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A Study On Generalized Mersenne Numbers

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Abstract

In this paper, we introduce the generalized Mersenne sequence and we deal with, in detail, two special cases, namely, Mersenne and Mersenne-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

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Keywords

Mersenne numbers, Mersenne-Lucas numbers, generalized Fibonacci numbers.

1. Introduction

A Mersenne number, denoted by M_n, is a number of the form $M_n = 2^n - 1$. The Mersenne sequence $\{M_n\}_{n\geq 0}$ can also be defined recursively by

$$M_n = 3M_{n-1} - 2M_{n-2}$$

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with initial conditions $M_0 = 0, M_1 = 1$. A Mersenne-Lucas number, denoted by H_n, is a number of the form $H_n = 2^n + 1$. The Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ can also be defined, by the second-order recurrence relation,

$$H_n = 3H_{n-1} - 2H_{n-2}$$

with initial conditions $H_0 = 2, H_1 = 3$. $\{M_n\}_{n\geq 0}$ is the sequence $A000225$ in the OEIS [21], whereas $\{H_n\}_{n\geq 0}$ is the id-number $A000051$ in OEIS. Note that Mersen-Lucas numbers are also called as Fermat numbers. In fact, there are two definitions of the Fermat numbers. The less common is a number of the form $2^n + 1$, the first few of which are $2, 3, 5, 9, 17, 33, ...$ (OEIS $A000051$). The much more commonly encountered Fermat numbers are a special case, given by the binomial number of the form $F_n = 2^{2^n} + 1$. The first few for $n = 0, 1, 2, ...$ are $3, 5, 17, 257, 65537, 4294967297, ...$ (OEIS $A000215$).

Mersenne sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence, see for example, [1,2,3,4,5,6,7,8,9,10,15,16,17,18,19,20,23,27,28].

A straightforward calculation shows that if M_n is a prime number, then n is a prime number, though not all M_n are prime. When M_n is a prime number, it is called Mersenne prime. The Mersenne numbers play a key role in an investigations on the prime numbers so, throughout the history, many researchers searched to find Mersenne primes. Some tests are very important for the search for Mersenne primes, mainly the Lucas-Lehmer test. There are other tests such as Pepin's test. For example, in [22, Theorem 3.3], Šolcová and Křižek proposed some Mersenne numbers that can stand as a base in Pepin's test.

There are papers (see, for instance [2,3,7,19]) that seek to describe the prime factors of M_n, where M_n is a composite number and n is a prime number. Moreover, some papers seek to describe prime divisors of Mersenne number M_n, where n cannot be a prime number (see for example [8,16,18,20,27]).

Generalizations of Mersenne numbers can be obtained in various ways (see for example [5,10,17,23]). Our generalizations of Mersenne numbers in section 2 are not Mersenne in the sense of [10,23].

The purpose of this article is to generalize and investigate these interesting sequence of numbers (Mersenne numbers). First, we recall some properties of Fibonacci numbers and its generalizations, namely generalized Fibonacci numbers.

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [12]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \qquad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [11,13,14] and Soykan [25].

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS[21]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r, s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \qquad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \qquad H_0 = 2, H_1 = r, \tag{1.3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1) = F_n$, Fibonacci sequence,

2. $H_n(2,1;1,1) = L_n$, Lucas sequence,

3. $G_n(0,1;2,1) = P_n$, Pell sequence,

4. $H_n(2,2;2,1) = Q_n$, Pell-Lucas sequence,

5. $G_n(0,1;1,2) = J_n$, Jacobsthal sequence,

6. $H_n(2,1;1,2) = j_n$, Jacobsthal-Lucas sequence.

The following theorem shows that the generalized Fibonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have the following:*

(a)

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

(b)

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Proof. For the proof, see Soykan [26, Theorem 3.2 and Theorem 3.3]. □

The following theorem presents sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers).

Theorem 2. *Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}. \quad (1.4)$$

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-s)^m - (n+1)H_m)x^nW_{j+mn} + (-s)^m (n+1)x^nW_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}.$$

(c) *If $(-s)^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

Proof. It is given in Soykan [26, Theorem 4.1]. □

Note that (1.4) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + x(H_m - (-s)^m x)W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 3. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.5}$$

Proof. For a proof, see [25, Lemma 1.1]. □

1.1 Binet's Formula for the Distinct Roots Case and Single Root Case

Let α and β be two roots of the quadratic equation

$$x^2 - rx - s = 0, \tag{1.6}$$

of which the left-hand side is called the characteristic polynomial (or the characteristic equation) of the recurrence relation (1.1). The following theorem presents the Binet's formula of the sequence W_n.

