

# Farsightedly Basic Networks

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## Abstract

We make two main contributions to the theory of network formation. First, we introduce a new notion of farsighted equilibrium in network formation. In particular, we introduce the notion of a *farsightedly basic network*, as well as the notion of a *farsighted basis*. Given the feasible set of networks, the rules governing network formation, and the preferences of individuals, a farsightedly basic network contained in the farsighted basis represents a possible final resting point (or absorbing state) of a network formation process in which agents behave farsightedly. Thus, the set of networks forming a farsighted basis contains networks which are likely to emerge and persist if individuals behave farsightedly. Second, we show that given any set of rules governing network formation and given any profile of individual preferences over the feasible set of networks, there exists a nonempty subset of networks forming a farsighted basis of the network formation process.

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# 1 Introduction

## *Overview*

In many economic and social situations, the totality of interactions between individuals and coalitions can be modeled as a network. In this paper, we analyze the problem of equilibrium in the process of network formation assuming that individuals are farsighted; that is, assuming that individuals are concerned with the eventual consequences of their immediate actions in forming connections and interacting with other individuals. To focus the analysis, we address the following question: given the rules governing network formation (i.e., the rules governing what connections and interactions are possible), and given the preferences of the individuals, what networks are likely to emerge and persist if individuals are farsighted? One possible approach to this question - an approach introduced by Page, Wooders, and Kamat (2001) and the approach taken here - is to think of each possible network representation of individual connections and interactions as a node in a larger network in which the arcs represent coalitional preferences over networks and possible coalitional moves from one network to another. We call such a network of networks (or network formation network) a *supernetwork*. Given the supernetwork representation of the network formation process, we then define a farsighted dominance relation on the networks composing the nodes of the supernetwork and using this supernetwork-determined notion of farsighted dominance, we address the issue of farsighted equilibrium in network formation. We make two main contributions. First, we introduce a new notion of farsighted equilibrium in network formation. In particular, we introduce the notion of a *farsightedly basic network*, as well as the notion of a *farsighted basis*. Given the structure of the supernetwork representing the rules governing network formation and the preferences of individuals, a farsightedly basic network contained in the farsighted basis represents a possible final resting point (or absorbing state) of the network formation process in which agents behave farsightedly. Thus, the farsighted basis of the supernetwork contains networks which are likely to emerge and persist if individuals behave farsightedly. Second, we show that all supernetworks possess a nonempty farsighted basis (i.e., possess a nonempty subset of networks forming a farsighted basis for the network formation process).

A key ingredient in establishing that all supernetworks possess a farsighted basis is the notion of an *inductive supernetwork*. A supernetwork is said to be inductive if given any farsighted domination path through the supernetwork there exists a network (i.e., a node in the supernetwork) which is reachable via a farsighted domination path from every network on the given path.<sup>1</sup> We show that all supernetworks are inductive. Then, by a straightforward application of a classical result due to Berge (1958), we are able to conclude that all supernetworks possess a farsighted basis. Formally, a set of networks is said to be a farsighted basis for the supernetwork, if given any two distinct networks in the basis there exists no finite farsighted domination path connecting the two networks and if for any network not contained in the basis,

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<sup>1</sup>A farsighted domination path is a sequence of networks,  $\{G_k\}_k$ , forming a path through the supernetwork such for all  $k$ , network  $G_k$  farsightedly dominates network  $G_{k-1}$ .

there exists a finite farsighted domination path from this network to some network contained in the basis. Thus, a farsighted basis is simply a von Neumann-Morgenstern stable set with respect to the relation defined via the farsighted domination paths on the set networks composing the nodes of the supernetwork.<sup>2</sup>

We also show that given any supernetwork, any farsighted basis is a subset of the largest consistent set of networks. Consistency with respect to farsighted dominance and the notion of a largest consistent set were introduced by Chwe (1994) in an abstract game setting. Here we extend the notion of farsighted consistency and the largest consistent set to network formation. Given agent preferences and the rules governing network formation, as represented via the supernetwork, a network is said to be farsightedly consistent if no agent or coalition of agents is willing to alter the network (via the addition, subtraction, or replacement of arcs) for fear that such an alteration might induce further network alterations by other agents or coalitions that in the end leave the initially deviating agent or coalition no better off - and possibly worse off. Our notion of a farsighted basis represents a refinement of our network rendition of the largest consistent set. Moreover, our result establishing that all supernetworks possess a farsighted basis, together with our result showing that any farsighted basis is a subset of the largest consistent set, imply that for any supernetwork the largest consistent set of networks is nonempty.

In order to illustrate the utility of our framework, we apply our supernetwork framework to a three-agent version of the co-author model of Jackson and Wolinsky (1996) and we compute the largest consistent set for two different supernetworks corresponding to the collection of directed co-author networks.<sup>3</sup> Jackson and Wolinsky show that, in general, a pairwise stable co-author network can be partitioned into fully intracommunity components, each of which has a different number of members (see Proposition 4, part (ii) in Jackson and Wolinsky (1996)).<sup>4</sup> In our example, a similar conclusion can be drawn: each farsightedly consistent network in each of the two co-author supernetworks can be partitioned into fully intracommunity components, each of which has a different number of members. Also, in both of our examples, each farsightedly consistent co-author network is Nash as well as Pareto optimal.

#### *Directed Networks vs Linking Networks*

We focus on directed networks, and in fact, we extend the definition of directed networks found in the literature. In a directed network, each arc possesses an orientation or direction: arc  $j$  connecting nodes  $i$  and  $i'$  must either go from node  $i$  to node  $i'$  or must go from node  $i'$  to node  $i$ .<sup>5</sup> In an undirected (or linking) network, arc

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<sup>2</sup>Put differently, a farsighted basis is a von Neumann-Morgenstern stable set with respect to the transitive closure of the farsighted dominance relation.

<sup>3</sup>It would have been preferable to have included a larger number of agents in our example. However, moving beyond three agents, the number of co-author networks in the supernetwork increases dramatically, making the computation very difficult. The computation of a farsighted basis remains an open question.

<sup>4</sup>A network is pairwise stable if for each pair of agents directly connected via an arc in the network both agents weakly prefer to remain directly connected, and if for each pair of agents not directly connected, a direct connection preferred by one of the agents makes the other agent strictly worse off (i.e., if one agent prefers to be directly connected, the other does not).

<sup>5</sup>We denote arc  $j$  going from node  $i$  to node  $i'$  via the ordered pair  $(j, (i, i'))$ , where  $(i, i')$  is also

$j$  would have no orientation and would simply indicate a connection or link between nodes  $i$  and  $i'$ . Under our extended definition of directed networks, nodes are allowed to be connected by multiple arcs. For example, nodes  $i$  and  $i'$  might be connected by arcs  $j$  and  $j'$ , with arc  $j$  running from node  $i$  to  $i'$  and arc  $j'$  running in the opposite direction (i.e., from node  $i'$  to node  $i$ ).<sup>6</sup> Thus, if node  $i$  represents a seller and node  $i'$  a buyer, then arc  $j$  might represent a contract offer by the seller to the buyer, while arc  $j'$  might represent the acceptance or rejection of that contract offer. Also, under our extended definition loops are allowed and arcs are allowed to be used multiple times in a given network.<sup>7</sup> For example, arc  $j$  might be used to connect nodes  $i$  and  $i'$  as well as nodes  $i'$  and  $i''$ . However, we do not allow arc  $j$  to go from node  $i$  to node  $i'$  multiple times in the same direction. By allowing arcs to possess direction and be used multiple times and by allowing nodes to be connected by multiple arcs, our extended definition makes possible the application of networks to a richer set of economic environments. Until now, most of the economic literature on networks has focused on linking networks (see for example, Jackson and Wolinsky (1996) and Dutta and Mutuswami (1997)).

Given a particular directed network, an agent or a coalition of agents can change the network to another network by simply adding, subtracting, or replacing arcs from the existing network in accordance with certain rules represented via the supernetwork.<sup>8</sup> For example, if the nodes in a network represent agents, then the rule for adding an arc  $j$  from node  $i$  to node  $i'$  might require that both agents  $i$  and  $i'$  agree to add arc  $j$ . Whereas the rule for subtracting arc  $j$ , from node  $i$  to node  $i'$ , might require that only agent  $i$  or agent  $i'$  agree to dissolve arc  $j$ . This particular set of rules has been used, for example, by Jackson and Wolinsky (1996). Other rules are possible. For example, the addition of an arc might require that a simple majority of the agents agree to the addition, while the removal an arc might require that a two-thirds majority agree to the removal. Given the flexibility of the supernetwork framework, any set rules governing network formation can be represented.

#### *Other Approaches and Related Literature*

While the literature on stability in networks is well established and growing (e.g., see Dutta and Mutuswami (1997), Jackson (2001), and Jackson and van den Nouweland (2001)), the literature on farsighted stability in network formation is in its infancy. As far as we know, Watts (2002) and Page, Wooders, and Kamat (2001) are the first papers in the literature to address non-myopic behavior in strategic network formation, while Page, Wooders, and Kamat (2001) is the first to address explicitly the issue of farsighted stability in network formation. Since then, other papers have appeared focusing on non-myopic behavior in network formation. Most notable are the papers by Deroian (2003), and Dutta et al (2003).

