x) = ce^{\lambda^+ x}\big(v_n'(x - x_n) + \lambda^+ v_n(x - x_n)\big) = F\big((v_n(x + r_i - x_n)e^{\lambda^+(x+r_i)})_i\big) \tag{2.75}$$

That is, for $y = x - x_n$,

$$\begin{aligned} c\big(v_n'(y) + \lambda^+ v_n(y)\big) &= e^{-\lambda^+(y+x_n)} F\big((v_n(y + r_i)e^{\lambda^+(y+x_n+r_i)})_i\big) \\ &= e^{-\lambda^+(y+x_n)}\big[F\big((v_n(y + r_i)e^{\lambda^+(y+x_n+r_i)})_i\big) - \nabla F(0).\big((v_n(y + r_i)e^{\lambda^+(y+x_n+r_i)})_i\big)\big] \\ &\quad + \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0) v_n(y + r_i)e^{\lambda^+ r_i}. \end{aligned}$$

From assumption $(E+)$ $ii)$, we then have

$$c\big(v_n'(y)+\lambda^+v_n(y)\big) \;=\; \sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0)v_n(y+r_i)e^{\lambda^+r_i}+O\left(e^{-\lambda^+(y+x_n)}\left|(v_n(y+r_i)e^{\lambda^+(y+x_n+r_i)})_i\right|^{1+\alpha}\right)$$

i.e,

$$c\big(v_n'(y)+\lambda^+v_n(y)\big)=\sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0)v_n(y+r_i)e^{\lambda^+r_i}+O\left(e^{\lambda^+\alpha(y+x_n)}\left|(v_n(y+r_i)e^{\lambda^+r_i})_i\right|^{1+\alpha}\right) \tag{2.76}$$

Step 2 : Passing to the limit $n\to+\infty$
Because of (2.74), we have

$$0<k_1\le v_n(y)\le k_2<+\infty \quad \text{for} \quad y\le -x_n \tag{2.77}$$

and for any compact set $K\subset\mathbb{R}$

$$e^{\lambda^+\alpha(y+x_n)}\left|(v_n(y+r_i)e^{\lambda^+r_i})_i\right|^{1+\alpha}\to 0 \quad \text{as} \quad n\to+\infty \quad (\text{because } x_n\to-\infty)$$

uniformly in $y\in K$. As $c\neq 0$, we get from (2.76) that there exists some $C_K>0$ (independent on n) such that

$$|v_n'|\le C_K \quad \text{on} \quad K.$$

Applying Ascoli's theorem, there exists a subsequence $v_{n'}$ such that

$$v_{n'}\longrightarrow v_\infty \quad \text{locally uniformly on} \quad \mathbb{R}.$$

Moreover v_∞ satisfies

$$c\big(v_\infty'(y)+\lambda^+v_\infty(y)\big)=\sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0)v_\infty(y+r_i)e^{\lambda^+r_i} \tag{2.78}$$

and (using (2.77))

$$k_1\le v_\infty\le k_2 \quad \text{on} \quad \mathbb{R}. \tag{2.79}$$

Step 3 : Applying Fourier transform
Applying Fourier transform to (2.78), implies that

$$\hat v_\infty(\xi)G(\xi)=0,$$

where $G(\xi)=c(i\xi+\lambda^+)-\displaystyle\sum_{j=0}^{N}\frac{\partial F}{\partial X_j}(0,...,0)e^{\lambda^+r_j}e^{i\xi r_j}$.

Step 3.1 **:** $G(\xi) = 0 \iff \xi = 0$
Clearly, if $\xi = 0$ then $G(\xi) = 0$ (because λ^+ solves (2.68)).
Assume that $G(\xi) = 0$ with $\xi \in \mathbb{R}$. Hence

$$c\lambda^+ = \sum_{j=0}^{N} \frac{\partial F}{\partial X_j}(0, ..., 0)e^{\lambda^+ r_j} \cos(\xi r_j) \tag{2.80}$$

and

$$c\xi = \sum_{j=0}^{N} \frac{\partial F}{\partial X_j}(0, ..., 0)e^{\lambda^+ r_j} \sin(\xi r_j). \tag{2.81}$$

Using the fact that $\frac{\partial F}{\partial X_j}(0) \geq 0$ for $j \neq 0$, we deduce from (2.68) and (2.80) that for all $j \in \{1, ..., N\}$, we have

$$\begin{cases} \frac{\partial F}{\partial X_j}(0, ..., 0) = 0 \\ \text{or} \\ \xi r_j = 0 \quad mod(2\pi) \quad \text{and} \quad \frac{\partial F}{\partial X_j}(0) > 0. \end{cases} \tag{2.82}$$

Substituting (2.82) in (2.81), taking into consideration that $r_0 = 0$, implies that $c\xi = 0$ and thus $\xi = 0$, because $c \neq 0$.

Step 3.2 **:** $v_\infty = const$
From step 3.1, we deduce that $supp\{\hat{v}\} \subset \{0\}$. Therefore,

$$\hat{v}(0) = \sum_{finite} c_k \delta_0^{(k)}.$$

Inverse Fourier transform implies that v_∞ is a polynomial. But v_∞ is bounded (see (2.79)), hence

$$v_\infty = const := A.$$

Consequently,

$$\frac{\phi(x + x_{n'})}{e^{\lambda^+(x+x_{n'})}} = v_{n'}(x) \to A.$$

□

6 Uniqueness of the profile and proof of Theorem 1.5

We prove, in this section, the uniqueness of the profile (under the assumption (D) or (E)). Under Assumption (D) we will use a Strong Maximum Principle, while under assumption (E) we will need the asymptotics joint to a Half Strong Maximum Principle (just on the half-line, see Lemma 6.1). We show, in a first subsection, three different kinds of Strong Maximum Principle satisfied by (2.14) when $c \neq 0$. In a second subsection, we prove the uniqueness of the profile and Theorem 1.5.

6.1 Different kinds of Strong Maximum Principle

Here, we prove three different kinds of Strong Maximum Principle for (2.14) when $c \neq 0$. We also add a technical lemma (Lemma 6.5) that allow us to compare two different solutions on $\mathbb{R}$ with at least one contact point.

We prove the Strong Maximum Principle (Lemma 6.1, 6.3 and 6.4) for $c > 0$. However, when $c < 0$, the corresponding results can be deduced from the case $c > 0$ using the transformation of Lemma 4.4.

Lemma 6.1. (Half Strong Maximum Principle)
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying assumption (A) and let ϕ_1, $\phi_2 : \mathbb{R} \to [0,1]$ be respectively a viscosity sub and a super-solution of (2.21). Assume that

$$\begin{cases} \phi_2 \geq \phi_1 \quad \text{on} \quad \mathbb{R} \\ \phi_2(0) = \phi_1(0). \end{cases}$$

If $c > 0$ (resp. $c < 0$), then

$$\phi_1 = \phi_2 \quad \text{for all} \quad x \leq 0 \quad (\text{resp. } x \geq 0).$$

Proof of Lemma 6.1
Assume that $c > 0$ and let $w(x) := \phi_2(x) - \phi_1(x)$. Since ϕ_2 is a super-solution and ϕ_1 is a sub-solution of (2.21), then using the Doubling of variable method we show that w is a viscosity super-solution of

$$cw'(x) \geq F((\phi_2(x + r_i))_{i=0,\ldots,N}) - F((\phi_1(x + r_i))_{i=0,\ldots,N}) \quad \text{on} \quad \mathbb{R}.$$

But F is non-decreasing w.r.t. X_i for all $i \neq 0$, thus we get

$$cw'(x) \geq F(\phi_1(x) + w(x), (\phi_1(x + r_i))_{i=1,\ldots,N}) - F((\phi_1(x + r_i))_{i=0,\ldots,N}).$$

Now, since F is globally Lipschitz, then

$$w'(x) \geq \frac{-L}{c} w(x), \tag{2.83}$$

with L is the Lipschitz constant of F.

Notice that $y(x) = w(x_0)e^{\frac{-L}{c}(x-x_0)}$ satisfies the equality in inequality (2.83) for any $x_0 \in \mathbb{R}$. As $y(x_0) = w(x_0)$, then using the comparison principle for the "ode" (2.83), we deduce that

$$w(x) \geq w(x_0)e^{\frac{-L}{c}(x-x_0)} \quad \text{for all} \quad x \geq x_0. \tag{2.84}$$

If $w(x_0) > 0$, hence $w(x) > 0$ for all $x \geq x_0$. This implies that

$$\phi_2 > \phi_1 \quad \text{for all} \quad x \geq x_0.$$

Finally, since $\phi_2(0) = \phi_1(0)$, then we deduce that

$$\phi_2 = \phi_1 \quad \text{for all} \quad x \leq 0,$$

(otherwise, if there is $x_1 < 0$ such that $\phi_2(x_1) > \phi_1(x_1)$, then from the above argument, we deduce that $\phi_2(0) > \phi_1(0)$, a contradiction). □

Lemma 6.2. (Strong Maximum Principle under $(D\pm)$ $ii)$)
Let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A). Let ϕ_1, $\phi_2 : \mathbb{R} \to [0,1]$ be respectively a viscosity sub and super-solution of (2.21) such that

$$\phi_2 \geq \phi_1 \quad on \quad \mathbb{R} \quad \text{and} \quad \phi_2(0) = \phi_1(0)$$

a) If F is increasing w.r.t. X_{i_0} for certain $i_0 \neq 0$ then

$$\phi_2(kr_{i_0}) = \phi_1(kr_{i_0}) \quad for\ all \quad k \in \mathbb{N}.$$

b) If we assume moreover that F satisfies $(D+)$ $ii)$ if $c > 0$, or $(D-)$ $ii)$ if $c < 0$, then

$$\phi_1(x) = \phi_2(x) \quad for\ all\ x \in \mathbb{R}.$$

Proof of Lemma 6.2

a) Assume for simplicity that $i_0 = 1$. Let ϕ_1, ϕ_2 be respectively a viscosity sub and a viscosity super-solution of (2.21). Then using the Doubling of variable method, we can show that the function $w = \phi_2 - \phi_1$ satisfies

$$cw'(x) \geq F((\phi_2(x + r_i))_{i=0,...,N}) - F((\phi_1(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \tag{2.85}$$

in the viscosity sense. As w is a viscosity super-solution of (2.85), $w(0) = 0$ and $w \geq 0$ on $\mathbb{R}$, we deduce that

$$0 \geq F((\phi_2(r_i))_{i=0,...,N}) - F((\phi_1(r_i))_{i=0,...,N}) \quad \text{at} \quad x = 0.$$

Thus using the fact that $\phi_2(0) = \phi_1(0)$ and that F is monotone w.r.t. X_i for all $i \neq 0$, we get

$$F((\phi_2(r_i))_{i=0,\dots,N}) = F((\phi_1(r_i))_{i=0,\dots,N}).$$

Next, since F is increasing w.r.t. X_1, we deduce that

$$\phi_2 = \phi_1 \quad \text{at} \quad x = r_1,$$

(otherwise, $F((\phi_2(r_i))_{i=0,\dots,N}) > F((\phi_1(r_i))_{i=0,\dots,N})$, because F is non-decreasing w.r.t. X_i for $i \neq 0$, 1 and increasing w.r.t. X_1). Therefore, upon repeating the above argument for $x = r_1$, we show that

$$\phi_2(kr_1) = \phi_1(kr_1) \quad \text{for all} \quad k \in \mathbb{N}.$$

b) Assume that $c > 0$ and that F satisfies $(D+)$ *ii*) (the other case being similar). By contradiction, suppose that there exists $x \in \mathbb{R}$ such that $\phi_1(x) < \phi_2(x)$. Let $k \in \mathbb{N}$ big enough such that $kr_{i_+} > x$. Then, using Lemma 6.1 (up to shift the functions), and the fact that $\phi_1(kr_{i_+}) = \phi_2(kr_{i_+})$, we get that $\phi_1(x) = \phi_2(x)$, which is a contradiction.

□

Lemma 6.3. (Comparison principle, under $(D\pm)$ *i*))
Assume that $c > 0$ (resp. $c < 0$) and let F satisfying (A) and $(D+)$ i) (resp. $(D-)$ i)). Let ϕ_1, ϕ_2 be respectively a viscosity sub and a viscosity super-solution of (2.21). Assume that $\phi_1(0) = \phi_2(0)$ and

$$\phi_1 \leq \phi_2 \quad \text{on} \quad [-r^*, 0] \quad \big(\text{resp. on } [0, r^*]\big),$$

then

$$\phi_1(x) \leq \phi_2(x) \quad \text{for all} \quad x \geq -r^* \quad \big(\text{resp. } x \leq r^*\big).$$

Proof of Lemma 6.3
Assume that $c > 0$ (the case $c < 0$ being similar). If $r^* = 0$, then the result follows from the comparison principle for ODEs.

Let us assume that $r^* > 0$. Since $\phi_1 \leq \phi_2$ on $[-r^*, 0]$ and $r_i < 0$ for all $i \neq 0$ (see assumption $(D+)$ *i*)), then for all $x \in [0, \min_{i\neq 0}(-r_i)]$, the function $w(x) := \phi_1(x) - \phi_2(x)$ satisfies (in the viscosity sense)

$$\begin{aligned} cw'(x) &\leq F((\phi_1(x + r_i))_{i=0,\dots,N}) - F((\phi_2(x + r_i))_{i=0,\dots,N}) \\ &\leq F(w(x) + \phi_2(x), (\phi_2(x + r_i))_{i\neq 0}) - F((\phi_2(x + r_i))_{i=0,\dots,N}) \\ &\leq L|w(x)| \quad \text{(because } F \text{ is } L\text{-Lipschitz).} \end{aligned}$$

where we have used in the second line the fact that $\phi_1(x+r_i) \leq \phi_2(x+r_i)$ for $i \neq 0$, because $-r^* \leq x + r_i \leq 0$ for all $i \neq 0$. But $w(0) = 0$ and $y \equiv 0$ is a solution of $cw' = L|w|$, then using the comparison principle of the "ode," we deduce that

$$w \leq 0 \quad \text{for all} \quad x \in [0, \min_{i\neq 0}(-r_i)].$$

This implies that

$$\phi_1 \leq \phi_2 \quad \text{for all} \quad x \in [0, \min_{i\neq 0}(-r_i)].$$

Finally, the result of this lemma ($\phi_1 \leq \phi_2$ for all $x \geq -r^*$) follows by repeating the above argument several times, each on the new extended interval. □

Lemma 6.4. (Strong Maximum principle under $(D\pm)$ $i)$)
Assume $c > 0$ (resp. $c < 0$) and let F satisfying (A) and $(D+)$ $i)$ (resp. $(D-)$ $i)$). Let ϕ_1, ϕ_2 be two solutions of (2.21) such that

$$\phi_1(0) = \phi_2(0) \quad \textit{and} \quad \phi_1 \leq \phi_2 \quad \textit{on} \quad \mathbb{R}.$$

Then

$$\phi_1(x) = \phi_2(x) \quad \textit{for all} \quad x \in \mathbb{R}.$$

Proof of Lemma 6.4
Let $c > 0$ (the case $c < 0$ is deduced from the case $c > 0$ using Lemma 4.4). Using Lemma 6.1, we deduce that

$$\phi_1 = \phi_2 \quad \text{for all} \quad x \leq 0.$$

Thus, it is sufficient to prove that $\phi_1 \geq \phi_2$ for all $x \geq 0$ (because $\phi_1 \leq \phi_2$ for $x \geq 0$). We have,

$$\phi_1(0) = \phi_2(0) \quad \text{and} \quad \phi_1 \geq \phi_2 \quad \text{on} \quad [-r^*, 0] \quad (\text{since } \phi_1 = \phi_2 \quad \forall x \leq 0),$$

and ϕ_2, ϕ_1 are respectively a viscosity sub and super-solution of (2.21). Hence using the comparison principle (Lemma 6.3), we deduce that

$$\phi_1 \geq \phi_2 \quad \text{for all} \quad x \geq -r^*.$$

Therefore, $\phi_1(x) = \phi_2(x)$ for all $x \in \mathbb{R}$. □

Lemma 6.5. (Ordering two solutions of (2.14) up to translation)
Assume that $c \neq 0$ and let $F : [0,1]^{N+1} \to \mathbb{R}$ satisfying (A) and (C). Let ϕ_1 and ϕ_2 be two solutions of (2.14). There exists a shift $a^* \in \mathbb{R}$ and some $x_0 \in [-r^*, r^*]$ such that $\phi_2^{a^*}(x) := \phi_2(x + a^*)$ and ϕ_1 satisfy

$$\begin{cases} \phi_2^{a^*} \geq \phi_1 \quad \textit{on} \quad \mathbb{R} \\ \phi_2^{a^*}(x_0) = \phi_1(x_0). \end{cases}$$

Proof of Lemma 6.5
The idea of the proof is to translate ϕ_2 and then to compare the translation with ϕ_1.

Step 1 : Family of solutions above ϕ_1
For $a \in \mathbb{R}$, let us define

$$\phi_2^a(x) := \phi_2(x+a).$$

For some $a > 0$ large enough, (because of the conditions at $\pm\infty$ in (2.14)), we have

$$\phi_2^{\overline{a}} \geq \phi_1 \quad \text{on} \quad [-r^*, r^*] \quad \text{for all} \quad \overline{a} \geq a,$$

and then using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\overline{a} \geq a$, we have

$$\phi_2^{\overline{a}} \geq \phi_1 \quad \text{on} \quad \mathbb{R}.$$

Step 2 : There exists a^* such that $\phi_2^{a^*}$ and ϕ_1 touch at $x_0 \in [-r^*, r^*]$
Let

$$a^* = \inf\{a \in \mathbb{R}, \quad \phi_2^{\overline{a}} \geq \phi_1 \quad \text{on} \quad \mathbb{R} \quad \text{for all} \quad \overline{a} \geq a\}.$$

Recall that $c \neq 0$ and then $\phi_i \in C^1(\mathbb{R})$ for $i = 1, 2$.

Assume by contradiction that

$$\inf_{[-r^*, r^*]} \left(\phi_2^{a^*} - \phi_1\right) \geq \delta > 0.$$

Then for all $0 \leq \varepsilon \leq \varepsilon_0$ with ε_0 small enough, we have

$$\phi_2^{a^*-\varepsilon} - \phi_1 \geq 0 \quad \text{on} \quad [-r^*, r^*].$$

Applying the comparison principle (Theorem 4.1 and Corollary 4.2), we get

$$\phi_2^{a^*-\varepsilon} - \phi_1 \geq 0 \quad \text{on} \quad \mathbb{R},$$

which is a contradiction with the definition of a^*. Thus

$$\inf_{[-r^*, r^*]} \phi_2^{a^*} - \phi_1 = 0.$$

Hence, there exists $x_0 \in [-r^*, r^*]$ such that

$$\phi_2^{a^*} = \phi_1 \quad \text{at} \quad x_0,$$

knowing that $\phi_2^{a^*}(x) \geq \phi_1(x)$ for all $x \in \mathbb{R}$. □

6.2 Proof of Theorem 1.5 (b)

We devote this subsection for the proof of the uniqueness of the profile which is done in several lemmas. The proof of Theorem 1.5 is given at the end of this subsection. All the proofs are made in the case $c > 0$ since the case $c < 0$ is similar (or is deduced using Lemma 4.4).

Lemma 6.6. (Uniqueness of the profile, under $(E+)$**)**
Assume that $c > 0$ *and let* F *satisfying* (A)*,* (C) *and* $(E+)$*. Let* $\phi : \mathbb{R} \to [0,1]$ *be a solution of (2.14), then* ϕ *is unique up to space translation. Moreover* ϕ *is non-decreasing.*

Proof of Lemma 6.6
Assume that $c > 0$ and let ϕ_1, $\phi_2 : \mathbb{R} \to [0,1]$ be two solutions of (2.14). The goal of the proof is to show that there exists a translation $\phi_2^{a^*}$ of ϕ_2 such that $\phi_2^{a^*} = \phi_1$. To simplify the notation we denote r_{i_+} (introduced in $(E+)$) by r_1.

Step 1 **: Constructing a translation and applying Lemma 6.1**
Using Lemma 6.5, there exists $a^* \in \mathbb{R}$ such that the translation $\phi_2^{a^*}$ of ϕ_2 satisfies :

$$\begin{cases} \phi_2^{a^*} \geq \phi_1 \quad \text{on} \quad \mathbb{R} \\ \phi_2^{a^*}(x_0) = \phi_1(x_0). \end{cases} \tag{2.86}$$

Since $c > 0$, then applying Lemma 6.1, we deduce that

$$\phi_2^{a^*} = \phi_1 \quad \text{for all} \quad x \leq x_0. \tag{2.87}$$

Step 2 **: Asymptotics of** ϕ_1 **and** $\phi_2^{a^*}$
Using Lemma 5.2 and Proposition 5.1, we get that there exists a subsequence (n') of $(n)_{n\in\mathbb{N}}$ (because $x_0 - nr_1 \to -\infty$ as $n \to +\infty$) and two constants $A_1, A_2 > 0$ such that

$$\frac{\phi_2^{a^*}(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0 - n'r_1 + x)}} \to A_1 \quad \text{locally uniformly on} \quad \mathbb{R}. \tag{2.88}$$

$$\frac{\phi_1(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0 - n'r_1 + x)}} \to A_2 \quad \text{locally uniformly on} \quad \mathbb{R}.$$

Using equation (2.87), we deduce that $A_1 = A_2 := A$.

Step 3 **: Exchange** ϕ_1 **and** ϕ_2
Applying Lemma 6.5, upon exchanging ϕ_1 and ϕ_2, we deduce that there exists $b^* \geq 0$ and y_0 such that

$$\begin{cases} \phi_1^{b^*}(x) := \phi_1(x + b^*) \geq \phi_2 \quad \text{on} \quad \mathbb{R} \\ \phi_1^{b^*}(y_0) = \phi_2(y_0). \end{cases}$$

Moreover, from Lemma 6.1, we get

$$\phi_1^{b^*}(x) = \phi_2 \quad \text{for all} \quad x \leq y_0 \quad (\text{since } c > 0).$$

Now, using and Lemma 5.2 and Proposition 5.1 and since $y_0 - n'r_1 \to -\infty$ as $n' \to +\infty$, we get the existence of a subsequence of (n') (still denoted by (n')) such that

$$\frac{\phi_1^{b^*}(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}}, \quad \frac{\phi_2(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B \quad \text{locally uniformly on} \quad \mathbb{R}. \tag{2.89}$$

Step 4 **: Conclusion,** $\phi_1 = \phi_2^{a^*}$
For any fixed $x \in \mathbb{R}$, we have

$$\frac{\phi_2(x_0 + a^* - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A, \tag{2.90}$$

$$\frac{\phi_1(x_0 - n'r_1 + x)}{e^{\lambda^+(x_0-n'r_1+x)}} \to A, \tag{2.91}$$

$$\frac{\phi_1(y_0 + b^* - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B \tag{2.92}$$

and

$$\frac{\phi_2(y_0 - n'r_1 + x)}{e^{\lambda^+(y_0-n'r_1+x)}} \to B. \tag{2.93}$$

For $x = y_0 + b^*$, equation (2.91) implies that

$$\frac{\phi_1(x_0 - n'r_1 + y_0 + b^*)}{e^{\lambda^+(x_0-n'r_1+y_0)}} \to Ae^{\lambda^+ b^*}.$$

Also, equation (2.92) with $x = x_0$ implies that

$$\frac{\phi_1(x_0 - n'r_1 + y_0 + b^*)}{e^{\lambda^+(x_0-n'r_1+y_0)}} \to B,$$

thus

$$Ae^{\lambda^+ b^*} = B.$$

Similarly, if we substitute $x = y_0$ in (2.90) and $x = x_0 + a^*$ in (2.93), we show that

$$A = Be^{\lambda^+ a^*}.$$

Therefore,

$$a^* = -b^*.$$

But

$$\phi_2^{a^*}(x) = \phi_2(x + a^*) \geq \phi_1(x)$$

and

$$\phi_1^{b^*}(x) = \phi_1(x + b^*) = \phi_1(x - a^*) \geq \phi_2(x),$$

hence we get

$$\phi_2(x + a^*) = \phi_1(x).$$

Moreover $\phi_2(x + a) \geq \phi_1(x)$ for all $a \geq a^*$, which shows that the profile is nondecreasing. □

Lemma 6.7. (Uniqueness of the profile, under $(D+)i)$ **or** $ii)$**)**
Assume that $c > 0$ and let F satisfying (A) and (C). Let $\phi : \mathbb{R} \to [0, 1]$ be a solution of (2.14). If, in addition, F satisfies $(D+)$ (i) or $ii)$, then ϕ is unique up to space translation.

Proof of Lemma 6.7
The proof follows from Lemma 6.5 and the Strong Maximum Principle (Lemma 6.4 or Lemma 6.2). □

Lemma 6.8. (Monotonicity of the profile)
Assume that $c > 0$ (resp. $c < 0$) and let $F : [0, 1]^{N+1} \to \mathbb{R}$ satisfying (A) and (C). Let $\phi : \mathbb{R} \to [0, 1]$ be a solution of (2.14). If F satisfies $(D+)$ $i)$ or $ii)$ or $(E+)$ (resp. $(D-)$ $i)$ or $ii)$ or $(E-)$), then $\phi' > 0$ on $\mathbb{R}$.

Proof of Lemma 6.8
Assume that $c > 0$ (the proof when $c < 0$ being similar) and let ϕ be a solution of (2.14).

Step 1 **:** ϕ **is non-decreasing**
The goal is to show that $\phi(x + a) \geq \phi(x)$ for all $a \geq 0$. As in the proof of Lemma 6.5, we deduce that for $a \geq 0$ large enough and for all $\overline{a} \geq a$, we have

$$\phi^{\overline{a}}(x) := \phi(x + \overline{a}) \geq \phi(x) \quad \text{on} \quad [-r^*, r^*].$$

Thus using the comparison principle (Theorem 4.1 and Corollary 4.2), we deduce that for all $\overline{a} \geq a$, we have

$$\phi^{\overline{a}}(x) \geq \phi(x) \quad \text{on} \quad \mathbb{R}.$$

Set

$$a^* = \inf\{a \geq 0, \ \ \phi^{\overline{a}}(x) \geq \phi(x) \quad \text{on} \quad \mathbb{R} \quad \text{for all} \quad \overline{a} \geq a\},$$

we want to prove that $a^* = 0$. By definition of a^*, there exists some x_0 such that

$$\begin{cases} \phi^{a^*} \geq \phi \quad \text{on} \quad \mathbb{R} \\ \phi^{a^*}(x_0) = \phi(x_0). \end{cases} \tag{2.94}$$

Case 1 **:** F **satisfies** $(E+)$
From Lemma 6.6, ϕ is nondecreasing and then $a^* = 0$.

Case 2 **:** F **satisfies** $(D+)$ $i)$ **or** $ii)$
Using (2.94) and the Strong Maximum Principle (Lemma 6.2 or Lemma 6.4), we get that $\phi^{a^*} = \phi$, i.e., ϕ is periodic of period a^*. But $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, thus $a^* = 0$.

Step 2 **:** ϕ **is increasing**
Let $a > 0$, we want to show that $\phi(x+a) > \phi(x)$. From Step 1, we have $\phi(x+a) \geq \phi(x)$. Assume that there exists x_0 such that

$$\phi(x_0 + a) = \phi(x_0).$$

Repeating the same argument, as in Step 1, under $(D+)$ $i)$ or $ii)$ or $(E+)$, we prove that $a = 0$, which is a contradiction. Thus

$$\phi(x+a) > \phi(x) \quad \text{on} \quad \mathbb{R} \quad \text{for any} \quad a > 0.$$

Step 3 **:** $\phi' > 0$
For $a > 0$, we define

$$w_a(x) = \frac{\phi(x+a) - \phi(x)}{a}.$$

Using the same arguments as in the proof of Lemma 6.1 (see (2.84)), we get that for all $x_0 \in \mathbb{R}$

$$w_a(x) \geq w_a(x_0) e^{\frac{-L}{c}(x-x_0)} \quad \text{for all} \quad x \geq x_0.$$

Passing to the limit $a \to 0$, we get that

$$\phi'(x) \geq \phi'(x_0) e^{\frac{-L}{c}(x-x_0)} \geq 0 \quad \text{for all} \quad x \geq x_0. \tag{2.95}$$

By contradiction, assume that there exists x_1 such that $\phi'(x_1) = 0$. This implies that

$$\phi'(x) = 0 \quad \text{for all} \quad x \leq x_1. \tag{2.96}$$

Indeed, if there exists $x_0 < x_1$ such that $\phi'(x_0) > 0$, then (2.95) implies that

$$\phi'(x_1) \geq \phi'(x_0) e^{\frac{-L}{c}(x_1 - x_0)} > 0,$$

which is a contradiction.

But ϕ is increasing so (2.96) is a contradiction and so $\phi' > 0$.

□

Proof of Theorem 1.5
(a) Uniqueness of the velocity

The proof of the uniqueness of the velocity is follows from Proposition 4.5 in Section 4.

(b) Uniqueness of the profile and strict monotonicity

The uniqueness and the strict monotonicity of the solution when $c > 0$ is done in Lemma 6.6, 6.7 and Lemma 6.8. However the case $c < 0$ is a consequence of Lemma 4.4 and the previous results. □

7 Construction of a monotone Lipschitz continuous periodic extension of F

We devote the Appendix A for the proof of Lemma 2.1. To this end, we need to start with two useful results about the orthogonal projection. For any convex set K in $\mathbb{R}^d$ and for any $y \in \mathbb{R}^d$, we call

$$Proj_{|_K}(y)$$

the orthogonal projection of y on K.

Lemma 7.1. (Characterization of the orthogonal projection)
Let $N \geq 1$ and $y = (y_1, ..., y_N) \in \mathbb{R}^N$. Then

$$Proj_{|_{[0,1]^N}}(y) = \left(\left(Proj_{|_{[0,1]}}(y_i) \right)_{i=1,...,N} \right).$$

Proof of Lemma 7.1
Let $y = (y_1, ..., y_N) \in \mathbb{R}^N$ and set $y_i^0 = Proj_{|_{[0,1]}}(y_i)$. By definition of the orthogonal projection, we have

$$(y_i - y_i^0, \overline{y}_i - y_i^0) \leq 0 \quad \forall\ \overline{y}_i \in [0,1].$$

This implies that

$$(y - y^0, \overline{y} - y^0) \leq 0 \quad \forall\ \overline{y} = (\overline{y}_1, ..., \overline{y}_N) \in [0,1]^N, \tag{2.97}$$

with $y^0 = (y_1^0, ..., y_N^0)$. But (2.97) is a characterization of the orthogonal projection of y on $[0,1]^N$, thus

$$y^0 = Proj_{|_{[0,1]^N}}(y).$$

□

Using the above lemma, one can easily check the following consequences :

Corollary 7.2. (Ordering and a kind of linearity)
Let $y = (y_1, ..., y_N),\ z = (z_1, ..., z_N) \in \mathbb{R}^N$ *and set* $e = (1, ..., 1) \in \mathbb{R}^N$. *Assume that*

$$y \geq z$$

in the sense that $y_i \geq z_i$ *for all* $i \in \{1, ..., N\}$. *Let* $Q_0 = [0,1]^N$, *then*
i) Order preservation
We have

$$Proj_{|Q_0}(y) \geq Proj_{|Q_0}(z).$$

ii) "Linearity"
We have

$$Proj_{|Q_0}(y + e) = Proj_{|Q_0 - e}(y) + e,$$

where $Q_0 - e = [-1, 0]^N$.

After these preliminary results, we now go back to the proof of Lemma 2.1.

Proof of Lemma 2.1.
The proof is splitted into two main steps. In the first step (the main part of the proof), we construct the extension $\tilde{F}$ of F on $[0,1] \times \mathbb{R}^N$. In the second step, we extend $\tilde{F}$ on the whole $\mathbb{R} \times \mathbb{R}^N$. The function $\tilde{F}$ that we want to construct must satisfy

$$\begin{cases} \tilde{F}_{|Q} = F & \text{for} \quad Q := [0,1]^{N+1} \\ \tilde{F}(X + E) = \tilde{F}(X) & \text{with} \quad E = (1, ..., 1) \in \mathbb{R}^{N+1}. \end{cases}$$

This implies that for any $y \in Q_0 = [0,1]^N$ and $e = (1, ..., 1) \in \mathbb{R}^N$, we have (see Figure 2.1)

$$\begin{cases} \tilde{F}(1, y + e) = \tilde{F}(0, y) = F(0, y) \\ \tilde{F}(0, y - e) = \tilde{F}(1, y) = F(1, y). \end{cases}$$

Step 1 **: extension on** $[0,1] \times \mathbb{R}^N$
Recall that $Q_0 = [0,1]^N$, $e = (1, ..., 1) \in \mathbb{R}^N$ then set

$$Q_{-1} := Q_0 - e \quad \text{and} \quad Q_1 := Q_0 + e.$$

We first define the auxiliary functions G_α on $[0,1] \times Q_\alpha$ for $\alpha = -1,\ 0,\ 1$. For $y = (y_1, ..., y_N) \in \mathbb{R}^N$, we set

$$\left| \begin{array}{lll} G_0(x, y) = F(x, y) & \text{for all} & (x, y) \in [0,1] \times Q_0 \\ G_{-1}(x, y) = F(1, y + e) - F(1, e) & \text{for all} & (x, y) \in [0,1] \times Q_{-1} \\ G_1(x, y) = F(0, y - e) - F(0, 0) & \text{for all} & (x, y) \in [0,1] \times Q_1. \end{array} \right. \tag{2.98}$$

By construction and using (assumption (A)), we notice that G_α is Lipschitz continuous and non-decreasing w.r.t. y_i for all $i \in \{1, ..., N\}$ on $[0,1] \times Q_\alpha$, for $\alpha = -1, 0, 1$. Moreover, we have

$$\left| \begin{array}{lll} G_{-1}(x,0) = 0 & \text{for all} & x \in [0,1] \\ G_1(x,e) = 0 & \text{for all} & x \in [0,1]. \end{array} \right. \tag{2.99}$$

Now, for every $y \in \mathbb{R}^N$, we set for each $\alpha = -1, 0, 1$,

$$Y_\alpha(y) = Proj_{|Q_\alpha}(y).$$

Then we define the extension G of F on $[0,1] \times \mathbb{R}^N$ by :

$$G(x,y) = G_0(x, Y_0(y)) + (1-x) G_{-1}(x, Y_{-1}(y)) + x G_1(x, Y_1(y)).$$

Clearly, because of (2.99), we have

$$G(x,y) = F(x,y) \quad \text{for any} \quad (x,y) \in [0,1] \times Q_0.$$

Step 1.1 : $G(0,z) = G(1,z+e)$ for any $z \in \mathbb{R}^N$.
From the definition of G, we have for any $z \in \mathbb{R}^N$

$$\begin{array}{l} G(1,z) = G_0(1, Y_0(z)) + G_1(1, Y_1(z)) \\ G(0,z) = G_0(0, Y_0(z)) + G_{-1}(0, Y_{-1}(z)). \end{array}$$

Therefore,

$$\begin{array}{rcl} G(1,z+e) & = & G_0(1, Y_0(z+e)) + G_1(1, Y_1(z+e)) \\ & = & G_0(1, Y_{-1}(z)+e) + G_1(1, Y_0(z)+e) \\ & = & F(1, Y_{-1}(z)+e) + F(0, Y_0(z)) - F(0,0) \\ & = & F(1, Y_{-1}(z)+e) + G_0(0, Y_0(z)) - F(1,e) \\ & = & G_0(0, Y_0(z)) + G_{-1}(0, Y_{-1}(z)) \\ & = & G(0,z), \end{array}$$

where the second equality follows from Corollary 7.2 ii), while the third follows from (2.98) and the fourth follows from the fact that $F(1,e) = F(0,0)$.
Step 1.2 : $G(x,y)$ is monotone in y_i
The result of this step follows from the fact that the orthogonal projection preserves the order (Corollary 7.2 i)) and that for any $\alpha = -1, 0, 1$, G_α is non-decreasing on $[0,1] \times Q_\alpha$ w.r.t. y_i for all $i \in \{1, ..., N\}$.
Step 1.3 : G is globally Lipschitz
Let (x,y), $(\overline{x}, \overline{y}) \in [0,1] \times \mathbb{R}^N$, then

$$\begin{array}{rcl} |G(x,y) - G(\overline{x},\overline{y})| & \leq & |G_0(x, Y_0(y)) - G_0(\overline{x}, Y_0(\overline{y}))| + |\overline{x} - x|.|G_{-1}(x, Y_{-1}(y))| \\ & + & |1-\overline{x}|.|G_{-1}(x, Y_{-1}(y)) - G_{-1}(\overline{x}, Y_{-1}(\overline{y}))| + |x - \overline{x}|.|G_1(x, Y_1(y))| \\ & + & |\overline{x}|.|G_1(x, Y_1(y)) - G_1(\overline{x}, Y_1(\overline{y}))|. \end{array}$$

Since for $\alpha = -1, 0, 1$, the functions G_α are Lipschitz continuous and bounded on $[0,1] \times Q_\alpha$ and using the fact that the orthogonal projection is 1-Lipschitz, we conclude that

$$|G(x,y) - G(\overline{x}, \overline{y})| \leq M|(x - \overline{x}, y - \overline{y})|,$$

where $M = L_0 + L_{-1} + L_1 + M_{-1} + M_1$, with L_α is the Lipschitz constant of G_α, M_α the L^∞ norm of G_α for $\alpha = -1, 0, 1$.

Step 2 : extension on $\mathbb{R} \times \mathbb{R}^N$

Let $k \in \mathbb{Z}$ and set

$$\tilde{F}(x + k, y + ke) = G(x,y) \quad \text{for all} \quad (x,y) \in [0,1] \times \mathbb{R}^N.$$

First of all, $\tilde{F}$ is well defined because of Step 1.1. Moreover by construction, we have the periodicity property

$$\tilde{F}(x + 1, y + e) = \tilde{F}(x,y) \quad \text{for any} \quad (x,y) \in \mathbb{R} \times \mathbb{R}^N.$$

In addition, $\tilde{F}$ is Lipschitz continuous, non-decreasing in each y_i for $i \in \{1, ..., N\}$.

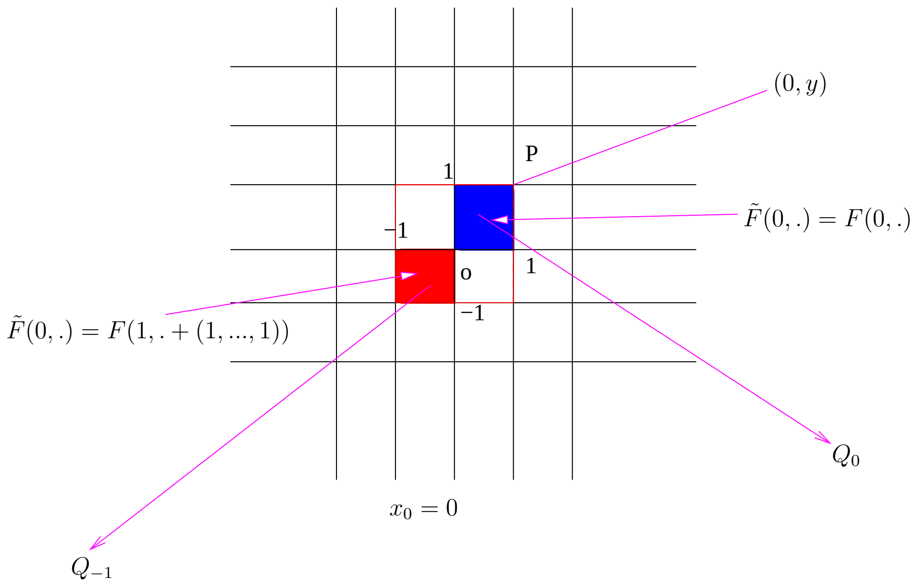

FIGURE 2.1 – A cut of $\{x_0 = 0\} \times \mathbb{R}^N$

□

8 Proof of miscellaneous properties of monotone functions

Appendix B is dedicated to the proof of some results about monotone functions, namely Lemma 2.10 and Lemma 2.11

We first prove Lemma 2.10.

Proof of Lemma 2.10
Assume for simplicity that $g = 0$ on $\mathring{I}$. Suppose to the contrary that there exists a closed interval $I_0 \subset \mathring{I}$, $\delta > 0$ and a subsequence $x_{n_j} \in I_0$ with $x_{n_j} \to x_0 \in I_0$ such that

$$|g_{n_j}(x_{n_j})| \geq \delta.$$

Assume that $g_{n_j}(x_{n_j}) \geq \delta$ (the case $g_{n_j}(x_{n_j}) \leq -\delta$ being similar). Let $\varepsilon > 0$ and consider a closed interval I_ε such that $I_0 \subset\subset I_\varepsilon \subset \mathring{I}$. Since $g_{n_j}(x)$ is non-decreasing in x, then

$$g_{n_j}(x) \geq \delta \quad \text{for all} \quad x \in \big(I_\varepsilon \backslash I_0\big) \cap \big(\{x \geq x_{n_j}\}\big) := I_+.$$

Choose $x_1 \in I_+$ such that $g_{n_j}(x_1) \to g(x_1)$ (g_n converges a.e. on I_+). Thus

$$0 = g(x_1) \geq \delta > 0,$$

a contradiction. □

Now, we give the proof of Lemma 2.11. To this end, we recall and prove the following result :

Lemma 8.1. (Properties of monotone functions)
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function.
i) Countable set of jumps :
The set

$$\mathcal{S} = \{x \ \text{ such that } \phi \text{ is discontinuous at } x\} \tag{2.100}$$

is at most countable.
ii) Density of test points :
Let $x_0 \in \mathbb{R}$, there exists a sequence of functions $\psi_n^+ \in C^\infty(\mathbb{R})$ (resp. $\psi_n^- \in C^\infty(\mathbb{R})$) and a real sequence $(x_n^+)_n$ (resp. $(x_n^-)_n$) such that

$$x_n^+ \to x_0 \quad (\textit{resp.} \quad x_n^- \to x_0)$$

and $\phi^* - \psi_n^+$ (resp. $\phi_* - \psi_n^-$) attains a local maximum (resp. a local minimum) at x_n^+ (resp. at x_n^-) for all n.

