

# **A Nonparametric Analysis of the Forward Rate Volatilities**

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# A Nonparametric Analysis of the Forward Rate Volatilities\*

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## Abstract

Heath, Jarrow, and Morton (1992) present a general framework for modeling the term structure of interest rates which nests most other models as special cases. In their framework, the dynamics of the term structure and the prices of derivative instruments depend only upon the initial term structure and the forward rate volatility functions. Despite their importance, there has been little empirical work studying the forward rate volatility functions. This paper begins to fill this gap by estimating some nonparametric models of the forward rate volatilities. In a univariate model, the form of the forward rate volatility function differs for different maturities, and for some maturities appears not to be a monotonic function of the level of the forward rate. In a bivariate model, a measure of the “slope” of the term structure seems to have an important impact on the volatility. These results differ from the simple models that have been proposed and used in the literature.

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# 1 Introduction

The “arbitrage” or “preference-free” approach to modeling the term structure of interest rates was pioneered by Ho and Lee (1986).<sup>1</sup> In this approach the initial term structure is used as an input to the model, and a no-arbitrage condition in the bond market is used to derive a restriction on the dynamics of the term structure. Heath, Jarrow, and Morton (HJM) (1990, 1992) generalize and extend the Ho-Lee approach, and present a very general framework for modeling the term structure of interest rates. Starting with the initial term structure of forward rates and the forward rate volatility functions, HJM derive the drifts of the forward rate processes under the equivalent martingale measure or risk neutral probability, and show that the drifts depend only on the initial term structure and the forward rate volatilities. As a result, the dynamics of the term structure and the prices of interest rate derivative instruments are completely determined by the initial term structure and the forward rate volatility functions. In the HJM framework, other term structure models (e.g., the Ho-Lee (1986), Hull-White (1990), and Cox, Ingersoll, Ross (1985) models) can be obtained as special cases of the general framework by making the appropriate choices of the forward rate volatility functions. The specification of the volatility structure also has a significant impact on the pricing of interest rate derivatives (Ritchken and Sankarasubramanian (1995b)).

For these reasons, the volatility functions play a crucial role in implementations of the HJM model. They (together with the initial term structure) completely determine the dynamics of the term structure, the numerical methods that may be used to compute the prices of interest rate derivatives,<sup>2</sup> the actual prices computed, and the ability of the model to fit market prices. However, surprisingly, to date there has been relatively little empirical work on the forward rate

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<sup>1</sup> In this approach prices do depend on preferences, but only through the initial term structure and the specified volatility functions. This avoids the direct modeling of preferences.

<sup>2</sup> Lattice and partial differential equation methods may only be used if the term structure is Markovian with respect to some finite set of state variables. This condition holds only for certain special cases of the volatility functions.

volatility functions. Much of the work that has been done has focused on a limited set of specifications, and used “implied” parameter approaches which estimate the model parameters by choosing them so that the prices generated by the model match or come close to the actual prices of some subset of the existing traded options. Different studies reach different conclusions, due to differences in the data and methodologies used.

Flesaker (1993) tests the ability of the constant volatility model (i.e., the Ho-Lee model) to fit option prices, and finds that the model is not consistent with the data. Amin and Morton (1994) study the ability of the HJM model with six different volatility specifications to match the prices of Eurodollar futures options. Although some models perform better than the others, there are systematic biases for all models. Cohen and Heath (1992) compare the performance of several forward rate models in predicting future market prices, and find that the proportional model performs best. Abken and Cohen (1994) use option prices to test several HJM forward rate volatility models, and find strong support for the exponential proportional model. Bühler, Uhrig-Homburg, Walter, and Weber (1998) compare the performance of some forward rate models for valuing interest rate options using German market data. For their data, the linear proportional model seems to work best. Amin and Ng (1997) use GARCH models to infer future volatility from the information in the Eurodollar futures options, and find that the square root model outperforms the other models they consider. In addition, Bliss and Ritchken (1995) test a general implication of the two-state-dimensional Markov model proposed in Ritchken and Sankarasubramanian (1995a). They find support for the restriction on the dimensionality of the state space.

This literature indicates that there is no consensus about what specifications of the volatility functions are reasonable, though there is some evidence in favor of the proportional models.<sup>3</sup> To use a parametric model, a specific functional form must be assumed. But because

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<sup>3</sup> Some approaches used by practitioners to estimate the forward rate volatility functions are described in

we have relatively little knowledge of the features of the forward rate volatility, imposing any functional restriction is itself almost certainly a misspecification. As we are still at the stage of exploring the features of the forward rate volatility, a reasonable way to start is to use nonparametric methods to estimate the forward rate volatility in order to provide guidance about what choices of the parametric models are reasonable.

In this paper, we use the interest rates implied from the prices of Eurodollar futures contracts and 1-month LIBOR futures contracts to construct time series of (estimates of) the “instantaneous” forward rates for a range of maturities, and use nonparametric methods to estimate the forward rate volatility functions. Specifically, in the context of a one-factor HJM model we use the Nadarya-Watson kernel regression estimator to estimate the relation between the forward rate volatility and the level of the forward rate. The results indicate relatively little dependence of the forward rate volatilities on the levels of the forward rates. For the short-term forward rates, the volatility is generally increasing in the level of the forward rate, and in the middle of the range of forward rates (where most of the data lie) the volatility is a convex function of the level of the forward rate. However, for the longer term forward rates the volatility appears not to be a monotonic function of the level of the forward rate, in that the estimated volatility decreases for high levels of the forward rate. For both short and long term forward rates, the dependence of the volatility on the level is relatively weak. These results are inconsistent with both the simple power function specifications of volatility that have been popular in the empirical literature studying the dynamics of the “short” rate, and the proportional specifications popular with practitioners.

We also use the kernel regression estimator to estimate a bivariate specification in which the volatility depends on both the level of the forward rate and the spread between long and short term forward rates. These results suggest that the forward rate volatility is increasing in the

spread between long and short term forward rates. Controlling for the effect of the spread, the volatility appears not to be a monotonic function of the level of the forward rate, in that once again the estimated volatility decreases for high levels of the forward rate. However, in the bivariate model, the relation between the volatility of forward rates and their levels is weaker than it is in the univariate results. Interpreted literally, these results are inconsistent with all simple models of the volatility functions that have been used or proposed. Interpreted more skeptically, they indicate little or no relation between the forward rate volatility and the level of the forward rate.

Nonparametric methods have recently been used in estimating the drift and diffusion functions of the “short” rate, and both our approach and results are related to this literature. Stanton (1997) uses kernel regression to estimate the drift and diffusion (i.e., volatility) functions of the short interest rate, and obtains an estimated diffusion function that is similar in shape to the diffusion functions we estimate for relatively short term forward rates. Boudoukh, Richardson, Stanton, and Whitelaw (1998) use kernel regression to estimate a two-factor model of the dynamics of the short rate and a proxy for the yield on a 10-year bond, and find that the diffusion functions depend on the spread between the 10-year yield and the short rate. Ait-Sahalia (1996a) estimates the diffusion function of the interest rate process by assuming a linear drift. He finds that the diffusion function increases linearly with the interest rate when the level is low, it then becomes exponentially positively related to the level of the interest rate and finally goes down at high levels of the interest rate. Ait-Sahalia (1996b) uses a semi-nonparametric estimator to estimate and test flexible specifications for the drift and diffusion functions of the spot rate process, and concludes that the drift is nonlinear. His point estimates also suggest that the volatility is not a monotonic function of the level of the spot rate.

The balance of the paper is organized as follows. Section 2 briefly reviews the HJM model, while Section 3 introduces the multivariate kernel regression estimator. Section 4 describes the data and the construction of the “instantaneous” forward rates. Section 5 reports the

results of the univariate kernel estimation, and Section 6 develops a bivariate model to estimate the diffusion process of the forward rate and reports the empirical results of the bivariate estimation. Section 7 summarizes and concludes.

## 2 The Heath-Jarrow-Morton Framework

To fix notation and clarify the exposition, this section briefly summarizes the HJM framework in the case of one Brownian motion or “factor.” HJM model the evolution of the term structure by focusing on the “instantaneous” forward rate at time  $t$  for time  $T$ , denoted  $f(t, T)$ .

The instantaneous forward rate is defined through the relation

$$P(t, T) = \exp \left[ - \int_t^T f(t, v) dv \right],$$

or

$$f(t, T) = - \frac{\partial \ln P(t, T)}{\partial T},$$

where  $P(t, T)$  is the price at time  $t$  of a default-free zero coupon bond paying one unit of account at time  $T$ .<sup>4</sup> The instantaneous spot interest rate at time  $t$ , denoted  $r(t)$ , is given by

$$r(t) = f(t, t).$$

The underlying probability model is a filtered probability space  $(\Omega, \mathcal{F}, \mathbf{F}, P)$  and a time index set  $\mathbf{T} = [0, T]$ , where  $\mathcal{F}_T = \mathcal{F}$  and  $\mathbf{F} = \{ \mathcal{F}_t : 0 \leq t \leq T \}$  is the standard filtration for a one-dimensional Brownian motion. For each date  $T$ , the forward rate  $f(t, T)$  evolves according to the Itô process

$$df(t, T) = \mathbf{m}(\mathbf{w}, t, T) dt + \mathbf{S}(\mathbf{w}, t, T) dB(t), \quad (1)$$

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<sup>4</sup> Equivalently, the instantaneous forward rate is  $\lim_{e \downarrow 0} f(t, T, T + e)$ , where  $f(t, T, T + e) \equiv -(1/e) \ln(P(t, T + e)/P(t, T))$ .

where  $B(t)$  is a Brownian motion,  $\mathbf{m}$  and  $\mathbf{S}$  are the “drift” and “volatility” (or diffusion) functions, respectively, and  $\mathbf{w} \in \Omega$  indicates the possible dependence of the drift and volatility functions on the entire history of the process. HJM show that there is a unique equivalent martingale measure or risk-neutral probability  $Q$ , and that under  $Q$  the drift is given by

$$\mathbf{m}^Q(\mathbf{w}, t, T) = \mathbf{S}(\mathbf{w}, t, T) \int_t^T \mathbf{S}(\mathbf{w}, t, v) dv.$$

Thus, under  $Q$ , the forward rate  $f(t, T)$  evolves according to the process

$$df(t, T) = \mathbf{m}^Q(\mathbf{w}, t, T) dt + \mathbf{S}(\mathbf{w}, t, T) dB(t). \quad (2)$$

A key feature of (1) and (2) is that the forward rate volatility functions are identical, i.e. they are unaffected by the change of probability. Thus, we may base estimation upon (1), the evolution of the process under the original probability  $P$ , even though it is (2), the evolution of the process under  $Q$ , which matters for the purposes of pricing interest rate derivative instruments. This means that we may estimate the volatility functions using a time series of forward rates generated under the original probability  $P$ .

