EIGENVALUES AND PERTURBED DOMAINS

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In the study of the global dynamics of certain types of partial differential equations, the stability properties of equilibria play a very important role. These properties often are closely related to the eigenvalues and eigenfuntions of a linear partial differential equation given by the linear variation from an equilibrium. If the partial differential equation is defined on a bounded domain, then one must investigate the dependence of the eigenvalues and eigenfunctions on the boundary conditions and perturbations of the domain. The purpose of these notes is to survey some of the results dealing with this latter problem for second order elliptic operators.

The first situation deals with regular perturbations; that is, the boundary of the original domain and the perturbed domain are C^k-close for some $k \geq 1$. By a change of coordinates onto the original domain, the regularity properties of eigenvalues and eigenfunctions are reduced to the study of the dependence of these quantities on variations in coefficients in the equation and in the boundary conditions. A differential calculus with respect to the domain is needed to discuss nonlinear problems; for example, bifurcation theory, generic hyperbolicity and transversality of stable and unstable manifolds with respect to the domain, maximization of functions over a domain with fixed volume, etc. Problems of this type are discussed in Section 1.

If the domain is irregular (that is, not regular), then the definition of eigenvalues and eigenfunctions for some boundary conditions is a nontrivial task. For example, if the domain is irregular and the boundary conditions are Dirichlet, then one must give first a precise definition of Dirichlet boundary conditions as well as what is meant by eigenvalues. This is discussed in Section 2 for the first eigenvalue and eigenfunction and it is stated that a maximum principle holds if the first eigenvalue is positive. This definition depends upon the domain and then it becomes important to give a topology on the domains in order to know the maximum principle remains true under small perturbations in this topology. It also is necessary to do the same thing for the complete spectrum of the operator.

In Section 3, for the Laplacian with Neumann boundary conditions, we present some results on the dependence of eigenvalues and eigenfunctions on exterior irregular perturbations of the domain, (including perturbations near points on the boundary) of the domain, dumbbell shaped domains, thin domains and more general perturbations.

1. Regular domain perturbations.

Let $\Omega_0 \subset \mathbb{R}^n$ be a bounded domain and let $\mathcal{K}_k(\Omega_0)$ be the collection of all regions Ω which are C^k-diffeomorphic to Ω_0. We introduce a topology by defining a sub-basis of the neighborhoods of a given Ω as

$$\{h \text{ is a small } C^k(\Omega, \mathbb{R}^n) - \text{neighborhood of the inclusion } i_\Omega : \Omega \subset \mathbb{R}^n\}.$$

When $\|h - i_\Omega\|_{C^k}$ is small, h is a C^k-imbedding of Ω into $\mathbb{R}^n$; that is, a C^k-diffeomorphism to its range $h(\Omega)$. Micheletti (1972) has shown that this topology is metrizable and that $\mathcal{K}_k(\Omega_0)$ may be considered a separable complete metric space. We say that $\Omega \in \mathcal{K}_k(\Omega_0)$ is a *C^k-regular perturbations* (or, sometimes, simply a *regular perturbation*) of a given domain Ω_0 if h is a small perturbation in $C^k(\Omega_0, \mathbb{R}^n)$ of the inclusion i_{Ω_0}.

Courant and Hilbert (1937) studied the effect of regular perturbations of the domain on the eigenvalues and eigenfunctions of boundary value problems for PDE. For example, if $\Omega_\epsilon \in \mathcal{C}_r(\Omega_0)$ is a continuous family of domains converging to Ω_0 as $\epsilon \to 0$ and $\lambda_k(\Omega_\epsilon)$, $k \geq 1$, denotes the ordered set of eigenvalues of the Laplacian $-\Delta_{BC}^{\Omega_\epsilon}$ with some boundary conditons BC and $\{\varphi_{k,\epsilon}\}$ is a set of normalized eigenvectors, they proved that the eigenvalues $\lambda_k(\Omega_\epsilon)$ and eigenfunctions $\{\varphi_{k,\epsilon}\}$ converge to those of $-\Delta_{BC}^{\Omega_0}$. The proof consisted of constructing the family of diffeomorphisms h_ϵ which map Ω_ϵ onto Ω_0 and reduce the problem to the study of a family of operators L_ϵ on Ω_0.

This result is very interesting, but it is desirable to have more information about the eigenvalues and eigenfunctions. For example, if the eigenvalues and eigenfunctions are smooth functions of ϵ, what are the Taylor series in ϵ? Other important problems arise which are concerned with the determination of those domains which are critical values of some function of the domain such as maximization of torsional rigidity with fixed area of the domain, minimization of the principle eigenvalue of the Laplacian with Dirichlet boundary conditions over domains with fixed volume, etc. To discuss such questions we need to have a differential calculus of boundary perturbations. Many people have been concerned about questions of this type. We are going to present the approach of Henry (1985), (1987), (1996) and refer the reader to Henry (1966) for extensive references.

If $F : \mathcal{K}_k(\Omega_0) \to Y$, Y a Banach space, then we can define the smoothness of F at Ω_0 in the following way. For any $\Omega \in \mathcal{K}_k(\Omega_0)$ which is close to Ω_0, there is an $h \in C^k(\Omega_0, \mathbb{R}^n)$ which is close to the inclusion i_{Ω_0} such that $\Omega = h(\Omega_0)$. Therefore, $F(\Omega) = F(h(\Omega_0)) \equiv (F \circ h)(\Omega_0)$. We say that F is C^r (resp. C^∞) (resp. analytic) if the map $h \mapsto F \circ h$ is C^k (resp. C^∞) (resp. analytic). In this sense, problems of perturbation of the boundary (or, of the domain of definition) of a boundary value problem is reduced to differential calculus in Banach spaces.

Consider a non-linear formal differential operator

$$F_\Omega(u)(x) = f(x, Lu(x)), \quad x \in \Omega,$$

where L is a constant coefficients linear differential operator of order m; say,

$$Lu = (u, u_{x_j}, 1 \leq j \leq n, u_{x_j x_k}, 1 \leq j, k \leq n, \dots)$$

and $f(x, \zeta)$ is a given smooth function. We may consider F_Ω as a map from $C^m(\Omega)$ to $C^0(\Omega)$ [or from $W_p^m(\Omega)$ to $L_p(\Omega)$] under appropriate hypotheses.

If $h : \Omega \to h(\Omega) \subset \mathbb{R}^n$ is a C^m-embedding, then a basic problem is concerned with the manner in which the function $F_{h(\Omega)}(u)$ depends upon h and in exhibiting explicit formulas for the derivatives with respect to h if the derivatives exist. Obtaining derivatives with respect to h by working directly on $F_{h(\Omega)}(u)$ leads to many difficulties. As Henry points out, the computation working directly on $F_{h(\Omega)}(u)$ is analogous to treating continuum mechanics with the Lagrange description where particles are labeled as to position at a given time (and in different coordinate systems). The Eulerian description in continuum mechanics labels the particles by a velocity functon of position and time in a fixed coordinate system. This suggests discussing properties of $F_{h(\Omega)}$ by considering functions which depend only upon the original domain Ω. This can be accomplished in the following way.

Any C^m-embedding $h : \Omega \to h(\Omega) \subset \mathbb{R}^n$ induces an isomorphism (pull-back) $h^* : C^r(h(\Omega)) \to C^r(\Omega)$ [or $h^* : W_p^r(h(\Omega)) \to W_p^r(\Omega)$] for $0 \leq r \leq m$ by

$$h^*\varphi = \varphi \circ h.$$

Henry observed that the function (analogous to the Eulerian description in continuum mechanics)

$$h^* F_{h(\Omega)} h^{*-1} : C^m(\Omega) \to C^0(\Omega) \text{ [or } W_p^m(\Omega) \to L_p(\Omega)]$$

acting in spaces which are independent of h could be of great assistance in the differential calculus with respect to the domain. The symbol h^{*-1} denotes $(h^{-1})^*$.

Let Ω be in $\mathcal{C}_r(\Omega_0)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \geq 0$, integers, a_α be functions on Ω, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $(\partial/\partial y)^\alpha = \prod_{j=1}^n (\partial/\partial y_j)^{\alpha_j}$ and define the linear operator

$$A_\Omega = \Sigma_{|\alpha| \leq m} a_\alpha(y)(\frac{\partial}{\partial y})^\alpha.$$

If we suppose that $\Omega(t)$ is a C^1-curve of domains in $\mathcal{C}_r(\Omega_0)$ represented by the diffeomorphisms $h(t, \cdot) = i_C + Vt + o(t)$ as $t \to 0$ and apply the above calculus to A acting on functions $u(t, \cdot)$, then several computations yield

$$\frac{\partial}{\partial t}(h^*(t, \cdot)A_{h(t,\Omega)}h^{*-1}(t, \cdot)u)|_{t=0} = A\frac{\partial u}{\partial t} + [V \cdot \nabla, A]u,$$

where $A = A_{\Omega_0}$, $[V \cdot \nabla, A] = V \cdot \nabla A - AV \cdot \nabla$ is the commutator of $V \cdot \nabla$ and A.

If the chain rule for differentiation were applied directly to the corresponding function $F_{\Omega(t)}u$, the special commutator structure as well as the higher order derivatives would not be easily recognizable. This seems to be the advantage of computing derivatives on a fixed domain. This point was noted by Peetre (1980) who related it to a Lie derivative. For operators in variational form, Courant and Hilbert (1937)) obtained an equivalent formula.

Henry has given many applications of this calculus to prove differentiability with respect to the domain for various quantities associated with a boundary value problem and to obtain explicit formulas for the first and sometimes second derivatives. We mention only a few without any proofs since they are very technical.

