Implementation in Mixed Nash Equilibrium*

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Abstract

A mechanism implements a social choice correspondence f in mixed Nash equilibrium if at any preference profile, the set of *all* pure and mixed Nash equilibrium outcomes coincides with the set of f-optimal alternatives at that preference profile. This definition generalizes Maskin's definition of Nash implementation in that it does not require each optimal alternative to be the outcome of a *pure* Nash equilibrium. We show that the condition of weak set-monotonicity, a weakening of Maskin's monotonicity, is necessary for implementation. We provide sufficient conditions for implementation and show that important social choice correspondences that are not Maskin monotonic can be implemented in mixed Nash equilibrium.

Keywords: implementation, Maskin monotonicity, pure and mixed Nash equilibrium, weak set-monotonicity, social choice correspondence.

JEL Classification Numbers: C72; D71

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1 Introduction

This paper studies the problem of implementation in mixed Nash equilibrium. According to our definition, a mechanism implements a social choice correspondence f in mixed Nash equilibrium if the set of all pure and mixed equilibrium outcomes corresponds to the set of f-optimal alternatives at each preference profile. Crucially, and unlike the classical definition of implementation, this definition of implementation does not give a predominant role to pure equilibria: an optimal alternative does not have to be the outcome of a pure Nash equilibrium. This sharply contrasts with most of the literature on Nash implementation, which does not consider equilibria in mixed strategies. Two notable exceptions are Maskin (1999) for Nash implementation and Serrano and Vohra (2007) for Bayesian implementation. These authors do consider mixed equilibria, but still require each f-optimal alternative to be the outcome of a *pure* equilibrium. Pure equilibria are yet again given a special status.

Perhaps, the emphasis on pure equilibria expresses a discomfort with the classical view of mixing as deliberate randomizations on the part of the players. However, it is now accepted that even if players do not randomize but choose definite actions, a mixed strategy may be viewed as a representation of the other players' uncertainty about a player's choice (e.g., see Aumann and Brandenburger, 1995). Additionally, almost all mixed equilibria can be viewed as pure Bayesian equilibria of nearby games of incomplete information, in which every player is uncertain about the exact preferences of the other players, as first suggested in the seminal work of Harsanyi (1973). This view acknowledges that games with commonly known preferences are an idealization, a limit of near-complete information games. This interpretation is particularly important for the theory of implementation in Nash equilibrium, whereby the assumption of common knowledge of preferences, especially on large domains, is at best a simplifying assumption.[1] Furthermore, recent evidence in the experimental literature suggest that equilibria

[1]The point that the assumption of common knowledge of preferences might be problematic is not new. For instance, Chung and Ely (2003) study the problem of *full* implementation of social choice functions under "near-complete information" and show that Maskin monotonicity is a necessary condition for implementation in undominated (pure) Nash equilibria. Their result sharply contrasts with Palfrey and Srivastava (1991), who have shown that almost all social choice functions are implementable in undominated (pure) Nash equilibria. Oury and Tercieux (2007) consider the problem of *partial* implementation of social choice functions under "almost complete" information and show that Maskin

in mixed strategies are good predictors of behavior in some classes of games e.g., coordination games and chicken games (see chapters 3 and 7 of Camerer, 2003). Since, for some preference profiles, a mechanism can induce one of those games, paying attention to mixed equilibria is important if we want to describe or predict players' behavior. In sum, we believe that there are no compelling reasons to give pure Nash equilibria a special status and modify the definition of implementation accordingly.

Our definition of mixed Nash implementation yields surprising results. We demonstrate that the condition of Maskin monotonicity is not necessary for full implementation in mixed Nash equilibrium. Intuitively, consider a profile of preferences and an alternative, say a, that is f-optimal at that profile of preferences. According to Maskin's definition of implementation, there must exist a pure Nash equilibrium with equilibrium outcome a. This implies that any alternative a player can trigger by unilateral deviations is less preferred than a. Thus, if we move to another profile of preferences where a does not fall down in the players' ranking, then a remains an equilibrium outcome and must be optimal at that new profile of preferences. Unlike Maskin's definition of implementation, our definition does not require a to be a pure equilibrium outcome. So, suppose that there exists a mixed equilibrium with a as an equilibrium outcome.[2] The key observation to make is that the mixed equilibrium induces a *lottery* over optimal alternatives. Thus, when we move to another profile of preferences where a does not fall down in the players' ranking, the original profile of strategies does not have to be an equilibrium at the new state. In fact, we show that a much weaker condition, *weak set-monotonicity*, is necessary for implementation in mixed Nash equilibrium. Weak set-monotonicity states that the set of optimal alternatives $f(\theta)$ at state θ is included in the set of optimal alternatives $f(\theta')$ at state θ', whenever the weak and strict lower contour sets at state θ of all alternatives in $f(\theta)$ are included in their respective weak and strict lower contour sets at state θ', for all players.

Weak set-monotonicity is a substantially weaker requirement than Maskin monotonicity. For instance, on the unrestricted domain of preferences, the strong Pareto and the strong core correspondences are weak set-monotonic, while they are not Maskin monotonic. Similarly, on the domain of strict preferences, the top-cycle and the uncov-

monotonicity is necessary for implementation in pure Nash equilibrium.

[2]More precisely, let σ^* be the mixed Nash equilibrium and $\mathbb{P}_{\sigma^*,g}$ the distribution over alternatives induced by the strategy profile σ^* and the allocation rule g. Then a belongs to the support of $\mathbb{P}_{\sigma^*,g}$.

ered set correspondences are weak set-monotonic, but not Maskin monotonic.

Furthermore, we show that weak set-monotonicity and no-veto power are sufficient for implementation on the domain of strict preferences. However, and somewhat surprisingly, the condition of weak set-monotonicity even coupled with the no-veto power condition is not sufficient for implementation in mixed Nash equilibrium in general. A mild strengthening of weak set-monotonicity that we call weak* set-monotonicity is required. Since no-veto power is not satisfied by important social choice correspondences like the strong Pareto and the strong core, we also present sufficient conditions that dispense with the no-veto power condition. (See Benoît and Ok, 2008, and Bochet, 2007.)

An important feature of our sufficiency proofs is the use of randomized mechanisms. Since we consider implementation in mixed Nash equilibrium, the use of randomized mechanisms seems natural. As for the players, the designer's randomization can be interpreted as the players' beliefs about the choice of alternatives. In the literature on (exact) Nash implementation, randomized mechanisms have been studied by Benoît and Ok (2008) and Bochet (2007). These authors restrict attention to mechanisms in which randomization by the designer can only occur out of equilibrium, and do not attempt to rule out mixed strategy equilibria with undesirable outcomes. On the contrary, we allow randomization among f-optimal alternatives at equilibrium and rule out mixed equilibria with outcomes that are not f-optimal. Our approach also differs from the use of randomized mechanisms in the literature on virtual implementation (e.g., see Matsushima, 1998, and Abreu and Sen, 1991), which heavily exploits the possibility of selecting undesirable alternatives with positive probability in equilibrium.

The paper is organized as follows. Section 2 presents a simple example illustrating our ideas. Section 3 introduces preliminary definitions and defines mixed Nash implementation. Section 4 presents the necessary condition of weak set-monotonicity, while sections 5 and 6 provides several sets of sufficient conditions. Section 7 applies our results to some well known social choice correspondences and section 8 concludes.

2 A Simple Example

This section illustrates our notion of mixed Nash implementation with the help of a simple example.

Example 1 There are two players, 1 and 2, two states of the world, θ and θ', and four alternatives, a, b, c, and d. Players have state-dependent preferences represented in the table below. Preferences are strict. For instance, player 1 ranks b first and a second in state θ, while a is ranked first and b last in state θ'.

θ		θ'	
1	2	1	2
b	c	a	c
a	a	d	d
c	b	c	a
d	d	b	b

The designer aims to implement the social choice correspondence f with $f(\theta) = \{a\}$ and $f(\theta') = \{a, b, c, d\}$. We say that an alternative x is f-optimal at state θ if $x \in f(\theta)$.

We first argue that the social choice correspondence f is not implementable in the sense of Maskin (1999). Maskin's definition of Nash implementation requires that for each f-optimal alternative at a given state, there exists a *pure* Nash equilibrium (of the game induced by the mechanism) corresponding to that alternative. So, for instance, at state θ', there must exist a pure Nash equilibrium with b as equilibrium outcome. Maskin requires furthermore that no such equilibrium must exist at state θ. However, if there exists a pure equilibrium with b as equilibrium outcome at state θ', then b will also be an equilibrium outcome at state θ, since b moves up in every players' ranking when going from state θ' to state θ. The correspondence f is thus not implementable in the sense of Maskin. In other words, the social choice correspondence f violates Maskin monotonicity, a necessary condition for implementation in the sense of Maskin.

