The Morishima Gross Elasticity of Substitution

Charles Blackorby, Daniel Primont and R. Robert Russell

No 787

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

The Morishima Gross Elasticity of Substitution

Charles Blackorby
University of Warwick

Daniel Primont
Southern Illinois University

R. Robert Russell
University of California, Riverside*

(January 2007)

Corresponding author: R. Robert Russell, Department of Economics, University of California, Riverside, CA 92521; rcubed@ucr.edu; 951-827-1585.

Abstract

We show that the Hotelling-Lau elasticity of substitution, an extension of the Allen-Uzawa elasticity to allow for optimal output-quantity (or utility) responses to changes in factor prices, inherits all of the failings of the Allen-Uzawa elasticity identified by Blackorby and Russell [1989 *AER*]. An analogous extension of the Morishima elasticity of substitution to allow for output quantity changes preserves the salient properties of the original Hicksian notion of elasticity of substitution.

JEL classification: D11, D24, D33.

Keywords: elasticity of substitution, relative factor shares, homotheticity.

Introduction

The two-variable elasticity of substitution was introduced by Hicks (1932) to study the evolution of relative factor shares in a growing economy. A logarithmic derivative of a quantity ratio with respect to a technical rate of substitution (with respect to a price ratio under the assumption of price-taking, cost-minimizing behavior), it is an intuitive measure of curvature of an isoquant and provides immediate information about the comparative statics of factor shares. Of two generalizations to encompass more than two inputs suggested by Allen and Hicks (1934), only one survived. That notion became known as the Allen-Uzawa elasticity of substitution after Uzawa (1962) (AUES) provided a much more elegant (and more general) formulation in the dual (in terms of derivatives of the cost function). Untold thousands of Allen-Uzawa elasticities have been estimated over the ensuing years to analyze substitutability and complementarity relationships among inputs and among consumption goods.

Blackorby and Russell (1981, 1989) later argued that the AUES preserves none of the salient properties of the original Hicksian notion and proposed an alternative elasticity, first formulated by Morishima (1967) (though independently discovered by Blackorby and Russell (1975])). The Morishima elasticity of substitution (MES) was shown to be the natural generalization of the original notion of Hicks when there are more than two inputs. The MES is gradually making its way into the empirical literature on substitutability and complementarity.

Both the AUES and the MES are computed using constant-output (compensated) demands. If the production function is homothetic (as was assumed by Hicks), this places no particular restriction on the resulting elasticities since they are then independent of output (as are optimal quantity ratios). When the technology is not homothetic, however, the use of compensated input demands is a real restriction, and these net elasticities may in fact be misleading because, as input prices vary, optimal output also changes, inducing scale effects on input quantity ratios. Following the original study of Hicks, we may be interested in the evolution of relative shares and other information about the quantity effects of price changes while optimally adjusting output.

In two recent papers, Bertoletti (2001, 2005) has resurrected a version of the AUES that allows output to adjust optimally, a concept that he calls the Hotelling-Lau elasticity of substitution (HLES) (first formulated in Lau (1978)). It is formulated by simply replacing the cost function with the profit function in the definition of of the AUES. In this note we argue that the HLES suffers from the same failings as the AUES and show that the Morishima gross elasticity of substitution (MGES), a natural extension of the MES to take account of optimal output adjustments, preserves the salient properties of the original Hicksian notion.[1]

There are n inputs, $x = (x_1, ..., x_n)$, that are employed to produce a scalar output y according to a production function, $y = f(x)$. The cost function is given by

$$c(y, w) = \min_x \{w \cdot x : f(x) \geq y\},$$

where w is a vector of input prices. The AUES for inputs i and j is defined by

$$\sigma_{ij}^{AU}(y, w) = \frac{c_{ij}(y, w)c(y, w)}{c_i(y, w)c_j(y, w)}$$

where subscripts of c denote partial derivatives. The MES is given by

$$\sigma_{ij}^{M}(y,w) = w_i \left(\frac{c_{ij}(y,w)}{c_j(y,w)} - \frac{c_{ii}(y,w)}{c_i(y,w)} \right).$$

