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Ba Chu and Soosung Hwang

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Ba Chu[1] and Soosung Hwang[2]

Cass Business School

[1]Faculty of Finance, Cass Business School, 106 Bunhill Row, London, EC1Y 8TZ, U.K. Tel: +44 (0)20 7040 8600. Fax: +44 (0)20 7040 8881. E-mail: B.M.Chu@city.ac.uk

[2]Corresponding author: Faculty of Finance, Cass Business School, 106 Bunhill Row, London, EC1Y 8TZ, U.K. Tel: +44 (0)20 7040 0109. Fax: +44 (0)20 7040 8881. E-mail: s.hwang@city.ac.uk. We would like to thank Giovanni Urga, Lorenzo Trapani and seminar participants on the Econometrics of Structural Breaks for their helpful comments.

Abstract

This study investigates the asymptotic properties of the least squares estimator (LSE) of an AR(1) process when the AR parameter of the true data generating process (DGP) has structural breaks which are generated by ergodic stationary processes. We further examine the special case where the process has some unit root sub-processes. In general, when there are structural breaks, (1) the rate of convergence to the limiting distribution becomes much slower than when there is no structural break, (2) the persistence level tends towards the largest sub-AR parameter, (3) the whole DGP appears to be a unit root process when some of its sub-processes are unit roots, and (4) the conventional DF test will be biased toward accepting the null of stationarity when the alternative that there exist some unit root sub-processes with probably long durations is true. The analysis is also extended to the case of an infinite number of structural breaks.

Keywords: AR(1), Unit Root, The Least Squares Estimator (LSE), Structural Breaks, Asymptotic Property, The Dicky Fuller (DF) Test

JEL Classification: C12, C13

1 Introduction

Many studies in the structural break literature investigate the effects of structural breaks in nonstationary processes such as trend stationary or unit root processes, and show that structural breaks in level or trend could cause an I(1) process to be even more persistent and thus impair the power of the conventional DF tests of unit root in distinguishing unit root and stationarity (see e.g., Perron, 1989; Leybourne et al., 1998; Kim et al., 2004; Hsu and Kuan, 2001). Despite a huge literature on the estimation and inference of structural breaks (e.g., Bai et al., 1998; Bai and Perron, 1998; Chong, 2001; Perron and Zhu, 2005), the asymptotic properties of stationary processes in the presence of multiple structural breaks have not yet been proposed. In particular, the effects of structural breaks in the persistence level of a process on its estimated persistence level have not yet been fully investigated. The structural breaks in the AR coefficient (SBAR) is a quite common phenomenon in many economic time series (e.g., Perron and Zhu (2005) show that there are structural breaks in the slopes of the stochastic trends of the logarithmic GDP time series for 10 different countries in the period between 1870 and 1986.)

Our study provides the asymptotic distributions of the least squares estimator (LSE) of the AR parameter when the AR sub-processes follow stationary AR(1) processes and/or unit root, highlighting an important topic in econometrics - unit root tests in the presence of structural breaks in the AR parameter. For this purpose, we shall use a simple zero mean AR(1) process as the main DGP. This DGP appears to be simple and restrictive, but the results could be intuitively more appealing. The asymptotic properties of the LSE of the AR parameter of a more generalised DGP are likely to be quite complicated in the presence of multiple structural breaks. If they become too complicated and thus simulations are required to investigate the asymptotic properties, the asymptotic results are likely to be less attractive. This could be a reason why the asymptotic properties of an ARMA process in the presence of multiple breaks have not yet been investigated in the literature.

When a zero mean AR(1) process has multiple structural breaks in the AR parameter, the LSE of the AR parameter obtained without considering the structural breaks tends towards the largest sub-AR parameter (in absolute term). Thus a short but highly persistent sub-process could make the entire process appear far more persistent. When some probably short sub-processes are unit roots, the asymptotic behaviour of the LSE of the AR parameter is dominated by these sub-unit root processes so that the entire process appear to be a unit root. In addition, we show analytically that the DF test could tend to accept the null of stationarity when the alternative that some sub-processes are unit roots is true; and the acceptance frequency depends on the number of structural breaks.

This paper is organized as follows. In the next section, we propose our DGP and the assumptions we use to derive the asymptotic distributions. Then, in section 3, given our DGP, we derive the asymptotic properties of the LSE of the AR parameter in the presence of structural breaks in the AR parameter. Section 4 offers the results of Monte Carlo simulations and the conclusions follow.

2 Data Generating Process and the Assumptions

In this study we examine the effects of structural breaks on the LSEs of AR(1) processes in two cases: (1) the break does not incur nonstationarity, (2) the break incurs nonstationarity in some sub-processes. The effects could be investigated using more general DGPs such as ARMA processes with seasonal dummies, but the asymptotic analysis is quite complicated in this case. Hence, we shall focus on a simple zero mean AR(1) process and reserve more complicated DGPs for future studies.

Let us consider the following zero mean AR(1) process with structural breaks in the AR parameter:

$$\begin{aligned} y_t &= \phi_t y_{t-1} + \xi_t \\ \phi_t &= (1 - I_t)\phi_{t-1} + I_t(\phi + \epsilon_t), \end{aligned} \tag{1}$$

where I_t is an indicator variable, i.e., $I_t = 1$ with the probability of p. Therefore, the AR parameter in (1) occasionally changes around ϕ, and the frequency of changes depends on p. When $|\phi| < 1$, the process is a special case of the so-called random-coefficient autoregressive process (RCAR) where $I_t = 1\ \forall t$. The asymptotic properties of RCAR process are not different from those of the standard AR(1) process in that $T^{1/2}(\hat{\phi} - \phi) = O_p(1)$ (see e.g., Nicholls and Quinn, 1982 or Tjøstheim, 1986). However, to our knowledge the asymptotic properties of the AR(1) process with an occasionally changing AR parameter are not yet fully investigated.

We assume that ξ_t is an ergodic martingale difference sequence (MDS) that satisfies $E[\xi_t|\mathcal{F}_{t-1}] = 0$, $E[\xi_t^2|\mathcal{F}_{t-1}] = \sigma^2$, where $\mathcal{F}_{t-1}$ is an information filtration generated by $\{c_0, z_0, \epsilon_0, \xi_0, ..., c_{t-1}, z_{t-1}, \epsilon_{t-1}, \xi_{t-1}\}$, and

$$E[|\xi_t|\mathbf{1}_{(|\xi_t|>a|\alpha_n|^{-1})}|\mathcal{F}_{t-1}] \xrightarrow{P} 0, \tag{2}$$

where the sequence $|\alpha_n| \longrightarrow |\alpha^*| < 1$ as $n \to \infty$ for some $a \in (0, 1]$. Supposing that all the moments of ξ_t are finite and $a = 1$, then by the Markov and Holder inequalities, we have

$$E[|\xi_t|\mathbf{1}_{(|\xi_t|>|\alpha_n|^{-1})}|\mathcal{F}_{t-1}] \leq |\alpha_n|^r E[|\xi_t|^r] = o(1)$$

as $n,\ r \longrightarrow \infty$. Thus, (2) is obviously satisfied. ϵ_t in (1) is an ergodic stationary process independent of ξ_t with the stationary distribution $F_\epsilon(\bullet)$ which has zero mean and variance σ_ϵ^2; the sample paths of ϵ_t satisfy $|\phi + \epsilon_t| \leq 1\ \forall t$.

