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A SCHWARZ-PICK LEMMA FOR THE MODULUS OF HOLOMORPHIC MAPPINGS FROM THE POLYDISK INTO THE UNIT BALL

SHAOYU DAI AND YIFEI PAN

Abstract. In this paper we prove a Schwarz-Pick lemma for the modulus of holomorphic mappings from the polydisk into the unit ball. This result extends some related results.


Keywords: holomorphic mappings; Schwarz-Pick lemma; the polydisk.

1. Introduction

Let $D$ be the unit disk in $\mathbb{C}$, $D^n$ and $B_n$ be the polydisk and the unit ball in $\mathbb{C}^n$ respectively. For $z = (z_1, \ldots, z_n)$ and $z' = (z'_1, \ldots, z'_n) \in \mathbb{C}^n$, denote $(z, z') = z_1\overline{z'_1} + \cdots + z_n\overline{z'_n}$ and $|z| = (z, z)^{1/2}$. Let $\Omega_{X,Y}$ be the class of all holomorphic mappings $f$ from $X$ into $Y$, where $X$ is a domain in $\mathbb{C}^n$ and $Y$ is a domain in $\mathbb{C}^m$. For $f \in \Omega_{X,Y}$ and $j = 1, \ldots, n$, define

$$|\nabla f|(z) = \sup_{\beta \in \mathbb{C}, |\beta|=1} \left( \lim_{t \to 0^+} \frac{|f|(z+t\beta)-|f|(z)}{t} \right), \quad z \in X; \tag{1.1}$$

$$|\nabla_j f|(z) = \sup_{\beta \in \mathbb{C}, |\beta|=1} \left( \lim_{t \to 0^+} \frac{|f|(z, z_1, \ldots, z_{j-1}, z_j+t\beta, z_{j+1}, \ldots, z_n)-|f|(z)}{t} \right), \quad z \in X, \tag{1.2}$$

where $f = (f_1, \ldots, f_m)$, $|f| = (|f_1|^2 + \cdots + |f_m|^2)^{1/2}$ and $z = (z_1, \ldots, z_n)$. Some calculation for $|\nabla f|$ and $|\nabla_j f|$ will be given in Section 2.

For $f \in \Omega_{D,D}$, the classical Schwarz-Pick lemma says that

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in D. \tag{1.3}$$

This inequality does not hold for $f \in \Omega_{D,B_m}$ with $m \geq 2$. For instance, the mapping $f(z) = \frac{1}{\sqrt{2}}(z, 1)$ satisfies

$$|f'(0)| = \sqrt{1-|f(0)|^2} > 1 - |f(0)|^2.$$

However Pavlović [3] found that (1.3) can also be written as

$$|\nabla f|(z) \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in D, \tag{1.4}$$

since (2.3). In [3], Pavlović proved that this form (1.4) can be extended to $\Omega_{D,B_m}$ and obtained the same inequality for $f \in \Omega_{D,B_m}$. Recently, we [1] proved that the form (1.4) also can be extended to $\Omega_{B_n,B_m}$ and obtained the following inequality for $f \in \Omega_{B_n,B_m}$:

$$|\nabla f|(z) \leq \frac{1-|f(z)|^2}{1-|z|^2}, \quad z \in B_n. \tag{1.5}$$

In view of the above results, it is interesting for us to consider that if there are some similar results for $f \in \Omega_{D^n,B_m}$. }

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For $f \in \Omega_{D^n, B_2}$, it is well known [1, 2] that
\[
\sum_{j=1}^{n} (1 - |z_j|^2)|f'_{z_j}(z)| \leq 1 - |f(z)|^2
\] (1.6)
for any $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$. This inequality does not hold for $f \in \Omega_{D^n, B_m}$ with $m \geq 2$. For instance, the mapping $f(z) = \frac{1}{\sqrt{3}}(z_1, z_2 + 0.1) \in \Omega_{D^2, B_2}$ satisfies
\[
\sum_{j=1}^{2} |f'_{z_j}(0)| = \frac{2}{\sqrt{3}} > 1 - |f(0)|^2.
\]
Similarly to (1.4), we find that (1.6) can be written as
\[
\sum_{j=1}^{n} (1 - |z_j|^2)|\nabla_j f|(z) \leq 1 - |f(z)|^2
\] (1.7)
for any $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$, since (2.5). In view of (1.4) and (1.5), the obvious question is that if the form (1.7) can not completely be extended to $\Omega_{D^n, B_m}$ with $m \geq 2$. The following example shows that the form (1.7) can not completely be extended to $\Omega_{D^n, B_m}$ with $m \geq 2$: the mapping $f(z) = \frac{1}{\sqrt{2}}(z_1, z_2) \in \Omega_{D^2, B_2}$ satisfies
\[
\sum_{j=1}^{2} |\nabla_j f|(0) = \sqrt{2} > 1 - |f(0)|^2,
\]
since $f(0) = 0$ and $|\nabla_j f|(0)| = |f'_{z_j}(0)|$ for $j = 1, 2$ by (2.4). However we find that the form (1.7) holds for $f \in \Omega_{D^n, B_m}$ at the point $z \in \mathbb{D}^n$ with $f(z) \neq 0$. Precisely:

