Czech Technical University in Prague
Faculty of Electrical Engineering
Department of Mathematics

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Technique of operator algebras in quantum structures

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Contents

1 Introduction

Theory of operator algebras is a vast discipline with a strong impact to many other branches of modern mathematics such as noncommutative geometry, quantum probability, theory of operator spaces, etc. Moreover, operator algebras provide the natural mathematical framework for classical as well as quantum physics [28]. In particular, they are used in an axiomatic formulation of quantum field theory [15]. This makes the theory of operator algebras interesting not only for mathematicians but also for theoretical physicists.

The aim of this work is the study of quantum structures by means of operator algebras. More specifically, in this vast field, our attention is focused on the well known Bell inequalities and a certain partial order on operator algebras called the star order.

Bell inequalities, first studied by Bell [6], provide an upper bound on the strength of correlations between measurements performed on physical systems. They play an important role in foundations of quantum physics especially in the discussion on the local hidden-variables theories. From the mathematical point of view, Bell inequalities describe correlations of noncommutative random variables which, in the C^*-algebraic approach, are modeled by self-adjoint elements in a C^*-algebra. Some results concerning Bell inequalities in the settings of operator algebras can be found, for example, in [10, 21, 29, 30].

Recently, Bell inequalities have been intensely studied especially in a context of quantum information theory [23]. The reason for this lies in a relation between a violation of Bell inequalities and entanglement [32]. The entanglement is one of the key ingredients of quantum information theory and so Bell inequalities have found significant applications in many areas of this theory such as quantum cryptography [2, 1], communication complexity [9], quantum game theory [20, 27], estimates of a bound for the dimension of the underlying Hilbert space [7, 8, 31], etc. All these aspects motivate effort to understand the structure of Bell inequalities and their violation. The deep results in this field have been recently obtained by applying the powerful techniques of operator space theory [19, 18, 24, 25].

The next goal of this thesis is the investigation of the star order introduced by Drazin [13]. This partial order has been studied mainly on matrix algebras where a number of interesting facts were obtained [4, 5, 16, 17, 22]. Recently, it has also been considered on the set of all bounded operators on a Hilbert space [3, 11]. This has not only brought new infinite dimensional results but it has also put older facts on the star order for matrices into a new perspective.

The star order is related to the well known Gudder order [14]. It turns out that the Gudder order, which can be interpreted as a logical order on bounded quantum observables, is in fact a restriction of the star order to the self-adjoint part of the set of all bounded operators on a Hilbert space. This observation gives, for example, the solution of preserver problem [12] and infimum and supremum problem [14, 26] for the star order on bounded self-adjoint operators.

2 Aims of the doctoral thesis

The first goal of this work is the study of the (CHSH version of) Bell inequality and its quantum form known as Cirel'son inequality. This is motivated by the results of Summers and Werner [29]. The main parts of this research can be formulated as follows.

- Generalize the Cirel'son inequality to (real and complex) linear spaces endowed with a pseudo inner product.
- Examine the structure of maximal violators of the Bell inequality in the context of *-algebras and Jordan algebras.

The second goal is the investigation of the star order on certain *-algebras. This can be regarded as a continuation of the research line given by the papers [3, 12, 14]. Our main effort is devoted to the following points.

- Analyze the star order on partial isometries.
- Investigate the infimum and supremum problem for the star order on *-algebras of all continuous complex-valued functions on a Hausdorff topological space.
- Explore (nonlinear) maps preserving the star order in both directions on von Neumann algebras.

3 Bell inequalities

In this section, we summarize our main contributions concerning Bell inequalities.

3.1 Bell inequalities and linear spaces

Let us recall some concepts and fix the notation. In the sequel, the symbol $\mathbb{F}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let X be a linear space over $\mathbb{F}$. By a *pseudo inner product* on X we mean a sesquilinear form $Q : X \times X \to \mathbb{F}$ such that, for all $x, y \in X$,

(i) $Q(x, y) = \overline{Q(y, x)}$,

(ii) $Q(x, x) \geq 0$.

A pseudo inner product $Q : X \times X \to \mathbb{F}$ induces a pseudonorm $\|\cdot\|_Q$ given by $\|x\|_Q = \sqrt{Q(x, x)}$, $x \in X$. The set of all elements $x \in X$ with $\|x\|_Q = 0$ will be denoted by N_Q.

