Pricing Multi-play Offers under Uncertainty and Competition

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Abstract - In a mature market, telecommunication operators try to differentiate themselves by marketing bundle offers. In this highly competitive context, operators should anticipate the strategies of their adversaries and guess the consumers' tastes, to maximize their benefits. To price their offers, operators have to deal with deep uncertainties on the other operators' cost structures, strategies, and on the consumers' preferences. We segment the market and estimate the comsumers' subjective prices, on each segment. Using Game theory, we define a pricing strategy to maximize the operator's payoff, under competition. Then, introducing dynamicity, we determine a strategy of line evolution for the operator, in order to learn the consumers' preferences, while minimizing regrets resulting from frequent line changes. Finally, to market bundles, numerous operators have to establish alliances. Focusing on Mobile Virtual Network Operators (MVNOs), we determine bargaining mechanisms, which might lead the various players to earn efficient and equitable guaranteed benefits.

Keywords : Bundles, Bayesian game, Simulation, Experts' theory, Bargaining

1 Introduction

In [4], the authors highlight the critical role of accurate preference modeling, on the efficency of systems based on Revenue Management techniques. In most of the literature, customers' choice models don't take into account interactions between prices and products. The problem is often modeled as an arrival process for each specific price, where the customer chooses to buy or not. Furthermore, dynamicity is hard to introduce and suppose generally that a single category of products is sold at a time.

To model customers' preferences, discrete choice models appear to be a good approach, especially multinomial logit ones (cf [2] and [4]). These models characterize customers' preferences towards a finite number of properties defining a specific product. The most famous one, due to Hotelling, has been massively used in the economic and marketing literature to handle problems of product differentiation. There exists other discrete choice models such as the model of vertical differentiation of Mussa and Rozen, the model of "ideal point", where the customers define their preferences using the existing distance between the product characteristics and the non-observable component of their personal tastes (cf [5]), or the Chuang and Sirbu's two parameter function (cf [1]).

These models are then used to price optimally the services. In [4], using a dynamic programming approach, a price set is defined at each time instant, to maximize the vendor's revenue. However, the model does allow neither customer interactions, neither interactions between operators and customers. On the contrary, in [20], a network owner and operators without their own network facilities set their decisions sequentially, in response to the other party's policy. Uncertainty on the consumers' and rival operators' behaviors are introduced in a stochastic programming framework, taking the view of one decision maker. Other approaches such [22], focus on the computation of leader-follower equilibrium for an industry involving two major stages of production and vertical competition between firms.

[2] considers the selection and pricing of product line and shows that at the optimum, in a n product line, the profit margins of the n products are equal. Besides, provided the line's length is limited, the vendor should better select those products, whose average margin is the highest. This article deals with the problem of cannibalization[1], but doesn't take into account competition between firms. [21] adresses the problem of determining optimal ordering and pricing policies in a finite horizon newsvendor model with unobservable lost sales. The retailer can proactively adjust order quantities to enhance the rate of learning. [12] introduces also a dynamic pricing example, with one seller attending consumers one by one, and selling them the same offer. The seller has only partial information on the consumers' tastes, but can increase his knowledge by observing the effects of price variations on the consumers' decisions.

In general, the demand is a linear function of price (cf [3]), including sometimes unknown parameters (cf [21]). The vendor can use various strategy, such as a skimming, i.e. introduce a luxurious version of a product, and a few months later a cheaper one. However, this approach is essentially fitted for monopolistic cases, or copyrighted offers. Other strategies are available, such as expanding the market shares, or targeting a specific market segment.

We suppose that the operator could include bundles of products in his line, as well as simple offers. By definition, a bundle is a bunch of offers sold at a given price. There exists essentially three kinds of bundling strategies :

- pure bundling, in which each set is composed of the same number of offers,
- mixed bundling, in which simple and composed offers, sold at different prices, co-exist,
- customized bundling (cf [9]), in which the consumer picks up m products between n proposed offers ($m \leq n$) and pays a pre-determined price. Numerous articles in the bundling literature, assume that

[1] We speak about cannibalization, when one offer makes another offer issued from the same or rival line, disappear.

the consumer's demand distribution is known by the vendor, and time-invariant. In reality, the sellers have partial knowledge about the demand distribution. The literature about bundles is vast (cf [8] and [9] for a review), and has evolved in two main approaches. The first one is an analysis of the bundle's components, which treats products as units of analysis. Complementarity between components results from the comparison of the product's influence on the bundle utility. The second one is based on the analysis of the bundle attributes. The complementarity is tested on the attributes composing the bundle. At first this approach was restricted to products sharing the same attributes, but extended in [7] to every categories.

The aim of the article is to define pricing and product line selection strategies, which might enable a telecommunication operator to maximize his benefit. One difficulty lies in the presence of multiplay offers, which might cannibalize other services. Furthermore, initially, the operator has only a partial knowledge of the consumers' tastes. However, the operator should market the offers, which are the best-suited to the consumers he wants to seduce. To perform such a goal, he might increase his information by making the product line evolve and observing his resulting profit evolution. Another difficult point, is to take into account the conflicting interests of all the players, whose final goal is to maximize their payoffs.

Marketing Multiplay offers is quite challenging for an operator. Indeed, usually he has to be present on all the markets, i.e. Mobile, Internet, Fix telephony and TV. Without network facilities, the operator has to develop alliances with other operators, to incorporate the lacking services in his offers. We mainly focus on the case of Mobile Virtual Network Operators (MVNOs), which establish alliances with Mobile operators to gain access to a mobile network.

In section 2, we introduce our customers' preference model, which enables us to estimate the densities of the consumers' reservation prices and to segment the market. Then, we determine the price equilibria, resulting from the bayesian game between operators, whose cost structures are unknown, and consumers, whose true reservation prices are not specified, in section 3. Game theory is used to model the conflicting interests of the adversaries. Furthermore, we introduce an original approach to compute the equilibria. In section 4, we allow the operators to make their product lines evolve dynamically, so as to gain information on the true customers' tastes. However, the evolution strategy should perform as well as if the operator had complete information about every player in the game, i.e. minimize his loss. Practically, we determine a randomized algorithm minimizing the operator's regret, i.e. the difference between the cumulated losses he would have suffered during a finite time slot, and the loss associated with the best offer set. Besides, this loss is not observed directely, and the operator has only access to a partial feedback (observation). Finally, in section 5, we specify MVNO agreement, i.e. whole pricing of global numbers of minutes that a Mobile operator might sell to an operator, without Mobile network facilities, leading to guaranteed benefits that might be seen as efficient and equitable.

2 Customer preference modeling

2.1 Basic definitions

Direct elicitation methods, in which the customers reveal their expected price, are not reliable (cf [8]). Indeed, the consumers can cheat, or commit judgement errors, which will biase demand functions and alter the accuracy of the model. Consequently, it is crucial to introduce the customers' errors and heterogeneity of demand.

Definition 1 *A choice set contains the offers that the operator and its rivals will put on the market, as well as the option of not buying anything at all. In the case of two competitive operators, the choice sets will be denoted $\mathcal{B}_{op_1}$ and $\mathcal{B}_{op_2}$, respectively, whereas the global choice set will be denoted $\mathcal{B} = \mathcal{B}_{op_1} \cup \mathcal{B}_{op_2}$.*

We recall the basic definition of a reservation price introduced by Kohli and Mahajan, and used in [7].

Definition 2 *The reservation price of the i^{th} consumer for the offer b, will be denoted $R_i(b) \geq 0$. It is the price of the offer b, for which the i^{th} consumer is indifferent between buying the offer b, or choosing any of the other offers in the global choice set.*

The aim of this section will be to estimate the reservation prices that each customer associates with each offer, in every possible global choice set, and to segment the market. We will essentially extend the results introduced by Chung and Rao (cf [7]) who consider only triple-play offers, to choice sets made of various size offers (i.e. simple, double and multiplay offers). A general discrete choice model, traducing the interdependences between the offers in the choice set, will be introduced[2].

2.2 The model

Definition of the system and its attributes.
The system is made of 4 components : Mobile phone, Fix phone, Internet and TV. Each component $J_k,\ k = 1, 2, 3, 4$, contains the set of offers (issued from simple, double or triple-plays), associated with this component.

At the same time, we define all the attributes composing the model. Each of which belongs to a specific class. There are three possible classes :

- the class A^1, contains the attributes fully comparable, i.e. appearing in every component of the system,
- the class A^2, gathers the attributes partially comparable, i.e. present in two or three components,

[2] A key point to note is that consumers'reservation prices are function of the global choice set considered.

Mobile	Fix	Internet	TV
$\mathbf{M_1^{(1)}}$	$\diamond$	$\diamond$	$\diamond$
$\diamond$	$\mathbf{F_1^{(1)}}$	$\mathbf{I_1^{(1)}}$	$\diamond$
$\mathbf{M_2^{(1)}}$	$\diamond$	$\mathbf{I_2^{(1)}}$	$\mathbf{TV_1^{(1)}}$
$\diamond$	$\mathbf{F_2^{(2)}}$	$\diamond$	$\diamond$
$\diamond$	$\diamond$	$\mathbf{I_3^{(2)}}$	$\diamond$
$\diamond$	$\mathbf{F_3^{(2)}}$	$\mathbf{I_4^{(2)}}$	$\mathbf{TV_2^{(2)}}$

TAB. 1 – Breaking down of the operators' offers on the system's components.

• in the class A^3, the attributes are non-comparable, i.e. appear in only one component of the system. $p_g,\ g = 1, 2, 3$, represents a class g attribute. We also introduce importance weights, $w_k,\ k = 1, 2, 3, 4$, measuring the relative importance that each customer gives to every component of the system. These coefficients are positive and sum up to 1, i.e. : $\sum_{k=1}^{4} w_k = 1$. Furthermore, each consumer i, affects a mark, $X^i_{j\ p_g} \in [\![1; 10]\!]$ to the presence of the attribute p_g in the offer j, belonging to one of the 4 system's components.

The weighted sum of the value of the attribute $p_g,\ g = 1, 2$ in the offer b, for the consumer i, can be written :

$$S^{ib}_{p_g} = \sum_{k=1}^{4} w_k \sum_{j \in J_k} \mathbf{1}_{(j,p_g,b)} X^i_{jp_g},\ g = 1, 2\ , \tag{1}$$

where, $\mathbf{1}_{(j,p_g,b)} = 1$, if the attribute p_g can be found in the j^{th} element of the k^{th} component of the system, provided it belongs to the offer b, and 0, otherwise.

The weighted dispersion of the value of the attribute $p_g,\ g = 1, 2$, in the bundle b, for the customer i, is of the form :

$$D^{ib}_{p_g} = \sum_{k=1}^{4} w_k \sum_{j \in J_k} \mathbf{1}_{(j,p_g,b)} \left(X^i_{jp_g} - \bar{X}^{ib}_{p_g} \right)^2,\ g = 1, 2, \tag{2}$$

where,

$$\bar{X}^{ib}_{p_g} = \frac{\sum_{k=1}^{4} \sum_{j \in J_k} \mathbf{1}_{(j,p_g,b)} X^i_{jp_g}}{\sum_{k=1}^{4} \sum_{j \in J_k} \mathbf{1}_{(j,p_g,b)}}\ ,\ g = 1, 2\ .$$

Furthermore, we suppose that $K \in \mathbb{N}$ categories of services are defined on the market.

Utility of an offer.

Our utility model is based on the random utility model of McFadden and Train ([10]). Let F, be the maximal number of segments. Each customer i belongs necessarily to a segment $f \in \mathcal{F} := \{1, 2, ..., F\}$.

If we suppose that the consumer i belongs to the segment f, based on the random utility model, we estimate his utility for the offer b, as follows :

$$U_{ib|i\in f}(P_b) \;=\; V_{ib|i\in f}(P_b) + \varepsilon_{ib},\ \forall i,\ \forall b \in \mathcal{B},\ \forall f \in \mathcal{F}\ . \tag{3}$$

ε_{ib} is an error term, traducing the bias introduced by the consumer i. $V_{ib|i\in f}$, is the "observed" value given by the consumer i to the offer b, conditionally to his belonging to the segment f.

The observed value takes the analytic expression :

$$\begin{aligned} V_{ib|i\in f}(P_b) \;&=\; \alpha_0^{fi} + \sum_{p_1\in A^1}[\beta_{p1}^{fi}S_{p_1}^{ib} + \gamma_{p_1}^{fi}D_{p_1}^{ib}] + \sum_{p_2\in A^2}[\beta_{p_2}^{fi}S_{p_2}^{ib} + \gamma_{p_2}^{fi}D_{p_2}^{fi}] + \sum_{p_3\in A^3}\alpha_{p_3}^{fi}C_{p_3}^{ib} + \alpha_P^{fi}P_b\ , \\ &=\; BV_{ib|i\in f} + \alpha_P^{fi}P_b\ . \end{aligned} \tag{4}$$

The sums and dispersion terms $S_{p_g}^{ib}$ and $D_{p_g}^{ib}$, $g = 1, 2$ are computed based on data. $C_{p_3}^{ib}$ can be fixed arbitrarily, and the term P_b contains the offer b's price.

In the case of a simple offer b_s, the function modeling the intrinseque value of the bundle takes the simplified form :

$$BV_{ib_s|i\in f} \;=\; \alpha_0^{fi} + \sum_{p_3\in A^3}\alpha_{p_3}^{fi}C_{p_3}^{ib_s}\ ,$$

whereas in the case of a double-play offer b_d, we get :

$$BV_{ib_d|i\in f} \;=\; \alpha_0^{fi} + \sum_{p_2\in A^2}[\beta_{p_2}^{fi}S_{p_2}^{ib_d} + \gamma_{p_2}^{fi}D_{p_2}^{ib_d}] + \sum_{p_3\in A^3}\alpha_{p_3}^{fi}C_{p_3}^{ib_d}\ .$$

The difficulty lies in the fact that the various coefficients are unknown. However, they can lead to quite useful interpretations in terms of marketing :

- α_0^{fi}, is an intercept parameter,
- $\beta_{p_g}^{ib} \geq 0$ called unbalancing parameter, means that the attribute p_g is desirable for the customer i, in the offer b, while a negative coefficient traduces the lack of interest of the customer in the presence of this attribute in the product,
- $\gamma_{p_g}^{ib} \geq 0$ called balancing parameter, means that the customer i sees the attribute p_g, in the bundle b, as complementary in the offer b, whereas a negative coefficient reveals its subsituability.

- Finally, the coefficient $\alpha_P^{if} \leq 0$ traduces the sensitivity of the customer i conditionally on his appartenance to the segment f, towards price.

Using conditional probability theory, the utility of the customer i for the offer b is :

$$V_{ib}(P_b) = \sum_{f=1}^{F}\psi_{fi}V_{ib|i\in f}(P_b)\ , \tag{5}$$

where ψ_{fi}, $f \in \mathcal{F}$, is the unknown probability that the customer i belongs to the segment f.

Estimation of the model's parameters.

To model the correlations between the error terms associated with the various offers in a choice set, we use the Nested Logit model. Solely the error term associated with the null choice is independent of the others. Then, the cumulative function of the error terms $(\varepsilon_{ib})_b$ takes the form :

$$F(\{\varepsilon_{ib}\}_{b=0}^{|\mathcal{B}|}) \;=\; \exp[-\exp(-\varepsilon_{i0}) - [\sum_{b=1}^{|\mathcal{B}|} \exp(-\frac{\varepsilon_{ib}}{\rho_{\mathbf{i1}}})]^{\rho_{\mathbf{i1}}}],\; 0 < \rho_{\mathbf{i1}} \leq 1\;. \tag{6}$$

The highest ρ_{i1} is, the more independence there is between error terms and as a first approximation (cf [10]), the less correlation there is between them. The error terms are simulated using acceptance-rejection method.
To estimate the coefficients of the equations (4), the probability of segment belongings defined in the equation (5), and the correlation coefficients introduced in (6), we use Monte Carlo Markov chain methods (MCMC), based on a priori, hyper a priori distributions. The algorithm used is an application of [7] to our problem. Briefly, it is based on an alternation of additionnal information generation, and Gibbs' sampling which re-estimate the parameters of interest using the new information.

