

Properties of Bias Corrected Realized Variance Under Alternative Sampling Schemes

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Abstract

In this paper I study the statistical properties of a bias corrected realized variance measure when high frequency asset prices are contaminated with market microstructure noise. The analysis is based on a pure jump process for asset prices and explicitly distinguishes among different sampling schemes, including calendar time, business time, and transaction time sampling. Two main findings emerge from the theoretical and empirical analysis. Firstly, based on the mean squared error criterion, a bias correction to realized variance allows for the more efficient use of higher frequency data than the conventional realized variance estimator. Secondly, sampling in business time or transaction time is generally superior to the common practice of calendar time sampling in that it leads to a further reduction in mean squared error. Using IBM transaction data, I estimate a 2.5 minute optimal sampling frequency for realized variance in calendar time which drops to about 12 seconds when a first order bias correction is applied. This results in a more than 65% reduction in mean squared error. If in addition prices are sampled in transaction time, a further reduction of about 20% can be achieved.

Keywords: realized variance; market microstructure noise; bias correction; pure jump process; diffusion limit; optimal sampling

JEL Classifications: G12, C14, C22

Introduction

With the increasing availability of high frequency financial data, model-free measurement of asset return volatility has recently attracted a lot of attention in the financial econometrics literature, with the so-called realized variance measure taking center stage. Under relatively mild conditions on the asset price process, realized variance – defined as the sum of squared intra-period returns – has been shown to provide a highly efficient and consistent estimator of the integrated variance as the sampling frequency increases (see for instance Andersen, Bollerslev, Diebold, and Labys, 2003; Barndorff-Nielsen and Shephard, 2004a). Yet, an important issue that stands in between the theory and practice of realized variance is the emergence of market microstructure noise at high sampling frequencies because this can lead to severe biases in the variance estimates. In some sense, the challenge is to balance the tension between the search for a model-free variance estimator that efficiently exploits the information contained in high frequency data on one hand, and the statistical complications that are introduced by the wide range of existing market microstructure effects on the other.

This paper contributes to this debate by providing an in-depth analysis of a modified realized variance measure that explicitly accounts for market microstructure effects through a non-parametric bias correction in the spirit of Newey and West (1987). In the context of realized variance, such a bias correction can be traced back to the work by French, Schwert, and Stambaugh (1987) and Zhou (1996) and has recently been revived by Hansen

and Lunde (2004). Both the theoretical and empirical analysis confirm the effectiveness of the bias correction in that it substantially reduces the bias and mean squared error of realized variance thereby yielding more accurate and reliable variance estimates, even when prices are sampled at high frequency and are subject to market microstructure noise. Even though these results are directly in line with the closely related work by Hansen and Lunde (2004), this paper stands to make several contributions to the existing literature. Firstly, the bias corrected realized variance measure is studied in the context of a *pure jump* model for asset prices. To date, the existing literature in this area has exclusively focused on diffusion-based models and the results derived here therefore complement existing ones and lead to new insights into the properties of realized variance. Secondly, the bias correction to realized variance is studied under *alternative sampling schemes*, including calendar time sampling, business time sampling, and transaction time sampling. While it is common practice to sample prices at regularly spaced intervals in calendar time using the previous tick or interpolation method, neither theory nor practice prevents the use of different sampling schemes. Interestingly, the results presented in this paper show that the choice of sampling scheme can have a marked impact on the statistical properties of realized variance and that, based on the mean squared error criterion, transaction time sampling is generally the superior scheme.

For the modeling of the price process, I adopt a framework that has recently been proposed by Oomen (2005). Here, the observed price process consists of an “efficient” martingale component that is modeled as a compound Poisson process (e.g. Press, 1967) plus a “market microstructure” component that is modeled as i.i.d. noise (e.g. Bandi and Russell, 2004b; Zhang, Mykland, and Ait-Sahalia, 2004). The virtue of such a pure jump specification is that it provides an ideal setting in which to analyze the properties of the price process on different time scales. Also, because the model is tractable and the joint characteristic function of returns can be derived in closed form, expressions for the bias and mean squared error of the bias corrected realized variance are readily available. These can then be used to (i) determine the optimal sampling frequency, (ii) measure the benefits of bias-correcting realized variance and (iii) measure the benefits of a particular sampling scheme. Another interesting feature of the pure jump model is that a so-called “diffusion limit” can be considered where the jump intensity increases and the jump size falls so that the resulting price path is made up of a growing number of smaller and smaller jumps. It is shown that, under such circumstances, the pure jump process converges in distribution to a class of popular diffusive processes that is widely used in the literature. Consequently, some of the results derived by Hansen and Lunde (2004) can be obtained as a limiting case within the framework adopted here.

The theoretical results presented in this paper illustrate the value of the bias correction to realized variance and the benefit of transaction time sampling. Intuitively, the bias correction permits the use of returns that are sampled at higher frequency leading to an efficiency gain that more than offsets the efficiency loss associated with the bias correction. The superiority of transaction time sampling is due to two reinforcing effects. First, in the current framework it turns out that the bias correction is more effective in transaction time than in calendar time so that returns can be sampled at higher frequency. Second, for an equal number of sampled returns, bias corrected realized variance attains a lower mean squared error when these returns are sampled regularly spaced on a transaction time scale instead of a calendar time scale. This is because returns sampled in transaction time

are essentially de-volatized through an appropriate deformation of the time scale. The empirical results are in agreement with this. Using IBM transaction data for the first 8 months of 2003, I estimate an optimal sampling frequency for realized variance of about 2.5 minutes in calendar time. When the proposed bias correction is applied, realized variance can be calculated using prices sampled at frequencies as high as 12 seconds with an associated reduction in mean squared error of more than 65%. Interestingly, these results are directly in line with those reported in Bandi and Russell (2004a) and Hansen and Lunde (2004) which both use very different methodologies. If, in addition to the bias correction, the price process is also sampled in transaction time, a further reduction in MSE of about 20% can be achieved. The results in this paper thus highlight the importance of the bias correction *and* the choice of sampling scheme.

The remainder of this paper is structured as follows. Section 2 review the pure jump model and the sampled schemes considered in this paper. This is then followed by a detailed analysis of the statistical properties of the bias corrected realized variance. Section 3 reports empirical results for IBM and S&P 500 Spider transaction data while section 4 concludes.

1 Bias corrected realized variance and alternative sampling schemes

This paper introduces market microstructure noise into a setting where the observed price follows a pure jump process. In particular, I adopt the modeling framework of Oomen (2005) and assume that the logarithmic asset price, $P(t)$, follows a compound Poisson process of the form:

$$P(t) = \sum_{j=1}^{M(t)} \varepsilon_j + \nu_{M(t)} = P^\varepsilon(t) + \nu_{M(t)} \quad (1)$$

where $\varepsilon_j \sim \text{iid } \mathcal{N}(0, \sigma_\varepsilon^2)$, $\nu_j \sim \text{iid } \mathcal{N}(0, \sigma_\nu^2)$, and $M(t)$ is a Poisson process with instantaneous jump intensity $\lambda(t)$, independent of ε and ν . No restrictions are imposed on the intensity process except that it is strictly positive and càdlàg. It is also assumed that $t \in [0, 1]$ where the unit interval represents one trading day.

The interpretation of the model in Eq. (1) is as follows: the observed price process $P(t)$ consists of (i) a martingale component, $P^\varepsilon(t)$, which can be thought of as tracking the evolution of efficient price process (i.e. the price process free of any market microstructure contamination) and (ii) a noise component, $\nu_{M(t)}$, which serves to capture the market microstructure effects. Accordingly, in the realized variance (RV) calculations below, the object of econometric interest is the integrated variance of the efficient price over the day, i.e. $\Sigma(1)$ where $\Sigma(t) = \Lambda(t)\sigma_\varepsilon^2$ and $\Lambda(t) = \int_0^t \lambda(u)du$. Because the focus in this paper is on transaction data, I interpret and refer to $\lambda(t)$ as instantaneous arrival frequency of trades with the process $M(t)$ counting the number of transactions¹ that have occurred up to time t . As such, it is natural to consider the process on two different time scales, namely a physical or calendar time scale t as well as a transaction time scale $M(t)$. This will be discussed in more detail later on.