Using Cauchy-Schwarz inequality, we generalize the Cirel'son inequality [10] to real and complex linear spaces endowed with a pseudo inner product.

Theorem 3.1. *Let X be a linear space over $\mathbb{F}$ equipped with a pseudo inner product Q. Then*

$$\frac{1}{2}\sup|Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| \leq \sqrt{2}, \tag{1}$$

where the supremum is taken over elements $a_i, b_i \in X$ $(i = 1, 2)$ such that $\|a_i\|_Q, \|b_i\|_Q \leq 1$.

An analysis of (1) turns out that if there are elements $a_i, b_i \in X$ $(i = 1, 2)$ in which the bound $\sqrt{2}$ is attained, then they have to satisfy surprising conditions. We mention here only the case of complex linear spaces (the real case is similar).

Theorem 3.2. *Suppose that X is a complex linear space endowed with a pseudo inner product Q. Let a_i, b_i $(i = 1, 2)$ be elements of X with pseudonorms $\|a_i\|_Q, \|b_i\|_Q \leq 1$. If*

$$\frac{1}{2}|Q(a_1, b_1 + b_2) + Q(a_2, b_1 - b_2)| = \sqrt{2},$$

then the following holds:

(i) $\|a_i\|_Q = \|b_i\|_Q = 1$.

(ii) $\operatorname{Re} Q(b_1, b_2) = 0$.

(iii) $\operatorname{Re} Q(a_1, a_2) = 0$.

(iv) *There is a complex unit γ and elements $n_1, n_2 \in N_Q$ such that*

$$\begin{aligned} a_1 &= \frac{\gamma}{\sqrt{2}}(b_1 + b_2) + n_1, \\ a_2 &= \frac{\gamma}{\sqrt{2}}(b_1 - b_2) + n_2. \end{aligned}$$

3.2 Maximal violation in *-algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be *-subalgebras of a unital *-algebra $\mathcal{C}$ and let φ be a state on $\mathcal{C}$. We say that the Bell inequality is *maximally violated in the state* φ if there are self-adjoint elements $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ with $a_i^2, b_i^2 \leq \mathbf{1}$ such that

$$\frac{1}{2} |\varphi(a_1(b_1 + b_2) + a_2(b_1 - b_2))| = \sqrt{2}.$$

The elements a_1, a_2, b_1, b_2 are called *maximal violators* of the Bell inequality in the state φ.

Since every state φ on a unital *-algebra $\mathcal{C}$ induces a pseudo inner product Q_φ on $\mathcal{C}$ given by $Q_\varphi(a, b) = \varphi(b^* a)$ for all $a, b \in \mathcal{C}$, we can apply Theorem 3.2 to describe the structure of maximal violators.

Theorem 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-subalgebras of a unital *-algebra $\mathcal{C}$. Let φ be a faithful state on $\mathcal{C}$. Suppose that $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ are maximal violators of the Bell inequality in the state φ. Then*

(i) $a_i^2 = b_i^2 = \mathbf{1}$,

(ii) $a_1 a_2 + a_2 a_1 = b_1 b_2 + b_2 b_1 = 0.$

Moreover, there is $\alpha \in \{-1, 1\}$ such that

$$\begin{aligned} a_1 &= \frac{\alpha}{\sqrt{2}}(b_1 + b_2), \\ a_2 &= \frac{\alpha}{\sqrt{2}}(b_1 - b_2). \end{aligned}$$

The previous theorem says that maximal violators of the Bell inequality in a faithful state form realizations of Pauli spin matrices. Recall that self-adjoint elements a_1 and a_2 of a unital *-algebra are called a *realization of Pauli spin matrices* if $a_1^2 = a_2^2 = \mathbf{1}$ and $a_1 a_2 + a_2 a_1 = 0$. Moreover, Theorem 3.3 implies that $\mathcal{A} \cap \mathcal{B}$ contains a unital *-subalgebra *-isomorphic to $M_2(\mathbb{C})$.

This means that the maximal violation in some faithful state is not compatible with mutual commutativity of subalgebras $\mathcal{A}$ and $\mathcal{B}$.