1.1. Torsional rigidity. The resistance to torsion of a cylindrical rod depends not only on the elastic constants of the material, but also on the geometry of the cross section $\Omega \subset \mathbb{R}^2$ through the torsional rigidity

$$R(\Omega) \equiv \int_\Omega |\nabla u|^2 = \int_\Omega u dx$$

where $u : \Omega \to \mathbb{R}$ is the solution of

$$\Delta u = -1, \quad u = 0 \text{ in } \partial\Omega.$$

Suppose that Ω is $C^{m,\alpha}$ regular, $m \geq 2, 0 < \alpha < 1$, and $t \mapsto h(t, \cdot) \in C^{m,\alpha}(\Omega, \mathbb{R}^2)$ is a C^1 curve near $t = 0$ with $h(0, \cdot) = i_\Omega$, $\dot{h}(0, \cdot) = V$. Let $v(t, y)$ be the solution of

$$\begin{aligned} \Delta_y v(t, y) &= -1 \quad \text{in } \Omega(t) = h(t, \Omega), \\ v(t, y) &= 0 \quad \text{in } \partial\Omega(t) = h(t, \partial\Omega). \end{aligned}$$

After several computations, Henry shows that the following results are true:

(1) $$\frac{d}{dt}R(\Omega(t)) = \int_{\partial\Omega(t)} V \cdot N_{\Omega(t)}(\frac{\partial v}{\partial N_{\Omega(t)}})^2,$$

(2) $$\begin{aligned} \frac{d^2}{dt^2}R(\Omega(t)) = &\int_{\partial\Omega(t)} (\frac{\partial v}{\partial N_{\Omega(t)}})^2(\frac{\partial \sigma}{\partial t} + \sigma\frac{\partial \sigma}{\partial N_{\Omega(t)}} + H\sigma^2) \\ &+ \int_{\partial\Omega(t)} 2\sigma\frac{\partial v}{\partial N_{\Omega(t)}}(\sigma\frac{\partial^2 v}{\partial N_{\Omega(t)}^2} + \frac{\partial \dot{v}}{\partial N_{\Omega(t)}}), \end{aligned}$$

where $N_{\Omega(t)}$ is the unit outward normal to $\Omega(t)$, $\sigma = V \cdot N_{\Omega(t)}$, $H = \operatorname{div} N_{N(t)}$ and $\dot{v} = \partial v/\partial t = -\sigma(dv/dN)$ on $\Omega(t)$.

Fixing the area of $\Omega(t)$ as a constant independent of t and evaluating (1) at $t = 0$, one can deduce that the disk D_2 is a critical point of $R(\Omega)$.

If $\Omega = D_2$ and the area of $\Omega(t)$ is a constant, then it is possible to show that

$$\frac{d^2}{dt^2} R(\Omega(t))|_{t=0} = -2\pi \Sigma_2^\infty |\sigma_k|^2 (k-1)$$

where $\sigma = \Sigma_{-\infty}^\infty \sigma_k e^{ik\theta}$, $0 \le \theta \le 2\pi$, in polar coordinates. Therefore, the disk D_2 is a maximum of the rigidity under the restriction that the area of Ω is constant. Serrin (1971) has shown that the disk is the only critical point in the class of connected, bounded C^2-regions.

1.2. Eigenvalues. The calculus also leads directly to formulas for the manner in which eigenvalues of boundary value problems depend upon the domain. For example, consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ in } \Omega, \quad u = 0 \text{ in } \partial\Omega, \tag{3}$$

where Ω is a C^2-regular domain. Suppose that $\lambda_0 = \lambda_0(\Omega)$ is a simple eigenvalue of (3) and let u_0 be the corresponding eigenfunction with $\int_\Omega u_0^2 = 1$. Using the implicit function theorem and the above calculus, it is shown that, for any $h \in C^2(\Omega, \mathbb{R}^n)$ in some C^2-neighborhood of the inclusion map $i_\Omega : \Omega \to \mathbb{R}^n$, there is a simple eigenvalue $\lambda(h(\Omega))$ near λ_0, the map $h \mapsto \lambda(h(\Omega))$ is analytic, and, if $h(t, \cdot)$ is a C^1-family of maps in this neighborhood with $h(0, \cdot) = i_\Omega$, $\dot{h}(0, \cdot) = V$, then

$$\frac{d}{dt}\lambda(h(t,\Omega))_{t=0} = -\int_{\partial\Omega} V \cdot N_\Omega (\frac{\partial u_0}{\partial N_\Omega})^2. \tag{4}$$

This derivative was computed formally, in some special cases, by Rayleigh (1894) (in the edition of Dover (1945, p.338, eq. 11)) and, for general two dimensional regions by Hadamard (1908). Garabedian and Schiffer (1952) did the general case.

Using (4) for $t = 0$, one deduces that, in the class of connected domains of fixed volume, if $\lambda(\Omega)$ is the principal eigenvalue, then the ball is the only critical point of $\lambda(\Omega)$ and is a minimizer.

Higher order derivatives may be computed but explicit expessions can only be given for special types of perturbations and special Ω. For an ellipse $\Omega(t)$ in $\mathbb{R}^2$ with semi-axes e^t, e^{-t}, the map $t \mapsto \lambda(\Omega(t))$ is an even function and

$$\lambda(\Omega(t)) = \lambda_0 + \frac{1}{2}\lambda_0(\lambda_0 - 2)t^2 + O(t^4).$$

The eccentricity of the ellipse is the solution ϵ of the equation $\sqrt{1-\epsilon^2} = e^{-2|t|}$. In terms of ϵ, we have

$$\lambda(\Omega(t(\epsilon))) = \lambda_0 + \frac{1}{32}\lambda_0(\lambda_0 - 2)(\epsilon^4 + \epsilon^6) + O(\epsilon^8)$$

as the eigenvalue near λ_0 in the ellipse of area π and eccentricity ϵ. Joseph (1967) has computed the series for this latter case, but obtained $(3/2)\lambda_0 - 5$ in place of $\lambda_0 - 2$. This does not change the qualitative properties of his results.

Other boundary conditions can be considered. For example, if we change the boundary conditions to Robin conditions

$$\frac{\partial u}{\partial N} + \beta(x)u = 0 \quad \text{in } \partial\Omega, \tag{5}$$

and assume that the Laplacian with conditions (5) has a simple eigenvalue λ_0 with normalized eigenfunction u_0 and assume perturbations of the domain Ω as above, then there is a unique eigenvalue $\lambda(\Omega(t))$ near λ_0 and

$$\frac{d}{dt}\lambda(\Omega(t))|_{t=0} = \int_{\partial\Omega} V \cdot N[|\nabla_{\partial\Omega} u_0|^2 - (\lambda_0 + \beta^2 - H\beta - \frac{\partial\beta}{\partial N_\Omega})u_0^2],$$

where $H = divN_\Omega$ is the mean curvature of $\partial\Omega$ and $\nabla_{\partial\Omega} u_0$ is the tangential component of the gradient of u_0.

Henry also has shown that the spectral projections on eigenspaces corresponding to multiple eigenvalues are smooth functions of the perturbation. This implies that the complete set of eigenvalues and eigenfunctions for a domain $h(\Omega)$ will converge to those of Ω as $h \to i_\Omega$ in the C^r-topology.

Many other applications of the calculus with respect to the boundary of the domain are given by Henry. For example, explicit formulas are give for the first few terms in the expansion of capacity in terms of the variations in the boundary. Similar results are given for Green's function for second order differential operators.

1.3. Bifurcation and generecity. For second order differential operators, Henry considers bifurcation problems near equilibrium points for nonlinear problems. In the case where the bifurcation corresponds to a simple zero eigenvalue with the first nontrivial term being quadratic (a codimension one singularity), he gives a complete description of the manner in which the number of solutions changes from zero to two leading to an unfolding of the singularity. In the case of a zero eigenvalue and the first nontrivial nonlinear term being a cubic (a codimension two singularity), he describes the number of solutions as a function of the domain together with another parameter in the differential equation leading to an unfolding of the singularity.

Recall that, in a complete metric space X, a property is said to be generic if it holds on a residual set; that is, on a set which is the countable intersection of open dense sets. In differential equations, it is very important to be able to assert that a given property is generic with respect to some parameters (which may be the vector field, the domain, etc.). Transversality theorems are the normal way to obtain such results. The classical transversality theorem deals with Fredholm maps of finite index (see, for example, Abraham and Robbin (1967)). In many of the problems dealing with perturbation of the boundary, the operator is Fredholm but has index $-\infty$. Therefore, to use transversality theory to discuss generic properties with respect to the boundary, a generalized transversality theorem is needed. An elegant and appropriate generalization has been given by Henry. We do not state the theorem and only mention some of the results that have been obtained by using this theorem.

Using his transversality theorem, Henry proved that, for a residual set of h near i_Ω, the eigenvalues of the Laplacian with Dirichlet, Robin or Neumann boundary conditions are simple. In the Neumann case, if the domain is not connected, then zero is a multiple eigenvalue. In this case, the simplicity of eigenvalues refers to all eigenvalues except zero. The case of Dirichlet boundary conditions also was proved by Uhlenbeck (1972). Henry has similar results for more general differential operators which generalize those of Micheletti (1973b).

When the domain Ω enjoys some symmetry properties and the perturbations are in some symmetry class, then it may or may not be possible to make perturbations which make the eigenvalues simple. On the other hand, it is reasonable to conjecture that there is an integer p, determined by the symmetries, for which perturbation in a residual set will yield eigenvalues of dimension at most p. Henry has an example with reflection symmetry for which $p = 1$; that is, the eigenvalues are still simple. Pereira (1989), (1995), (1996) has discussed more general situations for which the symmetry class generically has eigenvalues of multiplicity ≥ 2.

Let us now consider the equation

$$\Delta u + f(x, u, \nabla u) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ in } \partial\Omega, \tag{6}$$

where f is a C^2-function.

Using the generalized transversality theorem, Henry proved that there is a residual set of domains Ω for which all solutions of (6) are simple; that is, the linear variational equation about any equilibrium solution has only the zero solution. For the case in which $f(x, 0, 0) \equiv 0$, one can use the standard transversality theorem to prove this result. This latter case was considered by Saut and Temam (1979), but they inadvertently forgot to say that $f(x, 0, 0) \equiv 0$. Without this latter condition, the standard transversality theorem does not apply.

Henry also has some similar results for domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, for the case where the function f in (6) does not depend upon ∇u; that is, $f = f(x, u)$. Under some conditions on f to be listed below and for homogeneous Neumann boundary conditions (he also allows nonlinear Neumann boundary conditions), he proves, generically in Ω, that the equilibrium solutions are simple. Since the system in this case is gradient, this is the same as saying that the equilibrium points are hyperbolic. The conditions on f are

(i) There is a discrete set $\{c_j\} \subset \mathbb{R}$, possible empty, such that $f(x, c_j) = 0$ for all x and, at each such c_j, $f_u(x, c_j) \neq 0$ on a dense set of $\mathbb{R}^n$.