Note that the sequence of ergodic martingale differences includes the sequence of independent and identical distributed random variables as a special case. However, the results in our paper can be shown to hold under a fairly general assumption as used in Phillips (1988), namely that $\{\epsilon_t, \xi_t\}$ is strong mixing, though we do not use the strong mixing assumption for the innovation processes because our analytical results may not be necessarily improved given the complicated nature of this assumption.

We make a prior assumption that there are K different sub-processes with break times $[T\tau_1]+1$, $[T\tau_1]+[T\tau_2]+1$, ..., $\sum_{i=1}^{k}[T\tau_i]+1$, ..., and $\sum_{i=1}^{K-1}[T\tau_i]+1$, where $[T\tau_i]$ is the size of sub-sample i. Hence, the true trajectory of the process given in (1) is

$$\phi_{t_k} = (\phi + \epsilon_k), \text{ where } t_k \in \left[\sum_{i=1}^{k-1}[T\tau_i]+1,\ \sum_{i=1}^{k}[T\tau_i]\right] \tag{3}$$

and

$$\epsilon_k = \epsilon_{\sum_{i=1}^{k-1}[T\tau_i]+1} = \epsilon_{\sum_{i=1}^{k-1}[T\tau_i]+2} = ... = \epsilon_{\sum_{i=1}^{k}[T\tau_i]}.$$

The following assumptions are useful for further analysis.

Assumption 1 *For a given $K \in (0, \infty]$, the duration process τ_k is an ergodic process* *independent of* ϵ_t *and* ξ_t *. The sample paths of* τ_k *satisfy* $\tau_K = 1 - \sum_{k=1}^{K-1} \tau_k$*,* $E(\tau_k^2) = \sigma_\tau^2(K)$ *and* $E(\tau_k^3) = \lambda_\tau(K) > 0$*.* τ_k *has the stationary distribution* $F_\tau(\bullet)$*.*

As in Bai and Perron (1998), τ_k needs to be asymptotically distinct. Note that $0 \leq \tau_k \leq 1 \; \forall k$, and thus the non-central skewness of τ_k is always positive. Since τ_k generally decreases as K increases, $\sigma_\tau^2(K)$ and $\lambda_\tau(K)$ are decreasing functions of K.

Assumption 2 $p = O(T^{-1})$ *such that* $\lim_{T \to \infty} Tp = K - 1$*.*

Assumption 2 is useful for investigating the asymptotic properties of a process with a small number of structural breaks in large samples. Many studies such as Diebold and Inoue (2001), Leipus and Surgailis (2003), and Granger and Hyung (2004) allow $p \to 0$ to explain long memory with structural breaks in similar processes to equation (1). The assumption says that $T(k)$ tends to increase with T so that the number of structural breaks remains finite. However, as explained in Diebold and Inoue (2001) and Granger and Hyung (2004), this sample size-dependent probability may not reflect reality. In general, as T increases K is also expected to increase as in the following assumption.

Assumption 3 $K = O(T)$ *such that* $\lim_{T \to \infty} \frac{K-1}{T} = p$*.*

The probability of breaks however is usually small and thus K may be still small even if T increases to a large number. Thus the asymptotic results with Assumption 3 in this study should be interpreted with care. Nevertheless the assumption is useful for the investigation of the effects of structural breaks since it provides much simplified asymptotic results.

Assumption 4 *The changing points* $y_{\sum_{i=1}^{k}[T\tau_i]+1}$ *have finite first and second order moments, i.e.,* $E\left(y_{\sum_{i=1}^{k}[T\tau_i]+1}\right) < \infty$ *and* $E\left(y^2_{\sum_{i=1}^{k}[T\tau_i]+1}\right) < \infty$*.*

Assumption 4 is equivalent to $\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right| < \infty$ and $y^2_{\sum_{i=1}^{k}[T\tau_i]+1} < \infty$ almost sure. This follows from the Borel-Cantelli lemma. Since

$$\sum_{K=1}^{\infty} P\left\{\omega : \left(\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq K\right)\right\} < \sum_{K=1}^{\infty} \frac{E\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right|^2}{K^2} < \infty,$$

then

$$\begin{aligned} P\left\{\bigcap_{n=1}^{\infty}\bigcup_{K=n}^{\infty}\left[\omega : \left(\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq K\right)\right]\right\} &= P\left\{\omega : \left(\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right| \geq \infty\right)\right\} \\ &= 0. \end{aligned}$$

Thus, $\left|y_{\sum_{i=1}^{k}[T\tau_i]+1}\right| < \infty$ is almost sure (a.s.), and similarly we can prove that $y^2_{\sum_{i=1}^{k}[T\tau_i]+1} < \infty$ a.s.

3 The Asymptotic Behaviour of the LSE of the AR Parameter

Assume that our econometrician estimates the first order autocorrelation or the following misspecified zero mean AR(1) process for the data generating process in (1):

$$y_t = \varphi y_{t-1} + \eta_t, \tag{4}$$

where $\eta_t \sim (0, \sigma_\eta^2)$. Our concern here is the effects of the changing ϕ_{t_k} (or ε_k) on the asymptotic properties of $\hat{\varphi}$.

In this section, both stationarity and nonstationarity are analyzed. The stationarity condition of the process in (1) can be easily proved to be $\phi^2 + \sigma_\varepsilon^2 < 1$, which is not different from that of the RCAR. However, this condition is not sufficient to warrant that all sub-processes are stationary. As shown below, when one or more of sub-processes are unit roots, the asymptotic distribution becomes quite different from that of the process whose sub-processes are stationary; and the whole process will look like a unit root process.

3.1 Stationary Case: $|\phi + \varepsilon_k| < 1 \; \forall k$

We first investigate the case where all of the sub-AR processes are stationary.

