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Second-degree Price Discrimination by a Two-sided Monopoly Platform*

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Abstract

In this article we study second-degree price discrimination by a two-sided monopoly platform. We find novel distortions that arise due to the two-sidedness of the market. They make the standard result "no distortion at top and downward distortion at bottom" not holding. They generate a new type of non-responsiveness, different from the one found by Guesnerie and Laffont (1984). We also show that the platform may mitigate or remove non-responsiveness at one side by properly designing price discrimination on the other side. These findings help to address our central question, i.e., when price discrimination on one side substitutes for or complements price discrimination on the other side. As an application, we study the optimal mechanism design for an advertising platform mediating advertisers and consumers.

JEL codes: D4, D82, L5, M3

Key words: (second-degree) price discrimination, two-sided markets, non-responsiveness, type reversal, advertising

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1 Introduction

Many two-sided platforms mediating interactions between two different groups practice price discrimination against one or both groups of agents. However, little economic analysis has yet been put forward regarding second-degree price discrimination by a two-sided platform, despite the fact that second-degree price discrimination by a monopolist is one of the best-known applications of the principal-agent theory.[1]

For example, the world's largest on-demand streaming service, YouTube, launched its ad-free premium version 'YouTube Red' for a subscription fee in October 2015. Since YouTube advertisers pay different average-per-view costs depending on ad formats, ad amount and targeting, it suggests that YouTube now adopts price discrimination towards both advertisers and users. Other media platforms have also adopted ad-free premium services: e.g., Youku Tudou, China's biggest video site, allows subscribers to skip all ads at 20 RMB per month.[2]

Network neutrality regulation is another important example. The debate has primarily focused on whether a tiered-Internet should be allowed for Internet service providers (ISPs) vis-à-vis content providers while ISPs' menu pricing against residential consumers with different quality-price pairs remains uncontroversial.[3] Thus, we can conceptualize the ongoing network neutrality debate as whether society would benefit from introducing price discrimination on the side of content providers in the presence of price discrimination on the side of residential broadband Internet-service subscribers.

In this paper we adapt a canonical model of monopolistic screening à la Mussa and Rosen (1978) to a two-sided monopoly platform and study when price discrimination ('PD' for shorthand) on one side complements or substitutes for PD on the other side. Suppose that a platform considers introducing second-degree PD on one side (side B) in the presence of current PD on the other side (side A).[4] We say that PD on side B is complementary to (respectively, a substitute for) PD on side A if the platform's

[1] The seminal papers include Maskin and Riley (1984) and Mussa and Rosen (1978), and there is a vast literature on non-linear pricing. See Armstrong (2015) and Wilson (1993) for in-depth reviews.

[2] Amazon Kindle users can choose "Special Offers" to avoid ads for a price (\$15-\$20); Amazon's advertising fees vary depending on many factors such as product category, number of shipments and downloads, promotions.

[3] Aviv, Turner, and Williams (2016) estimate demand for residential broadband Internet subscribers facing a three-part tariff and shows that usage-based pricing can eliminates low-value traffic.

[4] We assume full participation of all agents on both sides. No PD means that both types of agents on each side are offered the same quality-price pair that satisfies both types' participation constraints on that side.

profit is higher (respectively, lower) with PD on both sides than with PD on side A only. When addressing the question, we pay particular attention to the insight that introducing PD on side B may affect not only the incentives of the agents on side B, but also the incentive constraints and implementable allocations on side A.

A central concept in our paper is *non-responsiveness* (Guesnerie and Laffont, 1984; Laffont and Martimort, 2002), which refers to a clash between the allocation that the principal desires to achieve and incentive compatible (or implementable) allocations. In a standard principal-agent model, this conflict may arise when the agent's type directly affects the principal's utility. For instance, suppose that the principal is a benevolent regulator who cares not only about economic efficiency in production cost of regulated firms but also about externality measured by the amount of pollution the firms emit. Incentive compatibility requires that low-cost firms should produce more than high-cost firms. However, if the higher cost results from greater efforts to reduce pollution, then the principal may want to induce the high-cost firms to produce more than the low-cost firms. Such a non-monotonic quantity schedule clashes with the monotonic incentive compatible schedule, which makes the principal adopt a pooling contract that is offered to both types of firms.

In this paper, we point out that non-responsiveness situations can frequently arise in two-sided platforms that offer intermediation services involving cross-group interactions. Furthermore, we show that a platform may mitigate or remove non-responsiveness at one side by properly designing price discrimination on the other side. Consider, for instance, a media platform that offers content to consumers who are exposed to advertisements delivered together. Suppose that there are rich (H type) and poor (L type) consumers. Without PD on the advertising side, the rich may suffer the higher average nuisance from advertisements so that they have a greater marginal willingness to pay to avoid ads than the poor. However, from an advertiser's perspective, rich consumers are more valuable than poor ones. Therefore, the platform may prefer that H type consumers have a greater exposure to ads than L types, which is impossible to implement without PD on the advertising side. The question is "Can the two-sided platform show more ads to rich consumers while still inducing self-selection?" The answer is positive, provided that there are high-end advertisements that annoy the rich consumers less than the poor ones. That is, by designing PD on the advertising side that assigns more weight to such high-end ads than to low-end ones, the platform can make viewing ads on average less displeasing to H types than to L types.

One assumption implicitly made in the above example is that, although rich consumers will pay more to avoid average ads, there must be some high-end ads that they find less offensive. Generally, we refer to this property as *type reversal.* Note also that in the given example the platform's screening instruments (called "qualities" in our model) on each side are complements from consumers' point of view.[5] In this paper provide a canonical model that allows for all possible combinations of type reversal or no reversal and complementarity or substitutability between the qualities on either side of a two-sided market.[6]

In our model, a monopolistic two-sided platform offers a menu of price-quality pairs to a continuum of agents of mass one on each side. The utility that an agent i of side k $(= A, B)$ obtains from interacting with an agent j of the other side $l \neq k$ (and $l = A, B$) depends on both agents' types and q_i^k (q_j^l), i.e., the quality that agent i (j) receives. We consider a two-type model: an agent on each side has either H or L type where an H type agent on side k is defined to have the greater *expected benefit* than an L type on side k from an increase in q_i^k when interacting with the agents on the other side. However, the H type does not necessarily have the greater benefit than the L type for interacting with *every* type of agents on the other side. We say that there is *type reversal* if the L type obtains the greater benefit than the H type when interacting with a particular type of agent. We further sub-categorize that there is *type reversal with a positive (negative) sorting* if the particular type on the other side is 'L' ('H'). The two qualities q_i^k and q_j^l can be complements or substitutes on a given side; also, they may affect the agent i's utility in a separate way so that the two qualities are neither complements not substitutes. The former case is referred to as non-separable case whereas the latter is called the separable case.

We have two sets of novel results. First, we consider the separable case and characterize the first-best and the second-best allocations. In the separable case, we find that the first-best quality schedule on side k is *non-monotonic* if the L type of side k generates sufficiently larger positive externalities to the other side than the H type of side k does. Asymmetric information creates the well-known own-side distortion but also new distortions due to the two-sidedness of the market as the information rent that a H type of a given side obtains can be affected by the quality schedule offered

[5]Suppose that the platform offers consumers the option to opt out of advertising at a fee; such a premium service is a higher quality product to consumers. The benefit from such a higher quality (i.e., from avoiding ads) increases with the amount of advertisements which is a higher "quality" on the advertising side.

[6]The qualities are complements (substitutes) on side k if the cross-derivative of $u^k(q_i^k, q_j^l)$ is positive (negative) for $k \neq l$ and $k, l = A, B$. This should not be confused with the complementarity (substitution) between PD of both sides.

to the other side. Consequently, the standard result of "no distortion at top and a downward distortion at bottom" does not hold any more. In addition, because of this new distortion, a non-responsiveness can occur even if the first-best quality schedule is monotonic—this is not possible in a one-sided market. In other words, two-sided interactions generate another source for non-responsiveness, different from the one identified by Guesnerie and Laffont (1984).