The profit function is defined by

$$\pi(p,w) = \max_{y,x} \{ py - w \cdot x : f(x) \geq y \},$$

where p is the output price. The HLES for inputs i and j is defined by

$$\sigma_{ij}^{HL}(p,w) = -\frac{\pi_{ij}(p,w)\pi(p,w)}{\pi_i(p,w)\pi_j(p,w)}. \tag{1}$$

The MGES is defined by

$$\sigma_{ij}^{MG}(p,w) = w_i \left(\frac{\pi_{ij}(p,w)}{\pi_j(p,w)} - \frac{\pi_{ii}(p,w)}{\pi_i(p,w)} \right). \tag{2}$$

The next section contains an example designed to persuade the reader that the proposed gross elasticity of substitution, HLES, suffers from all of the problems attributed to the AUES plus one additional problem, namely, that the gross and net elasticities are not the same in the case of a homothetic production function. This is inconsistent with Hicks's original concept of the elasticity of substitution. On the other hand, the GMES preserves these properties.

1 An Illustrative Example

We consider the technology given by the production function

$$y = f(x) = [\min\{x_1, g(x_2, x_3)\}]^b, \qquad 0 < b < 1, \tag{3}$$

where $g(\cdot)$ is homogeneous of degree one in (x_2, x_3). It can be shown that the cost function for the overall technology (3) is given by

$$c(y,w) = [w_1 + e(w_2, w_3)]\, y^{1/b}, \tag{4}$$

where

$$e(w_2, w_3) = \min\{w_2x_2 + w_3x_3 : g(x_2, x_3) \geq 1\} \tag{5}$$

is the unit cost function for $g(x_2, x_3)$ (see the Appendix). The Allen-Uzawa elasticity of substitution for inputs 2 and 3 is

$$\sigma_{23}^{AU}(y, w) = \frac{c_{23}(y, w)c(y, w)}{c_2(y, w)c_3(y, w)}.$$

Define $v = (w_2, w_3)$ so that $e(w_2, w_3) = e(v)$. For the cost function in (4), it can be shown (see the Appendix) that

$$\sigma_{23}^{AU}(y, w) = \frac{w_1 e_{23}(v)}{e_2(v)e_3(v)} + \frac{e_{23}(v)e(v)}{e_2(v)e_3(v)}. \tag{6}$$

Now suppose that the 2-3 aggregator function in (3) is given by the Cobb-Douglas form:

$$g(x_2, x_3) = x_2^a x_3^{1-a}. \tag{7}$$

It is straightforward (see the Appendix) to derive the unit cost function for (7). It is

$$e(v) = \left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}. \tag{8}$$

Take the appropriate partial derivatives of $e(\cdot)$ and use (6) to obtain

$$\sigma_{23}^{AU}(y, w) = a^a(1-a)^{1-a} w_1 w_2^{-a} w_3^{a-1} + 1. \tag{9}$$

Since the aggregator function is Cobb-Douglas one would expect that AUES for inputs 2 and 3 would be unity. However, the AUES in (9) can take on any value between one and infinity as input prices vary for any $a \in (0, 1)$. This was the key feature in the example provided by Blackorby and Russell (1989). (They set $a = 1/2$.)

We now compute the profit function for (3). It is defined by

$$\pi(p,w) = \max_y \{py - c(y,w)\}$$
$$= \max_y \left\{py - [w_1 + e(v)]\, y^{1/b}\right\}.$$

One can show (see the Appendix) that the profit function is given by

$$\pi(p,w) = \frac{B}{1+d}\left[p^{\frac{1}{1-b}}\right][w_1 + e(v)]^{1+d}, \tag{10}$$

where

$$1 + d = -\frac{b}{1-b} \qquad \text{and} \qquad B = (1+d)\left[b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}}\right]. \tag{11}$$

This is a special case of equation (11) in Bertoletti (2005).