Theorem 1 *Under Assumptions 1, 2 and 4, the asymptotic conditional distribution of the LSE of* φ *when the true DGP is (1) is given by*

$$\hat{\varphi} - \phi\Big|_{\{\tau_k, \varepsilon_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{\sum_{k=1}^K \varepsilon_k \frac{\tau_k}{1-\phi_k^2} W_{1k}^2(1)}{\sum_{k=1}^K \frac{\tau_k}{1-\phi_k^2} W_{1k}^2(1)} = \mathcal{D}, \tag{5}$$

where $\phi_k = \phi + \varepsilon_k$ *and* $W_{1k}(1)$ *is standard Brownian motion.*

Proof. See the Appendix. ■

By taking expectations, the limiting unconditional distribution of the bias in (5) is given by

$$\mathcal{L}(\hat{\varphi} - \phi | P) \Longrightarrow \int_{\substack{(0,1)^K \\ \sum_{k=1}^K \tau_k = 1}} \int_{\mathbb{R}^K} \mathbb{P}(\mathcal{D} | \{\tau_k, \varepsilon_k\}_{k=1}^K) \prod_{k=1}^K dF_\varepsilon(\varepsilon_k) dF_\tau(\tau_k).$$

Under certain conditions, the LSE $\widehat{\varphi}$ can be represented in terms of Bessel processes as follows.

Remark 1 *Define* $R_1(t) = \sqrt{\sum_1^K \tau_k \frac{\phi_k}{1-\phi_k^2} W_k^2(t)}$ *and* $R_2(t) = \sqrt{\sum_1^K \tau_k \frac{1}{1-\phi_k^2} W_k^2(t)}$ *respectively. Suppose that the sample paths of* ϕ_k *is are always positive, i.e.,* $1 > \phi_k > 0$ $\forall k$*. In view of Proposition 3.21 in Karatzas and Shreve (1991),* $R_1(t)$ *and* $R_2(t)$ *are mixed Bessel processes which are the solutions to the following stochastic differential equations:*

$$R_1(t) = \int_0^t \frac{K-1}{2R_1(s)} ds + B_1(t), \tag{6}$$
$$R_2(t) = \int_0^t \frac{K-1}{2R_2(s)} ds + B_2(t), \tag{7}$$

where $B_1(t)$ *and* $B_2(t)$ *are mixed Brownian motions that are defined as*

$$B_1(t) = \sum_{k=1}^K \int_0^t \frac{1}{R_1(s)} [\tau_k \frac{\phi_k}{1-\phi_k^2}]^2 W_k(s) dW_k(s),$$
$$B_2(t) = \sum_{k=1}^K \int_0^t \frac{1}{R_1(s)} [\tau_k \frac{1}{1-\phi_k^2}]^2 W_k(s) dW_k(s).$$

Thus

$$\hat{\varphi}\Big|_{\{\tau_k, \varepsilon_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{R_1^2(1)}{R_2^2(1)}. \tag{8}$$

When there is an infinite number of SBARs, the following proposition is obtained by applying Loeve's SLLN and Etemadi's SLLN for non-negative random variables in (Chow and Teicher, 1997).

Proposition 1 *With Assumption 3, if* $\lim_{K \longrightarrow \infty} \sum_1^K \frac{1}{k^\alpha} \left(\frac{\tau_k}{1-\phi_k^2} \right)^\alpha < \infty$ *for some* $\alpha \in (0, 2]$*, then*

$$\hat{\varphi} - \phi \overset{P}{\Longrightarrow} \frac{E\left[\frac{\varepsilon_k}{1-(\phi+\varepsilon_k)^2}\right]}{E\left[\frac{1}{1-(\phi+\varepsilon_k)^2}\right]}. \tag{9}$$

Therefore, as $K \to \infty$, the bias does not depend on break durations but on the means of $\frac{\varepsilon_k}{1-(\phi+\varepsilon_k)^2}$ *and* $\frac{1}{1-(\phi+\varepsilon_k)^2}$ *respectively.*

Proof. See the Appendix. ■

It is clear that when there is no structural break in the AR parameter, i.e., $\phi_k = \phi$, or $\varepsilon_k = 0\ \forall k$, then $\hat{\varphi} - \phi \overset{P}{\Longrightarrow} 0$. In other words, $\hat{\varphi}$ is the consistent estimate of ϕ as in the standard AR(1) process. However, since $\hat{\varphi}$ is a weighted average value of sub-AR parameters, the limit of $\hat{\varphi}$ may not be consistent with ϕ in the presence of SBARs. Theorem 1 shows that $\frac{\tau_k}{1-\phi_k^2}$ serves as the weights on the k-th sub-AR parameter, ϕ_k, and thus the LSE of the AR parameter is a weighted average value of the sub-AR parameters. The larger ϕ_k^2 is, *ceteris paribus*, the more weighted the sub-process is. A small value of ϕ_k, e.g., $0.1 > \phi_k > -0.1$, may not change its weight $\frac{\tau_k}{1-\phi_k^2}$ significantly. However, when $\frac{\tau_k}{1-\phi_k^2}$ increases with ϕ_k^2 and in particular ϕ_k^2 is very close to one, $\frac{\tau_k}{1-\phi_k^2}$ becomes extremely large. Thus regardless of signs of ϕ_k the sub-process with the largest ϕ_k is more weighted than the other sub-processes and thus $\hat{\varphi}$ tends towards the largest ϕ_k.

It is interesting to see the difference in the convergence rate between the AR(1) process with SBARs and the RCAR process. Koul and Schick (1996) show that the LSE of the AR parameter of a stationary RCAR process is $T^{1/2}$ consistent. However when the coefficient changes occasionally with a small probability, we find that the LSE becomes $O_p(1)$ and the limiting distribution of $\hat{\varphi} - \phi$ lies within certain ranges (depending on the distribution of ε_k).

3.2 Nonstationary Case: $|\phi + \varepsilon_k| = 1$ for at Least a k

We extend the analysis in the previous section to the special case where the set of sub-AR parameters $\{\phi_k\}_{k=1}^K$ contains at least one unit root. In other words, we allow both stationary and non-stationary sub-processes.

Theorem 2 *Under Assumptions 1, 2, 4 and when the set of sub-AR parameters contains at least one unit root, the asymptotic conditional distribution of the LSE of* φ *for the DGP as in (1) is given by*

$$\begin{aligned} \textit{Case 1} &: \ \widehat{\varphi} - 1 = o_p(1), \\ \textit{Case 2} &: \ T(\widehat{\varphi}-1)\Big|_{\{\tau_k,\varepsilon_k\}_{k=1}^K} \overset{W}{\Longrightarrow} \frac{-\sum_{k\in\widetilde{K}^S}\frac{\tau_k}{1+\phi_k}W_k^2(1)+\sum_{k\in\widetilde{K}^U}\left[\int_0^{\tau_k}W_k(s)dW_k(s)-W_{k-1}^2(\tau_{k-1})\right]}{\sum_{k\in\widetilde{K}^U}\int_0^{\tau_k}W_k^2(s)ds} \\ &= \mathcal{E}, \end{aligned} \tag{10}$$

where $\widetilde{K}^U$ *is the subset of* K *that contains unit roots, and* $\widetilde{K}^S$ *is the subset of* K *that contains stationary sub-processes.*