(ii) For any $c \in \mathbb{R} \setminus \{c_j\}$, the set $\{x \in \mathbb{R}^n : f = 0, f_x = 0, f_{xx} = 0, \text{ at } (x, c)\}$ has dimension $< n - 1$.

(iii) $\{(x, u) : u \notin \{c_j\}, f = 0, f_x = 0, f_{xx} = 0, f_u = 0, f_{xu} = 0, \text{ at } (x, u)\}$ has dimension $< n - 1$.

An example is $f(x, u) = r(u)s(x)$, where $s(x) \neq 0$ on a dense set and each zero of r is simple.

There also are some results in Henry which deal with the simplicity of solutions of nonlinear equations for which the boundary conditins are nonlinear.

2. Irregular domains and Dirichlet boundary conditions.

In the previous section, given an elliptic PDE on a regular domain with specified boundary conditions, we have discussed the effect on eigenvalues and eigenfunctions of regular perturbations of the domain. On the other hand, there are problems in PDE for which the original domain is irregular. If the original domain is irregular, then there first is the problem of the existence of solutions with specified boundary conditions. In this section, we discuss the work of Berestycki, Nirenberg and Varadhan (1994) in which they give a definition of the solution of an elliptic equation with Dirichlet boundary conditions on an irregular domain, as well as a definition of the first eigenvalue and show that the maximum principle holds if this eigenvalue is positive. Since this eigenvalue depends upon the domain, it is important to have a topology on the irregular domains which will imply the first eigenvalue remains positive under perturbations of the original domain and, therefore, conclude that the maximum principle holds on the perturbed domain. Such a topology has been given by Arrieta (1996) (1997).

2.1 General elliptic operators. Suppose that $a^{ij} \in C(\mathbb{R}^n)$, $b^i, c \in L^\infty(\mathbb{R}^n)$, $1 \leq i, j \leq n$, with $(\Sigma_i |b^i|^2)^{1/2} < \infty, |c| \leq b_0$ for some constant $b_0 \geq 0$,

$$c_0|\xi|^2 \leq \Sigma a^{ij}\xi_i\xi_j \leq C_0|\xi|^2$$

for some positive constants c_0, C_0 and define the elliptic differential operator

$$L = \Sigma_{i,j} a^{ij}\partial_{x_i x_j} + \Sigma_i b^i \partial_{x_i} + c. \tag{7}$$

If Ω is a bounded domain, we want to consider the Dirichlet problem

$$\begin{aligned} Lv &= f \ \text{ in } \ \Omega, \\ v &= 0 \ \text{ in } \ \partial\Omega, \end{aligned} \tag{8}$$

where $f \in L^n(\Omega)$ and $v = 0$ in $\partial\Omega$ means that $\lim_{x\to\partial\Omega} v(x) = 0$.

It is known that there exist domains Ω and operators L for which (8) has no solution. To overcome this difficulty, one must relax the manner in which the boundary conditions are to be satisfied. Berstycki, Nirenberg and Varadhan (1994) proceeded in the following very interesting way. Define the differential operator

$$M = L - c \tag{9}$$

and let H_j, $j \geq 1$, be a family of smooth domains, $\bar{H}_j \subset H_{j+1} \subset \Omega$, $j \geq 1$, $\cup_{j\geq 1} H_j = \Omega$. Since H_j is smooth, there is a unique solution u_j of the problem

$$Mu_j = -1 \ \text{ in } \ H_j, \quad u_j = 0 \ \text{ in } \ \partial H_j. \tag{10}$$

It can be shown that u_j is a nondecreasing sequence which converges to a function $u^{\Omega M}$ weakly in $W^{2,p}(J)$ and strongly in $C^1(J)$ for any compact set $J \subset \Omega$. Moreover, $u^{\Omega M}$ is a strong solution of $Mu = -1$ in Ω, $u^{\Omega M} > 0$ in Ω and only depends upon the domain Ω and the operator M and not on the sets H_j.

The Dirichlet condition $v = 0$ in (8) is replaced by the condition $v = 0(u^{\Omega M})$ where this means that $v(x^{(j)}) \to 0$ for any sequence $\{x^{(j)}\} \subset \Omega$ for which $x^{(j)} \to \partial\Omega$ and $u^{\Omega M}(x^{(j)}) \to 0$ as $j \to \infty$. Let us also use the notation $x^{(j)} \to \partial\Omega(u^{\Omega M})$ to denote that $x^{(j)} \to \partial\Omega$ and $u^{\Omega M}(x^{(j)}) \to 0$ as $j \to \infty$.

We say that the operator L satisfies the *Refined Maximum Principle (RMP) in* Ω if the condition, $Lw \geq 0$ in Ω for w bounded above and $\limsup_{j\to\infty} w(x^{(j)}) \leq 0$ if $x^{(j)} \to \partial\Omega(u^{\Omega M})$, implies that $w \leq 0$ in Ω.

The principal eigenvalue $\lambda(L, \Omega)$ of the operator L in a domain Ω is defined as

$$\lambda(L, \Omega) = \sup\{\mu : \exists\varphi > 0 \,\text{in}\, \Omega, (L + \mu)\varphi \leq 0 \,\text{in}\, \Omega\}. \tag{11}$$

The following very interesting results have been proved by Berestycki, Nirenberg and Varadhan (1994).

Theorem 1. *RMP holds for L if and only if $\lambda(L, \Omega) > 0$.*

Theorem 2. *If $\lambda(L, \Omega) > 0$, then there is a positive constant $A = A(\Omega, c_0, C_0, b, \lambda(L, \Omega))$ such that, for any $f \in L^n(\Omega)$, there is a unique solution v of*

$$\begin{aligned} Lv &= f \ \text{ in } \ \Omega, \\ v &= 0(u^{\Omega M}) \ \text{ in } \ \partial\Omega, \end{aligned} \tag{12}$$

and

$$\|v\|_{L^\infty(\Omega)} \leq A\|f\|_{L^n(\Omega)}. \tag{13}$$

Both of these results depend upon knowing that $\lambda(L, \Omega) > 0$. If we know that this condition is satisfied for Ω, how do we characterize the class of perturbations of Ω for which it will still be true? Arrieta (1996) has introduced a complete metric space of equivalence classes of bounded open sets in which $\lambda(L,\Omega)$, as well as the solution of (12), is continuous in Ω. We now describe this result.

Let $\Theta = \{\Omega \subset B_1 \subset \mathbb{R}^n : \Omega \text{ is open}\}$, where B_1 is the unit ball with center zero. For $\Omega \in \Theta$, if $\Gamma_{\Omega M} = \{x \in \partial\Omega : \exists\{x^{(j)}\} \subset \Omega, x^{(j)} \to x, u^{\Omega M}(x^{(j)}) \to 0 \,\text{as}\, j \to \infty\}$, then the set $\Omega^{*M} = \bar{\Omega} \setminus \Gamma_{\Omega M}$ is open. We say that $\Omega_1, \Omega_2 \in \Theta$ *are equivalent relative to the operator* M *and Dirichlet boundary conditions*, $\Omega_1 \sim_M \Omega_2$, if $\Omega_1^{*M} = \Omega_2^{*M}$. With this equivalence relation, following Arrieta (1996), we define $\tilde{\Theta}^M = \Theta/\sim_M$ and the metric

$$\begin{aligned} d^M_{L^\infty} &: \tilde{\Theta}^M \times \tilde{\Theta}^M \to \mathbb{R} \\ &(\Omega_1, \Omega_2) \mapsto d^M_{L^\infty}(\Omega_1,\Omega_2) = \|u^{\Omega_1 M} - u^{\Omega_2 M}\|_{L^\infty(B_1)}. \end{aligned}$$

Arrieta (1996) shows that $(\tilde{\Theta}^M, d^M_{L^\infty})$ is a complete metric space and also proves the following result.

Theorem 3. *If $\lambda(L,\Omega)$ is defined as in (11), then $\lambda(L,\Omega)$ and the corresponding eigenfunction are continuous in the metric $d^M_{L^\infty}$. If $\lambda(L,\Omega) > 0$, then so is the unique solution of (12).*

This result shows that, if the conditions of Theorems 1 and 2 hold for a given domain Ω_0, then they hold for an open neighborhood of Ω_0 in the space $(\tilde{\Theta}^M, d^M_{L^\infty})$.

2.2. Operators in divergence form. Several important questions arise with respect to the above metric imposed on the domains.

(1) Is it possible to show that the equivalence relation $\sim_M$ is independent of M for M in some class?
(2) In the definition of the metric, is it possible to replace $L^\infty(B_1)$ by $L^p(B_1)$ or $H^1(B_1)$?
(3) Is there a class of operators for which the metric for M and M^* are equivalent if they belong to this class?
(4) In these metrics for M in some class of operators, is it possible to obtain continuity of all of the spectrum?

Arrieta (1997) shows that the answers to these questions are mostly affirmative in the class of operators which can be described in divergence form. To describe the results, we need some notation. For a fixed constant $\nu > 0$, let

$$\begin{aligned} \mathcal{D} = \{L = &\Sigma_{i,j}\partial_{x_i}(a^{ij}\partial_{x_j}) + \Sigma_i b^i \partial_{x_i} + c, \\ & a^{ij} \in C^{0,1}(\mathbb{R}^n), b^i, c \in L^\infty(\mathbb{R}^n), 1 \le i,j \le n, \\ & \Sigma^n_{i,j=1} a^{ij}\xi_i\xi_j \ge \nu|\xi|^2\} \end{aligned}$$

$$\mathcal{D}_0 = \{L \in \mathcal{D} : c = 0\}$$

$$\mathcal{D}_{00} = \{L \in \mathcal{D}_0 : b_i = 0, 1 \le i \le n\}$$

Proposition 4. *The equivalence relation $\sim_M$ is independent of the operator $M \in \mathcal{D}_0$; that is, $\Omega^{*M} = \Omega^{*M^*}$ for every $M, M^* \in \mathcal{D}_0$.*

From Proposition 4, we can define $\Omega^* = \Omega^{*M}$ and $\tilde{\Theta} = \tilde{\Theta}^M$ for any $M \in \mathcal{D}_0$. From now on, when an open set Ω is considered, we can suppose that $\Omega = \Omega^*$ since the properties of an operator $L \in \mathcal{D}_0$ are the same on Ω and Ω^*. As we did for the metric $d^M_{L^\infty}$, we can define the metrics $d^M_{L^p}$, $1 \le p < \infty$, and $d^M_{H^1}$ on $\tilde{\Theta}$.

