'REVERSE HYSTERESIS': R&D INVESTMENT WITH STOCHASTIC INNOVATION

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Abstract

We consider optimal investment behavior for a firm facing both technological and economic uncertainty, in the context of a research project with unpredictable outcomes. The optimal investment strategy, in the form of a pair of trigger points for investment and abandonment, is derived. As in Dixit (1989), the investment trigger exceeds the Marshallian investment point. However the abandonment trigger may exceed the Marshallian exit point, in contrast to the Dixit result, giving rise to ‘reverse hysteresis.’ Thus the firm tends to abandon research rapidly as profitability declines, at times despite the existence of positive expected profits. The model also provides a unified framework encompassing two existing models as limiting cases.

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‘REVERSE HYSTERESIS’:
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1 Introduction

When a firm invests in a research project it faces two forms of uncertainty. In common with many other projects the economic value of the investment is uncertain. Since the return to a new product design or production process is derived from product market profits, the value of an invention is affected by fluctuations in market demand. However, in contrast to fixed capital investments, there is also technological uncertainty. Discovery occurs randomly, thus the relationship between the input of research effort and the creation of a marketable invention is uncertain.

In this paper we consider optimal investment behavior for a firm facing both technological and economic uncertainty. As in other real options models, the stochastic nature of product market returns gives rise to option values which must be taken into account in making optimal investment and abandonment decisions. In addition the discovery of a marketable invention is a Poisson arrival. Uncertainty in the relationship between inputs and outputs drives a wedge between the firm’s decision to invest in research and the outcome of this investment. As a result, the active firm faces a probability distribution over possible discovery dates. Thus, when the firm exercises its option to invest in research it gains a second option, that of making the discovery itself, whose exercise time occurs randomly rather than being a single date chosen explicitly by the firm.

The optimal investment strategy consists of a pair of trigger points for investment and abandonment. As in the Dixit (1989) product market model, sunk investment costs combined with uncertainty over market values cause the trigger point for investment to rise and that for abandonment to fall relative to their Marshallian equivalents, widening the region of hysteresis. When technological uncertainty is also present, however, a second option effect arises. This option value is due to the irreversibility of the discovery itself and raises both trigger points. At the investment trigger the two effects reinforce one another and research activity is further delayed compared with the Marshallian benchmark. At the abandonment trigger, however, the ‘discovery effect’ counteracts the sunk cost effect and the project is abandoned more rapidly than would otherwise be the case. When sunk costs are sufficiently small and the expected speed of discovery is high, the second effect dominates and abandonment takes place while expected profits are still positive, reversing the usual direction of hysteresis.

This finding is in stark contrast with the usual presumption that option effects cause firms to delay abandonment of an investment. The result can be explained as follows. Suppose that the firm is carrying out research at a time when the expected return is low, though positive. Discovery at this time incurs an opportunity cost of winning the prize at a later date when its value may be higher. While discovery has yet to occur the firm can retain the option over the invention by abandoning research at this point, resuming it later when conditions improve. Thus the model provides an alternative explanation to that of financial constraints for the abandonment of seemingly profitable projects.

In addition, the model can be shown to encompass two existing real options models, as well as generating a range of possible outcomes between the two. As the expected rate of discovery becomes negligible, the discovery effect is eliminated and the model becomes equivalent to the Dixit (1989) product market model. As the hazard rate tends to infinity, discovery occurs virtually as soon as research is commenced. The decision to invest in research becomes in effect the decision to make the discovery itself and the investment problem collapses to the McDonald and Siegel (1986) model of a single irreversible investment opportunity.

By incorporating technological uncertainty into a real options model of R&D investment, this paper combines two strands of the economic literature. A number of papers have modeled technological uncertainty in research as a Poisson arrival, including among others Loury (1979), Dasgupta and Stiglitz (1980), Lee and Wilde (1980), Reinganum (1983) and Dixit (1988). However, in these papers the return to successful research, or demand in the product market from which it is derived, is taken to be constant over time, thus ignoring the possibility of additional uncertainty over the economic value of the invention.

Real options models take account of economic uncertainty over the return to an investment but generally assume that the relationship between project inputs and outputs is deterministic, even when the investment under consideration is a research project. For example, Dixit and Pindyck (1994, pp. 319-326) consider a generalized two-stage investment project, where the first stage may be interpreted as research, in which the firm is able to invent immediately by paying a lump-sum cost. Lambrecht (1998) similarly models the creation of a patent as a single, deterministic step. Bar-Ilan and Strange (1998) allow an investment project to take time to build, but the length of the lag between commencement and completion is fixed. Thus, unlike here, the completion date of the project is known with certainty once investment is begun. Majd and Pindyck (1987) also consider sequential investment with time to build. Pindyck (1993) allows for technical uncertainty over the difficulty of completing a project, which can be seen as being analogous to an uncertain completion date. This information is endogenous to the investment process itself, being revealed only as the firm invests, thus tending to stimulate rather than hold back investment.

The structure of the paper is as follows. Section 2 describes the model and sets out optimality conditions for the firm's investment problem. A special case of the model in which the research technology involves no sunk costs is considered in section 3. Section 4 derives limiting results encompassing two existing real options models as special cases. Numerical simulations are presented in section 5. Section 6 concludes.

2 The Model

A single risk-neutral firm has the opportunity to invest in a research project, facing no actual or potential competitors in the area. There is both technological and economic uncertainty: discovery occurs randomly, and the value of the new technology follows a stochastic process. The firm's discount rate is given by the risk-free interest rate r, which is strictly positive, known and constant over time.