Let $(\mathcal{A}, \mathcal{B})$ be a pair of *-subalgebras of a unital *-algebra $\mathcal{C}$. A state φ on $\mathcal{C}$ is said to be *weakly uncoupled across* $(\mathcal{A}, \mathcal{B})$ if, for all $a \in \mathcal{A}$ and $b, c \in \mathcal{B}$, we have

$$\varphi(abc) = \varphi(bac).$$

Let us remark that if the algebras $\mathcal{A}$ and $\mathcal{B}$ commute then any state on $\mathcal{C}$ is weakly uncoupled. Therefore, the following theorem generalizes the result of Summers and Werner [29].

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be *-subalgebras of a unital *-algebra $\mathcal{C}$ containing the unit of $\mathcal{C}$. Let φ be a weakly uncoupled state on $\mathcal{C}$ across $(\mathcal{A}, \mathcal{B})$. If $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ are maximal violators of the Bell inequality in the state φ, then*

(i) $\varphi(a_i^2 c) = \varphi(c)$,

(ii) $\varphi(b_i^2 c) = \varphi(c)$,

(iii) $\varphi((a_1 a_2 + a_2 a_1)a) = \varphi((b_1 b_2 + b_2 b_1)b) = 0$

*for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$. Moreover, φ restricts to a tracial state on the unital *-subalgebras generated by $\{\mathbf{1}, a_1, a_2\}$ and $\{\mathbf{1}, b_1, b_2\}$, respectively.*

If φ in the previous theorem restricts to a faithful state on *-subalgebras $\mathcal{A}$ and $\mathcal{B}$, we obtain that the maximal violators are again realizations of Pauli spin matrices.

3.3 Maximal violations in Jordan algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Jordan subalgebras of a unital Jordan algebra $\mathcal{C}$ and let φ be a state on $\mathcal{C}$. We say that the Bell inequality is *maximally violated in the state* φ if there are elements $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ with $a_i^2, b_i^2 \leq \mathbf{1}$ such that

$$\frac{1}{2} \left|\varphi(a_1 \circ (b_1 + b_2) + a_2 \circ (b_1 - b_2))\right| = \sqrt{2}.$$

The elements a_1, a_2, b_1, b_2 are called *maximal violators* of the Bell inequality in the state φ.

Recall that two elements a, b in a Jordan algebra $\mathcal{A}$ are called *orthogonal* if $a \circ b = 0$. An element s of a unital Jordan algebra is called a *symmetry*, if $s^2 = \mathbf{1}$.

Theorem 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Jordan subalgebras of a unital Jordan algebra $\mathcal{C}$. Let φ be a faithful state on $\mathcal{C}$. Suppose that $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ are maximal violators of the Bell inequality in the state φ. Then a_1, a_2 and b_1, b_2 are orthogonal symmetries. Moreover, there is $\alpha \in \{-1, 1\}$ such that*

$$\begin{aligned} a_1 &= \frac{\alpha}{\sqrt{2}}(b_1 + b_2)\,, \\ a_2 &= \frac{\alpha}{\sqrt{2}}(b_1 - b_2)\,. \end{aligned}$$

In order to get an analogue of Theorem 3.4 we introduce the following notion. Let $\mathcal{A}$ and $\mathcal{B}$ be Jordan subalgebras of a unital Jordan algebra $\mathcal{C}$. We say that a state φ on $\mathcal{C}$ is *uncorrelated across* $\mathcal{A}$ and $\mathcal{B}$ if

$$\varphi(a \circ (b \circ c)) = \varphi(b \circ (a \circ c))\,,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{A} \cup \mathcal{B}$.

Theorem 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be Jordan subalgebras of a unital Jordan algebra $\mathcal{C}$. Let φ be a state on $\mathcal{C}$ uncorrelated across $\mathcal{A}$ and $\mathcal{B}$. Suppose that $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2)$ are maximal violators of the Bell inequality in the state φ. Then, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $c \in \mathcal{C}$,*

(i) $\varphi(a_i^2 \circ c) = \varphi(c)$,

(ii) $\varphi(b_i^2 \circ c) = \varphi(c)$,

(iii) $\varphi((a_1 \circ a_2) \circ a) = \varphi((b_1 \circ b_2) \circ b) = 0$,

(iv) $\varphi(a_i) = \varphi(b_i) = 0$.