The meaning of point *ii*) is that the set of points where ϕ^* is tested (in the sense of Definition 2.4) from above (resp. ϕ_* is tested from below) is dense in $\mathbb{R}$.

Proof of Lemma 8.1.
a) Proof of i) :
This is classical.
b) Proof of ii) for ϕ^* :
Let $x_0 \in \mathbb{R}$. We want to prove that there exists $\psi_n \in C^\infty(\mathbb{R})$ and $x_n \to x_0$ such that $\phi^* - \psi_n$ reaches a local maximum at x_n. For every $\varepsilon > 0$ and for any $b \in \mathbb{R}$, we define the test function

$$\psi_n^b = \frac{1}{\varepsilon}\left(x - \left(x_0 + \frac{1}{n}\right)\right)^2 + b,$$

then we set

$$\beta = \inf \mathcal{E} \quad \text{for} \quad \mathcal{E} = \left\{ b \in \mathbb{R}, \quad \psi_n^b(x) \geq \phi^*(x) \quad \forall\, x \in \left[x_0, x_0 + \frac{2}{n}\right] \right\}.$$

Indeed, since ϕ^* is locally bounded (because ϕ is a real non-decreasing function) and $\mathcal{E}$ is bounded from below (by definition of $\mathcal{E}$), then $\mathcal{E} \neq \emptyset$. From the definition of β, there always exists $x_n \in \left[x_0, x_0 + \frac{2}{n}\right]$ such that

$$\psi_n^\beta(x_n) = \phi^*(x_n) \quad \text{and} \quad \psi_n^\beta(x) \geq \phi^*(x) \quad \text{on} \quad I = \left[x_0, x_0 + \frac{2}{n}\right]. \tag{2.101}$$

We want to show that x_n belongs to the interior of I (at least for ε large enough). We have

$$\psi_n^\beta(x_0) = \frac{1}{\varepsilon n^2} + \beta > \beta = \psi_n^\beta\left(x_0 + \frac{2}{n}\right) \geq \phi^*\left(x_0 + \frac{2}{n}\right) \geq \phi^*(x_0), \tag{2.102}$$

the last two inequalities are true because of (2.101) and the fact that ϕ^* is non-decreasing respectively. Assuming

$$\frac{1}{\varepsilon} > n^2\left(\phi^*\left(x_0 + \frac{2}{n}\right) - \phi^*(x_0)\right),$$

we get

$$\begin{aligned} \psi_n^\beta\left(x_0 + \frac{2}{n}\right) &> \phi^*\left(x_0 + \frac{2}{n}\right) - \phi^*(x_0) + \beta \\ &\geq \phi^*\left(x_0 + \frac{2}{n}\right), \end{aligned}$$

where the last inequality follows from (2.102). This implies that $\phi^* - \psi_n^\beta$ reaches a local maximum at $x_n \in \left(x_0, x_0 + \frac{2}{n}\right)$ and $x_n \to x_0$ as $n \to +\infty$.
c) Proof of ii) for ϕ_* :
Applying argument b) for $\phi(x)$ replaced by $-\phi(-x)$, we get the result. □

Proof of Lemma 2.11.
We set

$$\mathcal{T} = \bigcup_{i=0}^{N} (\mathcal{S} - \{r_i\})$$

with $\mathcal{S}$ defined in (2.100). Using Lemma 8.1 i), we get that $\mathcal{T}$ is countable.
Step 1 : viscosity sense implies a.e. sense
Assume that ϕ is a viscosity solution of (2.27) (see Definition 2.4) and let $x_0 \in \mathbb{R}\backslash\mathcal{T}$. By definition, ϕ is continuous at x_0+r_i for all $i = 0, ..., N$. There exists two sequences of real numbers $(x_n^+)_n$ and $(x_n^-)_n$ such that ϕ^* is tested from above at x_n^+ and ϕ_* is tested from below at x_n^- by smooth functions (the sets of such points is dense in $\mathbb{R}$ (by Lemma 8.1, ii)), and such that

$$\lim_{n\to+\infty} x_n^{\pm} = x_0.$$

Moreover, from Definition 2.4, we have

$$0 \leq F((\phi^*(x_n^+ + r_i))_{i=0,\dots,N}) \tag{2.103}$$

and

$$0 \geq F((\phi_*(x_n^- + r_i))_{i=0,\dots,N}). \tag{2.104}$$

Now, using the fact that

$$\lim_{n\to+\infty} \phi^*(x_n^+ + r_i) = \phi(x_0 + r_i) \quad \text{for} \quad i = 0, ..., N.$$

and that F is Lipschitz continuous (see $(\tilde{A})$), we pass to the limit $n \to +\infty$ in (2.103), and we get

$$\begin{aligned} 0 &\leq \limsup_{n\to+\infty} F((\phi^*(x_n^+ + r_i))_{i=0,\dots,N}) \\ &\leq F((\phi(x_0 + r_i))_{i=0,\dots,N}). \end{aligned}$$

Similarly, we show that

$$\begin{aligned} 0 &\geq \liminf_{n\to+\infty} F((\phi_*(x_n^- + r_i))_{i=0,\dots,N}) \\ &\geq F((\phi(x_0 + r_i))_{i=0,\dots,N}). \end{aligned}$$

Thus

$$0 = F((\phi(x_0 + r_i))_{i=0,\dots,N}),$$

hence ϕ solves equation (2.27) at x_0. But $x_0 \in \mathbb{R}\backslash\mathcal{T}$ is arbitrary, thus ϕ solves (2.27) pointwisely on $\mathbb{R}\backslash\mathcal{T}$. Since $\mathcal{T}$ is countable, we get that ϕ satisfies (2.27) a.e..

Step 2 : a.e. sense implies viscosity sense
Let $x_0 \in \mathbb{R}$. We want to show that ϕ is a viscosity sub-solution at x_0. Let $\psi \in C^1$ such that $\phi \leq \psi$ with equality at x_0, and we want to prove that

$$0 \leq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}).$$

Case 1 : $x_0 \notin \mathcal{T}$
If $x_0 \notin \mathcal{T}$, then ϕ is continuous at $x_0 + r_i$ for all i. Because ϕ solves (2.27) a.e. on $\mathbb{R}$, then there exists a sequence $x_n \to x_0$ such that ϕ solves (2.27) at x_n. Hence we have

$$0 = F((\phi(x_n + r_i))_{i=0,\dots,N}).$$

Passing to the limit $n \to +\infty$, we get

$$0 \leq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}) = F((\phi(x_0 + r_i))_{i=0,\dots,N}).$$

Case 2 : $x_0 \in \mathcal{T}$
Now, assume that $x_0 \in \mathcal{T}$. Since $\mathcal{T}$ is countable, then choose $a_k > a_{k+1} > 0$ such that $a_k \to 0$ and $x_0 + a_k \notin \mathcal{T}$ for all k. Since $x_0 + a_k \notin \mathcal{T}$, then we deduce from Case 1 that

$$0 \leq F((\phi(x_0 + a_k + r_i))_{i=0,\dots,N}).$$

Now letting $a_k \to 0$, we get

$$\begin{aligned} 0 &\leq \limsup_{a_k \to 0} F((\phi(x_0 + a_k + r_i))_{i=0,\dots,N}) \\ &= F((\lim_{a_k \to 0} \phi(x_0 + a_k + r_i))_{i=0,\dots,N}) \\ &\leq F((\phi^*(x_0 + r_i))_{i=0,\dots,N}). \end{aligned}$$

Here, we use the fact that $\phi^*(x) = \lim_{k \to +\infty} \phi(x + a_k)$ for any $x \in \mathbb{R}$ (because ϕ is non-decreasing and $a_k > 0$ with $a_k \to 0$). Hence ϕ is a viscosity sub-solution of (2.27) at x_0.

Similarly, we show that ϕ is a viscosity super-solution at any point, and then ϕ is a viscosity solution. □

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Chapitre 3

Existence d'ondes progressives pour des dynamiques discrètes Lipschitz. Cas monostable comme une limite des cas bistables

Ce chapitre est un travail en collaboration avec R. Monneau [2].
Nous étudions la dynamique monostable discrète avec des non-linéarités Lipschitz générales. Cela inclut aussi les non-linéarités dégénérées. Dans le cas monostable positif, nous montrons l'existence d'une branche d'ondes progressives solutions pour des vitesses $c \geq c^+$, avec la non-existence de solutions pour $c < c^+$. Nous donnons aussi certaines des conditions suffisantes pour que $c^+ \geq 0$ et nous donnons un exemple quand $c^+ < 0$. Nous prouvons ainsi une borne inférieure de c^+, précisément nous montrons que $c^+ \geq c^*$, où c^* est associée à un problème linéarisé à l'infini. D'autre part, dans une condition KPP nous montrons que $c^+ \leq c^*$. Nous donnons aussi un exemple où $c^+ > c^*$.

Ce modèle de la dynamique discrète peut être considéré comme un modèle de Frenkel-Kontorova généralisé pour lequel nous pouvons également ajouter un paramètre de conduite de force σ. Nous montrons que σ peut varier dans un intervalle $[\sigma^-, \sigma^+]$. Pour $\sigma \in (\sigma^-, \sigma^+)$ cela correspond à un cas bistable, tandis que pour $\sigma = \sigma^+$ c'est un cas monostable positif, et pour $\sigma = \sigma^-$ c'est un cas monostable négatif. Nous étudions les fonctions de vitesse $c = c(\sigma)$ lorsque σ varie dans $[\sigma^-, \sigma^+]$. En particulier pour $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), nous trouvons des branches verticales d'ondes progressives solutions avec $c \geq c^+$ (resp. $c \leq c^-$).

Existence of traveling waves for Lipschitz discrete dynamics. Monostable case as a limit of bistable cases

M. Al Haj, R. Monneau

Abstract

We study discrete monostable dynamics with general Lipschitz non-linearities. This includes also degenerate non-linearities. In the positive monostable case, we show the existence of a branch of traveling waves solutions for velocities $c \geq c^+$, with non existence of solutions for $c < c^+$. We also give certain sufficient conditions to insure that $c^+ \geq 0$ and we give an example when $c^+ < 0$. We as well prove a lower bound of c^+, precisely we show that $c^+ \geq c^*$, where c^* is associated to a linearized problem at infinity. On the other hand, under a KPP condition we show that $c^+ \leq c^*$. We also give an example where $c^+ > c^*$.

This model of discrete dynamics can be seen as a generalized Frenkel-Kontorova model for which we can also add a driving force parameter σ. We show that σ can vary in an interval $[\sigma^-, \sigma^+]$. For $\sigma \in (\sigma^-, \sigma^+)$ this corresponds to a bistable case, while for $\sigma = \sigma^+$ this is a positive monostable case, and for $\sigma = \sigma^-$ this is a negative monostable case. We study the velocity function $c = c(\sigma)$ as σ varies in $[\sigma^-, \sigma^+]$. In particular for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), we find vertical branches of traveling waves solutions with $c \geq c^+$ (resp. $c \leq c^-$).

Our method of proof is new and relies on viscosity solutions. Moreover, the monostable case with $c = c^+$ is seen advantageously as a limit situation of the bistable case. For $c >> 1$, the traveling waves are constructed as perturbations of solutions of an associated ODE. Finally to fill the gap between $c = c^+$ and large c, we use certain hull functions that are associated to correctors of a homogenization problem.

Keywords : Traveling waves, degenerate monostable non-linearity, KPP non-linearity, bistable non-linearity, Frenkel-Kontorova model, viscosity solutions, Perron's method.

1 Introduction

1.1 General motivation

Our initial motivation was to study the classical fully overdamped Frenkel-Kontorova model, which is a system of ordinary differential equations

$$\frac{dX_i}{dt} = X_{i+1} - 2X_i + X_{i-1} + f(X_i) + \sigma, \tag{3.1}$$

where $X_i(t) \in \mathbb{R}$ denotes the position of a particle $i \in \mathbb{Z}$ at time t, $\dfrac{dX_i}{dt}$ is the velocity of this particle, f is the force created by a 1-periodic potential and σ represents the constant driving force. Such external force could be for example $f(x) = 1 - \cos(2\pi x) \geq 0$. This kind of system can be, for instance, used as a model of the motion of a dislocation defect in a crystal (see the book of Braun and Kivshar [25]). This motion is described by particular solutions of the form

$$X_i(t) = \phi(i + ct) \tag{3.2}$$

with

$$\phi' \geq 0 \quad \text{and} \quad \phi \text{ is bounded.} \tag{3.3}$$

Such a solution, ϕ, is called a traveling wave solution and c denotes its velocity of propagation. From (3.1) and (3.2), it is equivalent to look for solutions ϕ of

$$c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + f(\phi(z)) + \sigma \tag{3.4}$$

with $z = i + ct$. For such a model, and under certain conditions on f, we show the existence of traveling waves for each value of σ in an interval $[\sigma^-, \sigma^+]$ (see Theorem 1.7). We also get the whole picture (see Figure 3.4 for qualitative properties of this picture) of the velocity function $c = c(\sigma)$ with respect to the driving force σ, with vertical branches for $\sigma = \sigma^-$ or $\sigma = \sigma^+$.

When $f > 0 = f(0) = f(1)$ on $(0,1)$ and $\sigma = 0$, we can moreover normalize the limits of the profile ϕ as

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \tag{3.5}$$

This case is called a positive monostable case and is associated here to $\sigma^+ = 0$. Moreover, we can show the existence of a critical velocity c^+ such that the following holds. There exists a branch of traveling waves solutions for all velocity $c \geq c^+$ and there are no solutions for $c < c^+$.

The goal of this paper is to present similar results in a framework more general than (3.4). To this end, given a real function F (whose properties will be specified in Subsections 1.2 and 1.3), we consider the following generalized equation with $\sigma \in \mathbb{R}$

$$c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma, \tag{3.6}$$

where $N \geq 0$ and $r_i \in \mathbb{R}$ for $i = 0, ..., N$ such that

$$r_0 = 0 \quad \text{and} \quad r_i \neq r_j \ \text{ if } \ i \neq j, \tag{3.7}$$

which does not restrict the generality. In (3.6), we are looking for both the profile ϕ and the velocity c.

Equation (3.1) can be seen as a discretization of the following reaction diffusion equation

$$u_t = \Delta u + f(u). \tag{3.8}$$

In 1937, Fisher [52] and Kolmogorov, Petrovsky and Piskunov [83] studied the traveling waves for equation (3.8) which they proposed as a model describing the spreading of a gene throughout a population. Later, many works have been devoted for such equation that appears in biological models for developments of genes or populations dynamics and in combustion theory (see for instance, Aronson, Weinberger [6, 7] and Hadeler, Rothe [69]). For more developments and applications in biology of reaction diffusion equations, the reader may refer to [111] and to the references cited therein. There is also a considerable work on the existence, uniqueness and stability of traveling waves and their speed of propagation for the homogeneous KPP-Fisher non-linearity (see for example [70, 71, 72, 76, 131]). Such results have been shown also for the inhomogeneous, heterogeneous and random KPP-Fisher non-linearity (see [14, 15, 92]).

Traveling waves were studied also for discrete bistable reaction diffusion equations (see for instance [28, 34]). See also Chapter 2 and the literature Section 1.3. In the monostable case, we distinguish [77] (for nonlocal non-linearities with integer shifts) and [36, 86, 116, 124] (for problems with linear nonlocal part and with integer shifts also). See also [67] for particular monostable non-linearities with irrational shifts. We also refer to [65, 31, 68, 32, 33, 71, 130] for different positive monostable non-linearities. In the monostable case, we have to underline the work of Hudson and Zinner [77] (see also [130]), where they proved the existence of a branch of solutions $c \geq c^*$ for general Lipschitz non-linearities (with possibly an infinite number of neighbors $N = +\infty$, and possibly p types of different particles, while $p = 1$ in our study) but with integer shifts $r_i \in \mathbb{Z}$. However, they do not state the nonexistence of solutions for $c < c^*$. Their method of proof relies on an approximation of the equation on a bounded domain (applying Brouwer's fixed point theorem) and an homotopy argument starting from a known solution. The full result is then obtained as the size of the domain goes to infinity. Here we underline that our results hold for the fully nonlinear case with real shifts $r_i \in \mathbb{R}$.

Several approaches were used to construct traveling waves for discrete monostable dynamics. We already described the homotopy method of Hudson and Zinner [77]. In a second approach, Chen and Guo [32] proved the existence of a solution starting from an approximated problem. They constructed a fixed point solution of an integral reformulation (approximated on a bounded domain) using the monotone iteration method (with sub and supersolutions). This approach was also used to get the existence of a solution in [61, 33, 67, 68]. A third approach based on recursive method for monotone discrete in time dynamical systems was used by Wienberger et al. [86, 116]. See also [124], where this method is used to solve problems with a linear nonlocal part. In a fourth approach [65], Guo and Hamel used global space-time sub and supersolutions to prove the existence of a solution for periodic monostable equations.

There is also a wide literature about the uniqueness and the asymptotics at infinity of a solution for a monostable non-linearities, see for instance [31, 76] (for a degenerate case), [32, 33] and the references therein. Let us also mention that certain delayed reaction diffusion equations with some KPP-Fisher non-linearities do not admit traveling waves (see for example [61, 130]).

Finally, we mention that our method opens new possibilities to be adapted to more general problems. For example, we can think to adapt our approach to a case with possibly p types of different particles similar to [54]. The case with an infinite number of neighbors $N = +\infty$ could be also studied. We can also think to study fully nonlinear parabolic equations.

1.2 Main results in the monostable case

In this subsection, we consider equation (3.6) with $\sigma = 0$. We study the existence of traveling waves of equation (3.6) (with $\sigma = 0$) for positive degenerate monostable non-linearities and with conditions at infinity given by (3.5).

In order to present our results in this case, we have to introduce some assumptions on $F : [0,1]^{N+1} \to \mathbb{R}$.

Assumption ($A_{\mathbf{Lip}}$) :

- **i) Regularity :** $F \in \mathrm{Lip}([0,1]^{N+1})$.
- **ii) Monotonicity :** $F(X_0, X_1, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption ($P_{\mathbf{Lip}}$) :

Positive degenerate monostability :

Let $f(v) = F(v, ..., v)$ such that $f(0) = f(1) = 0, \quad f > 0 \;\text{ in }\; (0, 1)$.

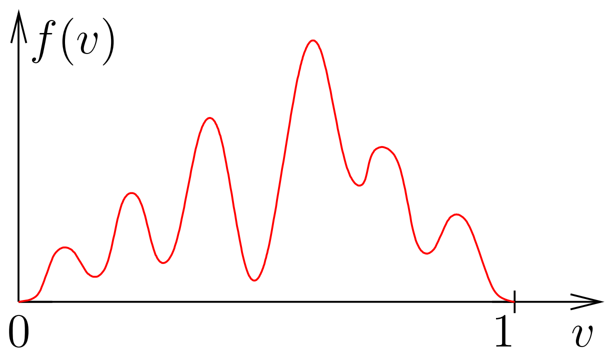

FIGURE 3.1 – Positive degenerate monostable non-linearity f

Our main result is :

Theorem 1.1. (Monostable case : existence of a branch of traveling waves) *Assume* (A_{Lip}) *and* (P_{Lip})*. Then there exists a real* c^+ *such that for all* $c \geq c^+$ *there exists a traveling wave* $\phi : \mathbb{R} \to \mathbb{R}$ *solution (in the viscosity sense (see Definition 2.1)) of*

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) \quad on \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 \quad and \quad \phi(+\infty) = 1. \end{cases} \tag{3.9}$$

On the contrary for $c < c^+$*, there is no solution of (3.9).*

Up to our knowledge, Theorem 1.1 is the first result for discrete dynamics with real shifts $r_i \in \mathbb{R}$ in the fully nonlinear case. Even when $r_i \in \mathbb{Z}$, the only result that we know for fully nonlinear dynamics is the one of Hudson and Zinner [77]. However, the nonexistence of solutions for $c < c^+$ is not addressed in [77].

See Figure 3.2 for an explicit Lipschitz non-linearity example for which our result (Theorem 1.1) is still true, even if $f'(0)$ is not defined. We also prove that the critical velocity c^+ is unstable in the following sense :

Proposition 1.2. (Instability of the minimal velocity c_F^+) *There exists a function* F *satisfying* (A_{Lip}) *and* (P_{Lip}) *with a minimal velocity* c_F^+ *such that there exists a sequence of functions* F_δ *(satisfying* (A_{Lip}) *and* (P_{Lip})*) with associated critical velocity* $c_{F_\delta}^+$ *satisfying*

$$F_\delta \to F \quad in \quad L^\infty([0,1]^{N+1})$$

when $\delta \to 0$*, but*

$$\liminf_{\delta \to 0} c_{F_\delta}^+ > c_F^+.$$

We believe that the critical velocity c^+ contains information about $f'(0)$; similar to classical result in [83] which asserts that the critical velocity of reaction diffusion equation (3.8) is $c^+ = 2\sqrt{f'(0)}$. This shows that when F is only Lipschitz, it becomes very difficult to capture c_F^+ and to show Theorem 1.1 (see its proof, Section 7).

Examples of functions F satisfying assumptions (A_{Lip}) and (P_{Lip}) are given for $N = 2$, $r_0 = 0$, $r_1 = -1$, $r_2 = 1$ by

$$F(X_0, X_1, X_2) = X_2 + X_1 - 2X_0 + g(X_0), \tag{3.10}$$

with for instance non-linearity $g(x) = x(1-x)$ or $g(x) = x^2(1-x)^2$.

In the next result, we prove that the critical velocity c^+ (given in Theorem 1.1) is non-negative for particular F, i.e. we need to assume some smoothness and strict monotonicity on F near $\{0\}^{N+1}$; and this is given in assumption (P_{C^1}) (which is stronger than (P_{Lip})) :

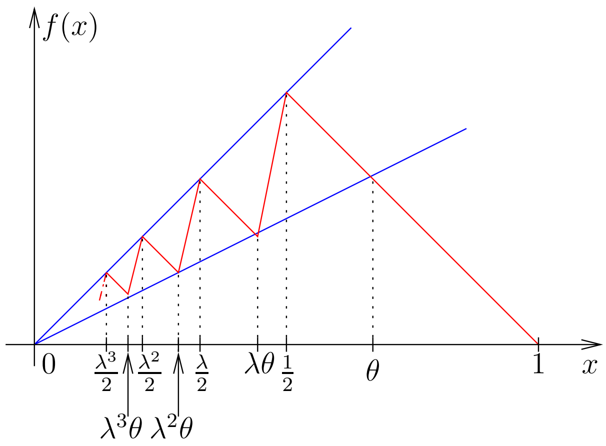

FIGURE 3.2 – Lipschitz positive degenerate monostable non-linearity ; the rest of the figure over $[0, \frac{\lambda^3}{2}]$ is completed by dilation of center 0 and ratio λ.

Assumption (P_{C^1}) :

Positive degenerate monostability :
Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$:
F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0, 1]^{N+1}$ and $f'(0) > 0$.

Proposition 1.3. (Non-negative c^+ for particular F)
Consider a function F satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1. Then we have $c^+ \geq 0$, if one of the three following conditions $i)$, $ii)$ or $iii)$ holds true :

$i)$ **Reflection symmetry of F**
Let $X = (X_i)_{i\in\{0,...,N\}} \in [0, 1]^{N+1}$. Assume that for all $i \in \{0, ..., N\}$ there exists $\bar{i} \in \{0, ..., N\}$ such that $r_{\bar{i}} = -r_i$; and

$$F(\overline{X}) = F(X) \quad \textit{for all} \quad X \in [0, 1]^{N+1},$$

where

$$\overline{X}_i = X_{\bar{i}} \quad \textit{for} \quad i \in \{0, ..., N\}.$$

$ii)$ **All the r_i's "shifts" are non-negative**
Assume that $r_i \geq 0$ for all $i \in \{0, ..., N\}$.

$iii)$ **Strict monotonicity**
Let

$$I = \Big\{ i \in \{1, ..., N\} \textit{ such that there exists } \bar{i} \in \{1, ..., N\} \textit{ with } r_{\bar{i}} = -r_i \Big\} \quad (3.11)$$

and assume that

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i \in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) > 0. \tag{3.12}$$

Notice that because of the monotonicity of F in X_j for $j \neq 0$, condition (3.12) is satisfied if

$$\frac{\partial F}{\partial X_0}(0) > 0.$$

Moreover, if

$$I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\bar{i}}}(0) \ \text{ for all } \ i \in I, \tag{3.13}$$

then condition (3.12) is equivalent to $f'(0) > 0$. In particular, under condition *i*) property (3.13) holds true. This shows that condition *iii*) is more general than condition *i*).

Remark that if we replace (P_{C^1}) by (P_{Lip}) assuming *i*), *ii*) or *iii*), we do not know if $c^+ \geq 0$.

Proposition 1.4. (Counter example with $c^+ < 0$)
There exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity (given in Theorem 1.1) is negative, i.e. $c^+ < 0$.

In the following proposition, we give a lower bound of the critical velocity c^+.

Proposition 1.5. (Lower bound for c^+)
Let F be a function satisfying (A_{Lip}) and (P_{C^1}). Let c^+ given by Theorem 1.1 and assume

$$\exists \ i_0 \in \{0, ..., N\} \ \ \textit{such that} \ \ r_{i_0} > 0 \ \ \textit{and} \ \ \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0, \tag{3.14}$$

then

$$c^+ \geq c^*,$$

where

$$c^* := \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} \quad \textit{with} \quad P(\lambda) := \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) e^{\lambda r_i}. \tag{3.15}$$

We can also get the result of Proposition 1.5 under conditions different from (3.14) (see Remark 8.2).
Here, it is natural to ask if we may have $c^+ = c^*$ in general or not. We give for instance in Lemma 8.4, an example of a non-linearity where we have $c^+ > c^*$ which answers the question. On the other hand, we can find a KPP type condition to insure the inequality $c^+ \leq c^*$, as show the following result :

Proposition 1.6. (KPP condition for $c^+ \leq c^*$)
Let F be a function satisfying (A_{Lip}) and (P_{Lip}). Let c^+ given by Theorem 1.1 and assume that F is differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. If moreover F satisfies the KPP condition :

$$F(X) \leq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)X_i \quad \text{for every} \quad X \in [0,1]^{N+1}, \tag{3.16}$$

then $c^+ \leq c^*$ with c^* defined in (3.15).

1.3 Main result on the velocity function

In this subsection, we consider equation (3.6) with a constant parameter $\sigma \in \mathbb{R}$ and $F : \mathbb{R}^{N+1} \to \mathbb{R}$. We are interested in the velocities c associated to σ (that we call roughly speaking the "velocity function").

For σ belonging to some interval $[\sigma^-, \sigma^+]$, we prove the existence of a traveling wave and we study the variation of its velocity c with respect to σ.

Let $E = (1,...,1)$, $\Theta = (\theta,...,\theta) \in \mathbb{R}^{N+1}$ and assume that the function F satisfies :

Assumption $(\tilde{A}_{C^1})$:

Regularity : F is globally Lipschitz continuous over $\mathbb{R}^{N+1}$ and C^1 over a neighborhood in $\mathbb{R}^{N+1}$ of the two intervals $]0,\Theta[$ and $]\Theta, E[$.

Monotonicity : $F(X_0,...,X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Periodicity : $F(X_0+1,...,X_N+1) = F(X_0,...,X_N)$ for every $X = (X_0,...,X_N) \in \mathbb{R}^{N+1}$.

Notice that, since F is periodic in E direction, then F is C^1 over a neighborhood of $\mathbb{R}E \backslash (\mathbb{Z}E \cup \mathbb{Z}\Theta)$.

Assumption $(\tilde{B}_{C^1})$:
Define $f(v) = F(v,...,v)$ such that :

Bistability : $f(0) = f(1)$ and there exists $\theta \in (0,1)$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0,\theta) \\ f' < 0 & \text{on} \quad (\theta,1). \end{cases}$$

See Figure 3.3 for an example of f satisfying $(\tilde{B}_{C^1})$. Notice that assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ holds true in particular for the Frenkel-Kontorova model for $\beta > 0$:

$$\frac{d}{dt}X_i = X_{i+1} + X_{i-1} - 2X_i - \beta \sin\left(2\pi\left(X_i + \frac{1}{4}\right)\right) + \sigma. \tag{3.17}$$

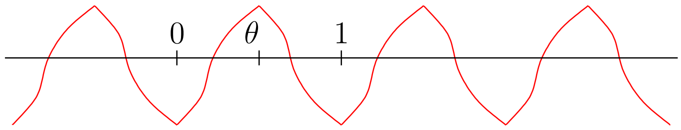

Figure 3.3 – Bistable non-linearity f

Theorem 1.7. (General case : traveling waves and the velocity function)
Under assumptions $(\tilde{A}_{C^1})$ *and* $(\tilde{B}_{C^1})$, *define* $\sigma^{\pm}$ *as*

$$\begin{cases} \sigma^+ = -\min f \\ \sigma^- = -\max f. \end{cases} \tag{3.18}$$

Associate for each $\sigma \in [\sigma^-, \sigma^+]$ *the solutions* $m_\sigma \in [\theta - 1, 0]$ *and* $b_\sigma \in [0, \theta]$ *of* $f(s) + \sigma = 0$. *Then consider the following equation*

$$\begin{cases} c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = m_\sigma \quad \text{and} \quad \phi(+\infty) = m_\sigma + 1, \end{cases} \tag{3.19}$$

1- Bistable case : traveling waves for $\sigma \in (\sigma^-, \sigma^+)$
We have

(i) **(Existence of a traveling wave)**
For any $\sigma \in (\sigma^-, \sigma^+)$, *there exists a unique real* $c := c(\sigma)$, *such that there exists a function* $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ *solution of (3.19) in the viscosity sense.*

(ii) **(Continuity and monotonicity of the velocity function)**
The map

$$\sigma \mapsto c(\sigma)$$

is continuous on (σ^-, σ^+) *and there exists a constant* $K > 0$ *such that the function* $c(\sigma)$ *is non-decreasing and satisfies*

$$\frac{dc}{d\sigma} \geq K|c| \quad \text{on} \quad (\sigma^-, \sigma^+)$$

in the viscosity sense. In addition, there exists real numbers $c^- \leq c^+$ *such that*

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \text{and} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+.$$

Moreover, either $c^- = 0 = c^+$ *or* $c^- < c^+$.

2- Monostable cases : vertical branches for $\sigma = \sigma^{\pm}$
We have

(i) **(Existence of traveling waves for $c \geq c^+$ when $\sigma = \sigma^+$)**
Let $\sigma = \sigma^+$, then for every $c \geq c^+$ there exists a traveling wave ϕ solution of

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^+ \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0 = m_{\sigma^+} \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{3.20}$$

Moreover, for any $c < c^+$, there is no solution ϕ of (3.20).
(ii) **(Existence of traveling waves for $c \leq c^-$ when $\sigma = \sigma^-$)**
Let $\sigma = \sigma^-$, then for every $c \leq c^-$, there exists a traveling wave ϕ solution of

$$\begin{cases} c\phi'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^- \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = \theta - 1 = m_{\sigma^-} \quad \text{and} \quad \phi(+\infty) = \theta. \end{cases} \tag{3.21}$$

Moreover, for any $c > c^-$, there is no solution ϕ of (3.21).

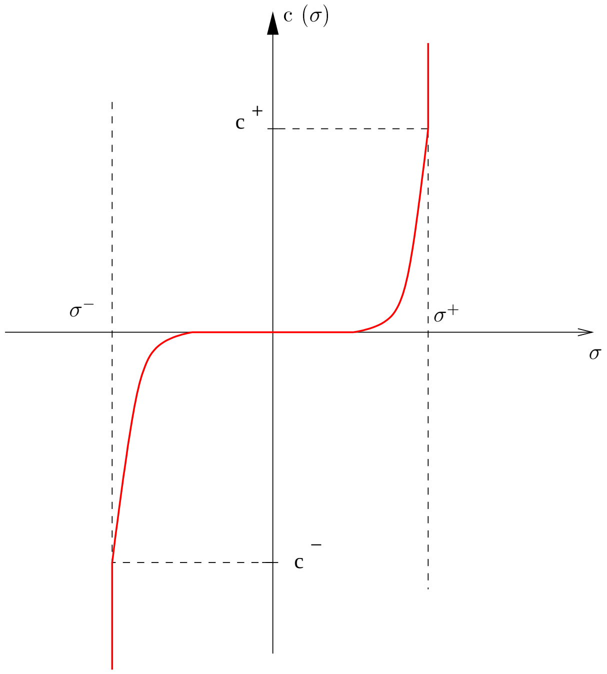

FIGURE 3.4 – Typical graph of the velocity function $c(\sigma)$ with vertical branches at $\sigma = \sigma^{\pm}$.

Note that for the Frenkel Kontorova model (3.17), we have $\sigma^{\pm} = \pm 1$ and $c^+ > 0 > c^-$ (cf. Lemma 8.1), and Figure 3.4 illustrates the graph of the velocity $c(\sigma)$ which has a plateau at the level $c = 0$ in particular if $|\sigma| < \beta - 1$ (see

Proposition 2.6).

In view of Theorem 1.7, we can ask the following :
Open question 1. For a general F, what is the precise behavior of the function $c(\sigma)$ close to the boundary of the plateau $c = 0$ and close to σ^+ and σ^-?
Open question 2. Can we construct a function F such that $c^+ = 0 = c^-$?
For indications in the direction of open question 1, see for instance [28] (discussion on page 4 after Theorem 1.2).

Remark 1.8. (sign of c^+ and c^-)
If we can apply Proposition 1.3 for $F + \sigma^+$, we deduce that $c^+ \geq 0$. Similarly, by symmetry (see Lemma 3.3), it is possible to introduce similar assumptions to conclude that $c^- \leq 0$.

Remark 1.9. (Existence of m_σ and b_σ for $\sigma \in [\sigma^-, \sigma^+]$)
Remark that under assumption $(\tilde{B}_{C^1})$ and from the definition of $\sigma^\pm$ (see (3.18)), the associated $m_\sigma \in [\theta - 1, 0]$ and $b_\sigma \in [0, \theta]$ exist uniquely for every $\sigma \in [\sigma^-, \sigma^+]$. This implies that the two maps $\sigma \to m_\sigma,\ b_\sigma$ are well defined.

Remark 1.10. (No solution of (3.19) when $\sigma \notin [\sigma^-, \sigma^+]$)
From the definition of $\sigma^\pm$ (see (3.18), we see that the function $f + \sigma = 0$ has no solution if $\sigma \notin [\sigma^-, \sigma^+]$. Moreover, if ϕ is a bounded solution of

$$c\phi'(z) = F(\phi(z + r_0), \phi(z + r_1), ..., \phi(z + r_N)) + \sigma \quad on \quad \mathbb{R}, \tag{3.22}$$

then $\phi(\pm\infty)$ should solve the equation $f + \sigma = 0$. Thus, we conclude that (3.22) does not admit a bounded solution if $\sigma \notin [\sigma^-, \sigma^+]$.

Notice that Theorem 1.1 is a generalization of Theorem 1.7-2 (i) for $\sigma = \sigma^+$. Also, notice that Theorem 1.7-1 (i) is already proved in Chapter 2 (see Chapter 2, Proposition 2.3).

As a notation, we set for a general function h :

$$F((h(z + r_i))_{i=0,...,N}) = F(h(z + r_0), h(z + r_1), ..., h(z + r_N))$$

and we define

$$r^* = \max_{i=0,...,N} |r_i|. \tag{3.23}$$

In the rest of the paper, we will use the notation introduced in Theorem 1.7.

1.4 Organization of the paper

Even if the main results of Subsections 1.2 and 1.3 are very different, the proofs are deeply related (because we use the results in the bistable case to deduce some results in the monostable case).

We recall, in Section 2, the notion of viscosity solutions and some useful results for monotone functions. Section 3 is devoted to the proof of existence of traveling waves solutions of (3.9) for $c >> 1$, which is applicable in particular for (3.20) and also for (3.21) when $c << -1$ (up to apply a suitable transformation). We use, in Section 4, results in Chapter 2 to prove the existence of a traveling wave and the uniqueness of the velocity for solutions of (3.19) as a function of the driving force $\sigma \in (\sigma^-, \sigma^+)$. In Section 5, we prove the continuity and monotonicity of the velocity function over (σ^-, σ^+) and we show that the velocity function attains finite limits $c^\pm$ at $\sigma^\pm$. We also prove, in this section, the existence of solutions of (3.20) (resp. (3.21)) for $c = c^+$ (resp. $c = c^-$). In Section 6, we fill the gap by proving the existence of solutions of (3.20) (resp. (3.21)) for every $c \geq c^+$ (resp. $c \leq c^-$). Moreover, we show that for any $c < c^+$ (resp. $c > c^-$) there is no solution of (3.20) (resp. (3.21)). We prove Theorem 1.7 at the end of Section 6.

Theorem 1.1 is proved in Section 7, which we split in two subsections. In Subsection 7.1, we recall an extension result to $\mathbb{R}^{N+1}$ of a non-linearity defined on $[0,1]^{N+1}$ and then we prove Theorem 1.1 in the special case where the non-linearity is smooth. In Subsection 7.2, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities, where the construction of the critical velocity c^+ follows the lines of the proof of the regular case, but requires a lot of work to adapt it to this very delicate situation. Section 8 is dedicated to properties of the critical velocity c^+. In Subsection 8.1, we prove that c^+ is non-negative under certain assumptions, namely Proposition 1.3. While in Subsection 8.2, we construct a counter-example for which $c^+ < 0$, i.e. Proposition 1.4 and we prove the instability result of Proposition 1.2. Subsection 8.3 is specified for the proof of Proposition 1.5 where we show that $c^+ \geq c^*$. In this subsection, we also show that $c^+ \leq c^*$ under a KPP type condition (precisely, we prove Proposition 1.6). We as well give an example (see Lemma 8.4) where $c^+ > c^*$.

Finally the Appendix (Section 9) is divided into three parts. We prove in Subsection 9.1 results about the passage to the limit in our equation and about the identification of the limits at infinity of the limit profile. In Subsection 9.2, we prove and state two kinds of results (which are used to prove that $c^+ \geq 0$) : first, extension by antisymmetry and antisymmetry-reflection (Propositions 9.4 and 9.7) and second, a comparison principle (Propositions 9.9 and 9.10). Lastly in Subsection 9.3, we prove a strong maximum principle (Proposition 9.11), a lower bound (Proposition 9.13) and a Harnack type inequality (Proposition 9.14) for a profile that we use to prove that $c^+ \geq c^*$ in Subsection 8.3.

1.5 Notations of our assumptions

In our paper, we introduce assumptions (A_{Lip}), (P_{Lip}) and (P_{C^1}) in Section 1.2, assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ in Section 1.3, assumptions $(\tilde{B}'_{C^1})$ and (B_{Lip}) in Section 4, assumptions $(\tilde{A}_{\text{Lip}})$, (A_{C^1}) and (P'_{C^1}) in Section 7.1 and assumption $(\tilde{B}_{m,b})$ in Section 9.1.

Generically, assumptions of type A holds for F, assumptions of type P are positivity assumptions on $f(v) = F(v, ..., v)$, and assumptions of type B are bistable assumptions for f.

Assumptions with *tilde* ($\tilde{}$) means that the functions F and f are considered on $\mathbb{R}^{N+1}$ and $\mathbb{R}$ respectively, and are assumed to be $(1, ..., 1)$-periodic and 1-periodic respectively. On the contrary, assumptions without *tilde* means assumptions for F and f on a finite box $[0, 1]^{N+1}$ and $[0, 1]$ respectively.

The subscript "Lip" means that we only require Lipschitz functions, while the subscript "C^1" means that we require C^1 functions (at least on some part of their domain of definition).

Finally, assumptions with *prime* ($'$) are (locally in the paper) variant of the assumptions without *prime*.

2 Preliminary results

We recall, in a first subsection, the definition of viscosity solutions, a stability result and Perron's method for constructing a solution. We state, in a second subsection, Helly's Lemma and the equivalence result between viscosity and almost everywhere solutions for non-decreasing functions. In a third subsection, we give an example with a discontinuous viscosity solution.

2.1 Viscosity solution

In the whole paper, we will use the notion of viscosity solutions that we introduce in this subsection. To this end, we recall that the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u are defined as

$$u^*(y) = \limsup_{x \to y} u(x) \quad \text{and} \quad u_*(y) = \liminf_{x \to y} u(x).$$

Definition 2.1. (Viscosity solution)
Let $I = I' = \mathbb{R}$ (or $I = (-r^*, +\infty)$ and $I' = (0, +\infty)$) and $u : I \to \mathbb{R}$ be a locally bounded function, $c \in \mathbb{R}$ and F defined on $\mathbb{R}^{N+1}$.

- The function u is a subsolution (resp. a supersolution) on I' of

$$cu'(x) = F((u(x + r_i))_{i=0,...,N}) + \sigma, \tag{3.24}$$

if u is upper semi-continuous (resp. lower semi-continuous) and if for all test function $\psi \in C^1(I)$ such that $u-\psi$ attains a local maximum (resp. a local minimum) at $x^* \in I'$, we have

$$c\psi'(x^*) \le F((u(x^*+r_i))_{i=0,\dots,N})+\sigma \quad \Big(\textit{resp. } c\psi'(x^*) \ge F((u(x^*+r_i))_{i=0,\dots,N})+\sigma\Big).$$

- *A function u is a viscosity solution of (3.24) on I' if u^* is a subsolution and u_* is a supersolution on I'.*

We also recall the stability result for viscosity solutions (see [11, Theorem 4.1] and [53, Proposition 2.4] for a similar proof).