The papers by Watts (2002), Deroian (2003), and Dutta et al (2003) differ from

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an ordered pair. Alternatively, if arc  $j$  goes from node  $i'$  to node  $i$ , we write  $(j, (i', i))$ .

<sup>6</sup>Under our extended definition, arc  $j'$  might also run in the same direction as arc  $j$ .

<sup>7</sup>A loop is an arc going *from* a given node *to* that same node. For example, given arc  $j$  and node  $i$ , the ordered pair  $(j, (i, i))$  is a loop.

<sup>8</sup>Put differently, agents can change one network to another network by adding, subtracting, or replacing ordered pairs,  $(j, (i, i'))$ , in accordance with certain rules.

our paper in at least two respects: (1) all three papers analyze the non-myopic formation of linking networks (rather than directed networks) and assume that network formation takes place one link at a time in accordance with a given set of rules usually requiring link addition to be bilateral while allowing link subtraction to be unilateral (thus, in all three of these papers a particular set of network formation rules is assumed); (2) all three papers utilize a notion of farsightedness and farsighted dominance that differs from the notion we use. Given the flexibility of the supernetwork framework, any set rules governing network formation can be represented. For example, the addition of an arc might require that a simple majority of the agents agree to the addition, while the removal an arc might require that a two-thirds majority agree to the removal.<sup>9</sup> Thus, under our approach, all types of individual or coalitional moves from one network to another are possible and moreover, under our approach, the future consequences of such moves are fully taken into account.<sup>10</sup>

## 2 Directed Networks

We begin by giving a formal definition of the class of directed networks we shall consider. Let  $N$  be a finite set of nodes, with typical element denoted by  $i$ , and let  $A$  be a finite set of arcs, with typical element denoted by  $j$ . Arcs represent potential connections between nodes, and depending on the application, nodes can represent economic agents or economic objects such as markets or firms.<sup>11</sup>

### **Definition 1** (*Directed Networks*)

*Given node set  $N$  and arc set  $A$ , a directed network,  $G$ , is a subset of  $A \times (N \times N)$ . We shall denote by  $\mathbb{N}(N, A)$  the collection of all directed networks given  $N$  and  $A$ .*

A directed network  $G \in \mathbb{N}(N, A)$  specifies how the nodes in  $N$  are connected via the arcs in  $A$ . Note that in a directed network order matters. In particular, if  $(j, (i, i')) \in G$ , this means that arc  $j$  goes from node  $i$  to node  $i'$ . Also, note that under our definition of a directed network, loops are allowed - that is, we allow an arc to go from a given node back to that given node. Finally, note that under our definition an arc can be used multiple times in a given network and multiple arcs can go from one node to another. However, our definition does not allow an arc  $j$  to go from a node  $i$  to a node  $i'$  multiple times.

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<sup>9</sup>Majority addition and two-thirds majority subtraction rules might arise naturally in agenda formation networks where agendas are represented by nodes and moves from one agenda to another are represented by arcs.

<sup>10</sup>Other important works on dynamic network formation which do not explicitly address the issue of farsightedness are, for example, Watts (2001) and Jackson and Watts (2001).

<sup>11</sup>Of course in a supernetwork, nodes represent networks.

The following notation is useful in describing networks. Given directed network  $G \subseteq A \times (N \times N)$ , let

$$\left. \begin{aligned} G(j) &:= \left\{ (i, i') \in N \times N : (j, (i, i')) \in G \right\}, \\ G(i) &:= \left\{ j \in A : (j, (i, i')) \in G \text{ or } (j, (i', i)) \in G \right\} \\ G(i, i') &:= \left\{ j \in A : (j, (i, i')) \in G \right\}, \\ G(j, i) &:= \left\{ i' \in N : (j, (i, i')) \in G \right\}. \end{aligned} \right\} \quad (1)$$

Thus,

$G(j)$  is the *set of node pairs* connected by arc  $j$  in network  $G$ ,  
 $G(i)$  is the *set of arcs* going from node  $i$  or coming to node  $i$  in network  $G$ ,  
 $G(i, i')$  is the *set of arcs* going from node  $i$  to node  $i'$  in network  $G$ ,  
and  
 $G(j, i)$  is the *set of nodes* which can be reached by arc  $j$  from node  $i$  in network  $G$ .

Note that if for some arc  $j \in A$ ,  $G(j)$  is empty, then arc  $j$  is not used in network  $G$ . Moreover, if for some node  $i \in N$ ,  $G(i)$  is empty then node  $i$  is not used in network  $G$ , and node  $i$  is said to be isolated relative to network  $G$ .

If in our definition of a directed network, we had required that  $G(j)$  be single-valued and nonempty for all arcs  $j \in A$ , then our definition would have been the same as that given by Rockafellar (1984).

Suppose that the node set  $N$  is given by  $N = \{i_1, i_2, \dots, i_5\}$ , while the arc set  $A$  is given by  $A = \{j_1, j_2, \dots, j_5, j_6, j_7\}$ . Consider the network,  $G$ , depicted in Figure 1.

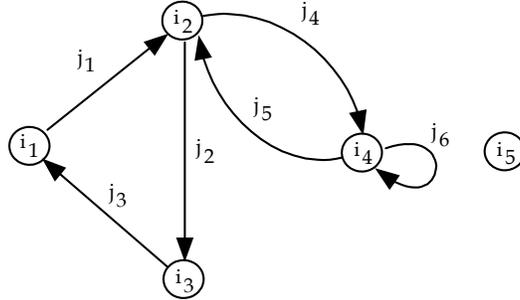


Figure 1: Network  $G$

In network  $G$ ,  $G(j_6) = \{(i_4, i_4)\}$ . Thus,  $(j_6, (i_4, i_4)) \in G$  is a loop. Also, in network  $G$ , arc  $j_7$  is not used. Thus,  $G(j_7) = \emptyset$ .<sup>12</sup> Finally, note that  $G(i_4) = \{j_4, j_5, j_6\}$ , while

<sup>12</sup>The fact that arc  $j_7$  is not used in network  $G$  can also be denoted by writing

$$j_7 \notin \text{proj}_A G,$$

$G(i_5) = \emptyset$ . Thus, node  $i_5$  is *isolated* relative to  $G$ .<sup>13</sup>

Consider the new network,  $G' \in \mathbb{N}(N, A)$  depicted in Figure 2.

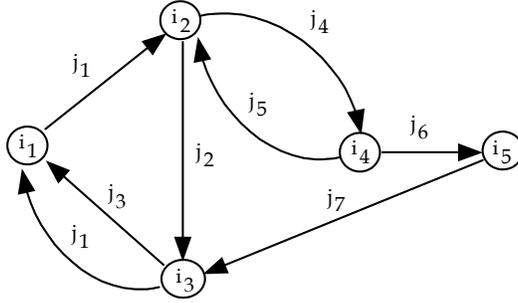


Figure 2: Network  $G'$

In network  $G'$ ,  $G'(j_1) = \{(i_1, i_2), (i_3, i_1)\}$ . Thus,  $(j_1, (i_1, i_2)) \in G'$  and  $(j_1, (i_3, i_1)) \in G'$ . Note that in network  $G'$ , node  $i_5$  is no longer isolated. In particular,  $G'(i_5) = \{j_6, j_7\}$ . Also, note that nodes  $i_2$  and  $i_4$  are connected by two different arcs pointed in opposite directions. Under our definition of a directed network it would be possible to create a new network from network  $G'$  by replacing arc  $j_5$  from  $i_4$  to  $i_2$  with arc  $j_4$  from  $i_4$  to  $i_2$ . However, it would *not* be possible under our definition to create a new network by replacing arc  $j_5$  from  $i_4$  to  $i_2$  with arc  $j_4$  from  $i_2$  to  $i_4$  - because our definition does not allow  $j_4$  to go from  $i_2$  to  $i_4$  multiple times. Finally, note that nodes  $i_1$  and  $i_3$  are also connected by two different arcs, but arcs pointed in the same direction. In particular,  $G(i_3, i_1) = \{j_1, j_3\}$ .

**Remark:**

Under our extended definition of a directed network, a directed graph or digraph, as it is sometimes called in the graph theory literature, can be viewed as a special case of a directed network. A directed graph consists of a pair,  $(N, E)$ , where  $N$  is a nonempty set of nodes or vertices and  $E$  is a nonempty set of ordered pairs of nodes. Given node set  $N$ , arc set  $A$ , and directed network  $G \in \mathbb{N}(N, A)$ , for each arc  $j \in A$ ,  $(N, G(j))$  is a directed graph where, recall from expression (1) above,  $G(j)$  is the set of ordered pairs of nodes connected by arc  $j$ , given by

$$G(j) := \{(i, i') \in N \times N : (j, (i, i')) \in G\}.$$

Thus, a directed network is a collection of directed graphs where each directed graph is labelled by a particular arc.

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where  $proj_A G$  denotes the projection onto  $A$  of the subset

$$G \subseteq A \times (N \times N)$$

representing the network.