In contrast with Maskin, we do *not* require that for each f-optimal alternative at a given state, there exists a pure Nash equilibrium corresponding to that alternative. We require instead that the set of f-optimal alternatives coincides with the set of *mixed* Nash equilibrium outcomes. So, at state θ', there must exist a mixed Nash equilibrium with b corresponding to an action profile in the support of the equilibrium.

We now argue that with our definition of implementation, the correspondence f is implementable. To see this, consider the mechanism where each player has two messages m_1 and m_2, and the allocation rule is represented in the table below.

	m_1	m_2
m_1	a	b
m_2	d	c

For example, if both players announce m_1, the chosen alternative is a. Now, at state θ, (m_1, m_1) is the unique Nash equilibrium, with outcome a. At state θ', both (m_1, m_1) and (m_2, m_2) are pure Nash equilibria, with outcomes a and c. Moreover, there exists a mixed Nash equilibrium that puts strictly positive probability on each action profile (since preferences are strict), hence on each outcome. Therefore, f is implementable in mixed Nash equilibrium, although it is not implementable in the sense of Maskin.

We conclude this section with two important observations. First, our notion of implementation in mixed Nash equilibrium is *ordinal*: the social choice correspondence f is implementable regardless of the cardinal representations chosen for the two players. Second, alternative d is f-optimal at state θ', and it moves down in player 1's ranking when moving from θ' to θ. This preference reversal guarantees the weak set-monotonicity of the correspondence f, which, as we shall see, is a necessary condition for implementation in mixed Nash equilibrium.

3 Preliminaries

An environment is a triplet $\langle N, X, \Theta \rangle$ where $N := \{1, \ldots, n\}$ is a set of n players, X a finite set of alternatives, and Θ a finite set of states of the world. Associated with each state θ is a preference profile $\succcurlyeq^\theta := (\succcurlyeq_1^\theta, \ldots, \succcurlyeq_n^\theta)$, where $\succcurlyeq_i^\theta$ is player i's preference relation over X at state θ. The asymmetric and symmetric parts of $\succcurlyeq_i^\theta$ are denoted $\succ_i^\theta$ and $\sim_i^\theta$, respectively.

We denote with $L_i(x, \theta) := \{y \in X : x \succcurlyeq_i^\theta y\}$ player i's lower contour set of x at state θ, and $SL_i(x, \theta) := \{y \in X : x \succ_i^\theta y\}$ the strict lower contour set. For any (i, θ) in $N \times \Theta$ and $Y \subseteq X$, define $\max_i^\theta Y$ as $\{x \in Y : x \succcurlyeq_i^\theta y \text{ for all } y \in Y\}$.

We assume that any preference relation $\succcurlyeq_i^\theta$ is representable by a utility function $u_i(\cdot, \theta) : X \to \mathbb{R}$, and that each player is an expected utility maximizer. We denote with $\mathcal{U}_i^\theta$ the set of all possible cardinal representations $u_i(\cdot,\theta)$ of $\succsim_i^\theta$ at state θ, and let $\mathcal{U}^\theta := \times_{i\in N}\mathcal{U}_i^\theta$.

A social choice correspondence $f : \Theta \to 2^X \setminus \{\emptyset\}$ associates with each state of the world θ, a non-empty subset of alternatives $f(\theta) \subseteq X$. Two classic conditions for Nash implementation are Maskin monotonicity and no-veto power. A social choice correspondence f is *Maskin monotonic* if for all (x,θ,θ') in $X \times \Theta \times \Theta$ with $x \in f(\theta)$, we have $x \in f(\theta')$ whenever $L_i(x,\theta) \subseteq L_i(x,\theta')$ for all $i \in N$. Maskin monotonicity is a necessary condition for Nash implementation (à la Maskin). A social choice correspondence f satisfies *no-veto power* if for all $\theta \in \Theta$, we have $x \in f(\theta)$ whenever $x \in \max_i^\theta X$ for all but at most one player $i \in N$. Maskin monotonicity and no-veto power are sufficient conditions for Nash implemetation (in the sense of Maskin).

Let $\Delta(X)$ be the set of all probability measures over X. A mechanism (or game form) is a pair $\langle (M_i)_{i\in N}, g\rangle$ with M_i the set of messages of player i, and $g : \times_{i\in N} M_i \to \Delta(X)$ the allocation rule. Let $M := \times_{j\in N} M_j$ and $M_{-i} := \times_{j\in N\setminus\{i\}} M_j$, with m and m_{-i} generic elements.

A mechanism $\langle (M_i)_{i\in N}, g\rangle$, a state θ and a profile of cardinal representations $(u_i(\cdot,\theta))_{i\in N}$ of $(\succsim_i^\theta)_{i\in N}$ induce a strategic-form game as follows. There is a set N of n players. The set of pure actions of player i is M_i, and player i's expected payoff when he plays m_i and his opponents play m_{-i} is

$$U_i(g(m_i, m_{-i}), \theta) := \sum_{x\in X} g(m_i, m_{-i})(x) u_i(x,\theta),$$

where $g(m_i, m_{-i})(x)$ is the probability that x is chosen by the mechanism when the profile of messages (m_i, m_{-i}) is announced. The induced strategic-form game is thus $G(\theta, u) := \langle N, (M_i, U_i(g(\cdot), \theta))_{i\in N}\rangle$. Let σ be a profile of mixed strategies. We denote with $\mathbb{P}_{\sigma,g}$ the probability distribution over alternatives in X induced by the allocation rule g and the profile of mixed strategies σ.[3]

Definition 1 *The mechanism $\langle (M_i)_{i\in N}, g\rangle$ implements the social choice correspondence f in mixed Nash equilibrium if for all $\theta \in \Theta$, for all cardinal representations $u(\cdot,\theta) \in \mathcal{U}^\theta$ of $\succsim^\theta$, the following two conditions hold:*

(i) For each $x \in f(\theta)$, there exists a Nash equilibrium σ^* of $G(\theta,u)$ such that x is in the support of $\mathbb{P}_{\sigma^*,g}$, and

[3]Formally, the probability $\mathbb{P}_{\sigma,g}(x)$ of $x \in X$ is $\sum_{m\in M} \sigma(m)g(m)(x)$ if M is countable.

(ii) if σ is a Nash equilibrium of $G(\theta, u)$, then the support of $\mathbb{P}_{\sigma,g}$ is included in $f(\theta)$.

Before proceeding, it is important to contrast our definition of implementation in mixed Nash equilibrium with Maskin (1999) definition of Nash implementation.

First, part (i) of Maskin's definition requires that for each $x \in f(\theta)$, there is a pure Nash equilibrium m^* of $G(\theta, u)$ with equilibrium outcome x, while part (ii) of his definition is identical to ours. In contrast with Maskin, we allow for mixed strategy Nash equilibria in part (i) and, thus, restore a natural symmetry between parts (i) and (ii). Yet, our definition respects the spirit of full implementation in that only optimal outcomes can be observed by the designer as equilibrium outcomes.

Second, as in Maskin, our concept of implementation is ordinal as all equilibrium outcomes have to be optimal, regardless of the cardinal representation chosen.

Third, we allow the designer to use randomized mechanisms. This is a natural assumption given that players can use mixed strategies. Indeed, although a randomized mechanism introduces some uncertainty about the alternative to be chosen, the concept of a mixed Nash equilibrium already encapsulates the idea that players are uncertain about the messages sent to the designer and, consequently, about the alternative to be chosen. We also stress that the randomization can only be among optimal alternatives in equilibrium. In the context of (exact) Nash implementation, Benoît and Ok (2008) and Bochet (2007) have already considered randomized mechanisms.[4] There are two important differences with our work. First, these authors restrict attention to mechanisms in which randomization only occurs out of equilibrium, while randomization can occur in equilibrium in our work, albeit only among optimal alternatives. Second, unlike us, they do not attempt to rule out mixed strategy equilibria with undesirable outcomes. Also, our work contrasts with the literature on virtual implementation (e.g., see Matsushima, 1998, and Abreu and Sen, 1991), which heavily exploits randomized mechanisms that select undesirable alternatives with positive probability in equilibrium. Unlike this literature, we focus on exact implementation: only f-optimal alternatives can be equilibrium outcomes.

Finally, from our definition of mixed Nash implementation, it is immediate to see that if a social choice correspondence is Nash implementable (i.e., à la Maskin), then

[4]Vartiainen (2007) also considers randomized mechanisms, but for the implementation of social choice correspondences in (pure) subgame perfect equilibrium on the domain of strict preferences.

it is implementable in mixed Nash equilibrium. The converse is false, as shown by Example 1 in Section 2. The goal of this paper is to characterize the social choice correspondences implementable in mixed Nash equilibrium. The next section provides a necessary condition.

4 A Necessary Condition

This section introduces a new condition, called *weak set-monotonicity*, which we show to be necessary for the implementation of social choice correspondences in mixed Nash equilibrium.