The metric $d^M_{L^p}$ is strictly weaker that the metric $d^M_{L^\infty}$. On the other hand, as noted by Arrieta (1997), even though the space $(\tilde{\Theta}, d^M_{L^\infty})$ is complete, the space $(\tilde{\Theta}, d^M_{L^p})$ is not complete for any $1 \le p < \infty$. As compensation, we have the following

Proposition 5. *For any $M \in \mathcal{D}_{00}$, the metrics $d^M_{L^p}$, $1 \le p < \infty$, and $d^M_{H^1}$ on $\tilde{\Theta}$ are equivalent.*

For any linear operator L, we let $\sigma(L)$ denote the spectrum of L and $\rho(L)$ the resolvent set of L. Regarding the convergence of the spectrum of an operator L in the metric $d^{L-c}_{L^p}$, Arrieta (1997) proves the following

Theorem 6. *Let $L \in \mathcal{D}$, $M = L - c \in \mathcal{D}_{00}$, and suppose that Ω_k, $k \ge 0$, is a sequence in $\tilde{\Theta}$ and define L_k to be the operator L with Dirichlet boundary conditions on Ω_k. If $d^M_{L^2}(\Omega_k, \Omega_0) \to 0$ as $k \to \infty$, then the following statements are true:*

(i) For any C^1-Jordan curve Γ in the complex plane such that $\Gamma \cap \sigma(L_0) = \emptyset$, there exists a $k_0 = k_0(\Gamma)$ such that $\Gamma \cap \sigma(L_k) = \emptyset$ for $k \ge k_0$. Moreover, if P_{Γ,L_k} is the spectral projection over the part of the spectrum inside Γ, then

$$\|P_{\Gamma,L_k} - P_{\Gamma,L_0}\|_{\mathcal{L}(L^2(B_1),H^1_0(B_1))} \to 0 \text{ as } k \to \infty.$$

(ii) If $R(\lambda, L_k)$ is the resolvent of L_k, then

$$\|R(\lambda, L_k) - R(\lambda, L_0)\|_{\mathcal{L}(L^2(B_1),H^1_0(B_1))} \to 0 \text{ as } k \to \infty$$

and the convergence is uniform in any compact $\Gamma \subset \rho(L_0)$.

Since the Laplace operator Δ is the simplest second order elliptic differential operator, it is natural to define a canonical metric d_2 by

$$d_2(\Omega_1, \Omega_2) = \|u^{\Omega_1\Delta} - u^{\Omega_2\Delta}\|_{L^2(B_1)}.$$

With this notation, Arrieta (1997) obtains the following interesting result.

Theorem 7. *Suppose that Ω_k, $k \ge 0$, is a sequence of domains in $\tilde{\Theta}$ and let $u^k = u^{\Omega_k\Delta}$. For any $L \in \mathcal{D}$, let L_k be the operator L_k with Dirichlet boundary conditions acting on Ω_k. For the following statements:*

(i) $d_2(\Omega_k, \Omega_0) \to 0$ as $k \to \infty$,

(ii) The spectrum of L_k aprroaches the spectrum of L_0 and the spectral projections of L_k approach the spectral projections of L_0 in $\mathcal{L}(L^2(B_1), H^1_0(B_1))$ as $k \to \infty$,

we have (i) implies (ii). Moreover, if L is self-adjoint, then both statements are equivalent.

Micheletti (1972-1976) has given results about the convergence of the spectrum of operators in the case of regular perturbations of the domain in the Courant metric. The Courant metric is stronger than the d_2 metric and therefore we can have convergence of the spectrum for more general domains.

Most of the results in the literature related to the behavior of the spectrum of an operator when the domain is perturbed put the emphasis on geometric conditions on the perturbations of the domain to guarantee the continuity of the spectrum (see the previous and the next section). For Dirichlet boundary condtions, the conditions in Theorem 7 are different from the conditions being imposed on the convergence properties of solutions of the simplest nontrivial elliptic equation $\Delta u = 1$ in the perturbed domains.

It is clear that it would be interesting to characterize, in some more analytic way, large classes of domains for which the condition (i) in Theorem 7 is satisfied. It would also be interesting to see if some similar theory is valid for other types of boundary conditions.

3. Neumann conditions and irregular perturbations.

If the perturbed domain depends upon a parameter ϵ in a metric space containing zero, then a family of domains Ω_ϵ is said to be an *irregular perturbation* of the domain Ω_0 if the measure of $\Omega_\epsilon \setminus \Omega_0$ approaches zero as $\epsilon \to 0$. The set of irregular perturbations contains but is more general that the set of regular perturbation of Ω_0 as defined in Section 1. For example, the domain Ω_ϵ could be a perturbation of Ω_0 which introduces an irregular bump at a point on the boundary of Ω_0. Another example could be a dumbbell shaped domain for which the connecting bar degenerates to a curve as $\epsilon \to 0$. A domain $\Omega_\epsilon \subset \mathbb{R}^n$ which degenerates to a domain $\Omega_0 \subset \mathbb{R}^m$ with $m < n$ (*thin domain*) also is an irregular perturbation.

In this section, we study the properties of eigenvalues and eigenfunctions of elliptic operators with Neumann boundary conditions as a function of external irregular perturbations of a bounded domain.

Problems of this type have independent interest and also play an important role in the dynamics of nonlinear equations. For example, if the nonlinear system is gradient, then the compact global attractor (that is, the maximal compact invariant set which attracts bounded sets uniformly) consists of the union of the unstable sets of the equilibrium. Knowing convergence properties of the eigenfunctions and eigenfunctions with respect to the domain leads, without too much difficulty, to results on the upper semicontinuity of attractors at the limit domain for parabolic equations. For hyperbolic equations, the upper semicontinuity is more difficult to prove. In some cases (for example, the variational case), it is easier to show upper semicontinuity directly. If each equilibrium is hyperbolic, then one can deduce continuity properties of the unstable manifolds and, as a consequence, deduce that the compact global attractors are Hausdorff continuous at the limit domain. We do not discuss this problem and refer the reader to Hale and Raugel (1992), (1995), Raugel (1995), Arrieta (2000).

In this section, we concentrate on Neumann boundary conditions for these types of perturbations. However, we begin with a few remarks about other types of boundary conditions.

If we assume Dirichlet boundary conditions, then it is possible to prove very general results. In fact, Babuška and Vyborny (1965) proved that the eigenvalues and eigenfunctions converge for a general $2m$-order elliptic operator with Dirichlet boundary conditions when the domains Ω_ϵ satisfy the following conditions:

(i) For all compact sets $K \subset \Omega_0$, there exists $\epsilon(K) \in (0, \epsilon_0)$ such that $K \subset \Omega_\epsilon$ for $\epsilon \in (0, \epsilon(K))$.

(ii) For each open set U with $\bar{\Omega}_0 \subset U$, there exists $\epsilon(U) \in (0, \epsilon_0)$ such that $\Omega_\epsilon \subset U$ for $\epsilon \in (0, \epsilon(U))$.

Other references dealing with these problems for Dirichlet boundary conditions are Courant and Hilbert (1937), Dancer (1988) (1990) (1996), Daners (1996), Lopes-Gomez (1996).

We will not discuss Robin boundary conditions and only mention that some references for this case are Dancer and Daners (1997), Daners (1996), Ozawa (1992), Ozawa and Roppongi (1995), Stummel (1976), Ward, Henshaw and Keller (1993), Ward and Keller (1991), (1993). Results related to convergence of eigenvalues and eigenvectors are more closely related to the Dirichlet problem than to the Neumann problem.

It was shown by an example in Courant and Hilbert (1937) that the eigenvalues of the Laplacian with Neumann boundary conditions may not be continuous if the perturbation of the domain is irregular. In the last few years, Neumann problems have received considerable attention by Arrieta (1991), (1995), (1996), (1997), Arrieta, Hale and Han (1991), Beale (1973), Brown, Hislop and Martinez (1995), Chavel and Feldman (1978), Ciuperca (1994), Hale and Raugel (1992), (1995), Hale and Vegas (1984), Hempel, Seco and Simon (1991), Hislop and Martinez (1991), Jimbo (1988), (1989), (1993), Jimbo and Morita (1992), Lobo-Hidalgo and Sanchez-Palencia (1979), Rauch and Taylor (1975), Raugel (1995), Vegas (1990), (1992), as well as others contained in the references of the above papers.

3.1. Perturbations near boundary points. It is instructive to begin with the counterexample of Courant and Hilbert (1937). Let $\Omega_\sigma = \{(x,y) : |x| < \sigma/2, |y| < \sigma/2\}$ of area $\sigma^2/4$ with center $(0,0)$. For any $\epsilon > 0$, $\tau > 0$, let $R_{\epsilon,\tau} = \{(x,y) : 0 < x < \epsilon, |y| < \tau/2\}$ and define

$$\Omega_{\epsilon,\tau} = \Omega_1 \cup (R_{\epsilon,\tau} + (1/2, 0)) \cup (\Omega_\epsilon + (1/2+\epsilon, 0)).$$

For $\tau = \epsilon^4$, the domain $\Omega_{\epsilon,\epsilon^4}$ can be viewed as a C^0-perturbation of Ω_0, but not a C^1-perturbation. Let $A_\epsilon : \mathcal{D}(A_\epsilon) \subset L^2(\Omega_{\epsilon,\epsilon^4}) \to L^2(\Omega_{\epsilon,\epsilon^4})$, $\mathcal{D}(A_\epsilon) = \{\varphi \in H^2(\Omega_{\epsilon,\epsilon^4}), \partial\varphi/\partial n = 0 \text{ on } \partial\Omega_{\epsilon,\epsilon^4}\}$, $A_\epsilon\varphi = -\Delta\varphi$. For all $\epsilon \geq 0$, 0 is an eigenvalue of A_ϵ. If λ_k^ϵ are the ordered eigenvalues of A_ϵ, then $\lambda_2^\epsilon > 0$ for $\epsilon > 0$. It is shown in Courant and Hilbert (1937) that $\lambda_2^\epsilon \to 0$ as $\epsilon \to 0$.