Theorem 4. *The general term of the sequence W_n can be presented by the following Binet formula:*

$$\begin{aligned} W_n &= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n &, \ \textit{if } \alpha \neq \beta \ \textit{(Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} &, \ \textit{if } \alpha = \beta \ \textit{(Single Root Case)} \end{cases} \\ &= \begin{cases} \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n &, \ \textit{if } \alpha \neq \beta \ \textit{(Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1} &, \ \textit{if } \alpha = \beta \ \textit{(Single Root Case)} \end{cases}. \end{aligned}$$

Proof. For a proof, see Soykan [25] and [26]. □

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \quad \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{1.7}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned}\alpha+\beta &= r,\\ \alpha\beta &= -s,\\ (\alpha-\beta)^2 &= (\alpha+\beta)^2-4\alpha\beta = r^2+4s.\end{aligned}$$

If $\Delta = r^2+4s \neq 0$ then $\alpha \neq \beta$ i.e., there are distinct roots of the quadratic equation (1.6) and if $\Delta = r^2+4s = 0$ then $\alpha = \beta$, i.e., there is a single root of the quadratic equation (1.6).

In the case $r^2+4s \neq 0$ so that $\alpha \neq \beta$, for all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$\begin{aligned}G_n &= \frac{\alpha^n}{(\alpha-\beta)}+\frac{\beta^n}{(\beta-\alpha)},\\ H_n &= \alpha^n+\beta^n,\end{aligned}$$

respectively. In the case $r^2+4s = 0$ so that $\alpha = \beta$, for all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$\begin{aligned}G_n &= n\alpha^{n-1},\\ H_n &= 2\alpha^n,\end{aligned}$$

respectively.

2 Generalized Mersenne Sequence

In this paper, we consider the case $r = 3, s = -2$. A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1,2,3,....$ Therefore, recurrence (2.1) holds for all integer n.

By Theorem 4, the Binet formula of generalized Mersenne numbers can be written as

$$W_n = \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n$$

where α and β are the roots of the quadratic equation $x^2-3x+2 = 0$. Moreover

$$\begin{aligned}\alpha &= 2\\ \beta &= 1\end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta &= 3, \\ \alpha\beta &= 2, \\ \alpha-\beta &= 1 \end{aligned}$$

So

$$W_n = (W_1 - W_0)2^n - (W_1 - 2W_0). \tag{2.2}$$

The first few generalized Mersenne numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Mersenne numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{3}{2}W_0 - \frac{1}{2}W_1$
2	$3W_1 - 2W_0$	$\frac{7}{4}W_0 - \frac{3}{4}W_1$
3	$7W_1 - 6W_0$	$\frac{15}{8}W_0 - \frac{7}{8}W_1$
4	$15W_1 - 14W_0$	$\frac{31}{16}W_0 - \frac{15}{16}W_1$
5	$31W_1 - 30W_0$	$\frac{63}{32}W_0 - \frac{31}{32}W_1$
6	$63W_1 - 62W_0$	$\frac{127}{64}W_0 - \frac{63}{64}W_1$
7	$127W_1 - 126W_0$	$\frac{255}{128}W_0 - \frac{127}{128}W_1$
8	$255W_1 - 254W_0$	$\frac{511}{256}W_0 - \frac{255}{256}W_1$
9	$511W_1 - 510W_0$	$\frac{1023}{512}W_0 - \frac{511}{512}W_1$
10	$1023W_1 - 1022W_0$	$\frac{2047}{1024}W_0 - \frac{1023}{1024}W_1$
11	$2047W_1 - 2046W_0$	$\frac{4095}{2048}W_0 - \frac{2047}{2048}W_1$
12	$4095W_1 - 4094W_0$	$\frac{8191}{4096}W_0 - \frac{4095}{4096}W_1$

Now we define two special cases of the sequence $\{W_n\}$. Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$M_n = 3M_{n-1} - 2M_{n-2}, \qquad M_0 = 0, M_1 = 1, \tag{2.3}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \qquad H_0 = 2, H_1 = 3, \tag{2.4}$$

The sequences $\{M_n\}_{n\geq 0}$ *and* $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively. Therefore, recurrences (2.3)-(2.4) hold for all integer n.