The pure jump specification of the efficient price process entertained here is non-standard in the litera-

ture and thus deserves some further discussion. To date, most – if not all – studies on RV use a continuous semi-martingale to describe the dynamics of the (efficient) price process, see for instance Andersen, Bollerslev, Diebold, and Labys (2003), Bandi and Russell (2004a,b), Barndorff-Nielsen and Shephard (2004a), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004), Hansen and Lunde (2004), Meddahi (2002), Zhang (2004), Zhang, Mykland, and Aït-Sahalia (2004). In such a setting it is well known that, in the absence of market microstructure noise, RV constitutes a consistent estimator of the quadratic variation and integrated variance under relatively mild conditions. Here, on the other hand, the price process is purely discontinuous with sample paths of finite variation and RV is now a consistent estimator of the quadratic variation but an inconsistent estimator of the integrated variance. Interestingly, however, it turns out that the pure jump model is closely related to the commonly used diffusion based models through a so-called “diffusion limit” which is defined here as follows:

Diffusion Limit \mathcal{D} : $\sigma_\varepsilon^2 \rightarrow 0, \lambda(t) \rightarrow \infty$ such that $\Sigma(t)$ remains unchanged

Intuitively, the diffusion limit characterizes the case where the jump intensity increases inversely proportional to a decrease in innovation variance while keeping the integrated variance $\Sigma(t)$ unchanged so that, in the limit, the resulting sample path is indistinguishable from a diffusive sample path. More specifically, the following result can be proved:

Proposition 1.1 : *In the diffusion limit, the efficient price process converges in distribution to a time changed Brownian motion with a time change equal to the integrated variance, i.e.*

$$P^\varepsilon(t) \xrightarrow{\mathcal{L}} W(\Sigma(t)) \quad \text{under } \mathcal{D}.$$

In the diffusion limit, the efficient price process has continuous sample paths of infinite variation.

Proof : See Appendix C.

The above proposition emphasizes the generality of the pure jump specification of the efficient price process in that the class of (univariate) continuous stochastic volatility martingales of Barndorff-Nielsen and Shephard (2004a) can be obtained as a limiting case within the present framework. It also highlights that the intensity process here can be viewed as the equivalent of the stochastic volatility process in diffusion based models. Further, from a practical viewpoint, the use of the diffusion limit greatly facilitates comparison of the results derived in this paper to those of say Hansen and Lunde (2004) which are derived in a diffusion setting.

The specification of the market microstructure noise in Eq. (1) is standard and directly in line with Bandi and Russell (2004b) and Zhang, Mykland, and Aït-Sahalia (2004) among others. Consistent with the impact of the ubiquitous bid-ask bounce in financial transaction data (e.g. Niederhoffer and Osborne, 1966), the i.i.d. noise component² “contaminates” the efficient price process and leads to negative first order serial correlation in returns, i.e.

$$R(t|\tau) = P(t) - P(t - \tau) = \sum_{j=M(t-\tau)}^{M(t)} \varepsilon_j + \nu_{M(t)} - \nu_{M(t-\tau)}.$$

From this it can also be seen that the magnitude of the noise component, relative to that of the efficient price innovation, increases with an increase in the sampling frequency (i.e. smaller τ). In this setting, market microstructure noise makes RV a biased estimator of the integrated variance of the efficient price process and several approaches have been suggested to deal with this. These include sparse sampling in an attempt to mitigate or minimize the impact of the noise (e.g Bandi and Russell, 2004b; Hansen and Lunde, 2004; Oomen, 2003, 2005), pre-filtering the price data in order to purge out serial correlation (Andersen, Bollerslev, Diebold, and Labys, 2003; Bollen and Inder, 2002), bias correction based on a model for the noise (Corsi, Zumbach, Müller, and Dacorogna, 2001), and sub-sampling Zhang (2004); Zhang, Mykland, and Aït-Sahalia (2004). French, Schwert, and Stambaugh (1987) and Zhou (1996) use a non-parametric bias correction in the spirit of Newey and West (1987). Such an approach has recently been revived by Hansen and Lunde (2004) and is also the focus in this paper. In particular, I consider the following bias-corrected realized variance measure (RVAC hereafter):

$$RVAC_N^z(q) = \sum_{i=1}^N R(t_i^z | \tau_i^z)^2 + \sum_{i=1}^N R(t_i^z | \tau_i^z) \sum_{k=1}^q (R(t_{i-k}^z | \tau_{i-k}^z) + R(t_{i+k}^z | \tau_{i+k}^z)), \quad (2)$$

where $\tau_i^z = t_i^z - t_{i-1}^z \geq 0$ and the set of sampling points $\{t_i^z\}_{i=0}^N$ is determined by a sampling scheme “z” (to be defined). The integrated variance of the efficient price process is thus estimated as the sum of squared returns (i.e. RV) plus a correction term that is based on the first q empirical autocovariances. While the statistical properties of RVAC, and various generalizations thereof, have been studied in detail by Barndorff-Nielsen, Hansen, Lunde, and Shephard (2004) and Hansen and Lunde (2004) the key contribution this paper makes is that here the impact of a particular sampling scheme is considered explicitly. As mentioned above, in the current framework two time scales are of particular interest, namely physical or calendar time scale t as well as a transaction time scale $M(t)$, where the latter gives rise to the following sequence of prices $\{p(k) = P(\inf M^{-1}(k)), k \in \{0, 1, \dots, M(1)\}\}$ with transaction returns defined as $r(k|h) = p(k) - p(k-h)$. Based on these time scales, various sampling schemes can be designed. Following Oomen (2005), I consider the following three:

Calendar Time Sampling (denoted as CTS_N) samples the sequence of prices $\{P(t_i^c)\}_{i=0}^N$ where $t_i^c = iN^{-1}$.

Business Time Sampling (denoted as BTS_N) samples the sequence of prices $\{P(t_i^b)\}_{i=0}^N$ where $t_i^b = \Lambda^{-1}(i\bar{\lambda})$ and $\bar{\lambda} = \Lambda(1)/N$.

Transaction Time Sampling (denoted as TTS_N) samples the sequence of prices $\{P(t_i^{tr})\}_{i=0}^N$ where $t_i^{tr} = \inf M^{-1}(i\bar{h})$ and $\bar{h} = M(1)/N$ is integer valued (or equivalently, $\{p(i\bar{h})\}_{i=0}^N$).

In words, CTS samples the price process at regular intervals in calendar time while BTS and TTS sample the price process at regular intervals on a physical time scale that is deformed by the expected and realized number of transactions respectively. Currently, CTS is by far the most widely used sampling scheme in the literature on RV. For instance, in order to calculate the daily RV it is common practice to sample the price process at frequencies

of between 5 and 30 minutes over the day. The idea of business time sampling dates back to the early work by Burns and Mitchell (1946) where economic data was mapped onto a timescale driven by the stages of the business cycle. In finance, the idea of business time first appeared in the seminal work by Clark (1973) on time deformation as a way to generate fat tailed distributions. Here, the defining feature of both BTS and TTS is that these schemes sample the price process more frequent when market activity (as measured by trading intensity) is high and less frequent when it is low.

A couple of further remarks are in order. Note that when the intensity process is constant, BTS is equivalent to CTS but not to TTS. Also, when the intensity process is latent, BTS is infeasible. In practice, a feasible BTS scheme, i.e. “FBTS”, can be based on an estimate of the intensity process, i.e. $\hat{\lambda}(t)$ for $t \in [0, 1]$. As discussed below, such an estimate can be obtained using standard non-parametric smoothing methods. An alternative FBTS scheme which I will consider here is one that samples the price process at multiples of $M(1)/N$ transactions, keeping in mind that $E[M(t)] = \Lambda(t)$ and $E[M(1)/N] = \bar{\lambda}$. Interestingly, this FBTS scheme is equivalent to TTS and can therefore be analyzed as such. Finally, it is pointed out that the number of sampled returns N can be chosen arbitrarily large for CTS and BTS whereas N is limited by the number of available transactions for TTS, i.e. $N \leq M(1)$.