Proposition 2.2. (Stability of viscosity solutions)
Consider a function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Assume that $(u_\varepsilon)_\varepsilon$ is a sequence of subsolutions (resp. supersolutions) of (3.24).

(i) Let

$$\overline{u}(x) = \limsup_{\varepsilon\to 0}{}^* u_\varepsilon(x) := \limsup_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y) \quad \textit{and} \quad \underline{u}(x) = \liminf_{\varepsilon\to 0}{}_* u_\varepsilon(x) := \liminf_{(\varepsilon,y)\to(0,x)} u_\varepsilon(y),$$

be the relaxed upper and lower semi-limits. If $\overline{u}$ (resp. $\underline{u}$) is finite, then $\overline{u}$ is a subsolution (resp. $\underline{u}$ is a supersolution) of (3.24).

(ii) Let $\mathcal{T}$ be a nonempty collection of subsolutions of (3.24) and set $U(x) = \sup_{u\in\mathcal{T}} u(x)$. If U^* is finite then U^* is a subsolution of (3.24). A similar result holds for supersolutions.

Next, we state Perron's method that we will use to construct a solution in Section 3.

Proposition 2.3. (Perron's method ([53, Proposition 2.8]))
Let $I=(-r^*,+\infty)$ and $I'=(0,+\infty)$ and F be a function satisfying $(\tilde{A}_{Lip})$ (introduced in Subsection 7.1). Let u and v defined on I satisfying

$$u \le v \quad \textit{on} \quad I,$$

such that u and v are respectively a sub and a supersolution of (3.24) on I'. Let $\mathcal{L}$ be the set of all functions $\tilde{v} : I \to \mathbb{R}$, such that $u \le \tilde{v}$ over I with $\tilde{v}$ supersolution of (3.24) on I'. For every $z \in I$, let

$$w(z) = \inf\{\tilde{v}(z) \quad \textit{such that} \quad \tilde{v} \in \mathcal{L}\}.$$

Then w is a solution of (3.24) over I' satisfying $u \le w \le v$ over I.

2.2 Some results for monotone functions

In this subsection, we state Helly's Lemma for the convergence of a sequence of non-decreasing functions. We also recall the result about the equivalence between the viscosity and almost everywhere solutions. These results will be used later in Sections 4, 5, 6 and 7.

Lemma 2.4. (Helly's Lemma, (see [5], Section 3.3, **page** 70**))**
Let $(g_n)_{n\in\mathbb{N}}$ *be a sequence of non-decreasing functions on* $[a,b]$ *verifying* $|g_n| \leq M$ *uniformly in* n*. Then there exists a subsequence* $(g_{n_j})_{j\in\mathbb{N}}$ *such that*

$$g_{n_j} \to g \quad a.e.\ on \quad [a,b],$$

with g *non-decreasing and* $|g| \leq M$*.*

Now, we state the lemma for non-decreasing functions about the equivalence between a viscosity and an almost everywhere solution.

Lemma 2.5. (Equivalence between viscosity and a.e. solutions)
Let F *satisfy assumption* $(\tilde{A}_{Lip})$ *(introduced in Subsection 7.1). Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing function. Then* ϕ *is a viscosity solution of*

$$0 = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma \quad on \quad \mathbb{R}, \tag{3.25}$$

if and only if ϕ *is an almost everywhere solution of the same equation.*

For the proof of Lemma 2.5, we refer the reader to Chapter 2, Lemma 2.11.

2.3 Example of discontinuous viscosity solution

We give in this section an example of a discontinuous viscosity solution.

Proposition 2.6. (Discontinuous viscosity solution)
Consider $\beta > 0$, $\sigma \in \mathbb{R}$ *and let* (c, ϕ) *be a solution of*

$$\begin{cases} c\phi'(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta \sin(2\pi\phi(z)) + \sigma \quad on \ \ \mathbb{R} \\ \phi \quad is\ non\text{-}decreasing \\ \phi(+\infty) - \phi(-\infty) = 1. \end{cases} \tag{3.26}$$

Then $\sigma^{\pm} = \pm\beta$*. Moreover, if* $|\sigma| < \beta - 1$*, then* $\phi \notin C^0$ *and* $c = 0$*.*

For the convenience of the reader we give the proof of this result (which is basically contained in Theorem 1.2 in Carpio et al. [28]).

Proof of Proposition 2.6
Clearly, we have $\sigma^{\pm} = \pm\beta$ (see Remark 1.10). Let $|\sigma| < \beta - 1$ and let us show that $\phi \notin C^0(\mathbb{R})$. Assume to the contrary that $\phi \in C^0(\mathbb{R})$.

Notice that because ϕ is non-decreasing and $\phi(+\infty) - \phi(-\infty) = 1$, we deduce that

$$\phi(z+1) - 2\phi(z) + \phi(z-1) \in [-1, 1].$$

Define now

$$\psi(z) = \phi(z+1) - 2\phi(z) + \phi(z-1) + \beta \sin(2\pi\phi(z)) + \sigma.$$

Because $\phi \in C^0$, then looking at the sup and inf of $\sin(2\pi\phi)$, we deduce that

$$\begin{cases} \sup_{\mathbb{R}} \psi \geq \beta + \sigma - 1 > 0 \\ \inf_{\mathbb{R}} \psi \leq -\beta + \sigma + 1 < 0, \end{cases}$$

where the strict inequalities follow from $|\sigma| < \beta - 1$. But $c\phi' = \psi$ which implies that $c\phi'$ changes sign. This is impossible because ϕ is non-decreasing. Therefore, $\phi \notin C^0(\mathbb{R})$, which implies that $c = 0$. □

3 Vertical branches for large velocities

We devote this section to the proof of existence of traveling waves solutions of (3.9) for $c >> 1$ and under the assumptions (A_{Lip}) and (P_{Lip}). The result is applicable in particular for function F defined on $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ with $\sigma = \sigma^+$, which can be always reduced to the case $\sigma^+ = 0$ by adding a constant to F, and hence we may get a solution for (3.20) for $c >> 1$. In this section, we show also the existence of solutions (3.21) for $c << -1$ which follows from the case $\sigma = \sigma^+$ using a transformation result (Lemma 3.3).

Proposition 3.1. (Existence of traveling waves for $c >> 1$)
Consider a function F satisfying (A_{Lip}) and (P_{Lip}). Then for $c >> 1$, there exists a traveling wave ϕ solution of (3.9).

Proof of Proposition 3.1
The strategy of the proof consists in constructing a sub and a supersolution for $c >> 1$ of a re-scaled form of the equation

$$c\phi'(y) = F(\phi(y + r_0), \phi(y + r_1), ..., \phi(y + r_N)) \quad \text{on} \quad \mathbb{R}, \tag{3.27}$$

and then in proving the existence of a solution using Perron's method.

Step 1 : re-scaling equation (3.27)
If ϕ is a solution of (3.27) (with $\phi(-\infty) = 0$, $\phi(+\infty) = 1$), then for every $z \in \mathbb{R}$, the function h defined as

$$h(z) := \phi(cz)$$

has to satisfy

$$h'(z) = F\left(\left(h\left(z+\frac{r_i}{c}\right)\right)_{i=0,\dots,N}\right) \quad \text{on} \quad \mathbb{R}. \tag{3.28}$$

Step 2 : sub and supersolution of (3.28)

In order to construct a sub and a supersolution of (3.28), we first mention some useful properties of the solution of the ODE

$$h_0' = F(h_0, ..., h_0) = f(h_0) \geq 0, \tag{3.29}$$

with $h_0(0) = \frac{1}{2}$.

Step 2.1 : existence and monotonicity of h_0

Since $f > 0$ on $(0,1)$ and f is Lipschitz over $[0,1]$ (see assumptions (A_{Lip}) and (P_{Lip})), then there exists a C^1 solution h_0 of (3.29) defined on $\mathbb{R}$, with values in $[0,1]$, satisfying

$$h_0' > 0 \quad \text{on} \quad \mathbb{R}. \tag{3.30}$$

Since the constant functions 0 and 1 are respectively a sub and a supersolution of (3.29) (since $f(0) = f(1) = 0$), then

$$0 \leq h_0(z) \leq 1.$$

We also easily deduce that

$$h_0(-\infty) = 0 \quad \text{and} \quad h_0(+\infty) = 1.$$

Step 2.2 : subsolution of (3.28)

Let $\varepsilon = \dfrac{1}{c}$ and $0 < \delta = \underline{M}\varepsilon$ for $\underline{M}$ chosen large, and c chosen such that $\underline{a} = 1-\delta > 0$. Then define

$$\underline{h}(z) = h_0(\underline{a}z) \quad \text{over} \quad \mathbb{R}.$$

Our goal is to prove that $\underline{h}$ is a subsolution of (3.28) on $\mathbb{R}$, taking advantage of the fact that (3.29) is a caricature of (3.28) for large c.

First notice that we have

$$\underline{h}\,(z+\varepsilon r_i) = h_0(\underline{a}z) + \underline{a}\varepsilon r_i \underline{L}_i \quad \text{with} \quad \underline{L}_i = \int_0^1 h_0'(\underline{a}z + \underline{a}\varepsilon r_i t)dt.$$

Because $F \in \text{Lip}([0,1]^{N+1})$ for some Lipschitz constant L, we get

$$\begin{aligned} F((\underline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\underline{a}z)) &= F((h_0(\underline{a}z) + \underline{a}\varepsilon r_i \underline{L}_i))_{i=0,\dots,N}) - F((h_0(\underline{a}z))_{i=0,\dots,N}) \\ &\geq -\varepsilon \underline{a} L \begin{vmatrix} r_0\underline{L}_0 \\ \vdots \\ r_N \underline{L}_N \end{vmatrix} \geq -\varepsilon \underline{a} L r^* \begin{vmatrix} \underline{L}_0 \\ \vdots \\ \underline{L}_N \end{vmatrix}, \end{aligned}$$

where $r^* = \max_{i=0,\dots,N} |r_i|$ (recall (3.23)).
We now estimate the $\underline{L}_i$'s.

Case 1 **:** $f \in C^1([0,1])$
If $f \in C^1([0,1])$, then for $z \in \mathbb{R}$, we have

$$h_0''(z) = f'(h_0(z))h_0'(z).$$

As $h_0' > 0$ on $\mathbb{R}$ and $f \in C^1([0,1])$, we get for $z \in \mathbb{R}$

$$(\ln(h_0'(z)))' = f'(h_0(z)),$$

where the absolute value of the right hand side is bounded by some constant $\mathcal{K}$. Hence, using the continuity of h_0', for any $b \in \mathbb{R}$ and for all $z \in \mathbb{R}$, we obtain

$$\ln\left(\frac{h_0'(z+b)}{h_0'(z)}\right) \leq \mathcal{K}|b|.$$

This implies that

$$h_0'(z+b) \leq h_0'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}. \tag{3.31}$$

Case 2 **:** $f \in \mathbf{Lip}([0,1])$
We want to show that (3.31) is still true if $f \in \mathrm{Lip}([0,1])$, and the point is to regularize by convolution the function f and then to pass to the limit. Using the extension result (cf. Lemma 7.1), there exists a function $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ and satisfying $(\tilde{A}_{\mathrm{Lip}})$. Moreover, the function $\tilde{f}(v) := \tilde{F}(v,...,v)$ is nothing but the periodic extension of f with period 1.

Let $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, where ρ is a mollifier and define the function $\tilde{f}_\varepsilon(x) := \tilde{f}\star\rho_\varepsilon(x)$. Then consider the ODE

$$\begin{cases} h_\varepsilon' = \tilde{f}_\varepsilon(h_\varepsilon) \\ h_\varepsilon(0) = \dfrac{1}{2}. \end{cases} \tag{3.32}$$

Since $\tilde{f}_\varepsilon$ is C^1, then there exists a unique regular solution h_ε defined over $\mathbb{R}$ and satisfies

$$h_\varepsilon'(z+b) \leq h_\varepsilon'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}. \tag{3.33}$$

Moreover, since $\tilde{f}_\varepsilon$ is periodic smooth, then there exists some C independent of ε such that

$$|h_\varepsilon'| \leq C \quad \text{on} \quad \mathbb{R}.$$

Therefore, using Ascoli's theorem and the extraction diagonal argument, h_ε converges locally uniformly to some h_1 that solves in the classical sense

$$\begin{cases} h_1' = \tilde{f}(h_1) \\ h_1(0) = \dfrac{1}{2}, \end{cases} \tag{3.34}$$

and

$$h_1'(z+b) \le h_1'(z)e^{\mathcal{K}|b|} \quad \text{for every} \quad z \in \mathbb{R}.$$

But the constant functions 0 and 1 are respectively sub and supersolution of (3.34), then

$$0 \le h_1 \le 1,$$

that is, h_1 is a solution of (3.29). Thus by uniqueness, we get that $h_1 = h_0$, and hence h_0 satisfies (3.31).

Consequences in both Case 1 and Case 2
Now, we go back to estimate the $\underline{L}_i$'s. Using (3.31), we get, for every $i \in \{0, ..., N\}$, that

$$0 \le \underline{L}_i = \int_0^1 h_0'(\underline{a}z + \underline{a}\varepsilon r_i t)dt \le h_0'(\underline{a}z)e^{\mathcal{K}\underline{a}\varepsilon|r_i|} \le h_0'(\underline{a}z)e^{\mathcal{K}\varepsilon r^*} =: \underline{K}h_0'(\underline{a}z),$$

using (3.31) for $b = \underline{a}\varepsilon r_i t$ and using the fact that $\underline{a} < 1$.

Therefore, for $L_1 := L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}$, we deduce that with $\delta = \underline{M}\varepsilon$

$$\begin{aligned} \underline{h}'(z) - F((\underline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) &= \underline{a}h_0'(\underline{a}z) - F((\underline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) \\ &= -\delta h_0'(\underline{a}z) - \Big(F((\underline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\underline{a}z))\Big) \\ &\le -\delta h_0'(\underline{a}z) + \varepsilon \underline{a} L r^* \begin{vmatrix} \underline{L}_0 \\ \vdots \\ \underline{L}_N \end{vmatrix} \\ &\le \varepsilon\left(-\underline{M} + \underline{a}L_1 r^* \underline{K}\right) h_0'(\underline{a}z) \le 0, \end{aligned}$$

where the last inequality follows once $\underline{M} \ge L_1 r^* \underline{K}$ (taking into account that $\underline{a} \le 1$). Therefore $\underline{h}$ is a subsolution of (3.28).

Step 2.3 : supersolution of (3.28)
The proof is similar to Step 2.2. Let $\varepsilon = \dfrac{1}{c}$ and $0 < \delta = \overline{M}\varepsilon$ with $\overline{M}$ chosen large, and c chosen such that $\overline{a} = 1 + \delta \le 2$. Then consider the function

$$\overline{h}(z) = h_0(\overline{a}z)$$

that we want to show to be a supersolution of (3.28). Clearly, we have

$$\overline{h}(z+\varepsilon r_i) = h_0(\overline{a}z) + \varepsilon\overline{a}r_i\overline{L}_i \quad \text{with} \quad \overline{L}_i = \int_0^1 h_0'(\overline{a}z + \varepsilon\overline{a}r_i t)$$

and

$$F((\overline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) - f(h_0(\overline{a}z)) \leq \varepsilon \overline{a} L \begin{vmatrix} r_0 \overline{L}_0 \\ \vdots \\ r_N \overline{L}_N \end{vmatrix} \leq 2\varepsilon L_1 r^* \overline{K} h_0'(\overline{a}z),$$

where for the last inequality we have used that $\overline{a} \leq 2$, $\overline{K} = e^{2\mathcal{K}\varepsilon r^*}$ and L_1 given in the above step. Finally, we deduce that

$$\begin{aligned} \overline{h}'(z) - F((\overline{h}(z+\varepsilon r_i))_{i=0,\dots,N}) &\geq \varepsilon\Big(\overline{M} - 2L_1 r^* \overline{K}\Big) h_0'(\overline{a}z) \\ &\geq 0, \end{aligned}$$

if we choose $\overline{M} \geq 2L_1 r^* \overline{K}$. Therefore $\overline{h}$ is a supersolution of (3.28).

Step 3 : constructing a solution u_s on a half line
Notice that

$$\overline{h}(0) = \underline{h}(0) = \theta \quad \text{and} \quad \overline{h}(z) > \underline{h}(z) \;\; \text{for all} \;\; z > 0, \tag{3.35}$$

but the supersolution $\overline{h}$ is not above the subsolution $\underline{h}$ on $\mathbb{R}$.

Step 3.1 : shifting $\underline{h}$
Define, for $s > 0$, the function

$$\underline{h}_s(z) = \underline{h}(z - s).$$

Using the definitions of $\overline{h}$, $\underline{h}$ and the properties of h_0, we can easily deduce that if $\overline{h}$ is fixed, then there exists a unique $k_s < 0$ such that

$$\overline{h}(k_s) = \underline{h}_s(k_s) \quad \text{and} \quad \overline{h} > \underline{h}_s \quad \text{over} \quad (k_s, +\infty)\,. \tag{3.36}$$

Indeed, solving the equality

$$h_0(\overline{a}z) = h_0(\underline{a}(z - s)),$$

we get the explicit expression

$$k_s = \frac{-\mu s}{1-\mu} \quad \text{with} \quad 0 < \mu = \frac{\underline{a}}{\overline{a}} < 1. \tag{3.37}$$

Step 3.2 : building a monotone solution for (3.28) on $(r^* + k_s, +\infty)$
Since $\overline{h}$ and $\underline{h}_s$ are respectively a super and a subsolution of (3.28) such that

$$\overline{h} > \underline{h}_s \quad \text{over} \quad (k_s, +\infty)\,,$$

then using Perron's method (Proposition 2.3), there exists a solution u_s of (3.28) on $(r^* + k_s, +\infty)$ such that $\underline{h}_s \le u_s \le \overline{h}$ on $(k_s, +\infty)$.

Step 3.3 : u_s is non-decreasing on $(k_s, +\infty)$.
Define for $x \in (k_s, +\infty)$ the function

$$\overline{u}(x) := \inf_{p \ge 0} u_s(x+p).$$

Clearly, since $\underline{h}_s(x) \le \underline{h}_s(x+p) \le u_s(x+p)$ for all $p \ge 0$ and $x \in (k_s, +\infty)$, we get $\underline{h}_s(x) \le \overline{u}(x) \le u_s(x) \le \overline{h}(x)$ for all $x \in (k_s, +\infty)$. On the other hand, for all $p \ge 0$, $u_s(x+p)$ is a solution of (3.28) over $(k_s + r^*, +\infty)$, then $(\overline{u})_*$ is supersolution of (3.28) over $(k_s + r^*, +\infty)$ (using Proposition 2.2 (ii)). But u_s is defined as the infimum of supersolutions (recall Proposition 2.3 for Perron's method), thus $u_s \le (\overline{u})_* \le \overline{u} \le u_s$ over $(k_s, +\infty)$. Therefore, for every $p \ge 0$,

$$u_s(x) = \overline{u}(x) \le u_s(x+p) \quad \text{over} \quad (k_s, +\infty),$$

and hence u_s is non-decreasing over $(k_s, +\infty)$.

Step 4 : passing to the limit $s \to +\infty$

Step 4.1 : setting
Let u_s be the solution of (3.28) over $(r^* + k_s, +\infty)$ constructed in Step 3. From equation (3.28) we deduce in particular that u_s is Lipschitz on $(r^* + k_s, +\infty)$ with

$$|u_s'| \le K_0 \quad \text{for a constant } K_0 \text{ independent of } s.$$

Now, for s large enough, we have $r^* + k_s < 0$. Hence, since $u_s(0) \le \overline{h}(0) = \theta$ and $u_s \ge \underline{h}_s$ on $(k_s, +\infty)$ with $\underline{h}_s(+\infty) = 1$, then there exists $z_s \ge 0$ such that $u_s(z_s) = \theta$.

Next, let $d_s = z_s - (k_s + r^*)$. We have from (3.37) that

$$\lim_{s \to +\infty} d_s = +\infty. \tag{3.38}$$

Step 4.2 : global non-decreasing solution of (3.28)
Let $v_s(x) := u_s(x + z_s)$ which is a solution of (3.28) on $(-d_s, +\infty)$ with $d_s \to +\infty$ as $s \to +\infty$. We have, $v_s(0) = \theta$ and

$$|v_s'| \le K_0 \quad \text{on } (-d_s + \infty).$$

Thus passing to the limit $s \to +\infty$, v_s converges (using Ascoli's Theorem) to some non-decreasing v solution of

$$\begin{cases} v'(z) = F\left(\left(v\left(z + \frac{r_i}{c}\right)\right)_{i=0,\dots,N}\right) \\ 0 \le v' \le K_0 \quad \text{on} \quad \mathbb{R} \\ 0 \le v \le 1 \quad \text{and} \quad v(0) = \theta. \end{cases} \tag{3.39}$$

Let $a = v(-\infty)$ or $v(+\infty)$. Then it is easy to see that $0 = f(a)$ which implies that

$$v(-\infty) = 0 \quad \text{and} \quad v(+\infty) = 1.$$

Therefore v is a solution of

$$\begin{cases} v'(z) = F\left(\left(v\left(z + \frac{r_i}{c}\right)\right)_{i=0,\dots,N}\right) & \text{on } \mathbb{R} \\ v \text{ is non-decreasing} \\ v(-\infty) = 0 \quad \text{and} \quad v(+\infty) = 1, \end{cases} \tag{3.40}$$

and this ends the proof. □

Lemma 3.2. (Vertical branches for $\sigma = \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Assume that $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$), then for $c >> 1$ (resp. $c << -1$), there exists a traveling wave solution of (3.20) *(resp.* (3.21)*).*

Proof of Lemma 3.2
Proving the existence of solution for $c >> 1$ when $\sigma = \sigma^+$ follows exactly from Proposition 3.1 where $\sigma^+ = 0$. However, the proof of the result for $c << -1$ when $\sigma = \sigma^-$ follows from the proof of the case $\sigma = \sigma^+$ and the transformation lemma below (Lemma 3.3). □

Lemma 3.3. (Transformation of solutions)
Let ϕ be a solution of

$$c\phi'(z) = F((\phi(z + r_i))_{i=0,\dots,N}) + \sigma^- \quad \textit{over} \quad \mathbb{R}, \tag{3.41}$$

then

$$\overline{\phi}(z) = \theta - \phi(-z)$$

is a solution of (3.41) with F, c, r_i and σ^- replaced respectively by

$$\begin{cases} \overline{F}(X_0, \dots, X_N) = -F((\theta - X_i)_{i=0,\dots,N}) \\ \overline{c} = -c, \quad \overline{r}_i = -r_i \quad \textit{and} \quad \overline{\sigma}^+ = -\sigma^- \end{cases} \tag{3.42}$$

Moreover, if F satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$ then $\overline{F}$ satisfies $(\tilde{A}_{C^1})$ and $(\tilde{B})$ (with $\overline{f}(v) = \overline{F}(v, \dots, v)$).

Proof of Lemma 3.3
The proof of Lemma 3.3 is straightforward. □

4 Existence of traveling waves for $\sigma \in (\sigma^-, \sigma^+)$

In this section we prove, for every $\sigma \in (\sigma^-, \sigma^+)$, the existence of a unique velocity $c = c(\sigma)$ and the existence of a traveling wave $\phi = \phi_\sigma$ solution of (3.19).

The main result of this section is :

Proposition 4.1. (Existence and uniqueness of $c = c(\sigma)$ for $\sigma \in (\sigma^-, \sigma^+)$)
Assume that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Then there exists a unique real $c(\sigma)$ (simply denoted by c_σ) such that there exists a function $\phi_\sigma : \mathbb{R} \to \mathbb{R}$ solution of (3.19) for $c = c_\sigma$ (in the viscosity sense). Moreover, this solution satisfies the following property : there is no $a > r^*$ (r^* is given in (3.23)) and $x \in \mathbb{R}$ such that

$$\phi_\sigma = b_\sigma \quad on \quad [x - a, x + a], \tag{3.43}$$

where b_σ, m_σ are defined in Theorem 1.7.

In order to prove Proposition 4.1, we introduce the following lemma :

Lemma 4.2. (Continuity and monotonicity of m_σ, b_σ over $[\sigma^-, \sigma^+]$)
Under the assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$, the two maps

$$\begin{array}{ccc} [\sigma^-, \sigma^+] & \to & [\theta - 1, 0] \\ \sigma & \mapsto & m_\sigma \end{array} \quad \text{and} \quad \begin{array}{ccc} [\sigma^-, \sigma^+] & \to & [0, \theta] \\ \sigma & \mapsto & b_\sigma, \end{array}$$

are continuous. Moreover, the map m_σ is increasing in σ, while b_σ is decreasing.

The proof of Lemma 4.2 is straightforward from the definition of $\sigma^\pm$ and from assumption $(\tilde{B}_{C^1})$.

Proof of Proposition 4.1
Let $\sigma \in (\sigma^-, \sigma^+)$. Let $m_\sigma \in (\theta - 1, 0)$ and $b_\sigma \in (0, \theta)$ (since $\sigma \neq \sigma^\pm$) be the solutions of $f(s) + \sigma = 0$. Because of assumption $(\tilde{B}_{C^1})$, the function $(f + \sigma)_{|[m_\sigma, m_\sigma+1]}$ is of bistable type, that is $f + \sigma$ satisfies

$$(\tilde{\boldsymbol{B}}'_{\boldsymbol{C^1}}) \left| \begin{array}{l} f(v) + \sigma = 0 \ \text{ for } \ v = m_\sigma, \ b_\sigma \ \text{ and } \ m_\sigma + 1 \\ (f + \sigma)_{|(m_\sigma, b_\sigma)} < 0, \quad (f + \sigma)_{|(b_\sigma, m_\sigma+1)} > 0 \ \text{ and } \ f'(b_\sigma) > 0. \end{array} \right.$$

Step 1 : existence of a traveling wave
Under assumption $(\tilde{A}_{C^1})$ and property $(\tilde{B}'_{C^1})$, the existence of a traveling wave ϕ_σ and a velocity c_σ solution of (3.19) follows from Proposition 2.3 in Chapter 2.

Step 2 : uniqueness of the velocity c_σ under (M)
Assume that F is decreasing close to $\{m_\sigma\}^{N+1}$ and $\{m_\sigma + 1\}^{N+1}$ in the direction

$E = (1, ..., 1)$. That is, there exists $\varepsilon > 0$ small such that F satisfies :

$$(\boldsymbol{M}) \left| \begin{array}{l} F(X + (a, ..., a)) < F(X) \text{ for all } a > 0 \text{ such that } X,\ X + (a, ..., a) \in [m_\sigma, m_\sigma + \varepsilon]^{N+1} \\ F(X + (a, ..., a)) < F(X) \text{ for all } a > 0 \text{ such that } X,\ X + (a, ..., a) \in [m_\sigma + 1 - \varepsilon, m_\sigma + 1]^{N+1}. \end{array} \right.$$

Then under assumptions $(\tilde{A}_{C^1})$ and (M), the velocity c_σ is unique, (as a consequence of Theorem 1.5 (a) in Chapter 2.

Step 3 : checking that F satisfies (M)
Since F is C^1 over a neighborhood of $\mathbb{R}E \backslash (\mathbb{Z}E \cup \mathbb{Z}\Theta)$, then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if $X,\ X + (a, ..., a) \in [m_\sigma, m_\sigma + \varepsilon]^{N+1}$, then

$$|\nabla F(X + t(a, ..., a)) - \nabla F(m_\sigma, ..., m_\sigma)| \leq \delta \tag{3.44}$$

for all $t \in [0, 1]$. Hence using (3.44), we get

$$\begin{aligned} F(X + (a, ..., a)) - F(X) - f'(m_\sigma)a &= \left(\int_0^1 dt \sum_{i=0}^{N} \left(\frac{\partial F}{\partial X_i}(X + t(a, ..., a)) - \frac{\partial F}{\partial X_i}(m_\sigma, ..., m_\sigma) \right) \right) a \\ &\leq (N + 1)a\delta. \end{aligned}$$

Now, since $f'(m_\sigma) < 0$, we deduce that

$$F(X + (a, ..., a)) - F(X) \leq (f'(m_\sigma) + (N + 1)\delta)a < 0$$

for $\delta > 0$ small enough. Similarly, we show that F is decreasing close to $\{m_\sigma + 1\}^{N+1}$.

Step 4 : verification of (3.43)
Assume that there exists $a > r^*$ and $x_0 \in \mathbb{R}$ such that

$$\phi_\sigma = b_\sigma \quad \text{on} \quad [x_0 - a, x_0 + a]. \tag{3.45}$$

Then proceeding as in Steps 4 and 5 in the proof of Proposition 2.3 in Chapter 2 (without any change in Steps 4.1, 4.2, 4.3 and Step 5), we get a contradiction. Indeed in the proof of Proposition 2.3, we were assuming that ϕ_σ is constant on a half line, but condition (3.45) is sufficient to conclude. □

In fact, we can prove the existence of traveling waves for the bistable non-linearity under weaker assumptions. In the following proposition, we prove a general existence result for traveling waves which we will use later in the proof of Theorem 1.1, Section 7.2, Step 1. This proposition will be used also to prove that $c^+ \geq 0$ (proof of Proposition 1.3), Section 8.1.

Assumption $(B_{\mathbf{Lip}})$ **:**
Let $f(v) := F(v, ..., v)$ and assume

Instability : $f(0) = 0 = f(1)$ and there exists $b \in (0,1)$ such that $f(b) = 0$, $f_{|(0,b)} < 0$ and $f_{|(b,1)} > 0$.

Strict monotonicity : There exists some $\eta > 0$ such that

$$F(X + (\omega, ..., \omega)) - F(X) \geq \eta\, \omega$$

for $\omega > 0$ small enough and for all X close enough to $(b, ..., b)$.

Proposition 4.3. (Existence of c for a Lipschitz bistable non-linearity)
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) and (B_{Lip}). Then there exist a real c and a function ϕ solution of (3.9) in the classical sense if $c \neq 0$ and almost everywhere if $c = 0$. Moreover, there is no $a > r^*$ and $x \in \mathbb{R}$ such that

$$\phi = b \quad on \quad [x - a, x + a]. \tag{3.46}$$

This result is the analogue of the existence result of Proposition 2.3 in Chapter 2 assuming that F is less regular near the instability b which is replaced by the strict monotonicity of F near b.

Proof of Proposition 4.3
As it is written above, the proof of Proposition 4.3 is a variant of the proof of Proposition 2.3 in Chapter 2. However, in this case, we obtain the contradiction using the strict monotonicity (Step 4.3) while the rest of the proof (Step 0 to Step 4.2 and Step 5) stays the same.
We now prove the contradiction using the strict monotonicity, revisiting Step 4.3 of the proof of Proposition 2.3 in Chapter 2.

Step 4.3 **: getting a contradiction**
We recall that we consider an approximation ϕ_p of the profile ϕ, for some parameter p going to zero, which satisfies

$$c_p \phi_p'(z) = F((\phi_p(x + r_i))_{i=0,...,N}).$$

We construct (see Proposition 2.3, Chapter 2) a local minimum x_p^* of ψ_p satisfying

$$0 < m_p = \psi_p(x_p^*),$$

where $\psi_p(x) = (\phi_p)_*(x + a) - (\phi_p)^*(x - a)$. Then it is possible to see as in Step 4.3 in the Proof of Proposition 2.3 in Chapter 2, that

$$0 \geq F((a_i)_{i=0,...,N}) - F((c_i)_{i=0,...,N}),$$

where

$$a_i = \begin{cases} k_i & \text{if} \quad r_i \le 0 \\ k_i + m_p & \text{if} \quad r_i > 0 \end{cases} \quad \text{and} \quad c_i = \begin{cases} k_i - m_p & \text{if} \quad r_i \le 0 \\ k_i & \text{if} \quad r_i > 0, \end{cases}$$

and

$$k_i = \begin{cases} (\phi_p)_* \left(x_p^* + a + r_i\right) & \text{if} \quad r_i \le 0 \\ (\phi_p)^* \left(x_p^* - a + r_i\right) & \text{if} \quad r_i > 0. \end{cases}$$

Here, the notation c_i is not ambiguous and has nothing to do with the velocity c_p. Since $a_i = c_i + m_p$ for every $i \in \{0, ..., N\}$, then

$$0 \ge F((c_i + m_p)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}).$$

Now, since $0 < m_p \to 0$ and $k_i \to b$ for all i (see Steps 4.1, 4.2 and 5 in the proof of Proposition 2.3, Chapter 2), then

$$c_i \to b \quad \text{for all } i \in \{0, ..., N\}.$$

Therefore, for p small enough, we have c_i close to b and $m_p > 0$ is small enough, thus using the strict monotonicity in (B_{Lip}), we deduce that

$$0 \ge F((c_i + m_p)_{i=0,\dots,N}) - F((c_i)_{i=0,\dots,N}) \ge \eta m_p > 0,$$

which is a contradiction.
Note that, the proof of (3.46) follows exactly as the proof of (3.43). □

5 Properties of the velocity

We split this section into two subsections. We dedicate a first subsection to the proof of monotonicity and continuity of the velocity function $c(\sigma)$ over (σ^-, σ^+). In a second subsection, we prove that the velocity function attains finite limits $c^\pm$ as σ goes to $\sigma^\pm$ respectively. We also prove the existence of traveling waves solutions of (3.20) (resp. (3.21)) for $c = c^+$ (resp. $c = c^-$).

5.1 Monotonicity and continuity of the velocity

This subsection consists in two results. The monotonicity (Proposition 5.1 and Lemma 5.4) and the continuity (Proposition 5.3) of the velocity function on (σ^-, σ^+).

We start with the following result.

Proposition 5.1. (Monotonicity of the velocity)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ *and let* $\sigma_1,\ \sigma_2 \in [\sigma^-, \sigma^+]$ *such that* $\sigma_1 < \sigma_2$. *Let* $i = 1, 2$ *and associate for each* $\sigma = \sigma_i$ *a solution* (c_i, ϕ_i) *of* (3.19). *Then*

$$c_1 \leq c_2.$$

Proof of Proposition 5.1
We argue by contradiction. Let $\sigma_1,\ \sigma_2 \in [\sigma^-, \sigma^+]$ such that $\sigma_1 < \sigma_2$. To simplify, we denote m_{σ_i} by m_i. Let

$$\phi_1(-\infty) = m_1 < m_2 = \phi_2(-\infty) < \phi_1(+\infty) = m_1 + 1 < m_2 + 1 = \phi_2(+\infty).$$

Assume to the contrary that $c_2 < c_1$. Let $a \in \mathbb{R}$ and define $\phi_2^a(x) = \phi_2(x+a)$. Hence, for $a \geq 0$ large enough fixed, we get

$$\phi_2^a \geq \phi_1 \quad \text{over} \quad \mathbb{R}.$$

Next, set

$$\left| \begin{array}{l} u_1(t,x) = \phi_1(x + c_1 t) \\ u_2(t,x) = \phi_2^a(x + c_2 t), \end{array} \right.$$

then for $j = 1, 2$, we have

$$\partial_t u_j(t,x) = F((u_j(t, x + r_i))_{i=0,\ldots,N}) + \sigma_j. \tag{3.47}$$

Moreover, at time $t = 0$, we have

$$u_2(0,x) = \phi_2^a(x) \geq \phi_1(x) = u_1(0,x) \quad \text{over} \quad \mathbb{R}.$$

Thus applying the comparison principle for equation (3.47) (see [53, Propositions 2.5 and 2.6]), we get

$$u_2(t,x) \geq u_1(t,x) \quad \text{for all} \ \ (t,x) \in [0, +\infty) \times \mathbb{R}.$$

Taking $x = y - c_2 t$, we get

$$\phi_2^a(y) \geq \phi_1(y + (c_1 - c_2)t) \quad \text{for all } t \geq 0 \text{ and } y \in \mathbb{R}.$$

Using that $c_1 > c_2$ and passing to the limit $t \to \infty$, we get

$$\phi_2^a(y) \geq \phi_1(+\infty) = m_1 + 1 \quad \text{for all } y \in \mathbb{R}.$$

But $\phi_2^a(-\infty) = m_2 < m_1 + 1$, hence a contradiction. Therefore $c_1 \leq c_2$. □

Then we have the straightforward consequence of Proposition 5.1.

Corollary 5.2. (Monotonicity and limits of $c(\sigma)$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$. For $\sigma \in (\sigma^-, \sigma^+)$, let $(c(\sigma), \phi_\sigma)$ be a solution of (3.19) given in Proposition 4.1. Then the velocity function is non-decreasing on (σ^-, σ^+). Moreover, the limits

$$\lim_{\sigma \to \sigma^-} c(\sigma) = c^- \quad \text{and} \quad \lim_{\sigma \to \sigma^+} c(\sigma) = c^+$$

exist and satisfy $-\infty \le c^- \le c^+ \le +\infty$.

Proposition 5.3. (Continuity of the velocity function)
Suppose that F satisfies $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma \in (\sigma^-, \sigma^+)$. Let $(c(\sigma), \phi_\sigma)$ be a solution of (3.19) given in Proposition 4.1. Then the map $\sigma \mapsto c(\sigma)$ is continuous on (σ^-, σ^+).

Proof of Proposition 5.3
Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and $c_0 := c(\sigma_0)$ be the associated velocity given in Proposition 4.1. Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma_0$ and let $c_n = c(\sigma_n)$. We want to show that $c_n \to c_0$. Assume that ϕ_0 and ϕ_n (for each n) are solutions of (3.19) associated respectively to σ_0 and σ_n (for each n).

Step 1 : passing to the limit $n \to +\infty$
As a consequence of the monotonicity of $c(\sigma)$ (Proposition 5.1) and the fact that $\sigma_0,\ \sigma_n \in (\sigma^-, \sigma^+)$ for all n, we get that c_n is bounded. Thus, up to a subsequence, we set $\overline{c} = \lim\limits_{n \to +\infty} c_n$.

Recall that (c_n, ϕ_n) solves

$$c_n \phi_n'(z) = F((\phi_n(z + r_i))_{i=0,\dots,N}) + \sigma_n$$

and $\theta - 1 < m_{\sigma_n} \le \phi_n \le m_{\sigma_n} + 1 < 1$.

Therefore, passing to the limit $n \to +\infty$ (see Lemma 9.1), ϕ_n converges to a function ϕ almost everywhere, and ϕ solves (in the viscosity sense)

$$\overline{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma_0. \tag{3.48}$$

Moreover, Theorem 9.3 implies that $(\overline{c}, \phi)$ solves (3.19) for $\sigma = \sigma_0$.

Step 2 : conclusion
From the uniqueness of the velocity on (σ^-, σ^+) (Proposition 4.1) and the fact that c_0 and $\overline{c}$ are associated to $\sigma_0 \in (\sigma^-, \sigma^+)$, we deduce that $\overline{c} = c_0$. From the uniqueness of the limit $\overline{c}$ (whatever is the subsequence $\sigma_n \to \sigma_0$), we deduce the continuity of the velocity function c. $\square$

Lemma 5.4. (Strict monotonicity)
Under the assumptions of Proposition 5.1, there exists a constant $K > 0$ such that $c(\sigma)$ satisfies

$$\frac{dc}{d\sigma} \geq K|c| \quad on \ \ (\sigma^-, \sigma^+) \tag{3.49}$$

in the viscosity sense.

Proof of Lemma 5.4
Clearly, if $c = 0$ then (3.49) holds true.
Let $\sigma_1, \ \sigma_2 \in (\sigma^-, \sigma^+)$ with $\sigma_1 < \sigma_2$ and, as in the proof of Proposition 5.1, let us call $c_1 \leq c_2$ the associated velocities and $\phi_1, \ \phi_2$ the corresponding profiles with $\phi_i(-\infty) = m_{\sigma_i}$ for $i = 1, \ 2$ and $m_{\sigma_1} < m_{\sigma_2}$. Recall also that $(c, \phi) = (c_i, \phi_i)$ solves for $\sigma = \sigma_i$ and $i = 1, \ 2$

$$c\phi' = F((\phi(x + r_i))_{i=0,\ldots,N}) + \sigma. \tag{3.50}$$

Suppose that $c_1 > 0$. Since $F \in \mathrm{Lip}(\mathbb{R}^{N+1})$ and ϕ_1 is bounded, then there exists some $C > 0$ such that

$$|F((\phi_1(x + r_i))_{i=0,\ldots,N})| \leq C.$$

Therefore

$$0 \leq \phi_1' \leq c_1^{-1}(|\sigma_1| + C).$$

Hence for $\delta = c_1(|\sigma_1| + C)^{-1}$, we have (using (3.50))

$$(c_1 + \delta(\sigma_2 - \sigma_1))\phi_1' \leq \sigma_2 + F((\phi_1(x + r_i))_{i=0,\ldots,N}).$$

But, this means that $(\overline{c}, \phi_1)$, with $\overline{c} = c_1 + \delta(\sigma_2 - \sigma_1)$, is a subsolution of (3.50) with $\sigma = \sigma_2$. Comparing $\phi_1(x + \overline{c}t)$ to $\phi_2(x + c_2 t)$ as in Proposition 5.1, we deduce that $\overline{c} \leq c_2$, that is,

$$\frac{c_2 - c_1}{\sigma_2 - \sigma_1} \geq c_1(|\sigma_1| + C)^{-1} =: Kc_1 \quad (\sigma_1 \in (\sigma^-, \sigma^+) \text{ bounded}). \tag{3.51}$$

Now letting $\sigma_1 \to \sigma_2$, and using the continuity of $c(\sigma)$, inequality (3.49) follows (in the sense of viscosity) in case $c > 0$. Similarly, we prove that $c(\sigma)$ verifies (3.49) for $c < 0$. □

5.2 Finite threshold velocities ($c^+ < +\infty$ and $c^- > -\infty$)

In this subsection, we show that $c^+ < +\infty$ (resp. $c^- > -\infty$) and we prove the existence of a solution for $c = c^+$ (resp. $c = c^-$) of (3.20) (resp. (3.21)).