Next, we present the first few values of the Mersenne and Mersenne-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
M_n	0	1	3	7	15	31	63	127	255	511	1023	2047	4095
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$	$-\frac{2047}{2048}$	$-\frac{4095}{4096}$
H_n	2	3	5	9	17	33	65	129	257	513	1025	2049	4097
H_n	2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$	$\frac{4097}{4096}$

For all integers n, Mersenne and Mersenne-Lucas (using initial conditions in Theorem 4) can be expressed using Binet's formulas as

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1, \end{aligned}$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 5. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Mersenne sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x}{1 - 3x + 2x^2}$$

Proof. In Lemma 3, take $r = 3, s = -2$. □

The previous Lemma gives the following results as particular examples.

Corollary 6. *Generated functions of Mersenne and Mersenne-Lucas numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} M_n x^n &= \frac{x}{1-3x+2x^2}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{2-3x}{1-3x+2x^2}, \end{aligned}$$

respectively.

Proof. In Lemma **??**, take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively. □

3 Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Mersenne sequence $\{W_n\}_{n\geq 0}$.

Theorem 7 (Simson Formula of Generalized Mersenne Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = -2^{n-1}\left(W_0 - W_1\right)\left(2W_0 - W_1\right). \tag{3.1}$$

Proof. For a proof of Eq. (3.1), see Soykan [24], just take $s = -2$. □

The previous theorem gives the following results as particular examples.

Corollary 8. *For all integers n, Mersenne and Mersenne-Lucas numbers are given as*

$$\begin{aligned} \begin{vmatrix} M_{n+1} & M_n \\ M_n & M_{n-1} \end{vmatrix} &= -2^{n-1}, \\ \begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= 2^{n-1}, \end{aligned}$$

respectively.

4 Some Identities

In this section, we obtain some identities of generalized Mersenne, Mersenne and Mersenne-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{M_n\}$.

Lemma 9. *The following equalities are true:*

$$\begin{aligned} 8W_n &= (15W_0 - 7W_1)M_{n+4} - (31W_0 - 15W_1)M_{n+3}, \qquad (4.1) \\ 4W_n &= (7W_0 - 3W_1)M_{n+3} - (15W_0 - 7W_1)M_{n+2}, \\ 2W_n &= (3W_0 - W_1)M_{n+2} - (7W_0 - 3W_1)M_{n+1}, \\ W_n &= W_0M_{n+1} + (-3W_0 + W_1)M_n, \\ W_n &= W_1M_n - 2W_0M_{n-1}, \end{aligned}$$

and

$$\begin{aligned}
8(W_0-W_1)(2W_0-W_1)M_n &= (6W_0-7W_1)W_{n+4}-(14W_0-15W_1)W_{n+3} \\
4(W_0-W_1)(2W_0-W_1)M_n &= (2W_0-3W_1)W_{n+3}-(6W_0-7W_1)W_{n+2} \\
2(W_0-W_1)(2W_0-W_1)M_n &= -W_1W_{n+2}-(2W_0-3W_1)W_{n+1} \\
(W_0-W_1)(2W_0-W_1)M_n &= -W_0W_{n+1}+W_1W_n \\
(W_0-W_1)(2W_0-W_1)M_n &= (-3W_0+W_1)W_n+2W_0W_{n-1}
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$W_n = a\times M_{n+4}+b\times M_{n+3}$$

and solving the system of equations

$$\begin{aligned}
W_0 &= a\times M_4+b\times M_3 \\
W_1 &= a\times M_5+b\times M_4
\end{aligned}$$

we find that $a=\frac{1}{8}(15W_0-7W_1), b=-\frac{1}{8}(31W_0-15W_1)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 10. *The following equalities are true:*