In order to analyze the properties of RVAC on the different time scales, the corresponding return moments are required. To that end, Eqs. (6) and (8) in Appendices A and B contain the closed form expressions of the characteristic function of returns (conditional on the intensity path λ) in calendar time and transaction time respectively. Appendix D lists the calendar time moment expressions required for the analysis in this paper. Here, the following notation is used:

$$\lambda_i = \Lambda(t_i) - \Lambda(t_i - \tau_i) \quad \text{and} \quad \lambda_{i,j} = \Lambda(t_j - \tau_j) - \Lambda(t_i), \quad (3)$$

where $t_j \geq t_i + \tau_j$ and λ_i thus denotes the integrated intensity over the i^{th} sampling interval and $\lambda_{i,j}$ denotes the integrated intensity between the end of the i^{th} sampling interval and the beginning of the j^{th} sampling interval. Without going into much detail, it is important to emphasize that returns in transaction time are normal with an MA(1) dependence structure induced by the noise component whereas returns in calendar time have an ARMA(1,1) dependence structure and can be highly non-normal depending on the specification of the intensity process (see Oomen, 2005, for an in-depth discussion of the statistical properties of the model). Another issue worth highlighting is the link among return moments under different sampling schemes. To illustrate this, fix N and consider the variance of a transaction return, i.e. $E[r(k\bar{h})^2] = \bar{h}\sigma_\varepsilon^2 + 2\sigma_\nu^2$ where $\bar{h} = M(1)/N$. Based on this, and without the use of the characteristic function, the corresponding conditional return moment in calendar time can be derived as follows:

$$\begin{aligned} E_\lambda[R(t_i^c|\tau_i^c)^2] &= E_\lambda[E_M[r(M(t_i^c)|M(t_i^c|\tau_i^c))]^2] \\ &= E_\lambda[(M(t_i^c|\tau_i^c)\sigma_\varepsilon^2 + 2\sigma_\nu^2)1_{\{M(t_i^c|\tau_i^c) \geq 1\}}] = \sum_{h=1}^{\infty} (h\sigma_\varepsilon^2 + 2\sigma_\nu^2) \frac{\lambda_i^h}{h!e^{-\lambda_i}} \\ &= \lambda_i\sigma_\varepsilon^2 + 2\sigma_\nu^2(1 - e^{-\lambda_i}) \end{aligned}$$

where $t_i^c = i/N$ and $M(t_i|\tau_i) = M(t_i) - M(t_i - \tau_i)$. Next, the variance of returns in business time can be obtained by simply replacing λ_i with $\bar{\lambda} = \Lambda(1)/N$, i.e.

$$E_\lambda[R(t_i^b|\tau_i^b)^2] = \bar{\lambda}\sigma_\varepsilon^2 + 2\sigma_\nu^2(1 - e^{-\bar{\lambda}})$$

Finally, assuming $\Lambda(1) = M(1)$, application of the diffusion limit to the above expression yields:

$$\lim_{\mathcal{D}} E_\lambda[R(t_i^b|\tau_i^b)^2] = \bar{h}\sigma_\varepsilon^2 + 2\sigma_\nu^2 = E[r(k|\bar{h})^2]$$

The above calculations illustrate a more general point, namely moments of returns in transaction time can be obtained as the diffusion limit of moments of returns in business time which, in turn, can be obtained as a special case of moments of returns in calendar time which, in turn, can be obtained as a probability weighted average of moments of returns in transaction time thereby completing the circle. Hence, for the analysis of the sampling schemes considered in this paper, it suffices to derive the relevant expressions in calendar time since the corresponding results in business time and transaction time follow directly from it.

1.1 Properties of bias corrected realized variance

In this section I investigate the statistical properties of the bias corrected realized variance measure in Eq. (2) using the pure jump process specified in Eq. (1) (CPP-MA hereafter). To simplify notation I denote $\Lambda(1)$ by Λ , $M(1)$ by M , $\Sigma(1)$ by Σ , and also define a so-called ‘‘noise ratio’’ $\gamma = \sigma_\nu^2/\sigma_\varepsilon^2$ which measures the relative magnitude of the market microstructure noise component.

1.1.1 The bias.

In the absence of market microstructure noise, it is well known that RV yields an unbiased estimate of the integrated variance Σ . In the presence of market microstructure noise this is not the case and RV is biased, in part, due to the induced return serial correlation. Based on the moment expressions in Appendix D, the bias under CTS can be expressed as:

$$E_\lambda[RVAC_N^c(q) - \Sigma] = \gamma\sigma_\varepsilon^2 \sum_{i=1}^N (1 - e^{-\lambda_i}) \left(2 - \sum_{k=1}^q (e^{-\lambda_{i-k,i}}(1 - e^{-\lambda_{i-k}}) + e^{-\lambda_{i,i+k}}(1 - e^{-\lambda_{i+k}})) \right).$$

Keep in mind that even though notation suppresses this, λ_i is always associated with a particular sampling scheme and sampling points, i.e. t_i^c and t_{i-1}^c in this case. Under BTS the bias expression simplifies to:

$$E_\lambda[RVAC_N^b(q) - \Sigma] = 2\gamma N\sigma_\varepsilon^2(1 - e^{-\bar{\lambda}})e^{-q\bar{\lambda}}.$$

Finally, using either the appropriate return moments in transaction time or by applying the diffusion limit to the above BTS expression, the bias under TTS is obtained as:

$$E[RVAC_N^{tr}(q) - \Sigma] = \begin{cases} 2\gamma N\sigma_\varepsilon^2 & \text{for } q = 0 \\ 0 & \text{for } q \geq 1 \end{cases}.$$

It is easy to see that for $q = 0$ the bias of RV is largest under TTS and smallest under CTS (see Oomen, 2005, for further discussion). On the other hand, a first order bias correction removes all bias under TTS while this is clearly not the case under CTS or BTS due to the ARMA dependence structure of returns on these time scales. When N is large, i.e. a high sampling frequency, the bias under BTS can be approximated as:

$$E_\lambda[RVAC_N^b(q) - \Sigma] \stackrel{N \text{ large}}{\approx} 2\gamma(1 - q\Lambda/N)\Sigma.$$

This illustrates that for the bias to remain constant, the order of the bias correction would need to grow proportional to the sampling frequency. If not, when $N \rightarrow \infty$, the bias will tend towards $2\gamma\Sigma$. Hence, in the current framework the bias is bounded but diverges to infinity in the diffusion limit (because σ_ν^2 is fixed, $\gamma \rightarrow \infty$). This latter observation is consistent with the results derived by Bandi and Russell (2004a) and Zhang, Mykland, and Ait-Sahalia (2004).

1.1.2 The mean squared error

In the absence of market microstructure noise, RV is unbiased and its MSE is equal to its variance. Oomen (2005) shows that in such a setting sampling as frequent as possible in transaction time is optimal in that this achieves the minimum MSE. Sampling in business time is second best, followed by calendar time sampling in last place. Also, the efficiency gain associated with TTS and BTS relative to CTS increases with an increase in the *variability* of the trade intensity process. Unfortunately, in the presence of market microstructure noise these relations break down and it turns out that none of the sampling schemes considered are uniformly superior across sampling frequencies and model specifications. Still, at and around the optimal sampling frequency, i.e. the frequency that minimizes the MSE of RV, TTS outperforms BTS and BTS outperforms CTS so that for all practical purposes the ordering in performance of the alternative sampling schemes is preserved. Below, I show that these findings extend to the bias corrected realized variance thereby further underlining the virtues of transaction time sampling.

Based on the return moments listed in Appendix D, it is straightforward to numerically compute the MSE of RVAC under CTS for given model parameters and intensity path. Further, in business time, the MSE expression for RVAC “simplifies” to:

$$\begin{aligned} E_\lambda[(RVAC_N^b(q) - \Sigma)^2] &= (4\sigma_\nu^2(e^{-\bar{\lambda}} + qe^{-\bar{\lambda}} - q)e^{-\bar{\lambda}q} + 2\sigma_\varepsilon^2\bar{\lambda}(1 + 2q) + 3\sigma_\varepsilon^2 + 8\sigma_\nu^2)\Sigma \\ &\quad - 8N\sigma_\nu^4(1 + e^{-\bar{\lambda}})^{-1}((e^{-\bar{\lambda}} - 1) + e^{-\bar{\lambda}q}(e^{-2\bar{\lambda}} - 2e^{-\bar{\lambda}(q+1)} - 1)) \\ &\quad 4N(N - 2q)\sigma_\nu^4(e^{-\bar{\lambda}} - 1)^2e^{-2\bar{\lambda}q} - 4N\sigma_\nu^4e^{-2\bar{\lambda}q}(1 + e^{-2\bar{\lambda}}) + R(q) \end{aligned} \quad (4)$$

where

$$\begin{aligned} R(q) &= 6q^2\sigma_\nu^4e^{-2q\bar{\lambda}}(1 - e^{-\bar{\lambda}})^2 + 2q(q + 1)\bar{\lambda}\sigma_\varepsilon^2\sigma_\nu^2e^{-\bar{\lambda}q}(1 - e^{-\bar{\lambda}}) - q(q + 1)\sigma_\varepsilon^4\bar{\lambda}^2 \\ &\quad - 4q\sigma_\nu^4(1 + e^{-\bar{\lambda}})^{-1}((1 - e^{-\bar{\lambda}}) + e^{-2q\bar{\lambda}}(3e^{-\bar{\lambda}} - e^{-3\bar{\lambda}} - 2)) - 4q\sigma_\nu^4e^{-q\bar{\lambda}}(1 - e^{-\bar{\lambda}}) \\ &\quad - 2\sigma_\nu^4(1 + e^{-\bar{\lambda}q})^2 + 4\sigma_\nu^4(e^{-2\bar{\lambda}q} - 1)(e^{-\bar{\lambda}} + 1)^{-2} + 4\sigma_\nu^4(e^{-\bar{\lambda}N} + 1) \end{aligned}$$