### 3. Estimation Approach

For estimation, we specialize (1) and consider forward rate dynamics of the form

$$df(t, t + \mathbf{t}) = \mathbf{m}(x(t), \mathbf{t}) dt + \mathbf{S}(x(t), \mathbf{t}) dB(t), \quad (3)$$

where  $x(t)$  is a vector of variables which (together with the maturity  $\mathbf{t} \equiv T - t$ ) determine the drift and volatility functions. In some of our analyses  $x(t)$  consists of only the forward rate  $f(t, t + \mathbf{t})$ , while in others it includes the spot rate  $r(t)$  or a “spread” between long and short-term forward rates. To estimate the process, we construct estimates (described below) of the instantaneous forward rates for various fixed maturities  $\mathbf{t} = 0.25, 0.5, 0.75, \dots, 3.75, \text{ and } 4.0$



years, and estimate a separate volatility function for each maturity. Thus, we estimate 16 different processes

$$df_t(t) = \mathbf{m}_t(x(t))dt + \mathbf{S}_t(x(t))dB(t),$$

though we do not report detailed results for all 16.

When  $x(t)$  consists of only the forward rate  $f(t, t + \mathbf{t})$ , equation (3) defines a Markov process. It is well known that in the HJM framework the drift under the martingale probability  $Q$  is “path dependent” except in certain special cases, which implies that the forward rate processes under  $Q$  are Markov only in certain special cases. However, (3) describes the process under the original probability  $P$ , and nothing in the HJM framework precludes the forward rates being Markov under  $P$ . In addition, Ait-Sahalia (1996c) tests, and fails to reject, the hypothesis that the slope of the term structure is a Markov process. Since the slope of the term structure is a forward rate, his results are consistent with the forward rate processes being Markov under  $P$ <sup>5</sup>. Most importantly, with the daily observation interval we use, misspecification of the drift function will have virtually no impact on our estimates of the forward rate volatility functions.

In the other specifications we use,  $x(t) = (f(t, t + \mathbf{t}), r(t))' = (f(t, t + \mathbf{t}), f(t, t))'$  or  $x(t) = (f(t, t + \mathbf{t}), f(t, t + \mathbf{t}_\ell) - f(t, t + \mathbf{t}_s))'$ , where  $\mathbf{t}_\ell$  is a “long” maturity,  $\mathbf{t}_s$  is a “short” maturity, and  $f(t, t + \mathbf{t}_\ell)$  and  $f(t, t + \mathbf{t}_s)$  are long and short maturity forward rates. In these cases, the process  $x$  is a Markov process. We assume that the transition function satisfies technical conditions sufficient for the process to be a diffusion process.<sup>6</sup> Thus, the estimation problem is to estimate the drift and “volatility” functions of either one or two-dimensional time-homogeneous processes of the form

$$dx(t) = \mathbf{m}(x(t))dt + \mathbf{S}(x(t))dB(t),$$

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<sup>5</sup> Of course, the failure to reject a hypothesis does not imply that it is true.

<sup>6</sup> See Arnold (1974), Section 2.5.

where we drop the index  $\mathbf{t}$  in order to simplify the notation. We assume that the drift and diffusion functions are such that the process is stationary.

### 3.1 The kernel regression estimator

The fact that  $x$  is a diffusion process implies that

$$E[x(t + \Delta) - x(t) | x(t)] = \mathbf{m}(x(t))\Delta + o(\Delta) \quad (4)$$

and

$$E[(x(t + \Delta) - x(t))^2 | x(t)] = \mathbf{s}^2(x(t))\Delta + o(\Delta), \quad (5)$$

where  $\Delta$  is a discrete (but arbitrarily small) time step in a sequence of observations of the process and  $o(\Delta)$  is the asymptotic order symbol used to denote a function  $\mathbf{z}$  such that

$\lim_{\Delta \downarrow 0} \mathbf{z}(\Delta)/\Delta = 0$ . Once estimates of the conditional mean and variance are obtained,

estimates of the drift and volatility or diffusion functions are obtained by “inverting” (4) and (5), yielding

$$\mathbf{m}(x(t)) = \frac{1}{\Delta} E[x(t + \Delta) - x(t) | x(t)] + \frac{o(\Delta)}{\Delta} \quad (6)$$

and

$$\mathbf{s}(x(t)) = \sqrt{\frac{1}{\Delta} E[(x(t + \Delta) - x(t))^2 | x(t)] + \frac{o(\Delta)}{\Delta}}. \quad (7)$$

This approach is justified by Banon (1978), and in the finance literature has recently been used by Stanton (1997).<sup>7</sup>

A kernel estimator of the stationary density of the process can be obtained as follows.

Let  $\{x^\Delta(t_i)\}_{i=1}^N$  be a sample of size  $N$  from the continuous time process  $x$ , observed at the discrete

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<sup>7</sup> Stanton (1997) also develops second- and third-order approximations of the drift and diffusion functions. Using daily data on the “short” rate, he finds that the three sets of estimators generate essentially the same results, indicating that the first-order estimators are reasonable choices. Although the conclusion is obtained based on the short rate data, the close relationship between the short rate and the forward rates suggest that the first order approximations will also work well with the forward rate data.

interval  $\Delta$ , where  $x^\Delta(t_i)$  is a vector in  $\mathfrak{R}^d$ . Furthermore, let  $z \in \mathfrak{R}^d$  be a point for which the estimate of the stationary density is desired, and let  $z_j$  denote the  $j$ th element of  $z$ . Letting  $\mathbf{p}(z)$  denote the stationary density, a kernel estimator is of the form

$$\hat{\mathbf{p}}(z) = \frac{1}{N \prod_{j=1}^d h_j} \sum_{i=1}^N \left\{ \prod_{j=1}^d K \left( \frac{z_j - x_j^\Delta(t_i)}{h_j} \right) \right\}, \quad (8)$$

where  $K$  is a kernel function satisfying the conditions

$$\begin{aligned} 0 &\leq K(u) \leq C < \infty, \\ K(u) &= K(-u), \\ \int_{\mathfrak{R}^d} K(u) du &= 1, \\ \int_{\mathfrak{R}^d} u^2 K(u) du &= 1, \\ \int_{\mathfrak{R}^d} u^n K(u) du &\leq \infty \quad \text{for } 0 \leq n < \infty, \end{aligned}$$

and  $d$  is the dimension of the process  $x$ , which in our applications is either one or two. The kernel function provides a method of weighting “nearby” observations so as to construct a smoothed histogram, which is the density estimator (8). As pointed out by Scott (1992), virtually all nonparametric estimators are asymptotically kernel methods. The Gaussian kernel

$$K(u) = \frac{1}{\sqrt{2\mathbf{p}}} \exp\left\{-\frac{1}{2} u^2\right\} \quad u \in (-\infty, \infty)$$

is convenient, and is the choice we use in the estimation below.<sup>8</sup> The parameter  $h_j$  in (8) is the bandwidth or smoothing parameter associated with the  $j$ th kernel. It determines the width of the kernel function around any point  $z$  and specifies the number of “neighboring points” of an

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<sup>8</sup> Commonly used kernel functions include the Epanechnikov kernel, the Triangle kernel, the Triweight kernel, the cosine arch kernel and the Gaussian kernel. The relative asymptotic efficiency of these kernels is very close (See Epanechnikov (1969) and Scott (1992) for details), so in most applications the choice of kernel function is not important. Both to make our study comparable to the current literature in finance and because it is convenient, we choose the Gaussian kernel.

observation (and their weights) that are to be considered in constructing the density estimator at the point  $z$ .

The Nadaraya-Watson kernel regression estimator we use is described in Silverman (1986) and Scott (1992). It can be obtained by using the standard relation between the conditional, joint, and marginal densities,

$$g(f(t+\Delta) - f(t) | x(t)) = \frac{g(f(t+\Delta) - f(t), x(t))}{g(x(t))}, \quad (9)$$

where  $g$  is used to denote the density functions. The conditional mean of interest is then obtained by performing the appropriate integration. We are interested in the conditional mean of the squared change  $(f(t+\Delta) - f(t))^2$ , i.e. we are interested in the conditional second moment  $E[(f(t+\Delta) - f(t))^2 | x(t) = x] = \mathbf{s}^2(x)\Delta$ . In this case, the kernel regression estimator becomes

$$\hat{\mathbf{s}}^2(x) = \frac{1}{\Delta} \frac{\sum_{i=1}^{N-1} (f(t+\Delta) - f(t))^2 \prod_{j=1}^d K\left(\frac{x_j - x_j^\Delta(t_i)}{h_j}\right)}{\sum_{i=1}^{N-1} \prod_{j=1}^d K\left(\frac{x_j - x_j^\Delta(t_i)}{h_j}\right)}.$$

Thus the kernel estimator for the diffusion is

$$\hat{\mathbf{S}}(x) = \sqrt{\frac{1}{\Delta} \frac{\sum_{i=1}^{N-1} (f(t+\Delta) - f(t))^2 \prod_{j=1}^d K\left(\frac{x_j - x_j^\Delta(t_i)}{h_j}\right)}{\sum_{i=1}^{N-1} \prod_{j=1}^d K\left(\frac{x_j - x_j^\Delta(t_i)}{h_j}\right)}} \quad (10)$$

In the univariate case when  $x(t)$  consists simply of the forward rate  $f(t)$ , this simplifies to

$$\hat{\mathbf{s}}(f) = \sqrt{\frac{1}{\Delta} \frac{\sum_{i=1}^{N-1} (f(t+\Delta) - f(t))^2 K\left(\frac{f - f^\Delta(t_i)}{h}\right)}{\sum_{i=1}^{N-1} K\left(\frac{f - f^\Delta(t_i)}{h}\right)}}. \quad (11)$$

### 3.2. Bandwidth

Härdle (1990) indicates that the optimal rate of convergence is of the order  $n^{-4/9}$ , when bandwidth is of the order  $n^{-1/5}$ . Various data-driven bandwidth choices can automatically achieve the optimal rate of convergence, without prior knowledge of the degree of differentiability of the estimator.

Previous studies also show that the performance of kernel estimators depends upon the choices of kernel and bandwidth  $(K, h)$ . But for practical problems, the bandwidth is of primary importance. Below we choose the bandwidth using biased cross-validation, a standard data-dependent choice.