$$\begin{aligned}
8W_n &= -(17W_0-9W_1)H_{n+4}+(33W_0-17W_1)H_{n+3} \\
4W_n &= -(9W_0-5W_1)H_{n+3}+(17W_0-9W_1)H_{n+2} \\
2W_n &= -(5W_0-3W_1)H_{n+2}+(9W_0-5W_1)H_{n+1} \\
W_n &= -(3W_0-2W_1)H_{n+1}+(5W_0-3W_1)H_n \\
W_n &= -(4W_0-3W_1)H_n+2(3W_0-2W_1)H_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
8(W_0-W_1)(2W_0-W_1)H_n &= -(10W_0-9W_1)W_{n+4}+(18W_0-17W_1)W_{n+3}, \\
4(W_0-W_1)(2W_0-W_1)H_n &= -(6W_0-5W_1)W_{n+3}+(10W_0-9W_1)W_{n+2}, \\
2(W_0-W_1)(2W_0-W_1)H_n &= -(4W_0-3W_1)W_{n+2}+(6W_0-5W_1)W_{n+1}, \\
(W_0-W_1)(2W_0-W_1)H_n &= -(3W_0-2W_1)W_{n+1}+(4W_0-3W_1)W_n, \\
(W_0-W_1)(2W_0-W_1)H_n &= (-5W_0+3W_1)W_n+2(3W_0-2W_1)W_{n-1}.
\end{aligned}$$

Next, we present a few basic relations between $\{M_n\}$ and $\{H_n\}$.

Lemma 11. *The following equalities are true:*

$$\begin{aligned} 8H_n &= 9M_{n+4} - 17M_{n+3} \\ 4H_n &= 5M_{n+3} - 9M_{n+2} \\ 2H_n &= 3M_{n+2} - 5M_{n+1} \\ H_n &= 2M_{n+1} - 3M_n \\ H_n &= 3M_n - 4M_{n-1} \end{aligned}$$

and

$$\begin{aligned} 8M_n &= 9H_{n+4} - 17H_{n+3}, \\ 4M_n &= 5H_{n+3} - 9H_{n+2}, \\ 2M_n &= 3H_{n+2} - 5H_{n+1}, \\ M_n &= 2H_{n+1} - 3H_n, \\ M_n &= 3H_n - 4H_{n-1}. \end{aligned}$$

We now present a few special identities for the generalized Mersenne sequence $\{W_n\}$.

Theorem 12. *(Catalan's identity of the generalized Mersenne sequence) For all integers n and m, the following identity holds:*

$$W_{n+m}W_{n-m} - W_n^2 = -2^{n-m}(2^m - 1)^2(W_0 - W_1)(2W_0 - W_1).$$

Proof. We use the identity (2.2). □

As special cases of the above theorem, we have the following corollary.

Corollary 13. *For all integers n and m, the following identities hold:*

(a) $M_{n+m}M_{n-m} - M_n^2 = -2^{n-m}(2^m - 1)^2$.

(b) $H_{n+m}H_{n-m} - H_n^2 = 2^{n-m}(2^m - 1)^2$.

Note that for $m = 1$ in Catalan's identity of the generalized Mersenne sequence, we get the Cassini identity for the generalized Mersenne sequnce.

Theorem 14. *(Cassini's identity of the generalized Mersenne sequence) For all integers n, the following identity holds:*

$$W_{n+1}W_{n-1} - W_n^2 = -2^{n-1}(W_0 - W_1)(2W_0 - W_1).$$

As special cases of the above theorem, we have the following corollary.

Corollary 15. *For all integers n, the following identities hold:*

(a) $M_{n+1}M_{n-1} - M_n^2 = -2^{n-1}$,

(b) $H_{n+1}H_{n-1} - H_n^2 = 2^{n-1}$.

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using (2.2). The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of generalized Mersenne sequence $\{W_n\}$.

Theorem 16. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$W_{m+1}W_n - W_mW_{n+1} = -(2^m - 2^n)(W_0 - W_1)(2W_0 - W_1).$$

(b) *(Gelin-Cesàro's identity)*

$$W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 = -2^{n-3}(W_0 - W_1)(2W_0 - W_1)((22 \times 2^{2n} - 53 \times 2^n + 22)W_1^2 + 2(11 \times 2^{2n} - 53 \times 2^n + 44)W_0^2 + (-44 \times 2^{2n} + 159 \times 2^n - 88)W_0W_1).$$

(c) *(Melham's identity)*

$$W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = 2 \times 2^n(W_0 - W_1)(2W_0 - W_1)(-(100 \times 2^n - 23)W_1 + 2(50 \times 2^n - 23)W_0).$$

Proof. Use the identity (2.2). □

As special cases of the above theorem, we have the following three corollaries. First one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of Mersenne sequence $\{M_n\}$.