The firm invests by setting up a research unit of fixed scale, sinking an irrecoverable set-up cost, K. Throughout any period of research activity the firm incurs a flow cost of C per unit time. Abandonment requires another sunk cost, L, to be paid immediately, while the set-up cost K must be incurred again if the project is to be resumed at a later date. When the firm engages in research activity it achieves the discovery according to a Poisson distribution with parameter or 'hazard rate' $h > 0$. Thus the conditional probability that the firm makes the breakthrough in a short time interval of length dt, given that it has not done so before this time, is hdt and the density function for the duration of research is given by the exponential distribution he^{-ht}. For ease of exposition the research program can be thought of as consisting of h independent lines of research, each with a hazard rate of unity and cost levels k, c, and l defined such that $K \equiv kh$, $C \equiv ch$ and $L \equiv lh$.[1] This formulation will allow the cost and hazard rate parameters to be changed in numerical simulations without affecting the expected value of the project, which would otherwise obscure the option value effects.

Following discovery the new technology is patented and sold for a lump-sum amount, π. Discovery is taken to be a single step resulting in the creation of a marketable product. Hence the Schumpeterian distinction between invention – the initial breakthrough or product design – and innovation – further creative steps required for mass production of a marketable good – is ignored. The market value of the patent evolves exogenously according to the following geometric Brownian motion (GBM) with drift [2]

$$d\pi = \mu\pi dt + \sigma\pi dz \tag{1}$$

where $\mu \in [0, r)$ is the drift parameter, measuring the expected growth rate of π,[3]
$\sigma > 0$ is the instantaneous standard deviation or volatility parameter, and
dz is the increment of a standard Wiener process, $dz \sim N(0, dt)$.

As a benchmark with which subsequent findings may be compared, we first consider the behavior of a myopic firm that ignores the option values. Implicitly, such a firm acts as though the volatility parameter σ in (1) were equal to zero. In the presence of uncertainty this firm is likely to start up and abandon research too frequently, incurring excessive sunk costs in doing so. The breakeven value of the prize, denoted π_B, is defined to be the point at which the expected gain from research exactly balances its flow cost

$$h\pi_B = C \equiv ch .$$

Thus,

$$\pi_B = c . \tag{2}$$

In the absence of sunk or fixed costs the firm will carry out research whenever the value of the prize exceeds this level. Taking account of fixed costs but ignoring sunkness, Marshallian theory tells us that the firm will take account of interest charges on fixed costs in assessing its profitability. Incorporating fixed costs in this way, the firm will invest at the Marshallian investment point, denoted π_{MH}, at which expected revenues equal the sum of flow costs and interest payments on the fixed investment cost as follows

$$h\pi_{MH} = C + rK \equiv ch + rkh .$$

Thus,

$$\pi_{MH} = c + rk . \tag{3}$$

The Marshallian abandonment point, π_{ML}, is similarly defined such that

$$h\pi_{ML} = C - rL \equiv ch - rlh .$$

Thus,

$$\pi_{ML} = c - rl . \tag{4}$$

We now derive optimality conditions for a firm that optimizes its investment strategy in the face of uncertainty. The firm assesses its value in each of two states, inactive (state 0) or active (state 1). The value of the inactive firm, $V_0(\pi)$, is simply the value of the call option to invest at a later date. The value of the active firm, $V_1(\pi)$, takes into account the flow costs and expected benefits of research as well as the put option to abandon the project in the future.

The value function in the idle state, $V_0(\pi)$, satisfies the following dynamic programming equation

$$V_0(\pi) = e^{-rdt}\{V_0(\pi) + E(dV_0(\pi))\}.$$

By Itô's Lemma, and using (1) to substitute for $E(d\pi)$ and $E((d\pi)^2)$, the following expression can be derived (ignoring terms of second and higher order in *dt*)

$$\frac{1}{2}\sigma^2\pi^2V_0''(\pi) + \mu\pi V_0'(\pi) - rV_0(\pi) = 0. \tag{5}$$

If π is very small, the probability of it rising to a level at which the firm would wish to invest is very small and, therefore, the option to invest is almost worthless. Thus, we can impose the end-point condition

$$V_0(\pi) \to 0 \text{ as } \pi \to 0.$$

Solving the differential equation subject to this condition yields

$$V_0(\pi) = B\pi^{\beta_0} \tag{6}$$

where $B > 0$ is an unknown constant, and

$$\beta_0 = \frac{1}{2}\left\{1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}\right\} > 1.$$

Similarly, the value of the active firm, $V_1(\pi)$, must satisfy the following dynamic programming equation

$$V_1(\pi) = (h\pi - ch)dt + e^{-(r+h)dt}\left(V_1(\pi) + E[dV_1(\pi)]\right)$$

from which the following differential equation can be derived

$$\frac{1}{2}\sigma^2\pi^2 V_1''(\pi) + \mu\pi V_1'(\pi) - (r+h)V_1(\pi) + h\pi - ch = 0\,. \tag{7}$$

If π is very large, the value of the option to shut down is tiny and the value of the active firm tends to the simple NPV of the research project. Solving the differential equation subject to this end-point condition yields

$$V_1(\pi) = A\pi^{-\alpha_1} + \frac{h\pi}{r+h-\mu} - \frac{ch}{r+h} \tag{8}$$

where $A > 0$ is an unknown constant, and

$$\alpha_1 = \frac{1}{2}\left\{\frac{2\mu}{\sigma^2} - 1 + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8(r+h)}{\sigma^2}}\right\} \;>\; 0.$$

The optimal investment strategy is described by two trigger points at with the firm switches between the two states. These are denoted π_H, the upper trigger point at which the firm commences research, and π_L, the lower trigger point at which research is abandoned. These critical values must satisfy the following value-matching and smooth-pasting conditions.[4] At π_H it must be the case that

$$V_0(\pi_H) = V_1(\pi_H) - kh$$

and

$$V_0'(\pi_H) = V_1'(\pi_H);$$

while at π_L we have

$$V_1(\pi_L) = V_0(\pi_L) - lh$$

and

$$V_1'(\pi_L) = V_0'(\pi_L).$$

Substituting from expressions (6) and (8) for the value functions $V_0(\pi)$ and $V_1(\pi)$ respectively, the following system of equations is obtained

$$A\pi_H^{-\alpha_1} + \frac{h\pi_H}{(r+h-\mu)} - \frac{ch}{(r+h)} = B\pi_H^{\beta_0} + kh \tag{9}$$

$$-A\alpha_1\pi_H^{-\alpha_1-1} + \frac{h}{(r+h-\mu)} = B\beta_0\pi_H^{\beta_0-1} \tag{10}$$

$$A\pi_L^{-\alpha_1} + \frac{h\pi_L}{(r+h-\mu)} - \frac{ch}{(r+h)} = B\pi_L^{\beta_0} - lh \tag{11}$$

$$-A\alpha_1\pi_L^{-\alpha_1-1} + \frac{h}{(r+h-\mu)} = B\beta_0\pi_L^{\beta_0-1}. \tag{12}$$