For large N , the term $R(q)$ is negligible. The derivation of the MSE expression in Eq. (4) is straightforward, but because it is lengthy and tedious it is omitted to conserve space. As already illustrated above, an interesting feature of the jump model is that moments of the price process under TTS can be derived as the diffusion limit of moments under BTS. As a consequence, the MSE of RVAC under TTS can be obtained as the diffusion limit of the MSE expression in Eq. (4), i.e.

$$\begin{aligned} E[(RVAC_N^{tr}(q) - \Sigma)^2] &= \lim_D E_\lambda[(RVAC_N^b(q) - \Sigma)^2] \\ &= \begin{cases} 2\sigma_\nu^4(2N^2 + 6N - 2) + 8\sigma_\nu^2\Sigma + 2\frac{\Sigma^2}{N} & q = 0 \\ 2\sigma_\nu^4(4N - 2q - 1) + 8\sigma_\nu^2\Sigma + 2\frac{\Sigma^2}{N}(1 + 2q) - \frac{\Sigma^2}{N^2}q(q + 1) & q \geq 1 \end{cases} \quad (5) \end{aligned}$$

The MSE expression in Eq. (5) generalizes the one derived by Hansen and Lunde (2004) for $q = 0, 1$ to higher order bias-corrections. It is noted, however, that because the market microstructure noise component is MA(1), a first order bias correction will generally be optimal in that it leads to the lowest overall MSE.

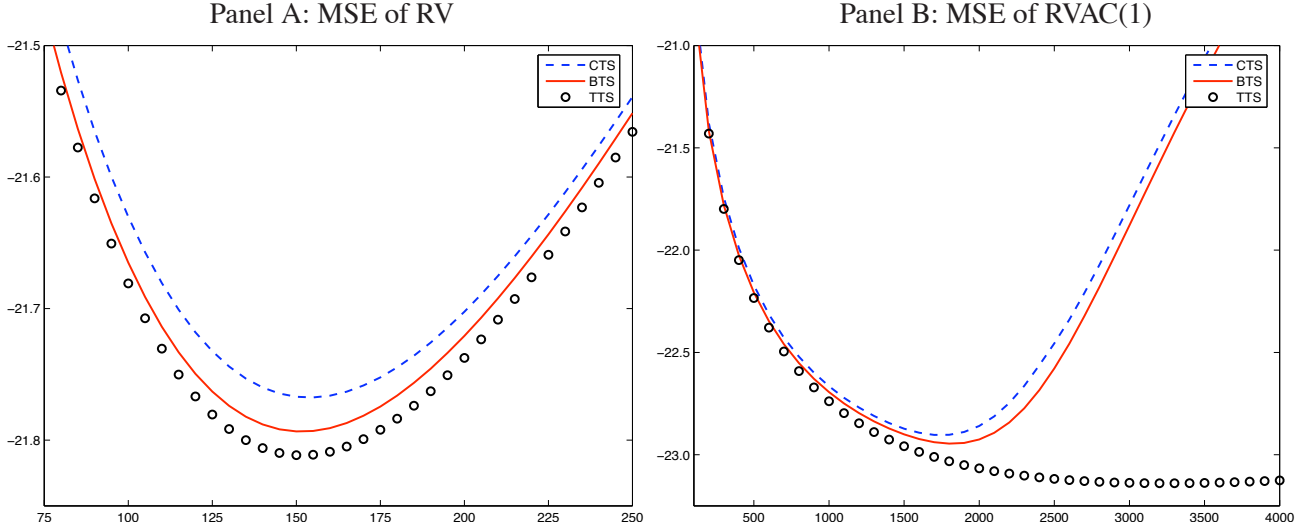
Based on the above expressions, it is now possible to determine the optimal sampling frequency, that is, the sampling frequency at which the conditional MSE of RVAC is minimized:

$$N_z^* = \arg \min_N E_\lambda[(RVAC_N^z(q) - \Sigma)^2], \quad \text{for } z \in \{c, b, tr\}.$$

Intuitively, N_z^* balances the trade-off between reducing the variance of the estimator by sampling at a higher frequency and reducing the market microstructure induced bias of the estimator by sampling at a lower frequency. For given parameters and sampling scheme, the optimal sampling frequency can be computed straightforwardly by numerically minimizing the MSE expression over N . Interestingly, for TTS it is easy to show that $N_{tr}^* \approx \sqrt{1 + 2q}\Lambda/2\gamma$ for $q \geq 1$ (when ignoring the term of order N^{-2} in the MSE expression) which confirms the intuition that the optimal sampling frequency increases with a decrease in noise ratio, an increase in bias correction, and an increase in the (expected) number of trades.

Without further specification of the model parameters and the intensity path it is very difficult to make qualitative comparisons of the bias or MSE of RVAC under the alternative sampling schemes, nor is it possible to unequivocally name a particular sampling scheme as superior. Therefore, in order to gain insights into the relative merits of each scheme, I provide some numerical illustrations that focus on the MSE of RV and RVAC(1). The CPP-MA model parameters are set to realistic values that are in line with the empirical results for IBM reported in Section 2.2 below, namely $\sigma_\epsilon^2 = 1.65e - 8$, $\gamma = 2$, and $\Lambda = M = 7,500$ implying an annualized daily return volatility of about 17.5%. For all calculations involving CTS, I use an intensity path $\{\lambda(t), t \in [0, 1]\}$ that is obtained as the average daily non-parametric smoothing estimates of the IBM trade intensity for August 2003 (mean-adjusted to ensure that $\Lambda = 7,500$). As expected, the resulting intensity path exhibits a distinct U-shaped pattern with a high at the market open and a low around lunch time of about 1.5 and 2/3 times its average over the day respectively. It is emphasized that in this setup the stochastic dependence in the arrival rates of trades is not accounted for and that the averaging of daily estimates will further smooth-out the intensity path. However,

Figure 1: Logarithmic MSE of RV and RVAC under alternative sampling schemes

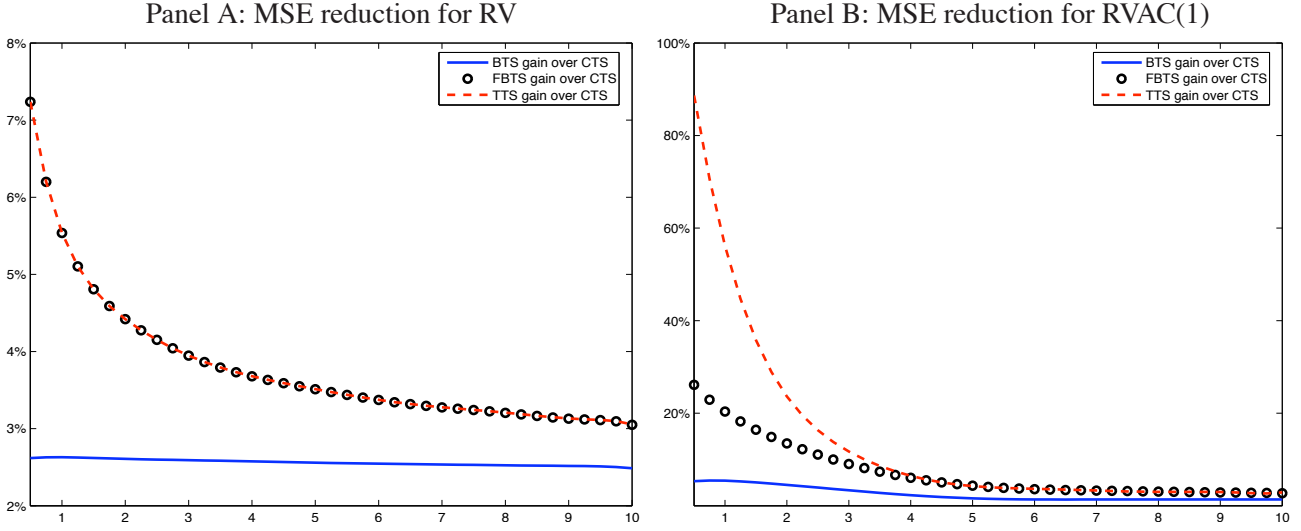


This figure plots the logarithmic MSE of RV (Panel A) and RVAC(1) (Panel B) as a function of the number of sampled returns N (horizontal axis) for calendar time sampling (dashed line), business time sampling (solid line), and transaction time sampling (circled-line). The CPP-MA model parameters are set as $\sigma_\varepsilon^2 = 1.65e - 8$, $\gamma = 2$, $\Lambda = M = 7,500$ with a U-shaped path for $\lambda(t)$ based on IBM estimates.

because it is found that the benefits of BTS and TTS increase with an increase in the variability of the intensity path, the results reported below are – if anything – likely to be biased in favor of CTS.