Corollary 17. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$M_{m+1}M_n - M_mM_{n+1} = 2^n - 2^m.$$

(b) *(Gelin-Cesàro's identity)*

$$M_{n+2}M_{n+1}M_{n-1}M_{n-2} - M_n^4 = 2^{n-3}(-22 \times 2^{2n} + 53 \times 2^n - 22).$$

(c) *(Melham's identity)*

$$M_{n+1}M_{n+2}M_{n+6} - M_{n+3}^3 = -2^{n+1}(100 \times 2^n - 23).$$

Second one presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of Mersenne-Lucas sequence $\{H_n\}$.

Corollary 18. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$H_{m+1}H_n - H_mH_{n+1} = 2^m - 2^n.$$

(b) *(Gelin-Cesàro's identity)*

$$H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = 2^{n-3}(22 \times 2^{2n} + 53 \times 2^n + 22).$$

(c) *(Melham's identity)*

$$H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = 2^{n+1}(100 \times 2^n + 23).$$

5 On the Recurrence Properties of Generalized Mersenne Sequence

Taking $r = 3, s = -2$ in Theorem 1 (a) and (b), we obtain the following Proposition.

Proposition 19. *For $n \in \mathbb{Z}$, generalized Mersenne numbers (the case $r = 3, s = -2$) have the following identity:*

$$\begin{aligned} W_{-n} &= -2^{-n}(W_n - H_nW_0) \\ &= \frac{-2^{-n}}{-W_1^2 - 2W_0^2 + 3W_0W_1}((2W_1 - 3W_0)W_0W_{n+1} - (W_1^2 - 2W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized Mersenne sequence at the positive index and the negative index: for Mersenne and Mersenne-Lucas numbers: take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and take $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively. Note that in this case $H_n = H_n$.

Corollary 20. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Mersenne sequence:*

$$M_{-n} = -\frac{1}{2^n}M_n = \frac{-2^n + 1}{2^n}.$$

(b) *Mersenne-Lucas sequence:*

$$H_{-n} = \frac{1}{2^n}H_n = \frac{2^n + 1}{2^n}.$$

6 The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}$

In this section, we present sum formulas of generalized Mersenne numbers. The following theorem presents sum formulas of generalized Mersenne numbers (the case $r = 3, s = -2$).

Theorem 21. *Let x be a real (or complex) number. For all integers m and j, for generalized Mersenne numbers we have the following sum formulas:*

(a) *if $2^m x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(2^m x - H_m)x^{n+1}W_{mn+j} + 2^m x^{n+1}W_{mn+j-m} + W_j - 2^m x W_{j-m}}{2^m x^2 - xH_m + 1}. \quad (6.1)$$

(b) *If $2^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(2^m(n+2)x - (n+1)H_m)x^n W_{mn+j} + 2^m(n+1)x^n W_{mn+j-m} - 2^m W_{j-m}}{2^{m+1}x - H_m}$$

(c) *If $2^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)(2^m(n+2)x^n - nx^{n-1}H_m)W_{mn+j} + 2^m n(n+1)x^{n-1}W_{mn+j-m}}{2^{m+1}}.$$

Proof. Take $r = 3, s = -2$ and $H_n = H_n$ in Theorem 2. □

Note that (6.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{(2^m x - H_m)x^{n+1}W_{mn+j} + 2^m x^{n+1}W_{mn+j-m} + x(H_m - 2^m x)W_j - 2^m x W_{j-m}}{2^m x^2 - xH_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 22. *For generalized Mersenne numbers (the case $r = 3, s = -2$) we have the following sum formulas:*

(a) $(m = 1,\ j = 0)$

If $2x^2 - 3x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{2}$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(2x-3)x^{n+1}W_n + 2x^{n+1}W_{n-1} + (W_1 - 3W_0)x + W_0}{2x^2 - 3x + 1},$$

and

if $2x^2 - 3x + 1 = 0$, i.e., $x = 1$ or $x = \frac{1}{2}$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(2(n+2)x - 3(n+1))x^n W_n + 2(n+1)x^n W_{n-1} + (W_1 - 3W_0)}{4x - 3}.$$

(b) $(m = 2,\ j = 0)$

If $4x^2 - 5x + 1 \neq 0$, i.e., $x \neq 1$, $x \neq \frac{1}{4}$, then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(4x - H_2)x^{n+1}W_{2n} + 4x^{n+1}W_{2n-2} + (3W_1 - 7W_0)x + W_0}{4x^2 - 5x + 1},$$

and

if $4x^2-5x+1=0,$ *i.e.,* $x=1$ *or* $x=\frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(8x-5+n(4x-5))x^n W_{2n} + 4(n+1)x^n W_{2n-2} + (3W_1-7W_0)}{8x-5}.$$

