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Abstract

We show that if a monopoly sector is imbedded in a general equilibrium framework and profits are taxed at one hundred percent, then unit (specific) taxation and ad valorem taxation are welfare-wise equivalent. This is contrary to all known claims. *Journal of Economic Literature* Classification Number:H21

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Unit Versus Ad Valorem Taxes—Monopoly In General Equilibrium

by

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1. Introduction

It is well-known that, in a competitive environment, unit (or specific) taxation and ad valorem taxation are equivalent. Cournot [1838, 1960] realized that the two tax systems needed different treatment in the case of monopoly. Wicksell [1896, 1959] argued that ad valorem taxes dominate unit taxation in a monopoly; a complete demonstration of this dominance was given by Suits and Musgrave [1955]. More specifically they demonstrated, if the consumer price and quantity of the monopoly good remained unchanged, that the government tax yield is higher with ad valorem taxes than under a regime of unit taxes. This follows because the profit-maximizing price of the monopolist is lower under ad valorem taxation than under unit taxation. Most recent work in this area has investigated forms of competition between pure monopoly and competition implicitly or explicitly accepting the above dominance argument. Delipalla and Keen [1992] examine different models of oligopoly with and without free entry to compare the two types of tax regimes while Lockwood [2004] shows, in a tax competition model, that tax competition is more intense with ad valorem taxes thus yielding a lower price in equilibrium.

Wicksell and Suits and Musgrave derived the above mentioned monopoly result in a partial equilibrium framework and claimed that ad valorem taxation was superior to unit taxation on welfare grounds. Recently, stronger and more explicit claims have been made: Skeath and Trandel [1994; p. 55] state that "in the monopoly case, given any unit excise tax, it is possible to find an ad valorem tax that Pareto dominates it."; Keen [1996; p. 9] states that "The conclusion—due to Skeath and Trandel [1994]—is thus strikingly unambiguous: with monopoly provision of a single good of fixed quality, consumers prefer ad valorem taxation because it leads to a lower price, firms prefer it because it leads to higher profits and government prefers it because it leads to higher revenue. There is no need to trade off the interests of these three groups: ad valorem taxation dominates specific."

It is this claimed welfare dominance of ad valorem taxes over unit taxation that we challenge in this paper. In the context of a general equilibrium model with a single monopoly sector, we prove that unit and ad valorem taxation are equivalent.

More specifically, we take a standard general equilibrium model in which a single monopoly sector has been imbedded.[1] In particular we adapt the model of Guesnerie and Laffont [1978] (hereafter GL) to pose this question. In order to study alternative tax schemes it must be the case that indirect taxation is an optimal instrument; we permit the government to employ a demogrant, a transfer (positive or negative) that is common

[1] Our results hold for any symmetric Cournot oligopoly as well.

to all consumers rather than individual lump-sum transfers. In addition, we need to decide how to treat the profits of the competitive and monopoly sector. The government can either levy one hundred percent profit taxation or the profits can be rebated to the individual consumers in proportion to the shares held in the companies. It is common in public economics to adopt the former assumption as it is usually an innocuous but simplifying assumption;[2] we follow this tradition here because it seems closest to the analysis of the partial equilibrium models whose results we challenge. In order for partial equilibrium models to be consistent with utility maximization, quasi-linear preferences are required and this means that the relevant demands are independent of the distribution of income.[3] Assuming one hundred percent taxation of profits is the closest that one can come to avoiding income effects in a general equilibrium framework. Nevertheless, this assumption does have strong implications and in our concluding remarks we discuss some of the additional problems that arise if monopoly profits are privately distributed. It should be noted however that if private ownership of the firms makes a substantial difference, it is because of income effects and the partial equilibrium models cannot cannot handle such changes in any case.

We first characterize the set of Pareto optima in the economy with unit taxation. We then convert each Pareto-optimal unit-tax equilibrium into an equivalent ad valorem tax equilibrium and ask if there exist any feasible ad valorem tax Pareto improvements from this equilibrium. We show that there none. We then reverse this procedure and characterize the set of Pareto optima with ad valorem taxes, convert these equilibria to unit tax equilibria, and then show that there are no possible Pareto improvements using unit taxes. Thus, we prove that ad valorem and unit taxation are equivalent. The set of Pareto optima under unit taxes is identical to the set of Pareto optima under ad valorem taxation.

The reasons that our results are in striking contrast to the literature are two-fold. The partial equilibrium problem analyzed by Wicksell and Suits-Musgrave ignores that, although tax revenues go up in the switch to ad valorem taxes, the profits of the monopolist go down. In a general equilibrium framework this reduction in profit must show up somewhere else. With one hundred percent taxation this simply means that government revenue from profit taxation goes down by the same amount that its revenue from indirect taxes went up. The surplus analysis of Skeath and Trandel arrives at an incorrect conclusion because the criteria they use do not translate into improvements in the sense of Pareto.[4]

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[2] A common alternative assumption is constant return-to-scale which simply eliminates profits; this will not work here because of the monopoly sector.

[3] See Chipman and Moore [1976] and Silverberg [1972].

[4] This is not surprising given these surplus arguments have already been shown to perform rather badly in both competitive and non competitive environments; see Blackorby and Donaldson [1985,1999]. However, even in the quasi-linear case, when most surplus arguments do work, the aggregate surplus is independent of the distribution of wealth. In this problem it is the distribution of wealth that is of crucial importance and ignoring it can lead to errors.