Panel A of Fig. 1 plots the logarithmic MSE of RV for the various sampling schemes as a function of the number of sampled returns, N . Panel B contains the corresponding results for RVAC(1). It is clear for both RV measures that TTS is the superior sampling scheme in terms of MSE for all the sampling frequencies considered. The commonly used CTS scheme performs worst. For RV it is interesting to note that the optimal sampling frequencies are roughly the same for all sampling schemes considered. With a first order bias correction added to RV, quite a different MSE pattern emerges. While TTS remains the superior sampling scheme, it now affords much higher sampling frequencies compared to CTS or BTS: the optimal sampling frequency roughly increases 10-fold under BTS and CTS but 20-fold under TTS. The reason that BTS outperforms CTS is due to the deformation of the time scale that leads to a constant jump intensity in business time. Nevertheless, returns are still sampled in physical time under both schemes and thus exhibit an ARMA(1,1) structure. On the other hand, the reason that TTS outperforms both CTS and BTS by such a large margin for RVAC(1) is the effectiveness of the bias correction: returns in transaction time are MA(1) and a first order correction removes all bias. Additional unreported simulations indicate that (i) the benefits of BTS and TTS relative to CTS increases with an increase in the variability of the intensity process and (ii) CTS can only lead to a lower MSE than BTS or TTS when the sampling frequency is chosen extremely high and well beyond its optimal value (this is in line with the results reported for RV by Oomen, 2005). Hence, for all practical purposes, it seems reasonable to conclude that based

Figure 2: Reduction in minimum MSE when using a sampling scheme different from CTS



This figure plots the reduction in minimum MSE (i.e. MSE attained at optimal sampling frequency N_z^*) against the noise ratio γ (horizontal axis) when instead of calendar time sampling, an alternative sampling scheme is adopted, i.e. business time sampling (solid line), feasible business time sampling (circled-line), or transaction time sampling (dashed line). The CPP-MA model parameters are set as $\sigma_\varepsilon^2 = 1.65e - 8$, $\Lambda = M = 7,500$ with a U-shaped path for $\lambda(t)$ based on IBM estimates.

on the MSE criterion TTS is superior, CTS is inferior, and BTS lies in between.

As a further illustration, Fig. 2 plots the reduction in the minimum MSE attained at the optimal sampling frequency when, instead of the worst performing CTS scheme, one of the alternative schemes is used to sample returns. Because the MSE gain associated with a particular sampling scheme is expected to vary with the level of market microstructure noise, I report the results across a range of values for γ between 0.5 and 10 (the empirical analysis below shows that a typical value for γ is 2 for IBM and 4 for S&P500 Spydres). In addition to BTS and TTS, I also consider a feasible-BTS scheme (FBTS hereafter) which samples returns in transaction time at the sampling frequency that is optimal in business time, i.e. N_b^* . The results in Panel A of Fig. 2 show that BTS outperforms CTS across the range of noise ratios considered with an average reduction in MSE of about 2.5%. For low and intermediate levels of market microstructure noise, the benefits to sampling in transaction time are somewhat larger. For instance, when $\gamma = 0.5$, the MSE of RV is reduced about 7% by simply sampling regularly spaced in transaction time as opposed to calendar time. For the bias corrected realized variance these results are more pronounced. Panel B of Fig. 2 shows that the MSE of RVAC(1) can be reduced by up to 90% when the noise ratio is low and returns are sampled in transaction time as opposed to calendar time. Keep in mind, however, that the number of returns used to construct RVAC(1) under TTS is typically much larger than under CTS or BTS (e.g. recall Panel B of Fig. 1 where $N_{tr}^* \gg N_b^*$). It is therefore also of interest to compare results of BTS and FBTS because both these schemes use the same number of return observations to construct the realized variance measure. Importantly, the results indicate that the reduction in MSE under FBTS can go

up to nearly 25% as opposed to about 5% under BTS. This difference in performance is now solely due to the sampling scheme and confirms the finding that TTS is clearly superior.

2 Optimal sampling and bias correction in practice

In this section I will discuss how the above methodology can be used in practice to (i) determine the optimal sampling frequency of the price process, (ii) measure the benefits of a bias correction to realized variance, and (iii) measure the benefits of sampling in business time or transaction time.

2.1 Data and parameter estimation

The empirical analysis below uses TAQ transaction data for IBM and S&P500 Spiders (SPY hereafter) over the period, January 2, 2003 through August 31, 2003. Transactions between 9.45 and 16.00 on any of the available exchanges are included. The data is filtered for outliers using the following algorithm: an intra-day return is classified as an outlier if (i) it is larger than 8 times the estimated standard deviation of returns for that day and (ii) the immediately subsequent return observation is of (roughly) the same magnitude and with opposite sign, i.e. an instantaneous price reversal. For more details on this algorithm see Oomen (2005). For IBM, 320 outliers were removed leaving a total of 1,224,127 transactions. For SPY, 1295 outliers were removed leaving a total of 4,048,665 transactions.

Estimation of the CPP-MA model parameters is done by moment matching in transaction time. In particular, Λ is estimated unbiasedly as the total number of transaction on a given day, i.e. M . Estimates of σ_ν^2 and σ_ε^2 are obtained by matching the population variance and first order autocovariance of transaction returns to their sample counterparts, i.e.

$$\begin{aligned} Cov(r(k|1), r(k-1|1)) &= -\sigma_\nu^2 \\ Var(r(k|1)) &= \sigma_\varepsilon^2 + 2\sigma_\nu^2 \end{aligned}$$

Although the sample autocovariance of transaction returns can in principle be positive, this does not occur on any day in the data set. Nor, does $2\hat{\sigma}_\nu^2$ ever exceed the sample variance of transaction returns. In order to calculate the MSE in calendar time, estimates of the intensity process $\{\lambda(t), t \in [0, 1]\}$ are also needed. To obtain these I use a non-parametric smoothing estimator (Diggle and Marron, 1988) that takes the form:

$$\hat{\lambda}(t) = \begin{cases} h^{-1} \sum_{i=0}^M K((t-t_i)/h) + K((t+t_i)/h) & t \in [0, h) \\ h^{-1} \sum_{i=0}^M K((t-t_i)/h) & t \in [h, 1-h] \\ h^{-1} \sum_{i=0}^M K((t-t_i)/h) + K((t+t_i-2)/h) & t \in (1-h, 1] \end{cases}$$

where $K(\cdot)$ is a kernel function, h is the bandwidth, and $\{t_i\}_{i=0}^M$ is the set of all transaction times for a particular day. Estimates of $\hat{\lambda}(t)$ close to the edges, i.e. $t \in [0, h)$ and $t \in (1-h, 1]$, are obtained using a ‘‘mirror’’ image adjustment to offset a downward bias that is induced by the use of a two-sided kernel (see Diggle and Marron,

1988, for more details). For the analysis in this paper I use a quartic kernel, i.e. $K(x) = 0.9375(1 - x^2)^2$ for $-1 \leq x \leq 1$ and 0 otherwise, and set the bandwidth $h = 0.075$ corresponding to a smoothing window of about 30 minutes forwards and backwards. Oomen (2005) reports extensive simulation results which confirm that the moment matching procedure and non-parametric smoothing approach work very well in that they deliver unbiased and accurate parameter estimates for realistic sample sizes.