A *spin system* in a unital Jordan algebra $\mathcal{A}$ is a collection $\mathcal{P}$ of at least two symmetries different from $\pm\mathbf{1}$ such that $s \circ t = 0$ whenever $s, t \in \mathcal{P}$ and $s \neq t$. If φ in the previous theorem restricts to a faithful state on *-subalgebras $\mathcal{A}$ and $\mathcal{B}$, we obtain that $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are spin systems.

4 Star order

In this section, we recapitulate the main results of our research dealing with the star order on proper *-algebras.

A *-algebra $\mathcal{A}$ is said to be *proper* if $a^*a = 0$ implies $a = 0$ for any $a \in \mathcal{A}$. Important examples of proper *-algebras are a C^*-algebra and a *-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space X.

The star order was introduced by Drazin [13] in a general context of so-called proper *-semigroups. Since a proper *-algebra $\mathcal{A}$ carries the multiplicative structure of a proper *-semigroup, we can define, following Drazin, a partial order on $\mathcal{A}$ as follows. Let $\mathcal{A}$ be a proper *-algebra. We say that $a \in \mathcal{A}$ is less than or equal to $b \in \mathcal{A}$ in the *star order*, written $a \preceq b$, if

$$a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$

We write $a \prec b$ if $a \preceq b$ and $a \neq b$.

4.1 Star order and partial isometries

Let $\mathcal{A}$ be a *-algebra. An element $a \in \mathcal{A}$ is called a *partial isometry* if $aa^*a = a$. Two projections e and f in $\mathcal{A}$ are said to be *equivalent*, written $e \sim f$, if there is a partial isometry $u \in \mathcal{A}$ such that $u^*u = e$ and $uu^* = f$.

Theorem 4.1. *Let $\mathcal{A}$ be a unital proper *-algebra. Suppose that $f_i \in \mathcal{A}$ $(i = 1, 2)$ are projections. Then the following conditions are equivalent:*

(i) $f_2 \sim f_1 \prec f_2$.

(ii) *There are partial isometries* $u_i \in \mathcal{A}$ *(*$i = 1, 2$*) such that* $u_1 \prec u_2$*,* $u_1 u_1^* = u_2^* u_2 = f_1$*, and* $u_2 u_2^* = f_2$*.*

The preceding result enables us to characterize infiniteness of C^*-algebras in Murray-von Neumann comparison theory. Let us recall that an element u of a unital C^*-algebra is called *coisometry* if $uu^* = \mathbf{1}$.

Corollary 4.2. *A unital* C^**-algebra* $\mathcal{C}$ *is infinite if and only if there are a partial isometry* $u_1 \in \mathcal{C}$ *and a coisometry* $u_2 \in \mathcal{C}$ *such that* $u_1 \prec u_2$ *and* $u_1 u_1^* = u_2^* u_2$*.*

4.2 Infimum and supremum problem

We consider the infimum and supremum problem for the star order on a proper *-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space X. Using topological arguments, we obtain the following results.

Theorem 4.3. *Let* $(f_\alpha)_{\alpha \in \Lambda}$ *be a family of elements of* $C(X)$*. The infimum* $\bigwedge_{\alpha \in \Lambda} f_\alpha$ *exists whenever* X *is locally connected or extremely disconnected.*

Theorem 4.4. *Let* X *be a locally connected or an extremely disconnected Hausdorff topological space. Suppose that* $(f_\alpha)_{\alpha \in \Lambda}$ *is a family of elements of* $C(X)$*. Then the following conditions are equivalent:*

(i) *There exists* $\bigvee_{\alpha \in \Lambda} f_\alpha$*.*

(ii) *There is* $h \in C(X)$ *such that* $f_\alpha \preceq h$ *for any* $\alpha \in \Lambda$*.*

Let $\mathcal{A}$ be an abelian C^*-algebra whose spectrum is locally connected or extremely disconnected. Applying the previous results, we get that the infimum of every subset of $\mathcal{A}$ exists. Moreover, the supremum of a subset M of $\mathcal{A}$ exists if and only if M has an upper bound.