Estimation of reservation prices and segmentation of the market.

Let $\mathcal{B} - \{b\}$, be the choice set $\mathcal{B}$ whose element b has been cancelled. Using the definition of the reservation price, we get that the reservation price of the customer i for the offer b can be implicitly defined as follows :

$$U_{ib}(P_b) = \max_{k\,\in\,\mathcal{B}-\{b\}} U_{ik}(P_k)\;. \tag{7}$$

More precisely,

$$\max_{k\in\mathcal{B}-\{b\}} U_{ik}(P_k) \;=\; \max\left\{\max_{k\in\mathcal{B},\,k\neq b,\,k\neq 0}\left\{U_{ik}(P_k);\frac{U_{i0}}{\rho_{i1}} - \sum_{k\in\mathcal{B},\,k\neq b,\,k\neq 0} \exp\left[\frac{U_{ik}(P_k)}{\rho_{i1}}\right]\right\}\right\}\;.$$

Then, applying (3) and (4) to (7), we get the following expression for the reservation price of the consumer i, for the offer b :

$$RP_{ib} \;=\; -\frac{BV_{ib} + \varepsilon_{ib} - \max_{k\in\mathcal{B}-\{b\}} U_{ik}(P_k)}{\alpha_P^{Fi}}\;.$$

However, the model parameters and the error terms are random variables. Hence, the reservation price of the customer i for the bundle b, in the choice set $\mathcal{B}$, can be estimated as follows :

$$\begin{aligned}\mathbf{E}[RP_{ib}] \;&=\; -\mathbf{E}\left[\frac{\alpha_0^{Fi}}{\alpha_P^{Fi}} + \sum_{p_1\in A^1}(\frac{\beta_{p_1}^{Fi}}{\alpha_P^{Fi}}S_{p_1}^{ib} + \frac{\gamma_{p_1}^{Fi}}{\alpha_P^{Fi}}D_{p_1}^{ib}) + \sum_{p_2\in A^2}(\frac{\beta_{p_2}^{Fi}}{\alpha_P^{Fi}}S_{p_2}^{ib} + \frac{\gamma_{p_2}^{Fi}}{\alpha_P^{Fi}}D_{p_2}^{ib}) + \sum_{p_3\in A^3}\frac{\alpha_{p_3}^{Fi}}{\alpha_P^{Fi}}C_{p_3}^{ib}\right]\\ &+\; \mathbf{E}\left[\frac{\max_{k\,\in\,\mathcal{B}-b} U_{ik}}{\alpha_P^{Fi}}\right]\;.\end{aligned} \tag{8}$$

To segment the market, we let the algorithm run on various models, where each model contains the maximum number of segments to use. Let F_i, be the maximal number of segments associated with the i^{th} model, $\mathcal{M}_i$. Let ϕ_i, be the values of the parameters obtained as outputs of the simulation.

Assume that the density of the model $\mathcal{M}_i$, is contained in : $f(\phi|\mathcal{M}_i, F_i)$. Bayesian inference tells us that to compare the i^{th} model to the others, we must compute the a posteriori probabilities :

$$\pi(\mathcal{M}_i|\phi) = \frac{\pi(\phi|\mathcal{M}_i)\pi(\mathcal{M}_i)}{\sum_{k=1}^{F} m(\phi|\mathcal{M}_k)\pi_k(\mathcal{M}_k)} .$$

$\pi(\mathcal{M}_i)$, is the a priori probability of the model $\mathcal{M}_i$, and $m(\phi|\mathcal{M}_i)$, is called marginal likelihood of the estimated coefficients, for the i^{th} model. Analytically, it can be written :

$$m(\phi|\mathcal{M}_i) = \int f(\phi|F_i, \mathcal{M}_i)\pi_i(F_i|\mathcal{M}_i)dF_i . \tag{9}$$

since the marginal likelihood is the normalizing constant of the a posteriori density, we get the basic expression of the marginal likelihood (cf [15]) :

$$m(\phi|\mathcal{M}_i) = \frac{f(\phi|\mathcal{M}_i, F_i)\pi(F_i|\mathcal{M}_i)}{\pi(F_i|\phi, \mathcal{M}_i)} .$$

Computing the identity at a specific value, $F_i^\star$, and taking the logarithm, we get :

$$\log m(\phi|\mathcal{M}_i) = \log\ f(\phi|\mathcal{M}_i, F_i^\star) + \log \pi(F_i^\star|\mathcal{M}_i) - \log \pi(F_i^\star|\phi, \mathcal{M}_i) .$$

This equation shows that the marginal likelihood relies on the a priori $\pi(F_i^\star|\mathcal{M}_i)$ and the a posteriori $\pi(F_i^\star|\phi, \mathcal{M}_i)$. This value can increase as the number of segments grows. Indeed, the last term can penalize the marginal likelihood, when new segments are added. The integral in the equation (9) is difficult to compute, all the more as the number of segments used grows (cf [15] and [31]). If we identify the model, with the number of segments to consider, the formal expression of the marginal likelihood becomes :

$$m(\phi) = \int f(\phi|F)\pi(F)dF .$$

Using Monte-Carlo simulation with $\pi(F|\phi)$ as importance function, the harmonic mean of the likelihood values becomes an estimate of marginal likelihood :

$$\hat{m}(\phi) = \left\{ \frac{1}{s}\sum_{k=1}^{s} \pi(\phi|F_k)^{-1} \right\}^{-1} ,\ s = 1, 2, ..., F . \tag{10}$$

As explained in [7], the likelihood obtained as output of the algorithm is of the form :

$$\begin{aligned}
\pi(\phi|F_k) &= \prod_{i\in\mathcal{N}} \sum_{f=1}^{k} \left\{ P_{i0|i\in k}^{C_i(0)} \prod_{b=1}^{|\mathcal{B}|} \left[P_{ib|b\neq 0, i\in k}\ (1 - P_{i0|i\in k})\right]^{C_i(b)} \psi_{ik} \right\} , \\
P_{ib|b\neq 0, i\in k} &= \frac{V_{ib|i\in k}}{\sum_{b'\in\mathcal{B}-\{0\}} \exp[V_{ib'|i\in k}]} , \\
P_{i0|i\in k} &= \frac{1}{1 + \exp[\rho_{i0} + \rho_{i1}\mathcal{V}]} ,\ \text{with}\ \mathcal{V} = \ln\left(\sum_{b=1}^{|\mathcal{B}|} \exp[\frac{V_{ib|i\in k}}{\rho_{i1}}]\right) .
\end{aligned}$$

$C_i(l) \in \{0; 1\}$, $\forall l \in \mathcal{B}$ models the consumer i's decisions maximizing his likelihood. The value of s, maximizing the estimate of the log marginal likelihood, coïncides with the optimal number of segments to consider.

Rate	Internet	non comparable
Security	Internet, Fix phone	partially comparable
Length of commitment	Internet, Fix, TV, Mobile	fully comparable
Illimited communication towards Fix phones	Fix, Mobile	partially comparable
Illimited communication towards Mobile phones	Fix, Mobile	partially comparable
Hotline 24h/24, 7d/7	Internet, Fix, TV, Mobile	fully comparable
Number of free TV channels	Mobile, TV	non comparable
Mobile card	Mobile	non comparable
Trust in the brand	Fix, Internet, TV, Mobile	fully comparable
Quality of Service / Reliability	Fix, Internet, TV, Mobile	fully comparable

Tab. 2 – Description of the main attributes of the model and their categories.

2.3 An application

We assume that 9 categories of services are available on the market :

• simple offers of **Fix telephony** (F), **Mobile phone** (M) and **Internet** (I),

• double-play offers : **Mobile phone-Internet** (M-I), **Mobile-TV** (M-TV), **Internet-Fix phone** (I-F), **Internet-TV** (I-TV),

• triple-play offers, i.e. bundles made of 3 distinct services, such as : **Fix-Internet-TV** (F-I-TV) and **Mobile-Internet-TV** (M-I-TV).

On the one side, the operator 1 markets a Mobile offer, a double-play offer of Fix phone and Internet, and a bundle made of Mobile, Internet and TV offers. On the other side, the operator 2, which doesn't own any mobile network, sells simple services of Fix telephony and Internet, and a triple-play offer of Fix phone, Internet and TV services. Hence, the customers have the choice between 6 different services. As shown in Tabular 1, each offer can be broken down in components belonging to a system's category. Note that that the operator 1's offers are denoted with a superscript indice (1), while the operator 2's services have a superscript indice (2).

In Tabular 2, we have stored the main attributes defining our model, and specify their categories.

Applying our extended preference model to a sample of consumers' data, we segment optimally the market in 4 segments. In Figure 1, we have drawn the estimated distributions of the balancing and unbalancing coefficients ($\hat{\text{Trust}}_f$) associated with the attribute of trust, on two market segments ($f = 1$ et $f = 3$). The densities are estimated using Fisher's coefficients, which characterize uniquely first and second order moments. As output of the preference model, we also get estimates of the densities modeling customers' price sensitivity, on each of the 4 market segments that we have previously detected (see Figure 2). Finally, we have drawn the estimated distributions of reservation prices on each of the 4 market segments, for the bundle offer of the operator 2 (see Figure 3). We also get M_f, the proportion of customers belonging to the market segment f.

3 Pricing under horizontal competition between the operators

Let $\mathcal{N}$, be the set of actors playing on the market. It is made of $N \in \mathbb{N}$ distinct consumers[3], and two rival operators. The model can easily be extended to the case of more than two operators. In this section, we assume that both operators define their product lines independently of one another.

3.1 A bayesian game between the operators and the customers

The type of the customer $i \in \{1, 2, ..., N\}$, is the number of the segment, to which he belongs. The customer i's type space is formed of all the possible segments :

$$T_i := \mathcal{F} , \forall i \in \{1, 2, ..., N\} . \tag{11}$$

However, the customer i knows to which segment he belongs, which means that he knows his *true* type.

The customer i's action space, C_i, is made of all the possible combinations of 6 elements, taking values in the binary set $\{0; 1\}$. We suppose that the consumer i has the choice between buying one of the offers, or nothing at all, which is equivalent to select the null option in the choice set.

$$C_i = \left\{ c_i = (c_i(1),\ c_i(2),\ ...,\ c_i(|\mathcal{B}|))^T \in \{0; 1\}^{|\mathcal{B}|} \right\} , \forall i \in \{1, 2, ..., N\} , \tag{12}$$

$c_i(b) \in \{0; 1\}$, is a binary variable such that $c_i(b) = 1$, whenever the customer i buys the offer b.

Assume that all the consumers know a mean market price (that is the mean of the prices at which the operators sold this kind of offer in the past) for each category of offers sold by the operator : $p_{\mathcal{M}}(b),\ \forall b \in \{1, 2, ..., K\}$.

For each possible type $t_i \in T_i$, of the player i, the subjective probability function must specify a probability distribution on the set T_{-i}, which represents what player i believes about the other players, provided his personal type is t_i.

$$\begin{aligned} p_i : \quad T_i &\rightarrow \Delta(T_{-i}) \\ t_i &\mapsto p_i(.|t_i) . \end{aligned} \tag{13}$$

Formally, for all combinations of types $t_{-i} \in T_{-i}$, $p_i(t_{-i}|t_i)$ represents the belief probability that the consumer i associates with the eventuality that t_{-i} should be the profile of types of the $(|\mathcal{N}| - 1)$ other players.

[3]If we know the total number of consumers, N, we can determine estimates of the number of consumers belonging to each market segment, using the section 2 results. Let N_{t_i}, be the amount of consumers belonging to segment t_i, $i \in \mathcal{F}$, then : $N_{t_i} = M_{t_i} N$, where M_{t_i} is the proportion of consumers belonging to t_i.

Hypothesis :

• We assume that for every customer, there exists a marginal *a priori* probability distribution modeling the probability that the consumer i might belong to the segment t_i : $\bar{p_i}(t_i) \in \Delta(T_i)$.

$$\bar{p_i}(t_i) \;=\; M_{t_i} \;,\; \forall t_i \in T_i \;,\; \forall i \in \mathcal{N} \;,$$

where M_{t_i}, is the proportion of consumers belonging to the segment t_i.

• In general, the prices at which the operator sells his offers do not really reflect their real costs (these costs include equipement costs, sometimes high costs of advertisement, interconnection agreements...). Consequentely, we assume that the operator k's real cost is a fraction (called lie-coefficient) of the mean market price. The operator k's type space then contains the set of all the possible lie-coefficients :

$$T_{\mathrm{op}_k} \;:=\; \left\{ s \in \mathcal{S} = \{\frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}\} \right\} \;. \tag{14}$$

The real cost of the offer b is then : $\delta_{\mathrm{op}_k}(b) \;:=\; t_{\mathrm{op}_k} \; p_{\mathcal{M}}(b)$. Furthermore, if the customer i belongs to the segment f, the probability that the operator's type might be s, is : $p_i(t_{\mathrm{op}_k} = s|t_i) \;=\; \mathbf{P}[S = s] \;=\; \frac{\hat{\mathrm{Trust}}_f(s)}{\sum_{s' \in \mathcal{S}} \hat{\mathrm{Trust}}_f(s')}$[4]. This probability quantifies the extent with which the customer i trusts in the operator.

• We suppose that consumers'types are independent of one another, and of the operators' beliefs.

The utility of the customer i, u_i, is a function from the space $C \times T$, in the real numbers, $\mathbb{R}$. We let :

$$\begin{aligned} u_i(c,t) \;&=\; \alpha_{t_i} \left(\sum_{j=1}^{2} \sum_{b=1}^{3} [X_{t_i}(b) - c_{\mathrm{op}_j}(b)] c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\mathrm{op}_j}(b)\}} \right) \\ &+\; (1 - \alpha_{t_i}) \left(\sum_{j=1}^{2} \sum_{b=1}^{3} X_{t_i}(b) c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\mathrm{op}_j}(b)\}} \right) \;, \\ &\text{with } X_{t_i}(b) \;\sim\; f_{t_i}(b), \; \forall i \in \{1, 2, ..., N\}, \; \forall b \in \mathcal{B} \;. \end{aligned} \tag{15}$$

This function traduces the phenomenon of cannibalization between the products in a same line, or products belonging to avderse lines. The consumer i's reservation price for the offer b, is drawn according to the estimated density of the reservation price towards this offer, on the segment to which the customer belongs : $f_{t_i}(b)$. The first part of the utility function represents the difference between the reservation price and the selling price of the offer, provided the customer i's reservation price is higher than the selling price of the offer and the customer i buys the offer ($c_i(b) \neq 0$). The second part of the equation takes into account the reservation prices of each offer in the choice set.

[4] The discretization of the probability density results in a homothetic transformation of the initial density, on the discrete support.

The parameter $\alpha_{t_i} \in [0;1]$ is the normalized positive part of the coefficient of price sensitivity, drawn according to the estimated density, on the segment t_i.

Judging by the mean market prices of the offers, the operator can choose one of the following 3 actions :

- increasing the price of an integer in the interval $[\![1;\xi]\!]$, where $\xi \in \mathbb{N}$, and $\xi \leq \min_{k \in \mathcal{B}} \{p_{\mathcal{M}}(b)\}$,
- decreasing the price of an integer in the interval $[\![-1;-\xi]\!]$,
- doing nothing.

Furthermore, 3 strategies are available for the operator. First, he can target a specific market segment. In this case, the operator must adapt the price to the targeted clients. Secondly, he can try to win as fast as possible the highest possible amount of market shares. In this case, he doesn't take into account the market segmentation, and tries to seduce the more clients possible. Finally, he can use a strategy based on price discrimination, i.e. sells the same offer at different prices on every segment.