**Theorem 1.** Let $f : \mathbb{D}^n \rightarrow \mathbb{B}_m$ be a holomorphic mapping with $m \geq 2$. Then
\[
\sum_{j=1}^{n} (1 - |z_j|^2)|\nabla_j f|(z) \leq 1 - |f(z)|^2, \quad \text{if } f(z) \neq 0
\] (1.8)
and
\[
\sum_{j=1}^{n} (1 - |z_j|^2)^2|\nabla_j f|(z)^2 \leq 1, \quad \text{if } f(z) = 0
\] (1.9)
for any $z = (z_1, \ldots, z_n) \in \mathbb{D}^n$.

The above theorem is the main result in this paper. Note that the inequality in (1.9) always holds whether if $f(z) = 0$ or $f(z) \neq 0$. When $f(z) \neq 0$, there is a better inequality, which is (1.8). Theorem 1 is coincident with (1.5) when $n = 1$. In addition, (1.8) and (1.9) are sharp. For example, the mapping $f(z) = \frac{1}{\sqrt{2}}(\frac{z_1}{1 - z_1}, \frac{z_2}{1 - z_2}) \in \Omega_{D^2, B_2}$ satisfies the equality in (1.8) at $z = 0$; the mapping $f(z) = \frac{1}{\sqrt{2}}(z_1, z_2) \in \Omega_{D^2, B_2}$ satisfies the equality in (1.9) at $z = 0$.

In Section 2, some calculation for $|\nabla f|$ and $|\nabla_j f|$ will be given. In Section 3, we will give the proof of Theorem 1 and some remarks for the equality cases in Theorem 1.

**2. SOME CALCULATION FOR $|\nabla f|$ AND $|\nabla_j f|$**

For $f \in \Omega_{X,Y}$ with $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$, by (1.11) we know that if $|f|(z) \neq 0$ then $f$ is $\mathbb{R}$-differentiable at $z$ and $\nabla f$ is the ordinary gradient; if $|f|(z) = 0$ then $f$ is not $\mathbb{R}$-differentiable at $z$ and $\nabla f$ is not the ordinary gradient. From Section 2 in [1], we have the following (2.1)-(2.3).

For $f \in \Omega_{X,Y}$,
\[
|\nabla f|(z) = \begin{cases} 
\frac{1}{|f(z)|} \left| \langle f'_{z_1}(z), f(z) \rangle, \ldots, \langle f'_{z_n}(z), f(z) \rangle \rangle \right|, & \text{if } f(z) \neq 0; \\
\sup_{\beta \in \mathbb{C}^n, |\beta|=1} |Df(z) \cdot \beta|, & \text{if } f(z) = 0,
\end{cases} 
\] (2.1)
where \( z = (z_1, \cdots, z_n) \in X \) and \( Df(z) \cdot \beta \) is the Fréchet derivative of \( f \) at \( z \) in the direction \( \beta \). Then for \( f \in \Omega_{X,Y} \) with \( X \subset \mathbb{C} \),

\[
|\nabla f|(z) = \begin{cases} \frac{1}{|f(z)|} |\langle f'(z), f(z) \rangle|, & \text{if } f(z) \neq 0; \\ |f'(z)|, & \text{if } f(z) = 0. \end{cases} \tag{2.2}
\]

In particular, for \( f \in \Omega_{X,Y} \) with \( X \subset \mathbb{C} \) and \( Y \subset \mathbb{C} \),

\[
|\nabla f|(z) = |f'(z)|. \tag{2.3}
\]

