ON PURIFICATION OF EQUILIBRIUM
IN BAYESIAN GAMES
AND EX-POST NASH EQUILIBRIUM

Edward Cartwright

And

Myrna Wooders

No 701

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

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Edward Cartwright
EUREQUA, Paris 1
Universite Paris-1 Pantheon-Sorbonne
106-112 Bd de l'Hopital
France - 75013 Paris
e.cartwright@tiscali.co.uk
http://myweb.tiscali.co.uk/ejcartwright/

Myrna Wooders
Department of Economics
University of Warwick
Coventry, CV4 7Al,
UK
M.Wooders@warwick.ac.uk
www.warwick.ac.uk/go/myrnawooders

First version: December 1, 2003, This version: February 19, 2004

Key words: purification, expost Nash, Bayesian games
JEL numbers: C7

Abstract

We demonstrate that if any realization of a strategy for a Bayesian game is, with high probability, an approximate (ε) Nash equilibrium of the induced game of complete information, then there is a purification of that strategy that is an approximate (α) equilibrium of the original Bayesian game. We also provide two examples demonstrating, amongst other things, that the bound α we obtain on the distance of the purification from satisfying the requirements for an exact equilibrium is tight.

1 Ex-post Nash equilibrium and purification of Bayesian equilibrium

First let us briefly recall some common terminology. In a Bayesian game each player is randomly assigned a type and, having been informed of his type but not those of the other players, each player chooses an action. A composition profile lists an assignment of types to players and an action choice for each player. Thus a composition profile represents a possible outcome in the game once all uncertainty is resolved. Ex-ante, a strategy vector induces a probability distribution over composition profiles.

Following Kalai (2002), we define a composition profile as ε-Nash if no player could gain by more than ε by changing his action. Given $\varepsilon \geq 0$ and $\rho \geq 0$ a strategy vector σ is (ε, ρ) *ex-post Nash* if the probability that it yields an ε-Nash composition profile is at least $1 - \rho$.

Given $\alpha \geq 0$, a strategy vector σ is a *Nash α-equilibrium* if, ex-ante, taking the strategy choices of the other players as given, no player can *expect* to improve upon his payoff by more than α from changing his strategy. An α-purification of a strategy σ is a pure strategy vector s that constitutes a Nash α-equilibrium and has support contained in the support of σ.[1]

In this paper we demonstrate that for any (ε, ρ) ex-post Nash strategy vector σ, in the support of σ there is a pure strategy vector s that is a Bayesian Nash α-equilibrium where $\alpha \leq (1 - \rho)\varepsilon + \rho D$ and D is an upper bound on payoffs. Note that, for this result, it is not required that the game has many players or that the game be semi-anonymous.

Recall that Kalai (2002) demonstrates that given $\varepsilon > 0$, with probability ρ, which can be made arbitrarily close to zero for games with sufficiently many players, every Nash equilibrium σ of a semi-anonymous Bayesian game Γ is (ε, ρ) ex-post Nash. Our result implies, following the results of Kalai (2002), that for any sufficiently large semi-anonymous game and for any Nash equilibrium of that game there exists an α purification. Importantly, we note that our result requires $\alpha > \varepsilon$; this implies that a strategy vector might not be a Nash ε-equilibrium even if, with very high probability, it yields composition profiles that are ε-Nash. We demonstrate this possibility with two examples.

2 A Bayesian game

There exists a finite set of *actions* $\mathcal{A}$ and finite set of *types* $\mathcal{T}$. We denote the set of possible *player compositions* by $\mathcal{C} \equiv \mathcal{A} \times \mathcal{T}$. A *Bayesian game* is given by a tuple (N, T, p, A, u) where:

$N = \{1, ..., n\}$ is a finite *player set*.

$T \equiv \mathcal{T}^N$ is a set of type profiles.

[1] For other purification results see, for example, the review in Khan and Sun (2002) and, for more recent results, Cartwright and Wooders (2003).

Function $p : T \rightarrow [0,1]$ is a *prior probability function* where $p(t)$ is the probability of type profile $t \in T$ and

$A \equiv \mathcal{A}^N$ is a set of actions profiles.

The players *utility functions* are described by vector $u = (u_1, ..., u_n)$. Letting $C \equiv \mathcal{C}^N$ denote the set of feasible composition profiles each u_i takes the form $u_i : C \rightarrow [0, D]$. Note that D gives an upper bound on player payoffs.