<sup>13</sup>If the loop  $(j_7, (i_5, i_5))$  were part of network  $G$  in Figure 1, then node  $i_5$  would no longer be considered isolated under our definition. Moreover, we would have  $G(i_5) = \{j_7\}$ .

### 3 Supernetworks

Let  $D$  denote a nonempty set of agents (or economic decision making units) with typical element denoted by  $d$ , and let  $\Gamma(D)$  denote the collection of all *nonempty* subsets (or coalitions) of  $D$  with typical element denoted by  $S$ .

Given collection of directed networks  $\mathbb{G} \subseteq \mathbb{N}(N, A)$ , we shall assume that each agent's preferences over networks in  $\mathbb{G}$  are specified via a network payoff function,

$$v_d(\cdot) : \mathbb{G} \rightarrow \mathbb{R}.$$

For each agent  $d \in D$  and each directed network  $G \in \mathbb{G}$ ,  $v_d(G)$  is the payoff to agent  $d$  in network  $G$ . Agent  $d$  then prefers network  $G'$  to network  $G$  if and only if

$$v_d(G') > v_d(G).$$

Moreover, coalition  $S' \in \Gamma(D)$  prefers network  $G'$  to network  $G$  if and only if

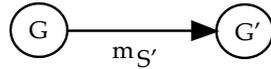
$$v_d(G') > v_d(G) \text{ for all } d \in S'.$$

Note that the payoff function of an agent depends on the entire network. Thus, the agent may be affected by directed links between other agents even when he himself has no direct or indirect connection with those agents. Intuitively, 'widespread' network externalities are allowed.

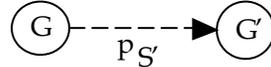
By viewing each network  $G$  in a given collection of directed networks  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  as a node in a larger network, we can give a precise network representation of the rules governing network formation as well as agents' preferences. To begin, let

$$\begin{aligned} \mathbb{M} &:= \{m_S : S \in \Gamma(D)\} \text{ denote the set of move arcs (or } m\text{-arcs for short),} \\ \mathbb{P} &:= \{p_S : S \in \Gamma(D)\} \text{ denote the set of preference arcs (or } p\text{-arcs for short),} \\ &\text{and} \\ \mathbb{A} &:= \mathbb{M} \cup \mathbb{P}. \end{aligned}$$

Given networks  $G$  and  $G'$  in  $\mathbb{G}$ , we shall denote by



(i.e., by an  $m$ -arc, belonging to coalition  $S'$ , going from node  $G$  to node  $G'$ ) the fact that coalition  $S' \in \Gamma(D)$  can change network  $G$  to network  $G'$  by adding, subtracting, or replacing arcs in network  $G$ . Moreover, we shall denote by



(i.e., by a  $p$ -arc, belonging to coalition  $S'$ , going from node  $G$  to node  $G'$ ) the fact that each agent in coalition  $S' \in \Gamma(D)$  prefers network  $G'$  to network  $G$ .

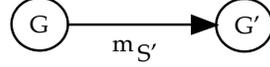
**Definition 2** (*Supernetworks*)

Given directed networks  $\mathbb{G} \subseteq \mathbb{N}(N, A)$ , agent payoff functions  $\{v_d(\cdot) : d \in D\}$ , and arc set  $\mathbb{A} := \mathbb{M} \cup \mathbb{P}$ , a supernetwork,  $\mathbf{G}$ , is a subset of  $\mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  such that for all networks  $G$  and  $G'$  in  $\mathbb{G}$  and for every coalition  $S' \in \Gamma(D)$ ,

- $(m_{S'}, (G, G')) \in \mathbf{G}$  if and only if coalition  $S'$  can change network  $G$  to network  $G'$ ,  $G' \neq G$ , by adding, subtracting, or replacing arcs in network  $G$ ,  
and  
 $(p_{S'}, (G, G')) \in \mathbf{G}$  if and only if  $v_d(G') > v_d(G)$  for all  $d \in S'$ .

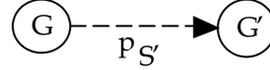
Thus, a supernetwork  $\mathbf{G}$  specifies how the networks in  $\mathbb{G}$  are connected via coalitional moves and coalitional preferences - and thus provides a *network representation* of agent preferences and the rules governing network formation. Given coalition  $S' \in \Gamma(D)$ ,  $m$ -arc  $m_{S'} \in \mathbb{M}$ , and  $p$ -arc  $p_{S'} \in \mathbb{P}$

$(m_{S'}, (G, G')) \in \mathbf{G}$  is denoted by



while

$(p_{S'}, (G, G')) \in \mathbf{G}$  is denoted by



**Remarks:**

(1) Under our definition of a supernetwork,  $m$ -arc loops and  $p$ -arc loops are ruled out. Thus, for any network  $G$  and coalition  $S'$ ,

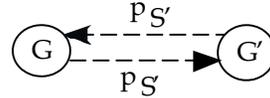
$$(m_{S'}, (G, G)) \notin \mathbf{G} \text{ and } (p_{S'}, (G, G)) \notin \mathbf{G}.$$

While  $m$ -arc loops are ruled out by definition, the absence of  $p$ -arc loops in supernetworks is due to the fact that each agent's preferences over networks are irreflexive. In particular, for each agent  $d \in D$  and each network  $G \in \mathbb{G}$ ,  $v_d(G) > v_d(G)$  is not possible. Thus,  $(p_{\{d\}}, (G, G)) \notin \mathbf{G}$ .

(2) The definition of agent preferences via the network payoff functions,

$$\{v_d(\cdot) : d \in D\},$$

also rules out the following types of  $p$ -arc connections:



Thus, for all coalitions  $S' \in \Gamma(D)$  and networks  $G$  and  $G'$  contained in  $\mathbb{G}$ ,

$$\text{if } (p_{S'}, (G, G')) \in \mathbf{G}, \text{ then } (p_{S'}, (G', G)) \notin \mathbf{G}.$$

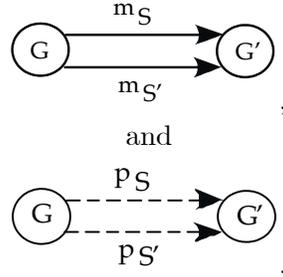
(3) For all coalitions  $S' \in \Gamma(D)$  and networks  $G$  and  $G'$  contained in  $\mathbb{G}$ , if

$$(p_{S'}, (G, G')) \in \mathbf{G}, \text{ then}$$

$$(p_S, (G, G')) \in \mathbf{G} \text{ for all subcoalitions } S \text{ of } S'.$$

(4) Under our definition of a supernetwork, multiple  $m$ -arcs, as well as multiple  $p$ -arcs, connecting networks  $G$  and  $G'$  in supernetwork  $\mathbf{G}$  are allowed. Thus, in supernetwork  $\mathbf{G}$  the following types of  $m$ -arc and  $p$ -arc connections are possible:

For coalitions  $S$  and  $S'$ , with  $S \neq S'$



However, multiple  $m$ -arcs, or multiple  $p$ -arcs, from network  $G \in \mathbf{G}$  to network  $G' \in \mathbf{G}$  belonging to the *same* coalition are not allowed - and moreover, are unnecessary. Allowing multiple arcs can be very useful in many applications. For example, multiple  $m$ -arcs (not belonging to the same coalition) connecting networks  $G$  and  $G'$  in a given supernetwork  $\mathbf{G}$  denote the fact that in supernetwork  $\mathbf{G}$  there is more than one way to get from network  $G$  to network  $G'$  - or put differently, there is more than one way to change network  $G$  to network  $G'$ .

(5) In many economic applications, the set of nodes,  $N$ , used in defining the networks in the collection  $\mathbb{G}$ , and the set of economic agents  $D$  are one and the same (i.e., in many applications  $N = D$ ).

## 4 Farsightedly Basic Networks

### 4.1 Farsighted Dominance

Given supernetwork  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ , we say that network  $G' \in \mathbb{G}$  farsightedly dominates network  $G \in \mathbb{G}$  if there is a finite sequence of networks,

$$G_0, G_1, \dots, G_h,$$

with  $G = G_0$ ,  $G' = G_h$ , and  $G_k \in \mathbb{G}$  for  $k = 0, 1, \dots, h$ , and a corresponding sequence of coalitions,

$$S_1, S_2, \dots, S_h,$$

such that for  $k = 1, 2, \dots, h$

$$(m_{S_k}, (G_{k-1}, G_k)) \in \mathbf{G},$$

and

$$(p_{S_k}, (G_{k-1}, G_h)) \in \mathbf{G}.$$

We shall denote by  $G \triangleleft\triangleleft G'$  the fact that network  $G' \in \mathbb{G}$  farsightedly dominates network  $G \in \mathbb{G}$ .

Figure 3 below provides a network representation of the farsighted dominance relation in terms of  $m$ -arcs and  $p$ -arcs. In Figure 3, network  $G_3$  farsightedly dominates network  $G_0$ .