The Hotelling-Lau elasticity of substitution is defined by

$$\sigma_{23}^{HL}(p,w) = -\frac{\pi_{23}(p,w)\pi(p,w)}{\pi_2(p,w)\pi_3(p,w)}. \tag{12}$$

For the profit function in (10), this becomes

$$\sigma_{23}^{HL}(p,w) = \frac{1-b}{b}\sigma_{23}^{AU}(y,w) - \frac{1}{b}, \tag{13}$$

where $\sigma_{23}^{AU}(y,w)$ was given in (6).[2] It is clear from (13) that all of the problems associated with the AES are inherited by the HLES. In addition, (13) demonstrates that the HLES is inconsistent with the very concept of the elasticity of substitution. The production function in the example is homothetic; hence, as emphasized by Kim [2000] and Stern [2004], the curvature is the same on every isoquant (as they are radial translates of one another), there are no scale effects on optimal quantity ratios, and the net and gross elasticities ought to be the same.

To make this point more explicit, suppose that the 2-3 aggregator function is Cobb-Douglas. Arguing, as in Blackorby-Russell (1989), that a reasonable value for $\sigma_{23}^{AU}(y, w)$ is unity, the Hotelling-Lau elasticity of substitution should be equal to

$$\begin{aligned}\sigma_{23}^{HL}(w) &= \frac{1-b}{b} - \frac{1}{b} \\ &= \frac{1-b-1}{b} \\ &= -1.\end{aligned}$$

However, when the aggregator is Cobb-Douglas (7), the Hotelling-Lau elasticity of substitution is given by

$$\sigma_{23}^{HL}(w) = \frac{1-b}{b}\left[a^a(1-a)^{1-a}w_1 w_2^{-a} w_3^{a-1} + 1\right] - \frac{1}{b},$$

which can take any value from $-1/b$ to infinity for any $a \in (0, 1)$ and $b \in (0, 1)$.

2 Properties of the MGES

Let x_i^* and x_j^* be the profit-maximizing quantities of inputs i and j. We are interested in calculating how the ratio of input quantities, x_i^*/x_j^*, changes in response to a change in the ratio of input prices, w_i/w_j. We begin by noting that, by Hotelling's Lemma,

$$\ln\left(\frac{x_i^*}{x_j^*}\right) = \ln\left(\frac{-\pi_i(p,w)}{-\pi_j(p,w)}\right) = \ln\left(\frac{\pi_i(p,w)}{\pi_j(p,w)}\right). \tag{14}$$

To differentiate (14) with respect to the log of w_i/w_j, we first note that

$$\pi(p,w) = w_j \hat{\pi}\left(p/w_j, w^{-j}/w_j\right), \tag{15}$$

where $w^{-j} = (w_1, ..., w_{j-1}, w_{j+1}, ..., w_n)$ and $\hat{\pi}\left(p/w_j, w^{-j}/w_j\right) = \pi\left(p/w_j, w/w_j\right)$. Using Hotelling's Lemma and (15) we get the MGES:

$$-\frac{\partial}{\partial \ln\left(w_i/w_j\right)} \ln\left(\frac{x_i^*}{x_j^*}\right) = \sigma_{ij}^{MG}(p, w) = w_i\left(\frac{\pi_{ij}(p, w)}{\pi_j(p, w)} - \frac{\pi_{ii}(p, w)}{\pi_i(p, w)}\right).$$

Thus, the MGES indicates how the income ratio, $S_{ij}(p, w) = w_i x_i^*/w_j x_j^*$, changes with a change in the input price ratio. In particular,

$$\frac{\partial \ln S_{ij}(p, w)}{\partial \ln\left(\dfrac{w_i}{w_j}\right)} = 1 - \sigma_{ij}^{MG}$$

It is also interesting to derive the relationship between the MES and the MGES. Let $x = h(y, w)$ and $x = x(p, w)$ be the cost-minimizing and profit-maximizing choices for the input vector. Also, $y = y(p, w)$ be the profit-maximizing output. Then

$$x(p, w) = h(y(p, w), w). \tag{16}$$

Differentiate (16) with respect to w_j to get

$$\frac{\partial x_i(p, w)}{\partial w_j} = \frac{\partial h_i(y, w)}{\partial w_j} + \frac{\partial h_i(y, w)}{\partial y}\frac{\partial y(p, w)}{\partial w_j}. \tag{17}$$