In order to prove that $c^+ < +\infty$ and $c^- > -\infty$, we need to start with the following useful lemma.

Lemma 5.5. (Bound on the velocity for $\sigma \in (\sigma^-, \sigma^+)$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Then there exists $0 < c_*^+ < +\infty$ (resp. $-\infty < c_*^- < 0$) such that the followings holds. Let $\sigma_0 \in (\sigma^-, \sigma^+)$ and c_{σ_0} be such that (c_{σ_0}, ϕ_0) is a solution of (3.19) with $\sigma = \sigma_0$. Then

$$-\infty < c_*^- \leq c_{\sigma_0} \leq c_*^+ < +\infty.$$

Proof of Lemma 5.5
Notice that from Proposition 3.1, there exists $0 < c_*^+ < +\infty$ (resp. $-\infty < c_*^- < 0$) such that for all $c_1 > c_*^+$ (resp. $c_2 < c_*^-$) there exists (c_1, ϕ_1) (resp. (c_2, ϕ_2)) solution of (3.19) for $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$).

We prove that $c_{\sigma_0} \leq c_*^+$ (the case $c_*^- \leq c_{\sigma_0}$ being similar). Assume to the contrary that $c_{\sigma_0} = c_1 > c_*^+$. Suppose that (c_1, ϕ_1) be a solution of (3.19) for $\sigma = \sigma^+$.

Let $\overline{\sigma}$ be such that

$$\sigma^- < \sigma_0 < \overline{\sigma} < \sigma^+ \tag{3.52}$$

and associate a solution $(\overline{c}, \overline{\phi})$ of (3.19) for $\sigma = \overline{\sigma}$. Using Proposition 5.1, we get that

$$\overline{c} \leq c_1 = c_{\sigma_0}.$$

Moreover, using (3.52) and the fact that $c_{\sigma_0} = c_1 > c_*^+ > 0$, we deduce from Lemma 5.4 that

$$c_1 = c_{\sigma_0} < \overline{c},$$

which is a contradiction. □

Then we have the straightforward result :

Corollary 5.6. (Finite limits of c as $\sigma \to \sigma^{\pm}$)
Consider a function F satisfying $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let c^-, c^+ given by Corollary 5.2 and c_*^+, c_*^- defined in Lemma 5.5. Then

$$-\infty < c_*^- \leq c^- \leq c^+ \leq c_*^+ < +\infty.$$

Lemma 5.7. (Existence of a solution of (3.20) for $c = c^{\pm}$)
Assume $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$). There exists a profile ϕ^+ (resp. ϕ^-) such that (c^+, ϕ^+) (resp. (c^-, ϕ^-)) solves (3.20) (resp. (3.21)).

Proof of Lemma 5.7

Step 0 : preliminary
Assume that $\sigma = \sigma^+$ and let us prove the existence of a solution of (3.20) for c^+ (proving the existence of solution of (3.20) for c^- in the case $\sigma = \sigma^-$ is treated similarly). The goal is to get a solution as a limit of the profiles as $\sigma \to \sigma^+$, recalling that $c^+ = \lim_{\sigma \to \sigma^+} c(\sigma)$.

Consider $\sigma \in (\sigma^-, \sigma^+)$ and let (c_σ, ϕ_σ) be a solution of (3.19), namely

$$\begin{cases} c_\sigma \phi'_\sigma(z) = F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma \quad \text{on} \quad \mathbb{R}. \\ \phi_\sigma \text{ is non-decreasing over } \mathbb{R} \\ \phi_\sigma(-\infty) = m_\sigma \quad \text{and} \quad \phi_\sigma(+\infty) = m_\sigma + 1. \end{cases} \tag{3.53}$$

As in the proof of Proposition 5.3, there exists some constant $M > 0$ independent on σ such that

$$|F(\phi_\sigma(z+r_0), \phi_\sigma(z+r_1), ..., \phi_\sigma(z+r_N)) + \sigma^+| \leq M \quad \text{for all} \quad \sigma \in (\sigma^-, \sigma^+).$$

Moreover, up to translate ϕ_σ, we can assume that (because $m_\sigma \to 0$ as $\sigma \to \sigma^+$)

$$(\phi_\sigma)_*(0) \leq \frac{1}{2} \leq \phi_\sigma^*(0). \tag{3.54}$$

Step 1 : passing to the limit $\sigma \to \sigma^+$
Applying Lemma 9.1, we deduce that there exists some function $\phi = \phi^+$ which satisfies, in viscosity sense

$$\begin{cases} c^+(\phi)'(z) = F(\phi(z+r_0), \phi(z+r_1), ..., \phi(z+r_N)) + \sigma^+ \quad \text{on} \quad \mathbb{R}. \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ 0 = m_{\sigma^+} \leq \phi \leq m_{\sigma^+} + 1 = 1. \end{cases} \tag{3.55}$$

Step 2 : limits of the profile ϕ
Passing to the limit in (3.54), we get

$$0 \leq \phi(-\infty) \leq \phi_*(0) \leq \frac{1}{2} \leq (\phi)^*(0) \leq \phi(+\infty) = 1.$$

Because $\phi(\pm\infty)$ solves

$$f(\phi(\pm\infty)) + \sigma^+ = 0,$$

the solution has to satisfy

$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

Therefore $\phi = \phi^+$ solves (3.20). □

6 Filling the gaps : traveling waves for $c \geq c^+$ and $c \leq c^-$

We prove, in this section, for each $c \geq c^+$ (resp. $c \leq c^-$) the existence of a solution of (3.20) (resp. (3.21)). We also prove that (3.20) (resp. (3.21)) admits no solution for any $c < c^+$ (resp. $c > c^-$).

Proposition 6.1. (Existence of solution for vertical branches of velocities) *Let F be a given function satisfying assumptions $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $c^+ < +\infty$ and $c^- > -\infty$ be given by Corollary 5.2. Then for every $c > c^+$ (resp. $c < c^-$), there exists a solution ϕ of (3.20) (resp. (3.21)).*

For $c = c^+$ or $c >> 1$ (resp. $c = c^-$ or $c << 1$) we already have the existence of a solution of ϕ of (3.20) (resp. (3.21)). Proposition 6.1 fills the gap for all $c \geq c^+$ (resp. $c \leq c^-$).

In order to prove Proposition 6.1, we will need the following preliminary result that is proved in [53].

Lemma 6.2. (Existence of a hull function ([53, Theorem 1.5 and Theorem 1.6 a1]))
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $\sigma \in \mathbb{R}$. There exists a unique $\lambda(\sigma, p) = \lambda_p(\sigma)$ such that there exists a locally bounded function $h_p : \mathbb{R} \to \mathbb{R}$ satisfying (in the viscosity sense) :

$$\begin{cases} \lambda_p h_p'(z) = F((h_p(z + pr_i))_{i=0,\dots,N}) + \sigma \quad on \quad \mathbb{R} \\ h_p(z+1) = h_p(z) + 1 \\ h_p'(y) \geq 0 \\ |h_p(z+z') - h_p(z) - z'| \leq 1 \quad for\ any\ z,\ z' \in \mathbb{R}. \end{cases} \tag{3.56}$$

Moreover, there exists a constant $K > 0$, independent on p and σ, such that

$$|\lambda_p - \sigma| \leq K(1 + p) \tag{3.57}$$

and the function

$$\begin{aligned} \lambda_p : \mathbb{R} &\to \mathbb{R} \\ \sigma &\to \lambda_p(\sigma) \end{aligned}$$

is continuous with $\lambda_p(\pm\infty) = \pm\infty$.

For the proof of Lemma 6.2, we refer the reader to [53, Theorems 1.5 and 1.6]. However, proving that $\lambda_p(\pm\infty) = \pm\infty$ follows from (3.57).

Corollary 6.3. (Existence of ϕ_p)
Let F be a given function satisfying assumption $(\tilde{A}_{C^1})$, $p > 0$ and $c \in (c^+, +\infty)$ fixed. Then there exists $\sigma = \sigma(c,p) \in \mathbb{R}$ such that there exists a function $\phi_p : \mathbb{R} \to \mathbb{R}$ that satisfies in the viscosity sense :

$$\begin{cases} c\phi_p'(z) = F((\phi_p(z+r_i))_{i=0,\dots,N}) + \sigma(c,p) \quad on \quad \mathbb{R} \\ \phi_p' \text{ non-decreasing} \\ \phi_p\left(z + \dfrac{1}{p}\right) = \phi_p(z) + 1. \end{cases} \tag{3.58}$$

Proof of Corollary 6.3
Let $\sigma = \sigma(c,p)$ such that

$$\lambda_p(\sigma) = cp \tag{3.59}$$

and define the function ϕ_p as :

$$\phi_p(x) = h_p(px), \tag{3.60}$$

where h_p is given by Lemma 6.2. This gives the result. □

Now, we give the proof of Proposition 6.1.

Proof of Proposition 6.1
Choose $\overline{c} > c^+$ and let $\delta_0 > 0$ such that

$$\overline{c} > c^+ + \delta_0.$$

Step 1 : preliminary
Choose $\eta > 0$ small and let $\sigma^+ - \eta \le \sigma_\eta < \sigma^+$. From Proposition 4.1, we know that for σ_η, there exits a solution $(c_{\sigma_\eta}, \phi_{\sigma_\eta})$ of (3.19) such that

$$c_{\sigma_\eta} \le c^+.$$

Moreover, as $c_{\sigma_\eta} = \lim_{p\to 0} c(\sigma_\eta, p)$ with $c(\sigma_\eta, p) = \dfrac{\lambda(\sigma_\eta, p)}{p}$ (see the proof of existence of Proposition 2.3, Chapter 2), then there exists p_η such that for all $0 < p \le p_\eta$, we have

$$|c(\sigma_\eta, p) - c_{\sigma_\eta}| \le \delta_0. \tag{3.61}$$

Thus, for $0 < p \le p_\eta$, we get

$$c(\sigma_\eta, p) \le c_{\sigma_\eta} + \delta_0 \le c^+ + \delta_0 < \overline{c}. \tag{3.62}$$

Moreover, since the map $\sigma \mapsto \lambda(\sigma, p) = c(\sigma, p)p$ is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2), then for such $0 < p \le p_\eta$, there exists $\overline{\sigma}_p \in \mathbb{R}$ and a function $\phi_p : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c(\overline{\sigma}_p, p) = \overline{c}$$

and $(\overline{c}, \phi_p)$ solves (3.58). Hence from (3.62), we get

$$c(\sigma_\eta, p) < c(\overline{\sigma}_p, p).$$

In addition, since $\lambda(\sigma, p)$ is non-decreasing with respect to σ, then

$$\overline{\sigma}_p > \sigma_\eta \geq \sigma^+ - \eta \quad \text{for } 0 < p \leq p_\eta. \tag{3.63}$$

Step 2 : passing to the limit $p \to 0$
Since $\lim_{p\to 0} \lambda(\overline{\sigma}_p, p) = \lim_{p\to 0} \overline{c}p = 0$, we deduce from (3.57) that there exists some $L_0 > 0$ independent of p such that

$$|\overline{\sigma}_p| \leq L_0 \quad \text{for } 0 < p \leq p_\eta. \tag{3.64}$$

Thus

$$\overline{\sigma}_p \to \overline{\sigma}_0 \ \text{ as } \ p \to 0 \quad \text{(up to a subsequence).}$$

Recall that ϕ_p is non-decreasing and that

$$\phi_p\left(x + \frac{1}{2p}\right) - \phi_p\left(x + \frac{-1}{2p}\right) = 1.$$

We can also assume that

$$\begin{cases} (\phi_p)^*(0) \geq \dfrac{1}{2} \\ (\phi_p)_*(0) \leq \dfrac{1}{2}. \end{cases}$$

Therefore, since $F \in \text{Lip}(\mathbb{R}^{N+1})$ and due to (3.64), we deduce (as in the proof of Lemma 2.8, Chapter 2) that there exists some $M > 0$ independent on n such that

$$|F((\phi_p(x + r_i)_{i=0,\dots,N})) + \overline{\sigma}_n| \leq M.$$

Applying arguments similar to the ones of the proof of Lemma 9.1, we see that ϕ_p converges to some ϕ almost everywhere and ϕ is a viscosity solution of

$$\begin{cases} \overline{c}\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \overline{\sigma}_0 \\ \phi \text{ non-decreasing and bounded} \\ \phi(+\infty) - \phi(-\infty) \leq 1, \end{cases} \tag{3.65}$$

and ϕ satisfies

$$\begin{cases} \phi^*(0) \geq \dfrac{1}{2} \\ \phi_*(0) \leq \dfrac{1}{2}. \end{cases}$$

In addition, we have

$$\overline{\sigma}_0 \geq \sigma^+ - \eta \quad \text{(because of (3.63))}.$$

But $\eta > 0$ is arbitrary, hence

$$\overline{\sigma}_0 \geq \sigma^+.$$

Moreover, since $\overline{\sigma}_0 \leq \sigma^+$ (otherwise, (3.65) admits no solution, see Remark 1.10), thus

$$\overline{\sigma}_0 = \sigma^+.$$

Finally, since $\phi(\pm\infty)$ solves $f + \sigma^+ = 0$, then we conclude that

$$\phi(-\infty) = m_{\sigma^+} = 0 \quad \text{and} \quad \phi(+\infty) = 1,$$

which ends the proof. □

Lemma 6.4. (Non-existence of solution for $c < c^+$ and $c > c^-$)
Consider a function F and assume $(\tilde{A}_{C^1})$ and $(\tilde{B}_{C^1})$. Let $\sigma = \sigma^+$ (resp. $\sigma = \sigma^-$) and $c^+ < +\infty$ (resp. $c^- > -\infty$) be given by Corollary 5.2. Let (c, ϕ) be a solution of (3.20) (resp. (3.21)), then $c \geq c^+$ (resp. $c \leq c^-$).

Proof of Lemma 6.4
Let $\sigma = \sigma^+$ and (c, ϕ) be a solution of (3.20). We want to prove that $c \geq c^+$ (similarly, we show that there is no solution of (3.21) for $c > c^-$ when $\sigma = \sigma^-$).

It is known from Theorem 1.7-1, that for every $\sigma \in (\sigma^-, \sigma^+)$, there exists $(c(\sigma), \phi_\sigma)$ solution of (3.19). Let $\sigma_n \in (\sigma^-, \sigma^+)$ be a sequence such that $\sigma_n \to \sigma^+$, $c(\sigma_n) \to c^+$ and $(c(\sigma_n), \phi_{\sigma_n})$ is a solution of (3.19). Since $\sigma_n < \sigma^+$, Proposition 5.1 implies that

$$c(\sigma_n) \leq c.$$

Therefore, passing to the limit $\sigma_n \to \sigma^+$, we get that

$$c^+ \leq c,$$

which ends the proof. □

Lemma 6.5. (Strict inequality between threshold velocities)
Consider a function F satisfying $(\tilde{A}_{C^1})$, $(\tilde{B}_{C^1})$ and let c^-, c^+ given by Corollary 5.6. If $c^- \neq 0$ or $c^+ \neq 0$, then

$$c^- < c^+.$$

Proof of Lemma 6.5
This is a straightforward consequence of (3.49).

□

Proof of Theorem 1.7
Theorem 1.7 is proved in several propositions and lemmata. In Propositions 4.1, 6.1, 5.3 and Lemma 5.4, we prove, for $\sigma \in (\sigma^-, \sigma^+)$, the existence of traveling waves and the monotonicity and the continuity of the velocity of propagation respectively. Existence of vertical branches of solutions (when $\sigma = \sigma^\pm$) is proved in Lemma 3.2, where we show the existence of traveling waves for $c >> 1$ and for $c << -1$; and in Corollary 5.6, Lemma 5.7, Proposition 6.1, Lemma 6.4 and Lemma 6.5, where we respectively show the existence of finite critical limits $c^\pm$ of the velocity function when σ goes to $\sigma^\pm$, the existence of solutions for the critical limits of velocity, fill the gap and prove the non-existence of solution when $c < c^+$ and $\sigma = \sigma^+$ or when $c > c^-$ and $\sigma = \sigma^-$, and finally prove the inequality between c^+ and c^-. □

7 Proof of Theorem 1.1

We devote this section to the proof of Theorem 1.1 and we split it into two subsections. We recall in a first subsection an extension result over $\mathbb{R}^{N+1}$. For pedagogical reasons, we also prove in this subsection the result of Theorem 1.1 in a simple case where the non-linearity F is assumed to be smooth (cf. Proposition 7.2). In a second subsection, we give the proof of Theorem 1.1 in full generality for Lipschitz non-linearities F.

To prove the result (in any case), we first show the existence of traveling waves for $c >> 1$ by applying Proposition 3.1. The next step is to define the critical velocity c^+ and then we prove, for all $c \geq c^+$, the existence of traveling wave solutions of system (3.9). Finally, We show the non-existence of solutions of (3.9) for any $c < c^+$.

7.1 Preliminary results

We start this subsection by recalling an extension result of the function F defined on $[0,1]^{N+1}$ into a function $\tilde{F}$ over $\mathbb{R}^{N+1}$. We also prove the result of Theorem 1.1 in a simple case.

Lemma 7.1. (Extension of F)
Consider a function F defined over $[0,1]^{N+1}$ and satisfying (A_{Lip}) such that $F(0,...,0) = F(1,...,1) = 0$. There exists an extension $\tilde{F}$ defined over $\mathbb{R}^{N+1}$ such that

$$\tilde{F}_{|_{[0,1]^{N+1}}} = F$$

and $\tilde{F}$ *satisfies*

Assumption ($\tilde{A}_{\text{Lip}}$) :

Regularity : $\tilde{F}$ *is globally Lipschitz continuous over* $\mathbb{R}^{N+1}$.

Monotonicity : $\tilde{F}(X_0, ..., X_N)$ *is non-decreasing w.r.t. each* X_i *for* $i \neq 0$.

Periodicity : $\tilde{F}(X_0+1, ..., X_N+1) = \tilde{F}(X_0, ..., X_N)$ *for every* $X = (X_0, ..., X_N) \in \mathbb{R}^{N+1}$.

Lemma 7.1 corresponds to Lemma 2.1 in Chapter 2 whose proof is given in the appendix A of Chapter 2.

Notice that the function $\tilde{f}(v) := \tilde{F}(v, ..., v)$ is nothing but a periodic extension of f on $\mathbb{R}$ with period 1, that is

$$\tilde{f}_{|_{[0,1]}} = f,$$

hence $\tilde{f}(0) = \tilde{f}(1) = 0$.

Notice also that ϕ is a solution of

$$\begin{cases} c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \ \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1 \end{cases}$$

if and only if ϕ is a solution of

$$\begin{cases} c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,...,N}) & \text{on } \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \ \mathbb{R} \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{3.66}$$

since $\tilde{F}_{|_{[0,1]^{N+1}}} = F$. In particular $\tilde{F}$ satisfies (P_{Lip}) if F satisfies (P_{Lip}).

In order to prove Theorem 1.1 in a special case when F is smooth (see Proposition 7.2), we need to introduce precise assumptions.

Assumption (A_{C^1}) :

Regularity : $F \in C^1([0,1]^{N+1})$.

Monotonicity : $F(X_0, ..., X_N)$ is non-decreasing w.r.t. each X_i for $i \neq 0$.

Assumption (P'_{C^1}) :

Positive degenerate monostability :

Let $f(v) = F(v, ..., v)$ such that $f(0) = 0 = f(1)$ and $f > 0$ in $(0, 1)$.

Smoothness near $\{0\}^{N+1}$ and $\{1\}^{N+1}$:

There exists $\delta > 0$ such that

$$\begin{cases} f' > 0 & \text{on} \quad (0, \delta) \\ f' < 0 & \text{on} \quad (1 - \delta, 1) \end{cases}$$

Proposition 7.2. (Vertical branch, simple case)
Consider a function F satisfying (A_{C^1}) and (P'_{C^1}). Then the result of Theorem 1.1 holds true.

Proof of Proposition 7.2
Note that $\sigma^+ = 0$ in this case.

Using Proposition 3.1, we deduce that for $c >> 1$, there exists a solution of (3.9). Next, from the extension lemma (Lemma 7.1), we see that if F satisfies (A_{C^1}) (which implies (A_{Lip})), then the extended function $\tilde{F}$ satisfies $(\tilde{A}_{C^1})$. Because of assumption (P'_{C^1}) and $\tilde{f}$ is 1-periodic with $\tilde{f} = f$ on $[0,1]$, there exists $\varepsilon_0 > 0$ small enough, such that for $-\varepsilon_0 < \sigma < 0$, $\tilde{f} + \sigma$ has a bistable shape over $(m_\sigma, m_\sigma + 1)$ where m_σ is defined exactly as in Theorem 1.7. Precisely, by bistable shape we mean that there exists m_σ and b_σ solutions of $\tilde{f} + \sigma = 0$ satisfying $-1 < m_\sigma < 0 < b_\sigma < m_\sigma + 1 < 1$ and

$$\begin{cases} \tilde{f} + \sigma < 0 \quad \text{on} \quad (m_\sigma, b_\sigma) \\ \tilde{f} + \sigma > 0 \quad \text{on} \quad (b_\sigma, m_\sigma + 1) \\ \tilde{f}'(b_\sigma) > 0 \;\text{ and }\; \tilde{f}'(m_\sigma) = \tilde{f}'(m_\sigma + 1) < 0. \end{cases}$$

For $\sigma \in (-\varepsilon_0, 0)$, using Proposition 4.1 (which stays true with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$ instead of $\sigma \in (\sigma^-, \sigma^+)$), we show the existence of a unique velocity c_σ such that there exists a profile ϕ_σ solution of system (3.19) with F replaced by $\tilde{F}$. From Propositions 5.1 and 5.3 (which stay true similarly for $(\tilde{B}_{C^1})$ replaced by (P'_{C^1}) and $\sigma \in (-\varepsilon_0, 0)$), we get that the map

$$\sigma \mapsto c_\sigma$$

is monotone continuous on $(-\varepsilon_0, 0)$ and we define as in Corollary 5.2 the critical velocity c^+ as

$$\lim_{\sigma \to 0^-} c_\sigma = c^+.$$

Again, up to replace $(\tilde{B}_{C^1})$ by (P'_{C^1}) and $\sigma \in (\sigma^-, \sigma^+)$ by $\sigma \in (-\varepsilon_0, 0)$, we can use Lemma 5.5, Corollary 5.6 and Lemma 5.7, and show that $c^+ < +\infty$ and that (3.9) admits a solution for $c = c^+$. We use Proposition 6.1 (again with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})) to fill the gap and get the existence of solution (c, ϕ) for each $c \geq c^+$. Finally, the non-existence of solutions for $c < c^+$ follows from Lemma 6.4 (with $(\tilde{B}_{C^1})$ replaced by (P'_{C^1})). □

Remark 7.3. (Another approach to prove the existence of vertical branch)
Instead of using the approach that we call "filling the gap" (using the (δ, σ)-problem and hull functions), it is also possible to propose a different proof that will be presented in a future version of the present work.

7.2 Proof of Theorem 1.1

This subsection is devoted for the proof of Theorem 1.1.

Proof of Theorem 1.1

Let us consider a general function $F : [0,1]^{N+1} \to \mathbb{R}$ and $f(v) = F(v,...,v)$ satisfying (A_{Lip}) and (P_{Lip}). We have to adapt the proof of Proposition 7.2 with a much lower regularity of F (here F is only Lipschitz). To this end, we will introduce an approximation F_δ of F.

Step 0 : a δ-approximation

Define for $X = (X_0,...,X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0),$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ \max\big(f(1-\delta) - L_0(v-(1-\delta)), 0\big) & \text{on} \quad [1-\delta, 1] \\ f & \text{on} \quad [\delta, 1-\delta], \end{cases}$$

with a constant $L_0 > 0$ satisfying $L_0 > 2\text{Lip}(F) =: 2L_F^\infty$. Clearly, we have $F_\delta(v,...,v) = f_\delta(v)$.

Set

$$\begin{cases} b_\delta = \delta - \dfrac{f(\delta)}{L_0} > 0 \\ m_\delta = 1 - \delta + \dfrac{f(1-\delta)}{L_0} < 1 \end{cases} \tag{3.67}$$

which satisfies

$$0 < b_\delta < \delta < 1 - \delta < m_\delta < 1, \tag{3.68}$$

and

$$f_\delta(b_\delta) = 0 = f_\delta(m_\delta) \quad \text{and} \quad f_\delta > 0 \;\; \text{on} \;\; (b_\delta, m_\delta).$$

Let $\tilde{F}$ and $\tilde{F}_\delta$ defined on $\mathbb{R}^{N+1}$ be the extension functions of F and F_δ (which are defined on $[0,1]^{N+1}$) respectively constructed by Lemma 7.1. Define $\tilde{f}_\delta(v) = \tilde{F}_\delta(v,...,v)$ and $\tilde{f}(v) = \tilde{F}(v,...,v)$, then $\tilde{f}_\delta$ and $\tilde{f}$ are 1-periodic with $(\tilde{f}_\delta)_{|[0,1]} = f_\delta$ and $(\tilde{f})_{|[0,1]} = f$. Moreover, since $\tilde{f}_\delta \le \tilde{f}$, we get that

$$\tilde{F}_\delta \le \tilde{F} \quad \text{over} \quad \mathbb{R}^{N+1}. \tag{3.69}$$

Now, for $\sigma < 0$ small fixed ($0 < -\sigma < \min_{[\delta,1-\delta]} f$), define $0 < b_{\delta,\sigma} < m_{\delta,\sigma} < 1$ such that

$$\begin{cases} \big(\tilde{f}_\delta + \sigma\big)(b_{\delta,\sigma}) = 0 = \big(\tilde{f}_\delta + \sigma\big)(m_{\delta,\sigma}) = \big(\tilde{f}_\delta + \sigma\big)(m_{\delta,\sigma} - 1) \\ \tilde{f}_\delta + \sigma < 0 \quad \text{on} \quad (m_{\delta,\sigma} - 1, b_{\delta,\sigma}) \\ \tilde{f}_\delta + \sigma > 0 \quad \text{on} \quad (b_{\delta,\sigma}, m_{\delta,\sigma}). \end{cases} \tag{3.70}$$

Notice that

$$\begin{cases} m_{\delta,\sigma} & \to m_\delta \\ b_{\delta,\sigma} & \to b_\delta \end{cases} \quad \text{as} \quad \sigma \to 0^-.$$

For simplicity, we will denote $\tilde{F}$, $\tilde{F}_\delta$, $\tilde{f}$ and $\tilde{f}_\delta$ by F, F_δ, f and f_δ respectively.

Step 1 : existence of a solution of the approximated non-linearity F_δ
From the definition of f_δ, we see that (for $0 < -\sigma < \min\limits_{[\delta,1-\delta]} f$)

$$b_\delta < b_{\delta,\sigma} < \delta. \tag{3.71}$$

Now, because of (3.71) and using the definition of F_δ with the fact that F is L_F^∞-Lipschitz, then for X close to $\{b_{\delta,\sigma}\}^{N+1}$ and $\omega > 0$ small enough, we get that

$$\begin{aligned} F_\delta(X+(\omega,...,\omega)) - F_\delta(X) &= F(X+(\omega,...,\omega)) - F(X) - f(X_0+\omega) + f(X_0) + f_\delta(X_0+\omega) - f_\delta(X_0) \\ &\geq -2\omega L_F^\infty + \omega L_0 \\ &= \omega(-2L_F^\infty + L_0) = \omega\eta > 0 \quad \text{(because of the condition on } L_0). \end{aligned} \tag{3.72}$$

Since $F_\delta + \sigma$ satisfies (A_{Lip}) and (B_{Lip}) with $[0,1]^{N+1}$ replaced by $[m_{\delta,\sigma}-1, m_{\delta,\sigma}]^{N+1}$ and b replaced by $b_{\delta,\sigma}$ (see (3.70) and (3.72)), then applying the result of Proposition 4.3 (but now on $[m_{\delta,\sigma}-1, m_{\delta,\sigma}]^{N+1}$), we deduce that there exists a solution $\phi_{\delta,\sigma}$ that solves in viscosity sense

$$\begin{cases} c_{\delta,\sigma}\phi'_{\delta,\sigma}(x) = F_\delta((\phi_{\delta,\sigma}(x+r_i))_{i=0,\dots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\sigma} \ \text{ is non-decreasing over } \mathbb{R} \\ \phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma}. \end{cases} \tag{3.73}$$

More precisely, we have used the fact that $F_\delta(\cdot + \{m_{\delta,\sigma}-1\}^{N+1}) + \sigma$ satisfies (A_{Lip}) and (P_{Lip}) on $[0,1]^{N+1}$ with b defined by $b_{\delta,\sigma} = b + m_{\delta,\sigma} - 1$, and Proposition 4.3 provides a profile $\phi : \mathbb{R} \to [0,1]$ such that $\phi + m_{\delta,\sigma} - 1 =: \phi_{\delta,\sigma}$.

Step 2 : $c_{\delta,\sigma}$ is non-decreasing in σ for δ fixed
Here, this is a variant of the proof of Proposition 5.1. Let $\delta > 0$ fixed, $-\min\limits_{[\delta,1-\delta]} f < \sigma_1 < \sigma_2 < 0$ and set $(c_{\delta,\sigma_1}, \phi_{\delta,\sigma_1})$, $(c_{\delta,\sigma_2}, \phi_{\delta,\sigma_2})$ be the associated solutions of (3.73) for σ_1 and σ_2 respectively.

We have

$$m_{\delta,\sigma_1} - 1 < m_{\delta,\sigma_2} - 1 < m_{\delta,\sigma_1} < m_{\delta,\sigma_2};$$

that is $\phi_{\delta,\sigma_1}(\pm\infty) < \phi_{\delta,\sigma_2}(\pm\infty)$, and $\phi_{\delta,\sigma_1}(+\infty) > \phi_{\delta,\sigma_2}(-\infty)$. Thus using the proof of Proposition 5.1, we deduce that $c_{\delta,\sigma_1} \leq c_{\delta,\sigma_2}$.

Step 3 : $c_{\delta,\sigma}$ is non-increasing in δ for σ fixed
For $\delta_2 > \delta_1 > 0$, fix σ such that $-\min\limits_{[\delta_1,1-\delta_1]} f < \sigma < 0$ and associate respectively the

two solutions $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ and $(c_{\delta_1,\sigma}, \phi_{\delta_1,\sigma})$ of (3.73). From the definition of F_δ, $m_{\delta,\sigma}$ and $b_{\delta,\sigma}$ (see Step 0), we see that

$$F_{\delta_2} \leq F_{\delta_1},$$

hence $(c_{\delta_2,\sigma}, \phi_{\delta_2,\sigma})$ is a subsolution of (3.73) for F_δ replaced by F_{δ_1}. Moreover, we also have that

$$m_{\delta_2,\sigma} - 1 < m_{\delta_1,\sigma} - 1 < m_{\delta_2,\sigma} < m_{\delta_1,\sigma},$$

hence $\phi_{\delta_2,\sigma}(\pm\infty) < \phi_{\delta_1,\sigma}(\pm\infty)$ and $\phi_{\delta_2,\sigma}(+\infty) > \phi_{\delta_1,\sigma}(-\infty)$. Using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that $c_{\delta_2,\sigma} \leq c_{\delta_1,\sigma}$.

Step 4 : passing to the limit $\sigma \to 0^- = \sigma^+$
For $\delta > 0$ fixed, let $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ be a solution of (3.73). Since $F_\delta \leq F$ (see Step 0), we deduce that $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a subsolution for (3.73), with F_δ replaced by F.

On the other hand, let us consider any solution ϕ_{c_0} of

$$\begin{cases} c_0\phi'_{c_0}(x) = F((\phi_{c_0}(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_{c_0} \quad \text{is non-decreasing over } \mathbb{R} \\ \phi_{c_0}(-\infty) = 0 \quad \text{and} \quad \phi_{c_0}(+\infty) = 1. \end{cases} \tag{3.74}$$

From Proposition 3.1, we know that such a solution does exist at least for $c_0 >> 1$.

Since $\phi_{\delta,\sigma}$ satisfies

$$\phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1 \quad \text{and} \quad \phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma},$$

then $\phi_{\delta,\sigma}(\pm\infty) < \phi_{c_0}(\pm\infty)$ and $\phi_{\delta,\sigma}(+\infty) > \phi_{c_0}(-\infty)$. Thus using the proof of Proposition 5.1 (which is still true for sub and supersolutions), we deduce that

$$c_{\delta,\sigma} \leq c_0 \quad \text{for all} \quad \sigma \in (-\min_{[\delta,1-\delta]} f, 0).$$

Since the map $\sigma \mapsto c_{\delta,\sigma}$ is non-decreasing, then

$$c_{\delta,\sigma} \to c_\delta^+ \quad \text{as} \quad \sigma \to 0^-.$$

Therefore, passing to the limit $\sigma \to 0^-$, using Lemma 9.1, $\phi_{\delta,\sigma}$ converges almost everywhere to some ϕ_δ that solves in the viscosity sense

$$\begin{cases} c_\delta^+\phi'_\delta(x) = F_\delta((\phi_\delta(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi_\delta \quad \text{is non-decreasing over } \mathbb{R} \\ m_\delta - 1 \leq \phi_\delta(-\infty) \quad \text{and} \quad \phi_\delta(+\infty) \leq m_\delta. \end{cases} \tag{3.75}$$

We can insure that ϕ_δ is non constant, assuming that

$$\begin{cases} (\phi_{\delta,\sigma})^*(0) \geq \dfrac{b_\delta + m_\delta}{2} \\ (\phi_{\delta,\sigma})_*(0) \leq \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

and this implies in addition that

$$\phi_\delta(-\infty) \leq b_\delta \quad \text{and} \quad \phi_\delta(+\infty) = m_\delta.$$

Step 5 : passing to the limit $\delta \to 0^+$
Since $c_{\delta,\sigma} \leq c_0$ for any $\delta > 0$ and $\sigma \in (-\min\limits_{[\delta,1-\delta]} f, 0)$, we get

$$c_\delta^+ \leq c_0 \quad \text{for all} \quad \delta \in \left(0, \frac{1}{2}\right). \tag{3.76}$$

Moreover, since $c_{\delta,\sigma}$ is non-increasing in δ, then c_δ^+ is non-increasing in δ. Hence from (3.76), we get

$$\lim_{\delta \to 0^+} c_\delta^+ = c^+ \leq c_0. \tag{3.77}$$

We can also assume, up to translation, that the solution ϕ_δ of (3.75) satisfies

$$\begin{cases} (\phi_\delta)^*(0) \geq \dfrac{1}{2} \\ (\phi_\delta)_*(0) \leq \dfrac{1}{2}. \end{cases}$$

Thus passing to the limit $\delta \to 0^+$, using again Lemma 9.1, then ϕ_δ converges, up to a subsequence, almost everywhere to some ϕ which solves in viscosity sense

$$\begin{cases} c^+\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi \ \text{ is non-decreasing over } \mathbb{R} \\ 0 \leq \phi(-\infty) \quad \text{and} \quad \phi(+\infty) \leq 1 \end{cases} \tag{3.78}$$

and satisfies

$$\begin{cases} (\phi)^*(0) \geq \dfrac{1}{2} \\ (\phi)_*(0) \leq \dfrac{1}{2}. \end{cases} \tag{3.79}$$

But $\phi(\pm\infty)$ is a solution of $\tilde{f} = 0$, then we get

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1.$$

This implies that if (c_0, ϕ_{c_0}) is a solution of (3.74), then $c_0 \geq c^+$ and moreover there exists such a solution $(c_0, \phi_{c_0}) = (c^+, \phi)$. We also recall that we have solutions of (3.74) for $c_0 >> 1$. Our goal now is to fill the gap and to show that we have solutions for all $c \geq c^+$.

Step 6 : filling the gap

This step is analogous to the proof of Proposition 6.1. Fix $\overline{c} > c^+$ and let $\beta_0 > 0$ such that

$$\overline{c} > c^+ + \beta_0. \tag{3.80}$$

Step 6.1 construction of a solution $(\overline{c}, \overline{\phi})$ associated to some $\overline{\sigma}$

Substep 6.1.1 : $c^+ = \lim\limits_{\delta \to 0^-} c^+_\delta$

We know from Steps 4, 5 that there exists a non trivial solution $(c^+_\delta, \phi_\delta)$ of (3.75) and that $c^+ = \lim\limits_{\delta \to 0^-} c^+_\delta$. Thus there exists some $\delta_0 > 0$ such that

$$|c^+_\delta - c^+| \leq \frac{\beta_0}{3} \quad \text{for all} \quad 0 < \delta \leq \delta_0. \tag{3.81}$$

Substep 6.1.2 : $c^+_\delta = \lim\limits_{\sigma \to 0^-} c_{\delta,\sigma}$

Similarly, we know from Steps 1, 4 that, for every $0 < \delta \leq \delta_0$, there exists a solution $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ of (3.73) and that $c^+_\delta = \lim\limits_{\sigma \to 0^-} c_{\delta,\sigma}$. Thus there exists some $\sigma_\delta > 0$ such that

$$|c_{\delta,\sigma} - c^+_\delta| \leq \frac{\beta_0}{3} \quad \text{for all} \quad 0 < -\sigma \leq \sigma_\delta. \tag{3.82}$$

Substep 6.1.3 : $c_{\delta,\sigma} = \lim\limits_{p \to 0^+} c_{\delta,\sigma,p}$

Based on the proof of Proposition 2.3 in Chapter 2, there exists (for every $0 < \delta \leq \delta_0$ and $0 < -\sigma \leq \sigma_\delta$ such that (3.82) holds true) a velocity $c_{\delta,\sigma,p}$, a profile $\phi_{\delta,\sigma,p}$ and some $p_{\delta,\sigma} > 0$ such that $c_{\delta,\sigma,p}$ converges up to a subsequence to $c_{\delta,\sigma}$ as $p \to 0$ and

$$|c_{\delta,\sigma,p} - c_{\delta,\sigma}| \leq \frac{\beta_0}{3} \quad \text{for all } p \text{ of the subsequence such that} \quad 0 < p \leq p_{\delta,\sigma}, \tag{3.83}$$

where $(c_{\delta,\sigma,p}, \phi_{\delta,\sigma,p})$ is a solution of

$$\begin{cases} c_{\delta,\sigma,p}(\phi_{\delta,\sigma,p})'(x) = F_\delta((\phi_{\delta,\sigma,p}(x + r_i))_{i=0,\ldots,N}) + \sigma \quad \text{on} \quad \mathbb{R} \\ (\phi_{\delta,\sigma,p})' \geq 0 \\ \phi_{\delta,\sigma,p}\left(x + \dfrac{1}{p}\right) = 1 + \phi_{\delta,\sigma,p}(x). \end{cases} \tag{3.84}$$

Substep 6.1.4 : construction of a solution $(\overline{c}, \overline{\phi})$ associated to some $\overline{\sigma}$

Since the map

$$\sigma \mapsto \lambda(\sigma, p) := p c_{\delta,\sigma,p} \tag{3.85}$$

is continuous with $\lambda(\pm\infty, p) = \pm\infty$ (see Lemma 6.2 applied to F_δ instead of F), then for every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$ such that (3.82) and (3.83) hold true, there exists $\overline{\sigma} = \overline{\sigma}_{\delta,p} \in \mathbb{R}$ and a function $\overline{\phi} = \phi_{\delta,\overline{\sigma},p} : \mathbb{R} \to \mathbb{R}$ (see Corollary 6.3) such that

$$c_{\delta,\overline{\sigma},p} = \overline{c}$$

and $(\overline{c}, \overline{\phi})$ solves

$$\begin{cases} \overline{c}\overline{\phi}'(x) = F_\delta((\overline{\phi}(x + r_i))_{i=0,\ldots,N}) + \overline{\sigma} \quad \text{on } \mathbb{R} \\ \overline{\phi}' \geq 0 \\ \overline{\phi}\left(x + \dfrac{1}{p}\right) = 1 + \overline{\phi}(x). \end{cases} \tag{3.86}$$

Substep 6.1.5 : consequence of Substeps 6.1.1-6.1.4
For every $0 < \delta \leq \delta_0$, $0 < -\sigma \leq \sigma_\delta$ and $0 < p \leq p_{\delta,\sigma}$, (3.80), (3.81), (3.82) and (3.83) hold true, thus we get

$$c_{\delta,\sigma,p} \leq c^+ + \beta_0 < \overline{c} = c_{\delta,\overline{\sigma},p}.$$

But the map $\sigma \mapsto c_{\delta,\sigma,p}$ is non-decreasing (see Lemma 6.2 and (3.85)), hence we obtain that

$$\sigma < \overline{\sigma} = \overline{\sigma}_{\delta,p}. \tag{3.87}$$

Step 6.2 : getting a profile for the original problem with velocity $\overline{c}$

Substep 6.2.0 : a priori estimate on $\overline{\sigma}$
The couple $(\overline{c}, \phi_{\delta,\overline{\sigma},p})$ is a solution of (3.86), thus for $p < 1$, we get

$$\phi_{\delta,\overline{\sigma},p}(x + 1) - \phi_{\delta,\overline{\sigma},p}(x) \leq 1;$$

and hence we can show that there exists a constant M_0 independent of p and δ such that

$$|F_\delta| \leq M_0.$$

Thus integrating (3.86) over $[0, 1]$, implies that there exists a constant $K > 0$ such that

$$|\overline{\sigma}| \leq K$$

for all $\delta < \frac{1}{2}$ and $p \leq 1$.