For some cases, a C^0-perturbation does not yield singular behavior of the eigenvalues of A_ϵ. In the example of Courant and Hilbert (1937), if $\tau = \epsilon^\beta$ and β is too small, this will be the case. A more trivial example can be obtained by eliminating the retangular square of size ϵ from the perturbation.

Arrieta, Hale and Han (1991) have given a complete description of the behavior of the eigenvalues and eigenfunctions of the Laplacian with Neumann boundary conditions for a general class of perturbations including the example above of Courant and Hilbert (1937). As we will see, the singular behavior of the eigenvalues relies on the way in which the original domain is perturbed as well as the relative sizes of the domains used as perturbation, but the shape of the perturbation is of no importance.

We now give a precise definition of the domain considered in Arrieta, Hale and Han (1991). Let Ω_0, D_1 be bounded, connected smooth domains such that

(H.1) There exist positive constants α, β such that

$$\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| < \alpha, |y| < \beta\} \cap \Omega_0 = \{(x,y) : -\alpha < x < 0, |y| < \beta\}$$

$$\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x < 2\alpha, |y| < \beta\} \cap D_1 = \{(x,y) : \alpha < x < 2\alpha, |y| < \beta\}$$

$$((0,\alpha) \times (-\beta,\beta)) \cap (\Omega_0 \cup D_1) = \emptyset$$

$$\{0\} \times (-\beta,\beta) \subset \partial\Omega_0, \quad \{\alpha\} \times (-\beta,\beta) \subset \partial D_1$$

(H.2) $\bar{\Omega}_0 \cap \bar{D}_1 = \emptyset$.

(H.3) For any connected set $R_1 \subset \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leq x \leq \alpha, |y| < \beta\}$, the set $\Omega_0 \cup D_1 \cup R_1$ is a bounded connected smooth domain in $\mathbb{R}^n$. Also, if $\Gamma_1^1 = \partial\Omega_0 \cap \partial R_1$, then $R_1 \cap \Gamma_1^1 \neq \emptyset$.

The set $(R_1 \setminus \Gamma_1^1) \cap D_1$ is a bounded connected domain with smooth boundary except probably at some points of Γ_1^1. Let $\eta > 0$ be a constant which will be fixed later. For $\epsilon > 0$ small, let

$$\begin{aligned} R_{\epsilon,\eta} &= \{(\epsilon x, \epsilon^\eta y) : (x,y) \in R_1\} \\ D_\epsilon &= \{(\epsilon x, \epsilon y) : (x,y) \in D_1\} \end{aligned} \tag{14}$$

There is an $\epsilon_0 > 0$ such that, for each $\epsilon \in (0,\epsilon_0)$, we have $\bar{\Omega}_0 \cap \bar{D}_\epsilon = \emptyset$ and $\bar{R}_{\epsilon,\eta} \cup \bar{D}_\epsilon \subset \{(x,y) : o \leq x < \alpha, |y| \leq \beta\}$.

The set $\Omega_\epsilon = \Omega_0 \cup R_{\epsilon,\eta} \cup D_\epsilon$ is a bounded open connected smooth domain.

Remark 8. The fact that $\partial\Omega_0$ is a piece of a hyperplane near $(0,0)$ is merely technical. It is shown in Arrieta, Hale and Han (1991) how to attach the perturbation near a point for arbitrary smooth domains Ω_0 and so all of the results below will be valid.

For each fixed $\epsilon_1 \in (0, \epsilon_0)$, the domain Ω_ϵ, for ϵ close to ϵ_1, is a C^1-perturbation of Ω_{ϵ_1}. Although this is not true at $\epsilon = 0$, we do have $\mu(\Omega_\epsilon \setminus \Omega_0) \to 0$ as $\epsilon \to 0$, where μ is Lebesgue measure. Let us also introduce the set S_γ by the relation

$$S_\gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : x^2 + |y|^2 \leq \gamma^2\} \cap \bar{\Omega}_0.$$

There is a γ_0 such that, for $0 < \gamma < \gamma_0$, we have $S_\gamma \subset \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : -\alpha < x \leq 0, |y| \leq \beta\}$.

For $0 \leq \epsilon < \epsilon_0$, we denote by $\{\omega_m^\epsilon, 1 \leq m < \infty\}$, a set of orthonormal eigenvectors corresponding to the ordered set of eigenvalues $\{\lambda_m^\epsilon, 1 \leq m < \infty\}$ of the Laplacian on Ω_ϵ with Neumann boundary conditions.

The following result regarding the second eigenvalue and eigenfunction is due to Arrieta, Hale and Han (1991).

Theorem 9. *Let $\Omega_\epsilon = \Omega_0 \cup R_{\epsilon,\eta} \cup D_\epsilon$ with $R_{\epsilon,\eta}, D_\epsilon$ defined by (14). For $\eta > (n+1)/(n-1)$, the following conditions hold:*

$$\lim_{\epsilon \to 0} \lambda_2^\epsilon = 0$$

$$\begin{cases} \lim_{\epsilon \to 0} \|\omega_2^\epsilon\|_{H^1(\Omega_0)} = 0 \\ \lim_{\epsilon \to 0} \|\omega_2^\epsilon\|_{H^2(R_{\epsilon,\eta})} = 0 \\ \lim_{\epsilon \to 0} \|\omega_2^\epsilon\|_{L^2(D_\epsilon)} = 1 \\ \lim_{\epsilon \to 0} \frac{1}{\mu(D_\epsilon)} (\int_{D_\epsilon} \omega_2^\epsilon)^2 = 1 \end{cases}$$

Furthermore, if Ω_0 is a C^∞-domain, then, for any integer $\ell \geq 1$ and any $\gamma \in (0, \gamma_0)$,

$$\lim_{\epsilon \to 0} \|\omega_2^\epsilon\|_{H^\ell(\Omega_0 \setminus S_\gamma)} = 0$$

Therefore, for any $\gamma \in (0, \gamma_0)$, the function ω_2^ϵ together with all derivatives up to order ℓ converge to zero pointwise in Ω_0 and uniformly in $\bar{\Omega}_0 \setminus S_\gamma$ as $\epsilon \to 0$.

The limit properties of the remainder of the eigenvalues and eigenfunctons is given in the following result.

Theorem 10. *Let $\Omega_\epsilon = \Omega_0 \cup R_{\epsilon,\eta} \cup D_\epsilon$ with $R_{\epsilon,\eta}, D_\epsilon$ defined by (14). For $\eta > (n+1)/(n-1)$, the following conditions hold:*

$$\lim_{\epsilon \to 0} \lambda_m^\epsilon = \lambda_{m-1}^0 \text{ for } m \geq 3$$

The corresponding eigenvectors can be chosen so that, for any sequence of positive numbers $\{\epsilon_k, 1 \leq k < \infty\}$ with $\epsilon_k \to 0$ as $k \to \infty$, there is a subsequence $\{\delta_k, 1 \leq k < \infty\}$ such that, for each $m \geq 3$, we have

$$\begin{cases} \lim_{k \to \infty} \|\omega_m^{\delta_k} - \omega_{m-1}^0\|_{H^1(\Omega_0)} = 0 \\ \lim_{\delta \to 0} \|\omega_m^{\delta_k}\|_{H^1(R_{\delta_k} \cup D_{\delta_k})} = 0 \end{cases}$$

Furthermore, if Ω_0 is a C^∞-domain, then, for any integer $\ell \geq 1$ and any $\gamma \in (0, \gamma_0)$,

$$\lim_{k \to \infty} \|\omega_m^{\delta_k} - \omega_{m-1}^0\|_{H^\ell(\Omega_0 \setminus S_\gamma)} = 0$$

Therefore, for any $\gamma \in (0, \gamma_0)$, the function $\omega_m^{\delta_k}, m \geq 3$, together with all derivatives up to order ℓ converge to ω_{m-1}^0 pointwise in Ω_0 and uniformly in $\overline{\Omega_0 \setminus S_\gamma}$ as $\epsilon \to 0$.

It is worth making a few remarks about these results. If we ignore the set $R_{\epsilon,\eta}$ and consider the eigenvalue problem on $\Omega_0 \cup D_\epsilon$, then there is no singular behavior in the eigenvalues. This is due to the fact that the only eigenvalue on the domain D_ϵ that remains bounded as $\epsilon \to 0$ is the eigenvalue zero. Theorems 7.1 and 7.2 assert that the double eigenvalue zero on the disconnected domain $\Omega_0 \cup D_\epsilon$ becomes two simple eigenvalues, zero and λ_2^ϵ with $\lambda_2^\epsilon \to 0$ as $\epsilon \to 0$ and the other eigenvalues converge to the eigenvalues of Ω_0 as $\epsilon \to 0$ provided that they remain bounded. Of course, this is under the restriction that $\eta > (n+1)/(n-1)$. If η is too small, then the eigenvalue problem on Ω_ϵ may not correspond so well to the one on the disconnected domain $\Omega_0 \cup D_\epsilon$.

Remark 11. The above result could have been stated in terms of spectral projections and then it would not be necessary to make a choice for the eigenfunctions.

Remark 12. Mixed boundary conditions as well as perturbations at a finite number of points also are discussed in Arrieta, Hale and Han (1991).

3.2. Dumbbell shaped domains. Let us now turn to the disucussion of dumbbell shaped domains. Jimbo (1988), (1989) seems to have been the first to discuss this problem in some generality for some special smooth domains in $\mathbb{R}^2$. For example, suppose that $\Omega_\epsilon = \Omega_0^L \cup \Omega_0^R \cup R_\epsilon$ is a smooth, connected domain in $\mathbb{R}^2$ for which $\Omega_0^L, \Omega_0^R, R_\epsilon$ are disjoint, Ω_0^L, Ω_0^R are smooth connected domains joined by a rectangular channel $R_\epsilon = L \times (0, \epsilon)$, $L = [0,1]$. Jimbo pointed out that the relevant limit problem should consist of the following three eigenvalue problems:

$$\Delta u = \mu u \ \text{ in } \ \Omega_0^R \cup \Omega_0^L, \quad \partial u/\partial n = 0 \ \text{ in } \ \partial\Omega_0^R \cup \Omega_0^L, \tag{15}$$

$$u_{xx} = \mu u \ \text{ in } \ L, \quad u = 0 \ \text{ in } \ \partial L. \tag{16}$$

We order the eigenvalues of the problems (15), (16) as

$$\mu_1^0 = \mu_2^0 = 0 > \mu_3^0 \geq \mu_4^0 \geq \ldots,$$

and let $\psi_1^0, \psi_2^0, \ldots$ be a corresponding set of normalized eigenfunctions. He proved the convergence of the eigenvalues and eigenfunctions on Ω_ϵ to those of (15), (16) as $\epsilon \to 0$.

