A law of scarcity for games.

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Abstract

The "law of scarcity" is that scarceness is rewarded; recall, for example, the diamonds and water paradox. In this paper, furthering research initiated in Kelso and Crawford (1982, *Econometrica* 50, 1483-1504) for matching models, we demonstrate a law of scarcity for cores and approximate cores of games.

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1 Introduction.

We show that under small group effectiveness (SGE), ensuring that all or almost all gains to collective activities can be realized by groups of players bounded in size, equal treatment payoff vectors in cores and ε-cores satisfy cyclic monotonicity, a property related to the strong axiom of revealed preference.[1] We also demonstrate a closely related comparative statics result that when the relative size of a group of players who are all similar to each other increases, then equal-treatment ε-core payoffs to members of that group will not significantly increase and may decrease. Following Wooders (1992b), we call these results laws of scarcity.[2] A game is described by certain parameters: (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. An equal treatment payoff vector assigns the same payoff to all players of the same approximate type. The conditions required on a game to obtain our results are that (i) each player has many close substitutes (a thickness condition) and (ii) that all or almost all gains to collective activities can be realized by groups of players bounded in size (SGE). The second condition may appear to be restrictive, but in fact, in the context of a "pregame," if there are sufficiently many players of each type, then per capita boundedness (PCB) – finiteness of the supremum of average payoff – and SGE are equivalent.[3]

Our results may be viewed as a contribution to literature on comparative statics properties of solutions of games. As noted by Crawford (1991), the first suggestion of the sort of results obtained in this paper may be in Shapley (1962), who showed that in a linear optimal-assignment problem the marginal product of a player on one side of a market weakly decreases when another agent is added to that side of the market and weakly increases when an agent is added to the other side of the market. Kelso and Crawford (1982), building on the model of Crawford and Knoer (1981), show that, for a many-to-one matching market with firms and workers, adding one or more firms to the market makes the firm-optimal stable outcome weakly better for all workers and adding one or more workers makes the firm-optimal stable outcome weakly better for all firms. Crawford (1991) extends these results to both sides of the market and to many-to-many matchings.[4] In contrast to this literature, our results

[1]See, for example, the discussion in Epstein (1981).

[2]Because of the similarity of the law of scarcity to the celebrated Law of Demand (cf. Hildenbrand 1994) the term "law of supply" was considered but rejected since, in the context of games, there appears to be no clear distinction between supply and demand. Depending on the details of the game, additional players may not supply anything except demand. For example in a market game with buyers and sellers demand and supply are interwoven. In addition, there is already an idea in the literature sometimes called the Law of Supply – the amount demanded of an input to production is inversely related to its price.

[3]Wooders (1994a), Theorem 4..

[4]And also to pair-wise stable outcomes but this is apparently not so directly related to our paper.

are not restricted to matching markets and treat all outcomes in equal treatment ε-cores. Moreover, we demonstrate cyclic monotonicity. Instead of the assumptions of "substitutability" of Kelso and Crawford (1982) we require our thickness condition and SGE. In addition, our results are limited to games with side payments – we discuss this restriction further in the concluding section.

Besides the matching literature, our results are related to prior results obtained within the context of a pregame. A pregame specifies a set of player types and a *single* worth function, assigning a worth to each finite list of attributes (repetitions allowed). Since there is only one worth function, all games derived from a pregame are related and, given the attributes of the members of a coalition, the payoff to that coalition is independent of the total player set in which the coalition is embedded; widespread externalities are not allowed. Note also that the domain of the worth function of a pregame is a topological space of player types – either a finite set or more generally a compact metric space of player types. These features of pregames have consequences; for example, the equivalence between SGE and PCB noted above for pregames does not hold in general for parameterized collections of games.[5] In contrast, our results apply to given games and, as in the earlier results for matching models, there is no requisite topological structure on the space of players types. While our results for a given game hold for all games in a collection described by the same parameters, there are no necessary relationships between games. For example, consider the collection of games where two-player coalitions are effective and there are only two types of players. This collection includes two-sided assignment games, such as marriage games and buyer-seller games, and also games where *any* two-player coalition is effective. There appears to be no way in which one pregame can accommodate all the games in the collection. Moreover, it is not required that games in a parameterized collection satisfy PCB. These considerations indicate that the framework of parameterized collections of games is significantly broader than that of a pregame.[6]