Invoking Hotelling/Shephard, we obtain

$$\pi_{ij}(p, w) = -\left[c_{ij}(y, w) + c_{iy}(y, w)\pi_{pj}(p, w)\right]$$

and

$$\pi_{ii}(p, w) = -\left[c_{ii}(y, w) + c_{iy}(y, w)\pi_{pi}(p, w)\right].$$

The Morishima gross elasticity of substitution is defined by

$$\sigma_{ij}^{MG}(p, w) = w_i\left(\frac{\pi_{ij}(p, w)}{\pi_j(p, w)} - \frac{\pi_{ii}(p, w)}{\pi_i(p, w)}\right).$$

Thus, it is clear that the MGES can be written as a function of the MES as follows:

$$\sigma_{ij}^{MG}(p,w)$$
$$= w_i \left(\frac{-\left[c_{ij}(y,w) + c_{iy}(y,w)\pi_{pj}(p,w)\right]}{\pi_j(p,w)} - \frac{-\left[c_{ii}(y,w) + c_{iy}(y,w)\pi_{pi}(p,w)\right]}{\pi_i(p,w)} \right)$$

$$= w_i \left(\frac{c_{ij}(y,w)}{c_j(y,w)} - \frac{c_{ii}(y,w)}{c_i(y,w)} \right) + w_i c_{iy}(y,w) \left(\frac{\pi_{pj}(p,w)}{\pi_j(p,w)} - \frac{\pi_{pi}(p,w)}{\pi_i(p,w)} \right)$$

$$= \sigma_{ij}^{M}(y,w) + w_i c_{iy}(y,w) \left(\frac{\pi_{pj}(p,w)}{\pi_j(p,w)} - \frac{\pi_{pi}(p,w)}{\pi_i(p,w)} \right). \quad (18)$$

We now show that the MGES and the MES are equal if and only if the production function is homothetic. It is apparent from (18) that the Morishima gross elasticity of substitution is equal to the Morishima (net) elasticity of substitution for all input pairs if and only if

$$\frac{\pi_{pj}(p,w)}{\pi_j(p,w)} - \frac{\pi_{pi}(p,w)}{\pi_i(p,w)} = 0, \quad i,j = 1,...,n$$

This is equivalent to the condition that

$$\frac{\partial}{\partial p} \left(\frac{\pi_i(p,w)}{\pi_j(p,w)} \right) = \frac{\pi_j(p,w)\pi_{ip}(p,w) - \pi_i(p,w)\pi_{jp}(p,w)}{\left[\pi_j(p,w)\right]^2}$$
$$= \frac{\pi_j(p,w)\pi_{pi}(p,w) - \pi_i(p,w)\pi_{pj}(p,w)}{\left[\pi_j(p,w)\right]^2}$$
$$= 0, \quad i,j = 1,...,n.$$

This is the well-known condition for separability of input prices from the output price. This separability condition is equivalent to homotheticity of the production function. Not surprisingly, for our example, the MES and MGES are equal.

3 Concluding Remarks

The original Hicksian elasticity of substitution is an insightful concept, formulated to answer specific economic questions. Generalizations of this notion should be faithful to the original conception, preserving those properties that infuse it with economic content. As shown by Blackorby and Russell [1981, 1989], the Allen-Uzawa elasticity of substitution preserves none of the salient properties of the original Hicksian notion: it "(i) is *not* a measure of the 'ease' of substitution, or curvature of the isoquant, (ii) provides *no* information about relative factor shares, ... and (iii) *cannot* be interpreted as a logarithmic derivative of a quantity ratio with respect to a price ratio ... " (Blackorby and Russell [1989, p. 883]).

The Hotelling-Lau elasticity of substitution, constructed by analogy to the Allen-Uzawa notion (substituting the profit function for the cost function) inherits the problems presented by the AUES. It is not a logarithmic derivative of a quantity ratio with respect to a price ratio—allowing output to change, and it does not provide comparative static content about relative factor incomes. In fact, it is not even a generalization of the AUES in any meaningful sense, since it does not reduce to the latter under the assumption of a homotheticity.

The Morishima gross elasticity of substitution, on the other hand, *does* provide immediate comparative-static information about the (qualitative and quantitative) effect of changes in relative prices on factor income ratios and *is* a logarithmic derivative of a quantity ratio with respect to a price ratio, allowing output to change. Moreover, the MGES reduces to the MES under the assumption of homotheticity.