Next, we consider the non-separable case and study how PD on a given side would affect the implementable allocations on the other side. We first characterize, as an intermediary step, the implementable allocations on side A given an arbitrary quality schedule on side B. We find that the implementable allocations on a given side k are equal to the set of monotonic quality schedules if one of the following conditions holds: (i) q_i^k and q_j^l are separable on side k, (ii) there is no type reversal on side k, (iii) there is no PD on side l with $l \neq k$. Then, we show that type reversal on side A can make a non-monotonic schedule implementable on side A when some appropriate PD is introduced onto side B. The intuition for this result is as follows. Basically, the implementability condition on side A means that, given a quality schedule on side B, an L type's gain from choosing q_L^A instead of q_H^A must be greater than that of an H type. Consider a non-monotonic schedule on both sides, i.e., $q_H^k < q_L^k$ for $k = A, B$. If there is type reversal with a positive sorting and the qualities are complements on side A, then an L type agent on side A can experience a much greater utility increase from choosing q_L^A instead of q_H^A than an H type, which makes $q_H^A < q_L^A$ implementable.[7]

Obtaining this insight, we then consider a symmetric two-sided platform with private information on both sides and study the implementable allocations with symmetric mechanisms. In the case of complements with a positive sorting (or substitutes with a negative sorting), we find that the implementable set includes all monotonic schedules, plus possibly a subset of non-monotonic schedules. Thus, PD on one side is likely to complement PD on the other side. By contrast, in the case of substitutes with a positive sorting (or complements with a negative sorting), the implementable set includes only monotonic schedule and possibly a strict subset of monotonic schedules. In this case, PD on one side is likely to substitute for PD on the other side.

Finally, we apply the insight obtained from the canonical model to an advertising platform that generates profits from consumers' content consumption and from advertisers' advertising to those content users. As we described earlier in the example, on the consumer side, we consider the type reversal with a negative sorting and

[7]As expected, this reasoning holds symmetrically for type reversal with a negative sorting when the qualities are substitutes.

complementarity between the qualities. In this circumstance, we show how the optimal profit-maximizing mechanism varies with the intensity of the type reversal on the consumer side. We find that, for a low intensity of type reversal, profit maximization requires the L type advertisers to advertise more than the H type advertisers, which clashes with the implementability condition on the advertising side. Thus, a pooling contract becomes optimal on the advertising side. This implies that a strict PD on the advertising side will reduce the platform's profit; PD on the advertising side substitutes for PD on the consumer side. By contrast, for a high intensity, profit maximization requires H type advertisers to advertise more than L type advertisers and can even require implementing a non-monotonic quality schedule on the consumer side (i.e., showing ads only to H type consumers). Then, PD on the advertising side is complementary to the PD on the consumer side as it not only allows implementation of a desirable discrimination on the advertising side but also a non-monotonic schedule on the consumer side.

■ **Related literature**

This article is related to several strands of literature. First, our paper is closely related to the second-degree PD in the principal-agent theory (e.g. Maskin and Riley, 1984; Mussa and Rosen, 1978) and to the concept of non-responsiveness. The non-responsiveness was developed by Guesnerie and Laffont (1984) and then was explored by Caillaud and Tirole (2004) in the context of financing an essential facility and by Jeon and Menicucci (2008) in the context of allocation of talent between the private sector and the science sector. To our knowledge, however, non-responsiveness has never been explored in the context of two-sided markets; our contribution is to identify a novel source for non-responsiveness that has to do with two-sidedness of the market.

Although the literature on two-sided platforms has been expanding rapidly,[8] there is little work that studies price discrimination in a two-sided market by explicitly addressing type-dependent interactions. One exception is Gomes and Pavan (2014) who consider heterogeneous agents on both sides in a centralized many-to-many matching setting. They provide conditions on the primitives under which the optimal matching rule has a threshold structure such that each agent on one side is matched with all agents on the other side above a threshold type. They also provide a precise characterization of the thresholds, but non-responsiveness was not studied. Unlike Gomes and Pavan, in our model all agents on one side interact with all agents on the other side;

[8]The literature is vast including Anderson and Coast (2005), Armstrong (2006), Caillaud and Jullien (2001, 2003), Hagiu (2006), Hagiu and Jullien (2011), Jeon and Rochet (2010), Rochet and Tirole (2003, 2006), Rysman (2009), and Weyl (2010).

in such setting we allow for a screening instrument on each side, study distortions in these instruments and further explore how type reversal together with complementarity/substitution between the instruments affects the implementable allocations. Choi, Jeon and Kim (2015) study second-degree PD of a two-sided monopoly platform in the context of network neutrality. However, they consider heterogeneous agents only on the content-provider side and assume homogeneous agents on the consumer side, while we consider heterogeneous agents on both sides. Moreover, there is no type reversal in their model while the type reversal and ensuing non-responsiveness are our theoretical driving forces. Böhme (2012) analyzes second-degree PD in a monopolistic screening model with network effects. He consider two types of agents in only one side who are heterogeneous regarding their intrinsic utility from joining the platform. By contrast, we consider two different types of agents on both sides, and each agent obtains different payoffs depending the type of the other side's agent matched and the qualities that he and the matched agent receive. Moreover, we focus on complementarity or substitution of two-sided price discrimination.

The rest of the article is organized as follows. We set up the canonical model in Section 2. In Sections 3-4, we consider the separable case and characterize the first-best and the second-best allocations. In Sections 5-6, we consider the non-separable case. In Section 5.1, we study the implementable allocations on side A for a given quality schedule on side B and in Section 5.2 we consider a symmetric two-sided market and study implementable allocations on both sides through symmetric mechanisms. In Section 6, we consider an advertising platform and show how the general insight gained from the canonical model can be applied to a more realistic situation. We conclude in Section 7. All mathematical proofs not covered in the text are relegated to Appendix A. We also offer further analysis of the symmetric two-sided market in the online appendix.

2 A canonical principal-agent model in two-sided markets

We consider a canonical principal-agent model (Mussa and Rosen, 1978 and Laffont and Martimort, 2002) and adapt it to a two-sided market where a monopoly platform as the principal designs a mechanism to mediate interactions between agents from two sides, $k = \text{A}, \text{B}$. On each side there is a mass one of agents. Let θ_i^k represent the type of agent i on side k. For simplicity we consider a two-type model. An agent has one of the two types, H or L, on each side, i.e., $\theta_i^A \in \{H, L\}$ and $\theta_j^B \in \{H, L\}$. Let $\nu_H^k \in (0, 1)$ represent the fraction of H-types on side k. Let $\nu_L^k \equiv 1 - \nu_H^k$. The platform chooses quality q_i^k for each agent i of side $k = \mathrm{A}, \mathrm{B}$. When an agent i of side k interacts with an agent j of side l with $k \neq l$ and $k, l = \mathrm{A}, \mathrm{B}$, the gross utility the agent i obtains can be represented as follows:

$$U_i^k(\theta_i^k, \theta_j^l, q_i^k, q_j^l) = \theta_{ij}^k u^k(q_i^k, q_j^l),$$

where the types interact in a multiplicative way with qualities as in Mussa and Rosen (1978) and θ_{ij}^k represents the consumption intensity the agent i of side k as a function of both agents' types. Compared to the price discrimination in one-sided market of Mussa and Rosen (1978), there are two additional interactions from the two-sidedness of the market: the type and the quality of the agent j on side l matter. This makes the model very rich but also easily involved even if we consider a two-type model.

To provide more tangible interpretation for the parameters and variables in above formulae, let us consider three applications.

- Net neutrality regulation: In the network neutrality debate, a monopoly ISP mediates a group of network subscribers ($k = A$) with a group of content providers ($k = B$). The parameter θ_{ij}^A in this setting measures consumer i's preference intensity when she consumes content provided by content provider j, which is then multiplied by her utility that depends on the consumer i's choice of her residential Internet quality, q_i^A, and the sending content provider j's quality, q_j^B. Similarly, θ_{ji}^B measures content provider j's preference intensity relating to the revenue which is also affected by the types of i and j.

- Advertising platform: In Section 6, we apply the model to an advertising platform. In the application, $q_i^A \in \{0, 1\}$ and $q_i^A = 1$ means no exposure to advertising like YouTube Red and $q_i^A = 0$ means exposure to advertising like standard YouTube. θ_{ij}^A captures consumer i's nuisance from advertiser j's advertisement. On side B, q_j^B represents advertising amount of advertiser j and θ_{ji}^B measures j's advertising revenue which is jointly affected by consumer i's type such as income and advertiser j's type such as the advertised product's characteristics.