### 3.3. Confidence Band

When the data exhibit time dependence, the traditional way of constructing confidence intervals is not consistent, and even the classical delete-1 jackknife or bootstrap method is not reliable. Given that our time series of forward rates displays strong time dependence, we use Moving Blocks Bootstrap (MBB) method suggested by Künsch (1989) and developed by Liu and Singh (1992).

Under the MBB scheme, some moving blocks  $\{B_1, B_2, \dots, B_{n-b+1}\}$  are constructed from the data dimension we choose. Each of the blocks contains  $b$  elements with the  $i$ th block containing data  $\{x^\Delta(t_i), x^\Delta(t_{i+1}), \dots, x^\Delta(t_{i+b-1})\}$ . We randomly draw a block (with replacement each time), delete this block from the data series, and calculate the estimate. This procedure is

repeated 5,000 times so that the 5,000 estimates are obtained to derive the estimated variance of the estimate. Then, the estimated variance of the (squared) forward rate volatility  $\hat{\mathbf{S}}^2(x)$  is

$$\hat{V}(\hat{\mathbf{S}}^2(x)) = \frac{b}{n(n-b+1)} \sum_{i=1}^{n-b+1} (J_i(x) - \hat{\mathbf{S}}^2(x))^2, \quad (12)$$

where

$$J_i = [n\hat{\mathbf{S}}^2(x) - (n-b)\hat{\mathbf{S}}_{-i}^2(x)]/b,$$

and  $\hat{\mathbf{S}}_{-i}^2(x)$  is the same estimate as in equation (11) except that data contained in block  $i$  are removed before the estimation is conducted. Confidence bands of the estimated forward rate volatility can then be constructed from the variance estimate.

Künsch (1989) indicates that to achieve the consistent result, the number of observations in a block should be larger than one for bootstrap method, but as the size of a block becomes larger, the bias of the estimator will increase. Following one of his examples for data with large sample size, we choose  $b = 4$ .

#### 4 Estimates of “Instantaneous” Forward Rates

We estimate the volatility functions using estimates of the “instantaneous” forward rates for a range of maturities. These estimates of the “instantaneous” forward rates are constructed from the daily settlement prices of the Eurodollar futures contracts (based on 3-month LIBOR, i.e., the London Interbank Offer Rate) and 1-month LIBOR futures contracts traded on the Chicago Mercantile Exchange. The Eurodollar futures contracts started trading in December 1981, when only 3 and 6-month contracts were available. Starting in 1990, the maturities of the Eurodollar contracts have extended out to at least 4 years, and currently extend out to 10 years. The 1-month Eurodollar futures contracts are also available from 1990. Some advantages of using Eurodollar futures data are discussed in Jegadeesh and Pennacchi (1996). Eurodollar and 1-month LIBOR futures contracts are very actively traded with a very

small bid-ask spread. Trading stops in all contracts at the same instant, at which time final settlement prices for all contracts are determined essentially simultaneously, eliminating concerns about the possible non-synchronicity of prices. In addition, because 3-month LIBOR is a common index for floating rate instruments such as interest rate swaps and floating rate notes, the Eurodollar contracts are widely used for hedging and arbitrage, linking the Eurodollar term structure to the term structure of swap rates. A further advantage pointed out by Amin and Morton (1994) is that Eurodollar and 1-month LIBOR futures contracts are cash-settled, which avoids some delivery and timing problems that are inherent in the Treasury bond and note futures contracts.

In the context of specific models, some previous studies (e.g., Grinblatt and Jegadeesh (1996)) determine the difference between the interest rate implied from the futures contracts and the actual implied forward rate, commonly known as the “convexity bias.” This bias is due primarily to the fact that the futures contracts settle gains and losses daily, while forward contracts are settled only at maturity. This, together with the asymmetric effect of the interest rate changes on bond prices, results in a gap between the interest rate implied from the futures contracts and the “true” implied forward rates. The former is usually a few basis points higher than the latter, and the difference increases with maturity. Burghardt and Hoskins (1994, 1995) document this relationship, and suggest an approximate procedure to adjust the implied futures interest rates that does not depend on any specific model.

To construct the estimates of the instantaneous forward rates, we start with the daily prices of the Eurodollar and 1-month LIBOR contracts from April 1990 to October 1998, compute the (continuously compounded) interest rates implied by the futures prices, and adjust the implied interest rates for the well known “convexity bias” to derive the forward rates with corresponding maturities. (see, e.g., Burghardt and Hoskins (1994, 1995)).

We then “chain” together the forward rates in order to build the term structure. Specifically, for day  $t$ , let  $t + s_1, t + s_2, \dots, t + s_\ell$  denote the last trading dates of the 1-month LIBOR contracts, and let  $t + \mathbf{t}_1, t + \mathbf{t}_2, \dots, t + \mathbf{t}_m$  denote the last trading dates of the Eurodollar contracts.<sup>9</sup> Starting from the last trading date  $t + s_1$  of the first 1-month LIBOR contract, we construct the forward rates  $f(t, t + s_1, t + s_2), f(t, t + s_1, t + s_3) \dots, f(t, t + s_1, t + \mathbf{t}_1)$ , where we stop using the 1-month LIBOR contracts at  $t + \mathbf{t}_1$ , the last trading date of the first Eurodollar contract.<sup>10</sup> From that date, we use the Eurodollar contracts to construct the forward rates  $f(t, t + s_1, t + \mathbf{t}_2), f(t, t + s_1, t + \mathbf{t}_3), \dots, f(t, t + s_1, t + \mathbf{t}_m)$ . The result of this process is a set of forward rates from  $t + s_1$  to  $t + s_2, \dots, t + \mathbf{t}_1, t + \mathbf{t}_2, \dots, t + \mathbf{t}_m$ . To these forward rates, we fit a cubic spline<sup>11</sup> to obtain the entire term structure from  $t + s_1$  to  $t + \mathbf{t}_m$ . Finally, we differentiated the spline function at the points  $t + 91.25$  days,  $t + 182.5$  days,  $\dots, t + 1460$  days, giving us 16 different series of daily instantaneous futures rates with maturities of 3 through 48 month, covering the time period from April 1990 to October 1998.<sup>12</sup>

Figure 1 shows the time series of the 6-month, 12-month, 24-month, and 48-month instantaneous forward rates, while Figure 2 shows the relationships between the daily changes and the levels of the time series. Some summary statistics of several selected time series of

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<sup>9</sup> The final settlement value of the Eurodollar and 1-month LIBOR contracts is based on either 3 or 1-month LIBOR quoted on the last trading date, for a deposit period beginning two business days after the last trading date. Thus, the forward rates we construct are actually for the times  $t + t_1 + 2$  business days,  $t + t_2 + 2$  business days, etc. The discussion in the text ignores this settlement convention in order to prevent the description from becoming needlessly complicated. However, the algorithms used to construct the forward rates incorporated the settlement conventions of the interbank market.

<sup>10</sup> This involves using at most three of the 1-month LIBOR futures contracts. Because both contracts stop trading two business days before the third Wednesday of the month, the last trading date of the first Eurodollar contract always coincides with the last trading date of one of the 1-month LIBOR contracts. One issue is that the maturity date of the deposit underlying a contract often does not coincide exactly with the last trading date of the next contract. In constructing the forward rates we assumed that it does. This has virtually no impact on the term structures we construct.

<sup>11</sup> We use the “natural” boundary condition that the second derivative of the spline function be zero at the endpoints.

<sup>12</sup> The shortest maturity was 91.25 days so that the forward rates we constructed would not be affected by



forward rates are provided in Table 1. We observe that while the mean increases with maturity, the standard deviation is a decreasing function of the maturity. The autocorrelation coefficients up to lag six of each series are all greater than 0.96, which shows that the data are highly serially dependent.

Figure 3 shows estimates of the stationary densities of the 6-month and 48-month forward rate series computed using the Gaussian kernel, with the bandwidth chosen using Biased Cross-validation. The two dashed curves are the 95% confidence interval of the marginal density obtained by Moving Blocks Bootstrap method. The 6-month density is more disperse than that of the 48-month and has thinner tails. This is consistent with the plots in Figure 2.

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the spline boundary condition.

## 5 Results of Univariate Estimation

### 5.1 Kernel regression estimates of the volatility functions

Panels A through D of Figure 4 report estimates of the diffusion function for the forward rates at maturities of 6, 12, 24, and 48 months. The estimates of the diffusion in Figure 4 ( $\hat{S}$ ) are obtained by carrying out the kernel estimation in equation (11) using the Gaussian kernel, with the bandwidth chosen by biased cross-validation.<sup>13</sup> Each of the panels shows the estimated diffusion function for the range covered by the data. The dotted lines show 95 percent confidence bands, constructed using the Moving Blocks Bootstrap method. We also performed the estimation for the other 12 series, so that estimates were obtained for maturities separated by 3 months, starting at  $t = 3$  months. The pattern of the estimates for the other maturities is consistent with the pattern seen in Figure 4, so the results for the other maturities are not reported. In Figure 4 we report results for the 6-month maturity rather than the 3-month maturity because of the possibility that our estimates of the 3-month “instantaneous” forward rate may be affected by the boundary condition used with the spline function used to construct the estimates.<sup>14</sup>

For the 6 and 12-month forward rates, the volatility function in general appears to be an increasing function of the forward rate, and generally appears to be convex, i.e., as forward rate becomes higher, the volatility increases at a greater rate. The convexity which is seen in the center of the graphs is broadly consistent with simple “power” volatility functions of the form  $S(f(t)) = af(t)^\xi$ , with  $\xi > 1$ , and thus is consistent with the volatility functions CKLS (1992) estimated for the “short” rate. However, markedly different volatility functions are obtained for the 24 and 48-month forward rates. Panels C and D each show a region near the middle of the

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<sup>13</sup> We have repeated the data analysis with the bandwidth choice of  $h = \widehat{SN}^{-1/5}$  which Stanton (1997) says he used, and obtained similar results. Monte Carlo evidence presented in Chapman and Pearson (1999) indicates that the choice of  $h = \widehat{SN}^{-1/5}$  works well in estimating the diffusion function, though they study larger sample sizes than that used in this paper.

<sup>14</sup> We also estimated the diffusion function for the 3-month “cash” Eurodollar rate (i.e., 3-month LIBOR) during the same period of time. The estimated volatility function is similar to that of the 6-month forward

graph in which the volatility is decreasing in the level of the forward rate. In both panels the volatility function changes from being a convex function of the forward rate, to a concave function, and back to a convex function.