Finally, our research has the advantage that the condition of SGE is defined to apply to a given game. In contrast, the conditions of PCB and SGE in the extant literature are defined by conditions on all games derived from a pregame or, in other words, by conditions on the entire pregame structure, rather than conditions on a given game. This is especially troublesome when SGE is defined in terms of limiting equal treatment payoffs for arbitrarily large games derived from pregames. We discuss this issue further below.

In the next section we define parameterized collections of games. In Section 3, the results are presented. Section 4 consists of an example, applying our results to a matching model with hospitals and interns. Section 5 relates the current paper to the

[5] See Wooders (1994b) or Kovalenkov and Wooders (1999b) for an example.

[6] A short survey discussing parameterized collections of games and their relationships to pregames appears in Wooders (1999).

literature and concludes the paper. An appendix demonstrates that all the bounds obtained are the *least* upper bounds for some sequences of games.

2 Definitions.

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *characteristic function*, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game* (*with side payments* or a *TU game*). Non-empty subsets of N are called *coalitions* or *groups.* A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\left\{S^k\right\}$ of S. We restrict our attention to superadditive games.

2.1 Parameterized collections of games.

$\delta-$substitute partitions. In our approach we approximate games with many players, all of whom may be distinct, by games with finite sets of player types.

Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real number. Informally, a δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. That is, if a player in a coalition is replaced by a δ-substitute, the payoff to that coalition changes by no more than δ per capita. Formally, given a partition $\{N[t] : t = 1, .., T\}$ of N, a permutation τ of N is *type consistent* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A *δ-substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S| .$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games may have the same partitions but have no other relationship to each other (in contrast to games derived from a pregame).

(δ,T)- type games. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, v) is a (δ, T)-*type game* if there is a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N. The set $N[t]$ is interpreted as an *approximate type.* Players in the same element of a δ-substitute partition are *δ-substitutes.* When $\delta = 0$, they are *exact substitutes.*

profiles. Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, v) be a game and let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β-effective B-bounded groups. The following notion formulates the idea of SGE in the context of parameterized collections of games. Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let β be a given non-negative real number, and let B be a given integer. A game (N, v) has *β-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $\left|S^k\right| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta |S|.$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups.*

parametrized collections of games $\Gamma((\delta, T), (\beta, B))$. Let T and B be positive integers, let δ and β be non-negative real numbers. Define

$$\Gamma((\delta, T), (\beta, B))$$

to be the collection of all (δ, T)-type games that have β-effective B-bounded groups.

2.2 Equal treatment ε-core.

the core and ε-cores. Let (N, v) be a game and let ε be a nonnegative real number. A payoff x is in the *ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon |S|$ for all $S \subset N$. Then $\varepsilon = 0$, the ε-core is the *core.*

the equal treatment ε-core. Given nonnegative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, v) relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, $x_i = x_j$.

With the definition of the equal treatment ε-core in hand, we can next address monotonicity properties and comparative statics for this concept. In the present paper we simply assume nonemptiness of the equal treatment ε-core of games. Under a non-restrictive condition of per capita boundedness, for $\varepsilon > 0$ this assumption is satisfied for all sufficiently large games in parameterized collections. Such a result appears in Kovalenkov and Wooders (1999b).

3 Results.

A technical lemma is required. For $x, y \in R^T$, let $x \cdot y$ denote the scalar product of x and y, i.e. $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1, 2$, let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Proof: Since (N, v) has β-effective B-bounded groups, there exists a partition $\left\{G^{1\ell}\right\}$ of S^1, such that $\left|G^{1\ell}\right| \leq B$ for any ℓ and $\sum_\ell v(G^{1\ell}) \geq v(S^1) - \beta \|f^1\|$. Let us denote the profiles of $G^{1\ell}$ by g^ℓ. Observe that $\sum_\ell g^\ell = f^1$.