- Privacy protection and targeted advertising: Consider consumer privacy protection design by an online-advertising platform who uses the information released from consumers to increase efficiency in targeted advertising. In this environment, q_i^A captures the level of privacy designed for consumer i and q_j^B the advertising amount by advertiser j, while the intensity measures typify the match-based preferences by consumers and advertisers.[9]

We assume that the utility function $u^k : \mathbb{R}^2_+ \to \mathbb{R}$ is strictly increasing in q^k, and concave in (q^k, q^l) with $k \neq l$ and $k, l = \mathrm{A}, \mathrm{B}$. Note that u^k may increase or decrease with q^l. For example, it is increasing in q^l in net neutrality application, but decreasing in q^l in the application to an advertising platform. Let u^k_m denote the partial derivative of u^k with respect to its m-th variable, for $m = 1, 2$. Moreover, we define u^k_{12} as follows:

$$u^k_{12}(q^k_i, q^l_j) \equiv \frac{\partial^2 u^k(q^k_i, q^l_j)}{\partial q^k_i \partial q^l_j}.$$

We assume that u^k_{12} has the same sign for each q^k_i and q^l_j; for a given side k the qualities are said to be independent if $u^k_{12} = 0$, complements if $u^k_{12} > 0$, and substitutes if $u^k_{12} < 0$. The costs of producing q^A_i and q^B_j are respectively denoted by $C^A(q^A_i)$ and $C^B(q^B_j)$. We assume that both cost functions are strictly increasing and convex.

Depending on the match of types, we may have the following four parameters of consumption intensity on side k:

$k \backslash l$	H	L
H	θ^k_{HH}	θ^k_{HL}
L	θ^k_{LH}	θ^k_{LL}

To give a standard meaning to the H and L types, we introduce the following notation and assumption:

Assumption 1. $\theta^k_H \equiv \nu^l_H \theta^k_{HH} + \nu^l_L \theta^k_{HL} > \theta^k_L \equiv \nu^l_H \theta^k_{LH} + \nu^l_L \theta^k_{LL}$ *with* $k \neq l$ *and* $k, l = A, B$.

Assumption 1 means that after a marginal increase in u^k, the H type on side k enjoys a higher increase in (expected) benefit than the L type on side k when interacting with all agents on the other side l. Under Assumption 1, we can further identify three sub-cases depending on the signs of $\theta^k_{HH} - \theta^k_{LH}$ and of $\theta^k_{HL} - \theta^k_{LL}$.[10]

[9] According to a recent settlement between the Federal Communications Commission (FCC) and Verizon Wireless in March 2016, the wireless company needs opt in from users in order to employ its tracking system so-called "supercookies" for targeted advertising. In the model, q^A_i can have two binary values for opt-in and opt-out (default) choices.

[10] We are implicitly assuming that $\theta^k_{HH} - \theta^k_{LH} \neq 0$ and $\theta^k_{HL} - \theta^k_{LL} \neq 0$ but this is immaterial and only for expositional brevity.

Definition (type reversal) We say that on side k, there is

$$\begin{cases} \text{no type reversal} & \text{if } \theta^k_{HH} - \theta^k_{LH} > 0 \text{ and } \theta^k_{HL} - \theta^k_{LL} > 0; \\ \text{type reversal with a positive sorting} & \text{if } \theta^k_{HH} - \theta^k_{LH} > 0 > \theta^k_{HL} - \theta^k_{LL}; \\ \text{type reversal with a negative sorting} & \text{if } \theta^k_{HL} - \theta^k_{LL} > 0 > \theta^k_{HH} - \theta^k_{LH}. \end{cases}$$

Type reversal arises on side k if an L type gets more benefit than an H type of the same side k when interacting with a *particular* type of the other side l, although an H type of side k gets more benefit in expected terms than an L type of side k when interacting with *all* the agents on side l. If this particular type is the L (H) type, then we have the type reversal with a positive (negative) sorting.

The platform offers a menu of quality-price pairs $\left\{(q^k_H, p^k_H), (q^k_L, p^k_L)\right\}$ on each side k $(= \mathrm{A}, \mathrm{B})$ where $p^k_H \in \mathbf{R}$ (for instance) is a fixed payment from a H-type agent to the platform. Let $\mathbf{q} \equiv (q^A_H, q^A_L, q^B_H, q^B_L) \in \mathbf{R}^4_+$ denote the vector of quality specifications. We assume that it is optimal for the platform to induce full participation of all agents on both sides. Therefore, no PD on side k means that the platform offers a single contract $(q^k_H, p^k_H) = (q^k_L, p^k_L)$ on side k which satisfies the participation constraints of both types.

Even with a two-type model of the multiplicative specification, our model is still characterized by quite a few parameters of $\Theta^k \equiv \{\theta^k_{HH}, \theta^k_{HL}, \theta^k_{LH},\ \theta^k_{LL}\}$, ν^k, and the utility function u^k is defined for each $k =$ A, B. For this reason, when necessary, we consider a simpler case by further specifying the model. In the case that $u^A_{12} = u^B_{12} = 0$, we have that u^A and u^B are separable in the sense that there exist four single variable functions u^A_A, u^A_B, u^B_B, and u^B_A such that

$$u^A(q^A, q^B) = u^A_A(q^A) + u^A_B(q^B) \quad \text{and} \quad u^B(q^B, q^A) = u^B_B(q^B) + u^B_A(q^A)$$

where the superscripts refer to the side of the agent whose utility is computed whereas the subscripts refer to the side of which the quality affects the utility of the agent. In Section 3-4, we focus on this separable case whereas in Sections 5-6 we consider the non-separable case.

3 First-best in the separable case

In this section we characterize the first-best quality schedule that maximizes the total surplus given as follows:

$$\begin{aligned}\Pi^{FB}(\mathbf{q}) = \nu_H^A \nu_H^B \left[\theta_{HH}^A u^A(q_H^A, q_H^B) + \theta_{HH}^B u^B(q_H^B, q_H^A)\right] \\ + \nu_H^A \nu_L^B \left[\theta_{HL}^A u^A(q_H^A, q_L^B) + \theta_{LH}^B u^B(q_L^B, q_H^A)\right] \\ + \nu_L^A \nu_H^B \left[\theta_{LH}^A u^A(q_L^A, q_H^B) + \theta_{HL}^B u^B(q_H^B, q_L^A)\right] \\ + \nu_L^A \nu_L^B \left[\theta_{LL}^A u^A(q_L^A, q_L^B) + \theta_{LL}^B u^B(q_L^B, q_L^A)\right] \\ - \nu_H^A C^A(q_H^A) - \nu_L^A C^A(q_L^A) - \nu_H^B C^B(q_H^B) - \nu_L^B C^B(q_L^B),\end{aligned}$$

where each of the first four lines represents the total surplus from each matching pattern (H, H), (H, L), (L, H) and (L, L) while the last line measures the total costs.[11]

Given our assumptions, Π^{FB} is concave and therefore the FOCs characterize the first-best quality schedule. In the separable case, the first-best quality schedule on side A i.e., $(q_H^{A,FB}, q_L^{A,FB})$ is determined by a system of the following two FOCs:

$$\theta_H^A u_A^{A\prime}(q_H^A) + (\nu_H^B \theta_{HH}^B + \nu_L^B \theta_{LH}^B) u_A^{B\prime}(q_H^A) = C^{A\prime}(q_H^A); \tag{1}$$

$$\theta_L^A u_A^{A\prime}(q_L^A) + (\nu_H^B \theta_{HL}^B + \nu_L^B \theta_{LL}^B) u_A^{B\prime}(q_L^A) = C^{A\prime}(q_L^A). \tag{2}$$

Note that none of (1) and (2) depend on the quality schedule on side B.

We find that the first-best quality schedule is non-monotonic on side A (i.e., $q_H^{A,FB} < q_L^{A,FB}$) if and only if

$$(\theta_H^A - \theta_L^A) u_A^{A\prime}(q_H^{A,FB}) + \left[\nu_H^B \left(\theta_{HH}^B - \theta_{HL}^B\right) + \nu_L^B \left(\theta_{LH}^B - \theta_{LL}^B\right)\right] u_A^{B\prime}(q_H^{A,FB}) < 0. \tag{3}$$

For a clearer interpretation of (3), let us consider the special case in which there exists a $\beta > 0$ such that

$$u_A^{B\prime}(q) = \beta u_A^{A\prime}(q) \qquad \text{for each } q > 0. \tag{4}$$

[11] An alternative cost function is

$$C^A(\nu_H^A q_H^A + \nu_L^A q_L^A) + C^B(\nu_H^B q_H^B + \nu_L^B q_L^B).$$

Qualitative results in this paper remain robust regardless of which cost function is chosen.