Although this system is sufficient to determine the four unknowns π_H, π_L, A and B, the equations are non-linear in the trigger points and analytic solutions cannot in general be found. The size of the investment trigger π_H relative to the Marshallian point π_{MH} can be determined qualitatively, as shall be seen below. However, no corresponding inequality expressing the magnitude of the abandonment trigger π_L relative to π_{ML} can be derived.

Following Dixit (1989), a function $F(\pi)$ describing the difference between the two value functions can be defined as follows

$$\begin{aligned} F(\pi) &= V_1(\pi) - V_0(\pi) \\ &= A\pi^{-\alpha_1} - B\pi^{\beta_0} + \frac{h\pi}{(r+h-\mu)} - \frac{ch}{(r+h)}. \end{aligned} \tag{13}$$

The value-matching and smooth-pasting conditions can then be rewritten as follows

$$F(\pi_H) = kh, \qquad F(\pi_L) = -lh, \tag{14}$$

$$F'(\pi_H) = 0, \qquad F'(\pi_L) = 0. \tag{15}$$

The signs of the second derivatives at π_H and π_L are given by

$$F''(\pi_H) < 0, \qquad F''(\pi_L) > 0. \tag{16}$$

Using equations (5) and (7), the following expression in $F(\pi)$ and $V_1(\pi)$ can be derived

$$\frac{1}{2}\sigma^2\pi^2F''(\pi) + \mu\pi F'(\pi) - rF(\pi) - hV_1(\pi) + h\pi - ch = 0. \tag{17}$$

Comparing this differential equation with the corresponding expression in Dixit (1989) there is an additional term, $hV_1(\pi)$, which changes the analysis significantly. This term captures the discovery effect; with probability *h* the discovery is made and the continuation value $V_1(\pi)$ is lost. This analysis yields the following results concerning the magnitudes of the investment trigger points.

Proposition 1: *The optimal investment trigger point under uncertainty,* π_H*, exceeds the corresponding Marshallian trigger point,* π_{MH}.

Proof: The proposition is proved by evaluating equation (17) at π_H. We know from (14) – (16) that $F(\pi_H) = kh$, $F'(\pi_H) = 0$ and $F''(\pi_H) < 0$. Since the firm can get a payoff of zero by never investing, the value of the active firm at the investment trigger $V_1(\pi_H)$ must be non-negative.

$$h\pi_H - ch = rF(\pi_H) - \mu\pi_H F'(\pi_H) - \frac{1}{2}\sigma^2 \pi_H{}^2 F''(\pi_H) + hV_1(\pi_H)$$
$$> rkh.$$

Dividing through by *h* and rearranging, the required inequality is obtained

$$\pi_H > c + rk = \pi_{MH}.$$

Q.E.D.

The direction of this inequality is qualitatively identical to the hysteresis result found in Dixit (1989): the optimal investment trigger π_H always lies above the Marshallian investment point π_{MH}. However, in this case the option effect due to sunk costs found in the Dixit paper is augmented by the additional term $hV_1(\pi_H)$. This further raises the level of π_H, changing the quantitative outcome of the model.

Proposition 2: *The location of the optimal abandonment trigger point under uncertainty,* π_L*, relative to the corresponding Marshallian trigger point,* π_{ML}*, is ambiguous.*

Proof: Evaluating equation (17) at π_L, we know from (14) – (16) that $F(\pi_L) = -lh$, $F'(\pi_L) = 0$ and $F''(\pi_L) > 0$. However, it can be seen from (8) that $V_1(\pi_L)$ may be either positive or negative: expected flow profits $h\pi_L - ch$ are likely to be small or negative, but the value of the option to quit given by $A\pi_L^{-\alpha_1}$ will be large and positive. Thus, the direction of the inequality cannot be determined in general. When $F''(\pi_L)$ is large the sunk cost effect dominates and the outcome is the same as in the Dixit model with $\pi_L < \pi_{ML}$. However, when $V_1(\pi_L)$ is positive and large the discovery effect dominates. In this case $\pi_L > \pi_{ML}$, reversing the usual direction of hysteresis. Q.E.D.

3 Model without sunk costs

In this section we analyze a special case of the model in which the sunk cost elements of the research technology are eliminated. By removing the hysteresis effects of sunk costs the impact of technological uncertainty and the discovery effect can be examined more clearly. This formulation also yields an explicit analytical result, allowing the roles of the underlying parameters to be examined in detail.

As sunk costs are eliminated the trigger points for investment and abandonment converge to a single point, which we shall denote π_0. However, the value-matching and smooth-pasting conditions will then be satisfied at any arbitrarily-chosen switching point and no longer determine the optimal investment strategy. To see this, reduce the simultaneous equation system (9) – (12) to two equations by eliminating the constant terms *A* and *B*, and then set $k = l = 0$. The resulting expressions are satisfied by any pair (π_H, π_L) such that $\pi_H = \pi_L$; i.e. the solution set is a ray from the origin. An additional first-order condition, given below, is needed to ensure that the trigger point maximizes firm value

$$\frac{\partial V_0}{\partial \pi_0} = \frac{\partial V_1}{\partial \pi_0} = 0. \tag{18}$$

The value of π_0 satisfying these equations, derived in appendix 1, is given by (19). This expression can also be derived as the limiting result of the system (9) – (12) as sunk costs are taken to zero.

$$\pi_0 = c\left\{\frac{(r+h-\mu)}{r+h}\frac{\beta_0\alpha_1}{(\beta_0-1)(\alpha_1+1)}\right\}. \tag{19}$$

Proposition 3: *The unique trigger point in the model without sunk costs,* π_0*, is greater than or equal to the breakeven level of the project, c. Specifically,* $\pi_0 = c$ *when* $h = 0$ *and* $\pi_0 > c$ *for* $h > 0$.