Since $f_t^2 \geq B$ for each t, it holds that $g^\ell \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^ℓ. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^ℓ, it holds that $\left|v(G^{1\ell}) - v(G^{2\ell})\right| \leq \delta \left\|g^\ell\right\|$. Since x^2 represents a payoff in the equal treatment ε-core of (S^2, v) and $G^{2\ell} \subset S^2$ has profile g^ℓ, the total payoff $x^2 \cdot g^\ell$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \left\|g^\ell\right\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^ℓ, it holds that $x^2 \cdot g^\ell \geq v(G^{2\ell}) - \varepsilon \left\|g^\ell\right\| \geq v(G^{1\ell}) - (\varepsilon + \delta) \left\|g^\ell\right\|$. Adding these inequalities we have $x^2 \cdot f^1 \geq \sum_\ell v(G^{1\ell}) - (\varepsilon + \delta) \|f^1\|$. It then follows that $x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta) \|f^1\|$.

Since x^1 represents a payoff in the equal treatment ε-core of (S^1, v), $x^1 \cdot f^1$ is feasible for (S^1, v), that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have $(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \|f^1\|$. ■

Now we can state and prove our main results.

3.1 Approximate cyclic monotonicity.

We give a bound on the amount by which approximate core payoffs for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on: δ, the extent players within each of T types may differ from being exact substitutes for each other; β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and on ε, a measure of the extent to which the ε-core differs from the core. Our result is stated both for absolute numbers and for proportions of players of each type. If exact cyclic monotonicity were satisfied, then the right hand sides of the equations (1) and (2) below could be set equal to zero.

Proposition 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), .., (S^K, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k

denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 + .. + f^K \right\| \quad (1)$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq K(\varepsilon + \delta + \beta). \quad (2)$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity.

Proof: From Lemma we have $(x^k - x^{k+1}) \cdot f^k \leq (\varepsilon + \delta + \beta) \left\| f^k \right\|$ for $k = 2, .., K-1$ and $(x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\| f^K \right\|$. Summing these inequalities we get (1).

Alternatively we have $(x^k - x^{k+1}) \cdot \frac{f^k}{\|f^k\|} \leq (\varepsilon + \delta + \beta)$ for $k = 1, .., K-1$ and $(x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain (2). ■

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 \right\| \text{ and } (x^1 - x^2) \cdot \left(\frac{f^1}{\|f^1\|} - \frac{f^2}{\|f^2\|}\right) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

3.2 Comparative Statics.

For $j = 1, .., T$ let us define $e^j \in \mathbf{R}^T$ such that $e_l^j = 1$ for $l = j$ and 0 otherwise. Our comparative statics results relate to changes in the abundances of players of a particular type.

Proposition 2. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff in the equal treatment ε-core of (S^k, v). Then the following holds:

(A) If $f^2 = f^1 + me^j$ for some positive integer m (i.e., the second game has more players of approximate type j but the same numbers of players of other types) then

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta) \frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} = (\varepsilon + \delta + \beta) \frac{2\|f^2\| - m}{m}.$$

(B) If $\frac{f^2}{\|f^2\|} = (1-\mu)\frac{f^1}{\|f^1\|} + \mu e^j$ for some $\mu \in (0,1)$ (i.e., the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta)\frac{2-\mu}{\mu}.$$

That is, approximately the equal treatment ε-core correspondence provides lower payoffs for players of a type that is more abundant.

Proof: (A): Applying Corollary we get $(x^2 - x^1) \cdot me^j \leq (\varepsilon + \delta + \beta)\, \|f^1 + f^2\|$. Since $\|f^2\| = \|f^1\| + m$, this inequality implies our first result.