The HSES can be written as a function of the AUES, but this simply compounds the problems with the latter, since the AUES provides no meaningful information about ease of substitution or the curvature of the isoquant.

The MGES can be written as a function of the MES, thus incorporating information about ease of substitution along an isoquant into the measure of substitution when output is allowed to vary. In short, the MGES is the "real" elasticity of gross substitutability.

Appendix

Derivation of (4): Given the problem,

$$c(y, w) = \min \left\{ w_1 x_1 + w_2 x_2 + w_3 x_3 : \min \{x_1, g(x_2, x_3)\} \geq y^{1/b} \right\},$$

let

$$\begin{aligned} \mathfrak{e}(y_{23}, v) &= \min \{w_2 x_2 + w_3 x_3 : g(x_2, x_3) \geq y_{23}\} \\ &= e(v) y_{23}. \end{aligned}$$

Then

$$\begin{aligned} c(y, w) &= \min_{x_1, y_{23}} \left\{ w_1 x_1 + e(v) y_{23} : \min \{x_1, y_{23}\} \geq y^{1/b} \right\} \\ &= [w_1 + e(v)]\, y^{1/b}. \end{aligned}$$

Derivation of (6): The relevant partial derivatives of c are (partial differentiation denoted by subscripts)

$$c_2(y, w) = e_2(v) y^{1/b}, \tag{19}$$

$$c_3(y, w) = e_3(v) y^{1/b}, \tag{20}$$

and

$$c_{23}(y, w) = e_{23}(v) y^{1/b}. \tag{21}$$

The Allen-Uzawa Elasticity of Substitution (AUES) for inputs 2 and 3 is given by

$$\sigma_{23}^{AU}(y, w) = \frac{c_{23}(y, w) c(y, w)}{c_2(y, w) c_3(y, w)}, \tag{22}$$

and in this case, using (19), (20), and (21), we get

$$\sigma_{23}^{AU}(y,w) = \frac{\left[e_{23}(v)y^{1/b}\right]\left[w_1 + e(v)\right]y^{1/b}}{\left[e_2(v)y^{1/b}\right]e_3(v)y^{1/b}}$$

$$= \frac{e_{23}(v)\left[w_1 + e(v)\right]}{e_2(v)e_3(v)}.$$

With a slight rearrangement,

$$\sigma_{23}^{AU}(y,w) = \frac{w_1 e_{23}(v)}{e_2(v)e_3(v)} + \frac{e_{23}(v)e(v)}{e_2(v)e_3(v)}. \tag{23}$$

Derivation of (8): Let $g(x_2, x_3) = x_2^a x_3^{1-a}$. Then

$$e(w_2, w_3) = \min\left\{w_1x_1 + w_2x_2 : x_2^a x_3^{1-a} \geq 1\right\}.$$

Form the Lagrangian:

$$\mathcal{L} = w_2x_2 + w_3x_3 - \lambda x_2^a x_3^{1-a}.$$

Then two of the first-order conditions are

$$\mathcal{L}_1 = w_2 - a\lambda x_2^{a-1}x_3^{1-a} = w_2 - a\lambda\frac{1}{x_2} = 0 \tag{24}$$

and (25)

$$\mathcal{L}_2 = w_3 - (1-a)\lambda x_2^a x_3^{-a} = w_3 - (1-a)\lambda\frac{1}{x_3} = 0, \tag{26}$$

where the second equality in (24) and (26) follows from $x_2^a x_3^{1-a} = 1$. The solutions for the input quantities are

$$x_2 = \frac{a}{w_2}\lambda \tag{27}$$

and (28)

$$x_3 = \frac{1-a}{w_3}\lambda. \tag{29}$$

It follows that

$$1 = x_2^a x_3^{1-a} = \left(\frac{a}{w_2}\lambda\right)^a \left(\frac{1-a}{w_3}\lambda\right)^{1-a}$$
$$= \left(\frac{a}{w_2}\right)^a \left(\frac{1-a}{w_3}\right)^{1-a} \lambda,$$

so that

$$\lambda = \left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}.$$

Put this result into (27) and (29) to get

$$x_2^* = \frac{a}{w_2}\left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}$$

and

$$x_3^* = \frac{1-a}{w_3}\left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}.$$