(c) $(m=2,\ j=1)$

If $4x^2-5x+1\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(4x-5)\,x^{n+1}W_{2n+1} + 4x^{n+1}W_{2n-1} + 2\,(W_1-3W_0)\,x + W_1}{4x^2-5x+1},$$

and

if $4x^2-5x+1=0,$ *i.e.,* $x=1$ *or* $x=\frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(8x-5+n(4x-5))x^n W_{2n+1} + 4(n+1)x^n W_{2n-1} + 2(W_1-3W_0)}{8x-5}.$$

(d) $(m=-1,\ j=0)$

If $x^2-3x+2\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 2,$ *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1} + (x-3)x^{n+1}W_{-n} - W_1 x + 2W_0}{x^2-3x+2},$$

and

if $x^2-3x+2=0,$ *i.e.,* $x=1$ *or* $x=2,$ *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(n+1)x^n W_{-n+1} + (2x-3+n(x-3))x^n W_{-n} - W_1}{(2x-3)}.$$

(e) $(m=-2,\ j=0)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{x^{n+1}W_{-2n+2} + (x-5)x^{n+1}W_{-2n} - W_2 x + 4W_0}{x^2-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{(n+1)x^n W_{-2n+2} + (2x-5+n(x-5))x^n W_{-2n} - W_2}{2x-5}.$$

(f) $(m=-2,\ j=1)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n}x^{k}W_{-2k+1}=\frac{x^{n+1}W_{-2n+3}+(x-5)x^{n+1}W_{-2n+1}-W_{3}x+4W_{1}}{x^{2}-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n}x^{k}W_{-2k+1}=\frac{(n+1)x^{n}W_{-2n+3}+(2x-5+n(x-5))x^{n}W_{-2n+1}-W_{3}}{2x-5}.$$

From the above proposition, we have the following corollary which gives sum formulas of Mersenne numbers (take $W_n=M_n$ with $M_0=0, M_1=1$).

Corollary 23. *For* $n\geq 0,$ *Mersenne numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $2x^2-3x+1\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{1}{2},$ *then*

$$\sum_{k=0}^{n}x^{k}M_{k}=\frac{(2x-3)x^{n+1}M_{n}+2x^{n+1}M_{n-1}+x}{2x^{2}-3x+1},$$

and

if $2x^2-3x+1=0,$ *i.e.,* $x=1$ *or* $x=\frac{1}{2},$ *then*

$$\sum_{k=0}^{n}x^{k}M_{k}=\frac{(2(n+2)x-3(n+1))x^{n}M_{n}+2(n+1)x^{n}M_{n-1}+1}{4x-3}.$$

(b) $(m=2,\ j=0)$

If $4x^2-5x+1\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n}x^{k}M_{2k}=\frac{(4x-H_{2})x^{n+1}M_{2n}+4x^{n+1}M_{2n-2}+3x}{4x^{2}-5x+1},$$

and

if $4x^2-5x+1=0,$ *i.e.,* $x=1$ *or* $x=\frac{1}{4},$ *then*

$$\sum_{k=0}^{n}x^{k}M_{2k}=\frac{(8x-5+n(4x-5))x^{n}M_{2n}+4(n+1)x^{n}M_{2n-2}+3}{8x-5}.$$

(c) $(m=2,\ j=1)$

If $4x^2-5x+1\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n}x^{k}M_{2k+1}=\frac{(4x-5)\,x^{n+1}M_{2n+1}+4x^{n+1}M_{2n-1}+2x+1}{4x^{2}-5x+1},$$

and

if $4x^2-5x+1=0,$ *i.e.,* $x=1$ *or* $x=\frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k M_{2k+1} = \frac{(8x-5+n(4x-5))x^n M_{2n+1}+4(n+1)x^n M_{2n-1}+2}{8x-5}.$$

(d) $(m=-1,\ j=0)$

If $x^2-3x+2\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 2,$ *then*

$$\sum_{k=0}^{n} x^k M_{-k} = \frac{x^{n+1}M_{-n+1}+(x-3)x^{n+1}M_{-n}-x}{x^2-3x+2},$$

and

if $x^2-3x+2=0,$ *i.e.,* $x=1$ *or* $x=2,$ *then*

$$\sum_{k=0}^{n} x^k M_{-k} = \frac{(n+1)x^n M_{-n+1}+(2x-3+n(x-3))x^n M_{-n}-1}{(2x-3)}.$$