These results are striking because they are inconsistent with *all* simple models of the forward rate volatility functions which have been suggested in the literature. While sampling variation cannot be ruled out as the explanation for the results, in both panels C and D there are points where the upper confidence band dips below a previously attained value of the lower confidence band, which suggests that there is a range in which the volatility function is decreasing. Moreover, the region in which the volatility appears to be decreasing in the forward rate is near the center of the distribution of forward rates, where there is a great deal of data. Thus, these results are not being driven by a relatively small number of data points.

Different results would be obtained with a different bandwidth choice, and the choice of a sufficiently large bandwidth would “smooth out” the bumps in Panels C and D, and make the volatility function almost flat. Such a bandwidth choice would be one mechanism of imposing a prior belief that the volatility function cannot be decreasing in the level of the forward rate. However, the relatively narrow bandwidth choice resulting from biased cross-validation gives a clearer picture of what the data say. In addition, the results in Chapman and Pearson (1999) suggest that relatively narrow bandwidth choices are reasonable in estimating the diffusion function. One possible explanation of the surprising results in Panels C and D is that the forward rate level is not the best descriptive factor, or it would work better with some other variables. We consider this possibility further in Section 6, where we allow the volatility function to depend on the “spread” between a long-term and a short-term forward rate.

## **5.2 Boundary Constraint**

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rate.

The method for estimating the forward rate volatility function described above does not guarantee that the volatility of the forward rate goes to zero as the forward rate approaches zero. Thus, it can result in estimates in which the forward rate volatility is positive when the forward rate is zero, which implies that the forward rate can be negative with positive probability. To avoid this problem, we use the approach of Stanton (1997) to force the diffusion function to be zero when the forward rate is zero. The results using the constrained estimator are provided in Panels A through D of Figure 5.

The estimated volatility functions of the 6 and 12-month forward rates are little affected by the imposition of the constraint. Comparing Panels A and B of Figure 5 to the corresponding panels in Figure 4, the shape of the volatility functions is unchanged, though of course the constrained volatility functions now start at the origin (not shown). In addition, because of the constraint the confidence bands are much narrower near the origin, and the estimated volatility function appears slightly smoother.

The impact of the constraint on the estimated volatility functions of the 24 and 48-month forward rates shown in Panels C and D is somewhat greater, though the qualitative character of the estimated functions remains unchanged. In both panels it remains true that the volatility function decreases for sufficiently high levels of the forward rate.

## **6 Bivariate Nonparametric Estimation**

### **6.1 The kernel regression estimator**

So far we have assumed that the forward rate volatility is a function of the maturity of the forward rate and its level. In this section we look for the other variables which would affect the forward rate volatility. One possible candidate is the forward spread, the difference between a long-term forward rate and a short-term forward rate. The yield spread has been used as a measure of the shape of the term structure and a reasonable forecast of the interest rate changes for decades. For example, Macaulay (1938) first noted that yield spreads tend to predict the

movement of long interest rates. Fama (1984), Campbell and Shiller (1991), and others analyze the predicting power of the yield spread for predicting the movement of short and long interest rates. The inclusion of the spread as a second variable is also suggested by other work (e.g., two-factor term structure models) in which the spread between long and short-term interest rates is important. If the volatility is actually influenced by the spread, and the spread is correlated with the levels of forward rates (as it must be), then apparently anomalous relations between the volatility and the level of the forward rate are possible. In addition, the results of Boudoukh, Richardson, Stanton, and Whitelaw (1998) suggest that the spread is a determinant of interest rate volatility. Here we examine effect of the forward rate spread on the forward rate volatility.

We perform bivariate kernel regressions incorporating the effect of the forward spread and that of the forward rate level. Specifically, for each of our 16 forward rate series we consider a time-homogeneous process of the form

$$df(t) = \mathbf{m}(f(t), s(t))dt + \mathbf{S}(f(t), s(t))dB(t).$$

When the regressors consist of the level of the forward rate and the spread, the estimator (11) specializes to

$$\hat{\mathbf{S}}^2(x_1, x_2) = \frac{\sum_{i=1}^{N-1} (f(t+\Delta) - f(t))^2 K\left(\frac{x_1 - f^\Delta(t_i)}{h_1}\right) K\left(\frac{x_2 - s^\Delta(t_i)}{h_2}\right)}{\sum_{i=1}^{N-1} K\left(\frac{x_1 - s^\Delta(t_i)}{h_1}\right) K\left(\frac{x_2 - s^\Delta(t_i)}{h_2}\right)} \quad (13)$$

where  $\{s^\Delta(t_i)\}_{i=1}^N = \{f^\Delta(t_i, t_i + \mathbf{t}_\ell) - f^\Delta(t_i, t_i + \mathbf{t}_s)\}_{i=1}^N$  is the sample of observations on the

spread. As indicated above, we use the Gaussian kernel  $K(u) = \frac{1}{\sqrt{2\mathbf{p}}} \exp\left(-\frac{1}{2}u^2\right)$ .

## 6.2 Description of the data

The four panels in Figure 6 consist of scatterplots showing the relationship between the 6-month, 12-month, 24-month and 48-month forward rates and the rate spread, respectively. We use the difference between 48-month and 6-month forward rates as our measure of the forward rate spread, which gives the largest possible spread among the forward rate series we use. Unsurprisingly, for the shorter maturity forward rates the spread is inversely related to the forward rate, and covers a relatively wide range. However, for the longer maturities the forward rates cover a relatively narrow range, and there appears to be little relationship between the spread and the level of the forward rate.

Panels A and B of Figure 7 plot the estimated joint density functions of the stationary distribution of the 6-month rate and the spread (again defined as the difference between the 48-month and 6-month forward rates), and the 48-month rate and the spread, respectively. In both cases, the density functions were estimated using the kernel estimator in equation (8). The density surface in Panel A has a simple structure, and reveals the negative correlation between the level of the 6-month forward rate and the spread. The density surface in Panel B is multimodal and much more irregular. Consistent with the scatterplot in Panel D of Figure 6, it does not reveal any obvious correlation between the 48-month forward rate and the spread.

### **6.3 Bivariate kernel regression results**

Panels A through D of Figure 8 show the estimated volatility functions of the 6, 12, 24, and 48 month forward rates as function of the levels of the forward rates and the forward rate spread. Similar to the confidence bands we constructed for the univariate estimates, confidence surfaces could be constructed for these bivariate regression estimates. However, these surfaces would overlap with the kernel regression surface and make it difficult to visualize and examine the significance of the bivariate results. One way to get around this is to first construct the regression function and confidence surfaces for the estimated bivariate volatility functions, and then display cross-sections of the three surfaces. The cross-section for a fixed level of the

forward rate will show the relation between the volatility and the spread, while the cross-section for a fixed level of the spread will show the relation between the volatility and the level of the forward rate. Panels A through D of Figures 9 and 10 show these cross-sections.

Panel A of Figure 8 plots the volatility of the 6-month forward rate as a function of the level of the forward rate and the forward rate spread, while Panel B shows the volatility of the 12-month forward rate as a function of its level and the forward rate spread. These graphs suggest that that the volatility of the 6-month forward rates is related to the spread, but that the 12-month volatility is at best only weakly related to the spread. In particular, for moderate levels of the forward rate, the 6-month volatility appears to be first sharply increasing, and then somewhat decreasing, in the level of the spread. In contrast to the univariate results in Figures 4 and 5, there does not appear to be any simple relation between the volatility of these forward rates and their levels.

This last statement may seem surprising, given the apparent relation between the volatility and the level of the forward rates for high values of the spread (along the “back” edge of the graphs in Panels A and B of Figure 8). However, one cannot reliably conclude anything from this apparent relationship, because Panels A and B of Figure 6 reveal that for the 6 and 12-month forward rates there are no realizations in which there are both high forward rates and high spreads. Thus, the apparent low volatility when both the forward rate and spread are high is not based on data in that region, but is only “extrapolated” from the other data points through the choice of the bandwidth. Examining Panels A and B of Figure 6, one can see that for moderate forward rates there have been a wide range of spreads, but that for high forward rates there have been only relatively low spreads, and for low forward rates there have been moderate to high spreads. Also, for low spreads there have been moderate to high forward rates.

Panels A and B of Figure 9 show the relations between the volatility and the forward rate spread, with the level of the forward rate fixed at its sample mean. The middle line in each of these graphs is a cross-section of the corresponding panel of Figure 8, while the upper and lower

lines are cross-sections of the confidence surfaces. These cross-sections indicate the forward rate volatility is increasing in the level of the spread. Panels A and B of Figure 10 show the relations between the volatility and the level of the forward rate, holding the spread fixed at its sample mean. Again, the middle line in each of these graphs is a cross-section of the corresponding panel of Figure 8, while the upper and lower lines are cross-sections of the confidence surfaces. These cross-sections reveal very little dependence of the volatility on the level of the spread. Comparing them to the corresponding panels of Figure 4, one can see that the estimated relation between the volatility and the level of the forward rate is weaker after controlling for the effect of the spread.

Panel C of Figure 8 plots the volatility of the 24-month forward rate as a function of the level of the forward rate and the forward rate spread, while Panel D shows the volatility of the 48-month forward rate as a function of its level and the forward rate spread. These graphs also display the feature that for moderate levels of the forward rate, the volatility appears to be first sharply increasing, and then decreasing, in the level of the spread. In fact, this phenomenon is more pronounced for the 24 and 48-month forward rates than it was for the 6 and 12-month forward rates. The maximum volatility occurs at a forward rate spread of about 2 percent. Also similar to the results for the shorter maturities, these graphs show no clear relation between the volatility and the level of the forward rate.

Panels C and D of Figures 9 and 10 show the cross-sections, holding fixed the level of the forward rate and the spread, respectively. As before, the middle line in each graph is a cross-section of the corresponding panel of Figure 8 with the other variable fixed at its sample mean, and the upper and lower lines are cross-sections of the confidence bands.

One difference between these two graphs and Panels A and B is that for low levels of the forward rate and moderate to high levels of the spread the volatility is increasing in the level of the spread, while in Panels A and B the relation is less clear. In addition, in Panels C and D the volatility is highest for moderate levels of the forward rate and the spread, not when the forward



rate is high and the spread is low. This, combined with the fact that most of the data lie near the diagonal running from the “back” corner to the “front” corner, explains why we obtain the “hump” for moderate levels of the forward rate in the univariate results shown Panels C and D of Figures 4 and 5. In essence, these figures showing the results of the univariate estimation are showing the value of the volatility surface on the diagonal running from high forward rates and low spreads to low forward rates and high spreads (from the “front” to the “back” of the graphs shown in Figure 8).