Substep 6.2.1 : passing to the limit $p \to 0$
Since $|f_\delta - f| \leq o_\delta(1)$, then we can assume, up to translation, that

$$\begin{cases} (\phi_{\delta,\overline{\sigma},p})^*(0) \geq \gamma_{\delta,\overline{\sigma}} \\ (\phi_{\delta,\overline{\sigma},p})_*(0) \leq \gamma_{\delta,\overline{\sigma}} \end{cases} \quad \text{with} \quad |f_\delta(\gamma_{\delta,\overline{\sigma}}) + \overline{\sigma}| \geq \frac{1}{4} osc(f), \tag{3.88}$$

with for instance $\gamma_{\delta,\overline{\sigma}} \in [m_\delta - 1, m_\delta]$. Hence using the proof of Lemma 9.1 and the last equality of (3.86), we pass to the limit $p \to 0$ and $\phi_{\delta,\overline{\sigma},p}$ converges up to subsequence to a non trivial (because of (3.88)) solution $\phi_{\delta,\overline{\sigma}_{\delta,0}}$ of

$$\begin{cases} \overline{c}\phi'_{\delta,\overline{\sigma}_{\delta,0}}(z) = F_\delta((\phi_{\delta,\overline{\sigma}_{\delta,0}}(z+r_i))_{i=0,\dots,N}) + \overline{\sigma}_{\delta,0} \quad \text{on} \quad \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}} \text{ is non-decreasing on } \mathbb{R} \\ \phi_{\delta,\overline{\sigma}_{\delta,0}}(+\infty) - \phi_{\delta,\overline{\sigma}_{\delta,0}}(-\infty) \leq 1, \end{cases} \tag{3.89}$$

where

$$\overline{\sigma}_{\delta,p} \to \overline{\sigma}_{\delta,0}$$

and $|\overline{\sigma}_{\delta,0}| \leq K$.

Substep 6.2.2 **: establishing** $\overline{\sigma}_{\delta,0} = 0$
Since $\sigma < \overline{\sigma}_{\delta,p}$ (see (3.87)), then we get $\sigma \leq \overline{\sigma}_{\delta,0}$. Thus passing to the limit $\sigma \to 0$, we get

$$\overline{\sigma}_{\delta,0} \geq 0,$$

without any change in equation (3.89). Moreover, since we have

$$0 = f_\delta(\phi_{\delta,\overline{\sigma}_{\delta,0}}(\pm\infty)) + \overline{\sigma}_{\delta,0}$$

and $f_\delta \geq 0$, then we get that

$$\overline{\sigma}_{\delta,0} = 0.$$

Therefore, because of (3.88), $\phi_\delta := \phi_{\delta,\overline{\sigma}_{\delta,0}=0}$ satisfies (3.75) with c_δ^+ replaced by $\overline{c}$.

Substep 6.2.3 **: passing to the limit** $\delta \to 0$
Up to translation, we assume that

$$\begin{cases} (\phi_\delta)^*(0) \geq \dfrac{b_\delta + m_\delta}{2} \\ (\phi_\delta)_*(0) \leq \dfrac{b_\delta + m_\delta}{2}, \end{cases}$$

Therefore, passing to the limit using once more Lemma 9.1, ϕ_δ converges up to a subsequence to a solution ϕ of (3.78) and (3.79), with c^+ replaced by $\overline{c}$. This ϕ is non trivial because of (3.79). Moreover, since $\phi(\pm\infty)$ solves $f = 0$, we deduce that ϕ is a solution of (3.9) associated for the velocity $\overline{c}$.

Step 7 : no solution for $c < c^+$
This step is analogous to Lemma 6.4. Let (c, ϕ) be a solution of (3.9). Then as a solution of (3.74), we can choose $(c_0, \phi_0) = (c, \phi)$. Therefore, the choice $c_0 = c$ in (3.77), implies that

$$c^+ \leq c,$$

and then there is no a solution of (3.9) for $c < c^+$. □

8 On the critical velocity

In a first subsection, we prove Proposition 1.3 which asserts that the critical velocity satisfies $c^+ \geq 0$ under additional assumptions. In a second subsection, we give an example (Proposition 1.4) that shows that we can have $c^+ < 0$ when the additional assumptions are not satisfied. We also prove the instability of the critical velocity, namely Proposition 1.2. We prove in a third subsection that $c^+ \geq c^*$, precisely Proposition 1.5. We also show, if F satisfies the KPP condition (3.16), that $c^* \geq c^+$ (see Proposition 1.6). In this subsection, we also give an example where $c^+ > c^*$ (Lemma 8.4).

8.1 Critical velocity c^+ is non-negative

This subsection is devoted for the proof of Proposition 1.3. Independently, we also show that $c^- < 0 < c^+$ for the Frenkel-Kontorova model (3.17).

Proof of Proposition 1.3
Let (c, ϕ) be a solution of (3.9) given in Theorem 1.1 with c fixed. Our goal is to show that $c \geq 0$; and hence $c^+ \geq 0$. We perform the proof in several steps.

Step 0 : preliminary
Define for $X = (X_0, ..., X_N) \in [0,1]^{N+1}$ and $\delta > 0$ small the function

$$F_\delta(X) = F(X) - f(X_0) + f_\delta(X_0), \tag{3.90}$$

where

$$f_\delta(v) = \begin{cases} f & \text{on} \quad [0, 1-\delta] \\ \max\left(f(1-\delta) - L_0(v - (1-\delta)), 0\right) & \text{on} \quad [1-\delta, 1], \end{cases}$$

with a constant $L_0 > 2\text{Lip}(F) > 0$ large enough. Let $\delta \in (0, \frac{1}{2})$ and set

$$1_\delta := 1 - \delta + \frac{f(1-\delta)}{L_0} < 1,$$

(where 1_δ was denoted by m_δ in the proof of Theorem 1.1).

Part I : antisymmetric extension of F_δ and proof for $ii)$
Using Proposition 9.4, there exists an antisymmetric extension G_δ on $[-1,1]^{N+1}$ of F_δ such that

$$\begin{cases} (G_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ G_\delta(-X) = -G_\delta(X) \quad \text{for all} \quad X \in [-1,1]^{N+1} \end{cases}$$

and satisfying (A_{Lip}) over $[-1,1]^{N+1}$ (since F_δ satisfies (A_{Lip}) over $[0,1]^{N+1}$). Moreover, still by Proposition 9.4, since F_δ is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$

(because of (P_{C^1}) and (3.90)) and $f'_\delta(0) = f'(0) > 0$, then there exists $\eta > 0$ such that for every X, $X + (a, ..., a) \in [-1, 1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$G_\delta(X + (a, ..., a)) - G_\delta(X) \geq \eta a. \tag{3.91}$$

In addition, the function $g_\delta(v) := G_\delta(v, ..., v)$ satisfies

$$\begin{cases} g_\delta(-1_\delta) = g_\delta(0) = g_\delta(1_\delta) = 0 \\ (g_\delta)_{|(-1_\delta,0)} < 0 \quad \text{and} \quad (g_\delta)_{|(0,1_\delta)} > 0, \end{cases} \tag{3.92}$$

(since we have $f_\delta(0) = 0 = f_\delta(1_\delta)$ and $f_\delta > 0$ on $(0, 1_\delta)$).

Step $I.1$: existence of traveling waves for G_δ
Clearly, since G_δ satisfies (3.92) and (3.91), then G_δ satisfies the assumption (B_{Lip}) with $[0, 1]^{N+1}$ replaced by over $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0. In addition, G_δ satisfies, by construction, the assumption (A_{Lip}) over $[-1_\delta, 1_\delta]^{N+1}$. Thus applying the result of Proposition 4.3 with $[0, 1]^{N+1}$ replaced by $[-1_\delta, 1_\delta]^{N+1}$ and b replaced by 0, we deduce that there exists a real c^0_δ and a function ϕ^0_δ solution of

$$\begin{cases} c^0_\delta(\phi^0_\delta)'(x) = G_\delta((\phi^0_\delta(x + r_i))_{i=0,\dots,N}) \quad \text{on} \quad \mathbb{R} \\ \phi^0_\delta \ \text{ is non-decreasing over } \mathbb{R} \\ \phi^0_\delta(-\infty) = -1_\delta \quad \text{and} \quad \phi^0_\delta(+\infty) = 1_\delta. \end{cases} \tag{3.93}$$

Step $I.2$: $c^0_\delta \geq 0$
We show in this step that c^0_δ is non-negative under $ii)$, i.e. assuming $r_i \geq 0$ for all $i \in \{0, ..., N\}$. Then $\psi(x) = -\phi^0_\delta(-x)$ satisfies

$$\begin{aligned} -c^0_\delta \psi'(y) &= -G_\delta((-\psi(y - r_i))_{i=0,\dots,N}) \\ &= G_\delta((\psi(y - r_i))_{i=0,\dots,N}) \\ &\leq G_\delta((\psi(y + r_i))_{i=0,\dots,N}), \end{aligned}$$

hence $(\overline{c} = -c^0_\delta, \psi)$ is a subsolution of (3.93). Using an argument similar to the computation of (3.72) for L_0 large enough (here $L_0 > 2\text{Lip}(F)$), we can show that G_δ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$, that is G_δ satisfies (3.154) and (3.156) (for $s = -1_\delta$ and $s' = 1_\delta$). Applying the comparison principle results (Proposition 9.9 and Proposition 9.10) and the ideas of the proof of Proposition 5.1, we deduce that

$$-c^0_\delta = \overline{c} \leq c^0_\delta,$$

that is

$$0 \leq c^0_\delta. \tag{3.94}$$

Step $I.3$: comparing c and c_δ^0
Recall that $(c_\delta^0, \phi_\delta^0)$ is a solution of (3.93). Moreover, since $G_\delta = F_\delta \leq F$ over $[0,1]^{N+1}$, then (c,ϕ) is a supersolution for (3.9), with F replaced by G_δ.

Since

$$\phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1$$

and $-1_\delta < 0 < 1_\delta < 1$, that is $\phi_\delta^0(\pm\infty) < \phi(\pm\infty)$ and $\phi_\delta^0(+\infty) > \phi(-\infty)$, then using the proof of Proposition 5.1 (which still true for sub and supersolutions), we deduce that

$$0 \leq c_\delta^0 \leq c.$$

Part II : extension of F_δ by antisymmetry-reflection and proof for $iii)$
In this part, we assume that F (and then F_δ) satisfies the strict monotonicity condition (3.12). Using Remark 9.8, we can assume that the set I defined in (3.11) satisfies

$$I = \{1, ..., N\},$$

i.e. for all $i \in \{1,...,N\}$, there exists $\bar{i} \in \{1,...,N\}$ such that $r_i = -r_{\bar{i}}$. Using now Proposition 9.7, there exists an extension $\overline{G}_\delta$ on $[-1,1]^{N+1}$ of F_δ such that

$$\begin{cases} (\overline{G}_\delta)_{|_{[0,1]^{N+1}}} = F_\delta \\ \overline{G}_\delta(-X) = -\overline{G}_\delta(X) \quad \text{for all} \quad X \in [-1,1]^{N+1} \end{cases}$$

and satisfying (A_{Lip}) over $[-1,1]^{N+1}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$, then (using Proposition 9.7) there exists $\eta > 0$ such that for every X, $X + (a,...,a) \in [-1,1]^{N+1}$ close to $\{0\}^{N+1}$ with $a > 0$ small, we have

$$\overline{G}_\delta(X + (a,...,a)) - \overline{G}_\delta(X) \geq \eta a. \tag{3.95}$$

In addition, the function $\overline{g}_\delta(v) := \overline{G}_\delta(v,...,v)$ satisfies

$$\begin{cases} \overline{g}_\delta(-1_\delta) = \overline{g}_\delta(0) = \overline{g}_\delta(1_\delta) = 0 \\ (\overline{g}_\delta)_{|_{(-1_\delta,0)}} < 0 \quad \text{and} \quad (\overline{g}_\delta)_{|_{(0,1_\delta)}} > 0. \end{cases} \tag{3.96}$$

Step $II.1$: existence of traveling waves for $\overline{G}_\delta$
This step is a variant of Step $I.1$ with G_δ replaced $\overline{G}_\delta$. Thus we deduce that there exists a real $\overline{c}_\delta^0$ and a function $\overline{\phi}_\delta^0$ solution of

$$\begin{cases} \overline{c}_\delta^0 (\overline{\phi}_\delta^0)'(x) = \overline{G}_\delta((\overline{\phi}_\delta^0(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \overline{\phi}_\delta^0 \quad \text{is non-decreasing over } \mathbb{R} \\ \overline{\phi}_\delta^0(-\infty) = -1_\delta \quad \text{and} \quad \overline{\phi}_\delta^0(+\infty) = 1_\delta. \end{cases} \tag{3.97}$$

Step $II.2$ **:** $\overline{c}_\delta^0 = 0$
Let $\psi(x) = -\overline{\phi}_\delta^0(-x)$, then

$$\begin{aligned}\overline{G}_\delta((\psi(x+r_i))_{i=0,\dots,N}) &= \overline{G}_\delta((-\overline{\phi}_\delta^0(-x-r_i))_{i=0,\dots,N}) \\ &= \overline{G}_\delta((-\overline{\overline{\phi}_\delta^0(-x+r_i)})_{i=0,\dots,N}) \\ &= -\overline{G}_\delta((\overline{\phi}_\delta^0(-x+r_i))_{i=0,\dots,N}) \\ &= -\overline{c}_\delta^0(\overline{\phi}_\delta^0)'(-x) = -\overline{c}_\delta^0\psi'(x).\end{aligned}$$

Thus $(-\overline{c}_\delta^0, \psi)$ is a solution of (3.97) with $\overline{c}_\delta^0$ replaced by $-\overline{c}_\delta^0$.

Similarly to Step $I.2$ (with $L_0 > 2\text{Lip}(F)$), we can show that $\overline{G}_\delta$ is decreasing close to $\{-1_\delta\}^{N+1}$ and $\{1_\delta\}^{N+1}$ inside $[-1_\delta, 1_\delta]^{N+1}$. By comparison principle, we get

$$\overline{c}_\delta^0 \le \overline{c} \quad \text{and} \quad \overline{c} \le \overline{c}_\delta^0,$$

which implies that

$$\overline{c}_\delta^0 = 0. \tag{3.98}$$

Step $II.3$ **: comparing** c **and** $\overline{c}_\delta^0$
This step is analogous to Step $I.3$ with $(c_\delta^0, \phi_\delta^0)$ replaced by $(\overline{c}_\delta^0, \overline{\phi}_\delta^0)$ and G_δ replaced by $\overline{G}_\delta$. Thus proceeding similarly we show that

$$0 = \overline{c}_\delta^0 \le c.$$

Part III **: proof for** $i)$
Under condition $i)$, we have

$$I = \{1, ..., N\} \quad \text{and} \quad \frac{\partial F}{\partial X_i}(0) = \frac{\partial F}{\partial X_{\overline{i}}}(0) \;\; \text{for all} \;\; i \in I,$$

thus condition (3.12) is equivalent to $f'(0) > 0$. Therefore, we can apply $iii)$ which shows that $c^+ \ge 0$. □

Lemma 8.1. (Sign of c^+ **and** c^- **for (FK) model** (3.17)**)**
Consider the Frenkel-Kontorova model with $\beta > 0$

$$c\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) - \beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \sigma,$$

with $\sigma \in [-\beta, \beta] = [\sigma^-, \sigma^+]$. *Let* $c^\pm$ *be the critical velocity associated to* $\sigma^\pm$. *Then*

$$c^- < 0 < c^+.$$

Proof of Lemma 8.1
Let $\sigma = \sigma^+ = \beta$ and let us show that $c^+ > 0$. Let ϕ be non-decreasing with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Integrating over the real line the equation

$$c^+\phi'(z) = \phi(z+1) + \phi(z-1) - 2\phi(z) + f(\phi(z)),$$

where $f(\phi(z)) = -\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta \geq 0$, we get that

$$c^+ = \int_{\mathbb{R}} \left(-\beta \sin\left(2\pi\left(\phi(z) + \frac{1}{4}\right)\right) + \beta\right) dz \geq 0.$$

Since $f > 0$ on $(0, 1)$, if $c^+ = 0$, then

$$\phi(z) = 0 \text{ or } 1 \quad \text{almost everywhere.}$$

This implies that

$$\Delta_1\phi(z) := \phi(z+1) + \phi(z-1) - 2\phi(z) = 0 \quad \text{almost everywhere.}$$

Consider now the set

$$\mathcal{A} = \{z \in \mathbb{R},\ \Delta_1\phi(z) \neq 0\},$$

which has measure zero. Thus the set $\mathcal{A}+\mathbb{Z}$ has also measure zero. Hence for a fixed $a \in \mathbb{R}\backslash(\mathcal{A}+\mathbb{Z}) \neq \emptyset$, we have

$$\Delta_1\phi(a+k) = 0 \quad \text{for every } k \in \mathbb{Z}.$$

This implies that there exists $\lambda, b \in \mathbb{R}$ (that may depend on a) such that

$$\phi(a+k) = \lambda k + b.$$

But ϕ is bounded, then $\lambda = 0$ and hence $\phi(a+k) = b$, which is a contradiction since $\phi(+\infty) \neq \phi(-\infty)$. Therefore $c^+ > 0$.

Similarly, for $\sigma = \sigma^- = -\beta$, we show that $c^- < 0$, since $f - 2\beta < 0$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
□

8.2 Instability of critical velocity

In this section, we show that the critical velocity c^+ given in Theorem 1.1 is unstable in the sense of Proposition 1.2, which we prove in this section.

Before proving Proposition 1.2, we give an example of a non-linearity F for which the associated critical velocity is negative. This example will be the proof of Proposition 1.4.

Proof of Proposition 1.4
The aim is to construct a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity satisfies $c^+ < 0$. To this end, we will construct a function $f \in \text{Lip}([0,1])$, which is linear in a neighborhood of zero with $f'(0) > 0$, such that there exists a couple (c, ϕ) with $c < 0$ solution of

$$\begin{cases} c\phi'(x) = \phi(x-1) - \phi(x) + f(\phi(x)) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{3.99}$$

Let $c = -\mu$ with $0 < \mu < 1$ and

$$\phi(x) = \begin{cases} \dfrac{1}{2}e^{\gamma x} & \text{on} \quad (-\infty, 0] \\ 1 - \dfrac{1}{2}e^{-\gamma x} & \text{on} \quad [0, +\infty) \end{cases}$$

with $\gamma > 0$. We claim that $\phi \in C^1(\mathbb{R})$ and $(-\mu, \phi)$ solves

$$\begin{cases} 0 < \phi(x) - \phi(x-1) - \mu\phi'(x) \quad \text{on} \quad \mathbb{R} \\ \phi' > 0 \\ \phi(-\infty) = 0 \quad \text{and} \quad \phi(+\infty) = 1, \end{cases} \tag{3.100}$$

which is possible to check for $0 < \gamma << 1$.

Therefore, it is sufficient to define the function f as

$$f(\phi(x)) := \phi(x) - \phi(x-1) - \mu\phi'(x) > 0 \quad \text{for all } x \in \mathbb{R}. \tag{3.101}$$

Notice that, when $x \to +\infty$, $\phi(+\infty) = 1$ and $\phi'(x) \to 0$, thus $f(1) = 0$. Similarly, we have $f(0) = 0$. Moreover, since $\phi \in C^{1,1}(\mathbb{R})$, we have that $f \in \text{Lip}((0,1))$. In fact, by a direct tedious calculation, one can deduce that

$$f(x) = \begin{cases} (1 - e^{-\gamma} - \mu\gamma)x & \text{for } x \in \left[0, \dfrac{1}{2}\right] \\ 1 + (1+\mu\gamma)(x-1) + \dfrac{e^{-\gamma}}{4(x-1)} & \text{for } x \in \left[\dfrac{1}{2}, 1 - \dfrac{1}{2}e^{-\gamma}\right] \\ (1 - e^{\gamma} + \mu\gamma)(x-1) & \text{for } x \in \left[1 - \dfrac{1}{2}e^{-\gamma}, 1\right], \end{cases}$$

and this implies that $f \in \text{Lip}([0,1])$ and $1 > f'(0) > 0$. We can even check that f is concave and C^1 except at the point $x = \frac{1}{2}$, where it is neither concave nor C^1.

Remark that to get more regular non-linearities, one can consider

$$f_\varepsilon(x) := \Big(\big(\phi(\cdot) - \phi(\cdot - 1) - \mu\phi'(\cdot)\big) \star \rho_\varepsilon\Big)(x), \tag{3.102}$$

where ρ_ε satisfies $\rho_\varepsilon \geq 0$, $\rho_\varepsilon(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ (ρ is a mollifier) and $supp\,\rho_\varepsilon \subset B_\varepsilon(0)$. However, in this case, $\rho_\varepsilon \star \phi$ is a solution of (3.99), with f replaced by f_ε, and then $f_\varepsilon \in C^\infty([0,1])$ with $f'_\varepsilon(0) > 0$. □

Now, we give the proof of the instability result, namely Proposition 1.2.

Proof of Proposition 1.2
We have seen, in Proposition 1.4, that there exists a function F satisfying (A_{Lip}) and (P_{C^1}) such that the associated critical velocity $c_F^+ := c^+$ satisfies

$$c_F^+ < 0. \tag{3.103}$$

Our goal is to build a sequence of functions F_δ with a critical velocity $c_{F_\delta}^+$ such that

$$F_\delta \to F \quad \text{in} \quad L^\infty([0,1]^{N+1})$$

as $\delta \to 0$, and prove that

$$\liminf_{\delta\to 0} c_{F_\delta}^+ > c_F^+. \tag{3.104}$$

Step 1 : construction of F_δ
Define for $X = (X_0, ..., X_N) \in [0,1]$ and $\delta > 0$ small the function

$$F_\delta(X) = F(X) - f(X_0) - f_\delta(X_0), \tag{3.105}$$

where

$$f_\delta(v) = \begin{cases} \max\big(f(\delta) + L_0(v-\delta), 0\big) & \text{on} \quad [0,\delta] \\ f & \text{on} \quad [\delta, 1], \end{cases} \tag{3.106}$$

with a constant $L_0 > 0$ satisfying $L_0 > 2\text{Lip}(F) =: 2L_F^\infty$.

By construction of f_δ, we clearly have

$$\|F_\delta - F\|_{L^\infty} \;=\; \|f - f_\delta\|_{L^\infty} \to 0 \quad \text{as} \quad \delta \to 0.$$

Step 2 : existence of $c_{F_\delta}^+$
Set

$$0_\delta = \delta - \frac{f(\delta)}{L_0} > 0,$$

(where 0_δ was denoted by b_δ in the proof of Theorem 1.1).

Since F_δ satisfies (A_{Lip}) and (P_{Lip}) with $[0,1]^{N+1}$ replaced by $[0_\delta,1]^{N+1}$, then applying the result of Theorem 1.1, we deduce that there exists a minimal velocity $c^+_{F_\delta}$ and a profile ϕ solution of

$$\begin{cases} c^+_{F_\delta}\phi'(z) = F_\delta(\phi(z+r_0),\phi(z+r_1),...,\phi(z+r_N)) \quad \text{on} \quad \mathbb{R} \\ \phi \text{ is non-decreasing over } \mathbb{R} \\ \phi(-\infty) = 0_\delta \quad \text{and} \quad \phi(+\infty) = 1. \end{cases} \tag{3.107}$$

Step 3 **: establishing** (3.104)
Our aim is to show that $c^+_{F_\delta} \geq 0$. Since F_δ is non-decreasing w.r.t. X_i for all $i \neq 0$, then for $X = (X_0, X') \in [0_\delta,1]^{N+1}$, we have

$$F_\delta(X_0,X') \geq F_\delta(X_0,0_\delta,...,0_\delta) := A(X_0).$$

Moreover, for X_0 $X_0 + h \in [0_\delta,\delta]$ with $h > 0$, we have

$$\begin{aligned} A(X_0+h) - A(X_0) &= F(X_0+h,0_\delta+h,...,0_\delta+h) - F(X_0,0_\delta,...,0_\delta) - f(X_0+h) \\ &\quad + f(X_0) + f_\delta(X_0+h) - f_\delta(X_0) \\ &\geq -2hL_F^\infty + hL_0 \\ &= h(L_0 - 2L_F^\infty) > 0, \end{aligned}$$

where we have used that F is L_F^∞-Lipschitz (in the second line) and that $L_0 > 2L_F^\infty$ in the last inequality. This implies that A is increasing over $[0_\delta,\delta]$, but $A(0_\delta) = F_\delta(0_\delta,0_\delta,...,0_\delta) = 0$. Hence, we get $A \geq 0$ over $[0_\delta,\delta]$.

Therefore, we deduce that

$$F_\delta \geq 0 \quad \text{over} \quad [0_\delta,\delta]\times[0_\delta,1]^N.$$

Now since $\phi(-\infty) = 0_\delta$, then for $z << -1$ very negative, we get that $\phi(z+r_0) = \phi(z) \in [0_\delta,\delta]$. Hence, for all $\phi(z) \in [0_\delta,\delta]$, we obtain from (3.107) that

$$c^+_{F_\delta}\phi'(z) = F_\delta(\phi(z+r_0),\phi(z+r_1),...,\phi(z+r_N)) \geq 0,$$

but $\phi' \geq 0$, thus we deduce that

$$c^+_{F_\delta} \geq 0.$$

This implies that (because of (3.103))

$$\liminf_{\delta\to 0} c^+_{F_\delta} \geq 0 > c^+_F.$$

Step 4 **: conclusion**
Let

$$\widehat{\phi}(x) = \frac{\phi(x) - 0_\delta}{1 - 0_\delta}, \quad c^+_{\widehat{F}_\delta} = (1-0_\delta)c^+_{F_\delta} \tag{3.108}$$

and

$$\widehat{F}_\delta((X_i)_{i=0,...,N}) = F_\delta(((1-0_\delta)X_i + 0_\delta)_{i=0,...,N}).$$

Then we have

$$\begin{cases} c^+_{\widehat{F}_\delta}\widehat{\phi}'(z) = \widehat{F}_\delta((\widehat{\phi}(z+r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R} \\ \widehat{\phi} \text{ is non-decreasing over } \mathbb{R} \\ \widehat{\phi}(-\infty) = 0 \quad \text{and} \quad \widehat{\phi}(+\infty) = 1 \end{cases} \tag{3.109}$$

and $c^+_{\widehat{F}_\delta}$ is the critical velocity associated to $\widehat{F}_\delta$ which is defined on $[0,1]^{N+1}$. Moreover, we still have $|\widehat{F}_\delta - F| \to 0$ as $\delta \to 0$ and $\widehat{F}_\delta$ satisfies (A_{Lip}) and (P_{Lip}) on $[0,1]^{N+1}$. In addition, since $0_\delta \to 0$ as $\delta \to 0$, then from (3.108) we still have

$$\liminf_{\delta\to 0} c^+_{\widehat{F}_\delta} = \liminf_{\delta\to 0} c^+_{F_\delta} \geq 0 > c^+_F.$$

Therefore, up to rename $\widehat{F}_\delta$ as F_δ, this ends the proof of Proposition 1.2. □

8.3 Lower bound for c^+

In this subsection, we prove a lower bound for the critical velocity c^+ given in Theorem 1.1. Precisely, we show in Proposition 1.5 that $c^+ \geq c^*$. In Lemma 8.4, we give an example where $c^+ > c^*$. In this subsection, we also prove that and $c^* \geq c^+$ under a KPP condition (see Proposition 1.6).

We start with the proof of Proposition 1.5

Proof of Proposition 1.5

Under assumptions (A_{Lip}) and (P_{Lip}), let c^+ given by Theorem 1.1. We want to show that $c^+ \geq c^*$ with c^* given in (3.15). Let $c \geq c^+$ such that $c \neq 0$ and let us prove that $c \geq c^*$. Associate for c a profile ϕ such that (c,ϕ) is a solution of (3.9) (this is always possible since $c \geq c^+$, see Theorem 1.1).

Step 1 : $\frac{\phi'(x)}{\phi(x)}$ is globally bounded

From Harnack inequality (3.184), we deduce that if $\phi(x_0) = 0$ at some point $x_0 \in \mathbb{R}$, then $\phi \equiv 0$ which is impossible for a solution of (3.9). Therefore $\phi > 0$.

We have

$$c\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)}F((\phi(x+r_i))_{i=0,...,N}).$$

We also know, using the monotonicity of F w.r.t. X_i for $i \neq 0$ and $F(0,...,0) = 0$, that

$$\begin{aligned} F(\phi(x),\phi(x+r_1),...,\phi(x+r_N)) &= F(\phi(x),\phi(x+r_1),...,\phi(x+r_N)) - F(0,...,0) \\ &\leq F(\phi(x),\phi(x+r^*),...,\phi(x+r^*)) - F(0,...,0), \end{aligned}$$

where $r^* = \max_{i=0,...,N} |r_i|$. Since F is Lipschitz (with constant Lipschitz L), then

$$F(\phi(x), \phi(x+r_1), ..., \phi(x+r_N)) \leq L \begin{vmatrix} \phi(x) \\ \phi(x+r^*) \\ \vdots \\ \phi(x+r^*) \end{vmatrix} \leq L_1 \phi(x+r^*) \quad \text{with} \quad L_1 = L \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}$$

and hence ($c \neq 0$)

$$0 \leq \frac{\phi'(x)}{\phi(x)} \leq \frac{1}{|c|} L_1 \frac{\phi(x+r^*)}{\phi(x)}.$$

From Proposition 9.14, we know that there exists a constant $\overline{\kappa}_0 > 1$ such that

$$\phi(x+r^*) \leq \overline{\kappa}_0 \phi(x), \tag{3.110}$$

therefore, we deduce that

$$0 \leq \frac{\phi'(x)}{\phi(x)} \leq \mathcal{M} := \frac{\overline{\kappa}_0 L_1}{|c|}. \tag{3.111}$$

Step 2 : proving that $c \geq c^*$

Since ϕ satisfies (3.111), then $\limsup_{x \to -\infty} \dfrac{\phi'(x)}{\phi(x)} = \lambda$ exists and $\lambda = \lim_{n \to +\infty} \dfrac{\phi'(x_n)}{\phi(x_n)}$ for some $x_n \to -\infty$ as $n \to +\infty$. Let

$$\phi_n(x) := \frac{\phi(x+x_n)}{\phi(x_n)} \geq 0,$$

then $\phi_n(0) = 1$ and ϕ_n satisfies

$$c\phi_n'(x) = \frac{1}{\phi(x_n)} F((\phi(x+x_n+r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R}. \tag{3.112}$$

Now, since for all i, $\phi(x+x_n+r_i) \to 0$ as $n \to +\infty$, $F(0,...,0) = 0$ and F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$, then we see that we can write for n large enough

$$c\phi_n'(x) = \sum_{i=0}^{N} \int_0^1 \frac{\partial F}{\partial X_i}(s\phi(x+x_n+r_i))\phi_n(x+r_i)ds \quad \text{on} \quad \mathbb{R}. \tag{3.113}$$

From (3.185), we deduce that for $k \in \mathbb{N}\backslash\{0\}$, we have

$$\phi(x_n + kr^*) \leq (\overline{\kappa}_0)^k \phi(x_n) \quad \text{and} \quad \phi(x+r^*) \leq \overline{\kappa}_0 \phi(x),$$

with $\overline{\kappa}_0 > 1$. Hence for $x \in [(k-1)r^*, kr^*]$, we get

$$0 \leq \phi_n(x) = \frac{\phi(x+x_n)}{\phi(x_n)} \leq (\overline{\kappa}_0)^k \leq (\overline{\kappa}_0)^{\frac{x}{r^*}+1} \leq \overline{\kappa}_0 e^{\mu x} \quad \text{with} \quad \mu = \frac{\ln \overline{\kappa}_0}{r^*}. \tag{3.114}$$

This implies that

$$0 \leq \phi_n(x) \leq \overline{\kappa}(x) := \overline{\kappa}_0 e^{\mu x^+}.$$

From (3.111), we have

$$0 \leq \frac{\phi_n'}{\phi_n} \leq \mathcal{M},$$

which implies that

$$0 \leq \phi_n'(x) \leq \mathcal{M}\overline{\kappa}(x). \tag{3.115}$$

Therefore, using Ascoli's Theorem and the extraction diagonal argument, we deduce that ϕ_n converges locally uniformly to some ϕ_∞ which satisfies (in the viscosity sense)

$$\begin{cases} c\phi_\infty'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\phi_\infty(x + r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_\infty' \geq 0 \\ \phi_\infty(0) = 1 \\ \phi_\infty(x + r^*) \leq \overline{\kappa}_0 \phi_\infty(x). \end{cases} \tag{3.116}$$

Therefore, using Lemma 8.3 below (with $a_0 = r^* > 0$), we deduce that

$$c \geq c^*. \tag{3.117}$$

Step 3 : conclusion ($c^+ \geq c^*$)
Since (3.117) holds true for any $c \geq c^*$ with $c \neq 0$, we deduce that $c^+ \geq c^*$. □

Remark 8.2. (About the assumption (3.14))
It is possible to show that Proposition 1.5 still holds true if we replace (3.14) by

$$\exists\ i_0 \in \{0, ..., N\} \quad \text{such that} \quad r_{i_0} < 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0$$

if

$$c^+ < 0.$$

In order to see it, we can prove a lower bound (analogue to Proposition 9.13) with

$$\psi_\varepsilon \geq \kappa\varepsilon \quad \text{on} \quad [\delta, R] \times [\delta, T_0]$$

for $\delta > 0$ (this lower bound is obtained with a variant of the strong maximum principle, Proposition 9.11).

From this, we can deduce a Harnack inequality for solution of (3.183) with $c < 0$ (analogue to Proposition 9.14). Again using this Harnack inequality, we can conclude that $c^+ \geq c^*$ as in the proof of Proposition 1.5.

Lemma 8.3. (Lower bound for c^+ for linear problem)
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that F satisfies (3.14) *and*

$$f'(0) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) > 0, \tag{3.118}$$

where we recall that $f(v) = F(v, ..., v)$. Let $c \neq 0$ and assume that there exists $a_0 > 0$ and $C_0 > 0$ such that ϕ is a solution of

$$\begin{cases} c\phi'(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\phi(x + r_i) \quad \text{on} \quad \mathbb{R} \\ \phi' \geq 0 \\ \phi > 0 \\ 1 \leq \frac{\phi(x + a_0)}{\phi(x)} \leq C_0 \quad \text{for all} \quad x \in \mathbb{R}. \end{cases} \tag{3.119}$$

Then

$$c \geq c^*,$$

where c^* is given in (3.15)*.*

Proof of Lemma 8.3
Step 0 : preliminary
Let $a \in (0, a_0)$ and let

$$K^* = \inf E \quad \text{with} \quad E = \{k \geq 1 \ \text{ such that } \ k\phi(x) \geq \phi(x + a) \quad \text{for all } x \in \mathbb{R}\}.$$

We deduce from (3.119) that $E \neq \emptyset$ because $C_0 \in E$. By definition of K^*, we have

$$K^*\phi(x) \geq \phi(x + a) \quad \text{for every } \ x \in \mathbb{R}. \tag{3.120}$$

We have $K^* \geq 1$. If $K^* = 1$, then ϕ is constant and the first equation of (3.119) gives

$$0 = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) = f'(0)$$

which is a contradiction with (3.118). Therefore $K^* > 1$, and there exists $\lambda > 0$ such that

$$K^* = e^{\lambda a}. \tag{3.121}$$

Again by definition of K^*, for every $\varepsilon > 0$, there exists $x_\varepsilon \in \mathbb{R}$ such that

$$(K^* - \varepsilon)\phi(x_\varepsilon) < \phi(x_\varepsilon + a). \tag{3.122}$$

Let

$$\phi_\varepsilon(x) := \frac{\phi(x+x_\varepsilon)}{\phi(x_\varepsilon)}.$$

Then $\phi_\varepsilon(0)=1$,

$$K^*\phi_\varepsilon(x) \geq \phi_\varepsilon(x+a) \tag{3.123}$$

and (3.122) can be rewritten as

$$(K^*-\varepsilon)\phi_\varepsilon(0) < \phi_\varepsilon(a). \tag{3.124}$$

Step 1 : passing to limit $\varepsilon \to 0$
Since $c \neq 0$, we can bound both ϕ_ε and ϕ'_ε on any bounded interval uniformly w.r.t. ε (as in Step 2 of the proof of Proposition 1.5). Therefore, using Ascoli Theorem and the extraction diagonal argument, we deduce that ϕ_ε converges to some ϕ_0 locally uniformly and ϕ_0 satisfies (in the viscosity sense)

$$\begin{cases} c\phi_0'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\phi_0(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi_0' \geq 0 \\ \phi_0(0)=1 \\ K^*\phi_0(0) \leq \phi_0(a) \quad (\text{using } (3.124)) \\ K^*\phi_0(x) \geq \phi_0(x+a) \quad (\text{using } (3.123)). \end{cases} \tag{3.125}$$

Now, let $w(x) = K^*\phi_0(x) - \phi_0(x+a)$. Then from (3.125), we deduce that w satisfies

$$\begin{cases} cw'(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)w(x+r_i) \quad \text{on} \quad \mathbb{R} \\ w \geq 0 \quad \text{on} \quad \mathbb{R} \\ w(0)=0. \end{cases} \tag{3.126}$$

Then using the half strong maximum principle Lemma 6.1 in Chapter 2, we get that $w(x)=0$ for all $cx \leq 0$, i.e.

$$k^*\phi_0(x) = \phi_0(x+a) \quad \text{for all} \quad cx \leq 0. \tag{3.127}$$

Step 2 : establishing $c \geq c^*$
Let

$$\phi_{0,n}(x) := \frac{\phi_0(x-cn)}{\phi_0(-cn)}.$$

Then $\phi_{0,n}(0)=1$. Moreover, using (3.127), we have

$$K^*\frac{\phi_0(x-cn)}{\phi_0(-cn)} = \frac{\phi_0(x-cn+a)}{\phi_0(-cn)} \quad \text{for all} \quad c(x-cn) \leq 0.$$

Hence

$$K^*\phi_{0,n}(x) = \phi_{0,n}(x+a) \quad \text{for all} \quad cx \le c^2 n. \tag{3.128}$$

Step 2.1 **: passing to the limit** $n \to +\infty$

As before, we can pass to the limit and show that $\phi_{0,n} \to \phi_{0,\infty}$ with

$$\begin{cases} c\phi'_{0,\infty}(x) = \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)\phi_{0,\infty}(x+r_i) \quad \text{on} \quad \mathbb{R} \\ \phi'_{0,\infty} \ge 0 \\ \phi_{0,\infty}(0) = 1. \end{cases} \tag{3.129}$$

Moreover, passing to the limit in (3.128), we deduce that

$$K^*\phi_{0,\infty}(x) = \phi_{0,\infty}(x+a) \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.130}$$

Step 2.2 **: conclusion**

Let

$$z(x) = \frac{\phi_{0,\infty}(x)}{e^{\lambda x}}.$$

Recall that $\phi_{0,\infty} \in C^1$ (because $c \neq 0$). Then $z \in C^1$ and satisfies

$$cz'(x) + c\lambda z(x) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x+r_i) \quad \text{on} \quad \mathbb{R}. \tag{3.131}$$

We also have

$$z(x+a) = \frac{\phi_{0,\infty}(x+a)}{e^{\lambda(x+a)}} = \frac{K^*\phi_{0,\infty}(x)}{e^{\lambda a}e^{\lambda x}} = z(x),$$

where we have used (3.130) and (3.121).

Because z is a-periodic (and continuous), there exists $x_0 \in \mathbb{R}$ such that z attain it's minimum at x_0. We claim that $z(x_0) \neq 0$. Indeed, if $z(x_0) = 0$, then we deduce from (3.131) that

$$\sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0+r_i) = 0.$$

Since $\frac{\partial F}{\partial X_i}(0,...,0) \ge 0$ for all $i = 1,...,N$ and F satisfies (3.14), we deduce that

$$z(x_0 + r_{i_0}) = 0.$$

Repeating the same process, we get that $z = 0$ on $x_0 + r_{i_0}\mathbb{N}$. Since z is a-periodic, then $z = 0$ on $x_0 + r_{i_0}\mathbb{N} + a\mathbb{Z} \equiv x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$.

Since $a \in (0, a_0)$ is arbitrary, then we can choose $a \in (0, a_0)$ such that $\frac{r_{i_0}}{a} \in \mathbb{R}\backslash\mathbb{Q}$. Therefore, $x_0 + a(\frac{r_{i_0}}{a}\mathbb{N} + \mathbb{Z})$ is dense in $\mathbb{R}$. This implies, since z is continuous, that

$$z = 0 \quad \text{on} \quad \mathbb{R},$$

which is a contradiction with $z(0) = 1$.

Therefore, $z(x_0) \neq 0$. Again, since $z(x_0) = \min z \geq 0$, then using (3.131), we get that

$$\begin{aligned} c\lambda z(x_0) &= \frac{\partial F}{\partial X_0}(0,...,0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0 + r_i) \\ &\geq \frac{\partial F}{\partial X_0}(0,...,0)e^{\lambda r_0} z(x_0) + \sum_{i=1}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i} z(x_0) \\ &= z(x_0) \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}. \end{aligned}$$

Using the fact that $z(x_0) \neq 0$, we deduce that

$$c\lambda \geq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}.$$

Recall that $\lambda > 0$. Therefore, we get

$$c \geq \frac{P(\lambda)}{\lambda} \geq \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda} = c^*,$$

where $P(\lambda) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)e^{\lambda r_i}$. This ends the proof. $\square$

Now, we give the proof of Proposition 1.6, where we show that $c^+ \leq c^*$ under a KPP type condition.