In the next section we introduce the notation, in the following we characterize the set of unit tax Pareto optima after which we convert the Pareto-optimal unit-tax equilibria into ad valorem equilibrium. In the following section we show that all such ad valorem equilibrium are also Pareto optimal. We reverse this procedure and show that ad valorem and unit taxation are equivalent.

2. Notation

The preferences of each of the H consumers are represented by an indirect utility function,

$$u_h = V^h(q_0, q, m) \quad \text{for} \quad h = 1, \dots, H \tag{2.1}$$

where $q_0 \in \mathbf{R}_{++}$ is the consumer price of the monopoly good, $q \in \mathbf{R}^N_{++}$ is the vector of consumer prices of the competitively supplied goods. The demogrant—a transfer that is common to all consumers—is given by m.[5] V^h is assumed to be twice continuously differentiable and strongly quasi-convex.[6] The demands are given by Roy's Theorem and the aggregate demand for the monopoly good is

$$x_0 = \sum_h \overset{*}{x}{}^h_0 = \overset{*}{x}_0(q_0, q, m) \tag{2.2}$$

and the aggregate demand vector for the competitively supplied commodities is given by

$$x = \sum_h x^h(q_0, q, m) \tag{2.3}$$

The consumer prices and producer prices are related by unit taxes, that is, $q_0 = p_0 + t_0$ and $q = p + t$, where p_0 and p are the producer prices of the monopoly good and competitive goods respectively. The unit tax on the monopoly good and the competitive goods are t_0 and t respectively.

As discussed in detail in GL the profit function of the monopolist (the function over which he optimizes) is not in general concave. Following them we assume that the solution to monopolist's profit maximization problem is locally unique and smooth. For the monopoly let

$$\begin{aligned} &\langle P_0^u(p, t_0, t, m),\, Y_0^u(p, t_0, t, m)\rangle := \\ &\qquad \operatorname{argmax}_{p_0^u, y_0^u}\{p_0 y_0^u - C(y_0^u, p) | y_0^u \geq \overset{*}{x}_0(q_0, q, m)\}, \end{aligned} \tag{2.4}$$

where the cost function of the monopolist is assumed to be differentially strongly concave.[7] We assume that P_0^u is single-valued and smooth and as in GL that

$$\nabla_{t_0} P_0^u \neq -1; \tag{2.5}$$

[5] There is also a public good g but, as it remains constant throughout the analysis, it is suppressed in the utility function.

[6] See Diewert, Avriel and Zang [1981].

[7] See Avriel, Diewert, Schaible, and Zang [1988].

that is, the monopolist cannot undo all changes by the tax authority of t_0. The input demands by the monopolist from the competitive sector is given by

$$y^m = \nabla_p C(y_0^u, p). \tag{2.6}$$

Monopoly profit with unit taxation is given by

$$\Pi^{mu}(p_0^u, p, y_0^u) = p_0^u y_0^u - p^T \nabla_p C(y_0^u, p) = p_0^u y_0^u - C(y_0^u, p). \tag{2.7}$$

The profit function of the competitive sector Π^c is assumed to be strongly convex and twice continuously differentiable.[8] Applying Hotelling's Lemma it can be re-written as

$$\Pi^c(p) = p^T y^c(p). \tag{2.8}$$

In addition we assume that

$$\operatorname{rank}\left[\, \nabla_p y^c(p) - \nabla_p y^m(p) \,\right] = N - 1. \tag{2.9}$$

This ensures that the zero eigenvector is unique up to a positive scalar multiple.[9] In addition the government produces a public good g from inputs y^g purchased from the competitive sector by

$$g \le F(y^g); \tag{2.10}$$

we assume that F is increasing and differentially strongly concave.

Equilibrium in this unit-tax economy is given by

$$-x + y^c - y^m - y^g \ge 0, \tag{2.11}$$

$$-\overset{*}{x}_0(q_0, q, m) + y_0^u \ge 0, \tag{2.12}$$

$$p_0^u - P_0^u(p, t_0, t, m) = 0, \tag{2.13}$$

and

$$F(y^g) - g \ge 0. \tag{2.14}$$

(2.11) is market clearing for the competitively produced commodities, (2.12) is market clearing for the monopoly good, and (2.13) ensures that the monopolist is profit maximizing whereas (2.14) requires the public-good producing firm to buy enough inputs to produce the fixed amount of the public good.

Given that the indirect utility functions are increasing in income, the budget constraints of all the consumers are binding at their optimal bundles. In addition, if the equilibrium conditions hold as equalities then from Walras Law it follows that the demogrant is

$$m = \frac{1}{H}\left[\Pi^{mu}(p_0^u, p, y_0^u) + t^T x + t_0 y_0 + \Pi^c(p) - p^T y^g\right]. \tag{2.15}$$

[8] See Avriel, Diewert, Schaible, and Zang [1988]. The assumption of a single competitive firm is without loss of generality; see Bliss [1975].

[9] In fact our regularity assumptions already guaranty this to be true almost everywhere.