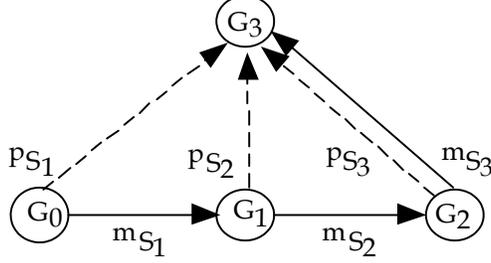


Figure 3:  $G_3$  farsightedly dominates  $G_0$

Note that what matters to the initially deviating coalition  $S_1$ , as well as coalitions  $S_2$  and  $S_3$ , is the ultimate network outcome  $G_3$ . Thus, the initially deviating coalition  $S_1$  will not be deterred even if

$$(p_{S_1}, (G_0, G_1)) \notin \mathbf{G}$$

as long as the ultimate network outcome  $G_3$  is preferred to  $G_0$ , that is, as long as  $G_3$  is such that

$$(p_{S_1}, (G_0, G_3)) \in \mathbf{G}.$$

## 4.2 Farsighted Domination Paths and Inductive Supernetworks

Given supernetwork  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ , we say that a sequence of networks  $\{G_k\}_k$  in  $\mathbb{G}$  is a *farsighted domination path*, written  $\triangleleft\triangleleft$ -path, *through supernetwork*  $\mathbf{G}$  if for any two consecutive networks  $G_{k-1}$  and  $G_k$ ,  $G_k$  farsightedly dominates  $G_{k-1}$ , that is, if

$$G_{k-1} \triangleleft\triangleleft G_k.$$

We can think of the farsighted dominance relation  $G_{k-1} \triangleleft\triangleleft G_k$  between networks  $G_k$  and  $G_{k-1}$  as defining a  $\triangleleft\triangleleft$ -arc *from* network  $G_{k-1}$  *to* network  $G_k$ . Given  $\triangleleft\triangleleft$ -path  $\{G_h\}_h$  through  $\mathbf{G}$ , the *length* of this path is defined to be the number of  $\triangleleft\triangleleft$ -arcs in the path. We say that network  $G_1 \in \mathbb{G}$  is  $\triangleleft\triangleleft$ -*reachable* from network  $G_0 \in \mathbb{G}$  in  $\mathbf{G}$  if there exists a finite  $\triangleleft\triangleleft$ -path in  $\mathbf{G}$  from  $G_0$  to  $G_1$  (i.e., a  $\triangleleft\triangleleft$ -path in  $\mathbf{G}$  from  $G_0$  to  $G_1$  of finite length). If network  $G_0 \in \mathbb{G}$  is  $\triangleleft\triangleleft$ -reachable from network  $G_0$  in  $\mathbf{G}$ , then we say that supernetwork  $\mathbf{G}$  contains a  $\triangleleft\triangleleft$ -*circuit*. Thus, a  $\triangleleft\triangleleft$ -circuit in  $\mathbf{G}$  starting at network  $G_0 \in \mathbb{G}$  is a finite  $\triangleleft\triangleleft$ -path from  $G_0$  to  $G_0$ . A  $\triangleleft\triangleleft$ -circuit of length 1 is called a  $\triangleleft\triangleleft$ -loop. Note that because preferences are irreflexive,  $\triangleleft\triangleleft$ -loops are in fact ruled out. However, because the farsighted dominance relation,  $\triangleleft\triangleleft$ , is *not transitive*, it is possible to have  $\triangleleft\triangleleft$ -circuits of length greater than 1.

Given supernetwork  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ , we can use the notion of  $\triangleleft\triangleleft$ -reachability to define a new relation on the nodes of the supernetwork. We shall write

$$G_1 \succeq G_0 \text{ if and only if } \begin{cases} G_1 \text{ is } \triangleleft\triangleleft \text{-reachable from } G_0, \text{ or} \\ G_1 = G_0. \end{cases} \quad (2)$$

The relation  $\supseteq$  is a weak ordering on the networks composing the nodes of the supernetwork  $\mathbf{G}$ . In particular,  $\supseteq$  is reflexive ( $G \supseteq G$ ) and  $\supseteq$  is transitive ( $G_2 \supseteq G_1$  and  $G_1 \supseteq G_0$  implies that  $G_2 \supseteq G_0$ ). The relation  $\supseteq$  is sometimes referred to as the transitive closure of the farsighted dominance relation,  $\triangleleft\triangleleft$ .

**Definition 3** ( *$\triangleleft\triangleleft$ -inductive Supernetworks*)

Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be a supernetwork.

Supernetwork  $\mathbf{G}$  is said to be  $\triangleleft\triangleleft$ -Inductive if given any  $\triangleleft\triangleleft$ -path,  $\{G_k\}_k$ , through  $\mathbf{G}$  there exists a network  $G^* \in \mathbb{G}$  such that

$$G^* \supseteq G_k \text{ for all } k.$$

Network  $G^*$  above is referred to as the majorant of the sequence.

Due to the finiteness of the node and arc sets,  $N$  and  $A$ , the collection of all directed networks,  $\mathbb{N}(N, A)$ , is a finite set. The finiteness of  $\mathbb{N}(N, A)$  implies that all supernetworks defined over  $\mathbb{N}(N, A)$  have a finite number of nodes, and this in turn implies that all supernetworks are  $\triangleleft\triangleleft$ -Inductive. To see this, let  $\{G_k\}_k$  be any  $\triangleleft\triangleleft$ -path through  $\mathbf{G}$ . If  $\{G_k\}_k$  is finite, then the last network in the sequence is a majorant. If  $\{G_k\}_k$  is infinite, then because the supernetwork is finite, the sequence contains at least one network which is repeated an infinite number of times, and this infinitely repeated network is a majorant. We summarize these observations in the following Theorem.

**Theorem 1** (*All supernetworks are  $\triangleleft\triangleleft$ -inductive*)

Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be any supernetwork. Then  $\mathbf{G}$  is  $\triangleleft\triangleleft$ -inductive.

### 4.3 Farsighted Bases

We begin with the definition.

**Definition 4** (*Farsighted Bases*)

Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be a supernetwork.

A subset  $\mathbb{B}$  of directed networks in  $\mathbb{G}$  is said to be a farsighted basis for supernetwork  $\mathbf{G}$  if

(a) (*internal  $\supseteq$ -stability*) whenever  $G_0$  and  $G_1$  are in  $\mathbb{B}$ , with  $G_0 \neq G_1$ , then neither  $G_1 \supseteq G_0$  nor  $G_0 \supseteq G_1$  hold, and

(b) (*external  $\supseteq$ -stability*) for any  $G_0 \notin \mathbb{B}$  there exists  $G_1 \in \mathbb{B}$  such that  $G_1 \supseteq G_0$ .

In other words, a nonempty subset of networks  $\mathbb{B}$  is a farsighted basis for supernetwork  $\mathbf{G}$  if  $G_0$  and  $G_1$  are in  $\mathbb{B}$ , with  $G_0 \neq G_1$ , then  $G_1$  is not reachable from  $G_0$ , nor is  $G_0$  reachable from  $G_1$ , and if  $G_0 \notin \mathbb{B}$ , then there exists  $G_1 \in \mathbb{B}$  reachable from  $G_0$ . Thus, a farsighted basis  $\mathbb{B}$  for supernetwork  $\mathbf{G}$  is simply a von Neumann-Morgenstern stable set with respect to the relation  $\triangleright$  determined by the supernetwork. Note that if a supernetwork possesses a farsighted basis, then by definition it is nonempty.

**Theorem 2** (*Existence of a Farsighted Basis for any Inductive Supernetwork, Berge (1958)*)

Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be any supernetwork. Then  $\mathbf{G}$  possesses a farsighted basis.

Theorem 2 is an immediate consequence of Theorem 1 above and a classical result due to Berge (1958) which, in our terminology, states that every inductive supernetwork possesses a farsighted basis.

## 5 Farsightedly Consistent Networks

In this section, we show that given any supernetwork, any farsighted basis is a subset of the largest consistent set of networks.

### 5.1 The Largest Consistent Set of Networks

A subset of directed networks  $\mathbb{F}$  is said to be farsightedly consistent if given any network  $G_0 \in \mathbb{F}$  and any  $m_{S_1}$ -deviation to network  $G_1 \in \mathbb{G}$  by coalition  $S_1$  (via adding, subtracting, or replacing arcs in accordance with  $\mathbf{G}$ ), there exists further deviations leading to some network  $G_2 \in \mathbb{F}$  where the initially deviating coalition  $S_1$  is not better off - and possibly worse off. A network  $G \in \mathbb{G}$  is said to be farsightedly consistent if  $G \in \mathbb{F}$  where  $\mathbb{F}$  is a farsightedly consistent set corresponding to supernetwork  $\mathbf{G}$ . Formally, we have the following definition.

**Definition 5** (*Farsightedly Consistent Sets*)

Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be a supernetwork.