Overall, the results shown in all four panels A through D are broadly consistent. In all of them, there is a clear relation between the volatility and the spread for moderate levels of the forward rate, with the maximum volatility occurs at a forward rate spread of about 2 percent. Also, the volatility is increasing in the level of the forward rate for low levels of the spread and moderate to high forward rates.

## 7 Conclusion

Heath, Jarrow, and Morton (1992) present a very general framework for modeling the term structure of interest rates and pricing interest rate derivative instruments. Despite the central role played by the forward rate volatility functions in the HJM framework, there has been relatively little work that directly estimates the stochastic process of the forward rate in a general framework. This paper begins to fill this gap by conducting a nonparametric analysis of the forward rate process volatilities.

Univariate kernel regression results indicate that for the short term forward rates the forward rate volatility is an increasing function of the forward rate. For these forward rates, the volatility is generally increasing in the level of the forward rate, and in the middle of the range of forward rates (where most of the data lie) the volatility is a convex function of the level of the forward rate. However, for the longer term forward rates the volatility appears not to be a monotonic function of the level of the forward rate, in that the estimated volatility decreases for high levels of the forward rate. For both short and long term forward rates, the dependence of the volatility on the level is relatively weak. These results are inconsistent with both the simple power function specifications of volatility that have been popular in the empirical literature studying the dynamics of the “short” rate, and the proportional specifications popular with practitioners.

The constrained estimation procedure of Stanton (1997) generates results consistent with the unconstrained estimates. The results for the univariate models suggest that no single simple specification can capture the volatility function for all maturities, and thus that it may be necessary to develop different diffusion or volatility models for forward rates with different maturities. Also, the estimated diffusion functions have some features that have not been captured by any of the previous studies. These results suggest that a more complicated parametric model may be needed to better explain the dynamics of the forward rate.

In the bivariate model, we estimate the relation between the volatility, the forward rate, and the forward spread. These results show that for low and moderate levels of the forward rate, the volatility is increasing in the spread. For high forward rates, the volatility no longer increases in the forward rate spread.

The results in this paper are strikingly different from the simple specifications of the volatility functions that have been proposed. Interpreted literally, they are inconsistent with all simple models of the volatility functions that have been used or proposed. A more skeptical interpretation would recognize that the results are based on a somewhat arbitrary choice of bandwidth, and that the finite sample properties of the standard error bands we compute are not known. Moreover, it might be hard to accept that the volatility of long term forward rates could actually decrease for high levels of the forward rate; to our knowledge, such a model has never been proposed. Interpreted in this very skeptical fashion, the results indicate that there is no important dependence of the forward rate volatilities on the levels of forward rates. Only the Gaussian HJM models and their special cases such as the Hull-White, Vasicek, and Ho-Lee models are even broadly consistent with these results.

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**Table 1**  
Summary Statistics

A. Means, standard deviations, and autocorrelations of the forward rates

Maturity	Mean	Standard Deviation	AR1	AR3	AR6
3 months	0.054	0.0137	.998	.994	.988
6 months	0.055	0.0131	.998	.992	.985
12 months	0.059	0.0126	.996	.989	.978
24 months	0.066	0.0115	.996	.988	.977
36 months	0.070	0.0111	.997	.989	.979
48 months	0.073	0.0110	.997	.991	.982

B. Means, standard deviations, and autocorrelations of the forward rate changes

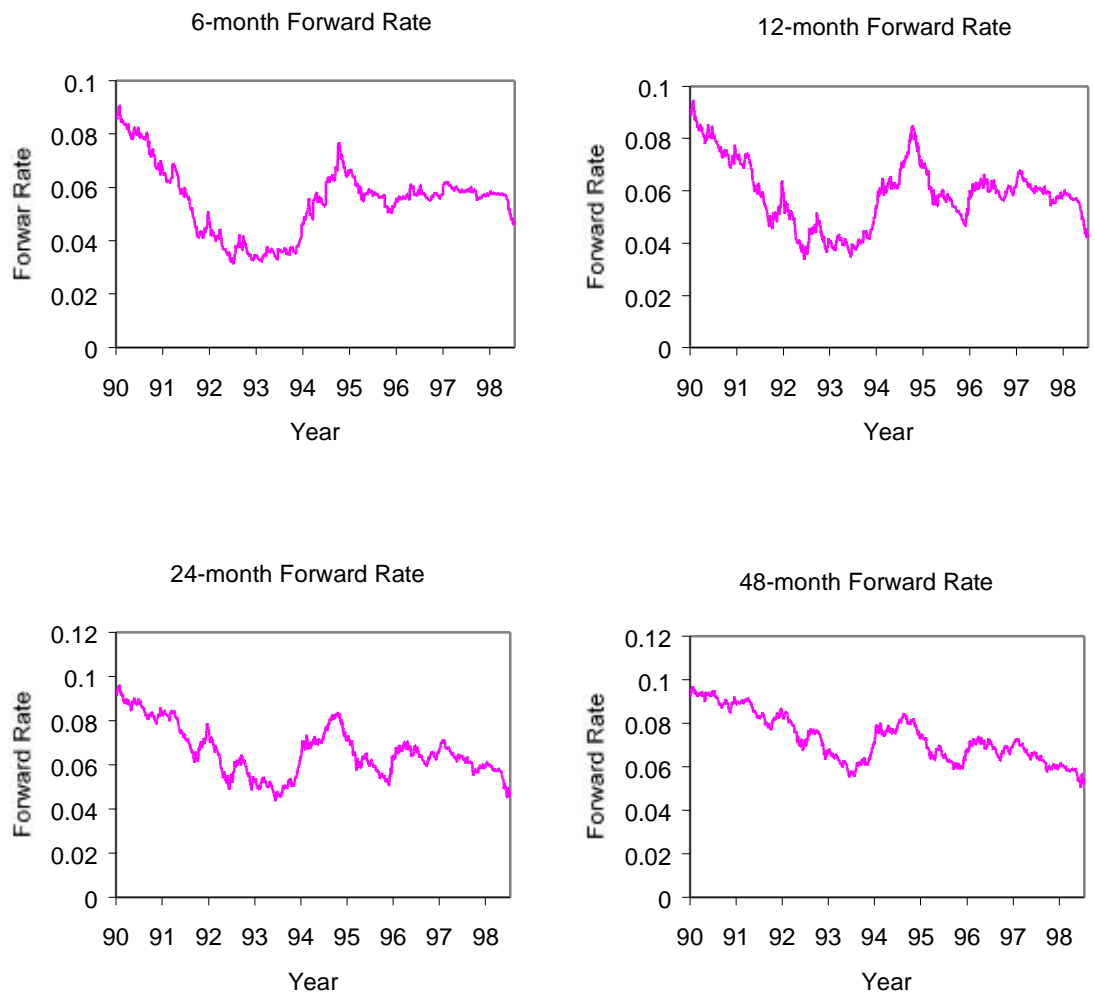
Maturity	Mean	Standard Deviation	AR1	AR3	AR6
3 months	-.00002	.00073	.103	.028	.018
6 months	-.00002	.00070	.080	-.007	-.006
12 months	-.00002	.00081	.094	-.002	-.017
24 months	-.00002	.00073	.124	-.002	-.003
36 months	-.00002	.00064	.121	-.025	-.046
48 months	-.00002	.00061	.086	-.013	-.044

C. Correlation coefficients of the forward rates

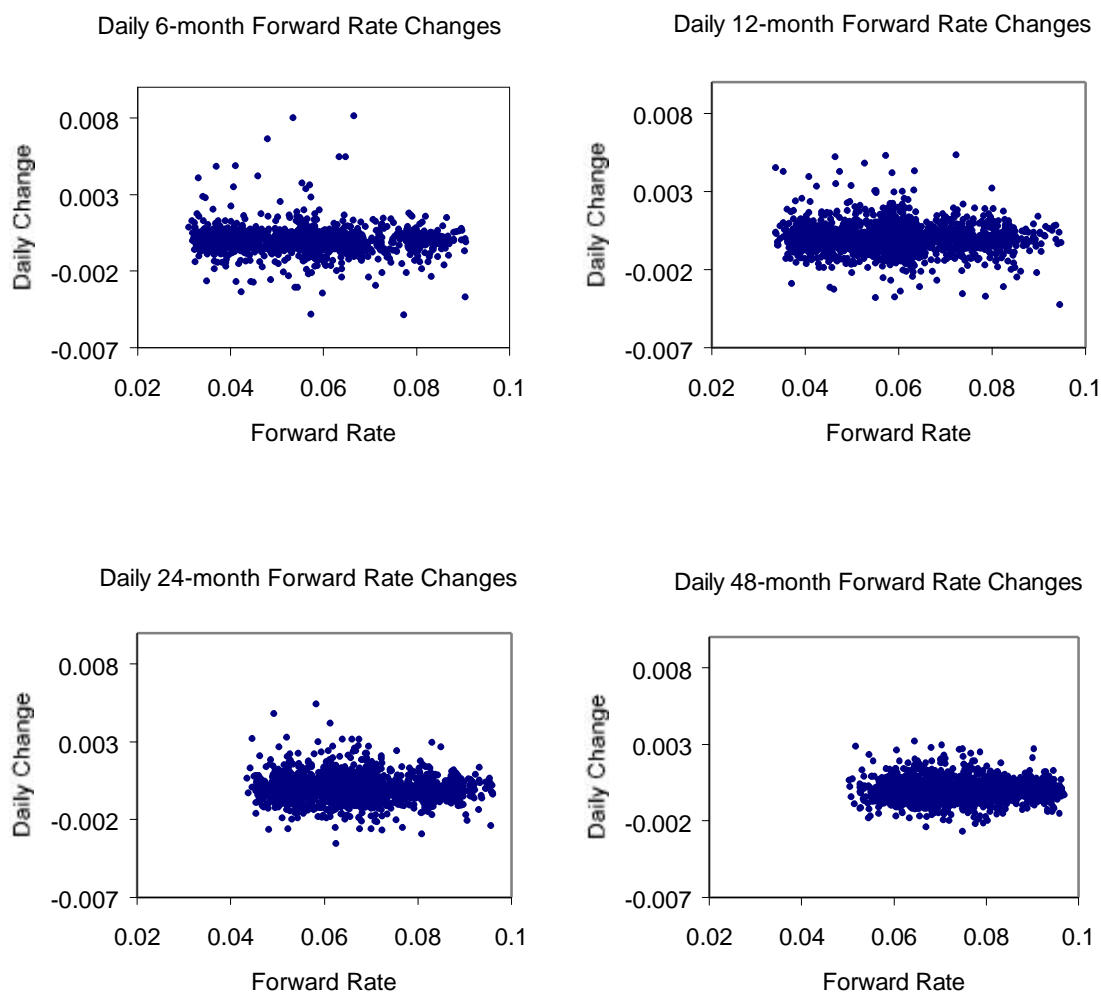
Maturity	6 months	12 months	24 months	36 months	48 months
3 months	.98	.91	.76	.61	.49
6 months		.97	.83	.67	.55
12 months			.92	.79	.68
24 months				.97	.91
36 months					.98

Means, standard deviations, and autocorrelation coefficients up to lag 6 of the daily Eurodollar instantaneous forward rates and their daily changes with selected maturities are computed. The correlation coefficients of several selected forward rate series are also provided. The forward rates are from April 1990 to October 1998. There are 2143 observations in each series.