From this it follows that

$$e(w_2, w_3) = w_2 x_2^* + w_3 x_3^*$$
$$= \left(\frac{w_2}{a}\right)^a \left(\frac{w_3}{1-a}\right)^{1-a}.$$

Derivation of (10): The profit maximization problem is:

$$\pi(p, w) = \max_y \{py - c(y, w)\}$$
$$= \max_y \left\{py - [w_1 + e(v)]\, y^{1/b}\right\}.$$

From the first-order condition,

$$p = \frac{1}{b}[w_1 + e(v)]\, y^{(1/b)-1},$$

or, rearranging,

$$y^{\frac{1-b}{b}} = pb\,[w_1 + e(v)]^{-1},$$

so that profit-maximizing output is given by

$$y^* = p^{\frac{b}{1-b}} b^{\frac{b}{1-b}} \left[w_1 + e(v)\right]^{\frac{-b}{1-b}} . \tag{30}$$

Note that

$$\left(y^*\right)^{1/b} = p^{\frac{1}{1-b}} b^{\frac{1}{1-b}} \left[w_1 + e(v)\right]^{\frac{-1}{1-b}} . \tag{31}$$

Putting (30) and (31) into the profit expression, $py - [w_1 + e(v)]\, y^{1/b}$, we arrive at

$$\pi(p, w) = pp^{\frac{b}{1-b}} b^{\frac{b}{1-b}} \left[w_1 + e(v)\right]^{\frac{-b}{1-b}} - \left[w_1 + e(v)\right] p^{\frac{1}{1-b}} b^{\frac{1}{1-b}} \left[w_1 + e(v)\right]^{\frac{-1}{1-b}}$$

$$= p^{\frac{1}{1-b}} b^{\frac{b}{1-b}} \left[w_1 + e(v)\right]^{\frac{-b}{1-b}} - p^{\frac{1}{1-b}} b^{\frac{1}{1-b}} \left[w_1 + e(v)\right]^{\frac{-b}{1-b}}$$

$$= \left[b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}}\right] \left[p^{\frac{1}{1-b}}\right] \left[w_1 + e(v)\right]^{\frac{-b}{1-b}} .$$

Rewrite this as

$$\pi(p, w) = \frac{B}{1+d} \left[p^{\frac{1}{1-b}}\right] \left[w_1 + e(v)\right]^{1+d} , \tag{32}$$

where

$$1 + d = -\frac{b}{1-b};$$

$$i.e, \qquad d = -1 - \frac{b}{1-b} = \frac{-1}{1-b}$$

and

$$B = (1+d) \left[b^{\frac{b}{1-b}} - b^{\frac{1}{1-b}}\right] .$$

Derivation of (13): Some partial derivatives of π in (10) are given by

$$\pi_2(p, w) = B \left[p^{\frac{1}{1-b}}\right] \left[w_1 + e(v)\right]^d e_2(v), \tag{33}$$

$$\pi_3(p,w) = B\left[p^{\frac{1}{1-b}}\right][w_1 + e(v)]^d\, e_3(v), \tag{34}$$

and

$$\begin{aligned}\pi_{23}(p,w) = dB\left[p^{\frac{1}{1-b}}\right][w_1 + e(v)]^{d-1}\, e_2(v)e_3(v) \\ + B\left[p^{\frac{1}{1-b}}\right][w_1 + e(v)]^d\, e_{23}(v).\end{aligned} \tag{35}$$