A subset  $\mathbb{F}$  of directed networks in  $\mathbb{G}$  is said to be farsightedly consistent in supernetwork  $\mathbf{G}$  if

$$\begin{aligned} & \text{for all } G_0 \in \mathbb{F}, \\ & (m_{S_1}, (G_0, G_1)) \in \mathbf{G} \text{ for some } G_1 \in \mathbb{G} \text{ and some coalition } S_1, \text{ implies that} \\ & \quad \text{there exists } G_2 \in \mathbb{F} \\ & \quad \text{with } G_2 = G_1 \text{ or } G_2 \triangleright \triangleright G_1 \text{ such that,} \\ & \quad (p_{S_1}, (G_0, G_2)) \notin \mathbf{G}. \end{aligned}$$

There can be many farsightedly consistent sets. We shall denote by  $\mathbb{F}^*$  is largest farsightedly consistent set (or simply, the *largest consistent set*). Thus, if  $\mathbb{F}$  is a farsightedly consistent set, then  $\mathbb{F} \subseteq \mathbb{F}^*$ .

Two questions arise in connection with the largest consistent set: (i) given super-network  $\mathbf{G}$ , does there exist a farsightedly consistent set of networks in  $\mathbf{G}$ , and (ii) is it nonempty? To establish existence, consider the mapping

$$\Lambda_{\mathbf{G}}(\cdot) : 2^{\mathbb{G}} \rightarrow 2^{\mathbb{G}},$$

where  $2^{\mathbb{G}}$  denote the collection of *all* subsets of  $\mathbb{G}$  (including the empty set). The mapping  $\Lambda_{\mathbf{G}}(\cdot)$  is defined as follows:

$$\begin{aligned} & \text{for subcollection of networks } \mathbb{H} \in 2^{\mathbb{G}} \text{ and network } G_0 \in \mathbb{G}, \\ & \quad G_0 \text{ is contained in } \Lambda_{\mathbf{G}}(\mathbb{H}) \\ & \quad \text{if and only if} \\ \forall G_1 \in \mathbb{G} \text{ such that } (m_S, (G_0, G_1)) \in \mathbf{G} \text{ for some coalition } S \in \Gamma(D) & \quad (3) \\ & \quad \exists \text{ a network } G_2 \in \mathbb{H} \text{ such that} \\ & \quad \text{(i) } G_2 = G_1 \text{ or } G_2 \triangleright \triangleright G_1, \text{ and} \\ & \quad \text{(ii) } (p_S, (G_0, G_2)) \notin \mathbf{G}, \text{ that is, } v_d(G_2) \leq v_d(G_0) \text{ for some } d \in S. \end{aligned}$$

First, note that a set  $\mathbb{F}$  is a farsightedly consistent set if and only if  $\mathbb{F}$  is a fixed point of the mapping  $\Lambda_{\mathbf{G}}(\cdot)$ . Second, note that  $\Lambda_{\mathbf{G}}(\cdot)$  is monotone increasing; that is, for any subcollections  $\mathbb{H}$  and  $\mathbb{E}$  of  $\mathbb{G}$ ,

$$\mathbb{H} \subseteq \mathbb{E} \text{ imply that } \Lambda_{\mathbf{G}}(\mathbb{H}) \subseteq \Lambda_{\mathbf{G}}(\mathbb{E}).$$

To see that  $\Lambda_{\mathbf{G}}(\cdot)$  has a unique, largest fixed point consider the following. Let

$$\Sigma = \left\{ \mathbb{H} \in 2^{\mathbb{G}} : \mathbb{H} \subseteq \Lambda_{\mathbf{G}}(\mathbb{H}) \right\}.$$

Note that  $\Sigma$  is nonempty since the empty subcollection  $\emptyset \in 2^{\mathbb{G}}$  is contained in  $\Sigma$ ; that is,  $\emptyset \subseteq \Lambda_{\mathbf{G}}(\emptyset)$ . Let  $\mathbb{F}^* = \bigcup_{\mathbb{H} \in \Sigma} \mathbb{H}$ . By the monotonicity of  $\Lambda_{\mathbf{G}}(\cdot)$  and the definition of  $\Sigma$ ,

$$\mathbb{H} \subseteq \Lambda_{\mathbf{G}}(\mathbb{H}) \subseteq \Lambda_{\mathbf{G}}(\mathbb{F}^*) \text{ for all } \mathbb{H} \in \Sigma.$$

Hence

$$\mathbb{F}^* = \bigcup_{\mathbb{H} \in \Sigma} \mathbb{H} \subseteq \bigcup_{\mathbb{H} \in \Sigma} \Lambda_{\mathbf{G}}(\mathbb{H}) \subseteq \Lambda_{\mathbf{G}}(\mathbb{F}^*).$$

Thus,

$$\mathbb{F}^* \subseteq \Lambda_{\mathbf{G}}(\mathbb{F}^*).$$

Moreover, by the monotonicity of  $\Lambda_{\mathbf{G}}(\cdot)$ ,

$$\Lambda_{\mathbf{G}}(\mathbb{F}^*) \subseteq \Lambda_{\mathbf{G}}(\Lambda_{\mathbf{G}}(\mathbb{F}^*)).$$

Thus we have,  $\Lambda_{\mathbf{G}}(\mathbb{F}^*) \in \Sigma$  and  $\Lambda_{\mathbf{G}}(\mathbb{F}^*) \subseteq \mathbb{F}^*$ , and we can conclude that

$$\mathbb{F}^* = \Lambda_{\mathbf{G}}(\mathbb{F}^*).$$

The set  $\mathbb{F}^*$  is the largest farsightedly consistent set because if  $\mathbb{F}'$  is any other fixed point (i.e., if  $\mathbb{F}' = \Lambda_{\mathbf{G}}(\mathbb{F}')$ ), then,  $\mathbb{F}' \in \Sigma$  and hence  $\mathbb{F}' \subseteq \mathbb{F}^*$ .

The method of proving existence above is based on Chwe (1994) and is similar to the method introduced by Roth (1975, 1977). The important point to take away from the proof is that in order to show that the largest consistent set is nonempty, it suffices to show that for some nonempty subset of networks  $\mathbb{H} \in 2^{\mathbb{G}}$ ,

$$\mathbb{H} \subseteq \Lambda_{\mathbf{G}}(\mathbb{H}).$$

## 5.2 Farsighted Bases and Nonemptiness of the Largest Consistent Set

We begin by defining a new relation on the collection of directed networks  $\mathbb{G} \subseteq \mathbb{N}(N, A)$ . Let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be a supernetwork and for any nonempty subset of networks  $\mathbb{H}$  define the relation  $\triangleright_{\mathbb{H}}$  on the  $\mathbb{G}$  as follows: for  $G^1$  and  $G^0$  in  $\mathbb{G}$ ,

$$G_1 \triangleright_{\mathbb{H}} G_0 \text{ if and only if } \begin{cases} \text{(a) } G_1 \in \mathbb{H} \cap P_{\triangleright\triangleright}(G_0), \text{ and} \\ \text{(b) } \mathbb{H} \cap P_{\triangleright\triangleright}(G_1) \subseteq P_{\triangleright\triangleright}(G_0), \end{cases} \quad (4)$$

where

$$P_{\triangleright\triangleright}(G_0) := \{G \in \mathbb{G} : G \triangleright\triangleright G_0\}.$$

The new relation  $\triangleright_{\mathbb{H}}$  is transitive, that is, if  $G_2 \triangleright_{\mathbb{H}} G_1$  and  $G_1 \triangleright_{\mathbb{H}} G_0$ , then  $G_2 \triangleright_{\mathbb{H}} G_0$  (see Page and Kamat (2003), Lemma 3.1).

We say that a subset of networks  $\mathbb{H}^*$  is  $\triangleright_{\mathbb{H}^*}$ -consistent if  $\mathbb{H}^*$  is a von Neumann-Morgenstern stable set with respect to the relation  $\triangleright_{\mathbb{H}^*}$  (i.e., if  $\mathbb{H}^*$  is  $\triangleright_{\mathbb{H}^*}$ -stable). Thus,  $\mathbb{H}^*$  is  $\triangleright_{\mathbb{H}^*}$ -consistent if,

- (a)  $\mathbb{H}^*$  is internally  $\triangleright_{\mathbb{H}^*}$ -stable; that is,  $G \in \mathbb{H}^*$  implies that  $\mathbb{H}^* \cap P_{\triangleright_{\mathbb{H}^*}}(G) = \emptyset$ , and
- (b)  $\mathbb{H}^*$  is externally  $\triangleright_{\mathbb{H}^*}$ -stable; that is,  $G \notin \mathbb{H}^*$  implies that  $\mathbb{H}^* \cap P_{\triangleright_{\mathbb{H}^*}}(G) \neq \emptyset$ ,

where

$$P_{\triangleright_{\mathbb{H}^*}}(G) := \{G' \in \mathbb{G} : G' \triangleright_{\mathbb{H}^*} G\}.$$

By Theorem 3.6 in Page and Kamat (2003), given any collection  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  of directed networks and given any supernetwork  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$ , there exists a nonempty set of networks  $\mathbb{H}^*$  that is  $\triangleright_{\mathbb{H}^*}$ -consistent. Moreover, by Theorem 4 in Berge (1958), if  $\mathbb{H}^*$  is  $\triangleright_{\mathbb{H}^*}$ -consistent, then  $\mathbb{H}^*$  is the unique  $\triangleright_{\mathbb{H}^*}$ -stable set.