Then, the condition (3) becomes

$$\underbrace{\left(\theta_H^A - \theta_L^A\right)}_{(\dagger)>0} + \underbrace{\beta\left[\nu_H^B\left(\theta_{HH}^B - \theta_{HL}^B\right) + \nu_L^B\left(\theta_{LH}^B - \theta_{LL}^B\right)\right]}_{(\ddagger)\gtreqless 0} < 0.$$

The first bracketed term denoted by (†) represents the change in the private benefit that a side A agent experiences when his type changes from L to H, which is positive under Assumption 1. By contrast, the term (‡) represents the change in the externality onto the agents on side B from the same type change. If $\nu_H^B\left(\theta_{HH}^B - \theta_{HL}^B\right) + \nu_L^B\left(\theta_{LH}^B - \theta_{LL}^B\right) > 0$, the H type of side A generates a greater positive or a smaller negative externality to side B than the L type of side A does. Hence, the first-best quality schedule is such that $q_H^{A,FB} > q_L^{A,FB}$. However, if the externality term (‡) is negative and important enough that the overall sign turns to become negative, we have $q_H^{A,FB} < q_L^{A,FB}$. Such case occurs when the L type agent on side A generates a sufficiently large positive (or a sufficiently small negative externality) to side B relative to the H type agent.

Proposition 1. *(First-best) Suppose that Assumption 1 holds and both u^A and u^B are separable (i.e., $u_{12}^A = u_{12}^B = 0$).*

(i) The first-best quality schedule on side A, $(q_H^{A,FB}, q_L^{A,FB})$ is determined by (1) and (2) independently of $(q_H^{B,FB}, q_L^{B,FB})$.

(ii) We have $q_H^{A,FB} < q_L^{A,FB}$ if and only if an H type's relative gain in terms of private benefit on side A *is smaller than an L type (of side A)'s relative contribution in terms of externality to side* B *(i.e., inequality (3) holds).*

(iii) Parallel statements can be made regarding $q_H^{B,FB}$ and $q_L^{B,FB}$.

4 Second-best in the separable case

In this section, we study the second-best mechanism in the separable case and identify the distortions generated by asymmetric information. By doing so, we can clearly identify two different sources for non-responsiveness: one is known from Guesnerie and Laffont (1984) and the other is new and arises due to the two-sidedness of the market.

The platform's optimization problem is given by:

$$\max_{\{(q_H^k,p_H^k),(q_L^k,p_L^k)\}} \nu_H^A\left[p_H^A - C^A(q_H^A)\right] + \nu_L^A\left[p_L^A - C^A(q_L^A)\right] + \nu_H^B\left[p_H^B - C^B(q_H^B)\right] + \nu_L^B\left[p_L^B - C^B(q_L^B)\right]$$

subject to

$$(\text{IR}_H^k) \qquad \nu_H^l \theta_{HH}^k u^k(q_H^k, q_H^l) + \nu_L^l \theta_{HL}^k u^k(q_H^k, q_L^l) - p_H^k \geq 0;$$
$$(\text{IR}_L^k) \qquad \nu_H^l \theta_{LH}^k u^k(q_L^k, q_H^l) + \nu_L^l \theta_{LL}^k u^k(q_L^k, q_L^l) - p_L^k \geq 0;$$

$$(\text{IC}_H^k)\ \nu_H^l \theta_{HH}^k u^k(q_H^k, q_H^l) + \nu_L^l \theta_{HL}^k u^k(q_H^k, q_L^l) - p_H^k \geq \nu_H^l \theta_{HH}^k u^k(q_L^k, q_H^l) + \nu_L^l \theta_{HL}^k u^k(q_L^k, q_L^l) - p_L^k; \tag{5}$$
$$(\text{IC}_L^k)\ \nu_H^l \theta_{LH}^k u^k(q_L^k, q_H^l) + \nu_L^l \theta_{LL}^k u^k(q_L^k, q_L^l) - p_L^k \geq \nu_H^l \theta_{LH}^k u^k(q_H^k, q_H^l) + \nu_L^l \theta_{LL}^k u^k(q_H^k, q_L^l) - p_H^k; \tag{6}$$

where $k, l = A, B$ and $k \neq l$.

We solve this problem when u^A and u^B are separable, under the following assumption:

$$u_l^k(0) = 0, \quad u_l^k \text{ is increasing and concave for each } k, l = A, B \tag{7}$$

Then the constraints get simplified as follows:

$$(\text{IR}_H^k) \qquad \theta_H^k u_k^k(q_H^k) + \nu_H^l \theta_{HH}^k u_l^k(q_H^l) + \nu_L^l \theta_{HL}^k u_l^k(q_L^l) - p_H^k \geq 0;$$
$$(\text{IR}_L^k) \qquad \theta_L^k u_k^k(q_L^k) + \nu_H^l \theta_{LH}^k u_l^k(q_H^l) + \nu_L^l \theta_{LL}^k u_l^k(q_L^l) - p_L^k \geq 0;$$

$$(\text{IC}_H^k)\ \theta_H^k u_k^k(q_H^k) - p_H^k \geq \theta_H^k u_k^k(q_L^k) - p_L^k;$$
$$(\text{IC}_L^k)\ \theta_L^k u_k^k(q_L^k) - p_L^k \geq \theta_L^k u_k^k(q_H^k) - p_H^k.$$

Notice that the two IC constraints (IC_H^A) and (IC_L^A) are independent of $\left(q_H^B, q_L^B\right)$ (since $u_{12}^A = 0$), but $\left(q_H^B, q_L^B\right)$ affects both IR constraints (IR_H^A) and (IR_L^A). By adding up the two incentive constraints on side k, we obtain the implementability condition on side k, which is equivalent to the monotonicity condition ($q_H^k \geq q_L^k$).

We focus on the standard case in which (IR_L^k) and (IC_H^k) are binding while (IR_H^k) is redundant.

Lemma 1. *When u^A and u^B are separable, under Assumption 1 and (7), suppose that for a given side k, there is no type reversal or there is type reversal with positive sorting. Then (IR_H^k) is redundant, (IR_L^k) and (IC_H^k) bind in the optimal mechanism, and (IC_L^k) is equivalent to $q_H^k \geq q_L^k$.*

Proof. See Appendix A. □

Lemma 1[12] allows to use (IR_L^k) and (IC_H^k) to pin down the agents' payments on

[12] The reason why we discard type reversal with negative sorting in Lemma 1 is the following. When (IR_L^k) and (IC_H^k) bind, (IR_H^k) is redundant if $(\theta_H^A - \theta_L^A)u_A^A(q_L^A) + \nu_H^B(\theta_{HH}^A - \theta_{LH}^A)u_B^A(q_H^B) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A)u_B^A(q_L^B) \geq 0$. However, if $\theta_{HH}^A < \theta_{LH}^A$ and $\theta_{HL}^A > \theta_{LL}^A$ then the above term can be negative if θ_{LH}^A

side k, and then we can write the expression for H-type's information rent as follows:

$$\Omega_H^k = (\theta_H^k - \theta_L^k)u_k^k(q_L^k) + \nu_H^l(\theta_{HH}^k - \theta_{LH}^k)u_l^k(q_H^l) + \nu_L^l(\theta_{HL}^k - \theta_{LL}^k)u_l^k(q_L^l).$$

Therefore, the platform's original problem is equivalent to maximizing the following objective subject to the monotonicity constraints $q_H^A \geq q_L^A$ and $q_H^B \geq q_L^B$:

$$\hat{\Pi}(\mathbf{q}) \equiv \Pi^{FB}(\mathbf{q}) - \nu_H^A\Omega_H^A - \nu_H^B\Omega_H^B. \tag{8}$$

Let $\hat{\mathbf{q}}$ denote the maximizer of $\hat{\Pi}$ when the monotonicity constraint is neglected. When we focus on side A, from the first-order conditions, we have:

$$\theta_H^A u_A^{A\prime}(\hat{q}_H^A) + \theta_{LH}^B u_A^{B\prime}(\hat{q}_H^A) = C^{A\prime}(q_H^A); \tag{9}$$

$$\theta_L^{Av} u_A^{A\prime}(\hat{q}_L^A) + \theta_{LL}^B u_A^{B\prime}(\hat{q}_L^A) = C^{A\prime}(q_L^A), \tag{10}$$

where $\theta_L^{Av} \equiv \theta_L^A - \left(\theta_H^A - \theta_L^A\right) v_H^A/\nu_L^A$ is the virtual valuation of an L-type of side A, which is smaller than θ_L^A under Assumption 1. Assume momentarily that $\hat{q}_H^A \geq \hat{q}_L^A$, which implies $q_H^{A,SB} = \hat{q}_H^A$ and $q_L^{A,SB} = \hat{q}_L^A$.