(B): From Lemma we have $(1-\mu)(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} \leq (1-\mu)(\varepsilon + \delta + \beta)$ and similarly $(x^2 - x^1) \cdot \frac{f^2}{\|f^2\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain $(x^2 - x^1) \cdot (\frac{f^2}{\|f^2\|} - (1-\mu)\frac{f^1}{\|f^1\|}) \leq (2-\mu)(\varepsilon + \delta + \beta)$. Thus we get that $(x^2 - x^1) \cdot \mu e^j \leq (2-\mu)(\varepsilon + \delta + \beta)$. This inequality implies our second result. ■

3.3 Remarks.

Remark 1. (B) of Proposition 2 is a strict generalization of (A). ((A) follows from (B) for $\mu = \frac{m}{\|f^2\|}$.) We choose to present (A) in addition to (B) since (A) may be more intuitive. Notice also that although (A) is an immediate consequence of Proposition 1, (B) formally does not follow from Proposition 1.

Remark 2. Note that the bounds on the closeness of all our results are computable for a given game and depend only on the parameters describing the game. In the Appendix we demonstrate that there are sequences of games for which the bounds provided are the *least* upper bounds.

Remark 3. For $\varepsilon = \beta = \delta = 0$ the bounds on the closeness of all our approximation results equals zero. Thus for games with finite number of player types and strictly effective small groups (e.g. for matching games with types) we get that the equal treatment core satisfies cyclic monotonicity and provides weakly lower payoffs for players of some type when this type becomes more abundant.

Remark 4. We have not demonstrated a nonemptiness result for equal treatment ε-cores of games in parameterized collections. Such a result appears in Kovalenkov and Wooders (1999b). The conditions in that paper are the same as those used in this paper. Thus, our results are not vacuous; they can be applied to all sufficiently large games and subgames in a parameterized collection.

Remark 5. The results stated all require that there be at least B players of each type in each game under consideration. With other notions of approximate cores, specifically, the ε-remainder core and the ε_1-remainder ε_2-core, which allow a small percentage of players to be ignored, it may only be required that there are many substitutes for most players in the game; we leave the details to the interested reader. See Kovalenkov and Wooders (1999a) for definitions and further references.

Remark 6. We leave it to the interested reader to show that results similar to those herein could be obtained for the strong ε-core. This approximate core notion requires that no group of agents can improve on a given payoff by ε in total, that is, given a game (N, v) and $\varepsilon \geq 0$, a payoff vector x is in the *strong ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon$ for all $S \subset N$. For strong ε-cores, the goodness of the approximation improves.

Remark 7. The reader may also wonder under what conditions all payoffs in ε-cores have the equal treatment property. For games derived from pregames, under the assumption of SGE (or PCB) ε-core payoffs have the property that almost all players of the same type receive nearly the average payoff for their type (Wooders (1980,1992b), Shubik and Wooders (1982)).[7] With the assumption that there are sufficiently many players of each approximate type, similar results can be demonstrated to hold under the conditions of our Propositions.

Remark 8. In the context of a pregame, as noted earlier, when there are sufficiently many players of each type in the games, then SGE and PCB are equivalent but, in the context of parameterized collections of games, this equivalence no longer holds. SGE, introduced in Wooders (1992a,b,1994a),[8] is a relaxation of "minimum efficient scale," MES (Wooders (1983)). MES dictates that *all* gains rather than *almost all* gains to improvement can be realized by groups of players bounded in size.[9] As indicated already by the techniques of Wooders (1980,1983), when there are sufficiently many players of each type present in the games, sequences of games derived from a pregame satisfying PCB can be approximated by games satisfying MES. (In fact, Shubik and Wooders (1982) suggestively call PCB *near minimum efficient scale.*) This is very

[7]A somewhat related result is demonstrated in Engl and Scotchmer (1996). Under the same condition of PCB and with an additional differentiability condition, Engl and Scotchmer show that per capita ε-core aggregate payoffs to sufficiently large coalitions can be approximated by equal-treatment payoffs. Their result, however, does not make any statement about payoffs to individual players. In contrast, the results of Wooders, alone and with Martin Shubik, demonstrate that an ε-core payoff vector assigns most (all but a small percentage) individual players of the same type approximately the average for their type.