This is because aggregating over consumer budgets and using the equilibrium conditions, (2.11) and (2.12), we obtain

$$\begin{aligned} Hm &= q_0 \overset{*}{x}_0 + q^T x \\ &\leq (p_0^u + t_0) y_0^u + (p+t)^T [y^c - y^m - y^g] \\ &= p_0^u y_0^u - p^T y^m + p^T y^c + t^T [y^c - y^m - y^g] - p^T y^g \\ &= \Pi^{mu}(p_0^u, p, y_0^u) + t^T [y^c - y^m - y^g] + t_0 y_0 + \Pi^c(p) - p^T y^g. \end{aligned} \tag{2.16}$$

Thus, at an equilibrium where (2.11) and (2.12) hold as equalities, the government budget just balances, *i.e.,* the revenue of the government from indirect and profit taxation, net of its expenditure on the public good, gets distributed as the demogrant to the consumers.

3. Unit-Tax Pareto Optima

In order to characterize the set of Pareto optima with unit taxation, we assume first that the equilibrium conditions, (2.11)-(2.14) hold with equality. From this equilibrium we calculate the directions of change in prices, taxes, and demogrant that could generate strict Pareto improvements,[10] preserve equilibrium, and ensure that the monopolist is maximizing profit and the public good producer can buy sufficient inputs to produce the public good.[11] If no such changes exist then the equilibrium with which we started is a Pareto optimum.

Assume that the equilibrium conditions, (2.11)-(2.14), hold with equality. Then, there exist strict Pareto-improving directions of change that are equilibrium-preserving and provide for the production of the public good if and only if

$$\begin{bmatrix} -x_0^1 & -x_0^1 & -x^{1T} & -x^{1T} & 1 & 0_N^T & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ -x_0^H & -x_0^H & -x^{HT} & -x^{HT} & 1 & 0_N^T & 0 \end{bmatrix} \begin{bmatrix} dp_0^u \\ dt_0 \\ dp \\ dt \\ dm \\ dy^g \\ dy_0^u \end{bmatrix} \gg 0 \tag{3.1}$$

[10] That is, changes that increase the utility of every consumer.

[11] Walras Law ensures that, at the new equilibria reached by such directions of change, the government's budgets are always balanced in the sense of (2.16).

and

$$\begin{bmatrix} -\nabla_{q_0}x & -\nabla_{q_0}x & -\nabla_q x + \nabla_p y^c - \nabla_p y^m & -\nabla_q x & -\nabla_m x & -I_N & -\nabla_{y_0^u} y^m \\ -\nabla_{q_0}x_0 & -\nabla_{q_0}x_0 & -\nabla_q^T x_0 & -\nabla_q^T x_0 & -\nabla_m x_0 & 0_N^T & 1 \\ 0 & 0 & 0_N^T & 0_N^T & 0 & \nabla_{y^g}^T F & 0 \end{bmatrix} \begin{bmatrix} dp_0^u \\ dt_0 \\ dp \\ dt \\ dm \\ dy^g \\ dy_0^u \end{bmatrix} \geq 0. \tag{3.2}$$

That the monopolist continues to maximize profits after the changes is guaranteed by

$$\begin{bmatrix} 1 & -\nabla_{t_0} P_0^u & -\nabla_p^T P_0 & -\nabla_t^T P_0 & -\nabla_m P_0^u & 0_N^T & 0 \end{bmatrix} \begin{bmatrix} dp_0^u \\ dt_0 \\ dp \\ dt \\ dm \\ dy^g \\ dy_0^u \end{bmatrix} = 0. \tag{3.3}$$

If there exists a vector $(dp_0^u, dt_0, dp, dt, dm, dy^g, dy_0^u)$ that satisfies (3.1)-(3.3), then the equilibrium in question is not a Pareto optimum because there are feasible changes in prices and quantities that make every individual better off. If there are no such feasible changes, then the equilibrium is a Pareto optimum. In order to characterize the set of all unit-tax Pareto optima, we employ Motzkin's Theorem.[12]

We are at a Pareto optimum, that is, there is not solution to (3.1)-(3.3), if and only if there exist $0 \neq \bar{s} \geq 0$, $(\bar{v}_0, \bar{v}^T) \geq 0$, $\bar{r} \geq 0$ and $\bar{\beta}$, such that

$$\sum_h \bar{s}_h \bar{x}_0^h = -\bar{v}^T \nabla_{q_0} \bar{x} - \bar{v}_0 \nabla_{q_0} \bar{x}_0 + \bar{\beta}, \tag{3.4}$$

$$\sum_h \bar{s}_h \bar{x}_0^h = -\bar{v}^T \nabla_{q_0} \bar{x} - \bar{v}_0 \nabla_{q_0} \bar{x}_0 - \bar{\beta} \nabla_{t_0} \bar{P}_0^u, \tag{3.5}$$

$$\sum_h \bar{s}_h \bar{x}^{hT} = -\bar{v}^T \nabla_q \bar{x} + \bar{v}^T [\nabla_p \bar{y}^c - \nabla_p \bar{y}^m] - \bar{v}_0 \nabla_q^T \bar{x}_0 - \bar{\beta} \nabla_p^T \bar{P}_0^u, \tag{3.6}$$

$$\sum_h \bar{s}_h \bar{x}^{hT} = -\bar{v}^T \nabla_q \bar{x} - \bar{v}_0 \nabla_q^T \bar{x}_0 - \bar{\beta} \nabla_t^T \bar{P}_0^u, \tag{3.7}$$

[12] Motzkin's Theorem states that either $Ax \gg 0$, $Bx \geq 0$, and $Cx = 0$ has a solution or $s^T A + v^T B + w^T C = 0$, where $0 \neq s \geq 0$ and $v \geq 0$ has a solution but not both. See Mangasarian [1969] for a formal statement and proof.

$$\sum_h \bar{s}_h = \bar{v}^T \nabla_m \bar{x} + \bar{v}_0 \nabla_m \bar{x}_0 + \bar{\beta} \nabla_m \bar{P}_0^u, \tag{3.8}$$

$$\bar{v}^T = \bar{r} \nabla_{y^g}^T \bar{F}, \tag{3.9}$$

and

$$\bar{v}^T \nabla_{y_0^u} \bar{y}^m = \bar{v}_0. \tag{3.10}$$

This system of equations, (3.4)-(3.10), characterizes all possible unit-tax Pareto optima.[13] However, the system can be simplified considerably as follows.