4.3 Preservers of the star order

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Let M and N be subsets of $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\varphi : M \to N$ is a *star order isomorphism* if φ is a bijection such that

$$a \preceq b \Leftrightarrow \varphi(a) \preceq \varphi(b)$$

for all $a, b \in M$. In the sequel, we shall denote $\mathcal{A}_n$ and $\mathcal{A}_{sa}$ the set of all normal elements of $\mathcal{A}$ and the set of all self-adjoint elements of $\mathcal{A}$, respectively. Recall that a bijection $\varphi : \mathcal{A} \to \mathcal{B}$ is called *Jordan *-isomorphism* if, for all $a \in \mathcal{A}$ and $b \in \mathcal{A}_{sa}$, $\varphi(a^*) = \varphi(a)^*$ and $\varphi(b^2) = \varphi(b)^2$.

Our effort is to investigate continuous star order isomorphisms between various subsets of von Neumann algebras. This is motivated by the result of Dolinar and Molnár [12] in which continuous star order isomorphisms of self-adjoint part of Type I_n factors, where $n \geq 3$, were described.

Theorem 4.5. *Let $\mathcal{A}$ be a von Neumann algebra without Type I_2 direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ be a continuous star order isomorphism. Suppose that there is an invertible central element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda)\, c$$

*for all $\lambda \in \mathbb{C}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(a) = \psi(f(a))c$$

for all $a \in \mathcal{A}_n$.

The following corollary describing certain continuous star order isomorphisms between abelian von Neumann algebras is an immediate consequence of the preceding theorem.

Corollary 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian von Neumann algebras. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism. Suppose* *that there is an invertible element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda)\, c$$

*for all $\lambda \in \mathbb{C}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(a) = \psi(f(a))c$$

for all $a \in \mathcal{A}$.

The next theorem is a version of Theorem 4.5 for star order isomorphisms between self-adjoint parts of a von Neumann algebras.

Theorem 4.7. *Let $\mathcal{A}$ be a von Neumann algebra without Type I_2 direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A}_{sa} \to \mathcal{B}_{sa}$ be a continuous star order isomorphism. Suppose that there is an invertible central self-adjoint element $c \in \mathcal{B}$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda)c$$

*for all $\lambda \in \mathbb{R}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(a) = \psi(f(a))c$$

for all $a \in \mathcal{A}_{sa}$.

Corollary 4.8. *Let $\mathcal{A}$ be a von Neumann algebra without Type I_2 direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A}_{sa} \to \mathcal{B}_{sa}$ be a continuous star order isomorphism. If*

$$\varphi(\lambda \mathbf{1}) = \lambda \mathbf{1}$$

*for all $\lambda \in \mathbb{R}$, then φ is the restriction of a Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ to $\mathcal{A}_{sa}$.*

The following result is another version of Theorem 4.5.

Theorem 4.9. *Let $\mathcal{A}$ be a von Neumann algebra without Type I_2 direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism. Suppose that there is an invertible central element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda)c$$

*for all $\lambda \in \mathbb{C}$. Then $\varphi(\mathcal{A}_n) \subseteq \mathcal{B}_n$, f is an injective function with $f(0) = 0$, and there is a unique Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(a) = \psi(f(a))c$$

for all $a \in \mathcal{A}_n$.

As a simple consequence of the previous theorem, we obtain the result concerning the automatic linearity of certain star order isomorphisms between von Neumann algebras. This result is an analogue of Corollary 4.8. Note that we do not assume that φ preserves the self-adjoint elements.

Corollary 4.10. *Let $\mathcal{A}$ be a von Neumann algebra without Type I_2 direct summand and let $\mathcal{B}$ be a von Neumann algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism such that*

$$\varphi(\lambda \mathbf{1}) = \lambda \mathbf{1}$$

for all $\lambda \in \mathbb{C}$. Suppose that

$$\varphi(a + ib) = \varphi(a) + i\varphi(b)$$

*for all $a, b \in \mathcal{A}_{sa}$. Then φ is a Jordan *-isomorphism.*

The preceding corollary provides a new characterization of Jordan *-isomorphisms in which the condition of linearity seems to be very relaxed.

5 Conclusion

The Cirel'son inequality has been generalized to real and complex linear spaces endowed with a pseudo inner product. Moreover, the structure of maximal violators of the (CHSH version of) Bell inequality has been studied in the context of *-algebras and Jordan algebras. It has been shown that maximal violators are closely related to Pauli spin matrices.

The investigation of the star order on partial isometries has led to a new characterization of infinite C^*-algebras. Furthermore, the infimum and supremum problem for the star order on an algebra $C(X)$ has been investigated. We have shown that every upper bounded subset of $C(X)$ has the infimum and the supremum whenever X is a locally connected or an extremely disconnected Hausdorff topological space.