To identify the distortions generated by asymmetric information, we compare (1) and (2) with (9) and (10). First, regarding the utility that the quality of side A generates to the same side, θ_L^A is replaced by θ_L^{Av}, which is well-known from the price discrimination in one-sided market. Second, regarding the utility that the quality of side A generates to the other side, $\nu_H^B\theta_{HH}^B + \nu_L^B\theta_{LH}^B$ is replaced with θ_{LH}^B and $\nu_H^B\theta_{HL}^B + \nu_L^B\theta_{LL}^B$ is replaced with θ_{LL}^B, which generates new distortion from the two-sidedness of the market. This occurs because the payment of type H on side B is determined not by (IR$_H^B$) but by (IC$_H^B$) (see Lemma 1). In particular, there is a distortion in $q_H^{A,SB}$ whenever $\theta_{HH}^B \neq \theta_{LH}^B$, and $q_H^{A,FB} \gtrless q_H^{A,SB}$ if and only if $\theta_{HH}^B \gtrless \theta_{LH}^B$. If an H type of side B obtains more benefit than an L type of the same side from interacting with an H type agent of side A, then there is a downward distortion in q_H^A. To make the same comparison for an L type, consider the special case in which (4) holds. Then $q_L^{A,SB} > q_L^{A,FB}$ if and only if

$$\beta\nu_H^B(\theta_{LL}^B - \theta_{HL}^B) > \theta_L^A - \theta_L^{Av} = \frac{\nu_H^A}{\nu_L^A}(\theta_H^A - \theta_L^A).$$

is sufficiently larger than θ_{HH}^A and q_H^B is sufficiently larger than q_L^B. For a similar reason, we consider u_B^A increasing: otherwise, the above term can be negative.

In summary, (i) no distortion at the top is not valid any more: there can be either an upward or a downward distortion at the top; (ii) a downward distortion at the bottom is likely but does not necessarily occur; under type reversal with positive sorting on side B such that $\theta^B_{LL} - \theta^B_{HL}$ is large enough, there is an upward distortion at the bottom on side A.

Now we turn to the monotonicity constraint. If $\widehat{q}^A_H < \widehat{q}^A_L$ holds, then $(q^{A,SB}_H, q^{A,SB}_L) \neq (\widehat{q}^A_H, \widehat{q}^A_L)$ and the monotonicity constraint binds in the optimum, that is we have $q^{A,SB}_H = q^{A,SB}_L$. This occurs if and only if the following condition holds:

$$(\theta^A_H - \theta^{Av}_L) u^{A\prime}_A(\widehat{q}^A_H) + \left[\theta^B_{LH} - \theta^B_{LL}\right] u^{B\prime}_A(\widehat{q}^A_H) < 0. \tag{11}$$

In the special case in which (4) holds, the condition becomes

$$\theta^A_H + \beta\theta^B_{LH} < \theta^{Av}_L + \beta\theta^B_{LL}.$$

In order to isolate effects, we decompose it as follows:

$$\underbrace{\theta^A_H - \theta^A_L + \beta(\nu^B_H\theta^B_{HH} + \nu^B_L\theta^B_{LH} - \nu^B_H\theta^B_{HL} - \nu^B_L\theta^B_{LL})}_{\text{First best term}}$$
$$< \underbrace{-\frac{\nu^A_H}{\nu^A_L}\left[\theta^A_H - \theta^A_L\right]}_{\text{distortion in one-side market}} + \underbrace{\nu^B_H\left[\left(\theta^B_{HH} - \theta^B_{LH}\right) - \left(\theta^B_{HL} - \theta^B_{LL}\right)\right]\beta}_{\text{distortion due to two-sidedness}} \tag{12}$$

If there were no distortion due to the two-sidedness of the market, then a necessary condition for (12) to be satisfied is that the first-best schedule is non-monotonic (i.e., the left hand side in (12) is negative). This is what happens in one-sided market as in Guesnerie and Laffont (1984). However, (12) can be satisfied even if the first-best schedule is monotonic as long as the distortion from the two-sidedness is strong enough. In case of pooling, we have that $q^{A,SB}_H = q^{A,SB}_L \equiv q^{A,SB}$ satisfies $\theta^A_L u^{A\prime}_A(q) + \theta^B_L u^{B\prime}_A(q) = C^A(q)$. If the optimal mechanism involves pooling on side A, then PD on side A becomes substitute for PD on side B as applying PD to both sides is not optimal.

We summarize thus far results as follows:

Proposition 2. *Suppose Assumption 1 and that both u^A and u^B are separable (i.e., $u^A_{12} = u^B_{12} = 0$) with u^k_l increasing and concave for each $k, l = A, B$, and u^A_A, u^B_A satisfy (4). In addition, assume that on side A there is no type reversal or there is type reversal with positive sorting as in Lemma 1.*

(i) (Quality distortions) The second best mechanism is such that $q^{A,SB}_H > q^{A,SB}_L$ if and

only if (12) is satisfied with the reverse inequality. In such a case the standard "no distortion at the top and a downward distortion at the bottom" result does not hold any more because of the new distortion due to two-sidedness of the market.

(a) $q_H^{A,FB} \gtreqless q_H^{A,SB}$ *if and only if* $\theta_{HH}^B \gtreqless \theta_{LH}^B$.

(b) $q_L^{A,SB} > q_L^{A,FB}$ *if and only if*

$$\beta\nu_H^B(\theta_{LL}^B - \theta_{HL}^B) > \theta_L^A - \theta_L^{Av} = \frac{\nu_H^A}{\nu_L^A}(\theta_H^A - \theta_L^A).$$

(ii) (Non-responsiveness) Pooling on side A becomes optimal if (12) holds, that is if $\beta\left[\theta_{LL}^B - \theta_{LH}^B\right]$ *is large enough. Pooling can occur even when the first-best quality schedule is monotonic because of the distortion from the two-sidedness of the market.*

(iii) If the optimal mechanism involves pooling on side k, then PD on side k is substitute for PD on side l ($\neq k$)*.*

5 Implementable allocations in the non-separable case

In this section we consider the non-separable case. As a preliminary step, in Section 5.1, we characterize the implementable allocations on side A with type reversal given an arbitrary quality schedule on side B. In Section 5.2, we consider a symmetric two-sided platform in which type reversal occurs on both sides and study the set of implementable allocations on both sides with symmetric mechanisms.

By summing the incentive constraints (5) and (6) and considering $k = A$, we find the implementability condition on side A as follows:

$$\begin{aligned}\Phi^A := \ & \nu_H^B\left[\theta_{HH}^A - \theta_{LH}^A\right]\left[u^A(q_H^A, q_H^B) - u^A(q_L^A, q_H^B)\right] \\ & + \nu_L^B\left[\theta_{HL}^A - \theta_{LL}^A\right]\left[u^A(q_H^A, q_L^B) - u^A(q_L^A, q_L^B)\right] \geq 0.\end{aligned} \tag{13}$$

5.1 Implementable allocations on side A given quality schedule on side B

Let us take (q_H^B, q_L^B) as given and assume a type reversal of positive sorting on side A, i.e., $(\theta_{HH}^A - \theta_{LH}^A) > 0 > (\theta_{HL}^A - \theta_{LL}^A)$. Later, we describe briefly how our results will extend to the case of negative sorting. Let F denote the set of (q_H^A, q_L^A) satisfying the implementability condition on side A (13) for a given pair of (q_H^B, q_L^B).[13] In order to

[13] Hence, F depends on (q_H^B, q_L^B) though our notation does not make it explicit.

describe F, we let

$$\begin{cases} M \text{ (from "monotonic")} & \text{denote the set of } (q_H^A, q_L^A) \text{ such that } q_H^A > q_L^A \geq 0; \\ N \text{ (from "non-monotonic")} & \text{denote the set of } (q_H^A, q_L^A) \text{ such that } 0 \leq q_H^A < q_L^A; \\ D \text{ (from "diagonal")} & \text{denote the set of } (q_H^A, q_L^A) \text{ such that } 0 \leq q_H^A = q_L^A. \end{cases}$$

Since $\Phi^A = 0$ at each point satisfying $q_H^A = q_L^A$, it is obvious that $D \subseteq F$. Moreover it is immediate to identify F if Φ^A is strictly monotonic with respect to q_H^A. Precisely, if Φ^A is strictly increasing in q_H^A then $F = M \cup D$; if Φ^A is strictly decreasing with respect to q_H^A, then $F = N \cup D$.

If u^A is separable, then

$$\Phi^A = \left(\theta_H^A - \theta_L^A\right)\left(u_A^A(q_H^A) - u_A^A(q_L^A)\right),$$

which is strictly increasing in q_H^A by Assumption 1, and thus $F = M \cup D$. This result does not depend on whether the type reversal occurs with a positive sorting or a negative sorting.