First, subtract (3.5) from (3.4), and then using (2.5) we obtain $\bar{\beta} = 0$.[14] Next, subtract (3.7) from (3.6) to obtain

$$\bar{v}^T [\nabla_p \bar{y}^c - \nabla_p \bar{y}^m] = 0. \tag{3.11}$$

The rank condition, (2.9), guarantees that the zero eigenvector of $[\nabla_p \bar{y}^c - \nabla_p \bar{y}^m]$ is unique up to positive scalar multiplication. Thus (3.11) implies that

$$\bar{v}^T = \bar{\mu} \bar{p}^T. \tag{3.12}$$

Then (3.9) implies in conjunction with (3.12), that

$$\bar{p}^T \bar{\mu} = \bar{r} \nabla_{y^g}^T \bar{F}, \tag{3.13}$$

which implies that there is production efficiency in the production of competitive goods in the economy. The competitive firms face prices $\bar{p}$ and the monopolist buys inputs from that sector at prices $\bar{p}$. From (3.13) we see that

$$\frac{\bar{p}_i}{\bar{p}_j} = \frac{F_i(\bar{y}^g)}{F_j(\bar{y}^g)} \quad \text{for} \quad i, j = 1, \ldots, N, \tag{3.14}$$

so that the public sector firm is setting its marginal rates of transformation equal to the ratios of producer prices of the competitive goods. Also, from (3.10) and (3.12), we have

$$\bar{\mu} \nabla_{y_0^u} \bar{C} = \bar{v}_0, \tag{3.15}$$

that is, the marginal cost of the monopolist is proportional to the social shadow price $\bar{v}_0$ (the shadow price that would have prevailed under public production of the monopoly good.) Thus, overall production efficiency is implied by (3.13) and (3.15).

[13] Note that we employ some abbreviations: $\bar{C} = C(\bar{y}_0^u, \bar{p})$; $\bar{F} = F(\bar{y}^g)$;$\bar{\Pi}^c = \Pi^c(\bar{p})$.

[14] This is analogous to results in GL and shows that the constraint requiring the monopolist to maximize profits is not binding. This means that in searching for a Pareto optimum the planner can do no better than use the monopoly power itself because it levies one hundred per cent profit taxes and can rebate these profits to the consumers as a demogrant. Alternatively, the monopoly constraint not binding implies that if we were to endow production of the monopoly good to the public sector (which means no price making behavior or any behavioral constraint in the production of that good), then the public sector firm does no better than the monopolist.

To summarize the equations characterizing a unit-tax Pareto optimum, (3.4)-(3.10), can be rewritten using $\bar{\beta} = 0$ and (3.13) as

$$\sum_h \bar{s}_h \bar{x}_0^h = -\bar{\mu}\bar{p}^T \nabla_{q_0}\bar{x} - \bar{v}_0 \nabla_{q_0}\bar{x}_0, \tag{3.16}$$

$$\sum_h \bar{s}_h \bar{x}^{hT} = -\bar{\mu}\bar{p}^T \nabla_q \bar{x} - \bar{v}_0 \nabla_q \bar{x}_0, \tag{3.17}$$

$$\sum_h \bar{s}_h = \bar{\mu}\bar{p}^T \nabla_m \bar{x} + \bar{v}_0 \nabla_m \bar{x}_0, \tag{3.18}$$

$$\bar{p}^T = \bar{r}\frac{\nabla_{y^g}\bar{F}}{\bar{\mu}}, \tag{3.19}$$

and

$$\bar{p}^T \nabla_{y_0^u} \bar{C} = \frac{\bar{v}_0}{\bar{\mu}}, \tag{3.20}$$

where $0 \neq \bar{s} \geq 0$, $(\bar{v}_0, \bar{v}^T) \geq 0$, and $\bar{r} > 0$.

4. Unit Tax Equilibrium as an Ad Valorem Tax Equilibrium.

First, we convert an arbitrary unit-tax equilibrium into an equivalent ad valorem tax equilibrium. A monopoly ad-valorem tax equilibrium with 100% taxation of all profits and redistribution of governmental revenue as a demogrant, R, is given by

$$-x(q, q_0, R) + y^c(p) - y^m(p, y_0^a) - y^g \geq 0, \tag{4.1}$$

$$-x_0(q_0, q, R) + y_0^a \geq 0, \tag{4.2}$$

$$p_0^a - P_0^a(\tau_0, p, \tau, R) = 0, \text{ and} \tag{4.3}$$

$$F(y^g) - g \geq 0. \tag{4.4}$$

where

$$q = (I_N + \boldsymbol{\tau})p, \qquad q_0 = (1 + \tau_0)p_0^a, \tag{4.5}$$

and

$$\begin{aligned} &\langle P_0^a(p, \tau_0, \tau, R), \& Y_0^a(p, \tau_0, \tau, R)\rangle := \\ &\text{argmax}_{p_0^a, y_0^a}\{p_0^a y_0^a - C(y_0^a, p) \big| y_0^a \geq x_0(p_0^a(1 + \tau_0), p^T(I_N + \boldsymbol{\tau}), R)\}. \end{aligned} \tag{4.6}$$