Definition 2 *A social choice correspondence f is weak set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$: (i) $L_i(x, \theta) \subseteq L_i(x, \theta')$ and (ii) $SL_i(x, \theta) \subseteq SL_i(x, \theta')$.*[5]

Weak set-monotonicity is a weakening of Maskin monotonicity. It requires that if for *all* players, the lower *and* strict lower contour sets of *all* alternatives in $f(\theta)$ do not shrink in moving from θ to θ', then the set of optimal alternatives $f(\theta')$ at θ' must be a superset of the set of optimal alternatives $f(\theta)$ at θ. As we shall see in Section 7, important correspondences, like the strong Pareto correspondence, the strong core correspondence, the top-cycle set and the uncovered set, are weak-set monotonic, while they fail to be Maskin monotonic.

Theorem 1 *If the social choice correspondence f is implementable in mixed Nash equilibrium, then it satisfies the weak set-monotonicity condition.*

Proof The proof is by contradiction on the contrapositive. Assume that the social choice correspondence f does not satisfy weak set-monotonicity and yet it is implementable in mixed Nash equilibrium by the mechanism $\langle M, g \rangle$.

Since f does not satisfy weak set-monotonicity, there exist x^*, θ, and θ' such that $x^* \in f(\theta) \setminus f(\theta')$, while $L_i(x, \theta) \subseteq L_i(x, \theta')$ and $SL_i(x, \theta) \subseteq SL_i(x, \theta')$ for all $x \in f(\theta)$, for all $i \in N$.

[5] Alternatively, a social choice correspondence f is weak set-monotonic if $x^* \in f(\theta) \setminus f(\theta')$ implies that there exists a pair (x, y) in $f(\theta) \times X$ and a player $i \in N$ such that either (1) $x \not\succsim_i^{\theta} y$ and $y \succ_i^{\theta'} x$, or (2) $x \succ_i^{\theta} y$ and $y \not\succsim_i^{\theta'} x$.

Claim. For each player $i \in N$, fix a cardinal representation $u_i(\cdot, \theta)$ of $\succsim_i^\theta$. We claim that there exists a cardinal representation $u_i(\cdot, \theta')$ of $\succsim_i^{\theta'}$ such that $u_i(x, \theta') \leq u_i(x, \theta)$ for all $x \in X$, $u_i(x, \theta') = u_i(x, \theta)$ for all $x \in f(\theta)$.

To prove our claim, consider any pair $(x, x') \in f(\theta) \times f(\theta)$ with $x \succsim_i^\theta x'$. Since $L_i(x, \theta) \subseteq L_i(x, \theta')$ for all $x \in f(\theta)$, we have that $x \succsim_i^{\theta'} x'$. Hence, we can associate with each alternative in $f(\theta)$ the same utility at θ' as at θ. Now, fix an $x \in f(\theta)$ and consider $y \in L_i(x, \theta)$. Since $L_i(x, \theta) \subseteq L_i(x, \theta')$, we must have $u_i(y, \theta') \leq u_i(x, \theta') = u_i(x, \theta)$. If $x \sim_i^\theta y$, then we can choose $u_i(y, \theta') \leq u_i(y, \theta) = u_i(x, \theta)$. If $x \succ_i^\theta y$, then we must have $x \succ_i^{\theta'} y$ since $SL_i(x, \theta) \subseteq SL_i(x, \theta')$; we can therefore choose $u_i(y, \theta')$ in the open set $(-\infty, u_i(y, \theta))$ and still represent $\succsim_i^{\theta'}$ by $u_i(\cdot, \theta')$. Finally, if $y \notin \cup_{x \in f(\theta)} L_i(x, \theta)$, we have that $u_i(y, \theta) > u_i(x, \theta)$ for all $x \in f(\theta)$. If $y \in L_i(x, \theta')$ for some $x \in f(\theta)$, then we can set $u_i(y, \theta') \leq u_i(x, \theta') = u_i(x, \theta) \leq \max_{x \in f(\theta)} u_i(x, \theta) < u_i(y, \theta)$. If $y \notin \cup_{x \in f(\theta)} L_i(x, \theta')$, then we can choose $u_i(y, \theta')$ in the open set $(\max_{x \in f(\theta)} u_i(x, \theta), u_i(y, \theta))$. This concludes the proof of our claim.

Before proceeding, we should stress the importance of the nestedness of the strict lower-contour sets in part (ii) of the definition of weak set-monotonicity. Let $x \in f(\theta)$ and assume that $x \sim_i^\theta y \succ_i^\theta z$ at state θ and $x \sim_i^{\theta'} z \succ_i^{\theta'} y$ at state θ'. Both alternatives z and y are in the lower contour set of x at θ and θ', but the strict lower-contour sets are not nested. Clearly, we cannot assign the same utility to x at θ and θ' and weakly decrease the utility of both y and z when moving from θ to θ'; the claim does not hold.

Since f is implementable and $x^* \in f(\theta)$, for any cardinal representation $u(\cdot, \theta)$ of $\succsim^\theta$, there exists an equilibrium σ^* of the game $G(\theta, u)$ with x^* in the support of $\mathbb{P}_{\sigma^*, g}$. Furthermore, since $x^* \notin f(\theta')$, for all cardinal representations $u(\cdot, \theta')$ of $\succsim^{\theta'}$, for all equilibria σ of $G(\theta', u)$, x^* does not belong to the support of $\mathbb{P}_{\sigma, g}$. In particular, this implies that σ^* is not an equilibrium at θ' for all cardinal representation $u(\cdot, \theta')$. Thus, there exist a player i, a message m_i^* in the support of σ_i^*, and a message m_i' such that

$$\sum_{m_{-i}} [U_i(g(m_i^*, m_{-i}), \theta) - U_i(g(m_i', m_{-i}), \theta)] \, \sigma_{-i}^*(m_{-i}) \geq 0$$

and

$$0 > \sum_{m_{-i}} [U_i(g(m_i^*, m_{-i}), \theta') - U_i(g(m_i', m_{-i}), \theta')] \, \sigma_{-i}^*(m_{-i}).$$

It follows that

$$\sum_{m_{-i}} [U_i(g(m_i^*, m_{-i}), \theta) - U_i(g(m_i^*, m_{-i}), \theta')]\, \sigma_{-i}^*(m_{-i}) \quad (1)$$
$$> \sum_{m_{-i}} [U_i(g(m_i', m_{-i}), \theta) - U_i(g(m_i', m_{-i}), \theta')]\, \sigma_{-i}^*(m_{-i})$$

Let us now consider the cardinal representations presented in the proof of the claim above; that is, where for all $i \in N$, $u_i(x, \theta') \leq u_i(x, \theta)$ for all $x \in X$ and $u_i(x, \theta) = u_i(x, \theta')$ for all $x \in f(\theta)$. Since f is implementable, we have that the support of $\mathbb{P}_{\sigma^*, g}$ is included in $f(\theta)$. Therefore, $U_i(g(m_i^*, m_{-i}), \theta) = U_i(g(m_i^*, m_{-i}), \theta')$ for all m_{-i} in the support of σ_{-i}^*. Hence, the left-hand side of the inequality (1) is zero. Furthermore, we have that $U_i(g(m_i', m_{-i}), \theta) \geq U_i(g(m_i', m_{-i}), \theta')$ for all m_{-i}. Hence, the right-hand side of (1) is non-negative, a contradiction. This completes the proof. □

Several remarks are worth making. First, Theorem 1 remains valid if we restrict ourself to deterministic mechanisms, so that weak-set monotonicity is a necessary condition for implementation in mixed Nash equilibrium, regardless of whether we consider deterministic or randomized mechanisms. Second, it is easy to verify that weak-set monotonicity is also a necessary condition for implementation in Nash equilibrium (à la Maskin) with randomized mechanisms. Third, while we have restricted attention to von Neumann-Morgenstern preferences, the condition of weak set-monotonicity remains necessary if we consider larger classes of preferences that include the von Neumann-Morgenstern preferences. Furthermore, if players have non-additive beliefs about the play of their opponents (modelled as simple capacities) and their preferences are represented as Choquet integral of Bernoulli utilities over capacities (as in the concept of *equilibrium under uncertainty* introduced by Eichberger and Kelsey, 2000), then our result also remains valid.[6] This suggests that our necessary condition does not critically depend on the restriction to von Neumann-Morgenstern preferences.

As customary in the large literature on implementation (see Jackson, 2001, and Maskin and Sjöström, 2002, for excellent surveys), it is natural to ask whether the condition of weak set-monotonicity is "almost" sufficient for implementation. The following example shows that this is not the case.

[6] This follows from Lemma 3.3 of Eichberger and Kelsey (2000) and the fact that the Choquet integral of $(\sigma_i^*, \sigma_{-i}^*)$ over $u_i(\cdot, \theta)$ is greater than over $u_i(\cdot, \theta')$ whenever $u_i(x, \theta) \geq u_i(x, \theta')$ for all $x \in X$.