[8]A condition closely related to SGE appears in Wooders and Zame (1987) There, to obtain one of their results, the authors assume that almost all gains to improvement can be realized by groups of players bounded in absolute size.. An equivalence between this condition and SGE is demonstrated in Wooders (1992b).

[9]MES has also been called *strict* small group effectiveness. For pregames with side payments it is equivalent to the "exhaustion of gains to scale" in Scotchmer and Wooders (1988) and to the "0-exhaustion of blocking opportunities" in Engl and Scotchmer (1996,1997).

useful in proving various results since games satisfying MES are especially tractable. It is noteworthy that the results of the current paper do not depend on PCB – a parameterized collection of games does not necessarily have bounded average payoffs. Consider, for example, the collection of games where all players are identical, two-player coalitions are effective, and the per-capita payoff to a two-person coalition in any game in the collection equals the number of players in the game. Clearly, without any loss, coalitions can be restricted to have no more than two players and, even though per capita payoffs are unbounded, our results apply to all the games in the collection. Thus, the crucial property is SGE.

4 An example.

Given the great importance of matching models (see, for example, Roth and Sotomayor (1990) for an excellent study and numerous references to related papers), we present an application of our results to a model of matching interns and hospitals. Our example is highly stylized. For a more complete discussion of the matching interns and hospitals problem, we refer the reader to Roth (1984).

The problem consists of the assignment of a set of interns $\mathcal{I} = \{1, .., i, .., I\}$ to hospitals. The set of hospitals is $\mathcal{H} = \{1, .., h, .., H\}$. The total player set N is given by $N = \mathcal{I} \bigcup \mathcal{H}$. Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ. Interns also have preferences over hospitals. We'll assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$). For simplicity, we'll assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's *quality.* More than one player may share the same rank in the ordering. In fact, we assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. Let us also assume that the rank assigned to any intern is between 0 and 1 and the rank assigned to any intern is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0, 1)$ and similarly, all hospitals have qualities in the interval $[1, 2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0, 2]$ into $2n$ intervals, $[0, \frac{1}{n}), .., [\frac{j-1}{n}, \frac{j}{n}), .., [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that there is a player with rank in the jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payoff vector in the ε-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x_j^2 - x_j^1) \leq (\varepsilon + \frac{1}{n})\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|}.$$

Of course this is not the most general application of our results – we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

It's remarkable that our results apply so easily. For this simple sort of example, it's probably the case that a sharper result can be obtained. This is beyond the scope of our current paper, however. Research in progress considers whether sharper results are obtainable with assortative matching of the kind illustrated by this example – that is, where players can be ordered so that players with higher ranks in the orderings are superior in terms of their marginal contributions to coalitions.

5 Concluding remarks.

The sort of comparative statics and monotonicity results of this paper for games derived from pregames satisfying MES was originally suggested in Wooders (1979). Under conditions equivalent to those of that paper, a proof was provided in Scotchmer and Wooders (1988). Wooders (1992a,b) extended the *monotonicity* analysis of Scotchmer and Wooders to hold for arbitrary changes in abundances of players of each type in games satisfying SGE and made the connection to the Law of Demand of economic theory (cf., Hildenbrand 1994). Engl and Scotchmer (1996,1997) extended the *comparative statics* analysis of Scotchmer and Wooders to hold for proportions of players of each type and further addressed the relationships between the law of scarcity and the Law of Demand.[10] All of these results, unlike the matching literature, require a fixed set of player types (or a fixed finite set of attributes of players and a single worth function defined over these attributes). The major difference between the results of these papers and those of the current paper are that our assumptions and results (a) treat more general collections of games and (b) apply to individual games. Moreover, our results apply uniformly to all games described by the same parameters; knowledge of an entire pregame is not necessary.[11]

[10] Versions of the comparative statics results of Engl and Scotchmer (1996) appeared earlier in a 1991 typescript and also in a 1992 Working Paper

[11] Since there may be many but a *finite* number of coalitions, in fact determining the required sizes of δ and T, β and B may be time-consuming but is possible. In contrast, to verify that a pregame

In the current paper, as in the matching literature initiated by Kelso and Crawford (1982), there is no restriction to a fixed set of player types, not even to a compact metric space of types. No topology on the space of player types is required. We perceive this as a major advantage of the Kelso and Crawford approach and of the current approach. Also, again as in the matching literature, our conditions are on the games themselves.