Proceeding as in (2.16), Walras' law and the equilibrium conditions (4.1) to (4.3) (when they hold as equalities) imply that[15]

$$\Pi^{ma} + \Pi^c + \tau^T \mathbf{p} x + \tau_0 p_0^a x_0 - p^T y^g = HR, \tag{4.7}$$

or

$$\Pi^{ma} + \Pi^c + \tau^T \mathbf{p}[y^c - y^m - y^g] + \tau_0 p_0^a y_0^a - p^T y^g = HR, \tag{4.8}$$

[15] Note that $\mathbf{p}$ is the vector p diagonalized to be a matrix. Similarly, $\boldsymbol{\tau}$ is a diagonalized version of τ.

where

$$\Pi^{ma}(p_0^a, p, y_0^a) = p_0^a y_0^a - C(y_0^a, p). \tag{4.9}$$

Now, consider any unit-tax Pareto optimal equilibrium with 100% taxation of all profits and redistribution of governmental revenue as a demogrant $\langle \bar{p}_0^u, \bar{t}_0, \bar{p}, \bar{t}, \bar{y}^g, \bar{y}_0^u, \bar{m} \rangle$ characterized by

$$-\bar{x} + \bar{y}^c - \bar{y}^m - \bar{y}^g \geq 0, \tag{4.10}$$

$$-\bar{x}_0(\bar{q}_0, \bar{q}, \bar{m}) + \bar{y}_0^u \geq 0, \tag{4.11}$$

$$\bar{p}_0^u - P_0^u(\bar{p}, \bar{t}_0, \bar{t}, \bar{m}) = 0, \tag{4.12}$$

and

$$F(\bar{y}^g) - g \geq 0. \tag{4.13}$$

The consumer prices are given by $\bar{q} = \bar{t} + \bar{p}$ and $\bar{q}_0 = \bar{t}_0 + \bar{p}_0^u$.

In order to represent it as an ad valorem tax equilibrium, we define $\bar{\tau}, \bar{\tau}_0, \bar{p}_0^a, \bar{y}_0^a, \bar{R}$ as follows:

(i) $\bar{\tau} = \bar{t}^T \bar{\mathbf{p}}^{-1}$ so that $\bar{q} = (I_N + \bar{\boldsymbol{\tau}})\bar{p}$;

(ii) $\bar{R} = \bar{m}$;

and

(iii) $\bar{y}_0^a$, $\bar{\tau}_0$, and $\bar{p}_0^a$ solve

$$\begin{aligned} \bar{p}_0^a(1 + \bar{\tau}_0) &= \bar{q}_0, \\ \bar{p}_0^a &= P_0^a(\bar{p}, \bar{\tau}_0, \bar{\tau}, \bar{R}), \\ \bar{y}_0^a &= Y_0^a(\bar{p}, \bar{\tau}_0, \bar{\tau}, \bar{R}). \end{aligned} \tag{4.14}$$

From this it follows that

$$x(p_0^u + \bar{t}_0, \bar{p} + \bar{t}, \bar{m}) = x(\bar{p}_0^a(1 + \bar{\tau}_0), \bar{p}^T(I_N + \bar{\boldsymbol{\tau}}), \bar{R}) \tag{4.15}$$

and

$$x_0(\bar{p}_0^u + \bar{t}_0, \bar{p} + \bar{t}, \bar{m}) = x_0(\bar{p}_0^a(1 + \bar{\tau}_0), \bar{p}^T(I_N + \bar{\boldsymbol{\tau}}), \bar{R}) \tag{4.16}$$

so that

$$\begin{aligned} \bar{y}_0^a = \bar{y}_0^u &=: \bar{y}_0 \text{ and} \\ \nabla_p C(\bar{y}_0, \bar{p}) &= \bar{y}^m. \end{aligned} \tag{4.17}$$

From the above definitions and (2.15), it follows that

$$\begin{aligned} &\Pi^{ma} + \Pi^c + \bar{\tau}^T \bar{\mathbf{p}}[\bar{y}^c - \bar{y}^m - \bar{y}^g] + \bar{\tau}_0 \bar{p}_0^a \bar{y}_0 - \bar{p}^T \bar{y}^g \\ &= \bar{p}_0^a \bar{y}_0 + \bar{\tau}_0 \bar{p}_0^a \bar{y}_0 - C(\bar{y}, \bar{p}) + \Pi^c + \bar{\tau}^T \bar{\mathbf{p}}[\bar{y}^c - \bar{y}^m - \bar{y}^g] - \bar{p}^T \bar{y}^g \\ &= \bar{p}_0^a(1 + \tau_0^a)\bar{y}_0 - C(\bar{y}, \bar{p}) + \Pi^c + \bar{t}^T[\bar{y}^c - \bar{y}^m - \bar{y}^g] - \bar{p}^T \bar{y}^g \\ &= (\bar{p}_0^u + \bar{t}_0^u)\bar{y}_0 - C(\bar{y}, \bar{p}) + \Pi^c + \bar{t}^T[\bar{y}^c - \bar{y}^m - \bar{y}^g] - \bar{p}^T \bar{y}^g \\ &= \Pi^{mu} + \Pi^c + \bar{t}[\bar{y}^c - \bar{y}^m - \bar{y}^g] + \bar{t}_0 \bar{y}_0 - \bar{p}\bar{y}^g \\ &= H\bar{m} = H\bar{R}. \end{aligned} \tag{4.18}$$