Example 2 There are three players, 1, 2 and 3, two states of the world, θ and θ', and three alternatives a, b and c. Preferences are represented in the table below.

θ			θ'		
1	2	3	1	2	3
b	a	c	b	$a \sim b$	c
c	b	a	c		a
a	c	b	a	c	b

The social choice correspondence is $f(\theta) = \{a\}$ and $f(\theta') = \{b\}$. It is weak set-monotonic since $SL_2(a,\theta) \nsubseteq SL_2(a,\theta')$ and $L_2(b,\theta') \nsubseteq L_2(b,\theta)$. However, it is not implementable in mixed Nash equilibrium. If it were implementable, then for any cardinal representation of $\succsim^{\theta}$, there would exist an equilibrium σ^* at θ such that the support of $\mathbb{P}_{\sigma^*,g}$ is $\{a\}$. Then, σ^* would also be an equilibrium at θ' for some cardinal representation of $\succsim^{\theta'}$, a contradiction. For instance, fix a cardinal representation $u(\cdot,\theta)$ at θ. Since players 1 and 3's preferences do not change from θ to θ', we can use the same cardinal representations at θ'. As for player 2, we can use $u_2(a,\theta') = u_2(a,\theta) = u_2(b,\theta') > u_2(b,\theta) > u_2(c,\theta) = u_2(c,\theta')$. The intuition is clear. Since players 1 and 3's preferences do not change from θ to θ' and a is top-ranked for player 2 at both states, any equilibrium at θ with outcome a remains an equilibrium at θ'. At state θ', there is no alternative that can be used to generate a profitable deviation for player 2.

Note, furthermore, that the social choice correspondence f in Example 2 satisfies the no-veto power condition. Hence, no-veto power together with weak set-monotonicity are not sufficient for Nash implementation. The next section provide sufficient conditions: a strengthening of weak-set monotonicity will be needed.

5 Sufficient Conditions

Before stating the main result of this section, we need to introduce two additional definitions. The first definition strengthens the notion of weak set-monotonicity.

Definition 3 *A social choice correspondence f is* weak* set-monotonic *if for all pairs $(\theta,\theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$: (i) $L_i(x,\theta) \subseteq L_i(x,\theta')$ and (ii) either $SL_i(x,\theta) \subseteq SL_i(x,\theta')$ or $x \in \max_i^{\theta'} X$.*

In Example 2, the social choice correspondence f is not weak* set-monotonic since $L_2(a,\theta) \subseteq L_2(a,\theta')$, $a \in \max_2^{\theta'}\{a,b,c\}$, and yet $a \notin f(\theta')$.

Clearly, if a social choice correspondence is Maskin monotonic, then it is weak* set-monotonic, and if it is weak* set-monotonic, then it is weak set-monotonic. Moreover, weak* set-monotonicity coincides with weak set-monotonicity if $\max_i^{\theta} X$ is a singleton for each $i \in N$, for each $\theta \in \Theta$. We refer to this domain of preferences as the *single-top preferences*. This mild domain restriction will prove useful in applications (see Section 7). Furthermore, on the domain of strict preferences (a subset of single-top preferences) $SL_i(x,\theta) = L_i(x,\theta) \setminus \{x\}$ for all x and θ, and weak* set-monotonicity is then equivalent to weak set-monotonicity.

The second condition we need is a restriction on the set of cardinal representations at each state. We assume that for each player $i \in N$, for each state $\theta \in \Theta$, the set of admissible cardinal representations is a compact subset $\overline{\mathcal{U}}_i^{\theta}$ of $\mathcal{U}_i^{\theta}$. It follows that there exists $\varepsilon > 0$ such that for all $i \in N$, for all θ, for each pair $(x,y) \in X \times X$ with $x \succ_i^{\theta} y$, and for all $u_i(\cdot,\theta) \in \overline{\mathcal{U}}_i^{\theta}$: $u_i(x,\theta) \geq (1-\varepsilon)u_i(y,\theta) + \varepsilon \max_{z \in X} u_i(z,\theta)$. Accordingly, we have to modify Definition 1 of mixed Nash implementation so as to include this restriction on the set of cardinal representations. We call this new notion of implementation, *mixed Nash C-implementation.*

Naturally, with C-implementation, the condition of weak set-monotonicity might fail to be necessary. For instance, consider the following example.

Example 3 There are two players, 1 and 2, two states of the world, θ and θ', and a unique cardinal representation at each state, indicated below:

θ		θ'	
1	2	1	2
d : 2	d : 5	d : 5	$c \sim d$: 2
c : 1.5	c : 2	a : 1	a : 1
a : 1	a : 1	c : 0.5	b : 0
b : -1	b : 0	b : -1	

For instance, at state θ, player 1's utility of d is 2, while player 2's utility is 5. The social choice correspondence is $f(\theta) = \{a\}$ and $f(\theta') = \{c\}$. It is not weak set-monotonic and yet it is implementable by the following mechanism, where $(1/2)a + (1/2)b$ denotes a 50-50 lottery on a and b.

	m_1	m_2
m_1	a	$(1/2)a + (1/2)b$
m_2	$(1/2)d + (1/2)b$	c

For "large" enough compact sets of cardinal representations, however, the condition of weak-set monotonicity remains necessary. For instance, fix $\delta > 0$, and let $\overline{\mathcal{U}}_i^\theta := \{u_i(x,\theta) \in [-K, K] \text{ for all } x \in X : x \succ_i^\theta y \Leftrightarrow u_i(x,\theta) \geq u_i(y,\theta) + \delta\}$ with K large enough but finite. The set $\overline{\mathcal{U}}_i^\theta$ of cardinal representations is clearly compact, and the proof of Theorem 1 carries over if $\delta < 2K/(|X|^2)$.[7]

We are now ready to present the main result of this section, which states that in any environment with at least three players, weak* set-monotonicity and no veto-power are sufficient conditions for implementation in mixed Nash equilibrium. The mechanism constructed in the proof is partly inspired by the mechanism in the appendix of Maskin (1999).[8]

Theorem 2 *Let $\langle N, X, \Theta\rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is weak* set-monotonic and satisfies no-veto power, then it is C-implementable in mixed Nash equilibrium.*

Proof Consider the following mechanism $\langle M, g\rangle$. For each player $i \in N$, the message space M_i is $\Theta \times \{\alpha^i : \alpha^i : X \times \Theta^2 \to X\} \times X \times \mathbb{Z}_{++}$. In words, each player announces a state of the world, a function from alternatives and pairs of states into alternatives, an alternative, and a strictly positive integer. A typical message m_i for player i is

[7]To see this, fix the following cardinal representation of $\succcurlyeq_i^\theta$: $u_i(x,\theta) = K$ for all $x \in \max_i^\theta X$, $u_i(x,\theta) = K - \delta|X|$ for all $x \in \max_i^\theta(X \setminus \max_i^\theta X)$, etc. This cardinal representation is clearly admissible and the difference in utilities between any two alternatives that are not indifferent is at least $\delta|X|$. It is then easy to see that, as required by the claim in the proof of Theorem 1, if we move to a state θ' where $L_i(x,\theta) \subseteq L_i(x,\theta')$ and $SL_i(x,\theta) \subseteq SL_i(x,\theta')$ for all $x \in f(\theta)$, for all $i \in N$, then there exists a cardinal representation $u_i(\cdot,\theta')$ with $u_i(x,\theta') \leq u_i(x,\theta)$ for all $x \in X$, and $u_i(x,\theta') = u_i(x,\theta)$ for all $x \in f(\theta)$. That is all we need for the proof of Theorem 1 to go through.

[8]The mechanism in the main body of Maskin (1999) does not deal with the issue of ruling out unwanted mixed Nash equilibria.

$(\theta^i, \alpha^i, x^i, \mathbf{z}^i)$. (Note that we denote any integer $\mathbf{z}$ in bold.) Let $M := \times_{i\in N} M_i$ with typical element m.

Let $\{f_1(\theta), \ldots, f_{K^\theta}(\theta)\} = f(\theta)$ be the set of f-optimal alternatives at state θ ; note that $K^\theta = |f(\theta)|$. Let $1 > \varepsilon > 0$ be such that for all $i \in N$, for all $\theta \in \Theta$, for each pair $(x, y) \in X \times X$ with $x \succ_i^\theta y$, and for all $u_i(\cdot, \theta) \in \overline{\mathcal{U}}_i^\theta$, we have $u_i(x, \theta) \geq (1-\varepsilon)u_i(y,\theta) + \varepsilon \max_{w\in X} u_i(w, \theta)$. Since $\overline{\mathcal{U}}_i^\theta$ is a compact subset of $\mathcal{U}_i^\theta$, such an ε exists. The allocation rule g is defined as follows.