In the case of a strategy of expansion, the action set of the operator j is of the form :

$$C_{\mathrm{op}_j} = \left\{ c_{\mathrm{op}_j} \in \mathbb{R}^3 \mid c_{\mathrm{op}_j}(b) \in [\![p_{\mathcal{M}}(b) - \xi; p_{\mathcal{M}}(b) + \xi]\!] \,,\ \forall b \in \mathcal{B}_{\mathrm{op}_j} \right\} . \tag{16}$$

But, if the operator j's action depends on market segments, it is denoted : $c_{\mathrm{op}_{j,k}},\ k \in \mathcal{F}$ and the action space contains all the possible combinations of actions, on each segment.

$$C_{\mathrm{op}_j} = \left\{ c_{\mathrm{op}_j} \in \mathbb{R}^{F * |\mathcal{B}_{\mathrm{op}_j}|} \mid c_{\mathrm{op}_{j,k}}(b) \in [\![p_{\mathcal{M}}(b) - \xi; p_{\mathcal{M}}(b) + \xi]\!] \,,\ \forall k \in \mathcal{F},\ \forall b \in \mathcal{B}_{\mathrm{op}_j} \right\} . \tag{17}$$

The operator j's utility then, takes one of the following three forms :

$$\begin{aligned}
\textbf{Targeted strategy}:\ \Phi_1 &:= \max_{k \in \{1,2,\dots,F\}} \sum_{b=1}^{3} \left(c_{\mathrm{op}_{j,k}}(b) - \delta_{\mathrm{op}_j}(b) \right) \left(\sum_{i=1}^{N} c_i(b) \mathbf{1}_{\{f_{t_i} = f_k\} \cap \{X_{t_i}(b) \geq c_{\mathrm{op}_{j,k}}(b)\}} \right) , \\
\textbf{Expansion of market shares}:\ \Phi_2 &:= \sum_{b=1}^{3} \left(c_{\mathrm{op}_j}(b) - \delta_{\mathrm{op}_j}(b) \right) \left(\sum_{i=1}^{N} c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\mathrm{op}_j}(b)\}} \right) , \\
\textbf{Price discrimination}:\ \Phi_3 &:= \sum_{k=1}^{F} \sum_{b=1}^{3} \left(c_{\mathrm{op}_{j,k}}(b) - \delta_{\mathrm{op}_j}(b) \right) \left(\sum_{i=1}^{N} c_i(b) \mathbf{1}_{\{f_{t_i} = f_k\} \cap \{X_{t_i}(b) \geq c_{\mathrm{op}_{j,k}}(b)\}} \right) .
\end{aligned} \tag{18}$$

Consequently[5], at the time instant $t \geq 0$, the operator j's utility is one of the 3 functions, $\Phi_k,\ k = 1,2,3$.

$$u_{\mathrm{op}_j}(c,t) \in \{\Phi_1,\ \Phi_2,\ \Phi_3\} \ ,\ \forall j = 1,2 \,. \tag{19}$$

[5]In the case of a discrimination strategy, the consumer i's utility should be slightly modified :

$$u_i(c,t) = \alpha_{t_i} \left(\sum_{j=1}^{2} \sum_{b=1}^{|\mathcal{B}_{\mathrm{op}_j}|} [X_{t_i}(b) - c_{\mathrm{op}_{j,t_i}}(b)] c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\mathrm{op}_{j,t_i}}(b)\}} \right) + (1 - \alpha_{t_i}) \left(\sum_{j=1}^{2} \sum_{b=1}^{|\mathcal{B}_{\mathrm{op}_j}|} X_{t_i}(b) c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\mathrm{op}_{j,t_i}}(b)\}} \right) ,$$

with $X_{t_i}(b) \sim f_{t_i}(b),\ \forall i \in \{1,2,...,N\},\ \forall b \in \mathcal{B}$.

The subjective probability distribution of the operator $j,\ j = 1, 2$, takes the form :

$$\begin{cases} p_{\mathrm{op}_j}(t_{-\mathrm{op}_j}|t_{\mathrm{op}_j}) \ = \ \prod_{i=1}^{N} \bar{p}_i(t_i)\ p_{\mathrm{op}_j}(t_{\mathrm{op}_k}) \ , \ k \neq j, \ k \in \{1;2\} \ , \\ p_{\mathrm{op}_j}(t_{\mathrm{op}_k}) \ = \ \frac{1}{|T_{\mathrm{op}_k}|} \ = \ \frac{1}{|\mathcal{S}|} \ , \ j,k = 1,2, \ j \neq k \ . \end{cases} \tag{20}$$

The operator j does not have any information about his rival. Hence, his subjective belief is distributed according to the uniform density on the lie-coefficient set.

3.2 Computation of Bayesian equilibria

Formally, the problem described above can be modeled as a bayesian game, denoted as :

$$\Gamma^b \ = \ (\mathcal{N}, (C_i)_{i\in\mathcal{N}}, (T_i)_{i\in\mathcal{N}}, (p_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}}) \ .$$

A randomized-strategy profile for the bayesian game Γ^b, is any σ belonging to the set $\times_{i\in\mathcal{N}} \ \times_{t_i\in T_i}$ $\Delta(C_i)$, which satisfy the following constraints :

$$\begin{cases} \sigma \ = \ ((\sigma_i(c_i|t_i))_{c_i\in C_i})_{t_i\in T_i, i\in\mathcal{N}} \ , \\ \sigma_i(c_i|t_i) \geq 0, \ \forall c_i \in C_i, \ \forall t_i \in T_i, \ \forall i \in \mathcal{N} \ , \\ \sum_{c_i\in C_i} \sigma_i(c_i|t_i) \ = \ 1, \ \forall t_i \in T_i, \ \forall i \in \mathcal{N} \ . \end{cases}$$

In such a strategy profile, $\sigma_i(c_i|t_i)$ represents the conditional probability that the player i would choose the action c_i, provided his type is t_i. In the strategy profile σ, the randomized-strategy associated with the type t_i of the player i, is :

$$\sigma_i(.|t_i) \ = \ (\sigma_i(c_i|t_i))_{c_i\in C_i} \ .$$

A bayesian equilibrium of the game Γ^b, is any randomized strategy-profile such that, for any player i, and for any type $t_i \in T_i$,

$$\sigma_i(.|t_i) \in \arg \max_{\tau_i\in\Delta(C_i)} \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i) \sum_{c\in C} \left(\prod_{j\in\mathcal{N}-i} \sigma_j(c_j|t_j) \right) \tau_i(c_i)\ u_i(c,t) \ . \tag{21}$$

3.3 Approximation of Bayesian equilibria using simulation

The utility functions are quite complex, and differ between the various players. In the article [11], Holenstein uses simulation to approximate the equilibria, in auction game, where the players' types and the action spaces of both players are continuous. The use of simulation in auction games is quite a novelty and few articles are available on the subject[6]. Our aim is to determine a best response based on a myopic approach, using Monte Carlo simulation and Hastings-Metropolis algorithm. For each

[6]Cai and al. use Monte Carlo methods to approximate Bayesian equilibria in sequential auction games, cf [19].

agent i, the strategies are initially simulated according to a uniform distribution on the action set, i.e. :

$$\sigma_i^{(0)}(c_i|t_i) \;=\; \frac{1}{|C_i|},\; i \in \{1,2,...,N,\mathrm{op}_1,\mathrm{op}_2\}\;,\; c_i \in C_i,\; t_i \in T_i\;.$$

Then, at each iteration of the algorithm, a player will try to maximize his utility for a type value, simulated according to a one trial multinomial law, $\mathcal{M}(1;p_1,p_2,...,p_F)$ for the clients, and a uniform law on the operator type space, $\mathcal{U}_{T_{\mathrm{op}_j}}$, $j=1,2$, for the operators. His type been fixed, the player's new strategy is a best response to the strategies of the other players.

Algorithm 1. Bayesian equilibria approximation

Let $t=0$, and intialize the temperature : $\mathrm{Temp}(0)=1$.

The players' strategies are initialized according to a uniform distribution on the action spaces :

$$\sigma_i^{(0)}(c_i|t_i) \;=\; \frac{1}{|C_i|},\; i \in \{1,2,...,N,\mathrm{op}_1,\mathrm{op}_2\}\;.$$

While the norm of the players' conditional strategies changes by more than a fixed constant,

for each player i **do**,

$*$ for each customer, sample a type according to the prior distribution : $t_i \;\sim\; \mathcal{M}(1;p_1,p_2,...,p_F),\; i=1,2,...,N$ [7]. The operator's type is distributed according to a uniform law on their type space : $t_{\mathrm{op}_j} \;\sim\; \mathcal{U}_{T_{\mathrm{op}_j}},\; j=1,2$.

$*$ Then, determine the best response given the other players' strategies (via **Algorithm 2**).

$*$ Update the player i's strategy profile.

End,

$t=t+1$, update the temperature which decreases according to a pre-determined law, giving : $\mathrm{Temp}^{(t+1)}$.

End.

The algorithm computing player i's best response, given his type and the strategies of the other players, is descibed below.

Algorithm 2. Determination of the best response

Initialize the player i's randomized-strategy profile : $\sigma_i^{(0)}(.|t_i)$, while the $(|\mathcal{N}|-1)$ other players' strategies remain unchanged. Compute the associated expected utility :

$$U_i^{(0)}(\sigma^{(0)}|t_i) \;=\; \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i) \sum_{c\in C} \left\{ \prod_{j\in N-\{i\}} \sigma_j^{(t)}(c_j|t_j) \right\} \sigma_i^{(0)}(c_i|t_i)\; u_i(c,t)\;,$$

where, t_i is the player i's type, drawn according to the initial marginal distribution, in the main algorithm.

For k **from** 0 **to** Maximum number of steps,

sample $z \sim \mathcal{U}_{[0;1]}$, and sample $\sigma_i^\star(.|t_i) \;\sim\; q\left(\sigma_i^\star(.|t_i) \mid \sigma_i^{(k)}(.|t_i)\right)$.

At each step, compute the expected utility :

$$U_i^\star(\sigma^\star|t_i) \;=\; \sum_{t_{-i}\in T_{-i}} p_i(t_{-i}|t_i) \sum_{c\in C} \left\{ \prod_{j\in N-\{i\}} \sigma_j^{(k)}(c_j|t_j) \right\} \sigma_i^\star(c_i|t_i)\; u_i(c,t)^8\;.$$

If $z < \min\left\{1;\exp\left\{-\left[\frac{(U_i^\star(\sigma^{(k)}|t_i)-U_i^\star(\sigma^\star|t_i))}{\mathrm{Temp}^{(t)}}\right]\right\}\right\}$, **then** :

$\sigma_i^{(k+1)}(.|t_i) \;=\; \sigma_i^\star(.|t_i)$ and $U_i^{(k+1)}(.|t_i) \;=\; U_i^\star(\sigma^\star|t_i)$.

Otherwise, $\sigma_i^{(k+1)}(.|t_i) \;=\; \sigma_i^{(k)}(.|t_i)$ and $U_i^{(k+1)}(.|t_i) \;=\; U_i^{(k)}(\sigma^{(k)}|t_i)$.

End.

End.

[7] p_f, is the probability of belonging to the segment f .

[8] In the expected utility, we have to take into account the other players' strategies, which have been updated at the t^{th} iteration step.

The parameters of the instrumental density q, are updated at each iteration step. If we suppose that the state space is big but finite, q is a discrete distribution, whose normalized weights are issued from a multi-dimensional normal density centered in the last probability vector accepted, symetrically distributed on the randomized action vectors' space. Hence, q can be idenitfied with a multinomial law, whose weights change at each iteration $k : q = \mathcal{M}(1; p_k(1), p_k(2), ..., p_k(|C_i|))$. Consequentely, for each player i, conditionally to his type, we draw a Markov chain.

Theorem 1 *Assuming the finite dimensionality of the action space and the irreductibility of the conditional chains for each player, we prove that the algorithm converges towards a bayesian equilibrium.*

Proof. The proof is inspired from [18]. Let $\mathcal{X}_i$, be the set of randomized action-vectors for player i. Let $\mathcal{B}^\star$, be the set of bayesian equilibria, associated with our game. Formally :

$$\mathcal{B}^\star = \{\sigma \in \times_{i\in\mathcal{N}} \Delta(C_i) \mid \sigma \text{ is a bayesian equilibrium}\} .$$

At the temperature Temp(t), suppose we have simulated the type $t_i \in T_i$. In the Metropolis' algorithm, for each player, conditionally to his type, we choose a stopping rule of the form :

$$A^{\text{player } i}_{s_1,s_2}(t_i) := \min\left\{1; \exp\left(-[\frac{U_i^\star(s_1|t_i) - U_i^\star(s_2|t_i)}{\text{Temp}^{(t)}}]\right)\right\} , \forall s_1, s_2 \in \Delta(C_i) . \tag{22}$$

Let a^+, be the positive part of the real number a.

$$a^+ = \begin{cases} a, \text{ if } a > 0, \\ 0, \text{ otherwise} . \end{cases}$$

Using this definition, the equation (22), can be simplified, to give :

$$A^{\text{player } i}_{s_1,s_2}(t_i) := \exp\left\{-[\frac{U_i^\star(s_1|t_i) - U_i^\star(s_2|t_i)}{\text{Temp}^{(t)}}]^+\right\} , s_1, s_2 \in \Delta(C_i) . \tag{23}$$

At the iteration k, the probability associated with the simulation of the probability vector s_2, starting at s_1, and conditionally to the type t_i for the player i, is of the form :

$$G^{\text{player } i}_{s_1,s_2}(t_i) = p_k(s_2) . \tag{24}$$

$p_k(s_2)$, is the probability at the iteration k, to draw the state s_2, provided the weights are symetrically distributed on the states in the neighbourhood of s_1. We note that $G^{\text{player } i}_{s_1,s_2}(t_i) = G^{\text{player } i}_{s_2,s_1}(t_i)$, since the weights are supposed symmetrically distributed around the origin.

Using (22) and (24), we get the expression of the transition probability from the state s_1 to the state s_2, for the player i, conditionally to his type t_i :

$$P^{\text{player } i}_{s_1,s_2}(t_i) = \begin{cases} G^{\text{player } i}_{s_1,s_2}(t_i)\, A^{\text{player } i}_{s_1,s_2}(t_i) , \text{ if } s_2 \neq s_1 , \\ 1 - \sum_{s\in\mathcal{X}_i - \{s_1\}} P^{\text{player } i}_{s_1,s}(t_i) . \end{cases} \tag{25}$$

Definition 3 *Let $P^{player\ i}(t_i)$, be the matrix containing the transition probabilities issued from the algorithm, which generate the Markov chain $\{X_k^{player\ i}(t_i)\}_k$, for the player i, conditionally to his type t_i, at the temperature $Temp(t)$.*

$$X_k \;=\; \left\{ \left(X_k^{player\ i}(t_i) \right)_k \,,\; t_i \in T_i,\; i \in \mathcal{N} \right\} \,,$$

contains the set of Markov chains, generated for each player, conditionally to each type.
We say that the algorithm converges with probability one, if, and only if :

$$\lim_{Temp \to^{+} 0} \; \lim_{k \to (Maximum\ number\ of\ iteration\ steps)} \mathbf{P}[(X_k \;\in\; \mathcal{B}^\star] \;=\; 1 \,. \tag{26}$$

Note that ideally the maximum number of iteration steps to get best reponses should be big enough to determine global maxima. The temperature parameter in the main algorithm might prevent us to get stuck in local extrema.
We recall results, enabling to prove the existence and unicity of a stationary distribution for the Markov chain. Remind that if a Markov chain is discrete in time and state space, and if it is furthermore irreductible, then all its states have the same periodicity. The following lemma introduces a sufficient condition to prove the aperiodicity of an irreductible chain.