Thus, the above defined $\bar{\tau}, \bar{\tau}_0, \bar{p}_0^a, \bar{y}_0^a, \bar{R}$ solve (4.1) to (4.9) and hence they define a monopoly ad-valorem tax equilibrium with 100% taxation of all profits and redistribution of governmental revenue as a demogrant. Hence, the Pareto optimal unit tax equilibrium described by (4.10) to (4.13), can also be interpreted as an ad valorem tax equilibrium.

Although we know that this is an ad valorem-tax equilibrium, we do not know if it is Pareto optimal with respect to ad valorem tax changes. In order to accomplish this, we assume that the inequalities in the ad valorem equilibrium, (4.1) to (4.6) hold with equality and ask if there exist ad valorem Pareto-improving, equilibrium-preserving directions of change. If there are no such changes, then every unit-tax Pareto optimum is also an ad valorem-tax Pareto optimum.

The instruments under the control of the tax authority are given by $\nu = \langle dp_0^a, d\tau_0, dp, d\tau, dR, dy^g, dy_0^a \rangle$, and must be chosen so that ν is Pareto improving, equilibrium preserving, ensures that wealth continues to be distributed in equal proportion, that the monopolist is maximizing profit, and that the requisite amount of the public good is being produced.

ν is a Pareto-improving direction of change if and only if

$$\begin{bmatrix} -\bar{x}_0^1(1+\bar{\tau}_0) & -\bar{x}_0^1 p_0^a & -\bar{x}^{1T}(I_N+\bar{\boldsymbol{\tau}}) & -\bar{x}^{1T}\bar{\mathbf{p}} & 1 & 0_N^T & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\bar{x}_0^H(1+\bar{\tau}_0) & -\bar{x}_0^H p_0^a & -\bar{x}^{HT}(I_N+\bar{\boldsymbol{\tau}}) & -\bar{x}^{HT}\bar{\mathbf{p}} & 1 & 0_N^T & 0 \end{bmatrix} \nu \gg 0; \tag{4.19}$$

ν is an equilibrium-preserving direction of change that guarantees the production of the public good if and only if

$$[\, M^a \quad N^a \,]\, \nu \geq 0 \tag{4.20}$$

where

$$M^a = \begin{bmatrix} -\nabla_{q_0}\bar{x}(1+\bar{\tau}_0) & -\nabla_{q_0}\bar{x}p_0^a & -\nabla_q\bar{x}(I_N+\bar{\boldsymbol{\tau}}) + [\nabla_p\bar{y}^c - \nabla_p\bar{y}^m] & -\nabla_q\bar{x}\bar{\mathbf{p}} \\ -\nabla_{q_0}\bar{x}_0(1+\bar{\tau}_0) & -\nabla_{q_0}\bar{x}_0 p_0^a & -\nabla_q^T\bar{x}_0(I_N+\bar{\boldsymbol{\tau}}) & -\nabla_q^T\bar{x}_0\bar{\mathbf{p}} \\ 0 & 0 & 0_N^T & 0_N^T \end{bmatrix} \tag{4.21}$$

and

$$N^a = \begin{bmatrix} -\nabla_R\bar{x} & -I_N & -\nabla_{y_0}\bar{y}^m \\ -\nabla_R\bar{x}_0 & 0_N^T & 1 \\ 0 & \nabla_{y^g}^T F & 0 \end{bmatrix}; \tag{4.22}$$

and ν is an ad valorem tax profit-maximizing preserving directions of change if and only if

$$\begin{bmatrix} 1 & -\nabla_{\tau_0}\bar{P}_0^a & -\nabla_p\bar{P}_0^a & -\nabla_\tau\bar{P}_0^a & -\nabla_R\bar{P}_0^a & 0_N^T & 0 \end{bmatrix} \nu = 0. \tag{4.23}$$

If there exist Pareto-improving equilibrium-preserving directions of change that satisfy the above, then ad valorem taxation Pareto dominates unit taxation. However, if there do not exist such directions of change then every unit tax Pareto optimum is also an ad valorem tax Pareto optimum. In fact, we show that the converse is true as well.

Theorem A: *If there is complete taxation of all profits–both monopoly and the competitive sector–and if government revenue is rebated equally by means of a demogrant, then every unit-tax Pareto optimum is also an ad valorem-tax Pareto optimum and conversely.*