Considering a positive sorting and complementarity between the qualities, we have

$$\frac{\partial \Phi^A}{\partial q_H^A} = \nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) u_1^A(q_H^A, q_H^B) + \nu_L^B \left(\theta_{HL}^A - \theta_{LL}^A\right) u_1^A(q_H^A, q_L^B).$$

Therefore, Φ^A is strictly increasing in q_H^A if $q_H^B \geq q_L^B$ (since in this case $u_1^A(q_H^A, q_H^B) \geq u_1^A(q_H^A, q_L^B)$ and Assumption 1 holds) or if $q_H^B < q_L^B$ and $|\nu_L^B \left(\theta_{HL}^A - \theta_{LL}^A\right)|$ is close to zero and/or the effect of complementarity is small. Conversely, if $q_H^B < q_L^B$, $|\nu_L^B \left(\theta_{HL}^A - \theta_{LL}^A\right)|$ is close to $\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right)$ (i.e., $\theta_H^A - \theta_L^A$ is close to zero) and the effect of complementarity is strong, then Φ^A is strictly decreasing with respect to q_H^A.

In the case of substitutes, we obtain opposite results: Φ^A is strictly increasing in q_H^A if $q_H^B \leq q_L^B$ (again $u_1^A(q_H^A, q_H^B) \geq u_1^A(q_H^A, q_L^B)$ and Assumption 1 holds), or if $q_H^B > q_L^B$ and $|\nu_L^B \left(\theta_{HL}^A - \theta_{LL}^A\right)|$ is close to zero and/or the effect of substitution is small. Conversely, if $q_H^B > q_L^B$, $|\nu_L^B \left(\theta_{HL}^A - \theta_{LL}^A\right)|$ is close to $\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right)$ and the effect of substitution is strong, then Φ^A is strictly decreasing with respect to q_H^A. The case of a negative sorting (i.e., $\theta_{HL}^A - \theta_{LL}^A > 0 > \theta_{HH}^A - \theta_{LH}^A$) is symmetric to that of positive sorting.

The following proposition summarizes the results.

Proposition 3. *Suppose that Assumption 1 holds, there is type reversal on side A, and quality schedule on side B $\left(q_H^B, q_L^B\right)$ is given.*

(i) If u^A is separable, the implementable set equals the set of the weakly monotonic

schedules (i.e., $F = M \cup D$) regardless of type reversal.

(ii) Suppose that on side A, qualities are complements (resp. substitutes) and type reversal occurs with a positive (resp. negative) sorting.

(a) The implementable set on side A is equal to the set of the weakly monotonic schedules (i.e., $F = M \cup D$) if $q_H^B \geq q_L^B$.

(b) The implementable set on side A is equal to the set of the weakly non-monotonic schedules $F = N \cup D$ if $q_H^B < q_L^B$, the complementarity (resp. the substitution) is sufficiently strong and $\theta_H^A - \theta_L^A$ is close to zero.

(iii) Suppose that qualities are substitutes (resp. complements) and type reversal occurs with a positive (resp. negative) sorting. Then, the same statements as above in (ii) can be made for (a) if $q_H^B \leq q_L^B$ (b) if $q_H^B > q_L^B$, the substitution (resp. the complementarity) is sufficiently strong and $\theta_H^A - \theta_L^A$ is close to zero.

Proof. See Appendix A. □

Proposition 3 identifies when implementing a non-monotonic schedule on side A requires a monotonic (or non-monotonic) schedule on side B. Let us provide the intuition. The implementability condition (13) means that given the quality schedule on side B, when $q_H^A < q_L^A$, the L type's utility gain from receiving q_L^A instead of q_H^A must be larger than the H type's gain from doing the same. Consequently, absent PD on side B, a non-monotonic schedule (i.e., $q_H^A < q_L^A$) is not implementable by the definition of the H and L types.

However, this is no longer the case if we introduce PD on side B under type reversal on side A. Consider type reversal with a positive sorting and suppose that the qualities are complements. If $q_H^B < q_L^B$, a non-monotonic schedule (i.e., $q_H^A < q_L^A$) becomes now implementable as an L type's utility gain can be larger than an H type's one when the quality increases from q_H^A to q_L^A. This is because an L type enjoys a high marginal utility from interacting with an L type when the qualities are complements and type reversal with a positive sorting arises ($\theta_{HL}^A < \theta_{LL}^A$). Symmetrically, if qualities are substitutes and there is type reversal with a positive sorting, implementing a non-monotonic schedule on side A requires $q_H^B > q_L^B$.

The discussions of the case with no type reversal and the above proposition give us sufficient conditions for F to equal the set of the weakly monotonic schedules:

Corollary 1. *The implementable set on side A is equal to the set of the weakly monotonic schedules if at least one of the following conditions is satisfied:*

(i) There is no type reversal on side A;

(ii) u^A is separable;

(iii) There is no PD on side B (i.e., $q_H^B = q_L^B$).

5.2 Implementable allocations on both sides: a symmetric model

Building upon what we have obtained from the preceding analysis, now we study implementable allocations on both sides. While we will consider an asymmetric two-sided market of advertising platform in Section 6, here we find it useful to examine a symmetric two-sided market because its simplicity helps isolate the main driving forces. The following notations are made for the symmetric model:

$$\begin{cases} \theta_{HH}^A = \theta_{HH}^B \equiv \theta_{HH}; \quad \theta_{HL}^A = \theta_{HL}^B \equiv \theta_{HL}; \quad \theta_{LH}^A = \theta_{LH}^B \equiv \theta_{LH}; \quad \theta_{LL}^A = \theta_{LL}^B \equiv \theta_{LL} \\ \theta_H^A = \theta_H^B \equiv \theta_H; \quad \theta_L^A = \theta_L^B \equiv \theta_L; \\ \nu_H^A = \nu_H^B \equiv \nu_H; \quad \nu_L^A = \nu_L^B \equiv \nu_L; \\ u^A = u^B \equiv u. \end{cases}$$

We focus on a symmetric mechanism with $q_H^A = q_H^B = q_H$ and $q_L^A = q_L^B = q_L$. The analysis of the optimal mechanism is relegated to online Appendix.

In the symmetric model with a symmetric mechanism, the implementability condition (13) becomes:

$$\nu_H(\theta_{HH} - \theta_{LH})[u(q_H, q_H) - u(q_L, q_H)] + \nu_L(\theta_{HL} - \theta_{LL})[u(q_H, q_L) - u(q_L, q_L)] \geq 0. \quad (14)$$

As the case of a negative sorting is symmetric to the case of a positive sorting, here we focus on a positive sorting. It is convenient to define $r \equiv \frac{\nu_L(\theta_{HL} - \theta_{LL})}{\nu_H(\theta_{HH} - \theta_{LH})}$, such that $r \in (-1, 0)$ under Assumption 1. Hence, the implementability condition in (14) can be written as

$$A + rB \geq 0 \quad (15)$$

where

$$A = \int_{q_L}^{q_H} u_1(t, q_H)dt \qquad \text{and} \qquad B = \int_{q_L}^{q_H} u_1(t, q_L)dt.$$

We below study the set of (q_H, q_L) that satisfies (15) by distinguishing the case of complements from that of substitutes.

In the case of type reversal with a negative sorting, the implementability condition in (14) can be written as

$$r'A + B \geq 0 \tag{16}$$

where $r' \equiv \frac{\nu_H(\theta_{HH}-\theta_{LH})}{\nu_L(\theta_{HL}-\theta_{LL})} \in (-1, 0)$.