The star order isomorphisms have been examined. We have completely described the structure of certain nonlinear continuous star order isomorphisms from the normal part of a von Neumann algebra without Type I_2 direct summand onto the normal part of another von Neumann algebra. We have also discussed several modifications of this result. Some interesting corollaries of these assertions have been mentioned.

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Resumé

Disertační práce se zabývá Bellovými nerovnostmi a částečným uspořádáním nazývaným *-uspořádání. Tyto dvě struktury jsou zkoumány pomocí teorie operátorových algeber.

Studium Bellových nerovností je zaměřeno na CHSH verzi Bellovy nerovnosti a její kvantovou formu nazývanou Cirel'sonova nerovnost. Cirel'sonova nerovnost je zobecněna do reálných a komplexních lineárních prostorů s pseudo skalárním součinem. Výsledky obdržené na této abstraktní úrovni jsou poté aplikovány na studium maximálního narušení (CHSH verze) Bellovy nerovnosti formulované v matematickém rámci *-algeber. Je ukázáno, že prvky maximálně narušující Bellovu nerovnost úzce souvisí s Pauliho spinovými maticemi. Tyto výsledky jsou zobecněny do neasociativního případu Jordanových algeber.

Další oblastí našeho zájmu je *-uspořádání. Toto uspořádání je uvažováno na vhodných *-algebrách. Jako důsledek naší analýzy *-uspořádání na částečných isometriích obdržíme novou charakterizaci nekonečných C^*-algeber. Poté se věnujeme problému existence infima a suprema v případě *-uspořádání na *-algebře $C(X)$ všech spojitých komplexních funkcí na Hausdorffově topologickém prostoru X. Je dokázáno, že pokud topologický prostor X je lokálně souvislý nebo extrémně nesouvislý, potom každá shora omezená (vzhledem k *-uspořádání) podmnožina algebry $C(X)$ má infimum a supremum. Jako důsledek tak dostaneme například existenci infima a suprema libovolné shora omezené podmnožiny abelovské von Neumannove algebry.

Nakonec je zkoumán problém spojitých (obecně nelinearních) zobrazení zachovávajících *-uspořádání. Je popsána struktura jistých spojitých bijektivních zobrazení mezi normálními částmi von Neumannových algeber, která zachovávají *-uspořádání. Různé varianty tohoto výsledku stejně jako jejich důsledky jsou diskutovány.

List of author's works relating to the doctoral thesis

Papers in impacted journals

- M. Bohata: *Star order on operator and function algebras*, Publ. Math. Debrecen **79** (2011), 211–229.
- M. Bohata and J. Hamhalter: *Maximal violation of Bell's inequalities and Pauli spin matrices*, J. Math. Phys. **50** (2009), 082101. [Autorship: 50%]
- M. Bohata and J. Hamhalter: *Bell's correlations and spin systems*, Found. Phys. **40** (2010), 1065–1075. [Autorship: 50%]
- M. Bohata and J. Hamhalter: *Nonlinear maps on von Neumann algebras preserving the star order*, accepted for publication in Lin. Multilin. Alg. [Autorship: 50%]

International conferences

- M. Bohata: *Gudder and star order on *-algebras*, Quantum Structures Boston 2010, Boston, USA.
- M. Bohata and J. Hamhalter: *Bell's inequalities and Pauli matrices*, Quantum Structure 2009, Kočovce, Slovakia. [Authorship: 50%]
- M. Bohata and J. Hamhalter: *Bell's inequalities and spin systems*, Quantum Structures Boston 2010, Boston, USA. [Autorship: 50%]
- M. Bohata and J. Hamhalter: *Star order on operator algebras*, 3rd Conference of Settat on Operator Algebras and Applications 2011, Settat, Marocco. [Autorship: 50%]

- M. Bohata and J. Hamhalter: *Star order isomorphisms on von Neumann algebras*, Quantum Structures Cagliari 2012, Cagliari, Italy. [Autorship: 50%]
- M. Bohata and J. Hamhalter: *Structures on operator algebras and Jordan isomorphisms*, Workshop on Operator Theory and Operator Algebras 2012, Lisbon, Portugal. [Autorship: 50%]