Proof. See the Appendix. ■

By taking expectations, the asymptotic unconditional distribution of the bias as in (10) is given by

$$\mathcal{L}(T(\hat{\varphi}-1)|P) \Longrightarrow \int_{\substack{(0,1)^K \\ \sum_{k=1}^K \tau_k = 1}} \int_{\mathbb{R}^K} \mathbb{P}(\mathcal{E}|\{\tau_k,\varepsilon_k\}_{k=1}^K)\prod_{k=1}^K dF_\varepsilon(\varepsilon_k)dF_\tau(\tau_k).$$

When some sub-AR processes are unit roots, the limit of the LSE of φ converges to 1 (*Case 1*) since the unit root processes dominate the stationary sub-processes. The convergence happens irrespective of the number of unit roots or the time periods of unit roots as far as τ_k is asymptotically distinct. Therefore, even if the stationarity condition is satisfied in this AR(1) process with SBARs, i.e., $\phi^2+\sigma_\varepsilon^2<1$, there is still a possibility that at least one sub-process is unit root and thus the entire process may look like a unit root.

The second result in Theorem 2 suggests that it is very likely that using the conventional DF test to test for the null of a unit root in the presence of structural breaks in the AR parameter could result in spurious rejection, a term used by Leybourne et al. (1998). This is because when the number of structural breaks is sufficiently large, the asymptotic conditional distribution in (10) is more likely to be skewed toward the negative side of the real line as seen in Proposition 2 below. Thus, the null hypothesis $H_0:\phi=1$ is rejected rather often; the DF test concludes that the whole process is stationary, but most of its sub-processes with probably long durations are actually unit roots. Furthermore, the test does not have power to differentiate between $H_0:\ \phi=1$ (i.e., a unit root without SBARs) and $H_1:\{|\phi_k|<1\forall k\in\widetilde{K}^S;\ \phi_k=1\forall k\in\widetilde{K}^U\}$ (i.e., a unit root with SBARs) since the rates of convergence are the same under the null and the alternative.

Case 2 can be further analyzed with Assumption 3.

Proposition 2 *If* $\frac{\sum_{k=1}^{K}[\frac{\tau_k}{1+\phi_k}]^2}{k^2}<\infty$ *and* $K\to\infty$ *so that* $\widetilde{K}^S\to\infty$ *and* $\widetilde{K}^U\to\infty$*. Under Assumption 3, Case 2 in Theorem 2 becomes*

$$T(\widehat{\varphi}-1)\Big|_{\{\tau_k,\phi_k\}_{k=1}^K}\overset{P}{\Longrightarrow}\frac{-\sum_{k\in\widetilde{K}^S,\ \widetilde{K}^S\longrightarrow\infty}\frac{\tau_k}{1+\phi_k}-\sum_{k\in\widetilde{K}^U,\widetilde{K}^U\longrightarrow\infty}\tau_{k-1}}{\sum_{k\in\widetilde{K}^U,\ \widetilde{K}^U\longrightarrow\infty}\frac{\tau_k^2}{2}}. \tag{11}$$

Proof. The proof follows from Etemadi's SLLN for non-negative random variables. ■

As the number of SBARs becomes large, $T(\widehat{\varphi}-1)$ becomes always negative. As the durations of unit root sub-processes τ_k $\forall k \in \widetilde{K}^U$ increase, the numerator ($\sum_{k\in\widetilde{K}^S,\ |\widetilde{K}^S|\longrightarrow\infty} \frac{\tau_k}{1+\phi_k}$ and $\sum_{k\in\widetilde{K}^U,|\widetilde{K}^U|\longrightarrow\infty} \tau_{k-1}$) decreases while the denominator increases, and thus the negative value of $T(\hat{\varphi}-1)$ approaches zero. This implies that the sub-unit root processes begin to dominate the stationary sub-processes. On the contrary, as the durations of stationary sub-processes τ_k $\forall k \in \widetilde{K}^S$ increase, *ceteris paribus*, $T(\widehat{\varphi}-1)$ decreases, and thus stationary sub-processes dominate the nonstationary sub-processes and the entire process appears to be more stationary.

For given τ_k when the stationary sub-AR process is negatively autocorrelated, i.e., $-1 < \phi_k < 0$, the value of $\sum_{k\in\widetilde{K}^S,\ |\widetilde{K}^S|\longrightarrow\infty} \frac{\tau_k}{1+\phi_k}$ becomes larger and $T(\hat{\varphi}-1)$ decreases (towards stationarity). On the other hand as the sub-AR parameters ϕ_k increase towards 1, *ceteris paribus*, $T(\hat{\varphi}-1)$ increases. This has an obvious implication that the entire process look like a unit root process as the sub-AR parameters approach unit roots, thus the spurious rejection of the DF test for unit root does not exist in this case. Finally when the durations of the sub-unit root processes are small (i.e., the denominator in (11) is small), $T(\widehat{\varphi}-1)$ could be a large negative number, thus the limiting distribution has long left tail.

4 Simulations

In order to better understand the asymptotics of the LSE of the AR parameter, we simulate the asymptotic distributions and the compare the results with the sample LS estimates we obtain by estimating (4). The simulations are designed as follows. AR(1) series are generated for the sample sizes of $T = 100, 200, 500, 1000,$ and 3000. For the numbers of breaks, we set $K-1=4,9,49$, and 99 where K is the number of sub-processes. The error term in the AR process follows standard normal, $\xi_t \sim N(0,1)$, while the sizes of structural breaks in AR parameter are drawn from two different normal distributions, i.e., $\epsilon_t \sim N(0,0.2^2)$ and $N(0,0.3^2)$. For the values of ϕ, we take 0.4, 0, and -0.4. For the asymptotic distributions we generate Brownian motions ($W_k(1)$) with 10000 i.i.d. standard normal variates. We repeat the procedure 10000 times to obtain the sample LS estimates and asymptotic distributions.

For the stationary case in Theorem 1, we truncate any $\phi_k \geq 1$ to $\phi_k = 0.999$. The pattern in table 1 shows that the tendency of the LSE of φ towards the largest sub-AR parameter increases as K increases. In addition the rate of convergence to the limiting distribution is rather slow. Even with 1000 observations, the sample LS estimates of $\widehat{\varphi}$ do not approach the limiting distributions in particular when σ_ε and the number of breaks are large. Again the asymptotic result in Proposition 1 requires much larger number of breaks; the cases of $K = 99$ still show large deviations from the analytical value in (9). Panel B reports that $\widehat{\varphi} - \phi$ tends to increase for positive ϕ while it tends to decrease for negative ϕ. On the other hand, when $\phi = 0$ and ϕ_k (or ε_k) is symmetric, we find that the effects of SBARs are symmetric.