We can now state our main result on the relationship between the farsighted basis  $\mathbb{B}$  (i.e., the basis with respect to  $\triangleright\triangleright$ ) and the largest consistent set  $\mathbb{F}^*$ .

### **Theorem 3** *(All Farsightedly Basic Networks Are Farsightedly Consistent)*

*Let  $\mathbb{G} \subseteq \mathbb{N}(N, A)$  be a collection of directed networks and let  $\mathbf{G} \subset \mathbb{A} \times (\mathbb{G} \times \mathbb{G})$  be a supernetwork and let  $\mathbb{B}$  be a farsighted basis for  $\mathbf{G}$ . Then the largest consistent set,  $\mathbb{F}^*$ , is nonempty and  $\mathbb{B} \subseteq \mathbb{F}^*$ . Moreover, the largest consistent set,  $\mathbb{F}^*$ , is externally stable with respect to farsighted dominance, that is, if network  $G$  is a node in supernetwork  $\mathbf{G}$  not contained in  $\mathbb{F}^*$ , then there exists a network  $G'$ , a node in supernetwork  $\mathbf{G}$ , contained in  $\mathbb{F}^*$  that farsightedly dominates  $G$  (i.e.,  $G' \triangleright\triangleright G$ ).*

**Proof.** By Theorem 3.7 in Page and Kamat (2003), the unique  $\triangleright_{\mathbb{H}^*}$ -consistent set  $\mathbb{H}^*$  is such that

$$\mathbb{H}^* \subseteq \Lambda_{\mathbf{G}}(\mathbb{H}^*).$$

Thus,  $\mathbb{F}^*$  is nonempty and

$$\mathbb{H}^* \subseteq \mathbb{F}^*.$$

Note that if  $G \notin \mathbb{F}^*$  then  $G \notin \mathbb{H}^*$ . Thus, there exists a network  $G' \in \mathbb{H}^*$ , and hence a network  $G' \in \mathbb{F}^*$  such that  $G' \triangleright_{\mathbb{H}^*} G$ . Given the definition of the relation  $\triangleright_{\mathbb{H}^*}$ ,  $G' \triangleright \triangleright G$ , and thus,  $\mathbb{F}^*$  is externally stable with respect to the relation  $\triangleright \triangleright$ .

To complete the proof we must show that  $\mathbb{B} \subseteq \mathbb{H}^*$ . Suppose network  $G_0$  is contained in  $\mathbb{B}$  but not contained in  $\mathbb{H}^*$ . Since  $\mathbb{H}^*$  is  $\triangleright_{\mathbb{H}^*}$ -stable,  $G_0 \notin \mathbb{H}^*$  implies that there exists

$$G_1 \in \mathbb{H}^* \cap P_{\triangleright_{\mathbb{H}^*}}(G_0).$$

$G_1 \in P_{\triangleright_{\mathbb{H}^*}}(G_0)$  implies that  $G_1 \in P_{\triangleright \triangleright}(G_0)$ . If  $G_1 \in \mathbb{B}$  we have a contradiction of the internal  $\triangleright$ -stability of  $\mathbb{B}$ . If  $G_1 \notin \mathbb{B}$ , then there exists a  $\triangleleft \triangleleft$ -path through the supernetwork from  $G_1$  to some  $G_2 \in \mathbb{B}$ . But then  $G_1 \in P_{\triangleright \triangleright}(G_0)$  together with this fact implies that there exists a  $\triangleleft \triangleleft$ -path through the supernetwork from  $G_0 \in \mathbb{B}$  to  $G_2 \in \mathbb{B}$ , also a contradiction of the internal  $\triangleright$ -stability of  $\mathbb{B}$ . ■

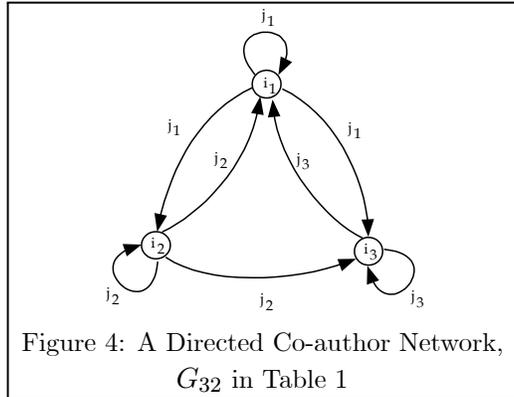
Thus, any farsightedly basic network is farsightedly consistent.

## 6 An Example: The Jackson-Wolinsky Co-author Model

In this section, we compute the largest consistent sets and the Nash equilibria for two supernetwork models of the Jackson and Wolinsky co-author problem. All of our computations are carried out using a *Mathematica* package developed by Kamat and Page (2001).

### 6.1 Co-author Networks

Consider a situation in which each of three researchers,  $i_1$ ,  $i_2$ , and  $i_3$ , is attempting to decide with which other researchers to collaborate. Figure 4 is a directed network representation of one possible configuration of research proposals and researcher collaborations.



In Figure 4, the loops,  $(j_1, (i_1, i_1))$ ,  $(j_2, (i_2, i_2))$ , and  $(j_3, (i_3, i_3))$ , at each of the nodes indicates that each researcher has undertaken a project on his own. The  $j_2$  arc from node  $i_2$  to  $i_3$  indicates that researcher  $i_2$  has *proposed* a research collaboration to researcher  $i_3$ , but since there is no reciprocating arc  $j_3$  from researcher  $i_3$  to researcher  $i_2$  no research collaboration is undertaken. Note that researchers  $i_1$  and  $i_2$  are connected by reciprocal arcs  $j_1$  and  $j_2$  indicating that a research collaboration has been undertaken. Thus, the reciprocating arcs indicate that  $i_1$  and  $i_2$  are co-authors. Note that  $i_1$  and  $i_3$  are also co-authors. Letting  $n_i$  denote the number of arcs from researcher  $i$  to any researcher  $i'$  (including  $i' = i$ ) - and therefore letting  $n_i$  denote the total number of research projects *and* research proposals undertaken by researcher  $i$  - we shall assume, as do Jackson and Wolinsky (1996), that the payoff to researcher  $i$  in any given co-author network is given by

$$\left[ \frac{1}{n_i} + \sum_{i'=co-author} \left( \frac{1}{n_i} + \frac{1}{n_{i'}} + \frac{1}{n_i n_{i'}} \right) \right] - c_i(n_i - 1),$$

where  $n_i - 1$  is the total number of research proposals and research collaborations and  $c_i(n_i - 1)$  is the direct costs of these proposals and collaborations.

Table 1 below, consisting of four panels, lists all possible co-author networks - assuming each researcher undertakes a project on his own. Denote the collection of all possible co-author networks by  $\mathbb{G}_{ca}$ .

(j <sub>3</sub> , (i <sub>3</sub> , i <sub>3</sub> ))				
	(j <sub>2</sub> , (i <sub>2</sub> , i <sub>2</sub> ))	( (j <sub>2</sub> , (i <sub>2</sub> , i <sub>2</sub> )), (j <sub>2</sub> , (i <sub>2</sub> , i <sub>1</sub> )) )	( (j <sub>2</sub> , (i <sub>2</sub> , i <sub>2</sub> )), (j <sub>2</sub> , (i <sub>2</sub> , i <sub>3</sub> )) )	( (j <sub>2</sub> , (i <sub>2</sub> , i <sub>2</sub> )), (j <sub>2</sub> , (i <sub>2</sub> , i <sub>1</sub> )), (j <sub>1</sub> , (i <sub>2</sub> , i <sub>3</sub> )) )
(j <sub>1</sub> , (i <sub>1</sub> , i <sub>1</sub> ))	G <sub>1</sub>	G <sub>2</sub>	G <sub>3</sub>	G <sub>4</sub>
( (j <sub>1</sub> , (i <sub>1</sub> , i <sub>1</sub> )), (j <sub>1</sub> , (i <sub>1</sub> , i <sub>2</sub> )) )	G <sub>5</sub>	G <sub>6</sub>	G <sub>7</sub>	G <sub>8</sub>
( (j <sub>1</sub> , (i <sub>1</sub> , i <sub>1</sub> )), (j <sub>1</sub> , (i <sub>1</sub> , i <sub>3</sub> )) )	G <sub>9</sub>	G <sub>10</sub>	G <sub>11</sub>	G <sub>12</sub>
( (j <sub>1</sub> , (i <sub>1</sub> , i <sub>1</sub> )), (j <sub>1</sub> , (i <sub>1</sub> , i <sub>2</sub> )), (j <sub>1</sub> , (i <sub>1</sub> , i <sub>3</sub> )) )	G <sub>13</sub>	G <sub>14</sub>	G <sub>15</sub>	G <sub>16</sub>