Proof: Using Motzkin's Theorem, there are no ad valorem directions of change satisfying (4.19) to (4.23) at the unit-tax optimum if and only if there exist $0 \neq \tilde{s}^T \geq 0$, $(\tilde{v}_0, \tilde{v}^T) \geq 0$, $\tilde{\beta}$, and $\tilde{r} \geq 0$ such that

$$\sum_h \tilde{s}_h \bar{x}_0^h (1+\bar{\tau}_0) = -\tilde{v}^T \nabla_{q_0} \bar{x}(1+\bar{\tau}_0) - \tilde{v}_0 \nabla_{q_0} \bar{x}_0 (1+\bar{\tau}_0) + \tilde{\beta}, \tag{4.24}$$

$$\sum_h \tilde{s}_h \bar{x}_0^h p_0^a = -\tilde{v}^T \nabla_{q_0} \bar{x} p_0^a - \tilde{v}_0 \nabla_{q_0} \bar{x}_0 p_0^a - \tilde{\beta} \nabla_{q_0} \bar{P}_0^a, \tag{4.25}$$

$$\sum_h \tilde{s}_h \bar{x}^{hT} (I_N + \bar{\boldsymbol{\tau}}) = -\tilde{v}^T \nabla_q \bar{x}(I_N + \bar{\boldsymbol{\tau}}) + \tilde{v}^T [\nabla_p \bar{y}^c - \nabla_p \bar{y}_m] - \tilde{v}_0 \nabla_q^T \bar{x}_0 (I_N + \bar{\boldsymbol{\tau}}) - \tilde{\beta} \nabla_p \bar{P}_0^a, \tag{4.26}$$

$$\sum_h \tilde{s}_h \bar{x}^{hT} \bar{\mathbf{p}} = -\tilde{v}^T \nabla_q \bar{x} \bar{\mathbf{p}} - \tilde{v}_0 \nabla_q \bar{x}_0 \bar{\mathbf{p}} - \tilde{\beta} \nabla_\tau \bar{P}_0^a, \tag{4.27}$$

$$\sum_h \tilde{s}_h = \tilde{v}^T \nabla_R \bar{x} + \tilde{v}_0 \nabla_R \bar{x}_0 + \tilde{\beta} \nabla_R \bar{P}_0^a, \tag{4.28}$$

$$\tilde{v}^T = \tilde{r} \nabla_{y^g}^T \bar{F}, \tag{4.29}$$

and

$$\tilde{v}^T \nabla_{y_0} \bar{y}^m = \tilde{v}_0. \tag{4.30}$$

Suppose then that (4.24)-(4.30) have a solution with $0 \neq \tilde{s} \geq 0$, $(\tilde{v}_0, \tilde{v}) \geq 0$, $\tilde{\beta}$ and $\tilde{r} \geq 0$. If this turns out to be correct then there are no ad valorem Pareto-improving equilibrium-preserving directions of change. If this assumptions leads to a contradiction then such changes exist and in that case ad valorem taxation would dominate unit taxation in this model. We first simplify (4.24)-(4.30) as follows.

First multiply (4.24) by $\bar{p}_0^a$ and (4.25) by $1 + \bar{\tau}_0$ and then subtract (4.25) from (4.24) to obtain

$$\tilde{\beta}(\bar{p}_0^a + (1+\bar{\tau}_0) \nabla_{\tau_0} \bar{P}_0^a) = 0. \tag{4.31}$$

Assuming that the monopolist cannot undo all taxes, (2.5), we have $\bar{p}_0^a + (1+\bar{\tau}_0) \nabla_{\tau_0} \bar{P}_0^a \neq 0$, which implies that $\tilde{\beta} = 0$.[16]

Next multiply (4.26) by $[I_N + \bar{\boldsymbol{\tau}}]^{-1}$ and subtract (4.27) from (4.26) after multiplying (4.27) by $\bar{\mathbf{p}}^{-1}$ to obtain after some manipulation

$$\tilde{v}^T [\nabla_p \bar{y}^c - \nabla_p \bar{y}^m] = 0 \tag{4.32}$$

[16] The consumer price, $\bar{q}_0$, is locally constant, $\bar{q}_0 = (1+\bar{\tau}_0)\bar{P}_0^a$. Hence, $dq = 0$ implies $\bar{p}_0^a + (1+\bar{\tau}_0)\nabla_{\bar{\tau}_0} \bar{P}_0^a = 0$.

which in turn implies, using the rank condition, that

$$\tilde{v}^T = \tilde{\mu}\bar{p}^T. \tag{4.33}$$

Now substitute (4.33) into (4.29) to obtain

$$\bar{p}^T = \tilde{r}\frac{\nabla_{y^g}\bar{F}}{\tilde{\mu}} \tag{4.34}$$

which shows that there is production efficiency in the competitive sector. Using (4.33), we can rewrite (4.30) as

$$\tilde{\mu}^T\nabla_{y_0}\bar{C} = \tilde{v}_0. \tag{4.35}$$

(4.35) together with (4.34) establishes overall production efficiency.

Using $\tilde{\beta} = 0$ and (4.33) we can rewrite (4.24)-(4.30) as

$$\sum_h \tilde{s}_h\bar{x}_0^h = -\tilde{\mu}\bar{p}^T\nabla_{q_0}\bar{x} - \tilde{v}_0\nabla_{q_0}\bar{x}_0, \tag{4.36}$$

$$\sum_h \tilde{s}_h\bar{x}^{hT} = -\tilde{\mu}\bar{p}^T\nabla_q\bar{x} - \tilde{v}_0\nabla_q\bar{x}_0, \tag{4.37}$$

$$\sum_h \tilde{s}_h = \tilde{\mu}\bar{p}^T\nabla_R\bar{x} + \bar{v}_0\nabla_R\bar{x}_0, \tag{4.38}$$

$$\bar{p}^T = \tilde{r}\frac{\nabla_{y^g}\bar{F}}{\tilde{\mu}}, \tag{4.39}$$

and

$$\nabla_{y_0}\bar{C} = \frac{\bar{v}_0}{\tilde{\mu}}. \tag{4.40}$$

If (4.36)-(4.40) have a solution satisfying the non-negativity and semi-positivity conditions stated at the beginning of the proof, then there are no ad valorem feasible Pareto improving directions of change and hence the unit tax Pareto optimum that we began with is also an ad valorem Pareto optimum.