The Hotelling-Lau Elasticity of Substitution is

$$\sigma_{23}^{HL}(p,w) = -\frac{\pi_{23}(p,w)\pi(p,w)}{\pi_2(p,w)\pi_3(p,w)}.$$

Because of (33), (34), and (35), this becomes

$$\begin{aligned}&\sigma_{23}^{HL}(p,w) \\ &= -\frac{\pi_{23}(p,w)\pi(p,w)}{\pi_2(p,w)\pi_3(p,w)} \\ &= -\frac{dB\,[w_1 + e(v)]^{d-1}\, e_2(v)e_3(v)\left[\frac{B}{1+d}[w_1 + e(v)]^{1+d}\right]}{\left[B\,[w_1 + e(v)]^d\, e_2(v)\right]\left[B\,[w_1 + e(v)]^d\, e_3(v)\right]} \\ &- \frac{B\,[w_1 + e(v)]^d\, e_{23}(v)\left[\frac{B}{1+d}[w_1 + e(v)]^{1+d}\right]}{\left[B\,[w_1 + e(v)]^d\, e_2(v)\right]\left[B\,[w_1 + e(v)]^d\, e_3(v)\right]}.\end{aligned}$$

After some simplification,

$$\sigma_{23}^{HL}(w) = -\frac{d}{1+d} - \frac{w_1 + e(v)}{1+d} \frac{e_{23}(v)}{e_2(v)e_3(v)}$$

$$= -\frac{d}{1+d} - \frac{1}{1+d} \left[\frac{w_1 e_{23}(v)}{e_2(v)e_3(v)} + \frac{e_{23}(v)e(v)}{e_2(v)e_3(v)} \right]$$

$$= -\frac{d}{1+d} - \frac{1}{1+d} \sigma_{23}^{AU}(y, w) \qquad \text{(using (23))}.$$

Since

$$d = \frac{-1}{1-b} \text{ and } 1 + d = \frac{-b}{1-b},$$

we can rewrite this result as

$$\sigma_{23}^{HL}(p, w) = \frac{-1}{b} + \frac{1-b}{b} \sigma_{23}^{AU}(y, w)$$

or

$$\sigma_{23}^{HL}(p, w) = \frac{1-b}{b} \sigma_{23}^{AU}(y, w) - \frac{1}{b}. \tag{36}$$

Notes

* We thank Paolo Bertoletti for drawing our attention to the issue addressed in this paper and for his comments on an earlier draft.

[1]The MGES was first formulated by Davis and Shumway (1996). Although the formulation that follows, like theirs, is for a single output, the concept can be straightforwardly extended to multiple outputs.

[2] This is a special case of equation (11) in Bertoletti (2005).

References

Allen, G. D.,and J. R. Hicks. "A Reconsideration of the Theory of Value, Part II." (1934). *Economica* 1, 196–219.

Bertoletti, P. "The Allen/Uzawa Elasticity of Substitution as a Measure of Gross Input Substitutability." (2001) *Rivista italiana degli economisti* 6, 87–94.

Bertoletti, P. "Elasticities of Substitution and Complementarity" A Synthesis." (2005). *Journal of Productivity Analysis* 24, 183–196.

Blackorby, C., and R. R. Russell. "The Partial Elasticity of Substitution." (1975). Discussion Paper No. 75-1, Department of Economics, University of California, San Diego.

Blackorby, C., and R. R. Russell. "The Morishima Elasticity of Substitution: Symmetry, Constancy, Separability, and Its Relationship to the Hicks and Allen Elasticities." (1981). *Review of Economic Studies* 48, 147–158.

Blackorby, C., and R. R. Russell. "Will the Real Elasticity of Substitution Please Stand Up?." (1989). *American Economic Review* 79, 882–888.

Davis, G. C., and C. R. Shumway. "To Tell the Truth about Interpreting the Morishima Elasticity of Substitution." (1996). *Canadian Journal of Agricultural Economics* 44, 173–182.

Hicks, J. R. *The Theory of Wages.* (1932). London: Macmillan.

Kim, H. Y. "The Antonelli Versus Hicks Elasticity of Complementarity and Inverse Input Demand Systems." (2000). *Australian Economic Papers* 39, 245–261.

Lau, L. "Applications of Profit Functions." In M. Fuss and D. McFadden (eds.), *Production Economics: A Dual Approach to Theory and Applications,* *Vol. I*. Amsterdam: North-Holland.

Stern, D. I. "Elasticities of Substitution and Complementarity." (2004). Rensselaer Working Papers in Economics, Number 0403, Rensselaer Polytechnic Institute.

Uzawa, H. "Production Functions with Constant Elasticities of Substitution." (1962). *Review of Economic Studies* 29, 291–299.