Arrieta, Hale and Han (1991) considered a more general type of dumbbell shaped domain for which the connecting channel R_ϵ could have a boundary which may not even be connected. Allowing this complicated type of channel is the main difference between this situation and the one considered by Jimbo (1989) and Hale and Vegas (1984).

We now give a precise definition of the perturbed domain. Let Ω_0^L, Ω_0^R be bounded connected smooth domains such that

(H.4) There exist positive constants α, β, γ such that

$$\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : -\alpha < x < \gamma, |y| < \beta\} \cap \Omega_0^L = \{(x,y) : -\alpha < x < 0, |y| < \beta\}$$

$$\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x < \gamma + \alpha, |y| < \beta\} \cap \Omega_0^R = \{(x,y) : \gamma < x < \gamma + \alpha, |y| < \beta\}$$

(H.5) $\bar{\Omega}_0^L \cap \bar{\Omega}_0^R = \emptyset$.

(H.6) For any connected set $R_1 \subset \{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 \leq x \leq \gamma, |y| < \beta\}$, the set $\Omega_0^L \cup \Omega_0^R \cup R_1$ is a bounded connected smooth domain in $\mathbb{R}^n$.

For $\epsilon > 0$ small, if we let

$$R_\epsilon = \{(x, \epsilon y) : (x,y) \in R_1\} \tag{17}$$

and define $\Omega_\epsilon = \Omega_0^L \cup R_\epsilon \cup \Omega_0^R$, then Ω_ϵ is a bounded open connected smooth domain.

Remark 13. As noted in Remark 8, the fact that $\partial\Omega_0^L$ and $\partial\Omega_0^R$ are pieces of a hyperplane near $(0,0)$ is merely technical.

Let $\{\mu_1^\epsilon = 0 > \mu_2^\epsilon \geq \mu_3^\epsilon \geq \ldots\}$ be the ordered set of eigenvalues of the Laplacian with Neumann boundary conditions on Ω_ϵ and let $\psi_1^\epsilon, \psi_2^\epsilon, \ldots$ be a corresponding set of normalized eigenfunctions. Arrieta, Hale and Han (1991) showed that $\mu_2^\epsilon \to 0$ as $\epsilon \to 0$ and that μ_3^ϵ is negative and bounded away from zero, which generalized a result of Hale and Vegas (1984). The methods used there as well as refinements of Arrieta (1991), (1995a) yield the following theorem.

Theorem 14. *If $\Omega_\epsilon = \Omega_0^L \cup R_\epsilon \cup \Omega_0^R$, where R_ϵ is defined by (17), then the following conclusions hold for any m:*

$$\lim_{\epsilon \to 0} \mu_m^\epsilon = \mu_m^0$$

and the corresponding eigenfunctions can be chosen so that

$$\lim_{\epsilon \to 0} \|\psi_m^\epsilon - \psi_m^0\|_{H^1(\Omega_0^L \cup \Omega_0^R)} = 0.$$

Remark 15. We remark there can be many different channels and many different open sets connected by these channels. The results will be the same except there are more eigenvalue problems for the limit as $\epsilon \to 0$.

3.3. Thin domains. Hale and Raugel (1992) have considered some properties of the dynamics of reaction diffusion equations on thin domains and, as a byproduct of the investigation, also have given results on the convergence of eigenvalues and eigenfunctions of the Laplacian with mixed boundary conditions. We describe a special case of their results for a particular case of a thin domain over a line segment for Neumann boundary conditions. For a more complete and more general discussion, see Raugel (1995).

Let

$$R_\epsilon = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < G(x,\epsilon)\}, \tag{18}$$

where the function $G \in C^1([0,1] \times [0,\epsilon_0])$ and satisfies

$$G(x,0) = 0, \;\; G_0(x) = \frac{\partial G}{\partial \epsilon}(x,0) > 0. \;\; x \in [0,1]. \tag{19}$$

Let $\{\lambda_m^\epsilon, m \geq 1\}$ be the ordered set of eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions and let $\{\varphi_m^\epsilon, m \geq 1\}$ be a corresponding set of normalized eigenfunctions. Hale and Raugel (1992) show that the appropriate limit problem as $\epsilon \to 0$ is the eigenvalue problem

$$\begin{aligned} -\frac{1}{G_0(x)}(G_0(x)u_x)_x &= \lambda u \;\text{ in }\; (0,1) \\ u_x &= 0 \;\text{ at }\; x = 0,1. \end{aligned} \tag{20}$$

If $\{\lambda_m^0, m \geq 1\}$ is the ordered set of eigenvalues of (20) and $\{\varphi_m^0, m \geq 1\}$ is a corresponding set of normalized eigenfunctions, they prove that the following statement is true.

Theorem 16. *For any integer m, there are positive constants $\epsilon_0(m)$, $C(m)$ such that, for every integer $n \le m$, $0 < \epsilon \le \epsilon_0$,*

$$|\lambda_n^\epsilon - \lambda_n^0| \le C(m)|\epsilon|,$$
$$\|\varphi_n^\epsilon - (\varphi_n^\epsilon, \epsilon^{-1/2}\varphi_n^0)\epsilon^{-1/2}\varphi_n^0\|_{H^1(R_\epsilon)}^2 \le C(m)|\epsilon|,$$

where $(\cdot,\cdot)$ is the L^2-inner product.

Results of this type permit the reduction of the two dimensional boundary value problem to a problem in one dimension.

Remark 17. If $\Gamma_\epsilon = \partial R_\epsilon \cap ((\{G(0,\epsilon)\} \times (0,1)) \cup (\{G(1,\epsilon)\} \times (0,1)))$, and one assumes Dirichlet boundary conditions on Γ_ϵ with Neumann on $\partial R_\epsilon \setminus \Gamma_\epsilon$, then the same conclusions as in Theorem 16 hold if we suppose that the limit problem satisfies Dirichlet boundary conditions.

It also is possible to consider nonlinear equations on thin domains and relate the flow to a nonlinear equation on the limit domain. For example, consider the equation

$$u_t = \Delta u + f(u) \quad \text{in } R_\epsilon$$

with homogeneous Neuman boundary conditions. Under natural conditions on f, this equation defines a flow which has a compact global attractor $\mathcal{A}_\epsilon$. If $\mathcal{A}_0$ is the compact global attractor for the equation

$$v_t = \frac{1}{G_0(x)}(G_0(x)v_x)_x + f(v)$$

with Neuman boundary conditions, the set of global attractors $\{\mathcal{A}_\epsilon, \epsilon > 0\} \cup \mathcal{A}_0$ is upper semicontinuous at $\epsilon = 0$.

Prizzi and Rybakowski (2001) have considered thin domains R_ϵ for which the function $G(x,\epsilon)$ can be multivalued. More specifically, suppose that Ω is an arbitrary smooth domain in $\mathbb{R}^2$. For any $\epsilon > 0$, let $T_\epsilon : (x,y) \in \mathbb{R}^2 \mapsto (x, \epsilon y) \in \mathbb{R}^2$ and define $R_\epsilon = T(\epsilon)\Omega$. The domain R_ϵ has a smooth boundary, but it need not be a graph over the x-axis and it may not be connected. However the domain converges to a line segment on the x-axis. Prizzi and Rybakowski (2001) prove that the corresponding limit differential equation is a differential equation in one space variable over a graph with the boundary conditions at each point on the graph being uniquely determined.

With the function $G(x,\epsilon)$ satisfying (19), the domain degenerates to the line in a nice uniform way. It does not allow, for example, the boundary to oscillate rapidly as $\epsilon \to 0$. Such problems are of interest in the theory of homogenization (see, for example, Bensoussan, Lions and Papanicolaou (1978), de Giorgi and Spagnolo (1973), Kesavan (1979)). Arrieta (1991), (1995) has allowed rapid oscillations as $\epsilon \to 0$. We now describe the results of Arrieta.

Consider a fixed function $a \in C^1([0,1],(0,\infty))$, let $k \in C^1([0,1] \times (0,\epsilon_0),(0,\infty))$, $k_\epsilon(x) = k(x,\epsilon)$ such that

$$\lim_{\epsilon\to 0} \epsilon|\frac{dk_\epsilon}{dx}(x)| = 0 \text{ uniformly in } (0,1),$$

define

$$g_\epsilon = a + k_\epsilon$$

and suppose that there are positive constants c_3, c_4 and a function $b \in C^1([0,1],(0,\infty))$ such that

$$c_3 \le g_\epsilon(x) \le c_4, \quad x \in [0,1],$$
$$\lim_{\epsilon\to 0} g_\epsilon = a \text{ weakly } L^2,$$
$$\lim_{\epsilon\to 0} \frac{1}{g_\epsilon} = b \text{ weakly } L^2.$$

With this notation, define the thin domain as

$$R_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < \epsilon g_\epsilon(x)\},$$

and let

$$\Gamma_\epsilon = \{(0, x_2) : 0 \le x_2 \le \epsilon g_\epsilon(0)\} \cup \{(1, x_2) : 0 \le x_2 \le g_\epsilon(1)\}.$$

Denote by $\{\lambda_n^\epsilon, n \ge 1\}$ the ordered set of eigenvalues and $\{\varphi_n^\epsilon, n \ge 1\}$ the corresponding eigenfunctions of the eigenvalue problem

$$\begin{aligned} -\Delta\varphi &= \lambda\varphi \quad \text{in } R_\epsilon \\ \varphi &= 0 \quad \text{in } \Gamma_\epsilon \\ \frac{\partial \varphi}{\partial n} &= 0 \quad \text{in } \partial R_\epsilon \setminus \Gamma_\epsilon. \end{aligned}$$

Denote by $\{\mu_n^\epsilon, n \ge 1\}$ the ordered set of eigenvalues and $\{\psi_n^\epsilon, n \ge 1\}$ the corresponding eigenfunctions of the eigenvalue problem

$$\begin{aligned} -\frac{1}{g_\epsilon}(g_\epsilon \psi_x)_x &= \mu\psi \quad \text{in } (0,1) \\ \psi &= 0 \quad \text{at } x = 0, 1. \end{aligned} \tag{21}$$

The following result is due to Arrieta (1991), (1995a).