2.2 Empirical results

For each month in the dataset, Table 1 reports the estimates of the model parameters (where $\hat{\gamma} = \hat{\sigma}_v^2 / \hat{\sigma}_\varepsilon^2$), together with the model-implied optimal sampling frequency and MSE for RV and RVAC(1) under the alternative sampling schemes. As before, I consider CTS, BTS, and TTS in addition to FBTS which samples in transaction time at the optimal sampling frequency under BTS, i.e. N_b^* . To compute the CTS statistics, I have used the daily intensity estimates averaged over the month. The optimal sampling frequency is reported in seconds, i.e. $22500/N$, so that results can be compared across time and securities. Because the optimal sampling frequency under CTS (FBTS) is very close (identical) to that under BTS, it is omitted to conserve space. The MSE for the BTS, FBTS, and TTS schemes is reported relative to the MSE under CTS (same as in Fig. 2).

Several interesting findings emerge from Table 1. First, for RV, it can be seen that the performance of FBTS and TTS is indistinguishable which suggests that $N_b^* \approx N_{tr}^*$ in line with Panel A of Fig. 1. Also, while the gains of sampling in business or transaction time are strictly positive for each month in the sample, their size is modest, i.e. about 2.5% and 4.5% for IBM and 5% and 6% for SPY respectively. Turning to RVAC, it is clear that the bias correction leads to a dramatic increase in the optimal sampling frequency together with a associated reduction in MSE irrespective of the sampling scheme or security. For instance, for IBM, the optimal sampling frequency increases from about 2.5 minutes to 12 seconds and reduces the MSE by more than 65% on average (i.e. 3.71 for RV and 1.19 for RVAC in calendar time). The gains associated with the choice of sampling scheme are of second order importance but still substantial. In particular, the best performing sampling scheme, i.e. TTS, reduces the MSE of RVAC(1) by about 20% for IBM and by about 10% for SPY. When interpreting these results, it is important to keep in mind that (i) the difference in performance between CTS and BTS isolates the benefits of sampling in business time as opposed to calendar time (because $N_c^* \approx N_b^*$), (ii) the difference in performance between BTS and FBTS isolates the benefit of sampling in transaction time as opposed to business time, and (iii) the difference in performance between FBTS and TTS isolates the benefit of sampling at higher frequency in transaction time since N_{tr}^* is typically larger than N_b^* . Finally, it is noted that the effectiveness of transaction time sampling appears weaker for SPY than it is for IBM. In line with the results presented in Fig. 2, this can be explained by the much higher level of market microstructure noise in the SPY data (i.e. $\gamma \approx 2$ for IBM and $\gamma \approx 4$ for SPY).

Table 1: CPP-MA model implied optimal sampling frequency and MSE of realized variance under alternative sampling schemes

		RV						RVAC(1)						
CPP-MA		N* in seconds			MSE			N* in seconds			MSE			
σ_ε^2	γ	Λ	C/BTS	TTS	CTS	BTS	FBTS	TTS	C/BTS	TTS	CTS	BTS	FBTS	TTS
<i>Panel A: IBM</i>														
Jan 03	1.63	2.70	8,583	163	165	5.04	-2.47%	-3.83%	12	8	1.56	-3.53%	-7.23%	-13.4%
Feb 03	1.95	2.25	7,699	156	158	5.48	-1.94%	-3.55%	12	9	1.74	-3.18%	-12.2%	-16.6%
Mar 03	2.15	1.99	8,408	136	136	6.90	-2.09%	-3.78%	11	5	2.15	-3.63%	-13.1%	-20.5%
Apr 03	1.57	2.16	7,772	150	153	3.52	-2.99%	-4.61%	12	9	1.12	-5.04%	-14.0%	-19.0%
May 03	1.32	1.80	7,391	138	140	2.04	-2.26%	-4.14%	12	6	0.66	-4.00%	-13.1%	-23.8%
Jun 03	1.62	1.80	7,053	142	144	2.92	-3.06%	-4.95%	13	6	0.97	-5.49%	-14.3%	-24.8%
Jul 03	1.79	1.44	6,216	133	134	2.59	-2.20%	-4.51%	13	7	0.90	-4.22%	-11.9%	-29.4%
Aug 03	1.36	1.14	5,907	118	118	1.20	-2.62%	-5.33%	13	4	0.44	-5.36%	-11.1%	-36.3%
Jan/Aug 03	1.67	1.91	7,379	142	144	3.71	-2.45%	-4.34%	12	7	1.19	-4.31%	-12.1%	-23.0%
<i>Panel B: S&P 500 Spydex</i>														
Jan 03	0.58	4.85	19,666	136	141	2.89	-5.34%	-6.03%	7	7	0.76	-4.32%	-7.11%	-7.11%
Feb 03	0.65	4.59	23,454	117	121	4.33	-5.17%	-5.85%	6	5	1.08	-5.22%	-8.10%	-8.46%
Mar 03	0.80	3.95	27,747	95	97	7.37	-4.75%	-5.47%	4	4	1.72	-6.01%	-8.78%	-10.7%
Apr 03	0.49	4.50	24,087	113	117	2.60	-6.45%	-7.12%	5	5	0.64	-6.74%	-9.59%	-10.1%
May 03	0.37	4.65	22,819	120	124	1.42	-6.10%	-6.78%	6	5	0.35	-5.51%	-8.36%	-8.57%
Jun 03	0.36	3.86	24,467	101	105	1.26	-6.13%	-6.88%	5	4	0.30	-8.17%	-10.7%	-12.8%
Jul 03	0.72	1.69	28,043	54	55	3.38	-3.69%	-4.99%	4	2	0.80	-7.68%	-11.9%	-32.3%
Aug 03	0.34	2.95	24,736	85	86	0.93	-4.34%	-5.25%	5	3	0.22	-7.89%	-13.5%	-16.9%
Jan/Aug 03	0.54	3.88	24,377	103	106	3.02	-5.25%	-6.05%	5	4	0.73	-6.44%	-9.75%	-13.4%

This table reports estimates of the CPP-MA model parameters, i.e. σ_ε^2 ($\times 1e8$), γ , and Λ , based on transaction returns over the month. For RV and RVAC(1) the table also reports the (i) the model-implied optimal sampling frequency in seconds, i.e. $22500/N_z^*$, for CTS/BTS and TTS and (ii) the model-implied minimum MSE ($\times 1e10$) under CTS and (iii) the reduction in minimum MSE under BTS, FBTS, and TTS relative to CTS.

3 Conclusion

This paper builds on the work by Oomen (2005), to investigate the statistical properties of a bias corrected realized variance measure in a setting where the observed price process is contaminated with market microstructure noise. While many of the empirical and theoretical results are in direct agreement with Hansen and Lunde (2004), this paper is distinguished from existing literature in two important ways. First, the analysis presented here is based on a pure jump model for asset prices as opposed to the commonly used diffusion-based models. As such, the results complement existing ones and lead to some new insights into the properties of realized variance. Second, the paper explicitly studies the impact that a particular choice of sampling scheme has on the properties of bias corrected realized variance.

The two main results which emerge from the empirical and theoretical analysis are (i) a first order bias correction leads to a dramatic reduction in mean squared error of realized variance since it allows for the use of higher frequency data and (ii) business time and transaction time sampling are generally superior to the common practice of calendar time sampling in that these lead to a further reduction of mean squared error. In the current framework, the benefit of transaction time sampling is particularly significant because a much higher sampling frequency can be afforded thanks to the effectiveness of the bias correction under this scheme. All in all, the results in this paper thus highlight the importance of both the bias correction *and* the choice of sampling scheme.

A Characteristic function of returns in transaction time

Let $V(h)$ denote the variance of transaction return $r(k|h)$ and $C(h, p, m)$ denote the covariance between $r(k|h)$ and $r(k+p+m|p)$. For the model in Eq. (1) it is easy to show that:

$$V(h) = h\sigma_\varepsilon^2 + 2\sigma_\nu^2 \quad \text{for } h \geq 1$$

$$C(h, p, m) = \begin{cases} -\sigma_\nu^2 & m = 0, h > 0, p > 0 \\ 0 & \text{otherwise} \end{cases}$$

Next, consider a sequence of four consecutive non-overlapping returns in transaction time, i.e. $\{r(k_i|h_i)\}_{i=1}^4$ where $k_{i+1} - h_{i+1} - k_i = m_i \geq 0$ for $i = 1, 2, 3$. Due to joint normality of transaction returns, their joint characteristic function takes the following standard form:

$$\begin{aligned} \ln \phi_{TT}(\bar{\xi}|\bar{h}, \bar{m}) &\equiv \ln E(\exp\{i\xi_1 r(k_1|h_1) + i\xi_2 r(k_2|h_2) + i\xi_3 r(k_3|h_3) + i\xi_4 r(k_4|h_4)\}) \\ &= -\frac{1}{2}(\xi_1^2 V(h_1) + \xi_2^2 V(h_2) + \xi_3^2 V(h_3) + \xi_4^2 V(h_4)) \\ &\quad -\xi_1 \xi_2 C(h_1, h_2, m_1) - \xi_1 \xi_3 C(h_1, h_3, h_2 + m_{1:2}) - \xi_1 \xi_4 C(h_1, h_4, h_{2:3} + m_{1:3}) \\ &\quad -\xi_2 \xi_3 C(h_2, h_3, m_2) - \xi_2 \xi_4 C(h_2, h_4, h_3 + m_{2:3}) - \xi_3 \xi_4 C(h_3, h_4, m_3) \end{aligned} \quad (6)$$

where $\bar{h} = \{h_1, h_2, h_3, h_4\}$, $\bar{\xi} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, $\bar{m} = \{m_1, m_2, m_3\}$, and $x_{a:b} = \sum_{i=a}^b x_i$.