Proof of Proposition 1.6

The goal is to prove that for any real $c > c^*$ ($c^* < +\infty$), we have $c^+ \leq c$.

For such c, we have $c > c^* = \inf_{\lambda > 0} \frac{P(\lambda)}{\lambda}$, hence there exists some $\lambda_0 > 0$ such that

$$c > \frac{P(\lambda_0)}{\lambda_0}.$$

This implies that $\phi(x) = e^{\lambda_0 x}$ satisfies

$$c\phi'(x) > G((\phi(x + r_i))_{i=0,...,N}), \tag{3.132}$$

where $G(X) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)X_i$. Let $\tilde{F}$ be the extension over $\mathbb{R}^{N+1}$ of F (given by Lemma 7.1). The goal is now to construct a supersolution of

$$cw'(x) = \tilde{F}((w(x + r_i))_{i=0,...,N}) \quad \text{on} \quad \mathbb{R}. \tag{3.133}$$

Step 1 **:** $\overline{\phi}(x) := \min(1, \phi(x))$ **is a supersolution of** (3.133)
We recall that $\phi(0) = 1$. Let $x < 0$, we have

$$\begin{cases} \overline{\phi}(x+r_i) = \phi(x+r_i) & \text{for} \quad r_i \leq 0 \\ \overline{\phi}(x+r_i) \leq \phi(x+r_i) & \text{for} \quad r_i > 0. \end{cases}$$

Since F is non-decreasing w.r.t. X_i for $i \neq 0$, then G satisfies the same property, hence

$$\begin{aligned} G((\phi(x+r_i))_{i=0,\dots,N}) &\geq G((\overline{\phi}(x+r_i))_{i=0,\dots,N}) \\ &\geq F((\overline{\phi}(x+r_i))_{i=0,\dots,N}), \end{aligned}$$

where we have used (3.16) and the fact that $0 \leq \overline{\phi}(x) \leq 1$. But $\overline{\phi}(x) = \phi(x)$ is a test function for $x < 0$ and ϕ satisfies (3.132), thus we get for $x < 0$:

$$c\overline{\phi}'(x) = c\phi'(x) > G((\phi(x+r_i))_{i=0,\dots,N}) \geq F((\overline{\phi}(x+r_i))_{i=0,\dots,N}).$$

Similarly for $x > 0$, we have

$$\begin{cases} \overline{\phi}(x+r_i) \leq 1 & \text{for} \quad r_i < 0 \\ \overline{\phi}(x+r_i) = 1 & \text{for} \quad r_i \geq 0. \end{cases}$$

Moreover, since $\overline{\phi}(x) = 1$ is a test function for $x > 0$, we get

$$c\overline{\phi}'(x) = 0 = F(1, ..., 1) \geq F((\overline{\phi}(x+r_i))_{i=0,\dots,N}).$$

Now for $x = 0$, we have $\overline{\phi}(0) = 1 = \phi(0)$ is a supersolution of (3.133) because there is no test function touching $\overline{\phi}$ from below at $x = 0$ (see Definition 2.1). Finally, since $0 \leq \overline{\phi}(x) \leq 1$, then $\tilde{F}((\overline{\phi}(x+r_i))_{i=0,\dots,N}) = F((\overline{\phi}(x+r_i))_{i=0,\dots,N})$ and hence $\overline{\phi}$ is a supersolution of (3.133).

Step 2 **: subsolution of** (3.133)
Let (c^+, ϕ^+) be a solution of (3.133) given by Theorem 1.1. We know, from the proof of Theorem 1.1 (see (3.73)), that

$$c^+ = \lim_{\delta \to 0} (\lim_{\sigma \to 0^-} c_{\delta,\sigma}) \quad \text{and} \quad \phi^+ = \lim_{\delta \to 0} (\lim_{\sigma \to 0^-} \phi_{\delta,\sigma})$$

where $\delta > 0$, $\sigma < 0$ are small enough and $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a solution of (with $\tilde{F}_\delta = F_\delta$)

$$c_{\delta,\sigma}\phi'_{\delta,\sigma}(x) = F_\delta((\phi_{\delta,\sigma}(x+r_i))_{i=0,\dots,N}) + \sigma$$

and $\phi_{\delta,\sigma}(-\infty) = m_{\delta,\sigma} - 1$, $\phi_{\delta,\sigma}(+\infty) = m_{\delta,\sigma}$ with $m_{\delta,\sigma} - 1 < 0 < m_{\delta,\sigma} < 1$.

Since $F_\delta = \tilde{F}_\delta \leq \tilde{F}$ (see (3.69)) and $\sigma < 0$, then we deduce that $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$ is a subsolution of (3.133) with (c, w) is replaced by $(c_{\delta,\sigma}, \phi_{\delta,\sigma})$.

Step 3 : establishing $c^+ \leq c^*$
Using the proof of Proposition 5.1, we deduce that

$$c_{\delta,\sigma} \leq c.$$

Passing to the limit $\sigma \to 0^-$ and then $\delta \to 0$ (as in the proof of Theorem 1.1), we deduce that

$$c^+ \leq c \quad \text{for all} \quad c > c^*. \tag{3.134}$$

This implies that

$$c^+ \leq c^*.$$

$\square$

Now, we give an example of non-linearity where we have $c^+ > c^*$.

Lemma 8.4. (Example with $c^+ > c^*$)
Consider the function $F^0 : [0,1]^3 \to \mathbb{R}$ defined as

$$F^0(X_0, X_{-1}, X_1) := g(X_1) + g(X_{-1}) - 2g(X_0) + f(X_0),$$

with $r_0 = 0$, $r_{\pm 1} = \pm 1$ and f, $g : [0,1] \to \mathbb{R}$ are C^1 over a neighborhood of 0, Lipschitz on $[0,1]$ and satisfying

$$\begin{cases} f(0) = f(1) = 0 \\ f > 0 \quad on \quad (0,1) \\ f'(0) > 0 \end{cases} \quad and \quad \begin{cases} g'(0) = 0 \\ g(1) = 1 + g(0) \\ g' \geq 0. \end{cases}$$

Let c^+ given by Theorem 1.1 (with F replaced by F^0), then

$$c^+ > c^*,$$

where c^* is defined in (3.15).

An example of such g is $g(x) = x - \frac{1}{2\pi}\sin(2\pi x)$.

Proof of Lemma 8.4
Since $g'(0) = 0$ and $f'(0) > 0$, then $P(\lambda) = f'(0) > 0$. Thus we get that $c^* = \inf_{\lambda>0} \frac{P(\lambda)}{\lambda} = 0$. By Proposition 1.3 *i*), we have that $c^+ \geq 0 = c^*$. We want to show that $c^+ \neq 0$.

Assume to the contrary that $c^+ = 0$ and let ϕ be a solution of (3.9) with F replaced by F^0. Using the equivalence between the viscosity solution and almost everywhere solutions (see Lemma 2.5), we deduce that ϕ is an almost everywhere solution of

$$0 = F((\phi(z + r_i))_{i=0,...,N}). \tag{3.135}$$

That is there exists a set $\mathcal{N}$ of measure zero such that for every $z \notin \mathcal{N}$, equation (3.135) holds true.

Let $\mathcal{N}_0 = \cup_{k\in\mathbb{Z}}(\mathcal{N} + k)$ and choose $z_0 \in \mathbb{R}\backslash\mathcal{N}_0$ (set $\mathcal{N}_0$ has also a zero measure), then equation (3.135) holds true for every $z_0 + k$ with $k \in \mathbb{Z}$. Hence

$$g(\phi(z_0+k+1))+g(\phi(z_0+k-1))-2g(\phi(z_0+k)) = -f(\phi(z_0+k)) \leq 0 \quad \text{for every} \quad k \in \mathbb{Z}. \tag{3.136}$$

Let h be the piecewise affine function which is affine on each interval $[k, k + 1]$ and satisfying $h(z_0 + k) = g(\phi(z_0 + k))$ with $k \in \mathbb{Z}$. Thus, it is easy to conclude using (3.136) that h is concave. Moreover, h is bounded because g is bounded on $[0, 1]$ and $0 \leq \phi \leq 1$. Therefore, h is constant. This implies that

$$g(\phi(z_0)) = g(\phi(z_0 + k)) = const \quad \text{for all} \quad k \in \mathbb{Z}.$$

Moreover, since $g' \geq 0$, $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$, we conclude that $g = const$ on $[0, 1]$, which is a contradiction with $g(1) = 1 + g(0)$. Hence, we get $c^+ > 0 = c^*$. □

9 Appendix : useful results

In this appendix, we present some techniques to pass to the limit in the equation, in a first subsection. In a second subsection, we prove some results that we use to prove that the critical velocity c^+ is non-negative. We prove, in a third subsection, a Harnack type inequality (Proposition 9.14) that we use to prove $c^+ \geq c^*$ in Subsection 8.3.

9.1 Results for passing to the limit

The main result of this subsection (Theorem 9.3) identifies the limits of a constructed profile. We also prove some results to pass to the limit in the equation, namely Lemma 9.1.

We start by introducing the following bistable notation :

Assumption ($\tilde{B}_{m,b}$) :
Let $f(x) := F(x, ..., x)$ and $m < b < m + 1$,

Bistability : $f(m) = 0 = f(b) = f(m + 1)$, $f < 0$ on (m, b) and $f > 0$ on $(b, m + 1)$.

Lemma 9.1. (Passing to the limit)
Consider a sequence of functions F_n satisfying $(\tilde{A}_{Lip})$ and $(\tilde{B}_{m_n,b_n})$ (with $m_n \in [0,1)$) such that

$$Lip(F_n) \leq C \quad \text{independent on } n. \tag{3.137}$$

Let (c_n, ϕ_n) be a solution of

$$\begin{cases} c_n\phi_n'(z) = F_n((\phi_n(z+r_i))_{i=0,\dots,N}) \quad \text{over} \quad \mathbb{R} \\ \phi_n \text{ is non-decreasing on } \mathbb{R} \\ \phi_n(-\infty) = m_n \quad \text{and} \quad \phi_n(+\infty) = m_n + 1. \end{cases} \tag{3.138}$$

Assume that

$$|\phi_n| \leq M \quad \text{for some} \;\; M > 0 \text{ independent of } n. \tag{3.139}$$

Assume moreover that there exists a real number c such that $c_n \to c$; and that $F_n \to F$ locally uniformly and $(m_n, b_n) \to (m, b)$ as $n \to +\infty$. Then, up to a subsequence, ϕ_n converges almost everywhere to some ϕ that solves in the viscosity sense

$$\begin{cases} c\phi'(z) = F((\phi(z+r_i))_{i=0,\dots,N}) \quad \text{over} \quad \mathbb{R} \\ \phi \text{ is non-decreasing on } \mathbb{R} \\ m \leq \phi(-\infty) \quad \text{and} \quad \phi(+\infty) \leq m + 1. \end{cases} \tag{3.140}$$

Moreover, either ϕ satisfies

$$m = \phi(-\infty) \quad \text{and} \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that

$$m = \phi^a(-\infty) \quad \text{and} \quad \phi^a(+\infty) = b$$

and

$$b = \phi^b(-\infty) \quad \text{and} \quad \phi^b(+\infty) = m + 1.$$

Proof of Lemma 9.1

Step 1 : passing to the limit
The proof of this result follows from Step 2 of the proof of Proposition 2.3 in Chapter 2. For the convenience of the reader, we give the proof here.

Because of (3.137) and since ϕ_n is bounded, we deduce that there exists a constant $M_0 > 0$ independent of n such that

$$|F_n((\phi_n(z+r_i))_{i=0,\dots,N})| \leq M_0 \quad \text{independent on } n. \tag{3.141}$$

Case 1 **:** $c \neq 0$
Since $|c_n| \geq \frac{|c|}{2}$ for n large, then

$$|\phi_n'| \leq \frac{2M_0}{c} \quad \text{for large} \quad n.$$

Thus ϕ_n is uniformly Lipschitz. Using Ascoli's Theorem and the diagonal extraction argument, we get that ϕ_n converges to ϕ (up to a subsequence) locally uniformly on $\mathbb{R}$. Moreover ϕ is non-decreasing and satisfies (by stability of viscosity solutions)

$$c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}). \tag{3.142}$$

We easily deduce (3.140).

Case 2 **:** $c = 0$
Since ϕ_n is monotone and bounded (uniformly in n), then using Helly's Lemma (Lemma 2.4) and the diagonal extraction argument, ϕ_n converges (up to a subsequence) to a non-decreasing ϕ a.e. Our goal is to show that

$$0 = F((\phi(x + r_i))_{i=0,\dots,N}). \tag{3.143}$$

Subcase 2.1 **:** $c_n = 0$ **for all** n
We first use the equivalence between viscosity solutions and almost everywhere solutions (Lemma 2.5) and then pass to the limit in (3.138) using Helly's lemma (Lemma 2.4). Hence, we get a solution ϕ of (3.143) almost everywhere. Again, we use Lemma 2.5 to conclude that ϕ is a viscosity solution of (3.143) and satisfies (3.140).

Subcase 2.2 **:** $c_n \neq 0$ **for all** n
We have

$$c_n \int_{b_1}^{b_2} (\phi_n)'(z)dz = \int_{b_1}^{b_2} \big(F_n((\phi_n(z + r_i))_{i=0,\dots,N})\big)dz$$

for every $b_1 < b_2$. That is,

$$c_n(\phi_n(b_2) - \phi_n(b_1)) = \int_{b_1}^{b_2} \big(F_n((\phi_n(z + r_i))_{i=0,\dots,N})\big)dz.$$

But, we have (3.141) and

$$F_n((\phi_n(z + r_i))_{i=0,\dots,N}) \to F((\phi(z + r_i))_{i=0,\dots,N}) \quad \text{a.e.}$$

Thus, using Lebesgue's dominated convergence theorem, we pass to the limit $n \to +\infty$ and get

$$0 = \int_{b_1}^{b_2} F((\phi(z + r_i))_{i=0,\dots,N})dz$$

which implies (since b_1 and b_2 are arbitrary) that

$$0 = F((\phi(z + r_i))_{i=0,\dots,N}) \quad \text{a.e.}$$

Then by Lemma 2.5, ϕ verifies (3.143) in the viscosity sense and satisfies (3.140).

Step 2 : limits of the profile
Since $\phi(\pm\infty)$ solves $f = 0$, then $\phi(\pm\infty) \in \{m, b, m+1\}$. Therefore, either ϕ satisfies

$$m = \phi(-\infty) \quad \text{and} \quad \phi(+\infty) = m + 1$$

or there exists two solutions ϕ^a and ϕ^b such that ϕ^a satisfies

$$m = \phi^a(-\infty) \quad \text{and} \quad \phi^a(+\infty) = b$$

and

$$\begin{cases} (\phi^a)_*(0) \leq \dfrac{m+b}{2} \\ (\phi^a)^*(0) \geq \dfrac{m+b}{2} \end{cases} \tag{3.144}$$

and ϕ^b satisfies

$$b = \phi^b(-\infty) \quad \text{and} \quad \phi^b(+\infty) = m + 1$$

and

$$\begin{cases} (\phi^b)_*(0) \leq \dfrac{m+1+b}{2} \\ (\phi^b)^*(0) \geq \dfrac{m+1+b}{2} \end{cases} \tag{3.145}$$

Solutions ϕ^a and ϕ^b can be obtained as limits of $\phi^a_n(x) = \phi_n(x + a_n)$ and $\phi^b_n(x) = \phi_n(x + b_n)$ for suitable shifts a_n, b_n such that ϕ^a_n and ϕ^b_n satisfies resp. (3.144) and (3.145). □

We recall now the existence result of traveling waves whose a slightly different statement is given in Proposition 2.3 in Chapter 2. In order to present the main result of this section, we need to introduce the following technical lemma.

Lemma 9.2. (Controlling the finite difference)
Consider F satisfying $(\tilde{A}_{C^1})$, $\sigma_0 \in (\sigma^-, \sigma^+)$ fixed and $\beta > 0$. Let $a > r^*$ (r^* is given by (3.23)) and $M_0 > 0$, then for all $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all function ϕ (viscosity) solution of

$$\begin{cases} c\phi'(x) = F((\phi(x + r_i))_{i=0,\dots,N}) + \sigma \quad \text{on } \mathbb{R} \\ \phi' \geq 0 \\ \phi(x+1) \leq \phi(x) + 1 \\ |c| \leq M_0 \\ |c\phi'| \leq M_0, \end{cases}$$

and for all $x_0 \in \mathbb{R}$ *satisfying*

$$\phi_*(x_0 + a) - \phi^*(x_0 - a) \leq \delta,$$

we have

$$dist\big(\alpha, \{m_\sigma, b_\sigma\} + \mathbb{Z}\big) < \varepsilon \quad \textit{for all} \;\; \alpha \in [\phi_*(x_0), \phi^*(x_0)].$$

Proof of Lemma 9.2

The proof of this lemma follows from a straightforward generalization of Proposition 3.2 in Chapter 2 for the function F replaced by $F+\sigma$ and $(0, b)$ replaced by (m_σ, b_σ) for $\sigma \in [\sigma_0 - \beta, \sigma_0 + \beta] \subset (\sigma^-, \sigma^+)$ and for some $\beta > 0$. We similarly show that for every $\varepsilon > 0$ there exists $\delta_\sigma(\varepsilon) > 0$ such that the result holds true.

However, we can show that $\delta_\sigma(\varepsilon) = \delta(\varepsilon)$ can be chosen independent of σ and the proof of this generalization follows exactly the same lines. Indeed, we proceed by contradiction assuming that the statement is false for a sequence $\sigma_n \in [\sigma_0-\beta, \sigma_0+\beta]$, and consider a sequence of solutions ϕ^n. The presence of σ_n does not create any additional difficulty in the passage to the limit in the equation. $\square$

Theorem 9.3. (Identification of the limits of the profile)

We work under the assumptions of Lemma 9.1 with $F_n = F + \sigma_n$, $m_n = m_{\sigma_n}$, $b_n = b_{\sigma_n}$ *and* F *satisfying* $(\tilde{A}_{C^1})$ *and* $(\tilde{B}_{C^1})$. *We assume moreover that the solution* (c_n, ϕ_n) *of* (3.138) *is given by Proposition 4.1 for* $\sigma_n \in (\sigma^-, \sigma^+)$. *Let* (c_∞, ϕ_∞) *be the solution of* (3.140) *constructed in Lemma 9.1. If* $\sigma_\infty \in (\sigma^-, \sigma^+)$, *then we have moreover*

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad \textit{and} \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

Proof of Theorem 9.3

Let (c_n, ϕ_n) be a solution of (3.138) given by Proposition 4.1 and (c_∞, ϕ_∞) be a solution of (3.140) for $\sigma_\infty \in (\sigma^-, \sigma^+)$, obtained by passing to the limit $n \to \infty$. Our aim is to show that

$$\phi_\infty(-\infty) = m_{\sigma_\infty} \quad \text{and} \quad \phi_\infty(+\infty) = m_{\sigma_\infty} + 1.$$

For $\varepsilon > 0$ small enough ($\varepsilon < \frac{1}{2}\min(b_{\sigma_n} - m_{\sigma_n}, m_{\sigma_n} + 1 - b_{\sigma_n})$), let $z_n,\ y_n \in \mathbb{R}$ such that

$$\begin{cases} (\phi_n)^*(z_n) \geq b_{\sigma_n} + \varepsilon \\ (\phi_n)_*(z_n) \leq b_{\sigma_n} + \varepsilon \end{cases} \tag{3.146}$$

and

$$\begin{cases} (\phi_n)^*(y_n) \geq b_{\sigma_n} - \varepsilon \\ (\phi_n)_*(y_n) \leq b_{\sigma_n} - \varepsilon. \end{cases} \tag{3.147}$$

Assume moreover that up to translate ϕ_n, we have

$$\begin{cases} (\phi_n)_*(0) \le b_{\sigma_n} \\ (\phi_n)^*(0) \ge b_{\sigma_n}. \end{cases}$$

For every $x \in \mathbb{R}$, set with $a > r^*$

$$\psi_n(x) := (\phi_n)_*(x+a) - (\phi_n)^*(x-a) \ge 0$$

and denote by

$$\overline{m}_n = \min_{[y_n,z_n]} \psi_n(x) = \psi_n(x_n) \ge 0,$$

for some $x_n \in [y_n, z_n]$ since ψ_n is lower semi-continuous.

We claim that $\overline{m}_n > 0$. Indeed, if $\overline{m}_n = 0$, then since $\psi_n(y_n),\ \psi_n(z_n) \ge \delta(\varepsilon) > 0$ (because of (3.146), (3.147) and using Lemma 9.2), we get

$$x_n \in (y_n, z_n).$$

Moreover, we have that

$$0 = \psi_n(x_n) = (\phi_n)_*(x_n+a) - (\phi_n)^*(x_n-a)$$

and ϕ_n is non-decreasing, hence

$$\phi_n = const \quad \text{over} \quad (x_n - a, x_n + a),$$

and ϕ_n solves $f + \sigma_n = 0$.
Now, since

$$b_{\sigma_n} - \varepsilon \le (\phi_n)^*(y_n) \le \phi_n(x_n) \le (\phi_n)_*(z_n) \le b_{\sigma_n} + \varepsilon,$$

we get that

$$\phi_n = b_{\sigma_n} \quad \text{over} \quad (x_n - a, x_n + a).$$

Therefore, for $r^* < \overline{a} < a$, we have

$$\phi_n = b_{\sigma_n} \quad \text{over} \quad [x_n - \overline{a}, x_n + \overline{a}],$$

which is in contradiction with Proposition 4.1. Therefore, $\overline{m}_n > 0$ and the proof of the identification of limits of the profile proceeds similarly as in the proof of Proposition 2.3 (Chapter 2), where now Step 5 is no longer necessary. In particular we avoid the case $\phi(\pm\infty) = b_{\sigma_\infty}$. □

9.2 Useful results used for the proof of $c^+ \geq 0$

This subsection is dedicated for the useful tools that we use to prove that the critical velocity is non-negative, i.e $c^+ \geq 0$.

Proposition 9.4. (Extension by antisymmetry)
Let F be a function defined over $Q = [0,1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0,...,0) = 0$. Then there exists an antisymmetric extension G defined over $[-1,1]^{N+1}$ such that

$$\begin{cases} G_{|_Q} = F \\ G(-X) = -G(X) \end{cases}$$

and G satisfies (A_{Lip}) over $[-1,1]^{N+1}$.

Moreover, if F is C^1 over a neigborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and $f'(0) > 0$ ($f(v) := F(v,...,v)$), then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0,...,X_N) \in [-1,1]^{N+1}$ such that X, $X+(a,...,a)$ are close enough to $\{0\}^{N+1}$, we have

$$G(X + (a,...,a)) - G(X) \geq \eta a. \tag{3.148}$$

Remark 9.5. (Reflection)
Note that if F is invariant by reflection symmetry, then it is possible to show that G also ; precisely, we mean that if $F(\overline{X}) = F(X)$ for $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$, then

$$G(\overline{X}) = G(X).$$

We recall before proving Proposition 9.4 the following properties of the orthogonal projection which can be easily shown :

Lemma 9.6. (Some properties of orthogonal projection)
Let $X = (X_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and call $Proj_{|_Q}(X)$ the orthogonal projection of X on $Q = [0,1]^{N+1}$. Then

$$Proj_{|_Q}(X) = (Proj_{|_{[0,1]}}(X_i))_{i=0,...,N}.$$

Moreover, we have

i) Order preservation
Let $Y = (Y_i)_{i=0,...,N} \in [-1,1]^{N+1}$ and assume that $X \geq Y$ in sense that $X_i \geq Y_i$ for all $i \in \{0,...,N\}$, then

$$Proj_{|_Q}(X) \geq Proj_{|_Q}(Y).$$

ii) "Antisymmetry"
Let $Q' = [-1,0]^{N+1} = -Q$, then

$$Proj_{|_{Q'}}(-X) = -Proj_{|_Q}(X).$$

Proof of Proposition 9.4
Let $X = (X_i)_{i=0,...,N} \in [-1,1]^{N+1}$, then define the extension function G by :

$$G(X) = F(Proj_{|_Q}(X)) - F(-Proj_{|_{Q'}}(X)), \qquad (3.149)$$

where we recall that $Q' = [-1,0]^{N+1}$. For $X \in Q$, we clearly have $G(X) = F(X)$.

Step 1 **:** $G(-X) = -G(X)$
We have

$$\begin{aligned} G(-X) &= F(Proj_{|_Q}(-X)) - F(-Proj_{|_{Q'}}(-X)) \\ &= F(-Proj_{|_{Q'}}(X)) - F(Proj_{|_Q}(X)) \\ &= -G(X), \end{aligned}$$

where we have used in the second line the antisymmetry in Lemma 9.6.

Step 2 **:** G **satisfies** $(A_{\mathbf{Lip}})$
Since F is globally Lipschitz and the orthogonal projection is 1-Lipschitz, then G is globally Lipschitz on $[-1,1]^{N+1}$.
We now prove that G is non-decreasing w.r.t. X_i for all $i \neq 0$. Let $X = (X_i)_{i=0,...,N}$, $Y = (Y_i)_{i=0,...,N} \in [-1,1]^{N+1}$ such that

$$\begin{cases} X_i \geq Y_i & \text{for all } i \in \{1,...,N\} \\ X_0 = Y_0, \end{cases}$$

and let us show that $G(X) \geq G(Y)$. In fact, since the orthogonal projection preserve the ordering (see Lemma 9.6) and since F is non-decreasing w.r.t. X_i for all $i \in \{1,...,N\}$, we conclude that G is non-decreasing w.r.t. X_i for all $i \in \{1,...,N\}$ over $[-1,1]^{N+1}$.

Step 3 **: checking** (3.148)
We first give some notations for the projection function. Consider $X = (X_0,...,X_N) \in [-1,1]^{N+1}$, then from Lemma 9.6, we have

$$Proj_{|_Q}(X) = (Proj_{|_{[0,1]}}(X_i))_{i=0,...,N} = \left(\begin{Bmatrix} X_i & \text{if} & X_i \geq 0 \\ 0 & \text{if} & X_i \leq 0 \end{Bmatrix} \right)_{i=0,...N} =: X^+.$$

Similarly, we have (with $Q' = -Q$)

$$Proj_{|_{Q'}}(X) = \left(\begin{Bmatrix} 0 & \text{if} & X_i \geq 0 \\ X_i & \text{if} & X_i \leq 0 \end{Bmatrix} \right)_{i=0,...N} =: X^-$$

We also define

$$Q_\Sigma = \left\{X = (X_0, ..., X_N) \in [-1,1]^{N+1}, \quad \sigma_i X_i \in [0,1] \quad \text{for} \quad i = 0, ..., N\right\},$$

where $\Sigma = (\sigma_0, ..., \sigma_N)$ and $\sigma_i = \pm 1$.

Now, we go back to the proof of (3.148) which is splitted in two cases. Let X, $X + (a, ..., a)$ close to $\{0\}^{N+1}$ with $a > 0$ small :

Case 1 **:** $X,\ X + (a, ..., a) \in Q_\Sigma$
From the definition of G (see (3.149)) and the notations introduced at the beginning of this step, we have

$$G(X+(a,...,a)) - G(X) = F((X+aE)^+) - F(X^+) - \Big(F(-(X+aE)^-) - F(-X^-)\Big),$$

where $E = (1, ..., 1)$. Thus, we get

$$G(X + (a, ..., a)) - G(X) = a\Theta.\nabla F(X^+) + o(|a\Theta|) + a\Gamma.\nabla F(-X^-) + o(|a\Gamma|),$$

where $a\Theta = (X + aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\Gamma = (X + aE)^- - X^-$ with $\Gamma = (\gamma_i)_{i=0,...,N}$, where

$$\gamma_i = \begin{cases} 0 & \text{if} \quad \sigma_i = 1 \\ 1 & \text{if} \quad \sigma_i = -1. \end{cases}$$

Hence, we obtain

$$\begin{aligned} G(X + (a, ..., a)) - G(X) \quad &= \quad a(\Theta + \Gamma).\nabla F(0) + a\Theta.(\nabla F(X^+) - \nabla F(0)) + o(a) \\ &+ \quad a\Gamma.(\nabla F(-X^-) - \nabla F(0)), \end{aligned}$$

but $a(\Theta + \Gamma) = (a, ..., a)$, therefore,

$$\begin{aligned} G(X + (a, ..., a)) - G(X) \quad &= \quad a\Big\{f'(0) + \Theta.(\nabla F(X^+) - \nabla F(0)) + o(1) \\ &+ \quad \Gamma.(\nabla F(-X^-) - \nabla F(0))\Big\}. \end{aligned}$$

Now, since F is C^1 over a neighborhood of X (X close to $\{0\}^{N+1}$), then we get

$$G(X + (a, ..., a)) - G(X) = a\left\{f'(0) + o(X^+) + o(X^-) + o(1)\right\} \geq a\frac{f'(0)}{2} > 0$$

for X close enough to $\{0\}^{N+1}$.

Case 2 **:** $X \in Q_\Sigma$ **and** $X + aE \in Q_{\widehat{\Sigma}}$
There exists an integer $p \geq 1$ such that

$$G(X + aE) - G(X) = \sum_{k=0}^{p} \Big(G(X + t_k E) - G(X + t_{k-1} E) \Big),$$

where $0 = t_0 < t_1 < ... < t_p = a$ such that for $k = 1, ..., p$, we have $X + [t_{k-1}, t_k]E \in Q_{\Sigma_k}$, with $\Sigma = \Sigma_0$ and $\widehat{\Sigma} = \Sigma_p$. Therefore, using Case 1 for each segment, we deduce that

$$G(X + aE) - G(X) \geq \eta a,$$

with $\eta = \frac{f'(0)}{2} > 0$.

□

We now introduce an extension by antisymmetry-reflection of F :

Proposition 9.7. (Extension by antisymmetry-reflection)
Let F be a function defined on $Q = [0,1]^{N+1}$ satisfying (A_{Lip}) and such that $F(0, ..., 0) = 0$. Let $X = (X_i)_{i=0,...,N} \in [0,1]^{N+1}$ and assume that

$$\text{for all } \; i \in \{1, ..., N\} \; \text{ there exists } \; \bar{i} \in \{1, ..., N\} \; \text{ such that } \; r_{\bar{i}} = -r_i. \tag{3.150}$$

Then there exists a function $\overline{G}$ defined on $[-1,1]^{N+1}$ which satisfies (A_{Lip}) on $[-1,1]^{N+1}$ such that

$$\begin{cases} \overline{G}_{|Q} = F \\ \overline{G}(-\overline{X}) = -\overline{G}(X) \quad \text{(antisymmetric-reflection)}, \end{cases}$$

where we recall that $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$.

Moreover, if F is C^1 over a neighborhood of $\{0\}^{N+1}$ and

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i=1}^{N} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0) \right) > 0, \tag{3.151}$$

then there exists $\eta > 0$ such that for every $a > 0$ small and $X = (X_0, ..., X_N) \in [-1,1]^{N+1}$ such that X, $X + (a, ..., a)$ are close enough to $\{0\}^{N+1}$, we have

$$\overline{G}(X + (a, ..., a)) - \overline{G}(X) \geq \eta a. \tag{3.152}$$

Remark 9.8. (On the reflection condition (3.150)**)**
Notice that we can always assume that the reflection condition (3.150) *is satisfied*

up to modify the function F. Indeed, if F does not satisfy the reflection condition (3.150), *i.e. we have*

$$\{i_1, ..., i_M\} = \{i \in \{1, ..., N\},\ \textit{such that}\ -r_i \notin \{r_j\}_{j=1,...,N}\}$$

with $M \geq 1$, then let us define

$$r_{N+j} = -r_{i_j} \quad \textit{for} \quad j = 1, ..., M.$$

Therefore, for each $i \in \{1, ..., N+M\}$ there exists $\bar{i} \in \{1, ..., N+M\}$ such that $r_{\bar{i}} = -r_i$. Now, for $\tilde{X} = (X, X')$ with $X' = (X_{N+1}, ..., X_{N+M})$, set

$$\tilde{F}(\tilde{X}) = F(X).$$

Thus $\tilde{F}$ satisfies (3.150) *with N replaced by $\tilde{N} = N + M$ and if moreover ϕ solves*

$$c\phi'(z) = F((\phi(z + r_i))_{i=0,...,N}),$$

then it solves $c\phi'(z) = \tilde{F}((\phi(z + r_i))_{i=0,...,\tilde{N}})$.

In addition, if F is C^1 in a neighborhood of $\{0\}^{N+1}$, then $\tilde{F}$ is C^1 in a neighborhood of $\{0\}^{\tilde{N}+1}$, and

$$\frac{\partial F}{\partial X_0}(0) + \sum_{i\in I} \min\left(\frac{\partial F}{\partial X_i}(0), \frac{\partial F}{\partial X_{\bar{i}}}(0)\right) = \frac{\partial \tilde{F}}{\partial X_0}(0) + \sum_{i=1}^{\tilde{N}} \min\left(\frac{\partial \tilde{F}}{\partial X_i}(0), \frac{\partial \tilde{F}}{\partial X_{\bar{i}}}(0)\right),$$

with $I = \Big\{i \in \{1, ..., N\}$ such that there exists $\bar{i} \in \{1, ..., N\}$ with $r_i = -r_{\bar{i}}\Big\}$.

Proof of Proposition 9.7

The proof is very similar to the proof of Proposition 9.4, so we give only a few details. Let $X \in [-1, 1]^{N+1}$, then define the extension function $\overline{G}$ by

$$\overline{G}(X) = F(Proj_{|_Q}(X)) - F(-Proj_{|_{Q'}}(\overline{X})), \tag{3.153}$$

where we recall that $\overline{X}_i = X_{\bar{i}}$ with $r_{\bar{i}} = -r_i$.

Step 1 **:** $\overline{G}(-\overline{X}) = -\overline{G}(X)$

We have

$$\begin{aligned}
\overline{G}(-\overline{X}) &= F(Proj_{|_Q}(-\overline{X})) - F(-Proj_{|_{Q'}}(\overline{-\overline{X}})) \\
&= F(-Proj_{|_{Q'}}(\overline{X})) - F(-Proj_{|_{Q'}}(-X)) \quad \text{(using Lemma 9.6 } ii\text{) and } \overline{-X} = -\overline{X}) \\
&= F(-Proj_{|_{Q'}}(\overline{X})) - F(Proj_{|_Q}(X)) \quad \text{(using again Lemma 9.6 } ii\text{))} \\
&= -\overline{G}(X).
\end{aligned}$$

Step 2 : $\overline{G}$ satisfies $(A_{\mathbf{Lip}})$ on $[-1,1]^{N+1}$
This step is an analogous of Step 2 in the proof of Proposition 9.4.

Step 3 : checking (3.152)
We have

$$\overline{G}(X) = F(X^+) - F(-(\overline{X})^-)$$

Let $\Sigma = (\sigma_0, \sigma_1, ..., \sigma_N)$ and define $\overline{\Sigma} = (\overline{\sigma}_0, \overline{\sigma}_1, ..., \overline{\sigma}_N)$ such that $\overline{\sigma}_i = \sigma_{\overline{i}}$ for all $i = 0, ..., N$; and then recall

$$Q_\Sigma = \left\{X = (X_0, ..., X_N) \in [-1,1]^{N+1} \quad \text{such that} \quad \sigma_i X_i \in [0,1] \quad \text{for} \quad i = 0, ..., N\right\}.$$

We have

$$X \in Q_\Sigma \Longleftrightarrow \overline{X} \in Q_{\overline{\Sigma}}.$$

Let $X,\ X + aE$ close enough to $\{0\}^{N+1}$ with $a > 0$ small and $E = (1, ..., 1)$.

Case 1 : $X,\ X + aE \in Q_\Sigma$
Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then (as in the proof of Proposition 9.4, Step 3) we have

$$\begin{aligned}
\overline{G}(X+aE) - \overline{G}(X) &= F((X+aE)^+) - F(X^+) - \Big(F(-(\overline{X+aE})^-) - F(-(\overline{X})^-)\Big) \\
&= F((X+aE)^+) - F(X^+) - \Big(F(-(\overline{X}+aE)^-) - F(-(\overline{X})^-)\Big) \\
&= a\Theta.\nabla F(X^+) + o(|a\Theta|) + a\overline{\Gamma}.\nabla F(-(\overline{X})^-) + o(|a\overline{\Gamma}|),
\end{aligned}$$

where $a\Theta = (X+aE)^+ - X^+$ with $\Theta = (\theta_i)_{i=0,...,N}$, where

$$\theta_i = \begin{cases} 1 & \text{if} \quad \sigma_i = 1 \\ 0 & \text{if} \quad \sigma_i = -1 \end{cases}$$

and $a\overline{\Gamma} = (\overline{X}+aE)^- - (\overline{X})^-$ with $\overline{\Gamma} = (\overline{\gamma}_i)_{i=0,...,N}$, where

$$\overline{\gamma}_i = \begin{cases} 0 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = 1 \\ 1 & \text{if} \quad \overline{\sigma}_i := \sigma_{\overline{i}} = -1. \end{cases}$$

Hence, using the fact that F is C^1, we get

$$\begin{aligned}
\overline{G}(X+aE) - \overline{G}(X) &= a\Big\{(\Theta + \overline{\Gamma}).\nabla F(0) + \Theta.(\nabla F(X^+) - \nabla F(0)) + o(1) \\
&\quad + \overline{\Gamma}.(\nabla F(-(\overline{X})^-) - \nabla F(0))\Big\} \\
&\geq \frac{a}{2}(\Theta + \overline{\Gamma}).\nabla F(0) > 0,
\end{aligned}$$

if $(\Theta+\overline{\Gamma}).\nabla F(0)>0$. This is true because

$$\begin{aligned}
(\Theta+\overline{\Gamma}).\nabla F(0) &= \sum_{i=0}^{N}\theta_i\frac{\partial F}{\partial X_i}(0)+\sum_{j=0}^{N}\overline{\gamma}_j\frac{\partial F}{\partial X_j}(0)\\
&= \sum_{i=0}^{N}\left(\theta_i\frac{\partial F}{\partial X_i}(0)+\overline{\gamma}_{\bar{i}}\frac{\partial F}{\partial X_{\bar{i}}}(0)\right)\\
&\geq \sum_{i=0}^{N}\left((\theta_i+\overline{\gamma}_{\bar{i}})\min\left(\frac{\partial F}{\partial X_i}(0),\frac{\partial F}{\partial X_{\bar{i}}}(0)\right)\right)\\
&= \frac{\partial F}{\partial X_0}(0)+\sum_{i=1}^{N}\min\left(\frac{\partial F}{\partial X_i}(0),\frac{\partial F}{\partial X_{\bar{i}}}(0)\right)>0 \quad \text{(using (3.151))},
\end{aligned}$$

where we have used in the fourth line the fact that $\theta_i+\overline{\gamma}_{\bar{i}}=1$ for all $i=0,...,N$, which follows from the definition of θ_i and $\overline{\gamma}_{\bar{i}}$ and the fact that $\sigma_i=\sigma_{\bar{\bar{i}}}$.

Case 2 **:** $X\in Q_\Sigma$ **and** $X+aE\in Q_{\widehat{\Sigma}}$
This case is exactly the same as Case 2 of Step 3 in the proof of Proposition 9.4. However, in this case, we can choose

$$\eta=\frac{a}{2}(\Theta+\overline{\Gamma}).\nabla F(0)>0.$$

□

Here, we recall two comparison principle results on half lines that we will also use to prove that $c^+\geq 0$.

Proposition 9.9. (Comparison principle on $[-r^*,+\infty)$**)**
Let $F:[s,s']^{N+1}\to\mathbb{R}$ *satisfying* (A_{Lip}) *over* $[s,s']^{N+1}$ *and assume that :*

$$\left|\begin{array}{l}\text{there exists } \eta_0>0 \text{ such that if}\\ X=(X_0,...,X_N),\ X+(\alpha,...,\alpha)\in[s'-\eta_0,s']^{N+1}\\ \text{then } F(X+(\alpha,...,\alpha))<F(X) \text{ if } \alpha>0.\end{array}\right. \qquad (3.154)$$

Let $u,\ v:[-r^*,+\infty)\to[s,s']$ *be respectively a sub and a supersolution of*

$$cu'(x)=F((u(x+r_i))_{i=0,...,N}) \quad on \quad (0,+\infty) \qquad (3.155)$$

in sense of Definition 2.1. Moreover, assume that

$$v\geq s'-\eta_0 \quad on \quad [-r^*,+\infty),$$

and that

$$u\leq v \quad on \quad [-r^*,0].$$

Then

$$u\leq v \text{ on } [-r^*,+\infty).$$

Similarly, we have the following proposition on the half line $(-\infty, -r^*]$:

Proposition 9.10. (Comparison principle on $(-\infty, -r^*]$)
Let $F : [s, s']^{N+1} \to \mathbb{R}$ satisfying (A_{Lip}) over $[s, s']^{N+1}$ and assume that :

$$\left| \begin{array}{l} \textit{there exists } \eta_1 > 0 \textit{ such that if} \\ X = (X_0, ..., X_N),\ X + (\alpha, ..., \alpha) \in [s, s + \eta_1]^{N+1} \\ \textit{then } F(X + (\alpha, ..., \alpha)) < F(X) \textit{ if } \alpha > 0. \end{array} \right. \tag{3.156}$$

Let u, $v : (-\infty, r^*] \to [s, s']$ be respectively a sub and a supersolution of

$$cu'(x) = F((u(x + r_i))_{i=0,...,N}) \quad on \quad (-\infty, 0) \tag{3.157}$$

in sense of Definition 2.1. Moreover, assume that

$$u \leq s + \eta_1 \quad on \quad (-\infty, r^*],$$

and that

$$u \leq v \quad on \quad [0, r^*].$$

Then

$$u \leq v \text{ on } (-\infty, r^*].$$

For the proof of Proposition 9.9 and Proposition 9.10, we refer the reader for Theorem 4.1 and Corollary 4.2 in Chapter 2 which is done for F defined on $[0, 1]^{N+1}$ instead of $[s, s']^{N+1}$.