Then by (2.2) and (2.2), we get that for \( f \in \Omega_{X,Y} \) and \( j = 1, \cdots, n \),

\[
|\nabla_j f|(z) = \begin{cases} \frac{1}{|f(z)|} |\langle f'_j(z), f(z) \rangle|, & \text{if } f(z) \neq 0; \\ |f'_j(z)|, & \text{if } f(z) = 0, \end{cases} \tag{2.4}
\]

where \( z = (z_1, \cdots, z_n) \in X \). Note that for the case that \( f(z) \neq 0 \), if \( f'_j(z) \) and \( f(z) \) are collinear, then \( |\nabla_j f|(z) = |f'_j(z)| \); if not, then \( |\nabla_j f|(z) \neq |f'_j(z)| \). In particular, for \( f \in \Omega_{X,Y} \) with \( Y \subset \mathbb{C} \),

\[
|\nabla_j f|(z) = |f'_j(z)|. \tag{2.5}
\]

### 3. Proof of Theorem 1

First we give one lemma.

**Lemma 1.** Let \( f(z) = \sum_{\alpha} a_{\alpha} z^\alpha \in \Omega_{B^n, B_m}, \) where \( z = (z_1, \cdots, z_n) \in \mathbb{C}^n, \) \( \alpha = (\alpha_1, \cdots, \alpha_n), \) \( z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n} \), \( f = (f_1, \cdots, f_m), \) \( f_j(z) = \sum_{\alpha} a_{j,\alpha} z^\alpha \) and \( a_{\alpha} = (a_{1,\alpha}, \cdots, a_{m,\alpha}). \) Then

\[
\sum_{\alpha} |a_{\alpha}|^2 \leq 1. \tag{3.1}
\]

**Proof.** For \( 0 < \sigma < 1 \), we have

\[
1 \geq \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(\sigma e^{i\theta_1}, \cdots, \sigma e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n
= \frac{1}{(2\pi)^n} \sum_{j=1}^m \int_0^{2\pi} \cdots \int_0^{2\pi} |f_j(\sigma e^{i\theta_1}, \cdots, \sigma e^{i\theta_n})|^2 d\theta_1 \cdots d\theta_n
= \sum_{j=1}^m \sum_{\alpha} |a_{j,\alpha}|^2 \sigma^{2|\alpha|}
= \sum_{\alpha} |a_{\alpha}|^2 \sigma^{2|\alpha|},
\]

where \( |\alpha| = \sum_{j=1}^n \alpha_j \). Letting \( \sigma \to 1 \) gives (3.1). \( \square \)

Now we give the proof of Theorem 1.

**Proof of Theorem 1.** First we prove the case that \( z = 0 \).

Therefore we need to prove that

\[
\begin{cases} \sum_{j=1}^n |\nabla_j f|(0)| \leq 1 - |f(0)|^2, & \text{if } f(0) \neq 0; \\ \sum_{j=1}^n |\nabla_j f|(0)|^2 \leq 1, & \text{if } f(0) = 0. \end{cases} \tag{3.2}
\]

By (2.4), it suffices to prove that

\[
\sum_{j=1}^n \left| \frac{f'_j(0)}{f(0)} \right| \leq 1 - |f(0)|^2, \text{ if } f(0) \neq 0 \tag{3.3}
\]
\[ \sum_{j=1}^{n} |f_{z_j}'(0)|^2 \leq 1, \quad \text{if} \quad f(0) = 0. \] (3.4)

Obviously, (3.4) holds by Lemma 1. For (3.3), let
\[ h(z) = \left\langle f(z), \frac{f(0)}{|f(0)|} \right\rangle, \quad z \in \mathbb{D}^n. \]
Then \( h(z) \) is a holomorphic function from \( \mathbb{D}^n \) into \( \mathbb{D} \), \( h(0) = |f(0)| \), and for \( j = 1, \cdots, n \),
\[ h_{z_j}'(0) = \left\langle f_{z_j}'(0), \frac{f(0)}{|f(0)|} \right\rangle, \quad (3.5) \]
where \( z = (z_1, \cdots, z_n) \). Applying (1.6) to \( h \) and by (3.5) we get
\[ \sum_{j=1}^{n} \left| \left\langle f_{z_j}'(0), \frac{f(0)}{|f(0)|} \right\rangle \right| = \sum_{j=1}^{n} |h_{z_j}'(0)| \leq 1 - |h(0)|^2 \]
\[ = 1 - |f(0)|^2. \]

Then (3.3) is proved. Therefore (3.2) is proved.

Now we prove the case that \( z = p \neq 0 \).