Lemma 1 *An irreductible Markov chain, with transition matrix P, is aperiodic, if there exists a state j in the state space $\mathcal{X}$, such that $P_{j,j} > 0$.*

Lemma 2 *If P is the transition matrix of a finite state space irreductible and aperiodic Markov chain. Then, the chain has a unique stationary distribution, π. π is consequentely the unique distribution satisfying conditions of stationarity, and the chain converges asymptotically towards this stationary distribution.*
Conditions of Stationarity :
π is an invariant or stationary distribution of the chain, if :

$$\pi \;=\; \pi\, P \,.$$

Which means that π is a left eigenvector, associated with the eigenvalue 1, for the Markov chain. Besides, if the distribution π satisfies the following condition, then it is stationary :

$$\pi(i)\; P_{i,j} \;=\; \pi(j)\; P_{j,i} \,,\; \forall i,j \in \mathcal{X} \,. \tag{27}$$

Assume that $G^{\text{player } i}(.|t_i)$ is irreductible, since $P_{s_1,s_1}^{\text{player } i}(.|t_i) > 0$, the associated Markov chain is irreductible and aperiodic. The lemma (2) then tells us, that there exists a unique stationary distribution.
Hence, if we determine an invariant distribution for the probability transition matrix, it is the limit distribution towards which the chain converges. Boltzmann's distribution is a probability distribution on the action space which puts most of the weights on the states, maximizing the objective function (here, the randomized vectors).

Definition 4 *The Boltzmann's distribution for the player i, conditionally to his type t_i, is defined as follows :*

$$q_{s_1}^{player\ i}(t_i) \;=\; \frac{\exp[\frac{U_i(s_1|t_i)}{Temp(t)}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{U_i(s|t_i)}{Temp(t)}]}\ ,\ \forall s_1\in\mathcal{X}_i\ . \tag{28}$$

We have already seen, using weight symmetry that, $P_{s_1,s_2}^{\text{player } i}(t_i) = P_{s_2,s_1}^{\text{player } i}(t_i)$, $\forall t_i \in t_i$, $\forall i \in \mathcal{N}$. Consequentely, to prove that the Boltzmann's distribution is invariant for our transition matrix, it is sufficient to consider the conditional probability of acceptance :

$$\begin{aligned} q_{s_1}^{\text{player } i}(t_i)\ A_{s_1,s_2}^{\text{player } i}(t_i) &= \frac{\exp[\frac{U_i(s_1|t_i)}{\text{Temp}(t)}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{U_i(s|t_i)}{\text{Temp}(t)}]}\ \exp[\frac{U_i(s_1|t_1)-U_i(s_2|t_1)}{\text{Temp}(t)}]^+\ , \\ &= \frac{\exp[\frac{U_i(s_2|t_i)}{\text{Temp}(t)}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{U_i(s|t_i)}{\text{Temp}(t)}]}\ \exp[\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\text{Temp}(t)}]\ \exp[-\left(\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\text{Temp}(t)}\right)^+]\ . \end{aligned} \tag{29}$$

But, for each real number a, $a = a^+ + a^- = a + (-a)^+$, hence :

$$\exp[-\left(\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\text{Temp}(t)}\right)^+] \;=\; \exp[-\left\{[\frac{U_i(s_1|t_i)-U_i(s_2|t_i)}{\text{Temp}(t)}] + [\frac{U_i(s_2|t_i)-U_i(s_1|t_i)}{\text{Temp}(t)}]^+\right\}]\ .$$

Using simplifications, we get that :

$$q_{s_1}^{\text{player } i}(t_i)\ A_{s_1,s_2}^{\text{player } i}(t_i) \;=\; q_{s_2}^{\text{player } i}(t_i)\ A_{s_2,s_1}^{\text{player } i}(t_i)\ .$$

Then, q is a stationary distribution for the Markov chain. For every player i, conditionally to each type $t_i \in T_i$, if we assume that $G^{\text{player } i}(.|t_i)$ is irreductible, then the Markov chain issued from our algorithm converges towards the stationary distribution, $q^{\text{player } i}(t_i)$.

We note $\mathcal{BR}(.|t_i)$, the set of player i's best responses, conditionally to his type t_i, at the time instant t, at the temperature Temp(t). This set contains the probability vectors maximizing the player i's expected utility, comditionally to his type t_i, the strategies of the other players being fixed. We note $U_i^\star(t_i)$, the player i's expected utility value, conditionally to his type t_i, for every element of the best response space, $\mathcal{BR}(.|t_i)$. Using these definitions, Boltzmann's measure at time instant t, becomes :

$$q_{s_1}^{\text{player } i}(t_i) = \frac{\exp[\frac{U_i(s_1|t_i)}{\text{Temp}^{(t)}}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{U_i(s|t_i)}{\text{Temp}^{(t)}}]}\ , \tag{30}$$

$$= \frac{\exp[\frac{-U_i^\star(t_i)+U_i(s_1|t_i)}{\text{Temp}(t)}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{-U_i^\star(t_i)+U_i(s|t_i)}{\text{Temp}(t)}]}\ \left(\mathbf{1}_{s_1\in\mathcal{BR}(.|t_i)} + \mathbf{1}_{s_1\in\mathcal{X}_i-\mathcal{BR}(.|t_i)}\right)\ , \tag{31}$$

$$= \frac{1}{\sum_{s\in\mathcal{X}_i}\exp[\frac{-U_i^\star(t_i)+U_i(s|t_i)}{\text{Temp}(t)}]}\ \mathbf{1}_{s_1\in\mathcal{BR}(.|t_i)} + \frac{\exp[\frac{-U_i^\star(t_i)+U_i(s_1|t_i)}{\text{Temp}^{(t)}}]}{\sum_{s\in\mathcal{X}_i}\exp[\frac{-U_i^\star(t_i)+U_i(s|t_i)}{\text{Temp}(t)}]}\ \mathbf{1}_{s_1\in\mathcal{X}_i-\mathcal{BR}(.|t_i)} \tag{32}$$

Op 2 vs **Op** 1	$\mathcal{T}$arget	$\mathcal{E}$xpansion	$\mathcal{D}$iscrimination
$\mathcal{T}$arget	$(111, 235, \mathbf{99}, \mathbf{59})$	$(58, 93, \mathbf{100}, \mathbf{145})$	$(69, 71, \mathbf{12}, \mathbf{42})$
$\mathcal{E}$xpansion	$(118, 67, \mathbf{1.22}, \mathbf{46.45})$	$(18, 18, \mathbf{221}, \mathbf{118})$	$(123, 124, \mathbf{235}, \mathbf{44})$
$\mathcal{D}$iscrimination	$(25, 31, \mathbf{28}, \mathbf{33})$	$(235, 125, \mathbf{145}, \mathbf{265})$	$(142, 365, \mathbf{86}, \mathbf{102})$

TAB. 3 – Comparaison of the various strategies used by the rival operators.

As the temperature decreases towards 0, the second term vanishes, since $U_i(s_1|t_i) \leq U_i^\star(t_i)$. Under this hypothesis, we have :

$$q_{s_1}^{\text{player } i}(t_i) \to \frac{1}{|\mathcal{BR}(.|t_i)|} \mathbf{1}_{s_1 \in \mathcal{BR}(.|t_i)} \ .$$

On the other hand, $\mathcal{BR}$, the set of the best responses obtained using our algorithm, is either empty, either identifiable with bayesian equilibria (provided it converges). Consequentely,

$$\lim_{\text{Temp}^{(t)} \to 0} \lim_{k \to \text{Nb max iter}} \mathbf{P}[X_k \in \mathcal{B}^\star] \geq \lim_{\text{Temp}^{(t)} \to 0} \lim_{k \to \text{Nb max iter}} \mathbf{P}[X_k \in \mathcal{BR}] \ = \ 1 \ .$$

□

Figure 4 shows the convergence of the *true* maximized utilities of both operators. The Theorem 1 enables us to prove the existence of equilibria for our game. However, using simulation, we note that the customers' utilities' shapes are flat, and consequentely, the equilibrium's unicity doesn't appear obviously. Consequentely, we suppose in the rest of the article, that there is equiprobability to choose one of the game equilibria.

3.4 Simulation Results

We have plotted the *true* expected utilities (i.e. conditional to their *true* types) of the various actors, as functions of confidence intervals' levels in Figure 5. By definition, a confidence interval's level measures the probability that a specific reservation price might belong to this segment. The expected utilities are not continuous in the confidence levels. For example, $U_{\text{op}_2}(\sigma, 0.0266) = 28$, whereas $U_{\text{op}_2}(\sigma, 0.0157) = 48.27$ and $U_{\text{op}_2}(\sigma, 0.0289) = 39$, shows that the true expected utility of the operator 2 is neither continuous at left, neither at right, with a precision of 0.2%.

In Tabular 3, we suppose that each operator selects one of the 3 available strategies. Assuming that the reservation prices are known with a fixed confidence level, we compute the expected utilities of two consumers and the rival operators (the two last coefficient in bold in the 4-dimension profit-vector), that is the amount of money they could expect to earn conditionally to their *true* types. A powerfull application of our model is the following : if an operator has access to consumers' data and guesses the selling strategy of its adversaries, then it is easy for him to determine which behavior might be the best to follow.

4 Learning customers' preferences

In this section, to predict the unknown evolution of the consumers' reservation prices, we use the theory of Experts ([12]). Basically, it deals with the prediction of individual sequential decisions. A forecaster playing against the environment, makes predicitons, whose performance should be compared to that of a set of reference experts. Randomized algorithms minimizing the forecaster's regret (of not having chosen the best expert) exist, and can be extended to problems of learning in repeated games.

Recall that the operators are unaware of the consumers' types, i.e. the segment to which they belong. The operator aims to define the set of offers, which is the best suited to the consumers' preferences. It is equivalent for the operators to learn, via simulation, as accurately as possible, the consumers' reservation prices. Intuitively, we could think of an exploration strategy, where every possible combination of services might be tested ([1]). However, the introduction of new products on the market is costly for the operator, in term of advertisments, equipements, marketing... Consequentely, the operator wants to determine the sequence of product lines, which would provide him a cumulated loss as small as the best line.

4.1 Definition of the operator's action set

The actions of the operators 1 and 2 will be denoted $I_t^{\mathrm{op}_1}$ and $I_t^{\mathrm{op}_2}$, respectively. The operator 1 (resp. 2) selects a choice set in one of the C_9^3 (resp. C_5^3) possible sets : $\mathcal{B}^t_{\mathrm{op}_k},\ k=1,2$, and fixes minimal time lengths for the contracts, associated with the offers considered. To be simple, we suppose that the minimal commitment lengths are either null (i.e. there is no special commitment), either of 6 months, either of 12 months, which are the most famous ones. $N_k,\ k=1,2$, is the set of possible actions that the operator k might use. $O_t^{i\to\mathrm{op}_k}(b)$, is a random variable containing the minimal time length, remaining at the time instant t, on the contract between the operator k and the consumer i, for the service b. The global action, i.e. the selection of choice sets and minimal contracts' lengths by the two operators, will be denoted : $I_t = \left(I_t^{\mathrm{op}_1}, I_t^{\mathrm{op}_2}\right)$.

The hidden information is stored in the vector : $Y_t = \left(Y_t^1,\ Y_t^2,\ t_{\mathrm{op}_1},\ t_{\mathrm{op}_2}\right)$.

- $Y_t^i,\ i=1,2$, is the consumer i's reservation prices associated with the choice sets selected by the operators $I_t = \left(I_t^{\mathrm{op}_1}, I_t^{\mathrm{op}_2}\right)$, at the time instant t.
- The lie-coefficients of the operators 1 and 2 are supposed time invariant.

4.2 Iterative resolution of the Bayesian game

Once the operators have defined their product line, a bayesian game occures between the various players since everyone tries to maximize his revenue. At the time instant t, the game function is denoted $J_{\mathcal{B}}(p^t_{\mathcal{M}})$. No analytical expression can be used to characterize it, since it is the result of the simulation algorithm, described in section 3. However, this function relies on the mean market prices, $p^t_{\mathcal{M}}$, which are updated at every repetition of the game.

Loss and feedback functions.

The instantaneous regret function of the operator $k,\ k=1,2$, at the time instant t, is defined as follows :

$$l_t^{\mathrm{op}_k}(I_t, Y_t, p_{\mathcal{M}}^t) \ = \ \sum_{i=1}^{N} \sum_{b=1}^{|\mathcal{B}_{\mathrm{op}_k}^t|} \Big(Y_t^i(b) - C_{\mathrm{op}_k}^t(b) \Big) \ \mathbf{1}_{\{C_i^t(b)=1\}} \ + \ t_{\mathrm{op}_k} \ p_{\mathcal{M}}^t(b) \ \mathbf{1}_{\{C_i^t(b)=0 \ \cup \ O_t^{i \to \mathrm{op}_k}(b) \neq 0\}} \ . \tag{33}$$

The instantaneous regret function relies on the operators' choices as well as, on the clients' reservation prices, which depend themselves on the selected choice sets. $p_{\mathcal{M}}^t(b)$, is the mean market price of the offer b. It can evolve, since at each iteration step, the operators takes decisions on their offers' current prices. We can then alleviate the notations, and write the loss function under the form : $l_t^{\mathrm{op}_k}(I_t, p_{\mathcal{M}}^t)$. Note that the choice of the rival operator appears implicitely in the consumers' choices : $C_i^t,\ i=1,2,...,N$. The operator k $(k=1,2)$ tries to achieve a cumulated loss almost as small as that of the best action, where the cumulated loss of each action is calculated by looking at what would have happened, if that action had been chosen throughout the whole repeated game. Indeed, the rival operator is non-oblivious, since the actions prescribed by the forecasting strategy alter the behavior of the opponents. More precisely, the operator k tries to minimize his cumulated regret, by choosing the sequence of decisions $(I_1^{\mathrm{op}_k},\ I_2^{\mathrm{op}_k},\ ...,\ I_n^{\mathrm{op}_k})$, asymptotically minimizing the following equation :

$$\sum_{t=1}^{n} l_t^{\mathrm{op}_k}(I_t, p_{\mathcal{M}}^t) \ - \ \min_{i \in \{1,2,...,N_k\}} l_t^{\mathrm{op}_k}\left((I_t^-, i), p_{\mathcal{M}}^t\right) \ , \tag{34}$$

The vector (I_t^-, i) corresponds to the vector I_t, whose k^{th} component has been replaced by i.

However, in practice, the operators do not know the loss functions, since they know neither to which segment the customers belong, nor their exact reservation prices. Fortunately, the operators have access to a partial feedback : the market shares captured at the time instant t, i.e. the number of consumers who have signed a contract with them *or* bought a without-commitment-offer :

$$h_t^{\mathrm{op}_k}\left((I_t^{\mathrm{op}_1}, I_t^{\mathrm{op}_2}), p_{\mathcal{M}}^t\right) \ = \ \sum_{b=1}^{K}\sum_{i=1}^{N} \mathbf{1}_{\{C_i^t(b)=1 \ \cup \ O_t^{i \to \mathrm{op}_k}(b) \neq 0\}},\ k=1,2,\ \forall t \geq 0 \ . \tag{35}$$

To begin with, we will suppose that the actors play according to Hannan consistent strategies, and the opponents react to the forecasting strategy.

Definition 5 *A strategy for the operator k, is said Hannan consistent, if, and only if :*

$$\sum_{t=1}^{n} \bar{l}_t^{op_k}(I_t, p_{\mathcal{M}}^t) \ - \ \min_{i \in \{1,2,...,N_k\}} l_t^{op_k}((I_t^-, i), p_{\mathcal{M}}^t) \ = \ o(n),\ n \to \infty \ ,\ \forall k=1,2. \tag{36}$$

The expected value, $\bar{l}$ is taken only with respect to the random variable $I_t^{op_k}$, which means that : $\bar{l}_t^{op_k}(I_t, p_{\mathcal{M}}^t) = \sum_{i=1}^{N_k} l_t^{op_k}\left((I_t^-, i), p_{\mathcal{M}}^t\right) p_t^{op_k}(i)$. More precisely, the cumulated regret of the operator k satisfies :

$$\limsup_{n \to \infty} \left(\frac{1}{n} \sum_{t=1}^{n} l_t^{op_k}(I_t, p_{\mathcal{M}}^t) - \frac{1}{n} \min_{i \in \{1,2,...,N_k\}} \sum_{t=1}^{n} l_t^{op_k}\left((I_t^-, i), p_{\mathcal{M}}^t\right) \right) \leq 0,\ a.\text{-}s.,\ k=1,2 \ . \tag{37}$$

The notion of correlated equilibrium is a fundamental one, introduced by Aumann. It means that in an average sense, no player has an incentive to divert from the recommendation, provided that all other players follow theirs. It generalizes Nash equilibria since the joint measure of the actions, is not necessairly a product distribution.