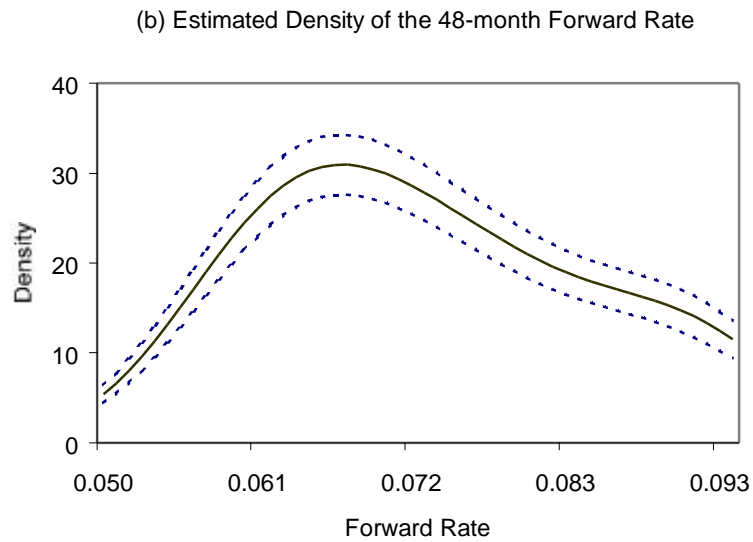
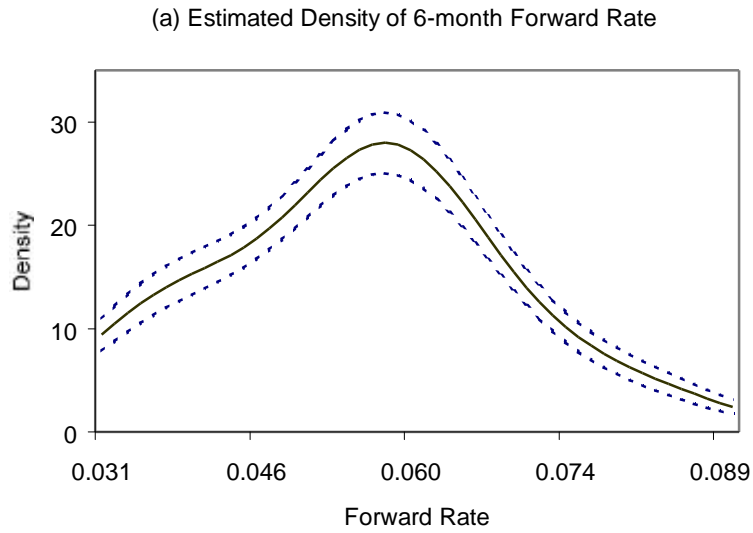




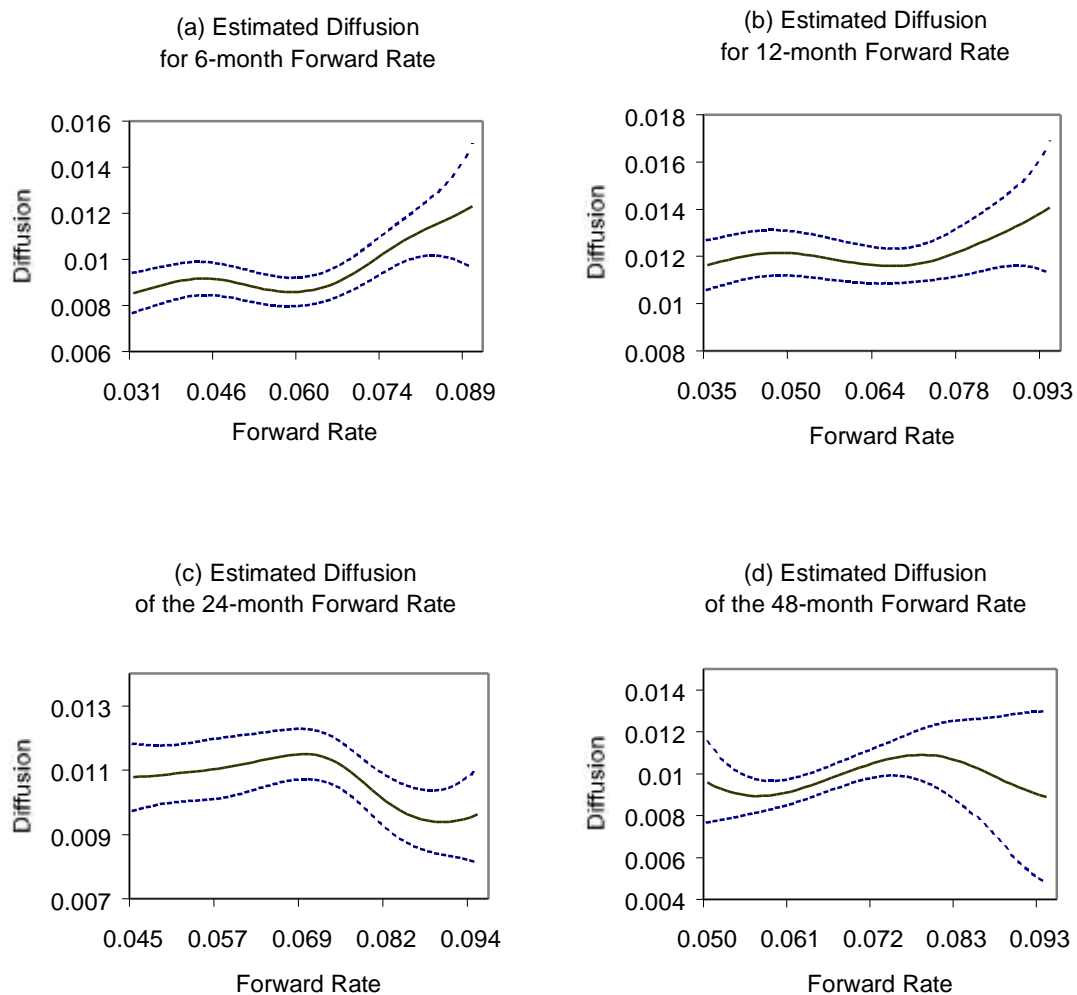
**Figure 1.** The time series of the 6-month, 12-month, 24-month and 48-month instantaneous forward rates. The instantaneous forward rates are derived from the daily Eurodollar futures prices from April 1990 to October 1998. There are 2143 observations.



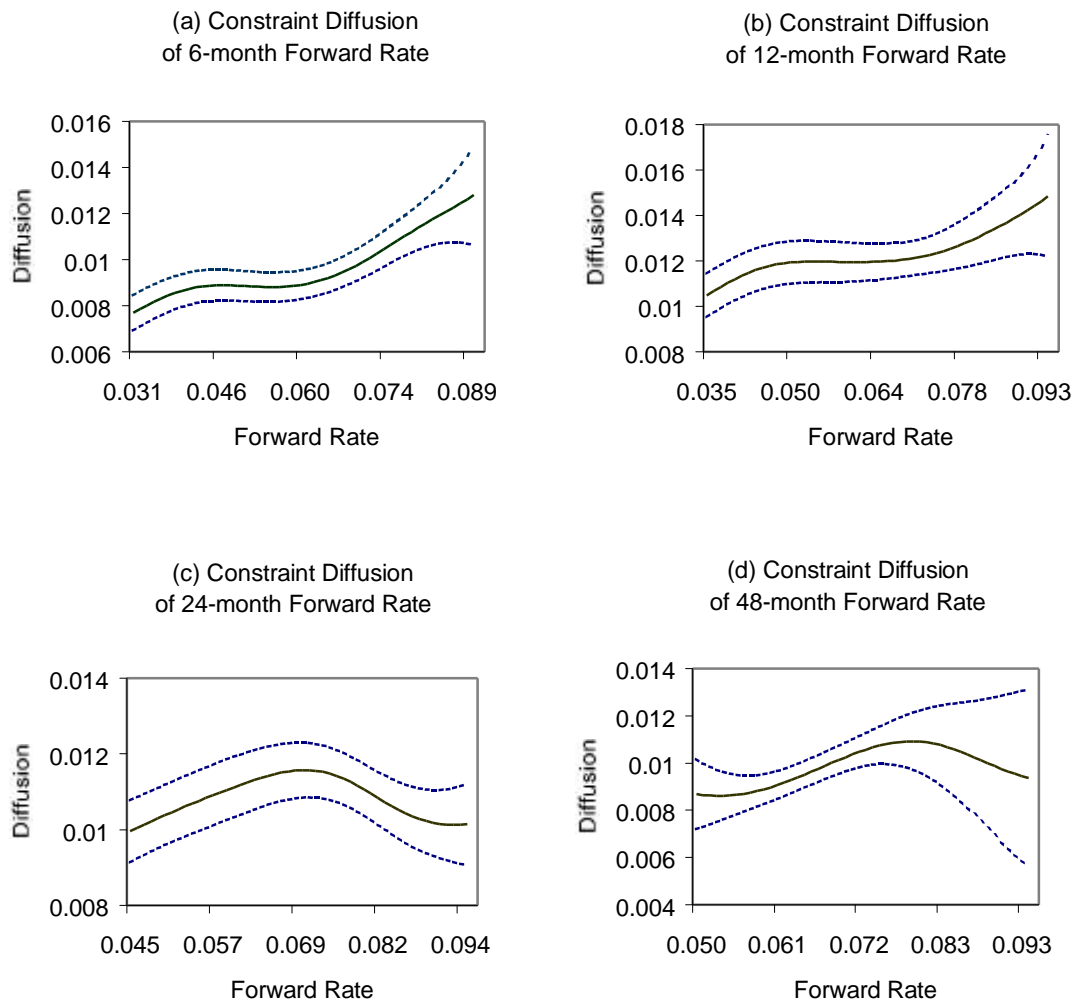
**Figure 2.** The relationship between the daily forward rate changes and the level of forward rate for the 6-month, 12-month, 24-month and 48-month forward rates respectively. These instantaneous forward rates are daily data from April 1990 to October 1998. There are 2143 observations.