Rule 1: If $m_i = (\theta, \alpha, x, 1)$ for all $i \in N$ (i.e., all agents make the same announcement m_i) and $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, then $g(m)$ is the "uniform" lottery over alternatives in $f(\theta)$; that is,

$$g(m) = \frac{1}{K^\theta} \sum_{k=1}^{K^\theta} f_k(\theta).$$

Rule 2: If there exists $j \in N$ such that $m_i = (\theta, \alpha, x, 1)$ for all $i \in N \setminus \{j\}$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, and $m_j = (\theta^j, \alpha^j, x^j, \mathbf{z}^j) \neq m_i$, then $g(m)$ is the lottery:

$$\frac{1}{K^\theta} \sum_{k=1}^{K^\theta} \left\{ \delta_k(m) \left[(1-\varepsilon_k(m))\alpha^j(f_k(\theta), \theta, \theta^j) + \varepsilon_k(m) x^j \right] + (1 - \delta_k(m)) f_k(\theta) \right\},$$

with

$$\delta_k(m) = \begin{cases} \delta & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in L_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin L_j(f_k(\theta), \theta) \end{cases}$$

for $1 > \delta > 0$, and

$$\varepsilon_k(m) = \begin{cases} \varepsilon & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \in SL_j(f_k(\theta), \theta) \\ 0 & \text{if } \alpha^j(f_k(\theta), \theta, \theta^j) \notin SL_j(f_k(\theta), \theta) \end{cases}.$$

That is, suppose all players but player j send the same message $(\theta, \alpha, x, 1)$ with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $k \in \{1, \ldots, K^\theta\}$. Let $m_j = (\theta^j, \alpha^j, x^j, \mathbf{z}^j)$ be the message sent by player j. If $\alpha^j(f_k(\theta), \theta, \theta^j)$ selects an alternative x in the strict lower-contour set $SL_j(f_k(\theta), \theta)$ of player j at state θ, then the designer replaces the outcome $f_k(\theta)$ from the uniform lottery with the lottery $\delta[(1-\varepsilon)x + \varepsilon x^j] + (1-\delta)f_k(\theta)$. If $\alpha^j(f_k(\theta), \theta, \theta^j)$ selects an alternative x in $L_j(f_k(\theta), \theta) \setminus SL_j(f_k(\theta), \theta)$ (i.e., player j is indifferent between x and

$f_k(\theta)$ at state θ), then the designer replaces the outcome $f_k(\theta)$ from the uniform lottery with the lottery $\delta x + (1-\delta) f_k(\theta)$. Otherwise, the designer does not replace the outcome $f_k(\theta)$ from the uniform lottery.

Rule 3: If neither rule 1 nor rule 2 applies, then $g\left((\theta^i, \alpha^i, x^i, \mathbf{z}^i)_{i\in N}\right) = x^{i^*}$, with i^* a player announcing the highest integer $\mathbf{z}^{i^*}$. (If more than one player i selects the highest integer, then g randomizes uniformly among their selected x^i).

Fix a state θ^*, and a cardinal representation $u_i \in \overline{\mathcal{U}}_i^{\theta^*}$ of $\succsim_i^{\theta^*}$ for each player i. Let u be the vector of cardinal representations.

We first show that for any $x \in f(\theta^*)$, there exists a Nash equilibrium σ^* of $G(\theta^*, u)$ such that x belongs to the support of $\mathbb{P}_{\sigma^*,g}$. Consider a profile of strategies σ^* such that $\sigma_i^* = (\theta^*, \alpha, x, 1)$ for all $i \in N$, so that rule 1 applies. The (pure strategy) profile σ^* is clearly a Nash equilibrium at state θ^*. By deviating, each player i can trigger rule 2, but none of these possible deviations are profitable. Any deviation can either induce a probability shift in the uniform lottery from $f_k(\theta^*)$ to a lottery with mass $(1-\varepsilon)$ on an alternative in $SL_i(f_k(\theta^*), \theta^*)$ and mass ε on x^j, or shift δ probability mass from $f_k(\theta^*)$ to an alternative indifferent to $f_k(\theta^*)$ (i.e., an alternative in $L_i(f_k(\theta^*), \theta^*) \setminus SL_i(f_k(\theta^*), \theta^*)$). By definition of ε, the former type of deviation is not profitable. Moreover, under σ^*, the support of $\mathbb{P}_{\sigma^*,g}$ is $f(\theta^*)$. Hence, for any $x \in f(\theta^*)$, there exists an equilibrium that implements x.

Conversely, we need to show that if σ^* is a mixed Nash equilibrium of $G(\theta^*, u)$, then the the support of $\mathbb{P}_{\sigma^*,g}$ is included in $f(\theta^*)$. Let m be a message profile and denote with $g^O(m)$ the set of alternatives that occur with strictly positive probability when m is played: $g^O(m) = \{x \in X : g(m)(x) > 0\}$. Let us partition the set of messages M into three subsets corresponding to the three allocation rules. First, let R_1 be the set of message profiles such that rule 1 applies, i.e., $R_1 = \{m : m_j = (\theta, \alpha, x, 1)$ for all $j \in N$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)\}$. Second, if all agents $j \neq i$ send some message $m_j = (\theta, \alpha, x, 1)$ with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, while agent i sends a different message $m_i = (\theta^i, \alpha^i, x^i, \mathbf{z}^i)$, then rule 2 applies and agent i is the only agent differentiating his message. Let R_2^i be the set of these message profiles. Let $R_2 = \cup_{i\in N} R_2^i$. Third, let R_3 be the set of message profiles such that rule 3 applies.

Consider an equilibrium σ^* of $G(\theta^*, u)$, and let M_i^* be the set of message profiles that occur with positive probability under σ_i^*. We need to show that $g^O(m^*) \subseteq f(\theta^*)$

for all $m^* \in M^* := \times_{i \in N} M_i^*$.

For any player $i \in N$, for all $m_i^* = (\theta^i, \alpha^i, x^i, \mathbf{z}^i) \in M_i^*$, define the (deviation) message $m_i^D(m_i^*) = (\theta^i, \alpha^D, x^D, \mathbf{z}^D)$, where: 1) α^D differs from α^i in at most the alternative associated with elements $(f_k(\theta), \theta, \theta^i)$ for all $\theta \in \Theta$, for all $k \in \{1, \ldots, K^\theta\}$, 2) $x^D \in \max_i^{\theta^*} X$, and 3) $\mathbf{z}^D > \mathbf{z}^i$ and for $1 > \mu \geq 0$, the integer $\mathbf{z}^D$ is chosen strictly larger than the integers $\mathbf{z}^j$ selected by all the other players $j \neq i$ in all messages $m_{-i}^* \in M_{-i}^*$, except possibly a set of message profiles $M_{-i}^\mu \subseteq M_{-i}^*$ having probability of being sent less than μ. (Note that μ can be chosen arbitrarily small, but not necessarily zero because other players may randomize over an infinite number of messages.) Consider the following deviation σ_i^D for player i from the equilibrium strategy σ_i^*:

$$\sigma_i^D(m_i) = \begin{cases} \sigma_i^*(m_i^*) & \text{if } m_i = m_i^D(m_i^*) \text{ for some } \ m_i^* \in M_i^* \\ 0 & \text{otherwise} \end{cases}.$$

First, note that under $(\sigma_i^D, \sigma_{-i}^*)$, the set of messages sent is a subset of $R_2^i \cup R_3$: either rule 2 applies and all players but player i send the same message or rule 3 applies. Second, whenever rule 3 applies, player i gets his preferred alternative at state θ^* with arbitrarily high probability $(1 - \mu)$. Third, suppose that under σ^*, there exists $m^* \in R_2^j$ with $j \neq i$. Under $(\sigma_i^D, \sigma_{-i}^*)$, with the same probability that m^* is played, $(m_i^D(m_i^*), m_{-i}^*) \in R_3$ is played (rule 3 applies) and with probability at most μ, the lottery $g((m_i^D(m_i^*), m_{-i}^*))$ under $(m_i^D(m_i^*), m_{-i}^*)$ might be less preferred by player i than the lottery $g(m^*)$. (With probability $1 - \mu$, $g((m_i^D(m_i^*), m_{-i}^*)) = \max_i^{\theta^*} X$.) Yet, since μ can be made arbitrarily small and utilities are bounded, the loss can be made arbitrarily small. Consequently, by setting $\alpha^D(f_k(\theta), \theta, \theta^i) \succcurlyeq_i^{\theta^*} \alpha^i(f_k(\theta), \theta, \theta^i)$ for all θ and all $k \in \{1, \ldots, K^\theta\}$, player i can guarantee himself a maximal loss of $\bar{u}$, arbitrarily small, in the event that $m^* \in \cup_{j \neq i} R_2^j$ under σ^*.

Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_1$; that is, for all $j \neq i$, $m_j^* = m_i^* = (\theta, \alpha, x, 1)$. In the event the message sent by all others is $m_j^* = m_i^*$, player i strictly gains from the deviation if $\alpha^D(f_k(\theta), \theta, \theta) \in L_i(f_k(\theta), \theta)$ and either (1) $\alpha^D(f_k(\theta), \theta, \theta) \succ_i^{\theta^*} f_k(\theta)$ or (2) $\alpha^D(f_k(\theta), \theta, \theta) \in SL_i(f_k(\theta), \theta)$, $\alpha^D(f_k(\theta), \theta, \theta) \succcurlyeq_i^{\theta^*} f_k(\theta)$ and $f_k(\theta) \notin \max_i^{\theta^*} X$. Since the expected gain in this event can be made greater than $\bar{u}$ by appropriately choosing μ, (1) and (2) cannot hold for any player i. It follows that for all i and all k, it must be that (1) $L_i(f_k(\theta), \theta) \subseteq L_i(f_k(\theta), \theta^*)$ and (2) either $SL_i(f_k(\theta), \theta) \subseteq SL_i(f_k(\theta), \theta^*)$

or $f_k(\theta) \in \max_i^{\theta^*} X$. Therefore, by the weak* set-monotonicity of f, we must have $f(\theta) \subseteq f(\theta^*)$. This shows that $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_1$.

Let us now suppose that there exists $(m_i^*, m_{-i}^*) \in R_2^i$; that is, for all $j \neq i$, $m_j^* = (\theta, \alpha, x, 1) \neq m_i^*$. In this case, any player $j \neq i$ strictly gains from the deviation σ_j^D whenever $\mathbf{z}^D$ is the largest integer, which occurs with probability of at least $1 - \mu$, unless $g^O(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$. Since μ can be made arbitrarily small, it must be $g^O(m_i^*, m_{-i}^*) \subseteq \max_j^{\theta^*} X$ for all $j \neq i$. Therefore, by no-veto power, it must be $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_2^i$.

It only remains to consider messages $(m_i^*, m_{-i}^*) \in R_3$. For such messages the argument is analogous to messages in R_2^i. For no player i to be able to profit from the deviation σ_i^D, it must be $g^O(m_i^*, m_{-i}^*) \subseteq \max_i^{\theta^*} X$ for all $i \in N$. Therefore, the condition of no-veto power implies $g^O(m_i^*, m_{-i}^*) \subseteq f(\theta^*)$ for all $(m_i^*, m_{-i}^*) \in R_3$. □

Some remarks are in order. First, our mechanism is a randomized mechanism. As we have already explained, we believe this is natural given that we consider the problem of implementation in mixed Nash equilibrium.

Second, our construction uses integer games. While we agree that integer games are not entirely satisfactory (see Jackson, 2001, for persuasive arguments), we are no different from the large literature on Nash implementation in having to resort to integer games in order to rule out unwanted (not f-optimal) mixed Nash equilibria.[9]

Third, Theorem 2 strongly relies on the condition of weak* set-monotonicity, a weakening of Maskin monotonicity, which is relatively easy to check in applications. We have not tried to look for necessary and sufficient conditions for mixed Nash implementation. We suspect that such a characterization will involve conditions that are hard to check in practice, as it is the case for Maskin's Nash implementation (e.g., condition μ of Moore and Repullo, 1990, condition M of Sjöström, 1991, condition β of Dutta and Sen, or strong monotonicity of Danilov, 1992). We do know, however, as the example below shows, that weak* set-monotonicity is not necessary for Nash implementation.

[9]Much of the literature on Nash implementation only requires that there are no unwanted pure Nash equilibria (the appendix in Maskin, 1999, is an exception). Then integer games can be replaced by modulo games. Modulo games, however, have (possibly unwanted) mixed equilibria.

Example 4 There are three players, 1, 2 and 3, three alternatives a, b and c, and two admissible profiles of preferences θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
a	b	b	$a \sim c$	$b \sim c$	b
c	c	a			a
b	a	c	b	a	c

The social choice correspondence is $f(\theta) = \{a, b, c\}$ and $f(\theta') = \{b, c\}$. It is not weak* set-monotonic, but it is implementable in mixed Nash equilibrium. To see that f is not weak* set-monotonic, note that $a \notin f(\theta')$ and yet: $L_i(x, \theta) \subseteq L_i(x, \theta')$ for all $x \in \{a, b, c\}$, for all $i \in \{1, 2, 3\}$; $SL_3(x, \theta) \subseteq SL_3(x, \theta')$ for all $x \in \{a, b, c\}$; $SL_1(c, \theta) \subseteq SL_1(c, \theta')$, $SL_1(b, \theta) \subseteq SL_1(b, \theta')$, and $SL_1(a, \theta) \nsubseteq SL_1(a, \theta')$, but $a \in \max_1^{\theta'} X$; $SL_2(a, \theta) \subseteq SL_2(a, \theta')$, $SL_2(c, \theta) \subseteq SL_2(c, \theta')$, and $SL_2(b, \theta) \nsubseteq SL_1(b, \theta')$, but $b \in \max_2^{\theta'} X$. To show that f is implementable in mixed Nash equilibrium, consider the mechanism in which players 1 and 2 have two messages each, m_1 and m_2, player 3 has no messages, and the allocation rule is represented below (player 1 is the row player):

	m_1	m_2
m_1	b	c
m_2	a	b

If the profile of preferences is θ', (m_1, m_2) is the unique pure Nash equilibrium of the game, with outcome c. There is also an equilibrium in which player 1 chooses m_1 and player 2 (appropriately) mixes over m_1 and m_2. There is no equilibrium in which both players mix. (Note that m_2 is weakly dominant for player 2 at state θ'.) On the other hand, it is clear that if the profile of preferences is θ, then there is an equilibrium in which both players totally mix between m_1 and m_2. Therefore, f is implementable in Nash equilibrium, although it is not weak* set-monotonic.

We now claim that the restriction to compact sets of cardinal representations and mixed Nash C-implementation in Theorem 2 can be relaxed (to mixed Nash implementation) if we strengthen the condition of weak* set-monotonicity to *strong set-monotonicity*. Theorem 3 formally states this result without proof.[10]

[10]The proof is obtained from the proof of Theorem 2 by setting $\varepsilon_k(m) = 0$ for all m.

Definition 4 *A social choice correspondence f is strong set-monotonic if for all pairs $(\theta, \theta') \in \Theta \times \Theta$, we have $f(\theta) \subseteq f(\theta')$ whenever for all $x \in f(\theta)$, for all $i \in N$, it is $L_i(x, \theta) \subseteq L_i(x, \theta')$.*

Note that on the domain of strict preferences, strong set-monotonicity coincides with weak and weak* set-monotonicity. Moreover, if a social choice correspondence is Maskin monotonic, then it is strong set-monotonic, and if it is strong set-monotonic, then it is weak-set monotonic.

Theorem 3 *Let $\langle N, X, \Theta \rangle$ be an environment with $n \geq 3$. If the social choice correspondence f is strong set-monotonic and satisfies no-veto power, then it is implementable in mixed Nash equilibrium.*

6 Dispensing with No-Veto Power

As pointed out by Benoît and Ok (2008) and Bochet (2007), the appeal of the no-veto power condition may be questioned in settings with a small number of agents. In the context of pure Nash implementation, and allowing for out-of-equilibrium randomness in the mechanism, they showed that no-veto power can be dispensed with, provided that some mild domain restrictions are imposed.[11] We now show that similar results can be obtained in the context of mixed Nash implementation.

Definition 5 (Bochet, 2007) *An environment $\langle N, X, \Theta \rangle$ satisfies* top-strict-difference *if for any $\theta \in \Theta$ and $x \in X$ such that $x \in \cap_{i \in I} \max_i^\theta X$ for $I \subseteq N$ with $|I| = n-1$, there exist $j, k \in N$ such that $\max_j^\theta X = \max_k^\theta X = \{x\}$.*

Top strict difference requires that if $n-1$ agents rank x at the top, then at least two agents must rank x strictly at the top.

[11] Bochet (2007) showed that with $n \geq 3$ Maskin monotonicity is sufficient for Nash implementation if preferences satisfy top-strict difference. Benoît and Ok (2008), again with $n \geq 3$, showed that Maskin monotonicity and weak unanimity of f are sufficient if preferences satisfy the top-coincidence condition. Both papers use out-of-equilibrium randomness in the mechanism, but limit themselves to rule out unwanted *pure* strategy equilibria.

Definition 6 (Benoît and Ok, 2008) *An environment* $\langle N, X, \Theta \rangle$ *satisfies the* top-coincidence *condition if for any* $\theta \in \Theta$ *and any* $I \subseteq N$ *with* $|I| = n - 1$*, the set* $\cap_{i \in I} \max_i^\theta X$ *is either empty or a singleton.*

Clearly, the top-coincidence condition is satisfied on the domain of single-top preferences. (Remember that on the domain of single-top preferences, each player has a single most preferred alternative at each state.)