■ **The case of substitutes**

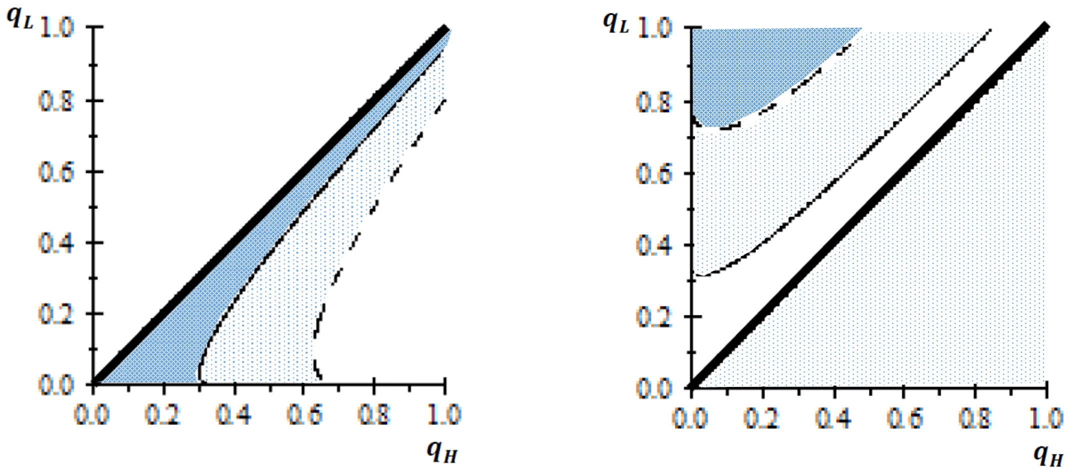
Consider the case in which the qualities are substitutes and there is type reversal with a positive sorting. If $q_H > q_L$, we have $u_1(t, q_H) < u_1(t, q_L)$ because of the substitution, which implies $B > A > 0$. Therefore, (15) is satisfied at (q_H, q_L) for $r \geq -A/B$ and is violated for $r < -A/B$. In particular, any pair (q_H, q_L) with $q_H > q_L$ satisfies the implementability condition (15) for $r = 0$, and no pair (q_H, q_L) such that $q_H > q_L$ satisfies the implementability condition (15) for $r = -1$, given $B > A$. Next if we consider $q_H < q_L$, we have $u_1(t, q_H) > u_1(t, q_L)$, implying $A < B < 0$. Hence, for any $r \in (-1, 0)$, no pair (q_H, q_L) with $q_H < q_L$ satisfies the implementability condition (15). Therefore the implementable set does not include any non-monotonic schedule. In addition, some monotonic schedules are also excluded from the implementable set if r is close to -1.

For illustrative purposes, consider an example of $u(q_H, q_L) = 4\sqrt{q_H} - q_H q_L + 4\sqrt{q_L}$, and assume that $q_H \in [0, 1]$, $q_L \in [0, 1]$ in order for u to be concave. Let $r^* = -\frac{6.88}{10}$. If $r \in (r^*, 0)$, then each (q_H, q_L) such that $q_H > q_L$ satisfies (15). Conversely, if $r \in (-1, r^*)$ then there exist some (q_H, q_L) with $q_H > q_L$ which do not satisfy (15). For instance, if $r = -\frac{8.5}{10}$ then (15) fails to hold for the points to the right of the dashed curve in Figure 1-(a); if $r = -\frac{9.5}{10}$, then (15) fails to hold for the points to the right of the thin curve in Figure 1-(a).

In the case of a negative sorting (i.e., $\theta_{HL} - \theta_{LL} > 0 > \theta_{HH} - \theta_{LH}$), the result is opposite following the similar reasoning. Suppose $q_H > q_L$. Because of the substitution, we have $u_1(t, q_H) < u_1(t, q_L)$ for any t, implying $B > A > 0$. Therefore, for any $r' \in (-1, 0)$ any pair (q_H, q_L) satisfying $q_H > q_L$ satisfies the implementability condition (16). Suppose now $q_H < q_L$. Because of the substitution, we have $u_1(t, q_H) > u_1(t, q_L)$ for any t, implying $A < B < 0$. Therefore, (16) is satisfied for $r' \leq -|B|/|A|$ and is violated for $r' > -|B|/|A|$. In particular, any pair (q_H, q_L) satisfying $q_H < q_L$ meets the implementability condition for $r' = -1$ and no pair (q_H, q_L) with $q_H < q_L$ satisfies the implementability condition for $r' = 0$. The previous arguments imply that the implementable set consists of all points below the 45 degree line (i.e., $q_H \geq q_L$), and possibly some points which are above the 45 degree line.

Recall from Corollary 1 that if there is PD only on one side, then the implementable set on that side coincides with all monotonic schedules. When PD occur on both sides,

[The set of implementable allocations with type reversal in the symmetric model]



(a) Substitutes with a positive sorting (or Complements with a negative sorting)

(b) Complements with a positive sorting (or Substitutes with a negative sorting)

Figure 1: Consider $u(q_H, q_L) = 4\sqrt{q_H} - q_H q_L + 4\sqrt{q_L}$. In Panel (a), the implementable allocations under a positive sorting when qualities are substitutes (or under a negative sorting when complements) comprise the points on and below the 45 degree line, $q_H \geq q_L$, except some points below the thin curve or the dashed curve depending on the value of r. In Panel (b), the implementable allocations under a positive sorting when qualities are complements (or under a negative sorting when substitutes) comprise all points on or below the 45 degree line, plus some points above the thin curve or the dashed curve depending on the value of r.

in the case of substitutes with a positive sorting, we see that with a symmetric mechanism the feasible set *shrinks*, as the platform cannot implement any non-monotonic schedule, and some monotonic schedules are not implementable either. Hence, PD on both sides are substitutes. By contrast, in the case of substitutes with a negative sorting, the PD on both sides *enlarges* the feasible set with respect to the case of PD on a single side: with a symmetric mechanism, the platform can implement any monotonic schedule, and possibly also some non-monotonic schedules on both sides. Therefore, PD on one side complements PD on the other side.

■ **The case of complements**

Consider now the case in which the qualities are complements and a positive sorting. Then, we can use the same reasoning applied to the analysis to the substitutes with a negative sorting. As a result, we find that the implementable set consists of all points below the 45 degree line (i.e., $q_H \geq q_L$), and possibly some points which are above the 45 degree line.

Let us revisit the example of $u(q_H, q_L) = 4\sqrt{q_H} + q_H q_L + 4\sqrt{q_L}$, with $q_H \in [0,1]$, $q_L \in [0,1]$. Let $r^* = -\frac{7.77}{10}$. If $r \in (r^*, 0)$, then no (q_H, q_L) with $q_L > q_H$ satisfies (15).

If $r \in (-1, r^*)$, then there exist some (q_H, q_L) such that $q_L > q_H$ which satisfy (15). For instance, if $r = -\frac{8.5}{10}$ then they are the points above the dashed curve in Figure 1-(b); if $r = -\frac{9.5}{10}$, then they are the points above the thin curve in Figure 1-(b).

In the case of a negative sorting, the result is similar to the case of substitutes with a positive sorting: the implementable set consists of only monotonic schedules and some monotonic schedules are excluded.

In summary, we have the followings:

Proposition 4. *Consider the symmetric two-sided market with private information on both sides and suppose that Assumption 1 holds.*

(i) For the case of substitutes and type reversal with a positive sorting (or complements with a negative sorting), the implementable set is a subset (possibly, a strict subset) of the set of weakly monotonic schedules. As $\theta_H - \theta_L$ tends to zero, the implementable set shrinks to the set of pooling schedules (i.e. $q_H = q_L$). Hence, PD on one side substitutes for PD on the other side.

(ii) For the case of complements and type reversal with a positive sorting (or substitutes with a negative sorting), the implementable set includes all weakly monotonic schedules and possibly some strictly non-monotonic schedules. As $\theta_H - \theta_L$ tends to zero, the implementable set expands to all monotonic and all non-monotonic schedules. Hence, PD on one side complements PD on the other side.

Remark (Optimal second-best schedule)**.** *Finding the second-best quality schedule can be complicated when the implementable set includes some non-monotonic schedules. For instance, consider the case of complements with a positive sorting. If $\widehat{q}_H \geq \widehat{q}_L$ holds, then we have $q_H^{SB} = \widehat{q}_H$ and $q_L^{SB} = \widehat{q}_L$. However, if $\widehat{q}_H < \widehat{q}_L$ holds, then $(\widehat{q}_H, \widehat{q}_L)$ can or cannot be implementable. If it is implementable and satisfies IR_H, then we have $q_H^{SB} = \widehat{q}_H$ and $q_L^{SB} = \widehat{q}_L$. Otherwise, we should compare the profit from the optimal pooling contract $\widehat{\pi}(q^p, q^p)$ and the highest profit from implementable non-monotonic schedule. When we solve for the latter, we should pay particular attention to IR_H as the best outcome from the implementable non-monotonic schedule may not satisfy IR_H. We further illustrate these points by analyzing a quadratic setting with complements (See the online appendix for more details).*

6 Application to an advertising platform

We here provide an application to demonstrate how our key insight plays out in a more realistic two-sided market of a media platform that mediates content users and advertisers via content. We consider private information on both sides.