Figure 1 shows the Gaussian kernel densities for the stationary case, i.e., $|\phi + \varepsilon_k| < 1$ $\forall k$ when $\phi = 0.4$. We find that there is a mass in the right tail of the limiting distribution. This is in part due to the truncation we impose to make the process stationary (i.e., $\phi_k \leq 0.999$). However the mass in limiting distribution looks much larger than we expect from the truncation. To investigate if the mass reflects the truncation, we allow ϕ_k to be larger than 1 (no truncation for $\phi_k \geq 1$). The last column of figure 1 shows a larger mass between 0.4 and 0.6 without the truncation. These results suggest that allowing the sub-AR processes to be nonstationary results in a higher tendency towards persistence, and the truncation reduces the mass. The mass reflects the extremely persistent sub-processes which dominate other less persistent sub-AR processes. This dominance is more apparent when the break size of the AR parameter is larger.

Finally we carry out a similar procedure for the nonstationary case in Theorem 2. When there is no sub-unit root process we make the largest sub-AR parameter be unit, and any sub-AR parameter whose $\phi_k > 1$ is truncated to 1. In addition, Theorem 2 requires $[T\tau_k] \rightarrow \infty$ as $T \rightarrow \infty$ and thus sub-sample sizes should not be small (asymptotic distinction in τ_k). Therefore we impose the restriction of $T\tau_k > 20\ \forall k$ in our simulations. Figure 2 shows that as the sample sizes increase a large mass begins to build up near 0, whose shape is very similar to those in Figure 1. This pattern is apparent in the case of $K-1=9$ rather than $K-1=49$. However, the mass near zero approaches the limiting distribution at the bottom of Figure 2 very slowly. The last column of Figure 1 suggests that the mass around 0 could become large much faster when there is no truncation on ϕ_k; when sub-AR processes with $\phi_k > 1$ are allowed the process more frequently looks like a unit root process.

5 Conclusion

These results suggest several conclusions. First, the LSE of the AR parameter has an asymmetric limiting distribution when there are structural breaks in the AR parameter of a zero mean stationary $AR(1)$ process. In the presence of structural breaks in the AR parameter, the LSE of the AR parameter tends towards the largest sub-AR parameters.

On the other hand, when there is at least one unit root process in sub-samples, the LSE of the AR parameter tends towards the unit root even if the condition for stationarity is satisfied. Hence, the unit root sub-processes could make the entire process look like unit root.

Second, our results suggest that when there are structural breaks in processes, the conventional statistics we use for inferences may not be appropriate. Because of slow convergence rates and biases in the persistence level, the conventional Gaussian t tests and the DF test of unit root are not very powerful for these types of processes.

Appendix

For the proof we use Corollary 1 of Chu and Hwang (2005), which shows for $|\phi| < 1$

$$\lim_{T\longrightarrow\infty} \frac{S_{[T\tau]}}{\sigma\sqrt{\frac{\phi^2}{1-\phi^2}}} \stackrel{W}{\Longrightarrow} W(1), \tag{12}$$

where $S_{[T\tau]} = \sum_{t=1}^{[T\tau]} \phi^t \xi_t$, $\forall \tau \in (0,1]$ and ξ_t is a sequence of martingale differences with $E[\xi_t|\mathcal{F}_{t-1}] = 0$ and $E[\xi_t^2|\mathcal{F}_{t-1}] = \sigma^2$.

Proof of Theorem 1

The LSE of φ is given as follows:

$$\begin{aligned} \hat{\varphi} &= \frac{\sum_{t=1}^{T} y_t y_{t-1}}{\sum_{t=1}^{T} y_{t-1}^2} \\ &= \frac{\sum_{k=1}^{K} \sum_{T(k-1)+1}^{T(k)} y_{t_k} y_{t_k-1}}{\sum_1^K \sum_{T(k-1)+1}^{T(k)} y_{t_k-1}^2} \\ &= \frac{\sum_1^K y_{T(k-1)+1} y_{T(k-1)} + \sum_1^K \sum_{T(k-1)+2}^{T(k)} \phi_k y_{t_k-1}^2 + \sum_1^K \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k}}{\sum_{k=1}^{K} y_{T(k-1)}^2 + \sum_1^K \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}^2} \end{aligned} \tag{13}$$

Applications of the result in (12) and the continuous mapping theorem together with Assumption 4 yield the following limits:

$$T^{-1}\sum_{t_k=T(k-1)+2}^{T(k)} \phi_k y_{t_k-1}^2 \overset{W}{\Longrightarrow} \frac{\sigma^2\tau_k C_k}{\phi_k} W_{1k}^2(1) + o_p(1). \tag{14}$$

$$T^{-1/2}\sum_{t_k=T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k} \overset{W}{\Longrightarrow} \sigma^2\phi_k^{-1}\sqrt{C_k}W_{1k}(1)W_{2k}(\tau_k) + o_p(1). \tag{15}$$

$$y_{T(k-1)}^2 \overset{W}{\Longrightarrow} \frac{\sigma^2}{1-\phi_k^2} W_k^2(1) + o_p(1). \tag{16}$$

$$T^{-1}\sum_{t_k=T(k-1)+2}^{T(k)} y_{t_k-1}^2 \overset{W}{\Longrightarrow} \sigma^2\phi_k^{-2}\tau_k C_k W_{1k}^2(1) + o_p(1). \tag{17}$$

In addition since

$$y_{T(k-1)+1}y_{T(k-1)} = y_{T(k-1)+1}\left[\phi_{k-1}^{[T\tau_{k-1}]-1} y_{T(k-2)+1} + \phi_{k-1}^{-1}\sum_{j=1}^{T(k-1)-T(k-2)-1}\phi_{k-1}^j \xi_{T(k-1)+1-j}\right],$$

it is straightforward to obtain

$$y_{T(k-1)+1}y_{T(k-1)} \overset{W}{\Longrightarrow} y_\infty \frac{\sigma}{\sqrt{1-\phi_{k-1}^2}} W_{1k}(1) + o_p(1). \tag{18}$$

Therefore we have

$$\hat{\varphi} = \frac{\sum_{k=1}^K (\phi+\varepsilon_k)\frac{\tau_k}{1-\phi_k^2}W_{1k}^2(1) + O_p(T^{-1/2})}{\sum_{k=1}^K \frac{\tau_k}{1-\phi_k^2}W_{1k}^2(1) + o_p(1)} = \phi + \frac{\sum_{k=1}^K \varepsilon_k \frac{\tau_k}{1-\phi_k^2}W_{1k}^2(1)}{\sum_{k=1}^K \frac{\tau_k}{1-\phi_k^2}W_{1k}^2(1)}, \tag{19}$$

since $\phi_k = \phi + \varepsilon_k$, and thus equation (5) follows.