Lemma 3 *At the time instant t, the joint measure $P_t \in \Delta(N_1 \otimes N_2)$ is a correlated equilibrium, iff :*

$$\begin{cases} \sum_{\{i|i_1=j\}} P_t(j,i_2)\left[l_t^{op_1}\left((j,i2),p_{\mathcal{M}}^t\right) - l_t^{op_1}\left((j',i_2),p_{\mathcal{M}}^t\right)\right] \leq 0,\ \forall j,\ j' \in \{1,...,N_1\}, \\ \sum_{\{i|i_2=k\}} P_t(i_1,k)\left[l_t^{op_2}\left((i_1,k),p_{\mathcal{M}}^t\right) - l_t^{op_2}\left((i_1,k'),p_{\mathcal{M}}^t\right)\right] \leq 0,\ \forall k,\ k' \in \{1,...,N_2\}\ . \end{cases} \tag{38}$$

The set of correlated equilibria at the time instant t, will be denoted : $\mathcal{C}_t$.

Let $\mathcal{H}_t$, be the Hannan set at the time instant t :

$$\mathcal{H}_t = \left\{ P_t \in \Delta(N_1 \otimes N_2) \mid \limsup_{t\to\infty} \sum_i P_t(i) l_t^{\mathrm{op}_k}(i,p_{\mathcal{M}}^t) - \sum_i P_t(i) l_t^{\mathrm{op}_k}\left((i^-,j),p_{\mathcal{M}}^t\right) \leq 0,\ \forall j \in \{1,2,...,N_k\},\ k=1,2 \right\}\ . \tag{39}$$

The marginal empirical frequencies of play for the operator k on the time slot $[\![0;n]\!]$, are defined as :

$$\hat{P}_n^{\mathrm{op}_k}(i) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{1}_{\{I_t^{\mathrm{op}_k}=i\}},\ i \in \{1,...,N_k\}\ ,$$

whereas the joint empirical frequency of both operators takes the form :

$$\hat{P}_n(i) = \hat{P}_n(i_1,i_2) = \frac{1}{n}\sum_{t=1}^{n} \mathbf{1}_{\{I_t^{\mathrm{op}_1}=i_1, I_t^{\mathrm{op}_2}=i_2\}},\ i=(i_1,i_2) \in \otimes_{k=1}^2\{1,...,N_k\}\ .$$

If both operator follows a Hannan consistent strategy, using the definition of the joint empirical frequency and the equations (37), we show that at the time instant t : $\hat{P}_t \in \mathcal{H}_t$.

Lemma 4 *The matrix of loss is defined as : $L_t^{op_k}(i,j) = l_t^{op_k}\left((i,j),p_{\mathcal{M}}^t\right)$, and the feedback matrix is of the form : $H_t^{op_k}(i,j) = h_t^{op_k}\left((i,j),p_{\mathcal{M}}^t\right)$, $\forall i,j \in \otimes_{k=1}^2\{1,...,N_k\},\ k=1,2$. In our game, we have proved by simulation, that at each time instant $t \geq 0$, $L_t^{op_k} = K_t^{op_k}\ H_t^{op_k}$, $k=1,2$.*

If we assume that the rival operators follow Hannan consistent strategies, at each time instant $t \geq 0$, the set of correlated equilibria of the repeated game is a subset of the Hannan set (in general, a proper one).

Using the general forecasters defined for Partial Monitoring, we show via simulation, that the joint empirical frequencies of the repeated game do not converge towards correlated equilibria.

The general forecaster for Partial Monitoring applied to a game between two operators is defined as follows, and inspired from [10] :

let the parameters $0 < \eta_t^k, \gamma_t^k < 1$, be real numbers [9],

we initialize the vectors of weights : $w_0^k = (0,0,...,0)$, $k=1,2$.

[9]More precisely, we take : $\eta_t^k = \left(\frac{N_k}{2N_k n \mu_k}\right)^{\frac{2}{3}}$, $\gamma_t^k = \left(\frac{(\mu_k N)^2 \ln N}{4n}\right)^{\frac{1}{3}}$, and $\mu_k = \max_{i,j}\left\{k_t^{\mathrm{op}_k}((i,j),p_{\mathcal{M}}^t)\right\}$. The proof of the efficiency of these parameters is given in [11].

$*$ The operator k's decision $I_t^{\mathrm{op}_k} \in \{1, ..., N_k\}$, is generated according to the discrete distribution :

$$\begin{cases} p_t^{\mathrm{op}_k}(i) \; = \; (1-\gamma_t^k) \, \dfrac{w_{t-1}^k(i)}{W_{t-1}^k} + \dfrac{\gamma_t^k}{N_k}, \; i \in \{1, ..., N_k\}, \; k = 1, 2 \; , \\ W_{t-1}^k \; = \; \displaystyle\sum_{i=1}^{N_k} w_{t-1}^k(i) \; . \end{cases}$$

$*$ The operator k gets the feedback : $h_t^{\mathrm{op}_k} \; = \; h_t^{\mathrm{op}_k}(I_t, p_{\mathcal{M}}^t)$. The estimated loss takes the form : $\tilde{l}_t^{\mathrm{op}_k}(i, p_{\mathcal{M}}^t) \; = \; \dfrac{k_t^{\mathrm{op}_k}(i_k, I_t^{\mathrm{op}_k}) \; h_t^{\mathrm{op}_k}\left((I_t^{\mathrm{op}_k}, i_{-k}), p_{\mathcal{M}}^t\right)}{p_t^{\mathrm{op}_k}(I_t^{\mathrm{op}_k})}$, $i = (i_1, i_2), \; k = 1, 2$.

Finally, we update the weights :

$$w_t^k(i) \; = \; w_{t-1}^k(i) \; \exp[-\eta_t^k \tilde{l}_t^{\mathrm{op}_k}(i, p_{\mathcal{M}}^t)] \; .$$

To prove the Lemma, it suffices to go back to the definition of a correlated equilibrium (6), with the additional difficulty that the game is time-varying. Indeed, the loss function is time dependent. Another important point to achieve a proper proof, is to note that the loss functions (33) are bounded. Otherwise, this would mean that the selling costs might increase towards infinity, which would imply that the clients would prefer not to buy an offer, since it would make their reservation prices vanish, and consequentely, their utilities would decrease towards $-\infty$.

We show by simulation, that the joint empirical frequencies are not correlated equilibria, since the Aumann's conditions (6) are not satisfied (see Figure 6).

4.3 Internal Regret

We introduce the notion of conditional instantaneous regret, which might lead us, provided the operators follow such strategies, to converge towards a correlated equilibrium.

$$\begin{aligned} r_{(j,j'),t}^{\mathrm{op}_k} &= p_t^{\mathrm{op}_k}(j) \left(l_t^{\mathrm{op}_k}(I_t, p_{\mathcal{M}}^t) - l_t^{\mathrm{op}k}\left((I_t^-, j'), p_{\mathcal{M}}^t\right) \right) \; = \; \mathbf{E}[\hat{r}_{(j,j'),t}^{\mathrm{op}_k}] \; , \\ \hat{r}_{(j,j'),t}^{\mathrm{op}_k} &= \mathbf{1}_{\{I_t^{\mathrm{op}_k} = j\}} \left(l_t^{\mathrm{op}_k}(I_t, p_{\mathcal{M}}^t) - l_t^{\mathrm{op}k}\left((I_t^-, j'), p_{\mathcal{M}}^t\right) \right) , \; \forall j, j' \in \{1, ..., N_k\} \; . \end{aligned}$$

The conditional regret $\sum_{t=1}^{n} \hat{r}_{(j,j'),t}^{\mathrm{op}_k}$ expresses how much better the operator k could have done, had he chosen action j' every time he played action j.

Theorem 2 *Assume that the game is played repeatedly so that the following assumption about the conditional regrets is statisfied :*

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \hat{r}_{(j,j'),t}^{op_k} \leq 0, \; \forall j, j' \in \{1, ..., N_k\}, \; k = 1, 2 \; . \tag{40}$$

Then, we could demonstrate that the distance between the empirical distributions and the set of correlated equilibria vanishes asymptotically :

$$\inf_{P_t \in \mathcal{C}_t} |P_t(i) - \hat{P}_t(i)| \to 0, \; t \to \infty, \; \forall i \in \otimes_{k=1}^2 \{1, ..., N_k\} \; .$$

Proof. We can reformulate the hypothesis (40) under the following expression :

$$\left\{\begin{array}{l} \limsup_{n\to\infty} \sum_{\{i|i_1=j\}} \hat{P}_n(i)\left(l_n^{\mathrm{op1}}(i) - l_n^{\mathrm{op1}}(j', i_2)\right) \leq 0,\ \forall j, j' \in \{1, ..., N_1\}\ , \\ \limsup_{n\to\infty} \sum_{\{i|i_2=k\}} \hat{P}_n(i)\left(l_n^{\mathrm{op2}}(i) - l_n^{\mathrm{op2}}(i_1, k')\right) \leq 0,\ \forall k, k' \in \{1, ..., N_2\}\ . \end{array}\right. \tag{41}$$

Let $P_t \in \mathcal{C}_t$, be a correlated equilibrium. Let's assume that there doesn't exist any $t > 0$ such that $P_t \in \mathcal{C}_t$. Then, since the randomized strategy space is compact[10], there exists a sub-sequence $\hat{P}_{t_k}$ of the sequence of probabilities $\hat{P}_t$ which converges towards a distribution $P^\star \notin \cap_{t\geq 0}\mathcal{C}_t$, such that : $\{P_{t_k}\}_k \to P^\star$, quand $t \to \infty$. But, $P^\star \notin \cap_{t\geq 0}\mathcal{C}_t$, means that for all $t \geq 0$, there exists an operator $k \in \{1, 2\}$, and an action pair $j, j' \in \{1, ..., N_k\}$, such that :

$$\sum_{\{i|i_k=j\}} P^\star(i)\left(l_t^{\mathrm{op}k}(i) - l_t^{\mathrm{op}k}(i^-, j')\right) > 0\ .$$

But, every converging sequence is a Cauchy's one, and every Cauchy's sequence admits a converging sub-sequence, which converges towards the same limit. This contradicts the initial hypothesis :

$$\forall t \geq 0,\ \limsup_{t\to\infty} \sum_{\{i|i_k=j\}} \hat{P}_t(i)\left(l_t^{\mathrm{op}k}(i) - l_t^{\mathrm{op}k}(i^-, j')\right) \leq 0\ .$$

Consequentely, we have proved that under the conditions (**??**), the empirical frequency of the operators converges towards a correlated equilibrium, for the distance :

$$\inf_{\{P_t\in\mathcal{C}_t\}} |P_t(i) - \hat{P}_t(i)| \to 0,\ \text{when } t \to \infty\ ,\ \forall i \in \otimes_{k=1}^2 N_k\ .$$

□

The couple (j, j') being fixed for the operator k, let :

$$V_{t,(j,j')} \ = \ \hat{r}^{\mathrm{op}_k}_{(j,j'),t} - \bar{r}^{\mathrm{op}_k}_{(j,j'),t}\ ,$$

with :

$$\left\{\begin{array}{lcl} \hat{r}^{\mathrm{op}_k}_{(j,j'),t} & = & \mathbf{1}_{\{I_t^{\mathrm{op}_k}=j\}}\left\{l_t^{\mathrm{op}_k}(I_t, p^t_{\mathcal{M}}) - l_t^{\mathrm{op}_k}\left((I_t^-, j'), p^t_{\mathcal{M}}\right)\right\}\ , \\ \bar{r}^{\mathrm{op}_k}_{(j,j'),t} & = & p_t^{\mathrm{op}_k}(j)\ \left\{l_t^{\mathrm{op}_k}(I_t, p^t_{\mathcal{M}}) - l_t^{\mathrm{op}_k}\left((I_t^-, j'), p^t_{\mathcal{M}}\right)\right\}\ . \end{array}\right.$$

Then,

$$\mathbf{E}[\hat{r}^{\mathrm{op}_k}_{(j,j'),t} - \bar{r}^{\mathrm{op}_k}_{(j,j'),t}] \ = \ 0\ ,$$

since $\mathbf{E}[\mathbf{1}_{\{I_t^{\mathrm{op}_k}=j\}}] = p_t^{\mathrm{op}_k}(j)$. Consequentely, $V_{t,(j,j')}$ is a sequence of martingale differences. We show that for all $t \geq 0$, and for all pair j, j', $V_{t,(j,j')} \in [0; c_{t,(j,j')}]$.
Indeed, we have that : $V_{t,(j,j')} \ \leq \ \sum_{b=1}^{3}\sum_{i=1}^{2} X_{t_i}^{(j)}(b) - \sum_{b=1}^{3}\sum_{i=1}^{2} X_{t_i}^{(j')}(b)$. But, the consumers' reservation prices depend

[10] The unit sphere **in finite dimension** is compact.

on the selected choice sets, and this reservation prices have been supposed to be bounded. Consequentely, we can improve the previous bounds :

$$V_{t,(j,j')} \;\leq\; \max_{j,j' \in \{1,2,\ldots,N_k\}} \sum_{i=1}^{2} \sum_{b=1}^{3} \left(X_{t_i}^{(j)}(b) - X_{t_i}^{(j')}(b) \right) < \infty \; .$$

We note this quantity : diff $=$ $\max_{j,j' \in \{1,2,\ldots,N_k\}} \sum_{i=1}^{2} \sum_{b=1}^{3} \left(X_{t_i}^{(j)}(b) - X_{t_i}^{(j')}(b) \right)$.

The Hoeffding-Azuma inequality tells us that :

$$\forall \alpha > 0, \mathbf{P}[\sum_{t=1}^{n} \left(\hat{r}^{\mathrm{op}k}_{(j,j'),t} - \bar{r}^{\mathrm{op}k}_{(j,j'),t} \right) > n\alpha] \leq \exp[-\frac{2n^2\alpha^2}{\sum_{t=1}^{n} c^2_{t,(j,j')}}] \; ,$$

which can be re-written as follows :

$$\forall \alpha > 0, \mathbf{P}[\sum_{t=1}^{n} \left(\hat{r}^{\mathrm{op}_k}_{(j,j'),t} - \bar{r}^{\mathrm{op}_k}_{(j,j'),t} \right) > n\alpha] \leq \exp[-\frac{2n^2\alpha^2}{n \; n^2 \; \mathrm{diff}^2}] \; .$$

Note that the second term of the inequality converges towards 0, as n grows towards infinity. Using Borel-Cantelli's lemma, we infer that :

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \left(\hat{r}^{\mathrm{op}_k}_{(j,j'),t} - \bar{r}^{\mathrm{op}_k}_{(j,j'),t} \right) = 0, \text{ with probability } 1 \; .$$

Hence, the joint empirical frequency of play, converges almost surely towards a correlated equilibrium.

How to compute small internal regrets ?