Proof: See appendix 2.

Corollary: When sunk costs are sufficiently small and the hazard rate h is sufficiently large, the abandonment trigger π_L exceeds c. In this case, the firm will abandon research while expected flow profits are positive, reversing the usual direction of hysteresis at this trigger point. This result is illustrated numerically in section 5.

4 Two polar cases

Two existing real options models can be expressed as polar cases of this model in which the hazard rate takes extreme values. When the hazard rate is negligible, the expected speed of discovery becomes extremely slow and the probability of losing the option to invest in the future becomes remote. The discovery effect is virtually eliminated and the option becomes essentially a perpetual one. As h tends to zero the model approaches the Dixit (1989) model of a firm's entry and exit decisions, in which the firm has a perpetual option to operate and current product market activity does not rule out further activity in the future.

As the hazard rate becomes very large, on the other hand, discovery becomes almost instantaneous. The decision to undertake research is, in effect, a decision to make the discovery and cash in the option once and for all. As h approaches infinity the model collapses to the McDonald and Siegel (1986) model of a single, irreversible investment opportunity with a constant investment cost.

Proposition 4: *The limiting case as* $h \to 0$ *is the Dixit (1989) model of a firm's optimal product market entry and exit decisions.*

Proof: In the system of equations (9) – (12) the hazard rate h is, in effect, a unit of account which scales the equations. The unknown constants A and B are scaled by h, as seen in the following expressions for A and B derived from equations (9) and (10)

$$A = \frac{h\pi_H^{\alpha_1}}{(\alpha_1 + \beta_0)} \left\{ \beta_0 \left(\frac{c}{r+h} + k \right) - (\beta_0 - 1) \frac{\pi_H}{r+h-\mu} \right\} = h\, a(h) \qquad (20)$$

$$B = \frac{h\pi_H^{-\beta_0}}{(\alpha_1+\beta_0)}\left\{(\alpha_1+1)\frac{\pi_H}{r+h-\mu} - \alpha_1\left(\frac{c}{r+h}+k\right)\right\} \ = \ h\, b(h) \tag{21}$$

where $\lim_{h\to 0} a(h)$ and $\lim_{h\to 0} b(h)$ are finite constants.

Substituting the expressions for *A* and *B*, canceling terms and taking limits as $h \to 0$, the system (9) – (12) can be rewritten in the following form

$$a\pi_H^{-\alpha_0} + \frac{\pi_H}{(r-\mu)} - \frac{c}{r} = b\pi_H^{\beta_0} + k \tag{22}$$

$$-\alpha_0 a\pi_H^{-\alpha_0-1} + \frac{1}{(r-\mu)} = \beta_0 b\pi_H^{\beta_0-1} \tag{23}$$

$$a\pi_L^{-\alpha_0} + \frac{\pi_L}{(r-\mu)} - \frac{c}{r} = b\pi_L^{\beta_0} - l \tag{24}$$

$$-\alpha_0 a\pi_L^{-\alpha_0-1} + \frac{1}{(r-\mu)} = \beta_0 b\pi_L^{\beta_0-1}. \tag{25}$$

where $\alpha_0 = \frac{1}{2}\left\{\frac{2\mu}{\sigma^2} - 1 + \sqrt{\left(1-\frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}\right\}$ and β_0 is as defined before.

The system (22) – (25) is equivalent to the fundamental set of equations derived in Dixit (1989), with π_H and π_L respectively taking the places of the critical values of the product market price, P_H and P_L. Thus, for a Dixit-style model in which *P* follows a GBM identical to that described by (1) above, and with entry, exit and operating costs given by *k*, *l* and *c* respectively, we can write

$$\lim_{h\to 0} \pi_H = P_H$$

and

$$\lim_{h\to 0} \pi_L = P_L.$$

Q.E.D.

Proposition 5: *The limiting case as h → ∞ is the McDonald and Siegel (1986) model of an irreversible investment decision with a constant investment cost K.*

Proof: In this case we use the original version of the model in which all cost parameters are constants (*C*, *K*, *L*), rather than being expressed in per-unit hazard rate terms (*c*, *k*, *l*). Substituting for the unknown constants *A* and *B* using (20) and (21) respectively, equation (11) can be written as follows[5]

$$\left(\frac{\pi_H}{\pi_L}\right)^{\alpha_1}\frac{1}{(\alpha_1+\beta_0)}\left\{\beta_0\left(\frac{C}{r+h}+K\right)-(\beta_0-1)\frac{h\pi_H}{r+h-\mu}\right\}+\frac{h\pi_L}{r+h-\mu}-\frac{C}{r+h}$$
$$=\left(\frac{\pi_H}{\pi_L}\right)^{-\beta_0}\frac{1}{(\alpha_1+\beta_0)}\left\{(\alpha_1+1)\frac{h\pi_H}{r+h-\mu}-\alpha_1\left(\frac{C}{r+h}+K\right)\right\}-L \tag{26}$$

When there is any degree of sunk costs (i.e. when it is *not* the case that $K = L = 0$), π_H strictly exceeds π_L, thus $\pi_H/\pi_L > 1$. Since α_1 is of order $\sqrt{h}$ and power terms dominate, it must be the case that

$$\beta_0\left(\frac{C}{r+h}+K\right)-(\beta_0-1)\frac{h\pi_H}{r+h-\mu}\rightarrow 0 \text{ as } h\rightarrow\infty.$$

Bearing in mind that $\frac{C}{r+h}\rightarrow 0$ and $\frac{h}{r+h-\mu}\rightarrow 1$ as $h\rightarrow\infty$, we can derive

$$\lim_{h\rightarrow\infty}\pi_H=\frac{\beta_0}{\beta_0-1}K \tag{27}$$

which is equivalent to the McDonald and Siegel (1986) model with a constant investment cost, as set out in Dixit and Pindyck (1994, pp. 136-142). Q.E.D.