The game is played as follows: According to the prior probability function p each player i is assigned a type t_i. Informed of his type (but not the types of the other players) each player chooses an action (possible using some randomization).

A *strategy* of player i is given by a vector $\sigma_i = (\sigma_i(a_i|t_1), ..., \sigma_i(a_i|t_{|\mathcal{T}|}))$ where $\sigma_i(a_i|t_i)$ is the probability player i would play action a_i if of type t_i. Let Σ denote the set of strategies. A *strategy vector* $\sigma \in \Sigma^N$ details the strategy of each player. We say that a strategy vector s is a *pure strategy vector* if and only if for each player i and each t_i there exists some a_i such that $s_i(a_i|t_i) = 1$. Let S denote the set of pure strategy vectors. We say that a set of pure strategy vectors $\{s^1, ..., s^M\} \subseteq S$ constitute a *support* for strategy vector σ if and only if there exists real numbers $\beta_1, ..., \beta_M$ where

1. $1 \geq \beta_m > 0$ for all m,
2. $\sum_m \beta_m = 1$ and
3. $\sigma_i(a_i|t_i) = \sum_m \beta_m s_i^m(a_i|t_i)$ for all i, a_i and t_i.

   Clearly every strategy vector $\sigma \in \Sigma$ has a support.

Given a strategy vector σ and a prior probability function p a *probability distribution over the set of composition profiles* C is induced. Thus, the *expected payoff* of a player can be calculated. Let $U_i : \Sigma^N \rightarrow [0, D]$ be the expected utility function of a player i where $U_i(\sigma) = E(u_i(c))$.

2.1 Equilibrium concepts

Given $\alpha \geq 0$, a strategy vector σ is a *Bayesian Nash α-equilibrium* if and only if:

$$U_i(\sigma_i, \sigma_{-i}|t_i) \geq U_i(\sigma'_i, \sigma_{-i}|t_i) - \alpha$$

for all $\sigma'_i \in \Sigma$, all $t_i \in \mathcal{T}$ and for all $i \in N$.[2] We say that a Bayesian Nash α equilibrium s is a *Bayesian Nash α-equilibrium in pure strategies* if s is a pure strategy vector.

Given $\varepsilon \geq 0$, a composition profile is ε *incentive compatible* for player i if

$$u_i(c) \geq u_i(a'_i, t_i, c_{-i}) - \varepsilon$$

[2] More formally we only require for $t_i \in \mathcal{T}$ where there is a positive probability that player i may be of type t_i.

for every action $a'_i \in \mathcal{A}$. A composition profile is ε *Nash* if it is ε incentive compatible for every player. Finally, a strategy profile is (ε, ρ) *ex-post Nash* if the probability that it yields an ε Nash composition profile is at least $1 - \rho$.

3 A purification Theorem for Bayesian games

First, we provide our main theorem. While we explicitly treat only the case of finite type spaces and finite action sets, the Theorem also holds when these spaces are countable, as in Cartwright and Wooders (2003).

Theorem: Take as given a Bayesian game Γ and small positive real numbers ε and ρ (both less than 1). If a strategy profile σ is (ε, ρ) ex-post Nash then in the support of σ there is a pure strategy vector s that is a Bayesian Nash α-equilibrium where $\alpha \leq (1 - \rho)\varepsilon + \rho D$.

Proof: Let σ^* be (ε, ρ) ex-post Nash and let $P \equiv \{s^1, ..., s^M\}$ be a support of σ^*. We proceed by contradiction. Thus, suppose that there exists no $s^m \in P$ such that s^m is a Bayesian Nash α-equilibrium for $\alpha = (1 - \rho)\varepsilon + \rho D$.[3]

We introduce some notation: Let C^* denote the set of ε Nash composition profiles of game Γ. Given a strategy vector σ' let $y(c, \sigma')$ denote the probability of composition profile c occurring. [2]

Take any $s^m \in P$. By our supposition, s^m is not a Bayesian Nash α equilibrium. Given that s is not a Bayesian Nash α-equilibrium it must be that the probability of a composition profile $c \notin C^*$ occurring is greater than ρ; that is,

$$\sum_{c \notin C^*} y(c, s^m) > \rho. \tag{1}$$

Suppose otherwise: with probability at least $1 - \rho$ an ε Nash composition profile arises; if a composition profile $c \notin C^*$ arises then each player can gain at most D by changing his action; thus, ex-ante the maximum a player can gain by changing his strategy is $(1 - \rho)\varepsilon + \rho D$ leading to the desired contradiction.