B Characteristic function of returns in calendar time

Consider a sequence of four consecutive non-overlapping returns in calendar time, i.e. $\{R(t_i|\tau_i)\}_{i=1}^4$ where $t_{i+1} - \tau_{i+1} - t_i \geq 0$ for $i = 1, 2, 3$. Conditional on the intensity process, the joint characteristic function of these returns can be expressed as follows:

$$\begin{aligned} \phi_{CT}(\bar{\xi}|\bar{t}, \bar{\tau}) &\equiv E_\lambda(\exp\{i\xi_1 R(t_1|\tau_1) + i\xi_2 R(t_2|\tau_2) + i\xi_3 R(t_3|\tau_3) + i\xi_4 R(t_4|\tau_4)\}) \\ &= \sum_{\bar{m}=0}^{\infty} \sum_{\bar{h}=0}^{\infty} \theta e^{-\lambda_{1:4}} \phi_{TT}(\bar{\xi}|\bar{h}, \bar{m}) \frac{\lambda_4^{h_4} \lambda_3^{h_3} \lambda_2^{h_2} \lambda_1^{h_1} \lambda_{3,4}^{m_3} \lambda_{2,3}^{m_2} \lambda_{1,2}^{m_1}}{h_4! h_3! h_2! h_1! m_3! m_2! m_1!} \end{aligned} \quad (7)$$

where $\bar{t} = \{t_1, t_2, t_3, t_4\}$, $\bar{\tau} = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, $\theta = e^{-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4}$, and $\lambda_{a,b} = \sum_{i=a}^{b-1} \lambda_{i,i+1}$. Using the explicit expression for ϕ_{TT} in Eq. (6), it is now possible to simplify the characteristic function (that is, not involving infinite summations) as:

$$\begin{aligned} \frac{\phi_{CT}(\bar{\xi}|\bar{t}, \bar{\tau})}{\theta} &= D_1 + D_2 + D_3 + D_4 + D_1 D_3 e^{\xi_1 \xi_3 \sigma_\nu^2 - \lambda_{1,3}} + D_1 D_4 e^{\xi_1 \xi_4 \sigma_\nu^2 - \lambda_{1,4}} + D_2 D_4 e^{\xi_2 \xi_4 \sigma_\nu^2 - \lambda_{2,4}} \\ &\quad + D_1 D_2 B_{1,2} + D_2 D_3 B_{2,3} + D_3 D_4 B_{3,4} + D_1 D_2 D_3 B_{1,2} B_{2,3} + D_1 D_2 D_4 B_{1,2} B_{2,4} \\ &\quad + D_1 D_3 D_4 B_{1,3} B_{3,4} + D_2 D_3 D_4 B_{2,3} B_{3,4} + D_1 D_2 D_3 D_4 B_{1,2} B_{2,3} B_{3,4} \end{aligned} \quad (8)$$

where $B_{a,b} = (1 - \exp\{-\lambda_{a,b}\} + \exp\{\xi_a \xi_b \sigma_\nu^2 - \lambda_{a,b}\})$ and $D_a = \exp\{-\xi_a^2 \sigma_\nu^2\} (\exp\{\lambda_a \exp\{-\frac{1}{2} \xi_a^2 \sigma_\varepsilon^2\}\} - 1)$. Below I will give a sketch of the way in which this expression can be derived. The key here is to decompose the infinite summation over \bar{h} in Eq. (7) as follows:

$$\begin{aligned} \sum_{\bar{h}=0}^{\infty} &= \sum_{h_1=1}^{\infty} + \sum_{h_2=1}^{\infty} + \sum_{h_3=1}^{\infty} + \sum_{h_4=1}^{\infty} + \sum_{h_1, h_2=1}^{\infty} + \sum_{h_1, h_3=1}^{\infty} + \sum_{h_2, h_3=1}^{\infty} + \sum_{h_1, h_4=1}^{\infty} + \sum_{h_2, h_4=1}^{\infty} + \sum_{h_3, h_4=1}^{\infty} \\ &+ \sum_{h_1, h_2, h_3=1}^{\infty} + \sum_{h_1, h_2, h_4=1}^{\infty} + \sum_{h_1, h_3, h_4=1}^{\infty} + \sum_{h_2, h_3, h_4=1}^{\infty} + \sum_{h_1, h_2, h_3, h_4=1}^{\infty} \end{aligned}$$

where for simplicity of notation I leave out all summations over arguments that run from 0 to 0 (e.g. $\sum_{h_1=0}^0$). The 15 components of the above decomposition each correspond to one term in Eq (8). For instance, consider the summation in Eq. (7) over \bar{m} and h_1 only:

$$\sum_{\bar{m}=0}^{\infty} \sum_{h_1=1}^{\infty} \Rightarrow \theta \sum_{h_1=1}^{\infty} e^{-\frac{1}{2} \xi_1^2 (h_1 \sigma_\varepsilon^2 + 2\sigma_\nu^2)} \frac{\lambda_1^{h_1}}{h_1!} = \theta D_1$$

Analogously, the summation in Eq. (7) over \bar{m} , h_1 , and h_2 can be worked out as follows:

$$\begin{aligned} \sum_{\bar{m}=0}^{\infty} \sum_{h_1=1, h_2=1}^{\infty} &\Rightarrow \theta e^{-\lambda_{1,2}} \sum_{m_1=0}^0 \sum_{h_1, h_2=1}^{\infty} e^{\xi_1 \xi_2 \sigma_\nu^2 - \frac{1}{2} (\xi_1^2 (h_1 \sigma_\varepsilon^2 + 2\sigma_\nu^2) + \xi_2^2 (h_2 \sigma_\varepsilon^2 + 2\sigma_\nu^2))} \frac{\lambda_2^{h_2}}{h_2!} \frac{\lambda_1^{h_1}}{h_1!} \frac{\lambda_{1,2}^{m_1}}{m_1!} \\ &+ \theta e^{-\lambda_{1,2}} \sum_{m_1=1}^{\infty} \sum_{h_1, h_2=1}^{\infty} e^{-\frac{1}{2} (\xi_1^2 (h_1 \sigma_\varepsilon^2 + 2\sigma_\nu^2) + \xi_2^2 (h_2 \sigma_\varepsilon^2 + 2\sigma_\nu^2))} \frac{\lambda_2^{h_2}}{h_2!} \frac{\lambda_1^{h_1}}{h_1!} \frac{\lambda_{1,2}^{m_1}}{m_1!} \\ &= \theta D_1 D_2 B_{1,2} \end{aligned}$$

The other terms can be worked out in a similar fashion; for instance, $\sum_{\bar{m}=0}^{\infty} \sum_{h_1, h_2, h_3=1}^{\infty} \Rightarrow D_1 D_2 D_3 B_{1,2} B_{2,3}$ and $\sum_{\bar{m}=0}^{\infty} \sum_{h_1, h_2, h_3, h_4=1}^{\infty} \Rightarrow D_1 D_2 D_3 D_4 B_{1,2} B_{2,3} B_{3,4}$.