9.3 Harnack Inequality for the profile

We prove in this subsection a Harnack inequality (Proposition 9.14) for the profile that we use in Subsection 8.3 to show that $c^+ \geq c^*$. The proof will use a strong maximum principle for a linear evolution problem that we also prove in this subsection.

Proposition 9.11. (Strong maximum principle for a linear evolution problem)
Let F be a function satisfying (A_{Lip}) and differentiable at $\{0\}^{N+1}$. Assume that

$$\exists\ i_0 \in \{0, ..., N\} \quad such\ that \quad r_{i_0} > 0 \quad and \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0. \tag{3.158}$$

Let $T > 0$ and $u : \mathbb{R} \times [0, T) \to [0, +\infty)$ be a lower semi-continuous function which is a supersolution of

$$u_t(x, t) = \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0) u(x + r_i, t) \quad for \quad (x, t) \in \mathbb{R} \times (0, T). \tag{3.159}$$

If $u(x_0,t_0)=0$ *for some* $(x_0,t_0)\in\mathbb{R}\times(0,T)$, *then*

$$u(x_0+kr_{i_0},t)=0 \quad \textit{for all} \;\; k\in\mathbb{N} \;\; \textit{and} \;\; 0\le t\le t_0.$$

Proof of Proposition 9.11
Let u be a supersolution of (3.159) such that $u\ge 0$ and assume that there exists some $(x_0,t_0)\in\mathbb{R}\times(0,T)$ such that $u(x_0,t_0)=0$.

Step 1 **:** $u(x_0,s)=0$ **for all** $0\le s\le t_0$

Step 1.1 **: preliminary**
Since u is a supersolution of (3.159) on $\mathbb{R}\times(0,T)$, then u satisfies in the viscosity sense

$$u_t(x,t)\ge\sum_{i=0}^{N}\frac{\partial F}{\partial X_i}(0,...,0)u(x+r_i,t) \quad \text{for all} \;\; (x,t)\in\mathbb{R}\times(0,T).$$

Because

$$\frac{\partial F}{\partial X_i}(0,...,0)\ge 0 \quad \text{for all} \quad i\ne 0 \tag{3.160}$$

and $\left|\frac{\partial F}{\partial X_0}(0,...,0)\right|\le L$, where L is the Lipschitz constant of F, we get in the viscosity sense (using $u\ge 0$) :

$$u_t(x,t)\ge -Lu(x,t) \quad \text{for all} \;\; (x,t)\in\mathbb{R}\times(0,T). \tag{3.161}$$

Step 1.2 **:** $u(x_0,\cdot)$ **is a viscosity supersolution of** (3.161) **on** $(0,T)$
We now set $v(t)=u(x_0,t)$. We claim that v satisfies in the viscosity sense

$$v_t\ge -Lv \quad \text{on} \quad (0,T). \tag{3.162}$$

In order to prove our claim, let ϕ be a test function such that

$$\begin{cases} v_*\ge\phi \quad \text{on} \quad (0,T) \\ v_*(t_0)=\phi(t_0) \quad \text{for some } t_0\in(0,T) \\ v_*>\phi \quad \text{for all} \quad t\ne t_0. \end{cases} \tag{3.163}$$

For every $\varepsilon>0$, define the function

$$\psi_\varepsilon(x,t):=\phi(t)-\frac{1}{\varepsilon}(x-x_0)^2.$$

Then

$$\psi_\varepsilon(x_0,t_0)=\phi(t_0)=v_*(t_0)=u_*(x_0,t_0).$$

Using the definition of ψ_ε and (3.163), we deduce that for any $r_\varepsilon > 0$ small enough such that $[t_0 - r_\varepsilon, t_0 + r_\varepsilon] \subset (0, T)$, we have

$$\begin{cases} \psi_\varepsilon(x_0 \pm r_\varepsilon, t) = \phi(t) - \dfrac{r_\varepsilon^2}{\varepsilon} \leq v_*(t) - \dfrac{r_\varepsilon^2}{\varepsilon} = u_*(x_0, t) - \dfrac{r_\varepsilon^2}{\varepsilon} < u_*(x_0, t) \\ \psi_\varepsilon(x, t_0 \pm r_\varepsilon) = \phi(t_0 \pm r_\varepsilon) - \dfrac{1}{\varepsilon}(x - x_0)^2 < v_*(t_0 \pm r_\varepsilon) = u_*(x_0, t_0 \pm r_\varepsilon) \end{cases}$$

Therefore, since u_* is lower semi-continuous, then for every $\varepsilon > 0$ there exists $c_\varepsilon \geq 0$ such that

$$\begin{aligned} \psi_\varepsilon - c_\varepsilon \leq u_* \quad &\text{on} \quad (x_0 - r_\varepsilon, x_0 + r_\varepsilon) \times (t_0 - r_\varepsilon, t_0 + r_\varepsilon) \\ = \ &\text{at } P_\varepsilon = (x_\varepsilon, t_\varepsilon) \in (x_0 - r_\varepsilon, x_0 + r_\varepsilon) \times (t_0 - r_\varepsilon, t_0 + r_\varepsilon), \end{aligned}$$

with $P_\varepsilon = (x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$ when $\varepsilon \to 0$ and $r_\varepsilon \to 0$.

Now, since u satisfies (3.161) in the viscosity sense and $\psi_\varepsilon - c_\varepsilon$ is a test function, then we deduce that

$$\phi_t(t_\varepsilon) = (\psi_\varepsilon)_t(P_\varepsilon) \geq -Lu_*(P_\varepsilon). \tag{3.164}$$

This implies that

$$\phi_t(t_0) \geq -L \liminf_{\varepsilon \to 0} u_*(P_\varepsilon) = -Lu_*(x_0, t_0) = -Lv_*(t_0).$$

Thus v satisfies (3.162) in the viscosity sense and hence $u(x_0, \cdot)$ satisfies (3.161) on $(0, T)$ in the viscosity sense.

Step 1.3 **: conclusion**
Let $0 \leq s_0 < t_0$ and set $w(t) = e^{-L(t-s_0)} v_*(s_0)$ which is a solution of $w_t = -Lw$. Because $v_*(s_0) \geq w^*(s_0)$, we deduce from the comparison principle that

$$v(t) \geq w(t) \quad \text{on} \quad [s_0, T). \tag{3.165}$$

In particular, evaluating (3.165) at $t = t_0$, we get

$$0 = v(t_0) \geq e^{-L(t_0 - s_0)} v_*(s_0),$$

which implies that

$$0 \geq v_*(s_0) = v(s_0) = u(x_0, s_0),$$

and this is true for any $s_0 \in [0, t_0]$. Because $u \geq 0$, we deduce that

$$u(x_0, s) = 0 \quad \text{for all} \quad 0 \leq s \leq t_0.$$

Step 2 **:** $u(x_0 + r_{i_0}, t_0) = 0$
Note that for the test function $\phi \equiv 0$, we have

$$\begin{cases} u(x, t) \geq \phi(x, t) & \text{for all} \quad (x, t) \in \mathbb{R} \times (0, T) \\ u(x_0, t_0) = \phi(x_0, t_0) & \text{for} \quad (x_0, t_0) \in \mathbb{R} \times (0, T). \end{cases}$$

Therefore, the supersolution viscosity inequality implies that

$$\begin{aligned} 0 = \phi_t(x_0,t_0) &\geq \sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0,...,0)u(x_0+r_i,t_0) \\ &\geq \frac{\partial F}{\partial X_0}(0,...,0)u(x_0,t_0) + \frac{\partial F}{\partial X_{i_0}}(0,...,0)u(x_0+r_{i_0},t_0), \end{aligned}$$

where we have used (3.160) and the fact that $u \geq 0$. Because $u(x_0,t_0) = 0$, we conclude that

$$0 \geq \frac{\partial F}{\partial X_{i_0}}(0,...,0)u(x_0+r_{i_0},t_0).$$

By assumption (3.158), we recall that $\frac{\partial F}{\partial X_{i_0}}(0,...,0) > 0$. Therefore, since $u \geq 0$, we deduce that

$$u(x_0+r_{i_0},t_0) = 0.$$

Step 3 **:** $u(x_0+kr_{i_0},s) = 0$ **for** $k \in \mathbb{N}$ **and** $0 \leq s \leq t_0$
Since $u(x_0+r_{i_0},t_0) = 0$, then by Step 2, we deduce that $u(x_0+kr_{i_0},t_0) = 0$ for $k \in \mathbb{N}$. Using Step 1, we get that $u(x_0+kr_{i_0},s) = 0$ for all $0 \leq s \leq t_0$ and $k \in \mathbb{N}$. □

Now, we give a lower bound for a solution of the nonlinear problem.

Lemma 9.12. (Existence of a solution for the nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$, (P_{Lip}) and let $\varepsilon \in (0,1]$. Then there exists $\psi : \mathbb{R}\times(0,+\infty) \to \mathbb{R}$ a viscosity solution of

$$\psi_t(x,t) = F((\psi(x+r_i,t))_{i=0,...,N}) \quad on \quad \mathbb{R}\times(0,+\infty) \tag{3.166}$$

with initial condition satisfying

$$\psi^*(\cdot,0) = \varepsilon H^* \quad and \quad \psi_*(\cdot,0) = \varepsilon H_*, \tag{3.167}$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function.

Proof of Lemma 9.12
The proof is done in steps.

Step 1 **: construction of** ψ_δ **solution of** (3.166)
Let $\delta > 0$ and define

$$H_\delta = \begin{cases} 0 & \text{if} \quad x \leq -\delta \\ \dfrac{x}{\delta}+1 & \text{if} \quad x \in [-\delta,0] \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

Then for every $x \in \mathbb{R}$, we have $H_\delta(x)$ is non-increasing as δ decreases to zero and we also have

$$H_\delta(x) \geq H(x).$$

Since for any given $\delta > 0$, the function H_δ is bounded uniformly continuous, then using [53, Corollary 2.9], we deduce that for every $\delta > 0$, there exists a unique continuous solution ψ_δ of (3.166) satisfying

$$\psi_\delta(x, 0) = \varepsilon H_\delta(x). \tag{3.168}$$

Step 2 : properties of ψ_δ
Since $H_\delta(x)$ is non-increasing when δ decreases to zero and $H_\delta(x) \geq 0$, then using the comparison principle (see [53, Proposition 2.5]), we deduce that ψ_δ is non-increasing as δ decreases to zero and $\psi_\delta(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0, +\infty)$.

Moreover, since $H_\delta(x+h) \geq H_\delta(x)$ for every $h \geq 0$ and $\delta > 0$ fixed, then by comparison principle ([53, Proposition 2.5]), we deduce that

$$\psi_\delta(x+h, t) \geq \psi_\delta(x,t),$$

i.e. ψ_δ is non-decreasing w.r.t. x. Also, since 0 and 1 are two solutions of (3.166) and $0 \leq \varepsilon H_\delta \leq 1$, then from the comparison principle we get that

$$0 \leq \psi_\delta \leq 1.$$

Now, let $C_0 = \sup_{[0,1]^{N+1}} |F|$ and for $h \geq 0$, we set $\psi_\delta^\pm(x,t) := \psi_\delta(x,h) \pm C_0 t$ for $t \geq 0$. Then ψ_δ^+ is a supersolution and ψ_δ^- is a subsolution of (3.166) with

$$\psi_\delta^-(x,0) \leq \psi_\delta(x,h) \leq \psi_\delta^+(x,0).$$

Hence, using the comparison principle, we get for all $t \geq 0$

$$\psi_\delta^-(x,t) \leq \psi_\delta(x,h+t) \leq \psi_\delta^+(x,t), \tag{3.169}$$

i.e.

$$\psi_\delta(x,h) - C_0 t \leq \psi_\delta(x,h+t) \leq \psi_\delta(x,h) + C_0 t.$$

Because this true for any $t,\ h \geq 0$, we deduce that

$$|\psi_\delta(x,t) - \psi_\delta(x,s)| \leq C_0 |t-s| \quad \text{for all} \quad x \in \mathbb{R},\ t, s \in [0, +\infty). \tag{3.170}$$

Step 3 : the limit $\delta \to 0$
Since ψ_δ is non-increasing as δ decreases to zero and $\psi_\delta(x,t) \geq 0$ for all $(x,t) \in \mathbb{R} \times (0, +\infty)$. Then ψ_δ^+ converges pointwisely to some function $\psi \geq 0$, as $\delta \to 0$.

Using the stability of viscosity solutions (Proposition 2.2 (ii), applied for $\sup -\psi_\delta$), we deduce that ψ_* is a supersolution of (3.166). Moreover, since ψ_δ is non-decreasing w.r.t. x and satisfies (3.170), then

$$\begin{cases} \psi \quad \text{is non-decreasing w.r.t.} \quad x \\ |\psi(x,t) - \psi(x,s)| \le C_0|t-s| \quad \text{for all} \quad x \in \mathbb{R},\ t,s \in [0,+\infty). \end{cases}$$

This implies that

$$\psi^* = \limsup_{\delta \to 0}{}^* \psi_\delta.$$

Hence, using Proposition 2.2 (i), we deduce that ψ^* is a subsolution of (3.166). Therefore, ψ solves (3.166) in the viscosity sense.

In addition, since $H_\eta(x) \ge H(x) \ge H_\delta(x-\delta)$, for every $\eta,\ \delta > 0$, then

$$\psi_\eta(x,t) \ge \psi_\delta(x-\delta,t) \quad \text{for every} \quad \eta,\ \delta > 0.$$

Passing to the limit $\eta \to 0$, we obtain

$$\psi(x,t) \ge \psi_\delta(x-\delta,t) \quad \text{for every} \quad \delta > 0,$$

this implies that for every $\delta > 0$, we have

$$\psi_\delta(x,t) \ge \psi(x,t) \ge \psi_\delta(x-\delta,t) \quad \text{for every } (x,t) \in \mathbb{R} \times [0,+\infty). \tag{3.171}$$

Moreover, we have $\psi_\delta \in C^0$ and

$$\begin{cases} \psi_\delta(x,0) = 0 = \psi_\delta(x-\delta,0) & \text{for} \quad x \le -\delta \\ \psi_\delta(x,0) = \varepsilon = \psi_\delta(x-\delta,0) & \text{for} \quad x \ge \delta. \end{cases}$$

Hence, for every $\delta > 0$, we get

$$\psi^*(x,0) = \psi_*(x,0) = \begin{cases} 0 & \text{for} \quad x \le -\delta \\ \varepsilon & \text{for} \quad x \ge \delta. \end{cases}$$

Therefore, we obtain that

$$\psi^*(x,0) = \psi_*(x,0) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \varepsilon & \text{for} \quad x > 0. \end{cases}$$

Using again (3.171), we get for $(x,t) = (0,0)$ that

$$\varepsilon \ge \psi^*(0,0) \ge \psi_*(0,0) \ge 0.$$

Finally, since ψ^* is upper semi-continuous and ψ_* is lower semi-continuous, we deduce that

$$\psi^*(x,0) = \varepsilon H^*(x) \quad \text{and} \quad \psi_*(x,0) = \varepsilon H_*(x).$$

□

Proposition 9.13. (Lower bound on a solution of the evolution nonlinear problem)
Consider a function F satisfying $(\tilde{A}_{Lip})$ and (P_{Lip}). Assume moreover that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$ and

$$\exists\ i_0 \in \{0, ..., N\} \quad \text{such that} \quad r_{i_0} > 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0. \tag{3.172}$$

Then there exists $\varepsilon_0 \in (0,1]$ and $T_0 > 0$ such that for all $\delta \in (0, T_0)$ and $R > 0$, there exists $\kappa = \kappa(\delta, R) > 0$ such that for every $0 < \varepsilon \leq \varepsilon_0$, the function $\psi = \psi_\varepsilon$ given by Lemma 9.12 with initial conditions (3.167) *satisfies*

$$\psi_\varepsilon(x,t) \geq \kappa\varepsilon \quad \text{for all} \quad (x,t) \in [-R,R] \times [\delta, T_0]. \tag{3.173}$$

Proof of Proposition 9.13
We first give an upper bound on the solution ψ of (3.166) and then we prove Proposition 9.13 by contradiction.

Step 0 : upper bound on ψ on $(0, 2T_0)$
Let

$$M(t) := \sup_{x \in \mathbb{R}} \psi(x,t).$$

Then $M(0) = \varepsilon$ (since $\psi^*(x,0) = \varepsilon H^*(x)$). Since ψ is a solution of (3.166) then, using the viscosity techniques, we can show that M^* is a subsolution, i.e. satisfies in the viscosity sense

$$v_t(t) \leq F(M^*(t), ..., M^*(t)) = f(M^*(t)).$$

Using the comparison principle for the ODE $x' = f(x)$, we deduce that

$$M^*(t) \leq M_0(t) \quad \text{over} \quad [0, \infty), \tag{3.174}$$

where M_0 is a solution of

$$\begin{cases} M_0'(t) = f(M_0(t)) \geq 0 & \text{for} \quad (0, +\infty) \\ M_0(0) = \varepsilon. \end{cases}$$

Now, because M_0 is non-decreasing, if $M_0(t) \leq 2\varepsilon$ then

$$M_0'(t) \leq \sup_{[0,2\varepsilon]} f \leq 2L_1\varepsilon,$$

where L_1 is the Lipschitz constant of f (because $f(0) = 0$). Thus we get

$$M_0(t) \leq \varepsilon + 2tL_1\varepsilon < 2\varepsilon \quad \text{if} \quad t < \frac{1}{2L_1}.$$

Therefore for

$$T_0 = \frac{1}{4L_1}, \tag{3.175}$$

we get $M^*(t) \leq M_0(t) \leq 2\varepsilon$ on $[0, 2T_0]$, which implies that $\psi_\varepsilon = \psi$ satisfies

$$\psi_\varepsilon(x,t) \leq 2\varepsilon \quad \text{for} \quad (x,t) \in \mathbb{R} \times [0, 2T_0]. \tag{3.176}$$

Step 1 **: establishing** (3.173)
Assume to that contrary that (3.173) is false. Then there exist $\delta \in (0, T_0)$ (with T_0 given in (3.175)), $R > 0$ and two sequences $\varepsilon_n \to 0$, $\kappa_n \to 0$ as $n \to +\infty$ and points

$$P_n = (x_n, t_n) \in [-R, R] \times [\delta, T_0] \tag{3.177}$$

such that

$$\psi_{\varepsilon_n}(P_n) \leq \kappa_n \varepsilon_n.$$

Define

$$\overline{\psi}_n(x,t) := \frac{1}{\varepsilon_n} \psi_{\varepsilon_n}(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0, 2T_0).$$

Then we have (using (3.176)),

$$\begin{cases} 0 \leq \overline{\psi}_n \leq 2 \quad \text{over} \quad \mathbb{R} \times [0, 2T_0) \\ \overline{\psi}_n(P_n) \leq \kappa_n \to 0 \\ (\overline{\psi}_n)_*(x, t=0) = H_*(x) \end{cases}$$

and

$$(\overline{\psi}_n)_t(x,t) = \frac{1}{\varepsilon_n} F(\varepsilon_n(\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}). \tag{3.178}$$

Step 1.1 **: uniform lower bound of** $\overline{\psi}_n$
Denote by $Z = (\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}$. Since F is C^1 over a neighborhood of $\{0\}^{N+1}$, then for ε_n small enough, we can show that

$$\begin{aligned} (\overline{\psi}_n)_t(x,t) &= \frac{1}{\varepsilon_n} F(\varepsilon_n(\overline{\psi}_n(x + r_i, t))_{i=0,\dots,N}) \\ &= \int_0^1 \frac{\partial F}{\partial X_0}(s\varepsilon_n Z)\overline{\psi}_n(x,t)ds + \sum_{i=1}^{N} \int_0^1 \frac{\partial F}{\partial X_i}(s\varepsilon_n Z)\overline{\psi}_n(x + r_i, t)ds \\ &\geq -L\overline{\psi}_n(x,t) + \frac{1}{2}\frac{\partial F}{\partial X_{i_0}}(0, \dots, 0)\overline{\psi}_n(x + r_{i_0}, t), \end{aligned}$$

where we have used the fact that $\overline{\psi}_n \geq 0$ and $\frac{\partial F}{\partial X_i} \geq 0$ for all $i \neq 0$. Hence $\overline{\psi}_n$ is a supersolution of the equation

$$w_t(x,t) = -Lw(x,t) + \frac{1}{2}\frac{\partial F}{\partial X_{i_0}}(0, \dots, 0)w(x + r_{i_0}, t). \tag{3.179}$$

Now, let

$$H_\eta(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \dfrac{1}{\eta} x & \text{if} \quad 0 \le x \le \eta \\ 1 & \text{if} \quad x \ge 1 \end{cases}$$

for $\eta > 0$ small. Since $\frac{\partial F}{\partial X_{i_0}}(0, ..., 0) \ge 0$, then by a simple calculation, we can show that the function

$$\phi(x,t) = e^{-Lt} H_\eta(x)$$

(with L the Lipschitz constant of F) is a subsolution of (3.179). Moreover, we have

$$(\overline{\psi}_n)_*(x, t = 0) = H_*(x) \ge H_\eta(x) = \phi(x, t = 0).$$

Therefore, using a comparison principle for (3.179), we deduce that

$$e^{-Lt} H_\eta(x) \le \overline{\psi}_n(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0, 2T_0). \tag{3.180}$$

Step 1.2 : passing to the limit and getting a contradiction
Since $\overline{\psi}_n(x,t)$ is uniformly bounded on $\mathbb{R} \times [0, 2T_0)$ and

$$(\overline{\psi}_n)_t(x,t) \ge \sum_{i=0}^{N} \int_0^1 \frac{\partial F}{\partial X_i}(s\varepsilon_n Z)\overline{\psi}_n(x + r_i, t) ds,$$

then using the fact that F is C^1 over a neighborhood of $\{0\}^{N+1}$ and $\varepsilon_n \to 0$, we deduce that $\overline{\psi}_\infty = \liminf_{n \to +\infty} {}_* \overline{\psi}_n$ satisfies in the viscosity sense on $\mathbb{R} \times [0, 2T_0)$

$$\begin{cases} (\overline{\psi}_\infty)_t(x,t) \ge \displaystyle\sum_{i=0}^{N} \frac{\partial F}{\partial X_i}(0, ..., 0)\overline{\psi}_\infty(x + r_i, t) \\ 0 \le \overline{\psi}_\infty \le 2 \end{cases}$$

and

$$e^{-Lt} H_\eta(x) \le \overline{\psi}_\infty(x,t) \quad \text{for all} \quad (x,t) \in \mathbb{R} \times [0, 2T_0). \tag{3.181}$$

In addition, we also have $P_n \to P_\infty = (x_\infty, t_\infty)$ in $[-R, R] \times [\delta, T_0]$, hence using the fact that $\overline{\psi}_n(P_n) \to 0$, we get

$$\overline{\psi}_\infty(P_\infty) = 0.$$

Using the strong maximum principle (Proposition 9.11) that holds for supersolutions, we deduce for $k \in \mathbb{N}$ that

$$\overline{\psi}_\infty(x_\infty + kr_{i_0}, t) = 0 \quad \text{for all} \quad 0 \le t \le t_\infty.$$

But $r_{i_0} > 0$, hence for $t = 0$, $k >> 1$ and using (3.181), we get

$$1 = H_\eta(x_\infty + kr_{i_0}) \leq \overline{\psi}_\infty(x_\infty + kr_{i_0}, 0) = 0,$$

which is a contradiction. □

In the following proposition, we give a Harnack type inequality.

Proposition 9.14. (Harnack inequality)
Let F be a function satisfying (A_{Lip}), (P_{Lip}) and assume that F is C^1 over a neighborhood of $\{0\}^{N+1}$ in $[0,1]^{N+1}$. Assume moreover that

$$\exists\ i_0 \in \{0, ..., N\} \quad \text{such that} \quad r_{i_0} > 0 \quad \text{and} \quad \frac{\partial F}{\partial X_{i_0}}(0, ..., 0) > 0. \tag{3.182}$$

Let (c, u) with $c \neq 0$ be a solution of

$$\begin{cases} cu'(x) = F((u(x + r_i))_{i=0,...,N}) & \text{on} \quad \mathbb{R} \\ u' \geq 0 \\ u(-\infty) = 0 \quad \text{and} \quad u(+\infty) = 1. \end{cases} \tag{3.183}$$

Then for every $\rho > 0$ there exists a constant $\overline{\kappa}_1 = \overline{\kappa}_1(\rho) > 1$ such that for every $x \in \mathbb{R}$, we have

$$\sup_{B_\rho(x)} u \leq \overline{\kappa}_1 \inf_{B_\rho(x)} u. \tag{3.184}$$

Moreover, there exists $\overline{\kappa}_0 > 1$ such that

$$u(x + r^*) \leq \overline{\kappa}_0 u(x), \tag{3.185}$$

where $r^* = \max_{i=0,...,N} |r_i|$.

Proof of Proposition 9.14
Let $\tilde{F}$ be the extension of F on $\mathbb{R}^{N+1}$ given by Lemma 7.1. Then define the function

$$\overline{u}(x, t) = u(x + ct),$$

where $u \in C^1$, because $c \neq 0$. Thus $\overline{u}$ satisfies

$$\overline{u}_t(x, t) = \tilde{F}((\overline{u}(x + r_i, t))_{i=0,...,N}) \quad \text{for all} \quad (x, t) \in \mathbb{R} \times (0, +\infty) \tag{3.186}$$

and

$$\overline{u}(x, 0) = u(x). \tag{3.187}$$

Let $x_0 \in \mathbb{R}$ such that $1 \geq u(x_0) > 0$. Since u is non-decreasing, then for all $x \in \mathbb{R}$ we have

$$\overline{u}(x, 0) \geq u(x_0) H(x - x_0), \tag{3.188}$$

where $H = 1_{[0,+\infty)}$ is the Heaviside function.

For $\varepsilon \in (0,1]$ that will be fixed later, let $\psi_\varepsilon = \psi$ be the solution given by Lemma 9.12 with initial conditions (3.167) and let

$$\overline{v}(x,t) = \psi_\varepsilon(x - x_0, t).$$

Now, using Proposition 9.13, we deduce that there exists some $\varepsilon_0 \in (0,1]$ and T_0 such that for all $\delta \in (0, T_0)$ and $R > 0$ there exists a constant $\kappa = \kappa(\delta, R) > 0$ such that if $\varepsilon \le \varepsilon_0$, then

$$\overline{v}(x,t) \ge \varepsilon\kappa \quad \text{for all} \quad (x,t) \in [x_0 - R, x_0 + R] \times [\delta, T_0]. \tag{3.189}$$

We now choose

$$\varepsilon = \min(\varepsilon_0, u(x_0)).$$

In particular, we have

$$\overline{u}(x,0) \ge \overline{v}^*(x,0) \quad \text{for all} \quad x \in \mathbb{R}.$$

Using the comparison principle (see [53, Proposition 2.5]), we deduce that

$$\overline{u} \ge \overline{v} \quad \text{for all} \quad (x,t) \in \mathbb{R} \times (0, +\infty). \tag{3.190}$$

From (3.189), we deduce that

$$\overline{u} \ge \kappa_1 u(x_0) \quad \text{on} \quad [x_0 - R, x_0 + R] \times [\delta, T_0], \tag{3.191}$$

with $\kappa_1 = \varepsilon_0 \kappa$ (using $\varepsilon \in (0,1]$, $u(x_0) \in (0,1]$ and the definition of ε). Because $\overline{u}(x,t) = u(x + ct)$, we conclude that

$$\inf_{(x,t)\in[x_0-R,x_0+R]\times[\delta,T_0]} u(x+ct) \ge \kappa_1 u(x_0).$$

Now, for any $r > 0$, we can find $R_r > 0$ large enough such that $B_r(x_0) \subset \overline{B}_{R_r}(x_0) + c[\delta, T_0]$. Therefore, since u is continuous and non-decreasing, then

$$u(x_0 - r) = \inf_{x \in B_r(x_0)} u(x) \ge \inf_{(x,t)\in[x_0-R,x_0+R]\times[\delta,T_0]} u(x+ct) \ge \kappa_1 u(x_0) \tag{3.192}$$

with $\kappa_1 = \kappa_1(r)$.

Let $\rho = \frac{r}{2}$ and choose y_0 such that $B_\rho(y_0) = (x_0 - r, x_0)$, i.e. $y_0 - \rho = x_0 - r$ and $y_0 + \rho = x_0$. Thus, using again the fact that u is non-decreasing, we get

$$\sup_{B_\rho(y_0)} u = u(y_0 + \rho) = u(x_0)$$

and

$$u(x_0 - r) = u(y_0 - \rho) = \inf_{B_\rho(y_0)} u.$$

Therefore, we deduce from (3.192) that

$$\sup_{B_\rho(y_0)} u \leq \overline{\kappa}_1 \inf_{B_\rho(y_0)} u \quad \text{with} \quad \overline{\kappa}_1 = \frac{1}{\kappa_1}. \tag{3.193}$$

Using (3.193) for $2\rho \geq r^*$ and $\overline{\kappa}_0 = \overline{\kappa}_1(r^*) = \big(\varepsilon_0 \kappa(\delta, R_{r^*})\big)^{-1}$, setting $z_0 = y_0 - \rho$ and using the monotonicity of u, we get

$$u(z_0+r^*) \leq u(z_0+2\rho) = u(y_0+\rho) = \sup_{B_\rho(y_0)} u \leq \overline{\kappa}_0 \Big(\inf_{B_\rho(y_0)} u \Big) = \overline{\kappa}_0 u(y_0-\rho) = \overline{\kappa}_0 u(z_0). \tag{3.194}$$

Finally, since x_0 is chosen arbitrary at the beginning of the reasoning, we deduce that (3.193) and (3.194) do hold for any y_0, z_0. This shows (3.184) and (3.185), and ends the proof. □

Chapitre 4

Convergence vers des murs de dislocations dans le cas périodique

Ce chapitre est un travail en collaboration avec Ł. Paszkowski [3].
Dans ce chapitre, nous nous intéressons à la convergence de l'accumulation des dislocations vers les murs de dislocations. Nous considérons le système dynamique généré par la force $f(x,y) = \frac{x(y^2-x^2)}{(y^2+x^2)^2}$, définie sur $\mathbb{R} \times \mathbb{Z}\backslash\{0\}$, qui décrit les phénomènes. Pour les données initiales $X^0 \in \Omega \cap \ell^\infty = \left\{X : |x_i - x_j| \leq \sqrt{3-2\sqrt{2}}\,|i-j|\right\} \cap \ell^\infty$, nous montrons l'existence d'une solution unique $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$. De plus, nous montrons que si X^0 est périodique, alors $X(t) = (x_j(t))_{j\in\mathbb{Z}}$ est périodique pour tout $t > 0$ et converge vers le barycentre des données initiales, c'est à dire $x_j(t) \to c = \frac{1}{N}\sum_{i=1}^N x_i^0$ pour chaque $j \in \mathbb{Z}$. Nous établissons également une ℓ^p contraction des solutions périodiques et effectuons des simulations numériques.

Convergence to walls of dislocations in the periodic case

M. Al Haj, Ł. Paszkowski

Abstract

In this paper we are interested in the convergence of accumulation of dislocations to walls of dislocations. We consider the dynamical system generated by the force $f(x,y) = \frac{x(y^2-x^2)}{(y^2+x^2)^2}$, defined over $\mathbb{R} \times \mathbb{Z}\backslash\{0\}$, that describes the phenomena. For initial data $X^0 \in \Omega \cap \ell^\infty = \left\{X : |x_i - x_j| \leq \sqrt{3-2\sqrt{2}}\,|i-j|\right\} \cap \ell^\infty$, we show the existence of unique solution $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$. Moreover, we prove that if X^0 is periodic, then $X(t) = (x_j(t))_{j\in\mathbb{Z}}$ is periodic for any $t > 0$ and converges to the barycenter of the initial data, i.e. $x_j(t) \to c = \frac{1}{N}\sum_{i=1}^N x_i^0$ for every $j \in \mathbb{Z}$. We also establish a ℓ^p contraction for periodic solutions and perform numerical simulations.

Keywords : Dynamical system, Cauchy Lipschitz theorem, comparison principle, periodic solution, viscosity solutions.

1 Introduction

It is well known, in real materials with dislocations, that we can observe several accumulation of dislocations in walls of dislocations or more general in cells with several walls. In this paper our aim is to investigate the dynamics of dislocations that interact together and converge to such walls of dislocations.

1.1 Presenting the problem

Let us consider a model describing horizontal motion of dislocation lines parallel to the z-axis. Considering the cross section of these lines, we can reduce the problem to its two-dimensional counterpart where each dislocation line is represented by its position $(x_i(t), i) \in \mathbb{R} \times \mathbb{Z}$. Finally, such horizontal evolution can be characterized as follows

$$x_i' = \sum_{j\neq i} f(x_j - x_i, j-i) \quad \text{for} \quad i \in \mathbb{Z}. \tag{4.1}$$

Here $f : \mathbb{R}\times\mathbb{Z}\backslash\{0\} \to \mathbb{R}$ is an anisotropic force of two-body interactions. An example of such a force, according to [41], is

$$f(x,y) = \frac{x(y^2-x^2)}{(y^2+x^2)^2}. \tag{4.2}$$

An important aspect of interatomic interactions is that atoms can attract each other at longer distances and repel at short distances aggregating into various bulk forms. Such behaviour, of course, depends on the form of the considered potentials.

One of the forces describing both long-range attraction and short-range repulsion between atoms is the interaction force given by (4.2). In such an example two particles attract each other if the vertical angle between them is less then $\frac{\pi}{4}$ and, on the other hand, repel each other if the angle is greater then $\frac{\pi}{4}$, see Figure 4.1 and Figure 4.2.

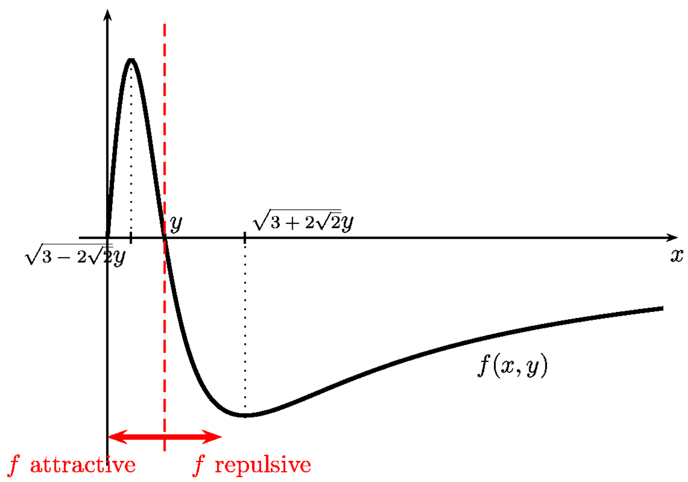

FIGURE 4.1 – Interaction force $f(x,y)$ as a function of the distance between two atoms for some fixed $y \in \mathbb{Z}\backslash\{0\}$ with the property $f(-x,y) = -f(x,y)$. A vertical angle between two particles corresponds to $\arctan(\frac{x}{y})$. Thus $\frac{\pi}{4}$ reads as $x = |y|$.

In the literature, however, there is a convention to express force in terms of energy potentials or commonly called *interatomic potentials*. Thus a general force acting on an atom can be seen as the negative derivative of some potential function with respect to its position : $f(r) = -\phi'(r)$.

The system of all particles acting together under the force defined in (4.2) can be rewritten in the following way

$$\begin{cases} \frac{d}{dt}X(t) = F(X(t)), \quad t > 0, \\ X(0) = X^0, \end{cases} \tag{4.3}$$

where $X(t) = (x_i(t))_{i\in\mathbb{Z}}, F(X) = (F_i(X))_{i\in\mathbb{Z}}$ and X^0 is some given initial position of dislocations. Moreover, $F_i(X)$ describes a resultant force acting on an i-th particle, *i.e.*

$$F_i(X) \stackrel{def}{=} \sum_{j\neq i} f(x_j - x_i, j - i) \quad \text{for each} \quad i \in \mathbb{Z}.$$

Since our aim is to study a long time behaviour of the dynamics of particles which converges to walls of dislocations, the property of the force f described in (4.2) forces us to consider the problem (4.3) with a special condition for the initial data. Namely, we assume

$$f(x,y) = \frac{x(y^2 - x^2)}{(y^2 + x^2)^2}, \tag{4.4a}$$

$$X^0 \in \Omega \cap \ell^\infty, \tag{4.4b}$$

where

$$\Omega = \left\{ X : |x_i - x_j| \leq \sqrt{3 - 2\sqrt{2}}\,|i - j| \right\}. \tag{4.5}$$

and $\ell^\infty = \ell^\infty(\mathbb{R})$ is the Banach space of all bounded sequences over $\mathbb{R}$ supplemented with the norm $\|\cdot\|_\infty = \sup_{n\in\mathbb{Z}} |x_n|$.

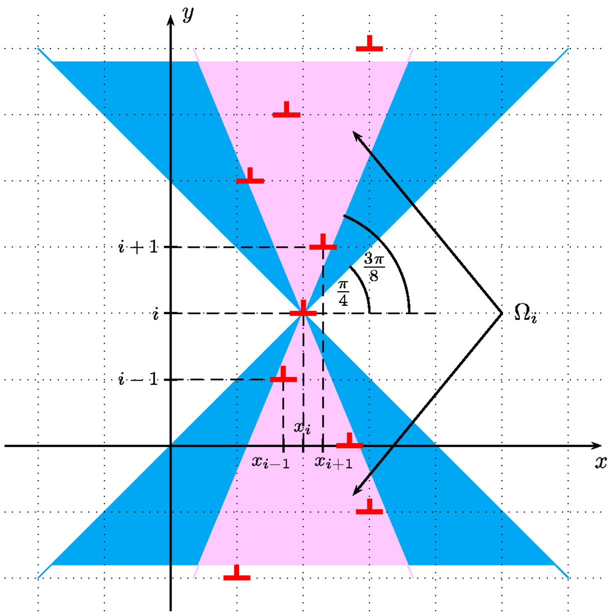

FIGURE 4.2 – A fixed particle x_i attracts all other particles if they are placed in a region marked in blue and pink. However, the force f is non-decreasing only if the particles are located in the region marked in pink. Such domain we call Ω_i and thus we can present Ω, defined in (4.5), as $\Omega = \cap_{i\in\mathbb{Z}}\Omega_i$.

Remark 1.1. (Sign of f when $X \in \Omega$)
If $X(t) = (x_i(t))_{i\in\mathbb{Z}} \in \Omega$, then, in particular, we have $(j-i)^2 \geq (x_j - x_i)^2$. This implies that if $X \in \Omega$, then (because of (4.2)) $f(x_j - x_i, j-i)$ has the sign as $x_j - x_i$.

Notice here that $\arctan\left(\sqrt{3-2\sqrt{2}}\right) = \frac{\pi}{8}$ guarantees that the force f restricted to Ω is not only attractive but also non-decreasing with respect to the first variable. Therefore, we are able to prove a comparison principle, which helps us to conclude *e.g.* existence of global-in-time solutions, which stays in Ω.

1.2 Main results

Our first result deals with the existence of solutions to the considered problem. More precisely, it reads as follows

Theorem 1.2. (Existence of a unique solution)
Let (4.4) hold. Then there exists a unique solution $X \in C^1([0,+\infty), \Omega \cap \ell^\infty)$ of the Cauchy problem (4.3). Moreover, if the initial data X^0 is N-periodic (i.e. $x_i^0 = x_{i+N}^0$, for every $i \in \mathbb{Z}$), then the solution remains N-periodic for every time $t > 0$.

The proof of the theorem consists in the application of the classical Cauchy-Lipschitz theorem and the comparison principle result. Notice that in general the locally Lipschitz condition with respect to the first variable of the function f is sufficient to obtain a unique local-in-time solution. In order to extend it to the global-in-time one we need to provide an apriori estimate, *e.g.* the comparison principle that ensures us that ℓ^∞-norm of the solution does not blow up.

However, if the function f satisfies the Lipschitz condition globally, which happens when f is defined by (4.4a), we immediately obtain a unique global-in-time solution by extending it with the universal step $T > 0$, see [26, Thm 7.3, p. 184]. Thus, in that case the comparison principle is needed only to ensure that the solution belongs to Ω for all times $t > 0$.

To prove the comparison principle for the problem (4.3), the monotonicity of a function f is a necessary assumption. Hence, the reason why we consider the initial condition in the special domain Ω is that the function f defined by (4.4a) is indeed monotone over that set.

Our second result is the long time behaviour of the dynamics of particles where we prove that dislocations accumulate creating so-called walls of dislocations. This result can be stated in the following way

Theorem 1.3. (Convergence to flat walls)
Let X be the N-periodic solution of the problem (4.3)-(4.4). Then it converges to a constant stationary solution of the problem (4.3)-(4.4) i.e. for every $i \in \mathbb{Z}$, we have $\lim_{t\to\infty} x_i(t) = c$, where $c = \frac{1}{N}\sum_{i=1}^N x_i^0$ is the barycenter of the initial data.

For the proof of the above theorem we refer to Section 4, and Section 6 for numerical experiments which show the convergence and more information.

We have also proved the following ℓ^p contraction for periodic solutions :

Proposition 1.4. **(l^p contraction)**
Let X and Y be two N-periodic solutions of the problem (4.3)-(4.4) *with N-periodic initial data X^0 and Y^0 respectively. Then the following estimate*

$$\|X(t) - Y(t)\|_p \leq \|X^0 - Y^0\|_p, \quad \textit{for all } t > 0$$

holds true provided $p \geq 2$.