The set $P = \{s^1, ..., s^M\}$ is a support for strategy vector σ and thus there exists real numbers $\beta_1, ..., \beta_M$ where (1)$1 \geq \beta_m > 0$ for all m, (2) $\sum_m \beta_m = 1$ and (3) $\sigma_i^*(a_i|t_i) = \sum_m \beta_m s_i^m(a_i|t_i)$ for all i, a_i and t_i. Thus,

$$y(c, \sigma^*) = \sum_m \beta_m y(c, s^m) \tag{2}$$

for all $c \in C$. Thus,

$$\sum_{c \notin C^*} y(c, \sigma^*) = \sum_{c \notin C^*} \left[\sum_m \beta_m y(c, s^m) \right] = \sum_m \beta_m \left(\sum_{c \notin C^*} y(c, s^m) \right) \tag{3}$$

[3] Note that if s^m is not a Bayesian Nash α-equilibrium then it cannot be a Bayesian Nash α' equilibrium for any $\alpha' < \alpha$.

Note, however that σ^* is (ε, ρ) ex-post Nash which by definition implies,

$$\sum_{c \notin C^*} y(c, \sigma^*) < \rho. \tag{4}$$

Clearly (1), (3) and (4) are incompatible if $\sum_m \beta_m = 1$. This gives the desired contradiction.■

A corollary of this result and results due to Kalai (2002) is that, given any $\varepsilon > 0$, for any sufficiently large semi-anonymous game and for any equilibrium σ of that game there exists a Nash ε-equilibrium in pure strategies in the support of σ.

4 Example 1: The bound on α is tight

In this section we provide an example to demonstrate that the bound obtained in Theorem 1 is tight. As a consequence, we show that the existence of an (ε, ρ) ex-post Nash strategy vector where ρ is arbitrarily small but greater than zero does not guarantee the existence of a Bayesian Nash ε equilibrium in pure strategies.

For notational simplicity, let there be $2n + 1$ players where n is an odd number. There are four types of player *Rich* (R), *Poor* (P), *High* (H) or *Low* (L). Only player 1's type is, however, random. We refer to player 1 as nature. Player 1 has type H with probability $\frac{1}{n}$ and type L with probability $1 - \frac{1}{n}$. Players $2, 3, ..., n + 1$ (called rich players) have type R with probability 1 and players $n+2, n+3, ..., 2n+1$ (called poor players) have type P with probability 1. Let h denote the type profile in which player 1 has type H, all rich players have type R and all poor players have type L. Let l denote the type profile where player 1 has type L, all rich players have type R and all poor players have type L. The prior probability distribution has the property that $p(h) = \frac{1}{n}$ and $p(l) = 1 - \frac{1}{n}$. Thus, half the players are always rich, half are always poor and nature is either type L or, with some small probability, type H.

Players choose one of two actions B or G. Given an action profile a and action a' let $w(R, a')$ be the number of players with type R who choose action a'. Thus, for example, $w(R, B)$ denotes the number of players who are rich and choose action B. Similarly, let $w(P, a')$ be the number of players with type P who choose action a'. The payoff function of each player is depends upon his type and the type of player 1.[4] Let $D > 1$ be some integer. A rich player i has payoff function,

$$\begin{aligned} u_i(a_i, a_{-i}, l) &= D - \frac{w(P, a_i)}{n} \\ u_i(a_i, a_{-i}, h) &= D \text{ if } w(P, a_i) \leq \frac{n}{2} \\ u_i(a_i, a_{-i}, h) &= 0 \text{ if } w(P, a_i) > \frac{n}{2}. \end{aligned}$$

[4] Note that it is sufficient to only detail the payoff of a player for type profiles (or composition profiles) that have positive probability.

Thus, if nature is L the payoff of a rich player depends negatively on the proportion of poor players who choose the same action as himself. If nature is H then his payoff is either D or 0 depending on whether or not half of the poor players have chosen the same action as himself. A poor player i has payoff function,

$$u_i(a_i, a_{-i}, t) = \frac{w(R, a_i)}{n}.$$

Thus, the payoff of a poor player depends positively on the proportion of rich players who choose the same action as himself. The payoff of a poor player does not depend on the type of nature. Finally, the payoff of nature is 1 for any composition profile.