Proof of Proposition 1

As $K \to \infty$, using Etemadi's SLLN for non-negative random variables we have

$$\left.\sum_{k=1}^{K} \frac{\phi_k}{1-\phi_k^2} W_{1k}^2(\tau_k)\right|_{\{\tau_k,\phi_k\}_{k=1}^K} \overset{a.s}{\Longrightarrow} \sum_{k=1}^{\infty} \frac{\phi_k \tau_k}{1-\phi_k^2},$$
$$\left.\sum_{k=1}^{K} \frac{1}{1-\phi_k^2} W_{1k}^2(\tau_k)\right|_{\{\tau_k,\phi_k\}_{k=1}^K} \overset{a.s}{\Longrightarrow} \sum_{k=1}^{\infty} \frac{\tau_k}{1-\phi_k^2}. \tag{20}$$

The result is then obtained by applying the SLLN for ergodic MDS under the assumption that $\sum_{k=1}^{K} \tau_k = 1$ and τ_k and ε_k are independent.

Proof of Theorem 2

Let $\widetilde{K}^U$ is the subset of K that contains unit root processes; $\phi_k = 1$, $k \in \widetilde{K}^U$, while $\widetilde{K}^S$ does not contain unit root processes. Then the LSE of φ is given by

$$\hat{\varphi} = \frac{\begin{array}{c}\sum_{k\in\widetilde{K}^U} \left(y_{T(k-1)+1} y_{T(k-1)} + \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}^2 + \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k}\right) + \\ \sum_{k\in\widetilde{K}^S} \left(y_{T(k-1)+1} y_{T(k-1)} + \sum_{T(k-1)+2}^{T(k)} \phi_k y_{t_k-1}^2 + \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k}\right)\end{array}}{\sum_{k\in\widetilde{K}^U} \left(y_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}^2 \right) + \sum_{k\in\widetilde{K}^S} \left(y_{T(k-1)}^2 + \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}^2 \right)}. \tag{21}$$

The asymptotic properties of the terms involving $k \in \widetilde{K}^S$ are the same as the results in the proof of Theorem 1, i.e.,

$$T^{-1/2} y_{T(k-1)+1} y_{T(k-1)} = o_p(1).$$
$$T^{-1} \sum_{t_k=T(k-1)+2}^{T(k)} y_{t_k-1}^2 \overset{W}{\Longrightarrow} \frac{\sigma^2 \tau_k}{1-\phi_k^2} W_{1k}^2(1).$$
$$T^{-1/2} \sum_{t_k=T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k} \overset{W}{\Longrightarrow} \frac{\sigma^2}{\sqrt{1-\phi_k^2}} W_{1k}(1) W_{2k}(\tau_k).$$
$$y_{T(k-1)}^2 \overset{W}{\Longrightarrow} \frac{\sigma^2}{1-\phi^2} W_k^2(1) + o_p(1).$$

Regarding the terms $k \in \widetilde{K}^U$, applications of Donsker's IP (see Theorem 14.1 of Billingsley (1999)) for MDS and the continuous mapping theorem yield

$$T^{-1/2} y_{T(k-1)+1} y_{T(k-1)} = o_p(1).$$
$$T^{-2} \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}^2 \overset{W}{\Longrightarrow} \sigma^2 \int_0^{\tau_k} W_{2k}^2(s) ds + O_p(T^{-1/2}).$$
$$T^{-1} \sum_{T(k-1)+2}^{T(k)} y_{t_k-1}\xi_{t_k} \overset{W}{\Longrightarrow} \sigma^2 \int_0^{\tau_k} W_{2k}(s) dW_{2k}(s) + O_p(T^{-1/2}).$$
$$T^{-1} y_{T(k-1)}^2 \overset{W}{\Longrightarrow} \sigma^2 W_{1k-1}^2(\tau_{k-1}) + O_p(T^{-1}).$$

Hence we have

$$\widehat{\varphi} \overset{W}{\Longrightarrow} \frac{\sum_{k\in\widetilde{K}^U} \int_0^{\tau_k} W_{2k}^2(s) ds + O_p(T^{-1/2})}{\sum_{k\in\widetilde{K}^U} \int_0^{\tau_k} W_{2k}^2(s) ds + O_p(T^{-1/2})} = 1.$$

An alternative expression of equation (21) is

$$\hat{\varphi}-1=\frac{\begin{array}{c}\sum_{k\in\widetilde{K}^U}\left(y_{T(k-1)+1}y_{T(k-1)}-y_{T(k-1)}^2+\sum_{T(k-1)+2}^{T(k)}y_{t_k-1}\xi_{t_k}\right)+\\ \sum_{k\in\widetilde{K}^S}\left(y_{T(k-1)+1}y_{T(k-1)}-y_{T(k-1)}^2+(\phi_k-1)\sum_{T(k-1)+2}^{T(k)}y_{t_k-1}^2+\sum_{T(k-1)+2}^{T(k)}y_{t_k-1}\xi_{t_k}\right)\end{array}}{\sum_{k\in\widetilde{K}^U}\left(y_{T(k-1)}^2+\sum_{T(k-1)+2}^{T(k)}y_{t_k-1}^2\right)+\sum_{k\in\widetilde{K}^S}\left(y_{T(k-1)}^2+\sum_{T(k-1)+2}^{T(k)}y_{t_k-1}^2\right)}. \tag{22}$$

Scaling $\hat{\varphi}-1$ by T, the limiting distribution of (22) is given by

$$T(\hat{\varphi}-1)|_{\{\tau_k,\varepsilon_k\}_{k=1}^K}\overset{W}{\Longrightarrow}\frac{-\sum_{k\in\widetilde{K}^S}\frac{\tau_k}{1+\phi_k}W_k^2(1)+\sum_{k\in\widetilde{K}^U}\int_0^{\tau_k}W_k(s)dW_k(s)-W_{k-1}^2(\tau_{k-1})+O_p(T^{-1/2})}{\sum_{k\in\widetilde{K}^U}\int_0^{\tau_k}W_k^2(s)ds+O_p(T^{-1/2})}.$$

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Table 1 Percentage Points of the Distributions of $\varphi - \phi$ for Various Number of Breaks, Break Sizes, and AR Parameters In the Presence of Structural Breaks in AR Parameters

The sizes of structural breaks in AR parameter are drawn from two different normal distributions, i.e., $N(0,0.2^2)$ and $N(0,0.3^2)$, while the error term in the AR process follows standard normal. For the values of ϕ, we take 0.4, 0, and -0.4. For the stationary case in Panel A, we truncate any ϕ=1 to ϕ=0.999.