Suppose that at the time instant $(t-1)$, the operator k selects an action according to the the discrete distribution : $p^{\mathrm{op}_k}_{t-1} \;=\; \left(p^{\mathrm{op}_k}_{t-1}(1), \ldots, p^{\mathrm{op}_k}_{t-1}(N_k) \right)$. For each couple of actions, (i,j), $i \neq j$, we define the probability vector $p^{\mathrm{op}_k, i \to j}_{t-1}$ as follows :

$$\begin{cases} p^{\mathrm{op}_k, i \to j}_{t-1}(l) = p^{\mathrm{op}_k}_{t-1}(l), \; \forall l \in \{1, \ldots, N_k\}, \; l \notin \{i,j\} \; , \\ p^{\mathrm{op}_k, i \to j}_{t-1}(i) = 0, \\ p^{\mathrm{op}_k, i \to j}_{t-1}(j) = p^{\mathrm{op}_k}_{t-1}(i) + p^{\mathrm{op}_k}_{t-1}(j) \; . \end{cases}$$

We search a probability distribution on the pairs (i,j) at the time instant t : $\Delta^{\mathrm{op}_k}_{(i,j),t}$, such that the expected conditional regret of the operator k might be as small as the one associated with the best modified strategy, i.e. :

$$\frac{1}{n} \sum_{t=1}^{n} \sum_{i \neq j} \tilde{l}^{\mathrm{op}_k}_t (p^{\mathrm{op}_k, i \to j}_t) \Delta^{\mathrm{op}_k}_{(i,j),t} \leq \min_{i \neq j} \frac{1}{n} \sum_{t=1}^{n} \tilde{l}^{\mathrm{op}_k}_t (p^{\mathrm{op}_k, i \to j}_t) + \varepsilon_n \; , \; \varepsilon_n \to 0 \; . \tag{42}$$

Proposition 1 *The following predictor for the operator k, is a good approximate of the inequality (42) :*

$$\Delta^{op_k}_{(i,j),t} \;=\; \frac{\exp[-\eta \; \sum_{s=1}^{(t-1)} \tilde{l}^{op_k}_s (p^{op_k, i \to j}_s)]}{\sum_{(k,l)|k \neq l} \exp[-\eta \sum_{s=1}^{(t-1)} \tilde{l}^{op_k}_s (p^{op_k, k \to l}_s)]} \; , \; \forall i,j \in \{1, \ldots, N_k\} \; , \; \forall t \geq 0 \; , \; k = 1, 2 \; . \tag{43}$$

Proof. Let,

$$W_t \;=\; \sum_{i=1}^{N_1 \otimes N_2} w_{i,t} \;=\; \sum_{i=1}^{N_1 \otimes N_2} \exp[-\eta \tilde{L}_{i,t}] \; ,$$

with, $\tilde{L}_{i,t} \ = \ \sum_{s=1}^{t} \tilde{l}_s^{\text{op}_k}(i_1, i_2)$.

We note that W_t can be written under the recursive form :

$$W_t \ = \ \sum_{i=1}^{N_1 \otimes N_2} \exp[-\eta \tilde{L}_{i,t-1}] \exp[-\eta \tilde{l}_{i,t}] \ = \ \sum_{i=1}^{N_1 \otimes N_2} w_{i,t-1} \exp[-\eta \tilde{l}_{i,t}] \ .$$

To begin :

$$\log \frac{W_t}{W_0} \ = \ \log \left(\sum_{i=1}^{N_1 \otimes N_2} \exp[-\eta \tilde{L}_{i,t}] \right) - \log\left(N_1(N_2-1)\right) \ \geq \ -\eta \tilde{L}_{j,t} - \log\left(N_1(N_2-1)\right) \ , \ \forall j \in N_1 \otimes N_2 \ . \quad (44)$$

Then,

$$\begin{aligned} \log \frac{W_t}{W_{t-1}} \ &= \ \log \sum_{i=1}^{N_1 \otimes N_2} \frac{w_{i,t-1}}{W_{t-1}} \exp[-\eta \tilde{l}_{i,t}] \ , \\ &\leq \ \log \sum_{i=1}^{N_1 \otimes N_2} \frac{w_{i,t-1}}{W_{t-1}} \left(1 - \eta \tilde{l}_{i,t} + (e-2)\eta^2 \tilde{l}_{i,t}^2\right) \ , \text{ since } \eta \in [0;1] \text{ and } \tilde{l}_{i,t} \leq 1 \text{ by hypothesis} \ , \\ &\leq \ \log \left(1 - \frac{\eta}{W_{t-1}} \sum_{i=1}^{N_1 \otimes N_2} \tilde{l}_{i,t} + \frac{(e-2)\eta^2}{W_{t-1}} \sum_{i=1}^{N_1 \otimes N_2} \tilde{l}_{i,t}^2 w_{i,t-1} \right), \text{ since } w_{i,t-1} \leq 1 \ , \\ &\leq \ -\eta \sum_{i=1}^{N_1 \otimes N_2} \frac{w_{i,t-1}}{W_{t-1}} \tilde{l}_{i,t} + \frac{(e-2)\eta^2}{W_{t-1}} \sum_{i=1}^{N_1 \otimes N_2} \tilde{l}_{i,t}^2 w_{i,t-1} \ . \end{aligned} \quad (45)$$

Combining the inequalities (44) and (45), we get that :

$$\sum_{i=1}^{N_1 \otimes N_2} \Delta_{i,t} \tilde{l}_{i,t} - \tilde{L}_{j,t} \leq \frac{1}{\eta} \log(W_t) + \frac{\eta(e-2)}{W_{t-1}} \sum_{i=1}^{N_1 \otimes N_2} w_{i,t-1} \tilde{l}_{i,t}^2 \ .$$

Finally, summing over all t, and dividing by n, we get that :

$$\frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{N_1 \otimes N_2} \Delta_{i,t} \tilde{l}_{i,t} - \frac{1}{n} \sum_{t=1}^{n} \tilde{L}_{j,t} \leq \frac{1}{n\eta} \sum_{t=1}^{n} \log(W_t) + \eta(e-2) \ .$$

W_t admits $N_1(N_2-1)$, as an upper bound. It is sufficient to choose a parameter η decreasing in $\frac{1}{n}$, to guaranteeing the convergence of the right part of the inequality towards 0.

□

In our model, the loss function is unknown by the operator 1, and the sums at the time instant s are of the form :

$$\tilde{l}_s^{\text{op}_1}(p_s^{\text{op}_1, i \to j}) = \sum_{k=1}^{\otimes_{k=1}^2 N_k} p_s^{\text{op}_1, i \to j}(k_1) \hat{l}_s^{\text{op}_1}(k_1, k_2) \ = \ \sum_{k=1}^{\otimes_{k=1}^2 N_k} p_s^{\text{op}_1, i \to j}(k_1) \frac{k_s^{\text{op}_1}(k_1, I_s(1)) h_s^{\text{op}_1}(I_s(1), k_2)}{p_{I_s(1),s}} \ ,$$

with, $I_s(1) \sim \ p_s^{\text{op}1}(I_s(1))$.

Using the same idea for the operator 2, we get :

$$\tilde{l}_s^{\text{op}2}(p_s^{\text{op}_2, i \to j}) = \sum_{k=1}^{\otimes_{k=1}^2 N_k} p_s^{\text{op}_2, i \to j}(k_2) \hat{l}_s^{\text{op}_2}(k_2, k_1) \ = \ \sum_{k=1}^{\otimes_{k=1}^2 N_k} p_s^{\text{op}_2, i \to j}(k_2) \frac{k_s^{\text{op}2}(k_2, I_s(2)) h_s^{\text{op}_2}(I_s(2), k_1)}{p_{I_s(2),s}} \ ,$$

with, $I_s(2) \sim \ p_s^{\text{op}2}(I_s(2))$.

For both operator, the new probability distribution results of the following fixed point equation, that should be solved recursively :

$$p_t^{\mathrm{op}_k} \;=\; \sum_{\{(i,j)|i\neq j\}} p_t^{\mathrm{op}_k, i\rightarrow j}\;\Delta^{\mathrm{op}_k}_{(i,j),t}\;,\; k=1,2\;, \tag{46}$$

At the top of the Figure 7, we have simulated the mean market prices' distribution of the 9 categories of services (see Application 2.3), at two different time instants, while at the bottom, we have drawn the dynamic evolution of the mean prices for two offer categories. In Figure 8, we have drawn the dynamic evolution of the *true* expected utilities of the operators (at the bottom). An immediate effect of the learning process is the increase of the operators' true expected utilities.

In Figure 9, we have plotted the joint empirical frequency evaluated in each possible action. We check that the distribution satisfies Aumann's conditions (6). Consequently, it is a correlated equilibrium.

5 MVNO Agreement

Many operators, having quite different cost structures, are in competition on the market. However, to stay in the stroke, the operators must diversify their services on every possible markets (Mobile, Fix telephony, Internet offers and TV), which increases various plays of alliances. A Mobile Virtual Network Operator (MVNO) is an operator who does have neither proper frequency spectra, neither mobile network. Consequentely, he is ought to develop alliances with traditional mobile operators, to sell Mobile services under his personal brand. Generally, MVNOs target customers, difficult to be fetched by traditional Mobile operators.

The game in which the operators are involved is highly uncertain. Indeed, the operators are not aware of the true reservation prices of the customers. Furthermore, they don't know the cost structures of their rivals and in general, MVNOs are unsure of the QoS levels their consumers would receive. We will assume that the operator 2 can extend his line, by including Mobile services, provided he agrees to establish an alliance with the operator 1, who owns a Mobile network.

5.1 Definition of the MVNO contract

The mobile operator 1 first fixes a global price, $p_{\mathcal{C}\mathrm{ontract}}$, and then a global number of minutes : q. It can be assimilated with wholesale selling. To simplify, we suppose that the global number of minutes sold can be identified with the maximum number of contracts, that the MVNO could define. We sum up the contract's definition below :

$$\mathrm{MVNO}_{\mathcal{C}\mathrm{ontract}} \;:=\; \{p_{\mathcal{C}\mathrm{ontract}}, q\}\;\;.$$

We assume that the global cost fixed by the operator 1, is bounded by values defined by a regulatory authority. Hence, the contract parameters belong to a bounded set :

$$(p_{\mathcal{C}\text{ontract}}, q) \in [\![p^{\star}_{\min}, p^{\star}_{\max}]\!] \times [\![1, N]\!] \cup (\infty, 0) . \tag{47}$$

By hypothesis, the values $(\infty, 0)$, are obtained if the operator 1 refuses to trade. Let $c_{\mathcal{C}\text{ontract}} \in \{0; 1\}$, be the variable containing the operator 2's decisions. It takes the value 1, if the operator 2 accepts to sign the Mobile operator 1's contract, and 0, otherwise. If the operator 2 accepts the contract, he can introduce a new offer called b', containing Mobile service, on the market. However, the MVNO can only sell a limited number of Mobile offers, since he has bought a finite number of minutes. The aim of this section will be to determine optimal mechanisms for the operators, which would lead them to earn guaranteed benefits efficient and equitable when interpersonal comparisions are made in terms of virtual utility scale (see [14]).
On the first side, Mobile operator 1 tries to maximize his profits under an expansion strategy[11] and the price, at which he sells the global amount of minutes.

$$u_{\text{op}_1}(c,t) = \sum_{b=1}^{|\mathcal{B}_{\text{op}_1}|} \left(c_{\text{op}_1}(b) - \delta_{\text{op}_1}(b)\right) \left(\sum_{i=1}^{N} c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\text{op}_1}(b)\}}\right) + p_{\mathcal{C}\text{ontract}} c_{\mathcal{C}\text{ontract}} . \tag{48}$$

On the other side, the operator 2 expect to target the most profitable customers' segment for him, with the help of his MVNO agreement. Indeed, MVNO's offers are intended to seduce a specific market segment, so the prices should be optimized to those specific customers. Nevertheless, the MVNO can establish only a limited number of contracts, and some consumers might be refused. We assume that the consumers' requests for the MVNO offer are ranked by increasing order, and that they are satisfied using the same process.

$$\begin{aligned} u_{\text{op}_2}(c,t) &= \sum_{b=1}^{|\mathcal{B}_{\text{op}_2}|} \left(c_{\text{op}_2}(b) - \delta_{\text{op}_2}(b)\right) \left(\sum_{i=1}^{N} c_i(b) \mathbf{1}_{\{X_{t_i}(b) \geq c_{\text{op}_2}(b)\}}\right) + \Big[\max_{s \in \mathcal{F}} \Big\{ \left(c_{\text{op}_{2,s}}(b') - \delta_{\text{op}_2}(b')\right) \\ &\quad \left(\sum_{i=1}^{N} c_i(b') \mathbf{1}_{\{\tilde{X}_{t_i}(b') \geq c_{\text{op}_{2,s}}(b')\} \cap \{f_{t_i} = f_s\} \cap \{\sum_{j=1}^{i} c_j(b') \leq q\}}\right) - p_{\mathcal{C}\text{ontract}} \Big\} \Big] c_{\mathcal{C}\text{ontract}} . \end{aligned} \tag{49}$$

The first term of the equation (49) represents the benefits issued from the selling of the operator 2's regular offers. Provided the operator 2 has accepted the MVNO agreement, the operator becomes a MVNO, and intends to expand his market shares, using a targeted approach (see the second term of the equation (49)). An important point to highlight is that the QoS levels that the Mobile operators 1 might deliver to the MVNO, are not necessarily the same as the level that the MVNO expects to receive. The MVNO can quantify this point only through the value of the reservation price, that the consumer i affects to the new offer b' : $\tilde{X}_{t_i}(b')$. However, the customers' reservation prices are estimated a priori, i.e. before the offers' marketing. Since the dispersion

[11] It is possible to use other strategies, such as target or discrimination ones, defined in section 3.

of such an offer is null, we will suppose that a noise is added to the reservation price, modeling the uncertainty that the MVNO should have on the received QoS :

$$\tilde{X}_{t_i}(b') \;=\; X_{t_i}(b') + w_M\omega, \; -\frac{X_{t_i}(b')}{w_M} \leq \omega \leq 0 \;,\; w_M \neq 0 \;,$$

ω is now supposed to be part of the Mobile operator 1's type.

The consumer i's utility is slightly modified to incorporate this new feature :

$$\begin{aligned} u_i(c,t) \;&=\; \alpha_{t_i}\left(\sum_{b=1}^{|\mathcal{B}_{\text{op}_1}|}\left[X_{t_i}(b) - c_{\text{op}_1}(b)\right] c_i(b)\mathbf{1}_{\{X_{t_i}(b)\geq c_{\text{op}_1}(b)\}}\right) \\ &+\; (1-\alpha_{t_i})\left(\sum_{b=1}^{|\mathcal{B}_{\text{op}_1}|} X_{t_i}(b)c_i(b)\mathbf{1}_{\{X_{t_i}(b)\geq c_{\text{op}_j}(b)\}}\right) \\ &+\; \Big\{\alpha_{t_i}\left[\tilde{X}_{t_i}(b') - c_{\text{op}_2}(b')\right] c_i(b')\mathbf{1}_{\{\tilde{X}_{t_i}(b')\geq c_{\text{op}_1}(b')\}\cap\{\sum_{j=1}^{i} c_j(b')\leq q\}} \\ &+\; (1-\alpha_{t_i})\tilde{X}_{t_i}(b')\; c_i(b')\mathbf{1}_{\{\tilde{X}_{t_i}(b')\geq c_{\text{op}_j}(b')\}\cap\{\sum_{j=1}^{i} c_j(b')\leq q(k)\}}\Big\} c_{\mathcal{C}\text{ontract}} \;. \end{aligned} \tag{50}$$

5.2 Selection of an efficient mechanism

The players' set $\mathcal{N}$, is now made of the consumers, the operator 2 and the Mobile operator 1. The set of coalitions will be denoted $\mathcal{C}$o. It contains all the non-empty subsets of $\mathcal{N}$:

$$\mathcal{C}\text{o} \;:=\; \{S | S \subseteq \mathcal{N}, S \neq \emptyset\} \;.$$

With each coalition, S, we associate D_S, the set of compatible decisions for the members of S, provided they cooperate with one another. We let T_S, be the admissible type combinations for the various players belonging to S :

$$T_S \;=\; \times_{i\in S} T_i \;.$$

A cooperative game under partial information is completely defined under the expression :

$$\Gamma \;=\; ((D_S)_{S\in\mathcal{C}\text{o}}, (T_i, u_i)_{i\in\mathcal{N}}, p) \;.$$

Where the joint belief probability p, is defined by :

$$p(t) \;=\; p_i(t_{\mathcal{N}-i}|t_i)p_i(t_i)^{12} \;,\; \text{with } p_i(t_i) = M_{t_i},\; i = 1,2,...,N,\; p_{\text{op}}(t_{\text{op}}) = \frac{1}{|T_{\text{op}}|} \;.$$

Let M_S, be the set of strategies for the players belonging to the coalition S from the type space T_S, taking values in the randomized action space, $\Delta(D_S)$. A mechanism, μ_S, is an application in the set M_S, satisfying

[12] Remind that the consumers' types are independant of one another and of the operator's type. But, the consumers' beliefs about the operator's type depend on their personal market segment.

the following normalizing constraints :

$$\begin{cases} \mu_S(d_S|t_S) \geq 0, \\ \sum_{c_S \in D_S} \mu_S(c_S|t_S) = 1, \ \forall d_S \in D_S, \ \forall t_S \in T_S \ . \end{cases} \tag{51}$$

In a cooperative game, the only mechanism that will be implemented, will be the one chosen by the grand coalition, $\mathcal{N}$. However, during bargaining, each coalition S selects a mechanism $\mu_S \in M_S$, that she threats to use if the other players refuse to cooperate with her members.