Substituting the limiting value of π_H it is clear that the value of the option to abandon research, given by $A\pi^{-\alpha_1}$, approaches zero as $h\rightarrow\infty$. Since discovery becomes instantaneous, occurring as soon as the investment takes place, abandonment of the uncompleted research project is no longer a realistic possibility and the option to do so has no value. In this case, the abandonment trigger π_L has no economic interpretation.

5 Numerical simulations

The system of equations (9) – (12) can be solved using numerical techniques to find solutions for π_H and π_L corresponding to a particular set of parameter values. In the simulations shown below the mathematical computation program *Matlab* was used find solutions to the system of equations. The parameter values used in the simulations are as follows. The parameters of the geometric Brownian motion governing the prize value π are $\mu = 0$ and $\sigma = 0.2$. The research technology requires an initial set-up cost of k = 0.5 and a flow cost of c = 1 per unit time, during which the firm has a probability of success given by the hazard rate h = 0.5. There is no exit cost (l = 0). The risk-free interest rate is r = 0.05. In each simulation the value of one parameter is varied while the rest are held constant at these levels.

Table 1 compares trigger points in the R&D model with the corresponding values found using the Marshallian approach and an equivalent Dixit-style model in which P follows an identical GBM and the entry, exit and operating costs are given by k, l and c respectively.

Table 1: Critical values in the Marshallian, Dixit and R&D models

	Marshallian model	Dixit model	R&D model
Entry point	$\pi_{MH} = 1.025$	$P_H = 1.31$	$\pi_H = 2.36$
No sunk costs	$\pi_B = c = 1.00$	$P_B = c = 1.00$	$\pi_0 = 1.54$
Exit point	$\pi_{ML} = 1.00$	$P_L = 0.80$	$\pi_L = 1.06$

As in Dixit (1989), the product market triggers P_H and P_L diverge rapidly from the breakeven level c compared with the Marshallian levels. Meanwhile the trigger points in the R&D model, π_H and π_L, are unambiguously higher than their Dixit counterparts. In consequence, the location of π_L, the trigger point at which research activity is abandoned, relative to the breakeven point is ambiguous in general. In this particular example, sunk costs are sufficiently small that the abandonment trigger, at 1.06, is greater than the breakeven point given by c = 1. Thus, in this case the usual direction of hysteresis at the lower trigger point is reversed and a project with a strictly positive expected value will be abandoned. The high level of π_0 relative to c shows that even in the absence of sunk costs there will be considerable delay before a research project is commenced.

Figures 1 to 4 illustrate the impact on the trigger points of varying the value of one parameter while all others are held constant at the values given above. Figure 1 shows the effect of varying the investment cost k over the range (0, 1], illustrating the 'reverse hysteresis' result. For sufficiently small values of k, π_L lies above the breakeven value given by c = 1. As k approaches zero, π_H and π_L converge to π_0 at a value of 1.54, a level considerably greater than c. As in Dixit (1989), trigger points diverge rapidly as sunk costs increase. At higher levels of k the abandonment trigger falls below the breakeven level and the usual direction of hysteresis is re-established.

Figure 2 shows the effect of changing the volatility parameter σ over the range (0, 0.2]. Option values rise with uncertainty, since the value of being able to curtail the downside of the distribution is greater when there is a wider dispersion of possible future outcomes. The investment trigger π_H is seen to rise steeply with σ, while the abandonment trigger falls slightly at first and then rises above the breakeven level as σ increases further. These results illustrate the different impacts of the two option effects at each trigger point. At π_H the two option effects combine to raise the trigger point as σ increases: the greater danger of being stranded with sunk costs and the possibility of higher returns in the more distant future both increase delay. At π_L, however, the two effects are in conflict. Initially as σ rises from a low level the sunk cost effect dominates and π_L falls slightly. At higher values of σ the discovery effect dominates, causing π_L to rise above the breakeven point.

Figure 3 shows the effect of varying the hazard rate h over the range [0, 0.5]. Note that due to the per-unit hazard rate cost formulation ($C = ch$, etc.), a change in h does not alter the expected value of the project. All three trigger points rise with h, steeply at first and then more gently as h increases further. A larger hazard rate increases the probability that the discovery will be made in the near rather than distant future, strengthening the discovery effect and raising the trigger points. Convergence of the R&D trigger points to their Dixit counterparts as $h \to 0$ is illustrated in figure 4, which shows values of h over the range [0, 0.2] only. As h becomes negligible, π_H approaches the Dixit entry point P_H = 1.3, π_L approaches the exit point P_L = 0.8, and π_0 approaches the breakeven level given by $c = 1$.

6 Concluding remarks

We have shown that when a firm has the opportunity to invest in a research project facing both technological and economic uncertainty, two option effects arise. The first is the standard hysteresis effect due to the presence of sunk investment costs. As in the Dixit (1989) product market model, this effect raises the trigger point for investment and lowers that for abandonment, resulting in hysteresis. The second option effect results from the irreversibility of discovery itself combined with technological uncertainty over its timing. This 'discovery effect' raises both trigger points. Thus at the investment trigger the discovery effect augments the effect of sunk costs, implying that the hurdle rate for research projects will be particularly high. At the abandonment trigger, by contrast, the two effects are in conflict raising the possibility of 'reverse hysteresis.' If sunk costs are small or the expected speed of discovery is high, the discovery effect dominates and the abandonment point exceeds the Marshallian level, reversing the standard hysteresis result. In extreme cases a firm may abandon a research project in a downturn even while its expected value remains positive.

By allowing for stochastic discovery, thus introducing uncertainty into the relationship between investment inputs and outputs, we have provided a general framework encompassing two existing real options models as polar cases. As the hazard rate for discovery tends to zero the model becomes equivalent to the Dixit (1989) model of a firm's optimal product market entry and exit decisions. As the hazard rate tends to infinity and discovery becomes instantaneous, the investment trigger approaches the McDonald and Siegel (1986) result for an irreversible investment opportunity with a constant investment cost. Thus, two existing models can be incorporated in an intuitively appealing form as extreme cases of this model, with a further range of possible outcomes between the two.