A. The Effects of Structural Breaks in AR Parameters for Various Number of Breaks, Break Sizes, and Number of Observations

AR Parameter	Number of Breaks	Break Size	Total Number of Observations	Percentage Points of the Distributions								
				1%	2.50%	5%	10%	50%	90%	95%	97.50%	99%
0.4	4	0.2	100	-0.343	-0.287	-0.237	-0.184	0.008	0.195	0.255	0.319	0.389
			200	-0.301	-0.247	-0.201	-0.149	0.013	0.185	0.241	0.297	0.368
			500	-0.266	-0.211	-0.170	-0.127	0.019	0.183	0.245	0.312	0.393
			1000	-0.249	-0.200	-0.162	-0.123	0.015	0.177	0.233	0.297	0.395
			Limit	-0.323	-0.265	-0.216	-0.160	0.020	0.215	0.283	0.355	0.473
		0.3	100	-0.438	-0.364	-0.296	-0.221	0.028	0.310	0.404	0.479	0.529
			200	-0.411	-0.328	-0.266	-0.193	0.033	0.327	0.432	0.511	0.556
			500	-0.386	-0.303	-0.238	-0.171	0.046	0.374	0.497	0.557	0.580
			1000	-0.371	-0.297	-0.234	-0.165	0.048	0.394	0.523	0.575	0.591
			Limit	-0.492	-0.396	-0.320	-0.235	0.067	0.503	0.588	0.596	0.598
0.4	49	0.2	100	-0.257	-0.213	-0.172	-0.132	0.008	0.136	0.169	0.195	0.232
			200	-0.175	-0.145	-0.118	-0.086	0.023	0.130	0.162	0.193	0.232
			500	-0.116	-0.093	-0.072	-0.049	0.037	0.132	0.165	0.202	0.263
			1000	-0.090	-0.070	-0.053	-0.032	0.044	0.135	0.175	0.221	0.308
			Limit	-0.116	-0.093	-0.068	-0.041	0.051	0.196	0.326	0.500	0.566
		0.3	100	-0.276	-0.224	-0.182	-0.140	0.022	0.171	0.215	0.249	0.294
			200	-0.191	-0.153	-0.122	-0.082	0.058	0.205	0.252	0.296	0.355
			500	-0.135	-0.098	-0.068	-0.033	0.101	0.274	0.335	0.385	0.440
			1000	-0.099	-0.065	-0.038	-0.004	0.130	0.345	0.413	0.463	0.507
			Limit	-0.123	-0.070	-0.024	0.030	0.394	0.581	0.588	0.592	0.594
0.4	99	0.2	200	-0.165	-0.138	-0.111	-0.083	0.013	0.106	0.131	0.154	0.180
			500	-0.102	-0.081	-0.063	-0.042	0.031	0.108	0.132	0.156	0.189
			1000	-0.072	-0.052	-0.038	-0.021	0.042	0.115	0.143	0.172	0.230
			Limit	-0.068	-0.048	-0.034	-0.014	0.057	0.221	0.413	0.512	0.562
		0.3	200	-0.177	-0.148	-0.117	-0.085	0.031	0.142	0.173	0.201	0.235
			500	-0.099	-0.075	-0.049	-0.021	0.079	0.199	0.240	0.281	0.328
			1000	-0.060	-0.038	-0.014	0.012	0.116	0.266	0.319	0.369	0.425
			Limit	-0.012	0.029	0.071	0.137	0.487	0.578	0.584	0.588	0.592

B. The Effects of Structural Breaks in AR Parameters for Different AR Parameters

AR Parameter	Number of Breaks	Break Size	Total Number of Observations	Percentage Points of the Distributions								
				1%	2.50%	5%	10%	50%	90%	95%	97.50%	99%
0.4	9	0.2	100	-0.303	-0.247	-0.205	-0.152	0.015	0.184	0.235	0.281	0.334
			200	-0.241	-0.202	-0.167	-0.124	0.025	0.180	0.231	0.281	0.363
			1000	-0.188	-0.149	-0.123	-0.090	0.031	0.174	0.225	0.295	0.403
			Limit	-0.259	-0.208	-0.166	-0.122	0.031	0.209	0.278	0.368	0.528
		0.3	100	-0.376	-0.304	-0.249	-0.182	0.044	0.292	0.374	0.432	0.497
			200	-0.321	-0.261	-0.207	-0.150	0.059	0.323	0.413	0.480	0.536
			1000	-0.262	-0.212	-0.164	-0.113	0.088	0.419	0.522	0.563	0.582
			Limit	-0.383	-0.292	-0.226	-0.154	0.108	0.570	0.591	0.595	0.598
0	9	0.2	100	-0.313	-0.260	-0.216	-0.170	0.001	0.172	0.223	0.267	0.312
			200	-0.266	-0.218	-0.182	-0.143	-0.001	0.145	0.190	0.228	0.267
			1000	-0.232	-0.187	-0.152	-0.114	0.001	0.119	0.153	0.188	0.228
			Limit	-0.310	-0.246	-0.201	-0.156	-0.002	0.152	0.200	0.247	0.312
		0.3	100	-0.453	-0.372	-0.303	-0.234	-0.002	0.228	0.298	0.358	0.448
			200	-0.434	-0.340	-0.273	-0.207	-0.001	0.213	0.284	0.349	0.451
			1000	-0.430	-0.329	-0.259	-0.191	-0.004	0.186	0.255	0.330	0.445
			Limit	-0.591	-0.425	-0.341	-0.250	0.001	0.254	0.335	0.424	0.597
-0.4	9	0.2	100	-0.351	-0.284	-0.236	-0.185	-0.016	0.154	0.203	0.244	0.302
			200	-0.341	-0.281	-0.227	-0.175	-0.021	0.120	0.162	0.196	0.240
			1000	-0.368	-0.277	-0.220	-0.168	-0.030	0.089	0.126	0.158	0.193
			Limit	-0.527	-0.363	-0.277	-0.206	-0.027	0.124	0.170	0.206	0.252
		0.3	100	-0.501	-0.442	-0.377	-0.300	-0.049	0.181	0.242	0.297	0.363
			200	-0.527	-0.476	-0.410	-0.323	-0.058	0.147	0.208	0.260	0.323
			1000	-0.580	-0.562	-0.517	-0.415	-0.086	0.105	0.159	0.204	0.261
			Limit	-0.597	-0.595	-0.591	-0.568	-0.110	0.153	0.228	0.288	0.367

Figure 1 Kernel Densities of $\widehat{\varphi}-\phi$ for Various Number of Samples and Break Sizes In the Presence of Structural Breaks in AR Parameter When $\phi = 0.4$ and $K - 1 = 9$