C Proofs

Proof of Proposition 1.1:³ Define $\sigma_n^2 = n^{-1} \sigma_\varepsilon^2$, $\lambda_n(t) = n\lambda(t)$ for given σ_ε^2 and $\lambda(t)$, $t \in [0, 1]$ so that the diffusion limit (\mathcal{D}) corresponds to the case where $n \rightarrow \infty$. Also define $P_n^\varepsilon(t) = \sum_{i=1}^{M_n(t)} \varepsilon_{ni}$ where $\varepsilon_{ni} \sim iid \mathcal{N}(0, \sigma_n^2)$ and $M_n(t) \sim Poisson(n\Lambda(t))$. It is evident that $P_n^\varepsilon(t)$ is a martingale, i.e. $E(P_n^\varepsilon(t+h) | \mathcal{F}_t) = \sum_{i=1}^{M_n(t)} \varepsilon_{ni} + E(\sum_{i=M_n(t)+1}^{M_n(t+h)} \varepsilon_{ni} | \mathcal{F}_t) = P_n^\varepsilon(t)$ using the law of iterated expectations. The required result can now be proved as an application of the central limit theorem for triangular arrays (see for instance Jacod and Shiryaev, 2003, Ch. VIII). For this, two conditions need to be verified, namely (i) $[P_n^\varepsilon(t)](t) \xrightarrow{p} \Lambda(t) \sigma_\varepsilon^2$ as $n \rightarrow \infty$ and (ii) $\lim_{a \uparrow \infty} \limsup_n \Pr(\sum_{i=1}^{M_n(t)} E(|\varepsilon_{ni}| 1_{\{|\varepsilon_{ni}| \geq a\}}) > \eta) = 0$ for all $\eta > 0$. Condition (i) can be verified as follows: $[P_n^\varepsilon(t)](t) = \sum_{i=1}^{M_n(t)} \varepsilon_{ni}^2 = n^{-1} M_n(t) \sigma_\varepsilon^2 + \sum_{i=1}^{M_n(t)} (\varepsilon_{ni}^2 - \sigma_n^2) \xrightarrow{p} \Sigma(t)$ as $n \rightarrow \infty$ by the (weak) law of large numbers. To verify condition (ii) redefine $\varepsilon_{ni} = \tilde{\varepsilon}_{ni} / \sqrt{n}$ where $\tilde{\varepsilon}_{ni} \sim iid \mathcal{N}(0, \sigma_\varepsilon^2)$ and note

that:

$$\begin{aligned}
\Pr\left(\sum_{i=1}^{M_n(t)} E(|\varepsilon_{ni}| 1_{\{|\varepsilon_{ni}| \geq a\}}) > \eta\right) &= \frac{1}{\sqrt{n}} \Pr\left(\sum_{i=1}^{M_n(t)} E(|\tilde{\varepsilon}_{ni}| 1_{\{|\tilde{\varepsilon}_{ni}| \geq a\sqrt{n}\}}) > \eta\right) \\
&\leq \frac{1}{\eta\sqrt{n}} E\left(\sum_{i=1}^{M_n(t)} E(|\tilde{\varepsilon}_{ni}| 1_{\{|\tilde{\varepsilon}_{ni}| \geq a\sqrt{n}\}})\right) = \frac{1}{\eta\sqrt{n}} n\Lambda(t) E(|\tilde{\varepsilon}_{n1}| 1_{\{|\tilde{\varepsilon}_{n1}| \geq a\sqrt{n}\}}) \\
&\leq \frac{1}{\eta\sqrt{n}} n\Lambda(t) E\left(\frac{|\tilde{\varepsilon}_{n1}|^2}{a\sqrt{n}} 1_{\{|\tilde{\varepsilon}_{n1}| \geq a\sqrt{n}\}}\right) = \frac{1}{a\eta} \Lambda(t) \int_{|s| > a\sqrt{n}} s^2 f(s) ds \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$ using Chebychev's inequality and the fact that $\tilde{\varepsilon}_{ni}$ are mutually independent and identically distributed and independent of $M_n(t)$. This completes the proof of the first part.

The second part can now be proved using the fact that a time changed Brownian motion has continuous sample paths if the time change is continuous. Because $\lambda(t)$ is strictly positive and càdlàg, $\Lambda(t)$ – and therefore $\Sigma(t)$ – is strictly increasing and continuous so that $W(\Sigma(t))$ has continuous sample paths. Further, because $W(\Sigma(t))$ is a non-trivial continuous local martingale, it must have paths of infinite variation (see for example Protter, 1990, Ch. 6). ■

D Moments of returns in calendar time

Based on the conditional characteristic function in Eq. (8), the following moments of returns in calendar time can be derived:

$$\begin{aligned}
E_\lambda[R(t_i|\tau_i)^2] &= \lambda_i \sigma_\varepsilon^2 + 2(1 - e^{-\lambda_i}) \sigma_\nu^2 \\
E_\lambda[R(t_i|\tau_i)^4] &= 3\lambda_i(1 + \lambda_i) \sigma_\varepsilon^4 + 12\sigma_\nu^4(1 - e^{-\lambda_i}) + 12\lambda_i \sigma_\varepsilon^2 \sigma_\nu^2 \\
E_\lambda[R(t_i|\tau_i) R(t_j|\tau_j)] &= -e^{-\lambda_{i,j}}(1 - e^{-\lambda_i})(1 - e^{-\lambda_j}) \sigma_\nu^2 \\
E_\lambda[R(t_i|\tau_i)^2 R(t_j|\tau_j)^2] &= \lambda_i \lambda_j \sigma_\varepsilon^4 + 2\lambda_i(1 - e^{-\lambda_j}) \sigma_\nu^2 \sigma_\varepsilon^2 + 2\lambda_j(1 - e^{-\lambda_i}) \sigma_\nu^2 \sigma_\varepsilon^2 \\
&\quad + 2(1 - e^{-\lambda_j})(2 + e^{-\lambda_{i,j}})(1 - e^{-\lambda_i}) \sigma_\nu^4 \\
E_\lambda[R(t_i|\tau_i)^3 R(t_j|\tau_j)] &= -3e^{-\lambda_{i,j}}(1 - e^{-\lambda_j})(\lambda_i \sigma_\varepsilon^2 + 2(1 - e^{-\lambda_i}) \sigma_\nu^2) \sigma_\nu^2 \\
E_\lambda[R(t_i|\tau_i) R(t_j|\tau_j)^2 R(t_k|\tau_k)] &= 2e^{-\lambda_{i,j} - \lambda_{jk}}(1 - e^{-\lambda_i})(1 - e^{-\lambda_j})(1 - e^{-\lambda_k}) \sigma_\nu^4 \\
E_\lambda[R(t_i|\tau_i)^2 R(t_j|\tau_j) R(t_k|\tau_k)] &= -e^{-\lambda_{jk}}(1 - e^{-\lambda_j})(1 - e^{-\lambda_k})(\lambda_i \sigma_\varepsilon^2 + 2(1 - e^{-\lambda_i}) \sigma_\nu^2) \sigma_\nu^2 \\
E_\lambda[R(t_i|\tau_i) R(t_j|\tau_j) R(t_k|\tau_k) R(t_m|\tau_m)] &= e^{-\lambda_{i,j} - \lambda_{km}}(1 - e^{-\lambda_i})(1 - e^{-\lambda_j})(1 - e^{-\lambda_k})(1 - e^{-\lambda_m}) \sigma_\nu^4
\end{aligned}$$

for $i < j < k < m$. Moments of returns in business time can be obtained from the above expressions by setting $\lambda_i = \bar{\lambda}$ and $\lambda_{i,j} = (j - i - 1) \bar{\lambda}$. Moments of returns in transaction time can be derived directly from the characteristic function in Eq. (6) or as the diffusion limit of the corresponding moments in business time.

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Notes

¹ Eq. (1) can be viewed equally well as a model for mid-quote data or tick data in which case $\lambda(t)$ is interpreted as the arrival frequency of quotes or ticks respectively. As such, the model is closely related to the literature on subordinated processes as initiated by Clark (1973) and, although sidestepped in this paper, the choice of subordinator is clearly an interesting issue in itself (see for example Ané and Geman, 2000; Jones, Kaul, and Lipson, 1994).

²In the current framework it is straightforward to allow for higher order dependence structure in the noise component, which could potentially serve to capture more complicated market microstructure. Also, it is possible to introduce a correlation between the noise component and the efficient price innovation so as to allow for non-iid noise in the spirit of Hansen and Lunde (2004). See Oomen (2005) for more details on this. The assumed independence between the intensity process and ε , which bars a leverage effect, is more difficult to relax. In a diffusion setting important progress has been made on this front by Bandi and Russell (2004a) and Barndorff-Nielsen and Shephard (2004b). In this paper I do not attempt to address this issue and analyze the model in the absence of leverage.

³ The first part of this proof is based on Johansen (2005)

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