Let \( p = (p_1, \cdots, p_n) \) and
\[ g(w) = f(\varphi(w)), \quad w = (w_1, \cdots, w_n) \in \mathbb{D}^n, \]
where \( \varphi(w) = (\varphi_1(w_1), \cdots, \varphi_n(w_n)) \), \( \varphi_j(w_j) = \frac{p_j - w_j}{p_j w_j} \) for \( j = 1, \cdots, n \). Then \( g(w) \) is a holomorphic mapping from \( \mathbb{D}^n \) into \( \mathbb{B}_m \), \( g(0) = f(p) \), and for \( j = 1, \cdots, n \),
\[ g_{w_j}'(0) = f_{z_j}'(p)(-1 + |p_j|^2). \] (3.6)

For the case that \( f(p) \neq 0 \), applying (3.3) to \( g \) and by (2.4), (3.6) we get
\[ \sum_{j=1}^{n} (1 - |p_j|^2) |\nabla_j |f(\cdot)|^2 = \sum_{j=1}^{n} (1 - |p_j|^2) \left| \left\langle f_{z_j}'(p), \frac{f(p)}{|f(p)|} \right\rangle \right| \]
\[ = \sum_{j=1}^{n} \left| \left\langle g_{w_j}'(0), \frac{g(0)}{|g(0)|} \right\rangle \right| \leq 1 - |g(0)|^2 \]
\[ = 1 - |f(p)|^2. \]

For the case that \( f(p) = 0 \), applying (3.4) to \( g \) and by (2.4), (3.6) we get
\[ \sum_{j=1}^{n} (1 - |p_j|^2)^2 |\nabla_j |f(\cdot)|^2 = \sum_{j=1}^{n} (1 - |p_j|^2)^2 |f_{z_j}'(p)|^2 \]
\[ = \sum_{j=1}^{n} |g_{w_j}'(0)|^2 \leq 1. \]

Then the theorem is proved. \( \square \)

In the following, we give some remarks for the equality cases in Theorem 1.

**Remark 1.** When \( n = 1 \), (1.8) and (1.9) reduce to (1.5). The equality case in (1.5) has been discussed in [1].
Remark 2. When \( n \geq 2 \), if the equality in (1.9) holds at some point \( p = (p_1, \cdots, p_n) \), then the structure of the expression of \( f \) will be controlled. Precisely:

\[
f(z) = \sum_{j=1}^{n} f'_j(p)\left(-1 + |p_j|^2\right)\frac{p_j - z_j}{1 - \overline{p_j}z_j}, \quad z \in \mathbb{D}^n,
\]

which is obvious by the proof of Theorem 1, Lemma 1 and (2.4).

Remark 3. When \( n \geq 2 \), if the equality in (1.8) holds at some point \( p = (p_1, \cdots, p_n) \), then the following discussion shows that the equality at \( p \) is not enough to control the structure of the expression of \( f \). By the proof of Theorem 1, we know that the key to the extremal problem of (1.8) at \( z = 0 \) is to solve the extremal problem of (1.6) at \( z = 0 \). That is: for \( h \in \Omega_{\mathbb{D}^n, \mathbb{D}} \), if \( \sum_{j=1}^{n} |h'_j(0)| = 1 - |h(0)|^2 \), then what the structure of the expression of \( h \) is. By the proof of (1.6) in [4], we only need to consider this problem: for \( h \in \Omega_{\mathbb{D}^n, \mathbb{D}} \) with \( h(0) = 0 \), if \( \sum_{j=1}^{n} |h'_j(0)| = 1 \), then what the structure of the expression of \( h \) is. However, the following examples show that the condition \( \sum_{j=1}^{n} |h'_j(0)| = 1 \) cannot control the higher order terms in the expansion of \( h \). Consequently, the structure of the expression of \( h \) cannot be controlled.

Examples:

\[
g(z) = \frac{1}{2}z_1 + \frac{1}{2}z_2 \in \Omega_{\mathbb{D}^2, \mathbb{D}}; \quad g'(z) = \frac{\frac{1}{2}z_1 + \frac{1}{2}z_2 - z_1z_2}{1 - \frac{1}{2}z_1 - \frac{1}{2}z_2} \in \Omega_{\mathbb{D}^2, \mathbb{D}}.
\]

Although the above two functions satisfy \( g(0) = g'(0) = 0 \), \( g'_1(0) = g'(z_1)(0) \), \( g'_2(0) = g'(z_2)(0) \) and \( \sum_{j=1}^{n} |g'_j(0)| = \sum_{j=1}^{n} |g'(z_j)(0)| = 1 \), the expression of \( g \) has no higher order terms and the expression of \( g' \) has some higher order terms.

References


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