First, consider the existence of a Bayesian Nash ε equilibrium in pure strategies. Given a strategy vector in which all rich players or all poor players play the same strategy there must exist at least one player who can gain by 1 or more by changing strategy. Thus, assume there is at least one rich player and one poor player playing G and one rich player and one poor player playing B. Given that n is odd there must always be a distinct number of poor players choosing action B as opposed to action G. It follows that ex-ante, given any pure strategy vector s, there must be at least one rich player i who can expect to gain at least

$$\frac{1}{n}\left(1 - \frac{1}{n}\right) + D\frac{1}{n} = \frac{1}{n} + \frac{1}{n}\left(D - \frac{1}{n}\right).$$

from changing strategy. Thus, there cannot exist a Bayesian Nash ε equilibrium in pure strategies for any $\varepsilon \leq \frac{1}{n} + \frac{1}{n}\left(D - \frac{1}{n}\right)$. In particular, there does not exist a Bayesian Nash ε equilibrium for $\varepsilon = \frac{1}{n}$.

Consider now the existence of a strategy vector that is ex-post stable. Let s' denote the strategy vector where $\frac{n-1}{2}$ rich players choose action B and $\frac{n+1}{2}$ choose action G and similarly $\frac{n-1}{2}$ poor players choose action B and $\frac{n+1}{2}$ choose action G. With probability $1 - \frac{1}{n}$ strategy vector s' will yield composition profile l that is $\frac{1}{n}$ incentive compatible for each player and is thus $\frac{1}{n}$ Nash. It follows that s' is $(\frac{1}{n}, \frac{1}{n})$ ex post Nash.

This example demonstrates that the bound obtained in Theorem 1 is tight. As a consequence, it also demonstrates how the existence of a strategy vector s that is (ε, ρ) ex post Nash for arbitrarily small ρ does not guarantee the existence of a Bayesian Nash ε equilibrium in pure strategies. The example does, however, draw on a strong discontinuity in the payoff function: (1) The type of nature can have a significant influence on payoffs and (2) A poor player can, through his choice of action, have a significant influence on the payoff of a rich player. Our second example, in the next section, does not exhibit such discontinuity in the payoff function.

5 Example 2: (ε, ρ) ex post Nash does not imply ε purification

There are $3n$ players where n is odd. There are four types of player *Poor* (R), *Rich* (R), *High* (H) and *Low* (L). Players $1, 2, ..., n$ (called rich) have type R with probability 1. Players $n+1, n+2, ..., 2n$ (called poor) have type P with probability 1. Players $2n+1, ..., 3n$ (called managers) have type H with probability $\frac{1}{n}$ and type L with probability $(1-\frac{1}{n})$. Managers are assigned types independently.

Given a type profile t let $h(t)$ denote the proportion of managers who are type high. If player i is rich then his payoff is given by,

$$\begin{aligned} u_i(a_i, a_{-i}, t) &= D - \frac{w(P, a_i)}{n} \text{ if } h(t) \leq \frac{2}{3} \\ u_i(a_i, a_{-i}, t) &= D - \frac{w(P, a_i)}{n} - (D-1)\left(\frac{h(t) - \frac{2}{3}}{\frac{1}{3}}\right)\frac{w(P, a_i)}{n} \text{ if } h(t) \geq \frac{2}{3} \end{aligned} \quad (5)$$

Thus, the payoff of a rich player depends negatively on the proportion of poor players playing the same action as himself. As the proportion of managers who have type H increases above $\frac{2}{3}$ then his payoff is influenced more by the actions of the poor players. If player i is poor then his payoff function is given by,

$$u_i(a_i, a_{-i}, t) = \frac{w(R, a_i)}{n} .$$

Thus, the payoff of a poor player depends positively on the proportion of rich players who choose the same action as himself. Let the payoff of a manager be 1 independent of the composition profile.[5]

As in the previous example we take there to be at least one rich player and one poor player playing G and one rich player and one poor player playing B. As n is odd the number of poor players playing G is distinct to the number playing B. Given that $\Pr[h(t) > 2/3] > 0$, for any pure strategy vector s there must be at least one rich player i who can expect, ex-ante, to gain by strictly more than $\frac{1}{n}$ if he changes strategy. Thus, there does not exist a Bayesian Nash ε equilibrium in pure strategies for any $\varepsilon \leq \frac{1}{n}$.