Definition 7 *A social choice correspondence* f *is* weakly unanimous *if for all* $\theta \in \Theta$*, we have* $x \in f(\theta)$ *whenever* $\{x\} = \cap_{i \in N} \max_i^\theta X$*.*

As argued by Benoît and Ok (2008), the top-coincidence condition is a fairly mild domain restriction, while weak unanimity is a much weaker condition than no-veto power. Clearly, if f satisfies no-veto power, then it is weakly unanimous, but the converse does not hold.

Theorem 4 *Let* $\langle N, X, \Theta \rangle$ *be an environment with* $n \geq 3$*. If the social choice correspondence* f *is weak* set-monotonic and either* (*a*) *the environment satisfies the top-coincidence condition and* f *is weakly unanimous, or* (*b*) *the environment satisfies the top-strict-difference condition, then* f *is* C*-implementable in mixed Nash equilibrium.*[12]

Proof For part (a) we use the same mechanism as in the proof of Theorem 2. For part (b) we slightly modify rule 3, replacing it with the following.

Rule 3': Let i^* be a player announcing the highest integer $\mathbf{z}^{i^*}$. If neither rule 1 nor rule 2 applies, then $g\left((\theta^i, \alpha^i, x^i, \mathbf{z}^i)_{i \in N}\right)$ is the random lottery that assigns probability $\left(1 - \frac{1}{\mathbf{z}^{i^*}}\right)$ to x^{i^*} and probability $\frac{1}{\mathbf{z}^{i^*}}$ to the uniform lottery over all alternatives in X.

The proof of both parts is very similar to the proof of Theorem 2; only two changes are needed.

The first, more substantial, change is for the case of a message realization $m^* \in R_2^i$. As in the proof of Theorem 2, we may conclude that all the alternatives in the support of $\mathbb{P}_{m^*,g}$ must belong to $\max_j^{\theta^*} X$ for all $j \in N \setminus \{i\}$. (Recall that $m_j^* = (\theta, \alpha, x, 1)$ for all

[12]Theorem 4 is stated for weak* set-monotonic correspondences and C-implementation, but also holds for strong set-monotonic correspondences and implementation. Only a modification like the one needed to prove Theorem 3 is required.

$j \in N \setminus \{i\}$, with $\alpha(f_k(\theta), \theta, \theta) = f_k(\theta)$ for all $f_k(\theta) \in f(\theta)$, and $m_i^* = (\theta^i, \alpha^i, x^i, \mathbf{z}^i) \neq m_j^*$.)

(a) By the top-coincidence condition, the support of $\mathbb{P}_{m^*,g}$ must then consist of a single alternative x^*. Hence it must be $x^* = f(\theta)$. Player i may deviate and send the message $m_i^D = (\theta^i, \alpha^D, x^D, \mathbf{z}^i)$, with $x^D \in \max_j^{\theta^*} X$ and with α^D differing from α^i only in the component $\alpha^D(f_k(\theta), \theta, \theta^i) \in L_i(f_k(\theta), \theta)$. For such a deviation not to be profitable it must be: (i) $x^* = f_k(\theta) \succcurlyeq_i^{\theta^*} \alpha^D(f_k(\theta), \theta, \theta^i)$ and (ii) if $\alpha^D(f_k(\theta), \theta, \theta^i) \in SL_i(f_k(\theta), \theta)$ then $x^* \succ_i^{\theta^*} \alpha^D(f_k(\theta), \theta, \theta^i)$ or $x^* \in \max_i^{\theta^*} X$. We can conclude that: (1) $L_i(x^*, \theta) \subseteq L_i(x^*, \theta^*)$ and (2) either $SL_i(x^*, \theta) \subseteq SL_i(x^*, \theta^*)$ or $x^* \in \max_i^{\theta*} X$. Furthermore, since $x^* \in \max_j^{\theta^*} X$ for all $j \in N \setminus \{i\}$, (1) and (2) hold for all j. Consequently, $f(\theta) = x^* \subseteq f(\theta^*)$ by the weak* set-monotonicity of f.

(b) By the top-strict-difference condition, there must be at least a $j \in N \setminus \{i\}$ such that $\max_j^{\theta^*} X$ is a singleton. Hence the support of $\mathbb{P}_{m^*,g}$ must consist of a single alternative x^* and it must be $x^* = f(\theta)$. The rest of the proof of this case is as the proof of part (a).

The second, minor, change in the proof is for the case of a message realization m^* such that rule 3 (rule 3') applies. Let i^* be a player announcing the highest integer $\mathbf{z}^{i^*}$. Since no player must be able to profitably gain from a deviation, it must be the case that $x^{i^*} \in \max_i^{\theta^*} X$ for all $i \in N$. (a) It follows from the top-coincidence condition and weak unanimity that $x \in f(\theta^*)$. (b) It follows from the top-strict-difference condition that for at least one agent j, we have $\max_j^{\theta^*} X = \{x^{i^*}\}$, then, by rule 3', setting $\mathbf{z}^j > \mathbf{z}^{i^*}$ and $x^j = x^{i^*}$ is a profitable message deviation for agent j. $\square$

7 Applications

This section contains a series of remarks in which we provide applications of our results to some important social choice rules.

Remark 1 The strong Pareto correspondence is defined as follows: $f^{PO}(\theta) := \{x \in X :$ there is no $y \in X$ such that $x \in L_i(y, \theta)$ for all $i \in N$ and $x \in SL_i(y, \theta)$ for at least one $i \in N\}$. The correspondence f^{PO} is weak set-monotonic, while it fails to be Maskin monotonic. To see that f^{PO} is weak set-monotonic, consider two states θ and θ' such that for all $i \in N$, for all $x \in f^{PO}(\theta)$, (i) $L_i(x,\theta) \subseteq L_i(x,\theta')$ and (ii) $SL_i(x,\theta) \subseteq SL_i(x,\theta')$. Suppose that $x^* \in f^{PO}(\theta)$, but $x^* \notin f^{PO}(\theta')$. At state θ', there must then exists $y \in X$ such that $y \notin SL_i(x^*,\theta')$ for all $i \in N$ and $y \notin L_i(x^*,\theta')$ for at least one $i \in N$. It follows that $y \notin SL_i(x^*,\theta)$ for all $i \in N$ and $y \notin L_i(x^*,\theta)$ for at least one $i \in N$, a contradiction with $x^* \in f^{PO}(\theta)$. Consequently, $f^{PO}(\theta) \subseteq f^{PO}(\theta')$ and f^{PO} is weak set-monotonic. Therefore, on the domain of single-top preferences, f^{PO} satisfies weak* set-monotonicity.[13] Moreover, while it violates no-veto power, f^{PO} satisfies weak unanimity. Hence, if we restrict attention to the domain of single-top preferences, Theorem 4 applies and f^{PO} is C-implementable in mixed Nash equilibrium. To see that f^{PO} fails no-veto power, suppose that at θ all players but one are indifferent between a and b, while the remaining player strictly prefers a over b. Suppose also that a and b are strictly preferred by all players to all other alternatives. Then, no-veto power requires b to be f-optimal while $b \notin f^{PO}(\theta)$. To see that the strong Pareto correspondence is not Maskin monotonic on the domain of single-top preferences, consider the following example. There are three players, 1, 2 and 3, and two states of the world θ and θ'. Preferences are given in the table below.

θ			θ'		
1	2	3	1	2	3
d	d	b	d	d	b
b	a	a	b	$a \sim b$	a
c	b	c	c		c
a	c	d	a	c	d

The strong Pareto correspondence is: $f^{PO}(\theta) = \{a,b,d\}$ and $f^{PO}(\theta') = \{b,d\}$. Maskin monotonicity does not hold since $L_2(a,\theta) \subseteq L_2(a,\theta')$ and yet $a \notin f^{PO}(\theta')$.[14]

Remark 2 A coalitional game is a quadruple $\langle N, X, \theta, v\rangle$, where N is the set of players, X is the finite set of alternatives, θ is a profile of preference relations, and $v : 2^N \setminus \{\emptyset\} \to 2^X$. An alternative x is weakly blocked by the coalition $S \subseteq N \setminus \{\emptyset\}$ if there is a $y \in v(S)$

[13]Recall that on the domain of single-top preferences, $\max_i^\theta X$ is a singleton for each $i \in N$, for $\theta \in \Theta$.