The model has a number of implications for policy towards research. First, the analysis suggests that hurdle rates for investment in research-intensive sectors are likely to be even higher than those used in other industries with equivalent levels of sunk costs and product market uncertainty. However, it should be noted that, in the absence of externalities, a social planner would take account of option values in the same way as the private firm and the high trigger points implied by this analysis are efficient. Hence the observation of a high hurdle rate for investment in research is not necessarily a basis for policy action without evidence of social externalities or other market imperfections.

In Dixit (1989) greater uncertainty raises the trigger point for investment but lowers that for abandonment, and so the long-run effect of uncertainty on economic activity is ambiguous. In this model greater economic uncertainty is likely to reduce the overall level of research activity, since the discovery effect raises both trigger points relative to their equivalent Dixit levels. Thus our analysis suggests that greater economic uncertainty is likely to reduce overall research activity and, to the extent that new technology is an important engine of economic growth, the resulting growth rate is likely to be lower.

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Appendix 1: Derivation of trigger point without sunk costs

In the absence of sunk costs the trigger points for investment and abandonment, π_H and π_L, converge to a single point, denoted π_0. At this point the value-matching and smooth-pasting conditions hold as usual. However, in the absence of sunk costs smooth-pasting is no longer an optimality condition but will hold at any arbitrarily-chosen switching point. Suppose two value functions intersect at two separate switching points, with the value-matching but not the smooth-pasting condition holding at each point. As the points of intersection converge to a unique switching point, the slopes of the value functions become equal and the two value-matching conditions are replaced by one value-matching and one smooth-pasting condition. A first-order condition must then be used to determine the optimal choice of π_0.

The value-matching and smooth-pasting conditions at π_0 are given respectively by

$$B\pi_0^{\beta_0} = A\pi_0^{-\alpha_1} + \frac{h\pi_0}{r+h-\mu} - \frac{ch}{r+h}$$
$$\beta_0 B\pi_0^{\beta_0-1} = -\alpha_1 A\pi_0^{-\alpha_1-1} + \frac{h}{r+h-\mu}.$$

Solving for the unknown constants A and B, the following expressions are derived

$$A = \frac{-h\pi_0^{\alpha_1}}{\beta_0+\alpha_1}\left\{(\beta_0-1)\frac{\pi_0}{r+h-\mu} - \beta_0\frac{c}{r+h}\right\}$$
$$B = \frac{h\pi_0^{-\beta_0}}{\beta_0+\alpha_1}\left\{(\alpha_1+1)\frac{\pi_0}{r+h-\mu} - \alpha_1\frac{c}{r+h}\right\}.$$

The value function $V_0(\pi)$ can then be written as

$$V_0(\pi) = B(\pi_0)\pi^{\beta_0}.$$

The first order condition with respect to π_0, ensuring optimality, is given by

$$\frac{\partial V_0}{\partial \pi_0} = 0$$

which, for any arbitrary non-zero value of π, requires

$$\frac{\partial B}{\partial \pi_0} = 0.$$

Thus,

$$\frac{dB}{d\pi_0} = \frac{h\pi_0^{-\beta_0-1}}{\beta_0+\alpha_1}\left\{(1-\beta_0)(\alpha_1+1)\frac{\pi_0}{r+h-\mu} + \beta_0\alpha_1\frac{c}{r+h}\right\} = 0.$$

Solving the first-order condition, the following expression for the optimal switching point π_0 is obtained. The second-order condition is negative, ensuring that the point is a maximum.

$$\pi_0 = c\frac{(r+h-\mu)}{(r+h)}\frac{\beta_0\alpha_1}{(\beta_0-1)(\alpha_1+1)}.$$

Appendix 2: Proof of proposition 3

We wish to prove that $\pi_0 \geq c$. The proof consists of two steps:

(i) demonstrating that $\pi_0 = c$ when h = 0, and

(ii) showing that $\pi_0(h)$ is a strictly increasing function, i.e. $\frac{\partial \pi_0}{\partial h}$ > 0.

(i) When h = 0, the expression for π_0 becomes

$$\pi_0 = c\frac{(r-\mu)}{r}\frac{\beta_0\alpha_0}{(\beta_0-1)(\alpha_0+1)}.$$

Substituting the relevant expressions for the roots and simplifying, we obtain

$$\frac{\beta_0\alpha_0}{(\beta_0-1)(\alpha_0+1)} = \frac{r}{(r-\mu)}.$$

Substituting into the expression given above for π_0, it can be determined that $\pi_0 = c$ when h = 0.

(ii) The partial derivative of π_0 with respect to h is given by

$$\frac{\partial \pi_0}{\partial h} = \frac{c\beta_0}{(\beta_0-1)[(r+h)(\alpha_1+1)]^2}\left\{\mu\alpha_1(\alpha_1+1)+(r+h)(r+h-\mu)\frac{\partial \alpha_1}{\partial h}\right\}.$$

We know that $\beta_0 > 1$ and, by definition, c > 0, thus

$$\operatorname{sgn}\frac{\partial \pi_0}{\partial h} = \operatorname{sgn}\left\{\mu\alpha_1(\alpha_1+1)+(r+h)(r+h-\mu)\frac{\partial \alpha_1}{\partial h}\right\}.$$

Since $\alpha_1 > 0$ and $r > \mu \geq 0$, a sufficient condition for $\frac{\partial \pi_0}{\partial h}$ > 0 is that $\frac{\partial \alpha_1}{\partial h}$ > 0, as is clearly the case.