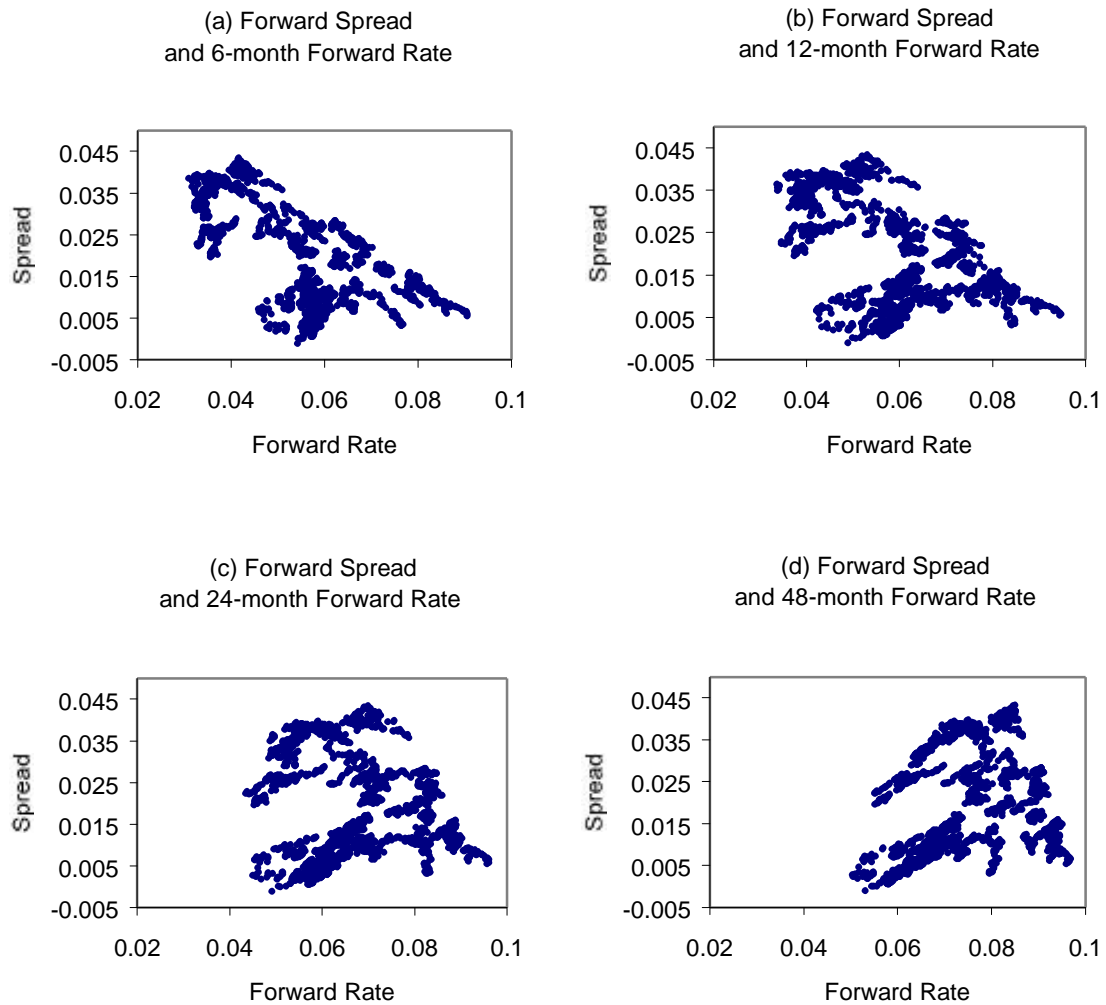
**Figure 3. Estimated kernel density.** The solid curves are the estimated kernel marginal density functions of the 6-month and 48-month forward rates, obtained by using daily data from April 1990 to October 1998. The dotted curves are the 95% confidence bands obtained by using the Moving Block Bootstrap method.



**Figure 4. Estimated naïve kernel diffusion.** The solid curves are the estimated diffusion functions of the 6-month, 12-month, 24-month and 48-month forward rates, obtained by using daily data from April 1990 to October 1998. The naïve kernel regression estimator is used without any restriction being imposed. The dotted curves are the 95% confidence bands obtained by using the Moving Block Bootstrap method.

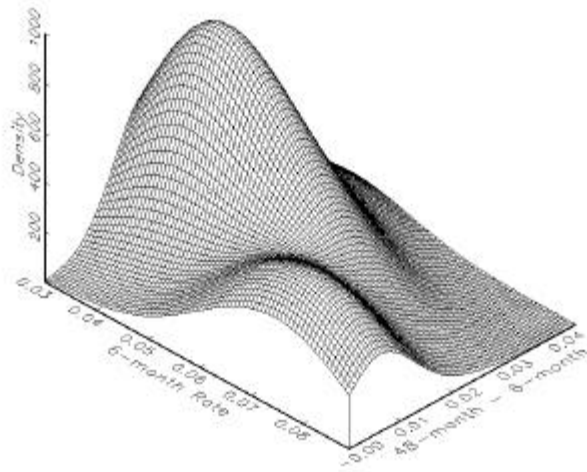


**Figure 5. Estimated constrained kernel diffusion.** The solid curves are the estimated diffusion functions of the 6-month, 12-month, 24-month and 48-month forward rates, obtained by using daily data from April 1990 to October 1998. The boundary condition is imposed so that the diffusions are zero when the forward rates become zero. The dotted curves are the 95% confidence bands obtained by using the Moving Block Bootstrap method.

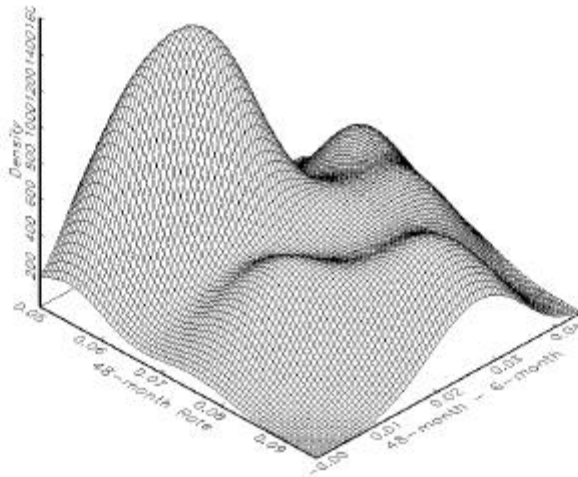


**Figure 6.** The relationship between the forward rate spread and the 6-month, 12-month, 24-month and 48-month forward rates respectively. The rate spread is the difference between the 48-month rate and the 6-month rate. These are daily data from April 1990 to October 1998. There are 2143 observations.

(a) Estimated Bivariate Density of the 6-month Forward Rate

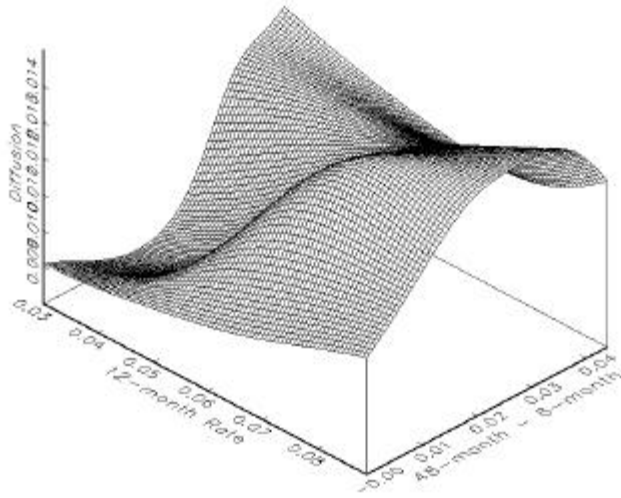


(b) Estimated Bivariate Density of the 48-month Forward Rate

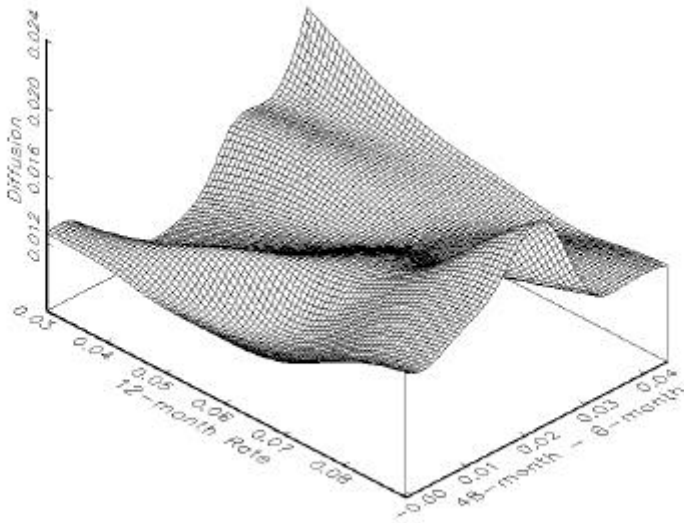


**Figure 7.** Estimated Bivariate Density Functions of the 6-month and 48-month forward rates, obtained by using daily data from April 1990 to October 1998. The density functions are determined by the forward rate and the rate spread, where the rate spread is the difference between the 48-month rate and the 6-month rate.