[14]In the unrestricted domain of preferences f^{PO} is not weak* set-monotonic. To see this, suppose alternative d is not available in the example. The strong Pareto correspondence is then $f(\theta) = \{a,b\}$ and $f(\theta') = \{b\}$. Weak* set-monotonicity fails, since it is $L_2(a,\theta) \subseteq L_2(a,\theta')$, $a \in \max_2^{\theta'}\{a,b,c\}$ and yet $a \notin f^{PO}(\theta')$. In fact, following the reasoning in Example 2, we can see that f^{PO} is not C-implementable in mixed Nash equilibrium in this modified example.

such that $x \in L_i(y,\theta)$ for all $i \in S$ and $x \in SL_i(y,\theta)$ for at least one $i \in S$. If there is an alternative that is not weakly blocked by any coalition in $2^N \setminus \{\emptyset\}$, then $\langle N, X, \theta, v\rangle$ is a game with a non-empty strong core. A coalitional environment with non-empty strong core is a quadruple $\langle N, X, \Theta, v\rangle$, where Θ is a set of preference relations such that $\langle N, X, \theta, v\rangle$ has a non-empty strong core for all $\theta \in \Theta$. The strong core correspondence f^{SC} is defined for all coalitional environments with non-empty strong core as follows: $f^{SC}(\theta) := \{x \in v(N) : \ x \text{ is not weakly blocked by any } \emptyset \neq S \subseteq N\}$. Using arguments that parallel the ones used for the strong Pareto correspondence, it can be verified that on the unrestricted domain of preferences the strong core correspondence f^{SC} is weak set-monotonic. If we restrict attention to the domain of single-top preferences f^{SC} also satisfies weak* monotonicity, while it is not Maskin monotonic on either domain of preferences. Like f^{PO}, the strong core correspondence violates no-veto power. However, f^{SC} satisfies weak unanimity and hence, by Theorem 4, is C-implementable in mixed Nash equilibrium in the domain of single-top preferences.

Remark 3 On the unrestricted domain of preferences, a Maskin monotonic social choice *function* - that is a correspondence f such that $f(\theta)$ is a singleton for all θ - must be constant (Saijo, 1988). This needs not be the case for a weak set-monotonic social choice function. To see this, define a partition $\{\Theta_1, \ldots, \Theta_K\}$ of Θ with $\Theta_1 := \{\theta \in \Theta : x_1 \in \max_i^\theta X\}$ and $\Theta_k := \{\theta \in \Theta \setminus (\cup_{k'<k}\Theta_{k'}) : x_k \in \max_i^\theta X\}$ for all $k > 1$ and let $f(\theta) = x_k$ for all $\theta \in \Theta_k$, for all k. The social choice function f is a selection of player i's dictatorial social choice correspondence and is not constant. Yet, it is weak set-monotonic.[15]

Remark 4 On the domain of strict preferences, the top-cycle correspondence, an important voting rule, is weak set-monotonic, while it is not Maskin monotonic. At each state θ, the top-cycle correspondence selects the smallest subset $f(\theta)$ of alternatives such that no alternative outside the set is (strictly) preferred by a strict majority of players to any alternative inside the set. Formally, we say that alternative x defeats alternative

[15]To get some intuition, consider two states θ and θ' such that for all players but player i, preferences are the same at θ and θ', player i's preferences differ only between x_2 and x_1: $x_2 \in \max_i^\theta X$, $x_2 \succ_i^\theta x_1$, $x_1 \in \max_i^{\theta'} X$ and $x_1 \sim_i^{\theta'} x_2$. Let $f(\theta) = x_2$ and $f(\theta') = x_1$. Maskin monotonicity and $f(\theta) = x_2$ would imply that $x_2 \in f(\theta')$; hence f is not Maskin monotonic. However, weak set-monotonicity does not imply that $x_2 \in f(\theta')$ since the strict lower contour sets of player i are not nested; f does not violate weak-set monotonicity.

y at state θ, written $x \gg^{\theta} y$, if the number of players who prefer x to y is strictly greater than the number of players who prefer y to x. The top-cycle correspondence is defined as:

$$f^{TC}(\theta) := \cap\{X' \subseteq X : x' \in X', x \in X \setminus X' \text{ implies } x' \gg^{\theta} x\}.$$

Note that if at state θ the alternative x is a Condorcet winner, then $f^{TC}(\theta) = \{x\}$. To prove that the top-cycle correspondence is weak set-monotonic, assume that $x^* \in f^{TC}(\theta) \setminus f^{TC}(\theta')$ while $L_i(x, \theta) \subseteq L_i(x, \theta')$ for all $x \in f^{TC}(\theta)$, for all $i \in N$. (Recall that when preferences are strict $L_i(x, \theta) \setminus \{x\} = SL_i(x, \theta)$ and, therefore, strong set-monotonicity coincides with weak set-monotonicity and weak* set-monotonicity.) Clearly, if x^* is a Condorcet winner at θ and, hence, $f^{TC}(\theta) = \{x^*\}$, then x^* is also a Condorcet winner at θ', hence $x^* \in f^{TC}(\theta')$, a contradiction. So, assume that x^* is not a Condorcet winner. There must then exist another $z \in f^{TC}(\theta)$ such that it is not the case that $z \gg^{\theta} x^*$. Since $L_i(x^*, \theta) \subseteq L_i(x^*, \theta')$ and $L_i(z, \theta) \subseteq L_i(z, \theta')$ for all $i \in N$, the ranking between x^* and z has not changed when moving from θ to θ', hence it must be $z \notin f^{TC}(\theta')$. Iterating the argument (with $x^* = z$), it follows that $f^{TC}(\theta) \cap f^{TC}(\theta')$ must be empty. Finally, since for any $x \in f^{TC}(\theta)$, $x \gg^{\theta} y$ for all $y \in X \setminus f^{TC}(\theta)$, it must be that $x \gg^{\theta'} y$ for all $y \in X \setminus f^{TC}(\theta)$ as the lower contour sets are nested. Therefore, $f^{TC}(\theta')$ must be empty, a contradiction with the non-emptyness of the top-cycle set at each state. Since f^{TC} satisfies no-veto power, it follows that Theorem 3 applies: on the domain of strict preferences the top-cycle correspondence is implementable in mixed Nash equilibrium. To see that f^{TC} is not Maskin monotonic, consider the following example with two states, three alternatives and three players.

θ			θ'		
1	2	3	1	2	3
a	c	b	a	c	c
b	a	c	b	a	b
c	b	a	c	b	a

We have that $f^{TC}(\theta) = \{a, b, c\}$ and $f^{TC}(\theta') = \{c\}$. Since $L_i(a, \theta) \subseteq L_i(a, \theta')$ for all $i \in N$, Maskin monotonicity and $f^{TC}(\theta) = \{a, b, c\}$ would require $a \in f^{TC}(\theta')$.

Remark 5 Another voting rule, the uncovered-set correspondence, is weak set-monotonic but not Maskin monotonic on the domain of strict preferences. The uncovered-set correspondence is defined as $f^{US} = f^{TC} \cap f^{PO}$. It follows from the weak set-monotonicity of

f^{TC} and f^{PO} that f^{US} is weak set-monotone and, hence, by Theorem 3, implementable in mixed Nash equilibrium.

Remark 6 The Borda and the Kramer voting rules fail to satisfy weak set-monotonicity.[16] It is simple to see that the example in Maskin (1999, page 30) shows that the Borda rule fails to satisfy not only Maskin monotonicity, but also weak set-monotonicity. For the Kramer rule, consider the example in the table below with five players, three alternatives and two states θ and θ'.

θ					θ'				
1	2	3	4	5	1	2	3	4	5
a	a	a	a	c	a	a	a	a	b
b	b	c	c	b	b	b	b	b	c
c	c	b	b	a	c	c	c	c	a

The Kramer rule selects a at state θ and b at state θ', a violation of weak set-monotonicity. So, our results are not so permissive so as to imply that all "reasonable" social choice correspondences are implementable in mixed Nash equilibrium.

8 Conclusions

In this paper, we have introduced the concept of mixed Nash implementation. According to our definition, a mechanism implements a social choice correspondence f in mixed Nash equilibrium if the set of all pure and mixed Nash equilibrium outcomes corresponds to the set of f-optimal alternatives at each preference profile. We have shown that weak set-monotonicity, a weakening of Maskin's monotonicity, is necessary for implementation in mixed Nash equilibrium. Moreover, we have provided several sets of sufficient conditions for implementation, which involve either mild domain restrictions and weak unanimity or no-veto power. Importantly, we have shown that important social choice correspondences that are not Maskin monotonic, like the strong Pareto, the strong core, the top-cycle and the uncovered set, may be implemented in mixed Nash equilibrium. Several open problems are left for future research. A first issue is to provide necessary

[16] The Kramer score of alternative x at state θ is $\max_{y \neq x} |\{i \in N : x \succ_i^\theta y\}|$. The Kramer rule selects the alternatives with the highest Kramer score at each state.

and sufficient conditions for implementation in mixed Nash equilibrium. A second problem is to find sufficient conditions for the case of two players. A third problem is to provide sufficient conditions which do not rely on randomized mechanism. Yet another issue is to extend the analysis to incomplete information, and study implementation in mixed Bayesian equilibrium.

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