$\left( \begin{array}{c} (j_3, (i_3, i_3)), \\ (j_3, (i_3, i_1)) \end{array} \right)$				
	$(j_2, (i_2, i_2))$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_3)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)), \\ (j_1, (i_2, i_3)) \end{array} \right)$
$(j_1, (i_1, i_1))$	$G_{17}$	$G_{18}$	$G_{19}$	$G_{20}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)) \end{array} \right)$	$G_{21}$	$G_{22}$	$G_{23}$	$G_{24}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{25}$	$G_{26}$	$G_{27}$	$G_{28}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{29}$	$G_{30}$	$G_{31}$	$G_{32}$

$\left( \begin{array}{c} (j_3, (i_3, i_3)), \\ (j_3, (i_3, i_2)) \end{array} \right)$				
	$(j_2, (i_2, i_2))$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_3)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)), \\ (j_1, (i_2, i_3)) \end{array} \right)$
$(j_1, (i_1, i_1))$	$G_{33}$	$G_{34}$	$G_{35}$	$G_{36}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)) \end{array} \right)$	$G_{37}$	$G_{38}$	$G_{39}$	$G_{40}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{41}$	$G_{42}$	$G_{43}$	$G_{44}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{45}$	$G_{46}$	$G_{47}$	$G_{48}$

$\left( \begin{array}{c} (j_3, (i_3, i_3)), \\ (j_3, (i_3, i_1)), \\ (j_3, (i_3, i_2)) \end{array} \right)$				
	$(j_2, (i_2, i_2))$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_3)) \end{array} \right)$	$\left( \begin{array}{c} (j_2, (i_2, i_2)), \\ (j_2, (i_2, i_1)), \\ (j_1, (i_2, i_3)) \end{array} \right)$
$(j_1, (i_1, i_1))$	$G_{49}$	$G_{50}$	$G_{51}$	$G_{52}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)) \end{array} \right)$	$G_{53}$	$G_{54}$	$G_{55}$	$G_{56}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{57}$	$G_{58}$	$G_{59}$	$G_{60}$
$\left( \begin{array}{c} (j_1, (i_1, i_1)), \\ (j_1, (i_1, i_2)), \\ (j_1, (i_1, i_3)) \end{array} \right)$	$G_{61}$	$G_{62}$	$G_{63}$	$G_{64}$

Table 1: All Possible Co-author Networks,  $\mathbb{G}_{ca}$

## 6.2 Co-author Network Payoffs and Preferences

We construct two possible supernetworks,  $\mathbf{G}_{ca1}$  and  $\mathbf{G}_{ca2}$ , over the collection  $\mathbb{G}_{ca}$  by varying the payoffs to each researcher of proposing to or collaborating with other researchers.<sup>14</sup> In supernetwork  $\mathbf{G}_{ca1}$  we shall assume that the payoffs to each researcher in any given co-author network  $G$  contained in the collection  $\mathbb{G}_{ca}$  are given by

$$v_{i_k}(G) = \left[ \frac{1}{n_{i_k}} + \sum_{i_{k'}=\text{co-author in } G} \left( \frac{1}{n_{i_k}} + \frac{1}{n_{i_{k'}}} + \frac{1}{n_{i_k}n_{i_{k'}}} \right) \right] - .3(n_{i_k} - 1), \quad (5)$$

where  $k = 1, 2, 3$ ,  $k' \in \{1, 2, 3\}$ ,  $k \neq k'$ . Table 2 below, consisting of four panels, summarizes the payoff possibilities in supernetwork  $\mathbf{G}_{ca1}$ . Each payoff vector - a 3-tuple whose components give the payoffs to each researcher - are subscripted by the identification number of the network generating the payoff vector (see Table 1 for networks and identification numbers).

$(1, 1, 1)_1$	$(1, .20, 1)_2$	$(1, .20, 1)_3$	$(1, -.27, 1)_4$
$(.20, 1, 1)_5$	$(1.45, 1.45, 1)_6$	$(.20, .20, 1)_7$	$(1.20, .73, 1)_8$
$(.20, 1, 1)_9$	$(.20, .20, 1)_{10}$	$(.20, .20, 1)_{11}$	$(.20, -.27, 1)_{12}$
$(-.27, 1, 1)_{13}$	$(.73, 1.20, 1)_{14}$	$(-.27, .20, 1)_{15}$	$(.51, .51, 1)_{16}$
$(1, 1, .20)_{17}$	$(1, .20, .20)_{18}$	$(1, .20, .20)_{19}$	$(1, -.27, .20)_{20}$
$(.20, 1, .20)_{21}$	$(1.45, 1.45, .20)_{22}$	$(.20, .20, .20)_{23}$	$(1.20, .73, .20)_{24}$
$(1.45, 1, 1.45)_{25}$	$(1.45, .20, 1.45)_{26}$	$(1.45, .20, 1.45)_{27}$	$(1.45, -.27, 1.45)_{28}$
$(.73, 1, 1.20)_{29}$	$(1.73, 1.20, 1.20)_{30}$	$(.73, .20, 1.20)_{31}$	$(1.51, .51, 1.20)_{32}$
$(1, 1, .20)_{33}$	$(1, .20, .20)_{34}$	$(1, 1.45, 1.45)_{35}$	$(1, .73, 1.20)_{36}$
$(.20, 1, .20)_{37}$	$(1.45, 1.45, .20)_{38}$	$(.20, 1.45, .20)_{39}$	$(1.20, 1.73, 1.20)_{40}$
$(.20, 1, .20)_{41}$	$(.20, .20, .20)_{42}$	$(.20, 1.45, 1.45)_{43}$	$(.20, .73, 1.20)_{44}$
$(-.27, 1, .20)_{45}$	$(.73, 1.20, .20)_{46}$	$(-.27, 1.45, 1.45)_{47}$	$(.51, 1.51, 1.20)_{48}$
$(1, 1, -.27)_{49}$	$(1, .20, -.27)_{50}$	$(1, 1.20, .73)_{51}$	$(1, .51, .51)_{52}$
$(.20, 1, -.27)_{53}$	$(1.45, 1.45, -.27)_{54}$	$(.20, 1.20, .73)_{55}$	$(1.20, 1.51, .51)_{56}$
$(.20, 1, .73)_{57}$	$(1.20, .20, .73)_{58}$	$(1.20, 1.20, 1.73)_{59}$	$(1.20, .51, 1.57)_{60}$
$(.51, 1, .51)_{61}$	$(1.51, 1.20, .51)_{62}$	$(.51, 1.20, 1.51)_{63}$	$(.51, .51, .51)_{64}$

Table 2: Network Payoffs in supernetwork  $\mathbf{G}_{ca1}$

For example, in co-author network  $G_{32}$  in supernetwork  $\mathbf{G}_{ca1}$ , the payoffs to researchers are given by

$$(v_{i_1}(G_{32}), v_{i_2}(G_{32}), v_{i_3}(G_{32})) = (1.51, .51, 1.20).$$

In network  $G_{32}$ , researcher  $i_2$  is involved in his own project, is involved in one research collaboration with researcher  $i_1$  (hence  $i_2$  and  $i_1$  are co-authors), and proposes a collaboration to researcher  $i_3$  - but because the proposal is not reciprocated, no

<sup>14</sup>By varying the payoffs, we change the configuration of the preference arcs and therefore change the supernetwork.

collaboration between  $i_2$  and  $i_3$  occurs. Thus,  $n_{i_2} = 3$  (2 projects and 1 proposal). Researcher  $i_2$ 's co-author,  $i_1$ , is involved in three research projects: one on his own, one with researcher  $i_1$ , and one with researcher  $i_3$  (hence  $n_{i_1} = 3$ ). Given payoff function (5), researcher  $i_2$ 's payoff is computed as follows:

$$v_{i_2}(G_{32}) = \frac{1}{3} + \left( \frac{1}{3} + \frac{1}{3} + \frac{1}{9} \right) - .6 \doteq .51.$$

Here,  $.6 = 2 \times .3$  is the direct cost to agent  $i_2$  of his research activities. Thus, his own project generates no direct cost, but does generate an indirect cost expressed as reduction in the average amount of time that can be spent on any one project.

Table 3 below, also consisting of three panels, summarizes the payoff possibilities in supernetwork  $\mathbf{G}_{ca2}$ . In supernetwork  $\mathbf{G}_{ca2}$ , we assume that payoffs are given by

$$v_{i_k}(G) = \begin{cases} \left[ \frac{1}{n_{i_k}} + \sum_{i_{k'}=\text{co-author in } G} \left( \frac{1}{n_{i_k}} + \frac{1}{n_{i_{k'}}} + \frac{1}{n_{i_k} n_{i_{k'}}} \right) \right] - 1.2, & \text{if } n_{i_k} - 1 = 2 \\ \left[ \frac{1}{n_{i_k}} + \sum_{i_{k'}=\text{co-author in } G} \left( \frac{1}{n_{i_k}} + \frac{1}{n_{i_{k'}}} + \frac{1}{n_{i_k} n_{i_{k'}}} \right) \right] - .3, & \text{if } n_{i_k} - 1 = 1. \end{cases} \quad (6)$$

Thus, in supernetwork  $\mathbf{G}_{ca2}$  the direct costs of research activities (proposals and collaborations) are increasing.