$U_i^\star(\mu, s_i|t_i)$, is the expected utility of player i, which results from the application of the mechanism $\mu \in M_{\mathcal{N}}$, if his type is t_i, but that he pretends to be of type s_i, the other players being honest :

$$U_i^\star(\mu, s_i|t_i) \ = \sum_{t_{N-i} \in T_{N-\{i\}}} p_i(t_{\mathcal{N}-i}|t_i) \sum_{d \in D} \mu(d|t_{\mathcal{N}-i}, s_i) \ u_i(d,t) \ . \tag{52}$$

If the player i chooses not to cheat, during the implementation of the mechanism μ, his expected utility is of the form :

$$U_i(\mu|t_i) \ = \sum_{t_{N-i} \in T_{N-\{i\}}} p_i(t_{\mathcal{N}-i}|t_i) \sum_{d \in D} \mu(d|t) \ u_i(d,t) \ . \tag{53}$$

A mechanism is said incentive compatible, if, and only if :

$$U_i(\mu|t_i) \geq U_i^\star(\mu, s_i|t_i), \ \forall i \in \mathcal{N}, \ \forall t_i \in T_i, \ \forall s_i \in T_i \ . \tag{54}$$

μ is incentive compatible, if, and only if, a Nash-Bayesian equilibrium for the players would be to report honestly their types during the implementation of the mechanism μ, provided they report their types confidentially and simultaneously.

A mechanism $\mu \in M_{\mathcal{N}}$, is incentive efficient, if, and only if, it is incentive compatible and there is no other incentive compatible mechanism $\hat{\mu}$, such that :

$$\begin{cases} U_i(\hat{\mu}|t_i) \geq U_i(\mu|t_i), \ \forall i \in \mathcal{N}, \ \forall t_i \in T_i, \\ \text{with } U_j(\hat{\mu}|t_j) > U_j(\mu|t_j), \ \text{for at least one type } t_j \text{ of the player } j \ . \end{cases} \tag{55}$$

If the players bargain with one another, they must finally agree on an incentive efficient mechanism.

We introduce $\tilde{A} := \{\alpha(s_i|t_i), \ s_i \in T_i, \ t_i \in T_i\}$, the set of all the possible vectors of shadow prices, issued from the incentive compatible constraints (trivially, $\alpha_i(t_i|t_i) = 0$). If the coalition chooses the action d and its type is t, λ and α being fixed, player i's virtual utility can be written under the form :

$$v_i(d,t)_{\lambda,\alpha} \ = \ \frac{\left[\lambda_i(t_i) + \sum_{s_i \in T_i} \alpha_i(s_i|t_i)\right] \ p_i(t_{N-\{i\}}|t_i) \ u_i(d,t) - \sum_{s_i \in T_i} \alpha_i(t_i|s_i) \ p_i(t_{N-\{i\}}|s_i) \ u_i(d, t_{N-\{i\}}|s_i)}{p(t)} \ . \tag{56}$$

Duality theory (cf [14]) enables us to say that the appropriate α to use, is solution of the dual problem for λ :

$$\min_{\alpha \in \tilde{A}} \sum_{t \in T} p(t) \ \max_{d \in D} \sum_{i \in \mathcal{N}} v_i(d,t)_{\lambda,\alpha} \ . \tag{57}$$

But, virtual utilities $v_i(d,t)_{\lambda,\alpha}$ are linear in α. Consequentely, the dual problem is merely a linear programming problem.

Let μ be an incentive efficient mechanism, $\lambda \in \Lambda^0$ and $\alpha \in \tilde{A}$, be such that μ is a solution of the primal problem for λ, and α a solution of the dual problem for λ. We say that the type s_i jeopardizes another type t_i of the player i, for the incentive efficient mechanism μ, if, and only if, the constraint that the type s_i will not gain anything by pretending to be of type t_i is binding, i.e. $U_i(\mu|s_i) = U_i^\star(\mu, t_i|s_i)$ and that his shadow price $\alpha_i(t_i|s_i)$ is positive. In some problems, the virtual utility can be seen as a signal containing information about the true types of the players (cf [14] and [17]). In our model, the customers can cheat, by pretending to belong to another segment, in order for example, to get disounted prices, in the case of a strategy of price discimination. On the contrary, the operators will pretend that their cost structures are superior to their true ones.

In the game, each coalition chooses a threat, in the case its complement would refuse to cooperate. Let $M = \times_{S \subseteq \mathcal{C}o} M_S$, be the set of all the mechanism combinations that the coalitions can choose as threats. $\mu = (\mu_S)_{S \in \mathcal{C}o} \in M$, includes specification of the mechanism μ_S, that each coalition $S \subseteq \mathcal{N}$, threats to use in the case that its complement $\mathcal{N} - S$, would refuse to cooperate. For each coalition S, $W_S(\mu,t)_{\lambda,\alpha}$, represents the sum of the virtual utilities that the players in S expect to receive if the combination of the types is t, and the coalitions S and $\mathcal{N} - S$, apply their threats. If $S \neq \mathcal{N}$, we get that :

$$W_S(\mu,t)_{\lambda,\alpha} \ = \sum_{d_S \in D_S} \sum_{d_{\mathcal{N}-S} \in D_{\mathcal{N}-S}} \mu_S(d_S|t_S) \ \mu_{\mathcal{N}-S}(d_{\mathcal{N}-S}|t_{\mathcal{N}-S}) \sum_{i \in S} v_i\left((d_S, d_{\mathcal{N}-S}), t\right)_{\lambda,\alpha} \ . \tag{58}$$

If $S = \mathcal{N}$, the expression (58) can be sum up as follows :

$$W_{\mathcal{N}}(\mu,t)_{\lambda,\alpha} \ = \sum_{d \in D} \mu(d|t) \sum_{i \in \mathcal{N}} v_i(d,t)_{\lambda,\alpha} \ . \tag{59}$$

$W(\mu,t)_{\lambda,\alpha} = (W_S(\mu,t)_{\lambda,\alpha})_{S \in \mathcal{C}o}$, is the characteristic function of the game. The Shapley value, for the player i, is :

$$\phi_i\left(W(\mu,t)_{\lambda,\alpha}\right) \ = \sum_{S \subseteq \mathcal{C}o, \ i \in S} \frac{(|S|-1)!(|\mathcal{N}|-|S|)!}{|\mathcal{N}|!} \left\{W_S(\mu,t)_{\lambda,\alpha} - W_{\mathcal{N}-S}(\mu,t)_{\lambda,\alpha}\right\} \ , \ \forall t \in T \ . \tag{60}$$

The allocation of guaranteed real utilities associated with the allocation of virtual utilities giving the Shapley value, satisfies the following equation (cf [14]) :

$$\begin{aligned} & \frac{\left\{\lambda_i(t_i) + \sum_{s_i \in T_i} \alpha_i(s_i|t_i)\right\} p_i(t_{\mathcal{N}-i}|t_i)\omega_i(t_i) - \sum_{s_i \in T_i} \alpha_i(t_i|s_i) p_i(t_{\mathcal{N}-i}|s_i)\omega_i(s_i)}{p(t)} \\ = & \sum_{t_{\mathcal{N}-i} \in T_{\mathcal{N}-i}} p_i(t_{\mathcal{N}-i}|t_i) \ \phi_i\left(W(\mu,t)_{\lambda,\alpha}\right), \ \forall i \in \mathcal{N}, \ \forall t_i \in T_i \ . \end{aligned} \tag{61}$$

Noise parameter value : ω	Guaranteed Benefit for Op_1	Guaranteed Benefit for Op_2 (MVNO)
0	10837	1330
0.5	−57.8	58.4
1	131	−1757
1.5	−377	0
2	86	5

TAB. 4 – Evaluation of the operators' guaranteed Gains for different noise values.

A vector $\omega \in \times_{i\in\mathcal{N}}\mathbf{R}^{|T_i|}$ which satisfy (61), is said to be guaranteed for λ, α, and μ. $\omega_i(t_i)$ is then a guaranteed value for the type t_i. The guaranteed values are gains of real utilities associated with an allocation that would give to each type of each player, its Shapley value, provided the players do interpersonal comparision of equity at the virtual utility scale.

We note that the threats μ_S plays a role on the allocation associated with the Shapley value only through the difference $(W_S - W_{\mathcal{N}-S})$, that each actor in S would like to maximize. Consequentely, we say that a mechanism $\mu \in M$, is a vector of rational threat with respect to λ and α, if :

$$\begin{aligned} & \sum_{t\in T} p(t)\left\{W_S(\mu,t)_{\lambda,\alpha} - W_{\mathcal{N}-S}(\mu,t)_{\lambda,\alpha}\right\} \\ = \quad & \max_{\nu_S\in M_S} \sum_{t\in T} p(t)\left\{W_S\left((\mu_{-S},\nu_S),t\right)_{\lambda,\alpha} - W_{\mathcal{N}-S}\left((\mu_{-S},\nu_S),t\right)_{\lambda,\alpha}\right\}\ ,\ \forall S \subseteq \mathcal{C}\mathrm{o}\ . \end{aligned} \tag{62}$$

In fact, the rival coalitions play a two-player zero sum game, when they choose their rational threats. A threat is a subtle mixture of on the one hand, a defensive aspect, and on the other hand, an offensive one. Indeed, for the coalition S, W_S contains the set of the defensive objectives, while $-W_{\mathcal{N}-S}$ represents the objective offensive function. The difference $(W_S - W_{\mathcal{N}-S})$ is then a combination of defensive and offensive purposes.

Theorem 3 *There exists at leat a solution to the bargaining game* Γ, μ.

The Theorem 3 proved in details in [14], guarantees the existence of an optimal bargaining strategy, in our game.

5.3 Simulation Results

We can then determine the guaranteed benefits of each player's type. Consequentely, the model advise the operator whether it is worth to enter the game, or not (if the benefits are negative, it might be quite risky). In Figure 11, we have drawn the guaranteed benefits of 2 consumers chosen at random, one Mobile operator and

Types=(lie-coef.,QoS level)	Guaranteed Benefits Op_{11}	Guaranteed Benefits Op_{12}	Guaranteed Benefits Op_{13}	Guaranteed Benefits Op_{2}
$(\mathbf{0.25}, \mathbf{0.25})$	0	0	0	0
$(\mathbf{1}, \mathbf{0.25})$	-10.2	-21.7	2.34	7.88
$(\mathbf{0.5}, \mathbf{1})$	-26.0	-74.2	-10.93	-14.03
$(\mathbf{1.5}, \mathbf{0.25})$	176.7	141.21	64.37	68.30

TAB. 5 – Operators' guaranteed benefits, in the case of 3 Mobile operators, in competition.

Operator 2, who might become a MVNO, as functions of their types.

In Figure 12, the probabilities of occurence of the agreement (number of minutes, global price) associated with the bargaining solution μ, are represented, the noise parameter being null. It is an attempt to help the operators to choose the decisions giving them afficient and equitable guaranteed benefits.

In Table 4, the evolution of the guaranteed benefits of the Mobile operator and the operator 2 are drawn, as functions of the noise parameter's value. In our setting, adding a noise parameter do not systematically increase the operator 1's benefit. This is due to the fact that operator 1 markets other offers, free from Mobile services.

In Table 5, we have extended our model to the case of 3 Mobile operators (Op_{11}, Op_{12}, Op_{13}) competing for the establishement of a MVNO contract with the operator 2. The QoS levels represents the deterioration of the QoS in the operator 2's Mobile traffic, due to the noise coefficient. We infer that the operators' benefits are better when they don't lie about their real costs and degrade the QoS of no more than 0.25.

5.4 Determination of prices' boundaries by a regulatory authority

As we have noted, the optimal mechanisms and the true expected utilities of the various players, may depend on the prices' bounds fixed by the regulatory authority.

The set of all the possible values of p tested, is denoted : $\mathcal{P}$. $U_i(\mu_p|t_i)$ represents the *true* expected utility of the player i, provided the prices' boundaries are of the form : $p^{\star}_{\min} = 0.1$ and $p^{\star}_{\max} = p,\ p \in \mathcal{P}$ [13]. The aim of the regulatory authority is to be equitable with all operators and to encourage concurrence. It will then have to cope with a multi-criteria optimization problem of the form :

$$\max_{p\in\mathcal{P}} \left\{U(\mu_p) \;=\; \left[U_{\text{op}_{11}}(\mu_p|t_{\text{op}_{11}}),\ U_{\text{op}_{12}}(\mu_p|t_{\text{op}_{12}}),\ U_{\text{op}_{13}}(\mu_p|t_{\text{op}_{13}}),\ U_{\text{op}_{2}}(\mu_p|t_{\text{op}_{2}})\right]\right\} \ . \tag{63}$$

The authority wants to determine the set of Pareto efficient prices' boundaries. Since the set $\mathcal{P}$ is discrete, the feasible set is explicitely given. Hence, to determine efficient solution, we proceed by pairwise vector comparison :

[13]Since the bounds of the interval are not integer values, we choose a step of 0.1 and pick all the corresponding real values falling between those two bounds.

Algorithm 3. Determination of Efficient solutions

Initialize $\mathcal{E} := \mathcal{P}$.
For i from 1 to $|\mathcal{P}| - 1$,
For j from $i + 1$ to $|\mathcal{P}|$,
If $U(\mu_i) \geq U(\mu_j)$ and there exists a $k \in \{1, 2, 3, 4\}$, such that $U_k(\mu_i|t_k) > U_k(\mu_j|t_k)$
$\mathcal{E} := \mathcal{E} - \{j\}$
Elseif $U(\mu_j) \geq U(\mu_i)$ and there exists a $k \in \{1, 2, 3, 4\}$, such that $U_k(\mu_j|t_k) > U_k(\mu_i|t_k)$
$\mathcal{E} := \mathcal{E} - \{i\}$
End
End

In Figure 5.4, we have computed the Pareto set, for 76 distinct values of p, the noise parameter being null.

6 Conclusion

We have developped original models to help operators maximize their benefits under uncertainty on their rivals and on the consumers' preferences. Judging by the simulations, the practical results seem promising. The approach might of course, be extended, by considering for example, a dynamic evolution of the operator's cost structures, or by introducing different risk attitudes in the utility functions.

The presence of multiple equilibria can be seen as annoying, since it forces us to associate probabilities with the selection of one of them. Consequentely, to avoid the computation of equilibria, another idea might be to use dynamic programming and more specifically, Markov decision processes (cf [20] and [23]).

To conclude, we introduce a few possible extensions. Indeed, once the customers have made their choices (i.e. bought some services), they generate traffic on the network. However, these traffics do not have the same constraints in terms of rates, Quality of Service (QoS)... For example, the consumers have not the same expectations for telephony's QoS, as for file downloadings. Which might lead us to introduce mechanisms of prioritarization, in the network. Traditionally, such problems have been tackled using queueing theory, with service differentiation. Besides, due to the success of some greedy applications such as streaming, optimal bandwidth sharing is becoming a fundamental subject, especially in the Internet.

The same problem might occur between operators sharing a common backbone. For example, one operator might want to advantage one subscribing operator, or some applications on others. Modeling these mechanisms requires a good knowledge of consumers' preferences, and economic strength relationships.

Another intersting point, is the management of consumers' unsatisfaction when there are no more offers left. This problem has been partially introduced in the section 5, where we have supposed that only a limited number of Mobile services can be sold by the MVNO. Nevertheless, no costs have been added to the consumers' loss, or to the selection of a "second choice" product. Problem of satisfaction also handle between operators, when the provider can't reach the QoS level, he has promised to send. For exemple, in Virtual Private Network, where the Quality of Service levels are specified in a contract (SLA, Service Level Agreement).