Theorem 18. *Let $(\lambda_n^\epsilon, \varphi_n^\epsilon)$, $(\mu_n^\epsilon, \psi_n^\epsilon)$ be the eigenpairs defined above. If $\eta_\epsilon = \sup\{\epsilon|g_\epsilon'(x)| : x \in [0,1]\}$, then, for any integer m, there are positive constants $\epsilon_0(m), C(m)$ such that, for $n \le m$, $\epsilon \in (0, \epsilon_0)$,*

$$\begin{aligned} 0 \le \mu_n^\epsilon - \lambda_n^\epsilon &\le C(m)\eta_\epsilon^2, \\ \|\varphi_n^\epsilon - (\varphi_n^\epsilon, \epsilon^{-1/2}\psi_n^\epsilon)_{L^2(R_\epsilon)}\epsilon^{-1/2}\psi_n^\epsilon\|_{H^1(R_\epsilon)}^2 &\le C(m)\eta_\epsilon^2. \end{aligned}$$

Theorem 18 gives the reduction of the problem to a one dimensional problem which still depends upon ϵ. It remains to find the appropriate limit problem for (21) as $\epsilon \to 0$. Arrieta (1991), (1995) shows that it should be the following eigenvalue problem

$$\begin{aligned} -\frac{1}{a}(\frac{1}{b}\xi_x)_x &= \nu\psi \quad \text{in } (0,1) \\ \xi &= 0 \quad \text{at } x = 0, 1. \end{aligned}$$

If we denote by $\{\nu_n, n \ge 1\}$ the ordered set of eigenvalues and $\{\xi_n, n \ge 1\}$ the corresponding normalized eigenfunctions of this eigenvalue problem, then we have the following result.

Theorem 19. *The following statements are true:*

$$\begin{aligned} &\lim_{\epsilon \to 0} \mu_m^\epsilon = \nu_m, \quad m \ge 1, \\ &\lim_{\epsilon \to 0} [\psi_m^\epsilon - D_\epsilon(\psi_m^\epsilon, \xi_m)\xi_m] = 0 \ \textit{strongly in } L^2((0,1), \ \textit{weakly in } H^1((0,1), \end{aligned}$$

where $D_\epsilon(f, g) = \int_0^1 g_\epsilon f g$.

An example for which the above result applies is the function $g_\epsilon(x) = 1 + \rho \sin(x\epsilon^{-\alpha})$, where $\rho \in (0,1)$, $\alpha > 0$. In this case, $a \equiv 1$ and $b \equiv (1 - \rho^2)^{-1/2} > 1$ and the eigenvalues of the two dimensional problem are close to the eigenvalues of the operator $(-1/b)\partial_x^2$ with Dirichlet boundary conditions.

3.4. General variations. For Neumann boundary conditions, Lobo-Hidalgo and Sanchez-Palencia (1979) proved that, if $\Omega_0 \subset \Omega_\epsilon$ and $m_n(\Omega_\epsilon \setminus \Omega_0) \to 0$ as $\epsilon \to 0$, where m_n is the Lebesgue measure in $\mathbb{R}^n$, then every point of the spectrum of $-\Delta_N^{\Omega_0}$ is approximated by points of the spectrum of $-\Delta_N^{\Omega_\epsilon}$, whereas the contrary statement is false in general; that is, there may be situations for which there are accumulation points of the spectrum of $-\Delta_N^{\Omega_\epsilon}$ which are not in the spectrum of $-\Delta_N^{\Omega_0}$.

Arrieta (1995b) has considered perturbed domains in this general setting and has given conditions for which one has convergence of eigenvalues and eigenfunctions. The conditions are stated in such a way as to lead to proofs of the results in this section as well as many more. To be specific, let $\Omega_0 \subset \Omega_\epsilon$ and $m_n(\Omega_\epsilon \setminus \Omega_0) \to 0$ as $\epsilon \to 0$, let $R_\epsilon = \Omega_\epsilon \setminus \bar{\Omega}_0$, $\Gamma_\epsilon = \partial\Omega_0 \cap \partial R_\epsilon$. For functions $V_\epsilon \in L^\infty(\Omega_\epsilon)$ with $\|V_\epsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C$ for some $C > 0$ independent of ϵ, consider the Schrödinger operators

$$\begin{aligned} A_N^{\Omega_\epsilon} &= -\Delta_N^{\Omega_\epsilon} + V_\epsilon \\ A_N^{\Omega_0} &= -\Delta_N^{\Omega_0} + V_\epsilon \\ A_{D(\Gamma_\epsilon)N}^{R_\epsilon} &= -\Delta_{D(\Gamma_\epsilon)N}^{R_\epsilon} + V_\epsilon \end{aligned}$$

where the superscript denotes the domain on which the operator is applied, the subscript N denotes homogeneous Neumann boundary conditions on the domain and the subscript $D(\Gamma_\epsilon)N$ denotes homogeneous Dirichlet conditions on Γ_ϵ and Neumann conditions on the remainder of the domain. We let $H^1_{\Gamma_\epsilon}(R_\epsilon)$ denote the space of H^1 functions which respect the Dirichlet boundary conditions on Γ_ϵ.

Without loss of generality, we may suppose that $V_\epsilon \geq 0$.

The objective is to show that eigenvalues and eigenfunctions of $A_N^{\Omega_\epsilon}$ behave as the eigenvalues and eigenfunctions of $A_N^{\Omega_0}$ and $A_{D(\Gamma_\epsilon)N}^{R_\epsilon}$. To achieve this, the following hypothesis is assumed:

(H) If $u_\epsilon \in H^1(\Omega_\epsilon)$ with $\|u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C_1$ for some positive constant C_1 independent of ϵ, then there exists $\bar{u}_\epsilon \in H^1_{\Gamma_\epsilon}(R_\epsilon)$ such that

$$\begin{aligned} &\lim_{\epsilon \to 0} \|u_\epsilon - \bar{u}_\epsilon\|_{L^2(R_\epsilon)} = 0 \\ &\|\nabla \bar{u}_\epsilon\|_{L^2(R_\epsilon)} \leq \|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon)} + o(1) \end{aligned}$$

Let $\{\lambda_{m,\Omega_\epsilon}, m \geq 1\}$, $\{\lambda_m(\Omega_0, \epsilon), m \geq 1\}$, $\{\tau_m(R_\epsilon), m \geq 1\}$ be the ordered set of eigenvalues counting multiplicity of the operators $A_N^{\Omega_\epsilon}$, $A_N^{\Omega_0}$, $A_{D(\Gamma_\epsilon)N}^{R_\epsilon}$ and let $\{\varphi_{m,\Omega_\epsilon}, m \geq 1\}$, $\{\psi_m(\Omega_0, \epsilon), m \geq 1\}$, $\{\psi_m(R_\epsilon), m \geq 1\}$ be a corresponding set of orthonormal eigenvectors. Let

$$\{\lambda_m^\epsilon, m \geq 1\} = \{\lambda_m(\Omega_0, \epsilon), m \geq 1\} \cup \{\tau_m(R_\epsilon), m \geq 1\}$$

be ordered (counting multiplicity), and define

$$\begin{aligned} &\varphi_m^\epsilon = \psi_i(\Omega_0, \epsilon) \text{ in } \Omega_0, \ = 0 \text{ in } R_\epsilon \text{ if } \lambda_m^\epsilon = \lambda_i(\Omega_0, \epsilon), \\ &\varphi_m^\epsilon = 0 \text{ in } \Omega_0, \ \psi_j(R_\epsilon) \text{ in } R_\epsilon, \text{ if } \lambda_m^\epsilon = \tau_j(R_\epsilon), \end{aligned}$$

Obviously, we have $\varphi_m^\epsilon \in H^1(\Omega_0) \cup H^1_{\Gamma_\epsilon}(R_\epsilon)$.

We say that $\sigma_\epsilon > 0$ divides the spectrum if there are positive constants δ, M, N such that, for $\epsilon \in (0, \epsilon_0)$, we have

$$[\sigma_\epsilon - \delta, \sigma_\epsilon + \delta] \cap \{\lambda_m^\epsilon, m \geq 1\} = \emptyset$$
$$\sigma_\epsilon \leq M$$
$$N(\sigma_\epsilon) \equiv \text{ Card}\{\lambda_i^\epsilon : \lambda_i^\epsilon \leq \sigma_\epsilon\} \leq N$$

If σ_ϵ divides the spectrum, then we can define the projection operator

$$P_{\sigma_\epsilon} : L^2(\Omega_\epsilon) \to [\varphi_1^\epsilon, \dots, \varphi_{N(\sigma_\epsilon)}^\epsilon]$$
$$g \mapsto \Sigma_{i=1}^{N(\sigma_\epsilon)} (g, \varphi_i^\epsilon)_{L^2(\Omega_\epsilon)} \varphi_i^\epsilon.$$

With this notation, Arrieta (1995b) proved the following result.

Theorem 19. *If condition (**H**) is satisfied, then*

(i) $\lim_{\epsilon\to 0}(\lambda_{m,\Omega_\epsilon} - \lambda_m^\epsilon) = 0, \quad m \geq 1,$

(ii) $\lim_{\epsilon\to 0} \|\varphi_{r_\epsilon,\Omega_\epsilon} - P_{\sigma_\epsilon}\varphi_{r_\epsilon,\Omega_\epsilon}\|_{H^1(\Omega_\epsilon\cup R_\epsilon)} = 0, \quad r_\epsilon = 1, 2, \dots, N(\sigma_\epsilon),$ *for any* σ_ϵ *which divides the spectrum.*

If we impose conditions on Ω_0 and Ω_ϵ which will ensure that the eigenvalues λ_m^ϵ do not accumulate at any finite point in $(0, \infty)$, then more can be proved. In fact, Arrieta (1995) obtained the following result.