1.3 Related results

Another possible model, first proposed in 1924 and repeatedly improved in subsequent years, involves the Lennard-Jones potential [84]

$$\phi(r) = 4\varepsilon\left[\left(\frac{r}{\sigma_0}\right)^{-12} - \left(\frac{r}{\sigma_0}\right)^{-6}\right],$$

where r is a distance between two atoms, ε is the depth (minimum) of the energy and σ_0 is the finite distance at which the interparticle potential is zero. Due to its computational simplicity and relatively good approximations, the Lennard-Jones potential is extensively used to describe the properties of gases and in computer simulations [84, 85].

There is no necessity to deal only with two-body potentials. One approach to represent the many-body potentials energy is to consider it as a sum of two-body, three-body, ..., N-body terms. An example of such constructed energy potential is the Stillinger-Weber potential [106] for semiconductor silicon containing only two- and three-body terms.

More facts about dislocations, examples of potentials used in various models, and numerical simulations performed on these models can be found in the book of Bulatov and Cai [27].

A similar model to ours, where a finite number of dislocations of different types occur (for instance positive and negative ones), was considered by El Hajj, Ibrahim and Monneau [41]. The authors studied horizontal motion of dislocation lines and they derived formally a two-dimensional mean field model called Groma-Balogh model. In the same paper, they also investigated a model with additional boundary conditions. They observed that positive dislocations move to the right, whereas the negative ones move to the left. In particular, numerical simulations of deformations of a slab under an external shear stress have been performed.

Related models called *individual cell-based* models occur not only in the theory of dislocations but also in the study of *e.g.* chemotherapy where x_i denotes a center of a tumour cell [40], chemotaxis [104] and many others. Moreover, particles may also evolve according to stochastic differential equations, see [22] and references therein for numerical simulations.

2 Comparison principle

In this section, we prove a comparison principle result for a general system of equations :

$$\frac{d}{dt}X(t) = G(X(t)), \quad t > 0, \tag{4.6}$$

where $X(t) = (x_i(t))_{i\in\mathbb{Z}}, G(X) = (G_i(X))_{i\in\mathbb{Z}}$ with $G_i(X) \overset{def}{=} \sum_{j\neq i} g(x_j - x_i, j-i)$ for each $i \in \mathbb{Z}$, and $g\colon \mathbb{R}\times\mathbb{Z}\backslash\{0\} \to \mathbb{R}$ is C^1 and globally nondecreasing with respect to first variable. We will apply this later in Section 3.

Lemma 2.1. (Comparison principle)
Let $T > 0$ and assume that $X, Y \in C^1([0,T), \ell^\infty)$ be two solutions of (4.6) with $X(0) = X^0$ and $Y(0) = Y^0$. Assume that $X^0 \leq Y^0$, then $X(t) \leq Y(t)$ for every $t \in [0,T)$.

Proof of Lemma 2.1 Notice that the assumption $X^0 = X(0) \leq Y(0) = Y^0$ reads as $x_n^0 := x_n(0) \leq y_n(0) =: y_n^0$ for every $n \in \mathbb{Z}$, and we shall prove that $x_n(t) \leq y_n(t)$ for every $n \in \mathbb{Z}$ and $t \in [0,T)$.

Define then new functions $Z(t) = (z_n(t))_{n\in\mathbb{Z}}$ and $M(t)$ as

$$z_n(t) = x_n(t) - y_n(t), \quad M(t) = \sup_{n\in\mathbb{Z}} z_n(t). \tag{4.7}$$

Since $X(t)$, $Y(t) \in \ell^\infty$ for all $t \in [0,T)$, then from the definition of M, we have

$$\forall_{t^*\in[0,T)}\, \exists_{n^*(t^*)}\ M(t^*) = z_{n^*(t^*)}(t^*), \tag{4.8}$$

where $n^*(t^*)$ may not be necessarily finite. Our goal is to show that

$$M(t) \leq 0 \quad \text{for all times } t \in [0,T). \tag{4.9}$$

The way to prove (4.9) is to show that for all $t \in [0,T)$, we have $M'(t) \leq 0$ in the viscosity sense. Then by a comparison principle or the Gronwall inequality we deduce (4.9).

Step 1 **:** $n^*(t^*) \in \mathbb{Z}$
Let $t^* \in [0,T)$ and consider a test function ϕ such that

$$\begin{cases} M(t) \leq \phi(t), \\ M(t^*) = \phi(t^*). \end{cases}$$

Then $M'(t^*) \le 0$ in the viscosity sense if $\phi'(t^*) \le 0$, see [37, Definition 2.2] for a definition of viscosity solutions.

From (4.7) and (4.8), we have

$$z_{n^*(t)}(t) \le M(t) \le \phi(t), \quad z_{n^*(t^*)}(t^*) = M(t^*) = \phi(t^*), \tag{4.10}$$

thus $\phi'(t) = \frac{d}{dt} z_{n^*(t)}(t)$ at $t = t^*$, since $z_{n^*(t)}, \phi$ are sufficiently smooth ($X,\ Y \in C^1([0,T), \ell^\infty)$).

If $\phi'(t) = \frac{d}{dt} z_{n^*(t)}(t) \le 0$ at $t = t^*$, then we have $M'(t^*) \le 0$ in the viscosity sense. Thus using the Gronwall inequality (which in the viscosity solutions framework is nothing else but the comparison principle), we deduce

$$M(t) \le M(0).$$

But $z_n(0) = x_n(0) - y_n(0) \le 0$ for all $n \in \mathbb{Z}$, thus $M(t) \le M(0) \le 0$.

Therefore our goal now is to show that indeed $\frac{d}{dt} z_{n^*(t)}(t) \le 0$ at $t = t^*$. Set $n^* = n^*(t^*)$, using the Taylor expansion of the function G, we have

$$\begin{aligned}\frac{dz_{n^*}(t)}{dt} &= \frac{dx_{n^*}(t)}{dt} - \frac{dy_{n^*}(t)}{dt} = G_{n^*}(X(t)) - G_{n^*}(Y(t)) \\ &= \sum_{m \in \mathbb{Z}} \partial_m G_{n^*}(\Theta(t))(x_m(t) - y_m(t)),\end{aligned}$$

where $\Theta(t) = \alpha X(t) + (1-\alpha) Y(t)$ for some $\alpha \in (0,1)$. Here $\partial_m G_n(X)$ is to be understood as

$$\partial_m G_n(X) := \frac{dG_n(X)}{dx_m}.$$

In particular for $t = t^*$, we obtain

$$\begin{aligned}\frac{dz_{n^*}(t^*)}{dt} &= \partial_{n^*} G_{n^*}(\Theta(t^*)) z_{n^*}(t^*) + \sum_{\substack{m \in \mathbb{Z} \\ m \ne n^*}} \partial_m G_{n^*}(\Theta(t^*))(x_m(t^*) - y_m(t^*)) \\ &\le \partial_{n^*} G_{n^*}(\Theta(t^*)) z_{n^*}(t^*) + z_{n^*}(t^*) \sum_{\substack{m \in \mathbb{Z} \\ m \ne n^*}} \partial_m G_{n^*}(\Theta(t^*)) \\ &= z_{n^*}(t^*) \sum_{m \in \mathbb{Z}} \partial_m G_{n^*}(\Theta(t^*)) = 0.\end{aligned} \tag{4.11}$$

The inequality in the middle line and the last equality in the above computations can be justified as follow.

First, we notice that for every $m \ne n^*$ and by the assumption on monotonicity of the function g, we have

$$\partial_m G_{n^*}(\Theta(t^*)) = g_x(\Theta_m(t) - \Theta_{n^*}(t), m - n^*)\alpha \ge 0.$$

Here g_x denotes the partial derivative of $g = g(x,y)$ with respect to the first variable x.

Second, we can calculate explicitly $\partial_{n^*} G_{n^*}(\Theta(t^*))$. Namely, by the structure of the function G, we get

$$\partial_{n^*} G_{n^*}(\Theta(t^*)) = -\sum_{m\neq n^*} g_x(\Theta_m(t) - \Theta_{n^*}(t), m - n^*)\alpha.$$

Summing up all the derivatives of G, we arrive at the last equality of (4.11).

Step 2 **:** $n^*(t^*) = +\infty$

Then there exists a subsequence n_k such that

$$M(t^*) = \sup_{n\in\mathbb{Z}} z_n(t^*) = \lim_{k\to+\infty} z_{n_k}(t^*). \tag{4.12}$$

Let us redefine the sequences up to shift the indices, we have

$$\left\{\begin{array}{lll} x_n^k(t) = x_{n+n_k}(t) \to & x_n^\infty \\ y_n^k(t) = y_{n+n_k}(t) \to & y_n^\infty \\ z_n^k(t) = z_{n+n_k}(t) \to & z_n^\infty = x_n^\infty - y_n^\infty \end{array}\right| \quad \text{as} \quad k\to+\infty.$$

The convergence of the sequences takes place up to subsequence of k, since x_n^k, y_n^k and z_n^k are bounded.

Moreover, we have from (4.12) that

$$\begin{aligned} M(t^*) = \lim_{k\to+\infty} z_{n_k}(t^*) &= \lim_{k\to+\infty} z_0^k(t^*) \\ &= z_0^\infty(t^*) \le \sup_{n\in\mathbb{Z}} z_n^\infty(t^*) \end{aligned}$$

and

$$\begin{aligned} M(t^*) &= \sup_{n\in\mathbb{Z}} z_n(t^*) \\ &\ge z_{n+n_k}(t^*) = z_n^k(t^*) \quad \text{for all } n\in\mathbb{Z}, \end{aligned}$$

i.e. $M(t^*) \ge z_n^\infty(t^*)$, and hence $M(t^*) \ge \sup_{n\in\mathbb{Z}} z_n^\infty(t^*)$.

Therefore,

$$M(t^*) = z_0^\infty(t^*) = \sup_{n\in\mathbb{Z}} z_n^\infty(t^*),$$

and hence $n^* = 0$. In addition, we have

$$z_n^\infty(0) = x_n^\infty(0) - y_n^\infty(0) \le 0.$$

Thus, applying the result of Step 1 for $z_n^\infty(t)$, we prove the desired result. □

3 Existence and uniqueness of solution

We give, in this section, the proof of Theorem (1.2) which combines the classical Cauchy-Lipschitz theorem [26, Thm 7.3, p. 184] and the comparison principle, Lemma 2.1.

Proof of Theorem 1.2
In the proof we argue in several steps.

Step 0 **: Properties of the function** f
Consider the function f be defined in (4.4a). Clearly, we have $f(\cdot,y) \in C^\infty(\mathbb{R})$ and $f(\pm\infty,y) = \mp 0$ for every $y \in \mathbb{Z}\backslash\{0\}$ fixed. Moreover, $f(\cdot,y)$ is antisymmetric and there exists $x_y = \sqrt{3-\sqrt{2}}|y|$ such that

$$f(x_y,y) = \max_{x\in\mathbb{R}} f(x,y), \quad f(-x_y,y) = \min_{x\in\mathbb{R}} f(x,y)$$

and $f(\cdot,y)$ is non-decreasing over $[-x_y,x_y]$, see for instance Figure 4.1.

Moreover, we see that for fixed $y \in \mathbb{Z}\backslash\{0\}$

$$\left|\frac{d}{dx}f(x,y)\right| \le \frac{d}{dx}f(0,y) = \frac{1}{y^2}. \tag{4.13}$$

Hence, f is globally Lipschitz continuous over $\mathbb{R}$ with $\frac{1}{y^2}$ Lipschitz constant depending on fixed y.

Step 1 **: Existence of a unique global solution for** (4.3)
Let $X = (x_i)_{i\in\mathbb{Z}},\ Y = (y_i)_{i\in\mathbb{Z}} \in \ell^\infty$. Using (4.13), we have

$$\begin{aligned}
\|F(X)-F(Y)\|_{\ell^\infty} &= \max_{i\in\mathbb{Z}} |F_i(X)-F_i(Y)| \\
&= \max_{i\in\mathbb{Z}} \Big|\sum_{j\neq i} f(x_j-x_i,j-i) - f(y_j-y_i,j-i)\Big| \\
&\le \max_{i\in\mathbb{Z}} \sum_{j\neq i} |f(x_j-x_i,j-i) - f(y_j-y_i,j-i)| \\
&\le \max_{i\in\mathbb{Z}} \sum_{j\neq i} \frac{1}{(j-i)^2}\Big(|x_j-y_j| + |x_i-y_i|\Big) \\
&\le 4\|X-Y\|_{\ell^\infty} \sum_{k=1}^{+\infty} \frac{1}{k^2}.
\end{aligned}$$

Thus

$$\|F(X)-F(Y)\|_{\ell^\infty} \le \frac{2}{3}\pi^2 \|X-Y\|_{\ell^\infty}. \tag{4.14}$$

Therefore, using the classical Cauchy-Lipschitz theorem [26, Thm 7.3, p. 184], there exists a unique solution $X \in C^1([0,+\infty),\ell^\infty)$ of (4.3).

Step 2 **: Invariance :** $X(t) \in \Omega$ **for every** $t \geq 0$
In this step we show that if $X(0) = X^0 \in \Omega$, then the solution $X(t)$ given in Step 1 satisfies

$$X(t) \in \Omega \quad \text{for every} \quad t \geq 0.$$

Step 2.1 **: Variant system of** (4.3)
For $y \in \mathbb{Z}\backslash\{0\}$ fixed, define a new function $\tilde{f} = \tilde{f}(x,y)$ as follows

$$\begin{cases} \tilde{f}(x,y) = f(x,y) & \text{for } x \in [-x_y, x_y], \\ \tilde{f}(x,y) = f(x_y,y) & \text{for all } x \geq x_y, \\ \tilde{f}(x,y) = f(-x_y,y) & \text{for all } x \leq -x_y. \end{cases} \tag{4.15}$$

Clearly, $\tilde{f}$ is Lipschitz and non-decreasing with respect to the first variable over the whole space. Moreover, since $\frac{d}{dx}f(\pm x_y, y) = 0$ for fixed $y \neq 0$, then $\tilde{f}$ is C^1 with respect to the first variable on $\mathbb{R}$.

Then we consider the following system

$$\begin{cases} \frac{d}{dt}\tilde{X}(t) = \tilde{F}(\tilde{X}(t)) \quad t \geq 0, \\ \tilde{X}(0) = X^0 \in \Omega \cap \ell^\infty, \end{cases} \tag{4.16}$$

where again $\tilde{X}(t) = (\tilde{x}_i(t))_{i\in\mathbb{Z}}$ and $\tilde{F}(\tilde{X}) = (\tilde{F}_i(\tilde{X}))_{i\in\mathbb{Z}}$, with

$$\tilde{F}_i(\tilde{X}(t)) := \sum_{j\neq i} \tilde{f}(\tilde{x}_j - \tilde{x}_i, j - i). \tag{4.17}$$

Similarly to Step 1 we show for every $\tilde{X} = (\tilde{x}_i)_{i\in\mathbb{Z}}$, $\tilde{Y} = (\tilde{y}_i)_{i\in\mathbb{Z}}$, that

$$\|\tilde{F}(\tilde{X}) - \tilde{F}(\tilde{Y})\|_{\ell^\infty} \leq \frac{2}{3}\pi^2 \|\tilde{X} - \tilde{Y}\|_{\ell^\infty}.$$

Therefore, using the classical Cauchy-Lipschitz theorem [26, Thm 7.3, p. 184], there exists a unique solution $\tilde{X} \in C^1([0,+\infty),\ell^\infty)$ of the variant problem (4.16).

Step 2.2 **:** $\tilde{X}(t) \in \Omega$ **for every** $t \geq 0$
We have $\tilde{X}(0) = X^0 \in \Omega$, i.e.

$$-\sqrt{3-2\sqrt{2}}|i-j| \leq x_i^0 - x_j^0 \leq \sqrt{3-2\sqrt{2}}|i-j|, \quad \forall i,j \in \mathbb{Z}.$$

Setting $m = i - j$, we obtain

$$\underline{x}_i^m(0) := x_{i-m}^0 - \sqrt{3-2\sqrt{2}}|m| \le x_i^0 \le x_{i-m}^0 + \sqrt{3-2\sqrt{2}}|m| =: \overline{x}_i^m(0),$$

for every $i, m \in \mathbb{Z}$.

Moreover, from the definition of the function $\tilde{F}$, see (4.17), it is clear that the problem (4.16) is invariant by translations. Hence,

$$\underline{X}^m = \left(\tilde{x}_{i-m}(t) - \sqrt{3-2\sqrt{2}}|m| =: \underline{x}_i^m\right)_{i\in\mathbb{Z}}$$

and

$$\overline{X}^m = \left(\tilde{x}_{i-m}(t) + \sqrt{3-2\sqrt{2}}|m| =: \overline{x}_i^m\right)_{i\in\mathbb{Z}}$$

are two solutions of (4.16) for each $m \in \mathbb{Z}$. Now, since $\tilde{f}$ is non-decreasing, we can apply the comparison principle, Lemma 2.1 (with $T = +\infty$), and deduce that for every $i, m \in \mathbb{Z}$

$$\underline{x}_i^m(t) \le \tilde{x}_i(t) \le \overline{x}_i^m(t) \quad \text{for } t > 0.$$

Hence, $\tilde{X}(t) \in \Omega$ for all $t > 0$.

Step 2.3 **:** $\tilde{X}$ **solves** (4.3)
We have that $\tilde{X}(t) \in \Omega$ for all $t > 0$. Thus

$$-x_y = -\sqrt{3-2\sqrt{2}}|i-j| \le \tilde{x}_i - \tilde{x}_j \le \sqrt{3-2\sqrt{2}}|i-j| = x_y, \quad \forall i, j \in \mathbb{Z}.$$

However, $\tilde{f}(\cdot, i-j)_{|[-x_y,x_y]} = f(\cdot, i-j)_{|[-x_y,x_y]}$, hence

$$\tilde{F}(\tilde{X}(t)) = F(\tilde{X}(t)) \quad \text{over} \quad t > 0.$$

Therefore, $\tilde{X}$ solves (4.3).

Step 2.4 **: Conclusion :** $X(t) \in \Omega$ **for every** $t \ge 0$
Since

$$\tilde{X}(0) = X^0 = X(0),$$

then by the uniqueness of the solution of system (4.3) (See Step 1), we get

$$X(t) = \tilde{X}(t) \in \Omega \quad \text{for every} \quad t \ge 0.$$

Thereupon, we have proved that $X(t)$ is the unique global solution of the problem (4.3)-(4.4).

Step 3 : X is periodic
Assume that $X(0) = X^0$ is N-periodic; i.e. $x_i^0 = x_{i+N}^0$ for every $i \in \mathbb{Z}$. Define $Y = (y_i)_{i\in\mathbb{Z}} = (x_{i+N})_{i\in\mathbb{Z}}$, where we recall that $X = (x_i)_{i\in\mathbb{Z}}$. Then X and Y are two solutions of (4.3) with $X(t)\ ,Y(t) \in \Omega$ for all $t \geq 0$. Moreover, we have $Y(0) = (x_{i+N}^0)_{i\in\mathbb{Z}} = X^0$ ($Y(0) \leq X^0$ and $X^0 \leq Y(0)$). Since f is non-decreasing over Ω, then using the comparison principle (Lemma 2.1 with $T = +\infty$), we deduce that

$$Y(t) = X(t) \quad \text{for every } t \geq 0,$$

i.e $x_i = x_{i+N}$ for every $t \geq 0$. □

4 Convergence to flat walls

The aim of this section is to prove that under the periodicity assumption imposed on the initial data, the solution constructed in Theorem 1.2 converges to a special stationary solution of the problem (4.3). Precisely, in this section, we prove Theorem 1.3.

Proof of Theorem 1.3

Step 0 : preliminary (reformulation of (4.3))
Let $X = (x_i)_{i\in\mathbb{Z}} \in C^1([0,\infty), \Omega \cap \ell^\infty)$ be a N-periodic (*i.e.* $x_{i+N} = x_i$) solution of (4.3). For $i = 1, ..., N$ we have

$$\begin{aligned}\frac{d}{dt}x_i(t) &= \sum_{j\neq i} f(x_j - x_i, j - i)\\ &= \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{k\in\mathbb{Z}} f(x_{j+kN} - x_i, j - i + kN).\end{aligned}$$

Using the periodicity of X ($x_{j+kN} = x_j$), we get that

$$\frac{d}{dt}x_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{k\in\mathbb{Z}} f(x_j - x_i, j - i + kN).$$

Hence, we can transform (4.1) into the following equation

$$\frac{d}{dt}x_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} g(x_j - x_i, j - i), \quad i = 1, \ldots, N, \tag{4.18}$$

where $g(x,y) = \sum\limits_{k\in\mathbb{Z}} f(x, y + kN)$. Moreover, since for every $y \neq 0$, the map $x \mapsto f(x,y)$ is Lipschitz with $\frac{1}{y^2}$ Lipschitz constant (see Step 0, in the proof of Theorem 1.2), $g(0,y) = f(0,y) = 0$ and $x \in \ell^\infty$, then

$$|g(x,y)| = \left|\sum_{k\in\mathbb{Z}} f(x, y+kN) - f(0, y+kN)\right| \leq \sum_{k\in\mathbb{Z}} \frac{1}{(y+kN)^2}|x| \leq \mathcal{M} \qquad (4.19)$$

for some $\mathcal{M} > 0$. Hence, g is uniformly bounded in x.

In order to prove the convergence of the solution, we set

$$M(t) = \frac{1}{2}\sum_{i=1}^{N} x_i^2(t) \qquad (4.20)$$

and we argue by steps.

Step 1 : M is non-increasing
Indeed, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N} x_i^2(t) &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} x_i(t)\, g(x_j(t) - x_i(t), j-i)\\
&= \sum_{i=1}^{N}\sum_{j=i+1}^{N} x_i(t)\, g(x_j(t) - x_i(t), j-i) + \sum_{i=1}^{N}\sum_{j=1}^{i-1} x_i(t)\, g(x_j(t) - x_i(t), j-i)\\
&= \sum_{i=1}^{N}\sum_{k=1}^{N-i} x_i(t)\, g(x_{i+k}(t) - x_i(t), k) - \sum_{j=1}^{N}\sum_{i=j+1}^{N} x_i(t)\, g(x_i(t) - x_j(t), i-j)\\
&= \sum_{i=1}^{N}\sum_{k=1}^{N-i} x_i(t)\, g(x_{i+k}(t) - x_i(t), k) - \sum_{j=1}^{N}\sum_{k=1}^{N-j} x_{j+k}(t)\, g(x_{j+k}(t) - x_j(t), k)\\
&= \sum_{i=1}^{N}\sum_{k=1}^{N-i} (x_i(t) - x_{i+k}(t))\, g(x_{i+k}(t) - x_i(t), k) \leq 0.
\end{aligned}$$

First, let us mention that due to the fact that the function $f = f(x,y)$, defined by (4.4a) is symmetric in y and antisymmetric in x, the function $g = g(x,y)$ possesses such property as well. Moreover, as a result of the boundedness of the function $g(\cdot, y)$ (which comes form the Lipschitz condition of $f(\cdot,y)$) and the fact that only finite sums are considered, we are allowed to use Fubini's theorem and change the order of summation. These facts justify the third equality.

The inequality is obtained by the fact that each single expression under the sums is nonpositive due to the definitions of the functions g, f and the fact that $X(t) \in \Omega \cap \ell^\infty$ for $t \geq 0$ (see Remark 1.1).

Finally, we conclude that $M(t) \to M_0$ as $t \to \infty$ since $M(t)$ is nonnegative and non-increasing.

Step 2 : limit of $X(t)$ as $t \to +\infty$
Let us define $X^n(t) := X(t+n)$. Then X^n is a solution of (4.3). Since $f(x,y)$ is Lipschitz with $\frac{1}{y^2}$ Lipschitz constant (independent if x), hence $\frac{d}{dt}X^n(t) \in \ell^\infty$ uniformly in n. Using Ascoli's theorem, up to some subsequence, $X^n(t) \to X^\infty(t)$ as $n \to \infty$ for every $t > 0$. Thus, we can write

$$M_0 = \lim_{n\to\infty} M(t+n) = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{N}\left(x_i^n(t)\right)^2 = \frac{1}{2}\sum_{i=1}^{N}\left(x_i^\infty(t)\right)^2. \tag{4.21}$$

Since X^n is a solution of (4.18) and $X^n(t) \in \Omega \cap \ell^\infty$, then the limit X^∞ is a classical solution and $X^\infty(t) \in \Omega \cap \ell^\infty$. Therefore, repeating all the computations performed in Step 1 for X^∞, we arrive at

$$\begin{aligned} 0 = \frac{d}{dt}M_0 &= \frac{d}{dt}\frac{1}{2}\sum_{i=1}^{N}\left(x_i^\infty(t)\right)^2 \\ &= \sum_{i=1}^{N}\sum_{k=1}^{N-i}(x_i^\infty(t) - x_{i+k}^\infty(t))\, g(x_{i+k}^\infty(t) - x_i^\infty(t), k). \end{aligned}$$

Since the solution lives in Ω, $g(x_{i+k}^\infty(t) - x_i^\infty(t), k)$ and $x_{i+k}^\infty(t) - x_i^\infty(t)$ have the same sign (see Remark 1.1), then

$$(x_i^\infty(t) - x_{i+k}^\infty(t))\, g(x_{i+k}^\infty(t) - x_i^\infty(t), k) \leq 0$$

for all $i \in \{1, .., N-1\}$ and $k \in \{1, ..., N-i\}$. Thus, either

$$x_i^\infty(t) = x_j^\infty(t) \quad \text{for all} \quad i = 1, \dots, N-1 \text{ and } j = i+1, \dots, N, \tag{4.22}$$

or we have $g = 0$. However, since $X^\infty(t) \in \Omega \cap \ell^\infty$, then $g = 0$ immediately implies (4.22) (see (4.4a) and the definition of g).

Therefore, we get from (4.22) that

$$x_1^\infty(t) = x_2^\infty(t) = \cdots = x_N^\infty(t).$$

Next, we plug X^∞ into the equation (4.18) to see that indeed $\frac{d}{dt}x_i^\infty(t) = 0$ (since $g(0,y) = 0$); thus, $x_i^\infty(t) = x_i^\infty(0) = c$, for all $i = 1, \dots, N$, and for some $c \in \mathbb{R}$. Moreover, we can write the explicit value of M_0, i.e.

$$M_0 = \frac{1}{2}Nc^2.$$

Now take another convergent subsequence, $X^m(t)$ of $X(t)$ such that $X^m(t) \to \overline{X}^\infty(t)$ as $m \to \infty$. Repeating all the calculations performed for the sequence X^n, we may show that $\overline{x}_i^\infty(t) = \overline{x}_i^\infty(0) = b$ for all $i = 1, \ldots, N$, $t \geq 0$ and some $b \in \mathbb{R}$. As before we conclude that

$$M_0 = \frac{1}{2} N b^2.$$

Thus, $b = c$, because we may assume, without loss of generality, that $b, c \geq 0$, since the problem (4.18) is invariant by translations and the initial data can be shifted to be positive.

This implies that the accumulation point of X is unique. Hence, $x_i(t) \to c$ as $t \to \infty$ for all $i = 1, \ldots, N$.

Step 3 : identification of the limit

In this step we prove that the barycenter is preserved in time, i.e.

$$\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i(0) \quad \text{for all} \quad t > 0,$$

which allows us to determine the value of the constant c.

We have

$$\frac{d}{dt} \sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_j - x_i, j - i).$$

But

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_j - x_i, j - i) = \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} g(x_j - x_i, j - i) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_i - x_j, i - j),$$

where we have changed the order of summation in the first equality (this is possible because g is a bounded function), and we replaced i and j in the second equality. Moreover, since $g(x, y)$ is antisymmetric w.r.t. x and symmetric w.r.t. y (because f antisymmetric and symmetric w.r.t. x and y respectively), then we get

$$\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_j - x_i, j - i) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_i - x_j, i - j) = -\sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_j - x_i, j - i).$$

Therefore, we deduce that

$$\frac{d}{dt} \sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} g(x_j - x_i, j - i) = 0,$$

and hence

$$\sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i(0).$$

Finally, since $x(t) \to c$ as $t \to \infty$, we conclude that

$$\sum_{i=1}^{N} x_i(0) = \lim_{t\to\infty} \sum_{i=1}^{N} x_i(t) = Nc,$$

i.e.

$$c = \frac{1}{N} \sum_{i=1}^{N} x_i(0) \tag{4.23}$$

thus, we have proved the desired result. □

5 From micro to macro model

We show in this section the ℓ^P contraction estimate of periodic solutions of (4.3)-(4.4), namely we give the proof of Proposition 1.4.

Proof of Proposition 1.4
Let $X = (x_i)_{i\in\mathbb{Z}}$ and $Y = (y_i)_{i\in\mathbb{Z}}$ be a N-periodic (*i.e.* $x_{i+N} = x_i$, $y_{i+N} = y_i$) solution of (4.3) of the class $C^1([0,\infty), \Omega \cap \ell^\infty)$. We proceed as in Section 4, Step 1. First, without loss of generality, we may transform (4.1) into the following equation

$$\frac{d}{dt} x_i(t) = \sum_{\substack{j=1 \\ j\neq i}}^{N} g(x_j - x_i, j - i), \quad i = 1, \dots, N,$$

with the function $g(x, y) = \sum_{k\in\mathbb{Z}} f(x, y + kN)$ uniformly bounded in x. Thus, we calculate

$$\frac{d}{dt}\frac{1}{p}\big\|X(t)-Y(t)\big\|_p^p = \sum_{i=1}^{N}\big|x_i(t)-y_i(t)\big|^{p-2}\big(x_i(t)-y_i(t)\big)\big(\dot{x}_i(t)-\dot{y}_i(t)\big) =$$

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\big|x_i(t)-y_i(t)\big|^{p-2}\big(x_i(t)-y_i(t)\big)\big(g(x_j(t)-x_i(t),j-i)-g(y_j(t)-y_i(t),j-i)\big)$$

$$= \sum_{i=1}^{N}\sum_{k=1}^{N-i}\underbrace{\Big(\big|x_i(t)-y_i(t)\big|^{p-2}\big(x_i(t)-y_i(t)\big)-\big|x_{i+k}(t)-y_{i+k}(t)\big|^{p-2}\big(x_{i+k}(t)-y_{i+k}(t)\big)\Big)}_{\mathcal{I}_1}$$

$$\cdot\underbrace{\Big(g(x_{i+k}(t)-x_i(t),k)-g(y_{i+k}(t)-y_i(t),k)\Big)}_{\mathcal{I}_2}\leq 0.$$

Let us mention here that due to the fact that the function $f = f(x, y)$, defined by (4.4a), is symmetric in y and antisymmetric in x, the function $g = g(x, y)$ possesses such property as well. Moreover, as a result of the boundedness of the function $g(\cdot, y)$ (which comes form the Lipschitz condition of $f(\cdot, y)$) and the fact that only finite sums are considered, we are allowed to use Fubini's theorem and change the order of summation. These facts justify the third equality.

Furthermore, we notice that for fixed $y \in \mathbb{Z}$ the function $f(x, y)$ is nondecreasing in the variable x provided $|x| \leq \sqrt{3-2\sqrt{2}}|y|$. Hence, by definition the function $g(x, y)$ is also nondecreasing in the variable x under the same condition on x. Suppose now that $\mathcal{I}_2 \leq 0$. This immediately implies, in view of the above information, that $x_{i+k}(t) - y_{i+k}(t) \leq x_i(t) - y_i(t)$; hence, $\mathcal{I}_1 \geq 0$. On the contrary, if $\mathcal{I}_2 \geq 0$, then $\mathcal{I}_1 \leq 0$. Hence, we conclude

$$\frac{d}{dt}\frac{1}{p}\|X(t)-Y(t)\|_p^p \leq 0,$$

which completes the proof. □

Corollary 5.1. (l^p contraction for a rescaling of x_i)
Let $p \geq 2$. Fix $\varepsilon > 0$ and let us define new variables in the following way

$$x_i(t) = \frac{1}{\varepsilon}u^{\varepsilon}(\varepsilon i, \varepsilon t), \quad \forall i \in \mathbb{Z}.$$

Then the above theorem reads as

$$\|u^{\varepsilon}(\cdot,\tau) - v^{\varepsilon}(\cdot,\tau)\|_p \leq \|u_0^{\varepsilon} - v_0^{\varepsilon}\|_p.$$

6 Numerical experiments

Here we present results of some numerical experiments to confirm the results obtained in Theorem 1.3. We construct an adaptive scheme as follows. Let $N > 0$ denote the total number of interacting particles. Let Δt denote a time-step and let us define an approximate solution of (4.3) by a solution $X^n = (X_1^n, \dots, X_N^n)$ of the following forward Euler scheme

$$X^{n+1} = X^n + \Delta t F(X^n) \stackrel{def}{=} S(X^n). \tag{4.24}$$

Lemma 6.1. (Monotonicity of the scheme)
The scheme derived in (4.24) *is monotone if and only if the time-step satisfies* $\Delta t \leq \frac{3}{\pi^2}$ *and the initial data* $X^0 \in \Omega$ *defined in* (4.5).

Proof of Lemma 6.1
To prove the monotonicity it is enough to show that $\partial_j S_i(X^n) \geq 0$ for all $i, j = 1, \dots, N$. First, we notice that due to Lemma 2.1 we get $X^n \in \Omega$ for all $n \in \mathbb{N}$.

Step 1 **:** $j \neq i$

$$\partial_j F_i(X^n) = \Delta t\, f_x(X_j^n - X_i^n, j - i) \geq 0, \tag{4.25}$$

since considered function f is non-decreasing with respect to the first variable.

Step 2 **:** $j = i$

$$\begin{aligned}
\partial_i F_i(X^n) &= 1 - \Delta t \sum_{\substack{j=1 \\ j \neq i}}^{N} f_x(X_j^n - X_i^n, j - i) \\
&\geq 1 - \Delta t \sum_{\substack{j=1 \\ j \neq i}}^{N} f_x(0, j - i) \\
&\geq 1 - \Delta t \sum_{\substack{j \in \mathbb{Z} \\ j \neq i}} \frac{1}{(j-i)^2} \\
&\geq 1 - \Delta t \frac{\pi^2}{3} \geq 0.
\end{aligned}$$

To justify the first inequality we use the properties of the function f described in the proof of Theorem 1.2 Step 1. Moreover, we extend the finite sum by its infinite version and we conclude its non-negativity. $\square$

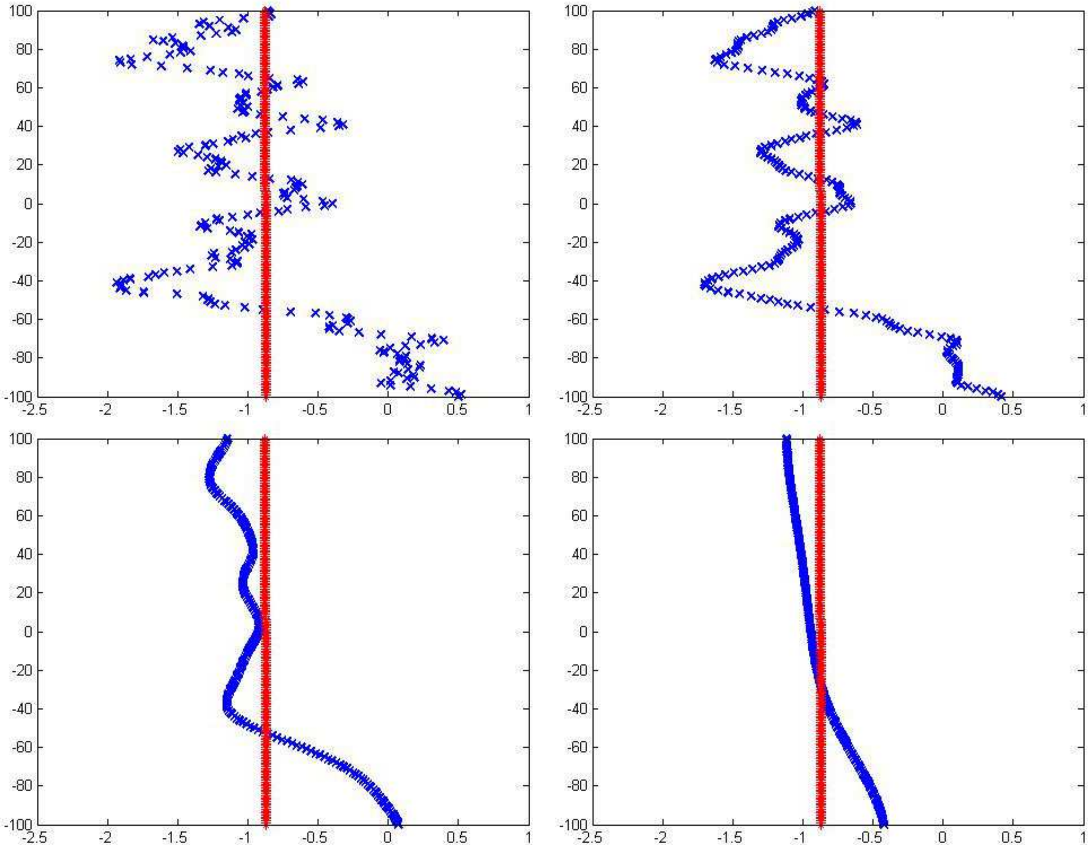

FIGURE 4.3 – Evolution of dislocations of (4.3) with initial data $X^0 \in \Omega$.

In our numerical experiments we assume the initial data $X^0 \in \Omega \cap \ell^\infty$ which is denoted by "**x**" on the left-upper plot in Figure 4.3. Furthermore, in every picture, by "*" we emphasised what the limit solution (by Theorem 1.3 the limit solution is at the barycenter of initial data) is. In Figure 4.3 we observe the evolution of dislocations which eventually converge.

However, we may also consider the initial data $X^0 \in \overline{\Omega}$ where

$$\overline{\Omega} = \left\{ X : \sqrt{3 - 2\sqrt{2}}\,|i - j| < |x_i - x_j| < |i - j| \right\}, \tag{4.26}$$

see the blue region in Figure 4.2. It is worth noticing that for such initial data, the force acting on dislocations is still attractive ; however, we do not have a comparison principle and we cannot guarantee that solution stays in $\overline{\Omega}$, but we can perform numerical experiments to see what happen with dislocations.

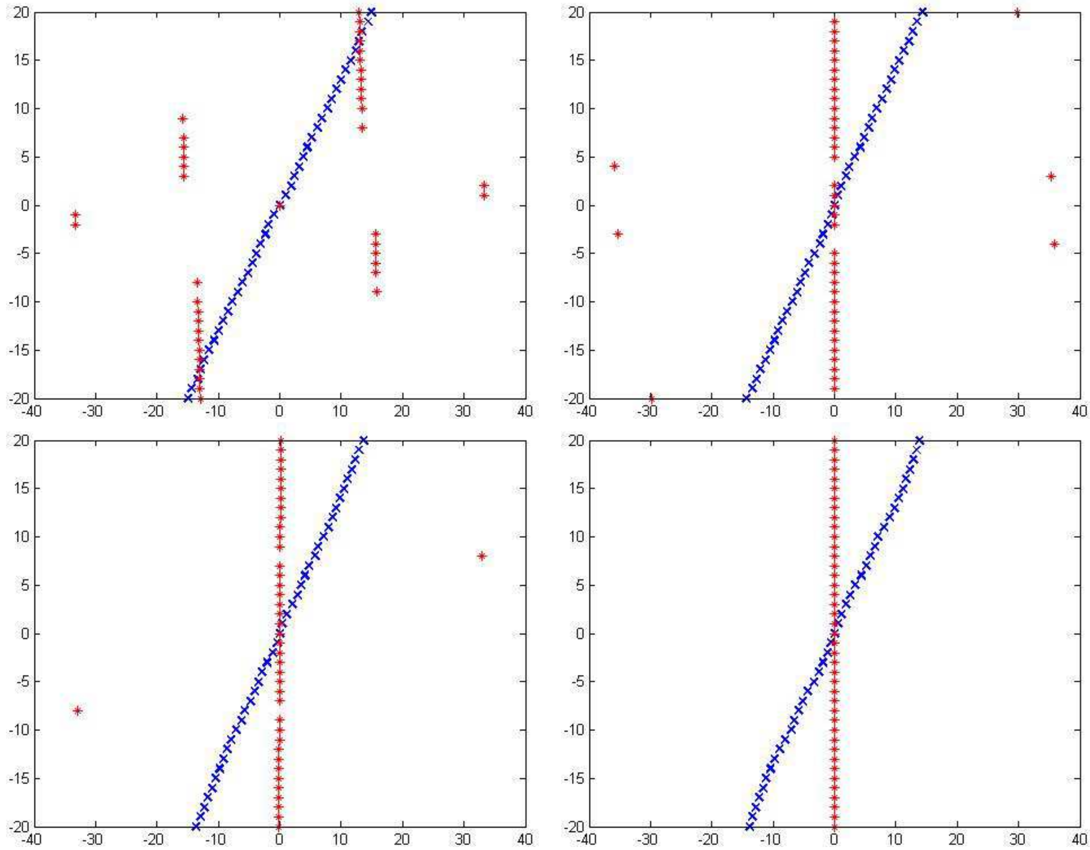

FIGURE 4.4 – Evolution of dislocations of (4.3) with initial data $X^0 \in \overline{\Omega}$. Each simulation starts with different initial data.

In the above pictures we can see that even small perturbation of initial data produces completely different solutions. The only one (right-lower plot in Figure 4.4) converged to a flat wall, while the remainder does not.

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