Let s' be the pure strategy vector whereby $\frac{n-1}{2}$ rich players choose action B and $\frac{n+1}{2}$ choose action G and similarly $\frac{n-1}{2}$ poor players choose action B and $\frac{n+1}{2}$ choose action G. With some probability $1-\rho'$ strategy vector s' will yield a composition profile c where $h(t) \leq 2/3$. When this occurs c is $\frac{1}{n}$ Nash. Thus, s' is $(\frac{1}{n}, \rho')$ ex post Nash. We shall now show that $\rho' \to 0$ as $n \to \infty$. Assuming,

[5] Intuitevely it may be that if managers have type H they prefer some policy or action that makes the payoff of rich players more sensitive to the actions of poor players. This is, however, not neccessary for the example.

for simplicity that n is divisible by 3, we obtain,[6]

$$\rho' = \sum_{x=\frac{2}{3}n}^{n} \binom{n}{x} \left(\frac{1}{n}\right)^x \left(1-\frac{1}{n}\right)^{n-x} = \Pr\left[F_{v_1,v_2} \leq \frac{v_2\frac{1}{n}}{v_1\left(1-\frac{1}{n}\right)}\right]$$

where F_{v_1,v_2} is the F distribution with parameters v_1 and v_2 and where $v_1 = \frac{4}{3}n$ and $v_2 = \frac{2}{3}n + 2$. Note that,

$$\frac{v_2\frac{1}{n}}{v_1\left(1-\frac{1}{n}\right)} = \frac{\frac{2}{3}+\frac{2}{n}}{\frac{4}{3}n-\frac{4}{3}} \to 0 \text{ as } n \to \infty.$$

Also note that $v_1, v_2 \to \infty$ as $n \to \infty$. It follows that $\rho' \to 0$ as $n \to \infty$. An alternative, if less formal, way of obtaining the same result is to note that if we let p denote the probability that a manager has type H then as $n \to \infty, p \to 0$ but $np = 1$. Thus, as n becomes large the binomial distribution determining the number of managers who have type H can be approximated by a Poisson distribution with parameter 1. It follows that the $\Pr[x \geq \frac{2}{3}n] \to 0$ as $n \to \infty$.

This example shows that the existence of a strategy vector s' that is (ε, ρ') ex post Nash is not sufficient for the existence of a Nash ε equilibrium in pure strategies. Note that this example does not rely on discontinuity in the payoff function. Indeed it is a semi-anonymous Bayesian game as defined by Kalai (2002).

6 Some concluding remarks

In this paper, we show that if a strategy profile σ for a Bayesian game is (ε, ρ) expost Nash, then there is a pure strategy profile s, with support contained in the support of σ, that is an α-equilibrium of the Bayesian game, where $\alpha \leq (1-\rho)\varepsilon + \rho D$, that is, there is an α purification of σ. Along with Kalai's (2002) result that every Bayesian Nash equilibrium of a semi-anonymous game with finite type and action sets is (ε, ρ) expost Nash,[7] our result implies that under the same conditions every equilibrium of the game has an α-purification. We remark that Cartwright and Wooders (2003) obtain a purification result in situations with a countable set of (Harsanyi) taste types, a countable set of actions and a compact metric space of crowding types where the crowding type

[6] A known result (see p110 of Johnson, Kotz qnd Kempis 1993) is that,

$$\sum_{x=r}^{n} \binom{n}{x} p^x q^{n-x} = \Pr\left[F_{v_1,v_2} \leq \frac{v_2 p}{v_1 q}\right]$$

where F_{v_1,v_2} is the F distribution with parameters $v_1 = 2r$ and $v_2 = 2(n-r+1)$.

[7] We note that Kalai's result follows from the law of large numbers. For our Theorem above and for those of Cartwright and Wooders (2003), the law of large numbers will not yield the results since we demonstrate that there is an approximate equilibrium in pure strategies where *every* player's action is close, in payoff, to a best response.

of a player describes those attributes or 'external' characteristics of a player that directly influence other players. As noted above, the Theorem of this paper also holds when both the set of types and the action sets are countable.

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