Theorem 20. *If the condition*

(**C**) *For any* $k \geq 1$*, there exists an integer* L_k *such that Card*$\{\lambda_m^\epsilon : \lambda_m^\epsilon \leq k\} \leq L_k$*,*

*is satisfied, then condition (**H**) is equivalent to the statements (i) and (ii) of Theorem 19.*

Arrieta (1995) shows that these results include all of the examples of the previous subsections.

References

Abraham, R. and J. Robbin (1967) *Transversal Mappings and Flows.* Benjamin.

Arrieta, J. (1991), Spectral properties of Schrödinger operators under perturbation of the domain. Ph.D. Dissertation, Georgia Tech.

Arrieta, J. (1995a) Neumann eigenvalue problems on exterior prturbations of the domain. *J. Differential Equations* **118**, 54-103.

Arrieta, J. (1995b) Rates of eigenvalues on a dumbbell domain. Simple eigenvalue case. *Trans. Am. Math. Soc.* **347**, 3503-3531.

Arrieta, J. (1996) Elliptic equations, principal eigenvalues and dependence on the domain. *Comm. PDE***21**, 971-991.

Arrieta, J. (1997) Domain dependence of elliptic operators in divergence form. *Resenhas IME-USP***3**, 107-123.

Arrieta, J. (2000) Spectral behavior and upper semicontinuity of attractors. In *Differential Equations, Vol.1* (Editors, B. Fiedler, K. Gröger, J. Sprekels), World Scientific. Notes on thin domains, 11/96.

Arrieta, J., J.K. Hale and W. Han (1991) Eigenvalue problems for nonsmoothly perturbed domains. *J. Differential Equations* **91**, 24-52.

Babuška, I. and R. Vyborny (1965) Continuous dependence of eigenvalues on the domains. *Czech. Math. J.* **15**, 169-178.

Beale, J.T. (1973) Scattering frequencies of resonators. *Pure Appl. Math.* **26**, 549-563.

Bensoussan, A., J.L. Lions and G. Papanicolaou (1978), *Asymptotic Methods in Periodic Structures*, North-Holland.
Berestycki, H., L. Nirenberg and S.R.S. Varadhan (1994) The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure. Appl. Math.* **47**, 47-92.
Brown, R., P.D. Hislop and A. Martinez (1995) Eigenvalues and resonances for domains with tubes: Neumann boundary conditions. *J. Differential Equations* **115**, 458-476.
Chavel, I. and E.A. Feldman (1978) Spectra of domains in compact manifolds. *J. Funct. Anal.* **30**, 196-222.
Courant, R. and D. Hilbert (1937) *Methods of Mathematical Physics, Vol. 1*, Wiley-Interscience, New York, 1953. German edition, 1937.
Ciuperca, I.S. (1998) Spectral properties of Schrödinger operators on domains with varying order of thinness. *J. Dyn. Diff. Eqns.* **10**, 73-108.
Dancer, E.N. (1988) The effect of domain shape on the number of positive solutions of certain nonlinear equations. *J. Differential Equations* **74**.
Dancer, E.N. (1990) The effect of domain shape on the number of positive solutions of certain nonlinear equations, II. *J. Differential Equations* **87**, 316-339.
Dancer, N. (1996) Domain variation for certain sets of solutions with applications. *Top. Meth. Nonlinear Anal.* Preprint.
Dancer, E.N. and D. Daners (1997) Domain perturbation of elliptic equations subject to Robin boundary conditions. *J. Differential Equations* **74)**, 86-132.
Daners, D. (1996) Domain perturbation for linear and nonlinear parabolic equations. *J. Differential Equations* **129**, 358-402.
de Giorgi, E. and S. Spagnolo (1973) Sulla convergenzia degli integrali dell'energia per operatori ellitici del 2 deg ordine. *Boll. U. Mat. Ital.* **8**, 391-411.
Garabedian, P. and M. Schiffer (1952), Convexity of domain functionals. *J. Analyse Math.* **2**, 281-368.
Hadamard, J. (1908), Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. *Œuvres de J. Hadamard* **2**(1968).
Hale, J.K., and G. Raugel (1992) Convergence in gradient-like systems and applications. *ZAMP* **43**, 63-124.
Hale, J.K., and G. Raugel (1995) A reaction-diffusion equation on a thin L-shaped domain. *Proc. Roy. Soc. Edinburgh* **125A**, 283-327.
Hale, J.K., and J. Vegas (1984) A nonlinear parabolic equation with varying domain. Arch. Rat. Mech. Anal. **2**, 99-123.
Hempel, R., I. Seco and B. Simon (1991) The essential spectrum of Neumann Laplacians on some bounded singular domains. *J. Funct. Anal.* **102**, 448-483.
Henry, D. (1985) Perturbation of the boundary for boundary value problems for partial differential equations. *Sem. Brasileiro Anal. ATS* **22**.
Henry, D. (1987) Generic properties of equilibrium solutions by perturbation of the boundary. In *Dynamics of Infinite Dimensional Systems* (Ed. Chow and Hale), NATO ASI Series **F 37**, 129-139. Springer-Verlag.
Henry, D. (1996) *Perturbation of the boundary in partial differential equations.* Lecture Notes (1996). To appear in Cambridge University Press.
Hislop, P.D., and A. Martinez (1991) Scattering resonances of a Helmholz resonator. *Indiana U. Math. J.* **40**, 767-788.
Jimbo, S. (1988) Singular perturbation of domains and the semilinear elliptic equation. *J. Fac. Sci. Univ. Tokyo* **35**, 27-76.

Jimbo, S. (1988) Singular perturbation of domains and the semilinear equation, II. *J. Differential Equations* **75**, 264-289.
Jimbo, S. (1989) The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary conditions. *J. Differential Equations* **77**, 322-350.
Jimbo, S. (1993) Perturbation formula of eigenvalues in a singularly perturbed domain. *J. Math. Soc. Japan* **45**, 339-356.
Jimbo, S., and Y. Morita (1992) Remarks on the behavior of certain eigenvalues on a singularly perturbed domain with several thin channels. *Comm. PDE* **17**, 523-552.
Joseph, D. (1967) Parameter and domain dependence. *Arch. Rat. Mech. Anal.* **24**.
Kesavan, S. (1979) Homogenization of elliptic eigenvalue problems: Part I. *Appl. Math. Optim.* **5**, 153-167.
Lobo-Hidalgo, M., and E. Sanchez-Palencia (1979) Sur certaines propriétés spectrales des perturbations du domaine dans les problèmes aux limites. *Comm. PDE* **4**, 1085-1098.
Lopez-Gomez, J. (1996) The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems. *J. Differential Equations* **127**, 263-294.
Micheletti, A.M. (1972) Perturbazione dello spettro dell'operatore di Laplace, in relazione ad una variazione del campo. *Ann. Scuola Norm. Sup. Pisa XXVI FASC. I*, 151-169.
Micheletti, A.M. (1973a) Metrica per famiglie de domini limitati e proprietá generiche degli autovalori. *Ann. Scuola Norm. Sup. Pisa XXVI fas. III*, 633-694.
Micheletti, A.M. (1973b) Perturbazione dello spettro di un operatoire ellittico tipo variazionale, in relazione ad una variazione del campo. *Annali Mat. Pura Appl.* **(IV)(XCVII)**, 261-281.
Micheletti, A.M. (1976) Perturbazione dello spettro di un operatore ellittico di tipo variazionale, in relazione ad una variazione del campo, II. *Recerche Mat.* **25**, 187-200.
Ozawa, S. (1981) Singular variations of domains and eigenvalues of the Laplacian. *Duke Math. J.* **48**, 769-778.
Ozawa, S. (1992) Singular variation of domain and spectra of the Laplacian with small Robin conditional boundary, I. *Osaka J. Math.* **29**, 837-850.
Ozawa, S., and S. Roppongi (1995) Singular variation of domains and continuity property of eigenfunction for some semi-linear elliptic equations, II. *Kodai Math. J.* **18**, 315-327.
Pereira, A.L. (1989) Auto valores do Laplaciano im regiõ es simé tricas. Ph.D. Thesis, Univ. São Paulo, Brasil.
Pereira, A.L. (1996) Appendix to Henry (1996).
Pereira, A.L. (1995) Eigenvalues of the Laplacian on symmetric regions. *NoDEA* **2**, 63-109.
Peetre, J. (1980) On Hadamard's variational formula. *J. Differential Equations* **36**, 335-346.
Prizzi, M. and K.P. Rybakowski (2001) The effect of domain squezzing upon the dynamics of reaction-diffusion equations. *J. Differential Eqns.* **173**, 271-320.
Rauch, J., and M. Taylor (1975) Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.* **18**, 27-59.
Raugel, G. (1995) Dynamics of partial differential equations on thin domains. In *Dynamical Systems (Ed. R. Johnson, Lect. Notes Math.* **1609**, 208-315.
Rayleigh, J.W.S. (1894), *Theory of Sound*, Dover (1945).
Saut, J.C., and R. Temam (1979) Generic properties of nonlinear boundary value problems. *Comm. PDE* **4**, 293-319.
Serrin, J. (1971) A symmetry problem of potential theory. *Arch. Rat. Mech. Anal.* **43**, 304-318.
Stummel, F. (1976) Perturbation of domains in elliptic boundary value problems. In *Applications of Methods of Functional Analysis to Problems in Mechanics (Eds. P. Germain and B. Nayroles). Lecture Notes Math.* **503**, 110-135.
Uhlenbeck, K. (1972) Eigenfunctions of Laplace operators. *Amer. J. Math.* **98**, 1073-1076.

Vegas, J. (1990) A functional-analytic framework for the study of elliptic operators on variable domains. *Proc. Roy. Soc. Edinburgh* **116A**, 367-380.
Ward, M.J., W.D. Henshaw and J.B. Keller (1993) Summing logarithmic expansions for singularly perturbed eigenvalue problems. *SIAM J. Appl. Math.* **53**, 799-828.
Ward, M.J., and J.B. Keller (1991) Nonlinear eigenvalue problems under strong localized perturbations with applications to chemical reactors. *Stud. Appl. Math.* **85**, 1-28.
Ward, M.J., and J.B. Keller (1993) Strong localized perturbations of eigenvalue problems. *SIAM J. Appl. Math.* **53**, 770-798