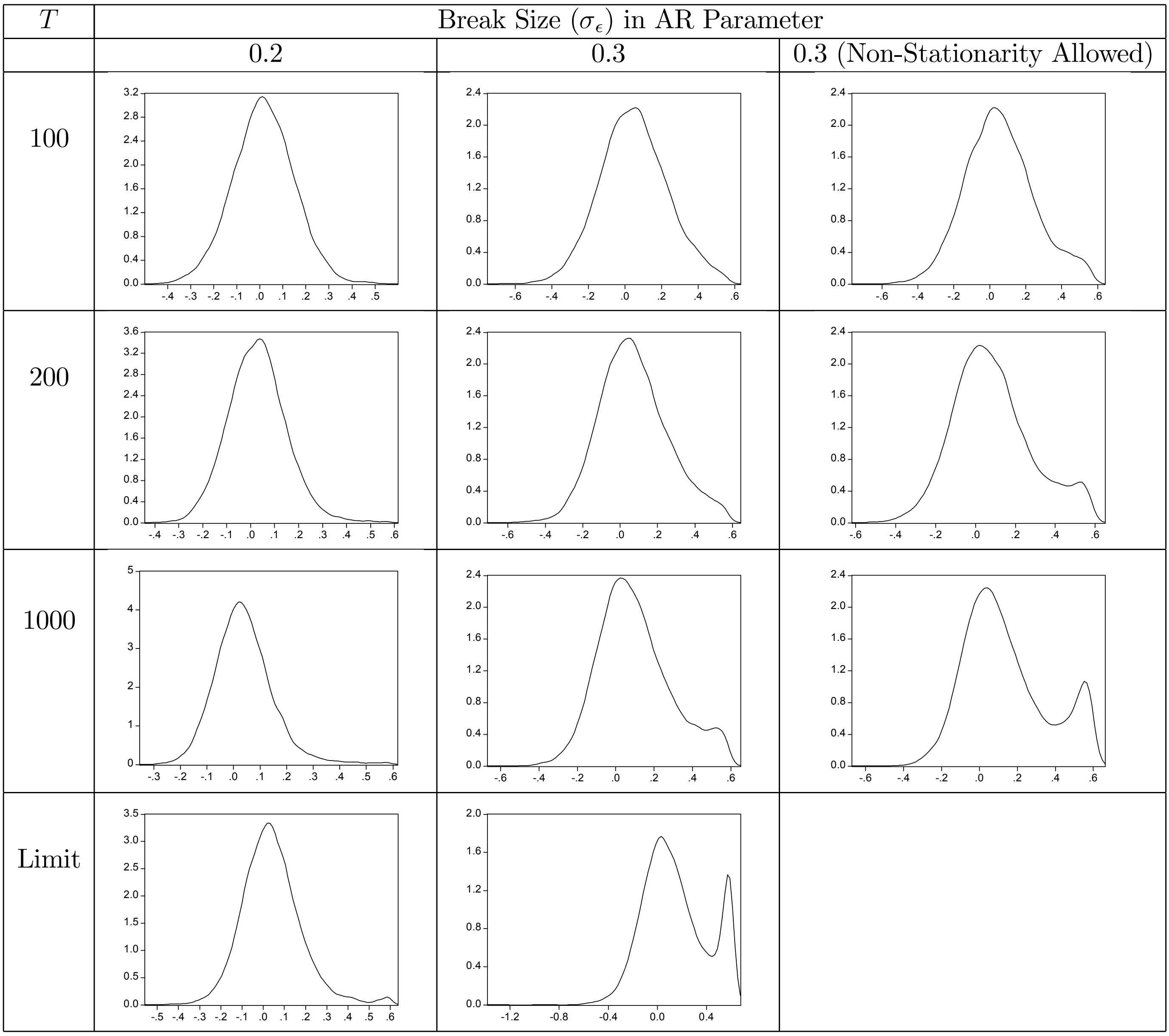
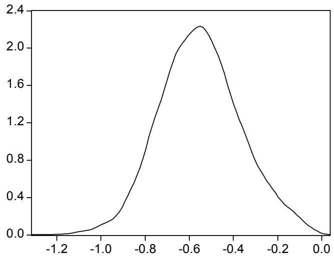
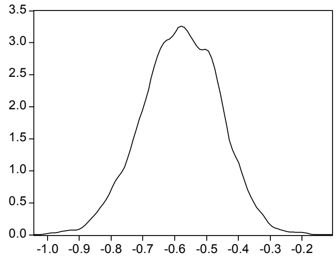
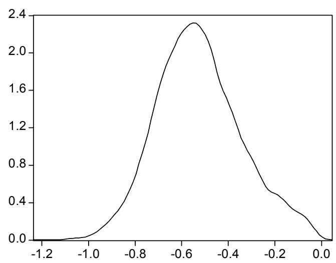
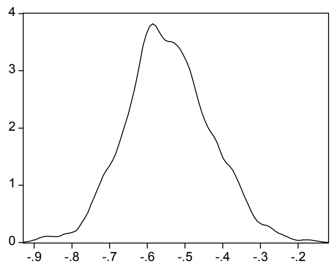
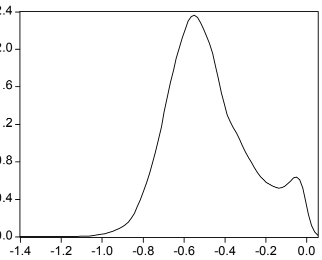
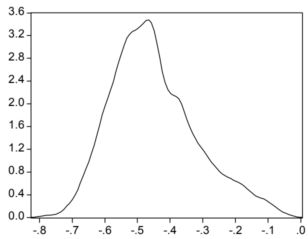
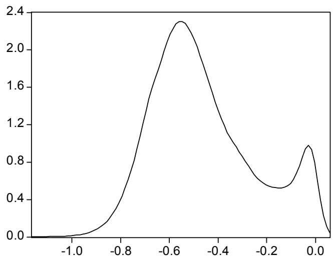
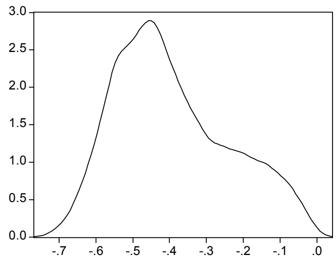
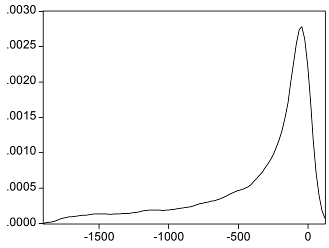
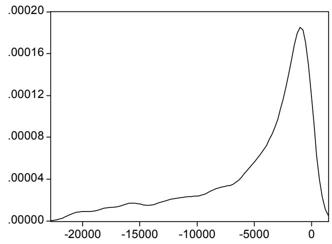


Figure 2 Kernel Densities of $\widehat{\varphi}-1$ for Various Number of Samples and Break Sizes In the Presence of Structural Breaks in AR Parameter When At Least One ϕ_k is Unit Root and $\sigma_\epsilon = 0.3$

T	Number of Breaks	
	$K-1=9$	$K-1=49$
100		
200		
1000		
3000		
Limit (Case 2) $T(\hat{\varphi}-1)$		
The limiting kernel distributions are obtained only using largest 7000 estimates to show the mass around 0 since their left tails are too long.		

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