Finally, returning again to the Kelso and Crawford (1982) and Crawford (1991) models and results, these authors allow the games under consideration to have non-transferable utility. Determination of conditions on payoff sets of NTU games or on preferences of agents in economic models sufficient to ensure comparative statics and monotonicity results similar to those of Kelso and Crawford (1982) and Crawford (1991) but for games generally is an open problem.

6 Appendix.

We present some sequences of games such that the bounds we obtained in our results are the *least* upper bounds.

I). Let us concentrate first on the central case $\delta = \beta = 0$. Consider a game (N, v) where any player can get only 1 unit or less in any coalition and there are no gains to forming coalitions. This game has strictly effective 1-bounded groups and all agents are identical. Formally, however, we may partition the set of players into many types. Thus $(N, v) \in \Gamma((0,\tau),(0,1))$ for any integer τ, $1 \leq \tau \leq |N|$. Notice also that for any $\varepsilon \geq 0$ the ε-core of the game is nonempty and very simple: it includes all payoff vectors that are feasible and provide at least $1-\varepsilon$ for each of the players. All the games that we are going to construct will be subgames of a game (N, v).

a). For the bound in Lemma we can present even a single game with two payoffs vectors that realize this bound. Namely, let $\tau = 1$ (all players are of one type) and let us consider any two subgames S^1, S^2 with the same number of players and the equal treatment payoffs $x^1 = 1$ and $x^2 = 1 - \varepsilon$. Then $(x^1 - x^2) \cdot f^1 = \varepsilon \|f^1\|$.

b). For the bound in Proposition 1, for $K \leq |N|$ and some nonnegative integer $l \leq |N| - K$, let us consider $\tau = K$ and the subgroups $S^1, .., S^K$ with the profiles $f^1, .., f^K$ where $f^k_t = l + 1$ for $t = k$ and 1 otherwise. Let also consider payoff vectors x^k where $x^k_t = 1$ for $t = k$ and $1 - \varepsilon$ otherwise. Then $(x^i - x^j) \cdot f^1 = \varepsilon l$ for any $i \neq j$. Hence

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K = \varepsilon l K = \varepsilon \left\| f^1 + f^2 + .. + f^K \right\| \frac{l}{l+K}$$

$$\text{and } (x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} = K\varepsilon \frac{l}{l+K}.$$

satisfies SGE or PCB requires consideration of an *infinite* number of payoff sets or, equivalently, a limiting set of equal treatment payoffs.

It is straightforward to verify that for any fixed K both our bounds in Proposition 2 will be the least upper bounds for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

c). For the bound in Proposition 2 it is enough to concentrate on (A) since it is a special case of the result (B). For $|N| \geq 2$ let us consider $\tau = 2$ and $l \leq |N| - 2$. Then consider the subgroups S^1, S^2 with the profiles $f^1 = (1, 1)$ and $f^2 = (l + 1, 1)$ and payoff vectors $x^1 = (1 - \varepsilon, 1)$ and $x^2 = (1, 1)$. Then

$$(x_1^2 - x_1^1) = \varepsilon = \varepsilon \frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} \frac{l}{l + 4}.$$

It follows that both our bounds in Proposition 2 will be the least upper bounds for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

II). It is easy to modify our example to allow for non-zero δ and β in a such a way that we will have the same profiles as in Part I, but will use the payoffs of $1 + \delta + \beta$ and $1 - \varepsilon$ instead of 1 and $1 - \varepsilon$. This will lead us to the appearance of $\varepsilon + \delta + \beta$ on the places of ε in all bound in Part I. We leave it as a simple exercise for the interested reader.

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