(e) $(m=-2,\ j=0)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k} = \frac{x^{n+1}M_{-2n+2}+(x-5)x^{n+1}M_{-2n}-3x}{x^2-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k} = \frac{(n+1)x^n M_{-2n+2}+(2x-5+n(x-5))x^n M_{-2n}-3}{2x-5}.$$

(f) $(m=-2,\ j=1)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1} = \frac{x^{n+1}M_{-2n+3}+(x-5)x^{n+1}M_{-2n+1}-7x+4}{x^2-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n} x^k M_{-2k+1} = \frac{(n+1)x^n M_{-2n+3}+(2x-5+n(x-5))x^n M_{-2n+1}-7}{2x-5}.$$

Taking $W_n=H_n$ with $H_0=2, H_1=3$ in the last proposition, we have the following corollary which presents sum formulas of Mersenne-Lucas numbers.

Corollary 24. *For $n \geq 0$, Mersenne-Lucas numbers have the following properties:*

(a) $(m = 1,\ j = 0)$

If $2x^2 - 3x + 1 \neq 0,\ i.e.,\ x \neq 1,\ x \neq \frac{1}{2},$ *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(2x-3)x^{n+1}H_n + 2x^{n+1}H_{n-1} - 3x + 2}{2x^2 - 3x + 1},$$

and

if $2x^2 - 3x + 1 = 0,\ i.e.,\ x = 1$ *or* $x = \frac{1}{2},$ *then*

$$\sum_{k=0}^{n} x^k H_k = \frac{(2(n+2)x - 3(n+1))x^n H_n + 2(n+1)x^n H_{n-1} - 3}{4x - 3}.$$

(b) $(m = 2,\ j = 0)$

If $4x^2 - 5x + 1 \neq 0,\ i.e.,\ x \neq 1,\ x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(4x - H_2)x^{n+1}H_{2n} + 4x^{n+1}H_{2n-2} - 5x + 2}{4x^2 - 5x + 1},$$

and

if $4x^2 - 5x + 1 = 0,\ i.e.,\ x = 1$ *or* $x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{2k} = \frac{(8x - 5 + n(4x-5))x^n H_{2n} + 4(n+1)x^n H_{2n-2} - 5}{8x - 5}.$$

(c) $(m = 2,\ j = 1)$

If $4x^2 - 5x + 1 \neq 0,\ i.e.,\ x \neq 1,\ x \neq \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(4x-5)\,x^{n+1}H_{2n+1} + 4x^{n+1}H_{2n-1} - 6x + 3}{4x^2 - 5x + 1},$$

and

if $4x^2 - 5x + 1 = 0,\ i.e.,\ x = 1$ *or* $x = \frac{1}{4},$ *then*

$$\sum_{k=0}^{n} x^k H_{2k+1} = \frac{(8x - 5 + n(4x-5))x^n H_{2n+1} + 4(n+1)x^n H_{2n-1} - 6}{8x - 5}.$$

(d) $(m = -1,\ j = 0)$

If $x^2 - 3x + 2 \neq 0,\ i.e.,\ x \neq 1,\ x \neq 2,$ *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{x^{n+1}H_{-n+1} + (x-3)x^{n+1}H_{-n} - 3x + 4}{x^2 - 3x + 2},$$

and

if $x^2 - 3x + 2 = 0,\ i.e.,\ x = 1$ *or* $x = 2,$ *then*

$$\sum_{k=0}^{n} x^k H_{-k} = \frac{(n+1)x^n H_{-n+1} + (2x - 3 + n(x-3))x^n H_{-n} - 3}{(2x-3)}.$$

(e) $(m=-2,\ j=0)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{x^{n+1}H_{-2n+2}+(x-5)x^{n+1}H_{-2n}-5x+8}{x^2-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k} = \frac{(n+1)x^n H_{-2n+2}+(2x-5+n(x-5))x^n H_{-2n}-5}{2x-5}.$$

(f) $(m=-2,\ j=1)$

If $x^2-5x+4\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{x^{n+1}H_{-2n+3}+(x-5)x^{n+1}H_{-2n+1}-9x+12}{x^2-5x+4},$$

and

if $x^2-5x+4=0,$ *i.e.,* $x=1$ *or* $x=4,$ *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1} = \frac{(n+1)x^n H_{-2n+3}+(2x-5+n(x-5))x^n H_{-2n+1}-9}{2x-5}.$$