$(1, 1, 1)_1$	$(1, .20, 1)_2$	$(1, .20, 1)_3$	$(1, -.87, 1)_4$
$(.20, 1, 1)_5$	$(1.45, 1.45, 1)_6$	$(.20, .20, 1)_7$	$(1.20, .13, 1)_8$
$(.20, 1, 1)_9$	$(.20, .20, 1)_{10}$	$(.20, .20, 1)_{11}$	$(.20, -.87, 1)_{12}$
$(-.20, 1, 1)_{13}$	$(.13, 1.20, 1)_{14}$	$(-.87, .20, 1)_{15}$	$(-.09, -.09, 1)_{16}$
$(1, 1, .20)_{17}$	$(1, .20, .20)_{18}$	$(1, .20, .20)_{19}$	$(1, -.87, .20)_{20}$
$(.20, 1, .20)_{21}$	$(1.45, 1.45, .20)_{22}$	$(.20, .20, .20)_{23}$	$(1.20, .13, .20)_{24}$
$(1.45, 1, 1.45)_{25}$	$(1.45, .20, 1.45)_{26}$	$(1.45, .20, 1.45)_{27}$	$(1.45, -.87, 1.45)_{28}$
$(.13, 1, 1.20)_{29}$	$(1.13, 1.20, 1.20)_{30}$	$(.13, .20, 1.20)_{31}$	$(.91, -.09, 1.20)_{32}$
$(1, 1, .20)_{33}$	$(1, .20, .20)_{34}$	$(1, 1.45, 1.45)_{35}$	$(1, .13, 1.20)_{36}$
$(.20, 1, .20)_{37}$	$(1.45, 1.45, .20)_{38}$	$(.20, 1.45, .20)_{39}$	$(1.20, 1.13, 1.20)_{40}$
$(.20, 1, .20)_{41}$	$(.20, .20, .20)_{42}$	$(.20, 1.45, 1.45)_{43}$	$(.20, .13, 1.20)_{44}$
$(-.87, 1, .20)_{45}$	$(.13, 1.20, .20)_{46}$	$(-.87, 1.45, 1.45)_{47}$	$(-.09, .91, 1.20)_{48}$
$(1, 1, -.87)_{49}$	$(1, .20, -.87)_{50}$	$(1, 1.20, .13)_{51}$	$(1, -.09, -.09)_{52}$
$(.20, 1, -.87)_{53}$	$(1.45, 1.45, -.87)_{54}$	$(.20, 1.20, .13)_{55}$	$(1.20, .91, -.09)_{56}$
$(.20, 1, .13)_{57}$	$(1.20, .20, .13)_{58}$	$(1.20, 1.20, 1.13)_{59}$	$(1.20, -.09, .91)_{60}$
$(-.09, 1, -.09)_{61}$	$(.91, 1.20, -.09)_{62}$	$(-.09, 1.20, .91)_{63}$	$(.69, .69, .69)_{64}$

Table 3: Network Payoffs in supernetwork  $G_{ca2}$

### 6.3 Computational Results

In Table 4 below we list the largest consistent sets corresponding to supernetworks  $\mathbf{G}_{ca1}$  and  $\mathbf{G}_{ca2}$ . We also list the Nash networks and the Pareto efficient networks for

each of our supernetworks.

	$\mathbf{G}_{ca1}$	$\mathbf{G}_{ca2}$
Largest Consistent Set $\mathbb{F}^*$	$\{G_6, G_{25}, G_{35}, G_{30}, G_{40}, G_{59}\}$	$\{G_6, G_{25}, G_{35}\}$
Nash Networks	$\{G_1, G_6, G_{25}, G_{35}, G_{30}, G_{40}, G_{59}\}$	$\{G_1, G_6, G_{25}, G_{35}\}$
Pareto Efficient Networks	$\{G_6, G_{25}, G_{35}, G_{30}, G_{40}, G_{59}\}$	$\{G_1, G_6, G_{25}, G_{35}\}$

Table 4: Summary of Computational Results

Figure 5 depicts the all co-author networks contained in Table 4.

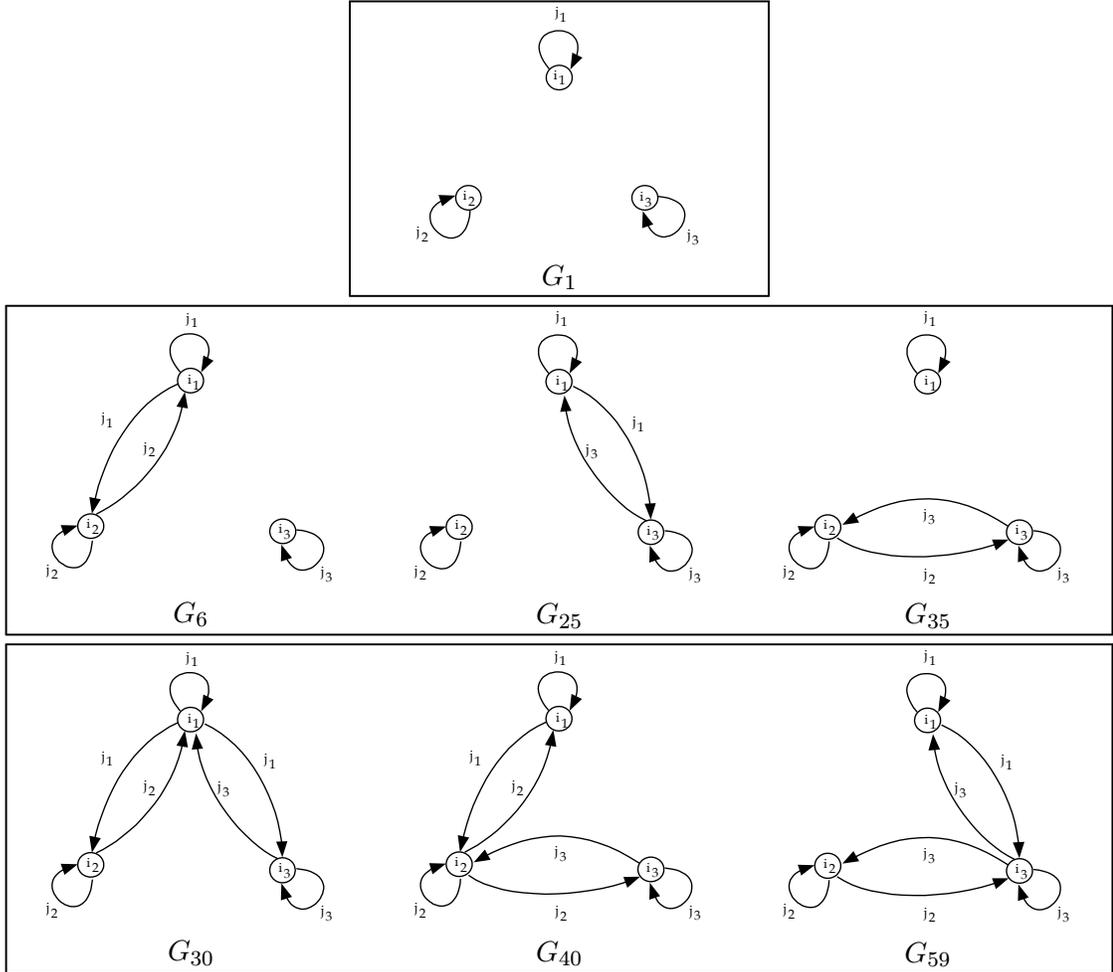


Figure 5: Farsightedly Consistent Networks Relative to Information Sharing Supernetwork  $G_1$

Note that in each supernetwork, the largest consistent set is a subset of the Nash set. Also, note that in each supernetwork, the farsightedly consistent networks as well as the Nash networks are such that all research proposals are reciprocated - thus, in all the farsightedly consistent networks and Nash networks researchers either collaborate or work alone. Finally, note that in supernetwork  $\mathbf{G}_{ca2}$  where the costs of research activities are increasing, the farsightedly consistent networks as well as the

Nash networks are such that no researcher is involved in more than 1 collaboration - it is simply too costly.

We conclude the co-author example by considering the issue of Pareto efficiency. Define the set of Pareto efficient networks relative to the collection  $\mathbb{G}_{ca}$  as follows:

$$\{G \in \mathbb{G}_{ca} : \text{there does not exist } G' \text{ such that } v_d(G') > v_d(G) \text{ for all } d \in D\}.$$

By examination of the payoffs in Tables 2 and 3, we conclude that in both supernetworks  $\mathbf{G}_{ca1}$  and  $\mathbf{G}_{ca2}$ , the set of farsightedly consistent networks is contained in the set of Pareto efficient networks. In fact, in supernetwork  $\mathbf{G}_{ca1}$ , the largest consistent set equals the Pareto efficient set. In supernetwork  $\mathbf{G}_{ca2}$ , only Pareto efficient, Nash network  $G_1$  is not contained in the largest consistent set. But note that  $G_1$  is weakly Pareto dominated by each of the farsightedly consistent networks,  $G_6, G_{25}$ , and  $G_{35}$ .

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