7 References

[1] KEPHART J.O., BROOKS C.H., DAS R., MACKIE-MASON J.K., GAZZALE R., DURFEE E.H., *Pricing Information Bundles in a Dynamic Environment*, in Proceedings of the ACM Conference on Electronic Commerce, Oct. 2001.

[2] AYDIN G., RYAN J.K., *Product Line Selection and Pricing Under the Multinomial Logit Choice Model*, Avr. 2000.

[3] KHOUJA M., ROBBINS S.S., *Optimal pricing and quantity of products with two offerings*, European Journal of Operational Research, 2005.

[4] TALLURI K., VAN RYZIN G., *Revenue Management Under General Discrete Choice Model of Consumer Behavior*, Management Science, Vol. 50, pp.15-33, jan. 2004.

[5] BERRY S., PAKES A., *The Pure Characteristics Discrete Choice Model with Application to Price Indices*, version préliminaire, juin 2001.

1992.

[6] KOHLI, RAJEEV, MAHAJAN V., *A Reservation-Price Model for Optimal Pricing of Multiattribute Products in Conjoint Analysis*, Journal of Marketing Research, 28, 1991.

[7] CHUNG J., RAO R. V., *A General Choice Model for Bundles with Multiple-Category Products : Application to Market Segmentation and Optimal Pricing for Bundles*, Journal of Marketing Research, Vol. XL, $115 - 130$, 2003.

[8] JEDIDI K., JAGPAL S., MANCHANDA P., *Measuring Heterogeneous Reservation Prices for Product Bundles*, Marketing Science, Vol.22, n1, pp.$107 - 130$, 2003.

[9] HITT L., CHEN P.-Y., *Bundling with Customer Self-Selection : A Simple Approach to Bundling Low-Marginal-Cost Goods*, Management Science, Vol.51, n10, pp.$1481 - 1493$, 2005.

[10] TRAIN K., *Discrete Choice Methods with Simulation*, Cambridge University Press, 2002.

[11] HOLENSTEIN R., *Using Sampling to Compute Bayes-Nash Equilibrium in Auction Games*, 2005.

[12] CESA-BIANCHI N., LUGOSI G., *Prediction, Learning, And Games*, Cambridge University Press, 2006.

[13] MYERSON R., *Game Theory, Analysis of Conflict*, Harvard University Press, 6$^{\text{ème}}$ Edition, 2004.

[14] MYERSON R., *Cooperative Games with Incomplete Information*, International Journal of Game Theory, 1984.

[15] CHIB S., JELIAZKOV I., *Marginal Likelihood From the Metropolis-Hastings Output*, Journal of the American Statistical Association, Vol96, n 453, 2001.

[16] TASSI, *Méthodes Statistiques*, Economica.

[17] HAGGSTROM O., *Finite Markov Chains and Algorithmic Applications*, 2001.

[18] VILALOBOS M., COELLO COELLO C., HERNANDEZ-LERMA O., *Asymptotic convergence of a simulated annealing algorithm for multiobjective optimization problems*, Math. Meth. Oper. Res., Vol64, pp.353 – 362, 2006.

[19] CAI G., WURMAN P., *Monte Carlo Approximation in Incomplete Information, Sequential Auction Games*, Decision Support Systems, Vol39, 2005.

[20] AUDESTAD J.-A., GAIVORONSKI A., WERNER A., *Extending the stochastic programming framework for the modeling of several decision makers : pricing and competition in the telecommunication sector*, Ann. Oper. Res., vol.142, 2006.

[21] BISI A., DADA M., *Dynamic Learning, Pricing, and Ordering by a Censored Newsvendor*, Naval Research Logistics, Vol.54, 2007.

[22] LELENO J., SHERALI H., *A Leader-Follower Model and Analysis for a Two-Stage Network of Oligopolies*, Annals of Operations Research, Vol.34, 1992.

[23] FUDENBERG D., LEVINE D., *The Theory of Learning in Games*, Economics Learning and Social Evolution series, 1998.

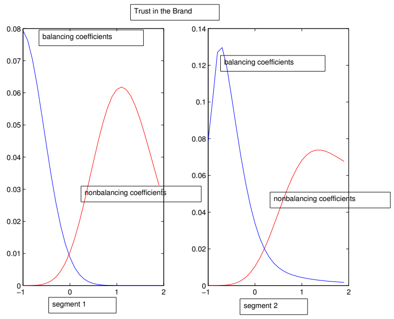

FIG. 1 – Trust in the Brand.

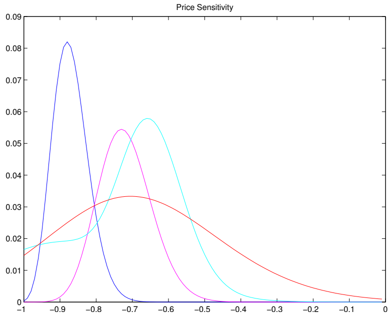

FIG. 2 – Price sensitivity.

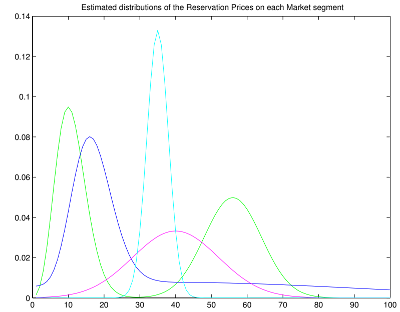

FIG. 3 – Estimated distributions of the reservation prices.

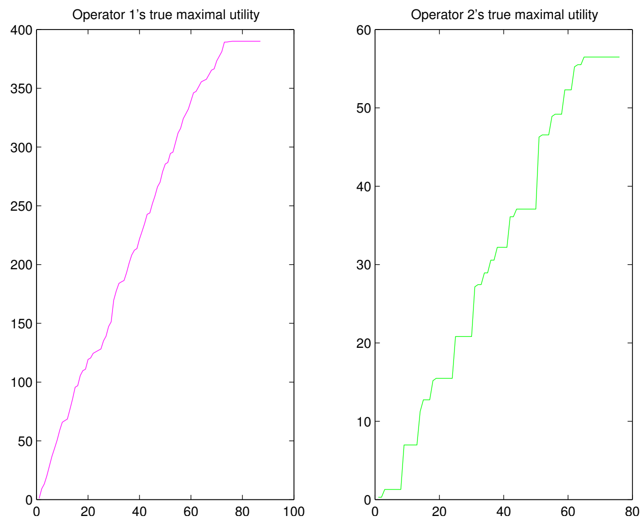

FIG. 4 – Convergence of the Bayesian equilibrium approximation algorithm.

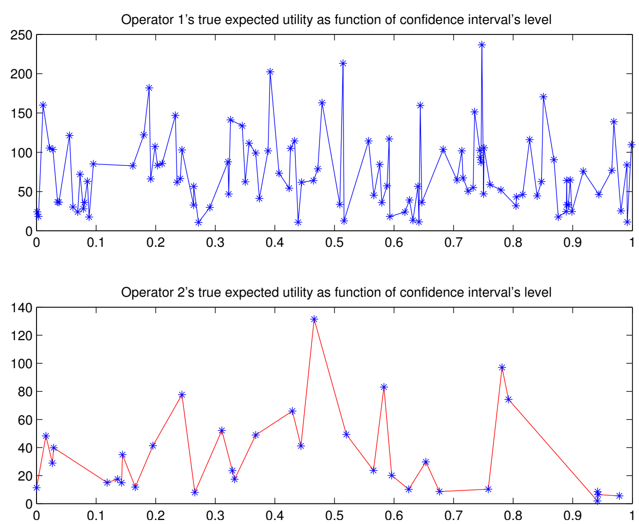

FIG. 5 – True expected utilities as functions of confidence interval's level.

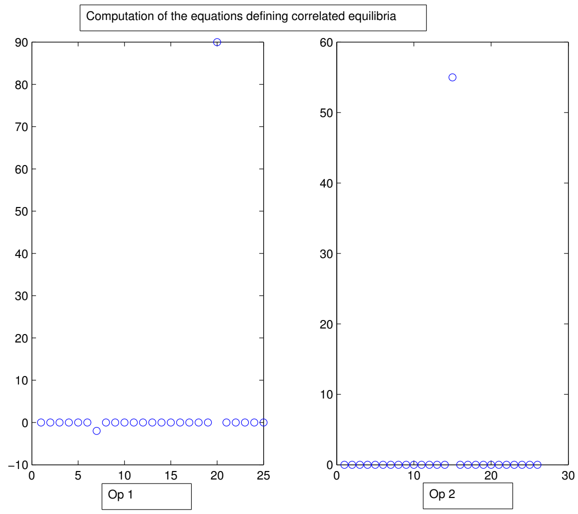

FIG. 6 – Aumann's conditions unsatisified for external regret minimization.

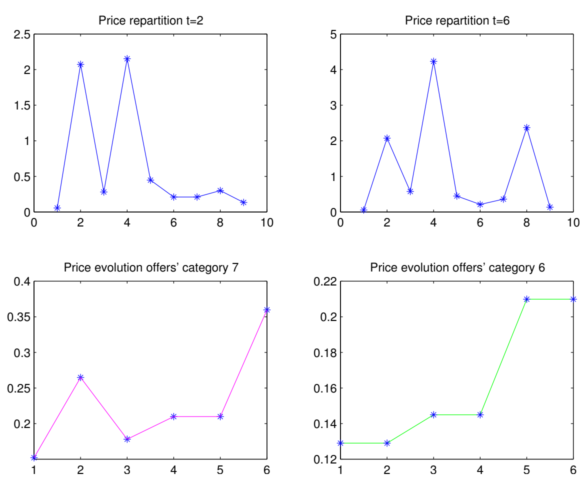

FIG. 7 – Dynamic evolution of the mean market prices.

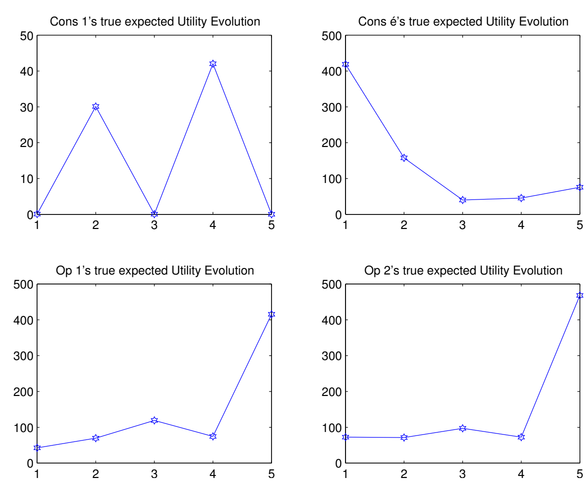

FIG. 8 – Dynamic evolution of the true expected utilities.

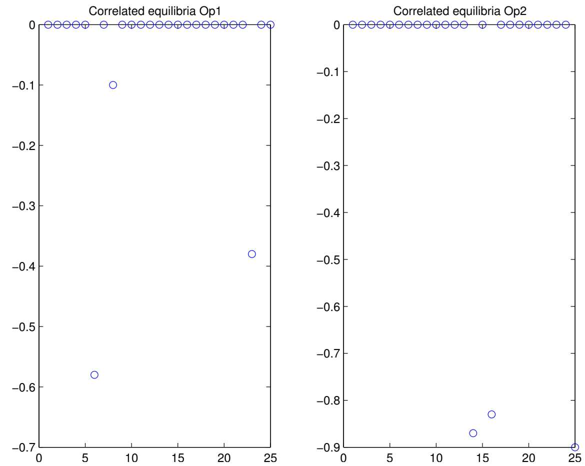

FIG. 9 – Joint Empirical Frequency satisfies Aumann's conditions.

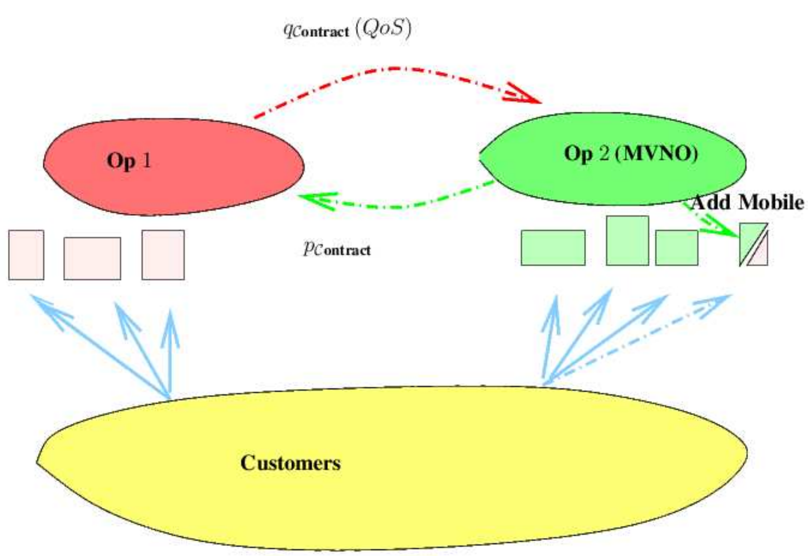

FIG. 10 – Principle of a MVNO agreement.

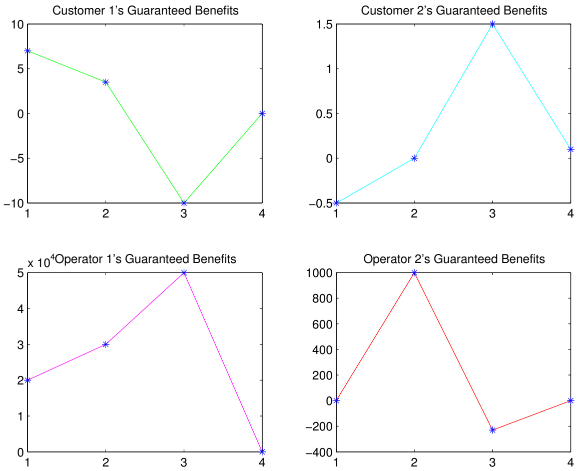

FIG. 11 – Guaranteed Benefits.

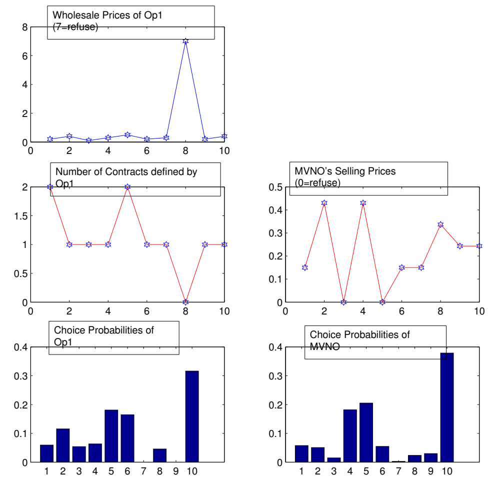

FIG. 12 – Probabilites of MVNO agreement's selection for Op1 and Op2.

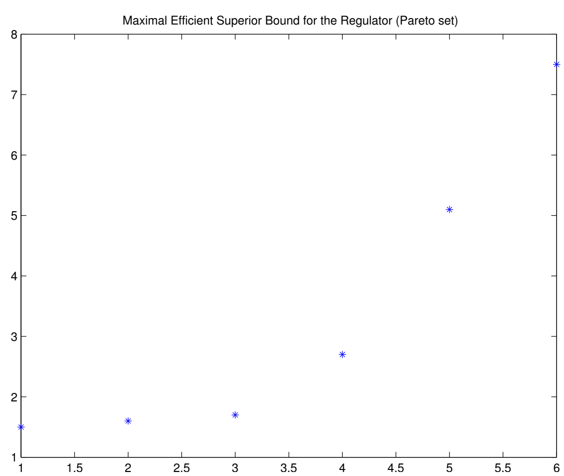

FIG. 13 – Efficient upper bounds for the MVNO agreement, fixed by a regulatory authority.