Figure 1: Effect of the sunk R&D entry cost *k*

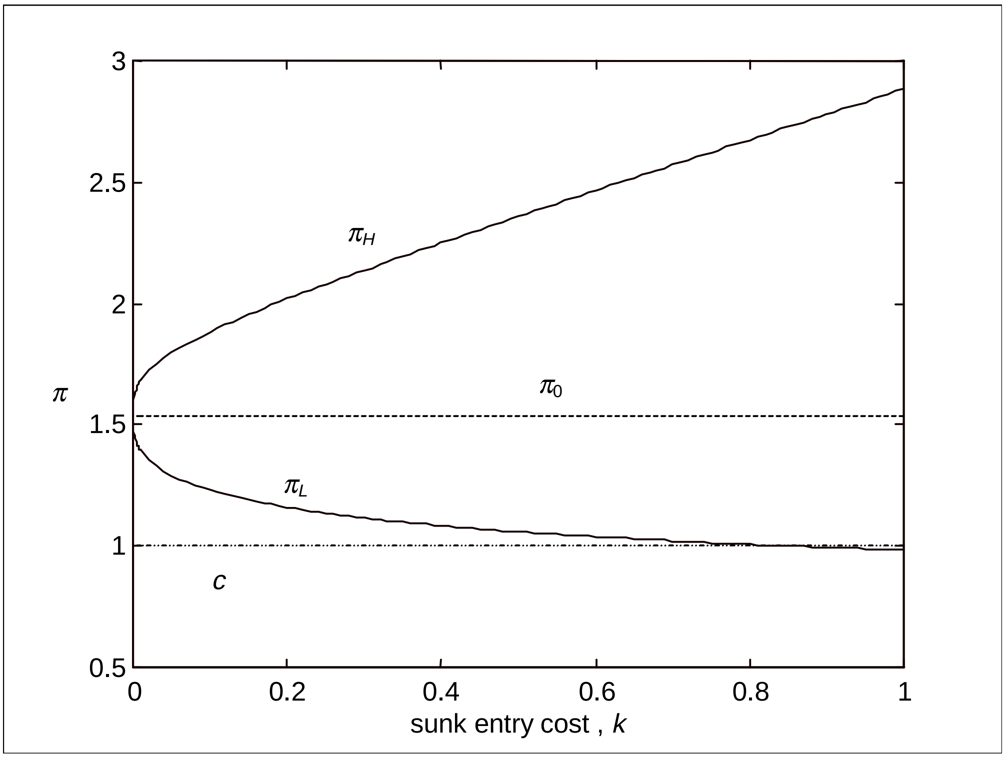


Figure 2: Effect of the volatility parameter, σ

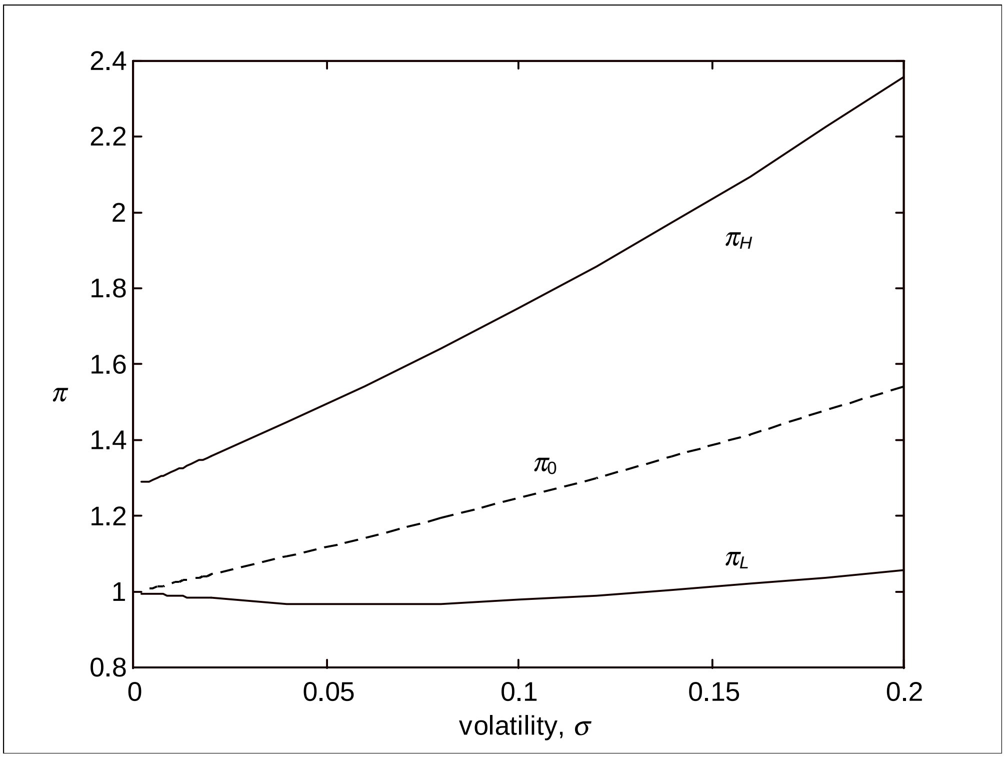


Figure 3: Effect of the hazard rate *h*

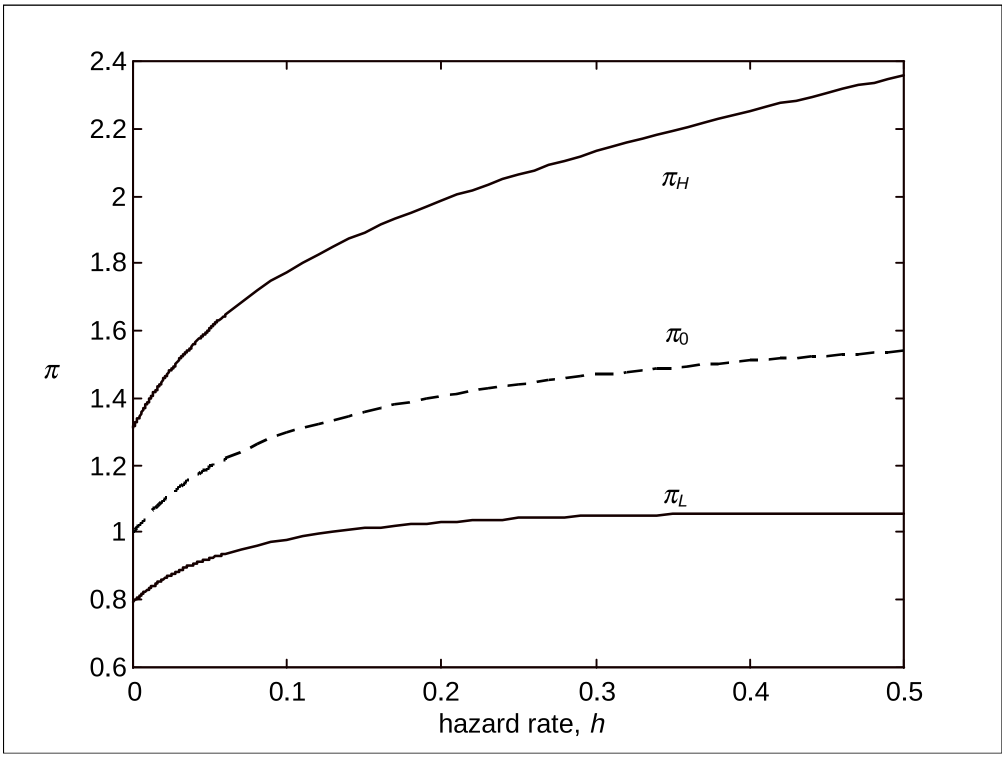
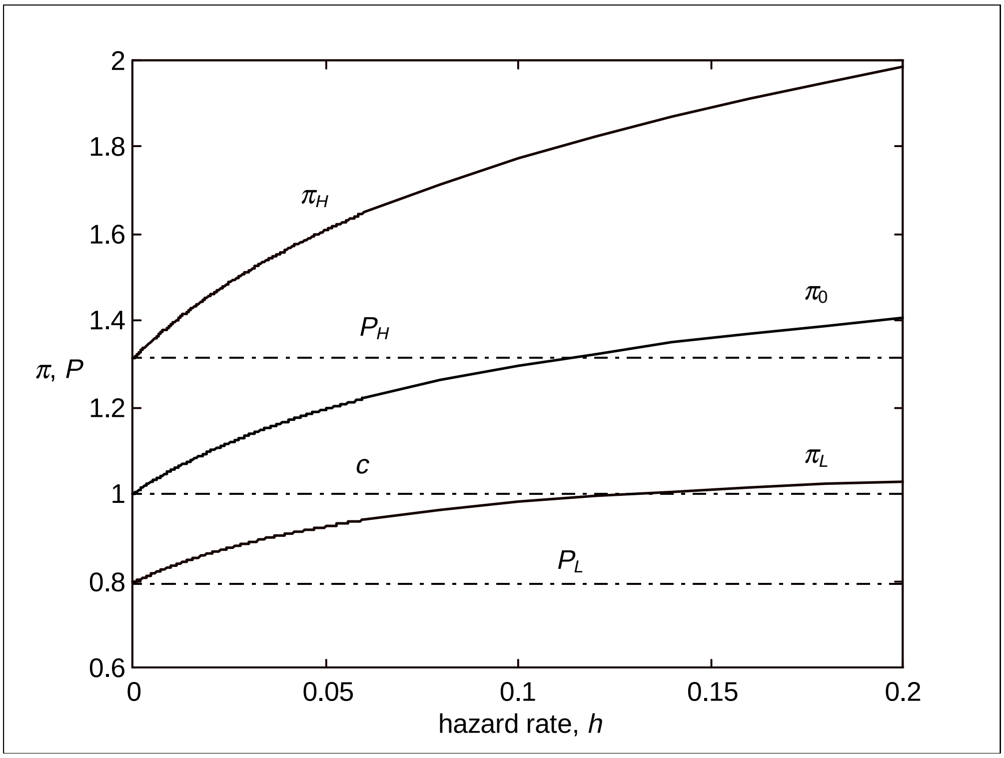


Figure 4: Convergence to the Dixit model as $h \to 0$



[1] Note that the number of research lines is not a choice variable for the firm. Given that individual lines are independent, the hazard rate of the entire project is given by $h.1 = h$.

[2] Alternatively, one could assume that the product market price, per-period profit or some underlying demand variable follows such a process, with π then calculated as the expected NPV of profits over the life of the patent. However, the simpler approach adopted in the paper can be justified as follows. If per-period profit follows GBM and avoidable costs are so low relative to revenues that the firm never wishes to cease production, the expected NPV is directly proportional to current profit. The factor of proportionality, λ, depends upon the trend growth rate μ, the discount rate r and the duration of the patent. If a variable x follows GBM then, by Itô's lemma, λx also follows GBM.

[3] The restriction that $\mu < r$, commonly found in real options models, is necessary to ensure that there is a positive opportunity cost to holding the option so that it will not be held indefinitely. The requirement that μ is non-negative is made for mathematical convenience; since the model is concerned with the effects of uncertainty, not expected trends, the results are not affected by this assumption.

[4] Expressed generally, value-matching requires the value of the firm in the two states to be equal at an optimal trigger point π^*, taking into account the sunk cost incurred in switching between the two. The smooth-pasting condition requires the value functions to meet smoothly at the trigger point. The necessity of this condition can be explained as follows. If instead there were a kink at this point, a deviation from the supposedly optimal policy raises the firm's expected payoff. By delaying for a small interval of time after the Brownian motion process first reaches π^*, the next step $d\pi$ is observed. If the kink is convex, the firm may obtain a higher expected payoff by entering if and only if π has moved (strictly) above π^*, since an average of points on either side of the kink give it a higher expected value than the kink π^* itself. If the kink is concave, on the other hand, second order conditions are violated. Continuation along the original value function would yield a higher payoff than switching to the alternative function, thus switching at π^* cannot be optimal. Further explanation of this condition can be found in appendix C of chapter four in Dixit and Pindyck (1994).

[5] Note that an identical expression for $\lim_{h\to\infty} \pi_H$ can be derived using equation (12).