(a) Estimated Bivariate Diffusion of the 6-month Forward Rate



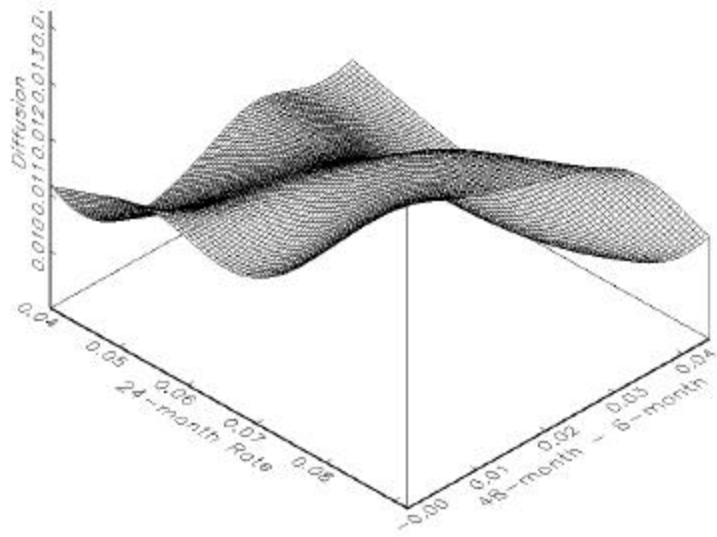
(b) Estimated Bivariate Diffusion of the 12-month Forward Rate



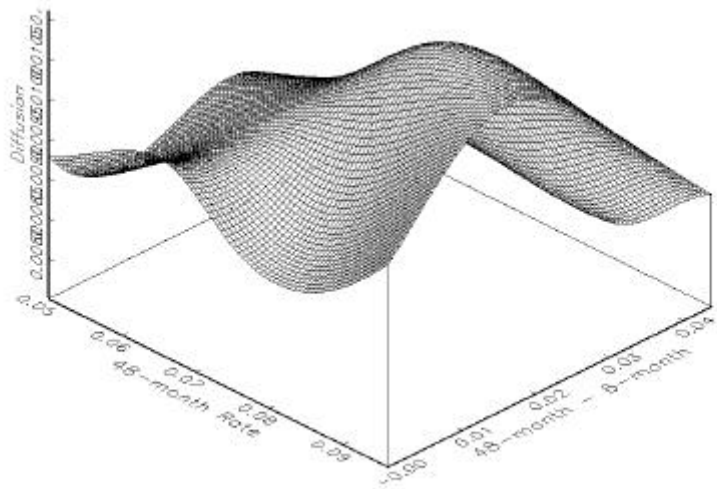
**Figure 8.** Estimated bivariate diffusion functions of the 6-month, 12-month, 24-month and 48-month forward rates, obtained by using daily data from April 1990 to October 1998. The diffusion functions are determined by the forward rate and the rate spread, where the rate spread is the difference between the 48-month rate and the 6-month rate.

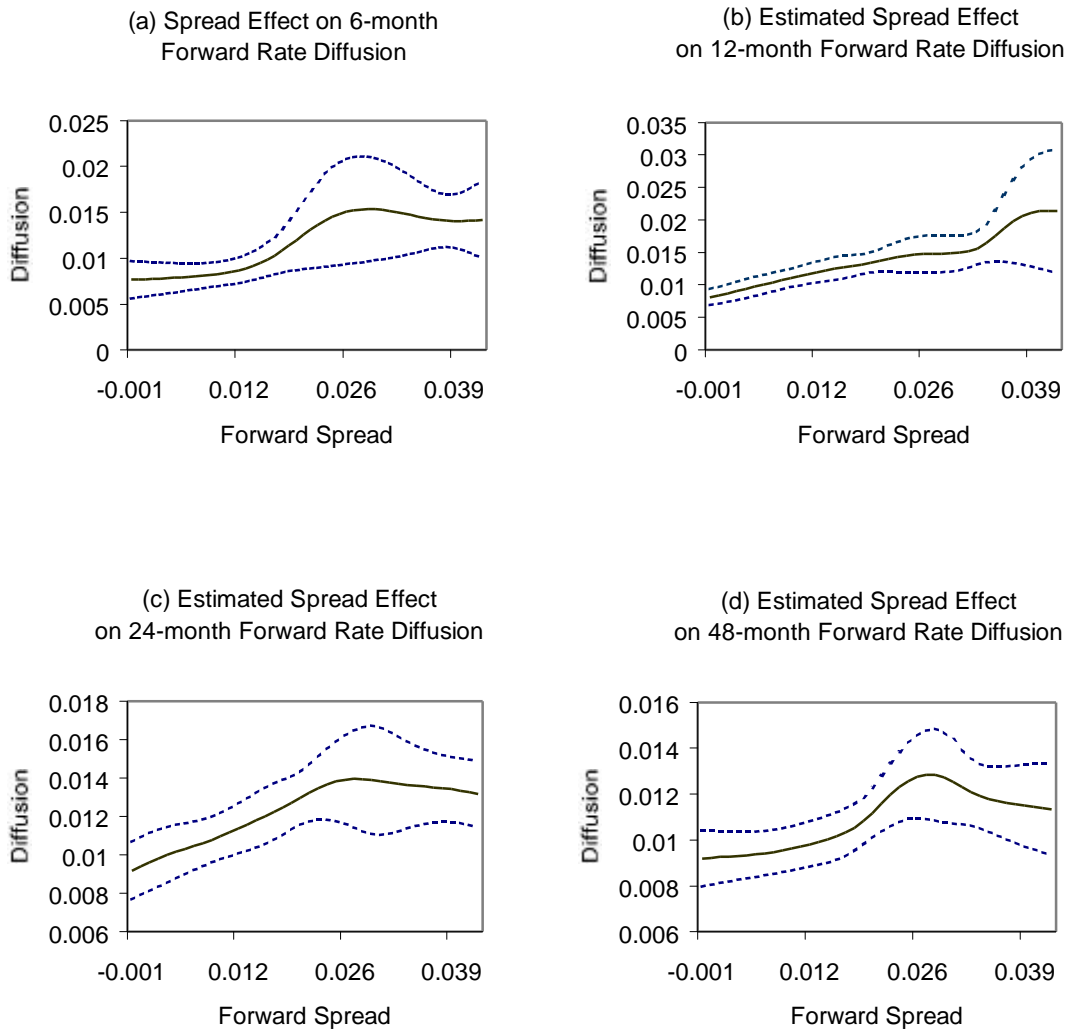


(c) Estimated Bivariate Diffusion of the 24-month Forward Rate

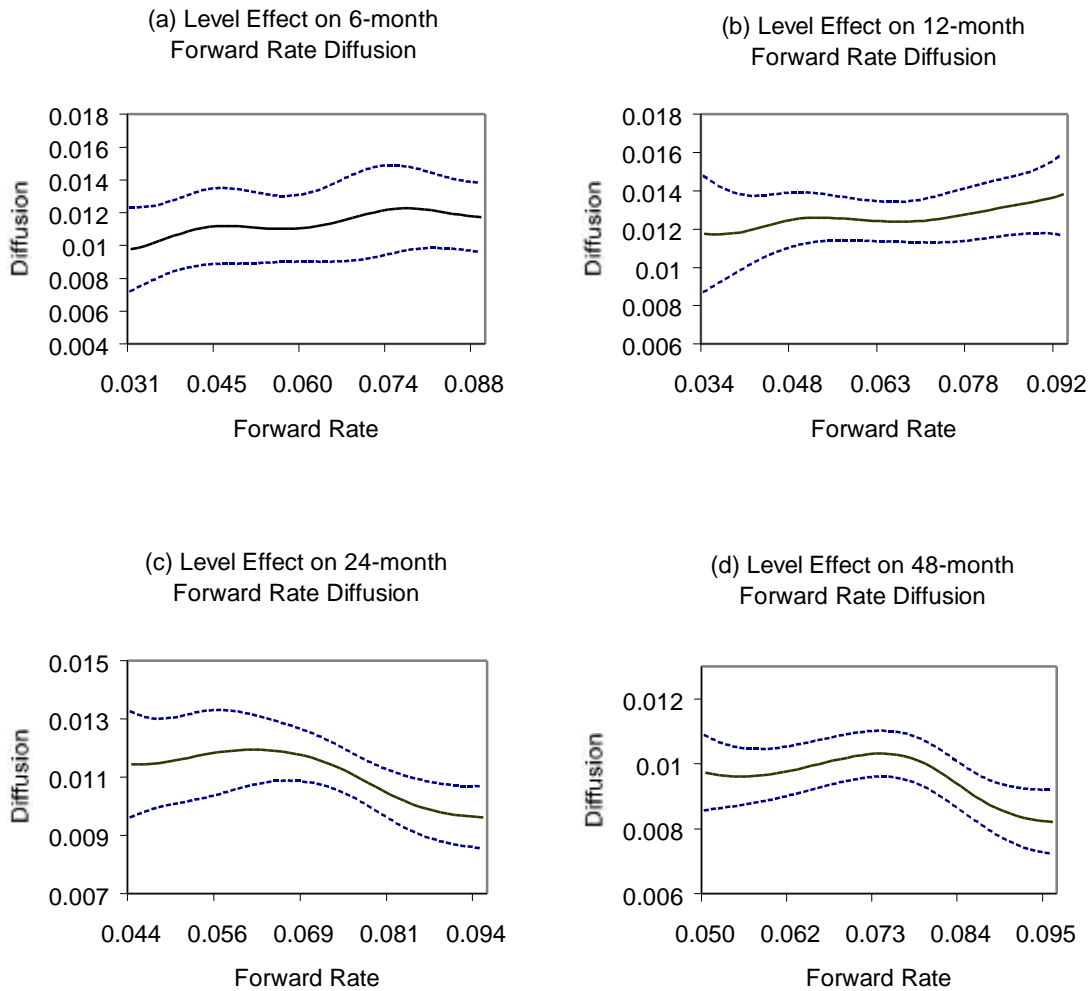


(d) Estimated Bivariate Diffusion of the 48-month Forward Rate





**Figure 9. Forward rate spread effect.** The solid curves are the cross-sections of bivariate kernel diffusion and their confidence surfaces, for the 6-month, 12-month, 24-month and 48-month forward rates. They are taken along the average forward rate of the corresponding data series. The spread is the difference between the 48-month forward rate and the 6-month forward rate. The dotted curves are the cross-sections of 95% confidence surfaces for the bivariate kernel diffusion estimates along the average forward rate levels. They are obtained by using the Moving Block Bootstrap method.



**Figure 10. Forward rate level effect.** The solid curves are the cross-sections of bivariate kernel diffusion and their confidence surfaces, for the 6-month, 12-month, 24-month and 48-month forward rates. They are taken along the average forward spread level. The spread is the difference between the 48-month forward rate and the 6-month forward rate. The dotted curves are the cross-sections of 95% confidence surfaces for the bivariate kernel diffusion estimates along the average forward spread level. They are obtained by using the Moving Block Bootstrap method.