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 25. *For* $n\geq 0,$ *Mersenne numbers and Mersenne-Lucas numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} M_k = -(n-1)M_n + 2(n+1)M_{n-1} + 1.$

(b) $\sum_{k=0}^{n} M_{2k} = \frac{1}{3}(-(n-3)M_{2n} + 4(n+1)M_{2n-2} + 3).$

(c) $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(-(n-3)M_{2n+1} + 4(n+1)M_{2n-1} + 2).$

(d) $\sum_{k=0}^{n} M_{-k} = -(n+1)M_{-n+1} + (2n+1)M_{-n} + 1.$

(e) $\sum_{k=0}^{n} M_{-2k} = \frac{1}{3}(-(n+1)M_{-2n+2} + (4n+3)M_{-2n} + 3).$

(f) $\sum_{k=0}^{n} M_{-2k+1} = \frac{1}{3}(-(n+1)M_{-2n+3} + (4n+3)M_{-2n+1} + 7).$

2.

(a) $\sum_{k=0}^{n} H_k = -(n-1)H_n + 2(n+1)H_{n-1} - 3.$

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} - 5).$

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} - 6).$

(d) $\sum_{k=0}^{n} H_{-k} = -(n+1)H_{-n+1} + (2n+1)H_{-n} + 3.$

(e) $\sum_{k=0}^{n} H_{-2k} = \frac{1}{3}(-(n+1)H_{-2n+2} + (4n+3)H_{-2n} + 5).$

(f) $\sum_{k=0}^{n} H_{-2k+1} = \frac{1}{3}(-(n+1)H_{-2n+3} + (4n+3)H_{-2n+1} + 9).$

7 Matrices Related with Generalized Mersenne Numbers

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. Then, we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = M_n$ in (7.1) we have

$$\begin{pmatrix} M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} M_n \\ M_{n-1} \end{pmatrix}. \tag{7.2}$$

We also define

$$M_n = \begin{pmatrix} M_{n+1} & -2M_n \\ M_n & -2M_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -2W_n \\ W_n & -2W_{n-1} \end{pmatrix}.$$

Theorem 26. *For all integers m, n, we have*

(a) $M_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n M_m = M_m C_n$.

Proof. Take $r = 3, s = -2$ in Soykan [25, Theorem 5.1.]. □

Corollary 27. *For all integers n, we have the following formulas for the Mersenne and Mersenne-Lucas numbers.*

(a) *Mersenne Numbers.*

$$A^n = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} M_{n+1} & -2M_n \\ M_n & -2M_{n-1} \end{pmatrix}.$$

(b) *Mersenne-Lucas Numbers.*

$$A^n = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 3H_{n+1} - 4H_n & -2(2H_{n+1} - 3H_n) \\ 2H_{n+1} - 3H_n & -2(2H_n - 3H_{n-1}) \end{pmatrix}.$$

Proof.

(a) It is given in Theorem 26 (a).

(b) Note that, from Lemma 11, we have

$$M_n = 2H_{n+1} - 3H_n.$$

Using the last equation and (a), we get required result. □

Theorem 28. *For all integers m, n, we have*

$$W_{n+m} = W_n M_{m+1} - 2W_{n-1} M_m. \tag{7.3}$$

Proof. Take $r = 3, s = -2$ in Soykan [25, Theorem 5.2.]. □

By Lemma 9, we know that

$$(W_0 - W_1)(2W_0 - W_1)M_n = -W_0 W_{n+1} + W_1 W_n,$$

so (7.3) can be written in the following form

$$(W_0 - W_1)(2W_0 - W_1)W_{n+m} = W_n(-W_0 W_{m+2} + W_1 W_{m+1}) - 2W_{n-1}(-W_0 W_{m+1} + W_1 W_m).$$

Corollary 29. *For all integers m, n, we have*

$$\begin{aligned} M_{n+m} &= M_n M_{m+1} - 2M_{n-1} M_m, \\ H_{n+m} &= H_n M_{m+1} - 2H_{n-1} M_m, \end{aligned}$$

and

$$H_{n+m} = H_n(2H_{m+2} - 3H_{m+1}) - 2H_{n-1}(2H_{m+1} - 3H_m).$$

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