Looking at the unit-tax Pareto optimum, (3.16)-(3.20), it is obvious that setting $\tilde{s} = \bar{s}$, $\tilde{v}_0 = \bar{v}_0$, $\tilde{\mu} = \bar{\mu}$, and $\tilde{r} = \bar{r}$ solve equations (4.36)-(4.40) and hence there are no ad valorem Pareto-improving equilibrium-preserving changes. Thus, we have shown that every unit-tax Pareto optimum is also an ad valorem-tax Pareto optimum.

We can also show that every ad valorem-tax Pareto optimum is a unit-tax Pareto optimum. Suppose $\langle\bar{p}_0^a, \bar{\tau}_0, \bar{p}, \bar{\tau}, \bar{y}^g, \bar{y}_0^a, \bar{R}\rangle$ is an ad valorem Pareto optimum and it therefore solves (4.36)-(4.40). The consumer prices are $\bar{q} = (I_N + \bar{\boldsymbol{\tau}})\bar{p}$ and $\bar{q}_0 = (1 + \bar{\tau}_0)p_0^a$. Define $\bar{t} = \bar{\mathbf{p}}\bar{\tau}$ so that $\bar{q} = \bar{t} + \bar{p}$; $\bar{m} = \bar{R}$; and $\bar{t}_0, \bar{p}_0^u$, and $\bar{y}_0^u$ such that they solve

$$\begin{aligned}
\bar{q}_0 &= \bar{p}_0^u + \bar{t}_0, \\
\bar{p}_0^u &= P_0^u(\bar{p}, \bar{t}, \bar{t}_0, \bar{m}), \text{ and} \\
\bar{y}_0^u &= Y_0^u(\bar{p}, \bar{t}, \bar{t}_0, \bar{m}).
\end{aligned} \tag{4.41}$$

It can be seen that $\langle \bar{p}_0^u, \bar{t}_0, \bar{p}, \bar{t}, \bar{y}^g, \bar{y}_0^u, \bar{m} \rangle$ is a unit-tax equilibrium. Since we are at an ad valorem optimum, there exist $0 \neq \tilde{s}^T \geq 0$, $(\tilde{v}_0, \tilde{v}^T) \geq 0$, $\tilde{\mu} > 0$, and $\tilde{r} \geq 0$ which solve (4.36) to (4.40). The multipliers $\bar{s} = \tilde{s}$, $\bar{v}_0 = \tilde{v}_0$, $\bar{\mu} = \tilde{\mu}$, and $\bar{r} = \tilde{r}$ solve equations (3.16) to (3.20). Thus, the ad valorem-tax optimum is also a unit-tax optimum. ■

5. Conclusion

We have shown in a general equilibrium model with monopoly that ad valorem and unit taxation are equivalent given that the government has the power to levy one hundred per cent profit taxes and rebates its revenue, after paying for the public good inputs, to the consumers as a demogrant. Although this probably comes as a surprise to some readers it is easy to understand once one observes (4.18). Moving from unit to ad valorem taxation, the monopolist's profit maximizing price falls; as the consumer price and quantity are fixed, this means that the government's indirect tax revenue from the monopolist rises. However, the profits of the monopolist fall by exactly the same amount so that government revenue remains constant. Thus, the demogrant to each consumer remains unchanged. In short, the particular set of policy instruments considered in the model above ensures that the equilibrium allocations attainable under a unit-tax regime and an ad valorem-tax regime are identical.

Although Wicksell and Suits and Musgrave stated that ad valorem taxation welfare dominated unit taxes, they did not seriously attempt to justify this claim. Skeath and Trandel and Keen did claim that ad valorem taxes Pareto dominated unit taxes. Their error comes from believing that standard surplus argument could be trusted to yield actual Pareto improvements; this is of course almost always incorrect.

Some readers may be bothered by the assumption of one hundred per cent profit taxation. However, letting there be private ownership of the monopoly presents an entirely new set of (as yet) unresolved problems in addition to leading possibly to production inefficiencies. Suppose that the government continues to distribute its revenue equally by a demogrant but that each consumer owns a share θ_h of the monopoly. Now, when the profit of the monopolist changes due to the change to ad valorem taxes so does the private income of consumer h. If θ_h is different from $1/H$ then the consumer income in the ad valorem-tax situation will be different than in the unit-tax situation. Hence, the unit-tax equilibrium cannot be reproduced as an ad valorem-tax equilibrium without changing the distribution of wealth. This may be clearer if one thinks of the special case of a representative consumer where all preferences satisfy the Gorman polar form and have parallel Engel curves.[17] In that case the marginal expenditure patterns of all agents are identical. Then, switching from unit to ad valorem taxation yields the same equilibrium prices and aggregate quantities under the two regimes. However, some individuals are worse off and some are better off. Neither Pareto dominates the other. In the general private ownership case, one would like to know if the utility possibility frontiers of the two tax regimes are identical, cross one another, or dominate; as argued above, this cannot be demonstrated by the means employed here.

[17] See Gorman [1953, 1961].

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