6.1 The Model

There is a mass one of consumers on side A and a mass one of advertisers on side B. Agents on each side have two different types H and L. To reduce the number of parameters, we consider the equal population of each type on both sides, i.e., $\nu_H^A = \nu_H^B = 1/2$. On side A the platform offers a menu of quality-price pairs (q_H, p_H^A) and (q_L, p_L^A), with $(q_H, q_L) \in \{0,1\}^2$ where '1' means high quality or no nuisance from advertising and '0' means low quality or nuisance from advertising.[14] On side B, the platform offers a menu of advertising levels and prices: (a_H, p_H^B) and (a_L, p_L^B), with $\{a_H, a_L\} \in \mathbb{R}_+^2$. Each consumer earns a constant utility $u_0 > 0$ from consuming the content offered by the platform if he does not receive any advertising. Consumer i suffers disutility from advertiser j's ads which is given by $\alpha_{ij}\psi(a_j)$ with $\alpha_{ij} > 0$ where we assume $\psi(\cdot)\,(\geq 0)$ is increasing. Then, consumer i's gross utility when $q_i = 0$ is given as

$$\begin{cases} u_0 - \frac{1}{2}\alpha_{HH}\psi(a_H) - \frac{1}{2}\alpha_{HL}\psi(a_L), & \text{if } \theta_i^A = H; \\ u_0 - \frac{1}{2}\alpha_{LH}\psi(a_H) - \frac{1}{2}\alpha_{LL}\psi(a_L), & \text{if } \theta_i^A = L. \end{cases}$$

In a similar manner let $\beta_{ji}R(a_j)$ with $\beta_{ji} > 0$ represent the revenue that advertiser j earns from consumer i when $q_i = 0$ where $R(\cdot)\,(\geq 0)$ is increasing with the advertising amount. Then, advertiser j's expected revenue from joining the platform is given by

$$\begin{cases} \frac{1}{2}\beta_{HH}R(a_H) + \frac{1}{2}\beta_{HL}R(a_H), & \text{if } \theta_j^B = H; \\ \frac{1}{2}\beta_{LH}R(a_L) + \frac{1}{2}\beta_{LL}R(a_L), & \text{if } \theta_j^B = L. \end{cases}$$

Then we impose the following assumptions on the parameters for two-sided interactions.

[14] Targeted advertising is not considered here in the sense that each consumer i commonly receives all advertising ($q_i = 0$) or no advertising ($q_i = 1$).

Assumption 2.

$$\begin{cases} (i)\ \alpha_{HH} + \alpha_{HL} > \alpha_{LH} + \alpha_{LL} \\ (ii)\ \alpha_{HH} < \alpha_{LH}, \quad \alpha_{HL} > \alpha_{LL} \\ (iii)\ \beta_{HH} > \beta_{LH}, \quad \beta_{HL} > \beta_{LL} \\ (iv)\ \beta_{HH} > \beta_{HL}, \quad \beta_{LH} > \beta_{LL}. \end{cases}$$

The first inequality (i) means that an H type consumer suffers more from nuisance than an L type in expected terms, which is equivalent to Assumption 1 applied to side A. The two inequalities in the second line (ii) introduce type reversal on side A. Conditional on receiving the ads from H type advertisers, an H type consumer's nuisance is smaller than an L type consumer's nuisance. Against L type advertisers, by contrast, the opposite holds. The inequalities in the third line (iii) means that an H type advertiser generates more revenue than an L type no matter what the consumer type they interact with: in other words, there is no type reversal on the advertiser side. Hence, this assumption is stronger than Assumption 1 applied to side B. Lastly, (iv) means that an H type consumer is more valuable than an L type consumer in terms of advertising revenue for both types of advertisers.

In terms of the taxonomy introduced in the canonical model, the type reversal on side A is of a negative sorting because we have $\theta^A_{HH} - \theta^A_{LH} = \alpha_{HH} - \alpha_{LH} < 0$ and $\theta^A_{HL} - \theta^A_{LL} = \alpha_{HL} - \alpha_{LL} > 0$. In addition, the two qualities (q, a) are complements on side A: from $u^A(q,a) = -\psi(a) \cdot (1-q)$, we have $u^A_2(1,a) - u^A_2(0,a) = \psi'(a) > 0$.[15]

We assume that the platform is not viable without selling advertising, which means $(q_H, q_L) = (1,1)$ is never optimal. In what follows, we characterize the optimal contracts for the linear specification of $\psi(a) = R(a) = a$. We restrict our attention to non-negative consumer prices of $p^A_H \geq 0,\ p^A_L \geq 0$ because a negative price may induce consumers to take the money and run without consumption.

6.2 The optimal mechanism

Consider the general case in which the platform can propose a menu (including a pooling contract) on each side. On the advertising side B, we have $a_H \geq a_L$ from the implementability condition; in addition, the binding IR^B_L and IC^B_H imply

$$p^B_L = \frac{1}{2}\left(\beta_{LH}(1-q_H) + \beta_{LL}(1-q_L)\right) a_L, \tag{17}$$

[15] $u^B(\cdot)$ can be written as follows: $u^B(q,a) = R(a) \cdot (1-q)$.

$$p_H^B = \frac{1}{2}\left(\beta_{HH}(1-q_H) + \beta_{HL}(1-q_L)\right)a_H - \frac{1}{2}\left((\beta_{HH}-\beta_{LH})(1-q_H) + (\beta_{HL}-\beta_{LL})(1-q_L)\right)a_L. \tag{18}$$

On side A, we have the following four constraints:

$$\begin{aligned}
(\mathrm{IC}_L^A) \qquad & u_0 - \frac{1}{2}\alpha_{LL}(1-q_L)a_L - \frac{1}{2}\alpha_{LH}(1-q_L)a_H - p_L^A \\
\geq\ & u_0 - \frac{1}{2}\alpha_{LL}(1-q_H)a_L - \frac{1}{2}\alpha_{LH}(1-q_H)a_H - p_H^A; \\
(\mathrm{IC}_H^A) \qquad & u_0 - \frac{1}{2}\alpha_{HL}(1-q_H)a_L - \frac{1}{2}\alpha_{HH}(1-q_H)a_H - p_H^A \\
\geq\ & u_0 - \frac{1}{2}\alpha_{HL}(1-q_L)a_L - \frac{1}{2}\alpha_{HH}(1-q_L)a_H - p_L^A; \\
(\mathrm{IR}_L^A) \qquad & u_0 - \frac{1}{2}\alpha_{LL}(1-q_L)a_L - \frac{1}{2}\alpha_{LH}(1-q_L)a_H - p_L^A \geq 0; \\
(\mathrm{IR}_H^A) \qquad & u_0 - \frac{1}{2}\alpha_{HL}(1-q_H)a_L - \frac{1}{2}\alpha_{HH}(1-q_H)a_H - p_H^A \geq 0.
\end{aligned}$$

Adding IC_L^A to IC_H^A leads to the inequality

$$(q_H - q_L)(a_L - \rho a_H) \geq 0, \tag{19}$$

where $\rho \equiv \frac{\alpha_{LH}-\alpha_{HH}}{\alpha_{HL}-\alpha_{LL}} \in (0,1)$. Given a negative sorting and complementarity between qualities, according to Proposition 3, implementing a non-monotonic schedule on side A ($q_H \leq q_L$) requires a monotonic schedule $a_H > a_L$ on side B (in fact, a sufficiently monotonic schedule in this application, i.e., $\rho a_H > a_L$).

Suppose that IR_L^A binds at $q_L = 0$ which pins down p_L^A equal to $p_L^A = u_0 - \frac{1}{2}\alpha_{LL}a_L - \frac{1}{2}\alpha_{LH}a_H$. As $p_L^A \geq 0$ must hold, $\alpha_{LL}a_L + \alpha_{LH}a_H \leq 2u_0$ must be satisfied. It means that the upper limit of advertising levels consistent with the binding IR_L^A, $p_L^A \geq 0$ and the implementability condition $a_H \geq a_L$ is represented by the line EA in Figure 2. Similarly, the binding IR_H^A at $q_H = 0$ with $p_H^A \geq 0$ requires $\alpha_{HL}a_L + \alpha_{HH}a_H \leq 2u_0$; the corresponding upper limit of advertising levels is represented by the line CD in Figure 2. Type reversal with a negative sorting implies that EA crosses CD from above as a_H increases.

■ **Allocation** $(q_H, q_L) = (0, 0)$

When $(q_H, q_L) = (0, 0)$, IC_H^A and IC_L^A do not impose any restriction on (a_H, a_L) but it implies $p_H^A = p_L^A = p^A$ and the binding participation constraint on side A is determined by the sign of $a_L - \rho a_H$. Precisely, if $a_L < \rho a_H$ then IR_L^A binds, and p^A is equal to $u_0 - \frac{1}{2}\alpha_{LL}a_L - \frac{1}{2}\alpha_{LH}a_H$. Conversely, if $a_L \geq \rho a_H$ then IR_H^A binds, and p^A is equal to $u_0 - \frac{1}{2}\alpha_{HL}a_L - \frac{1}{2}\alpha_{HH}a_H$. Given (17) and (18), we find that the platform's

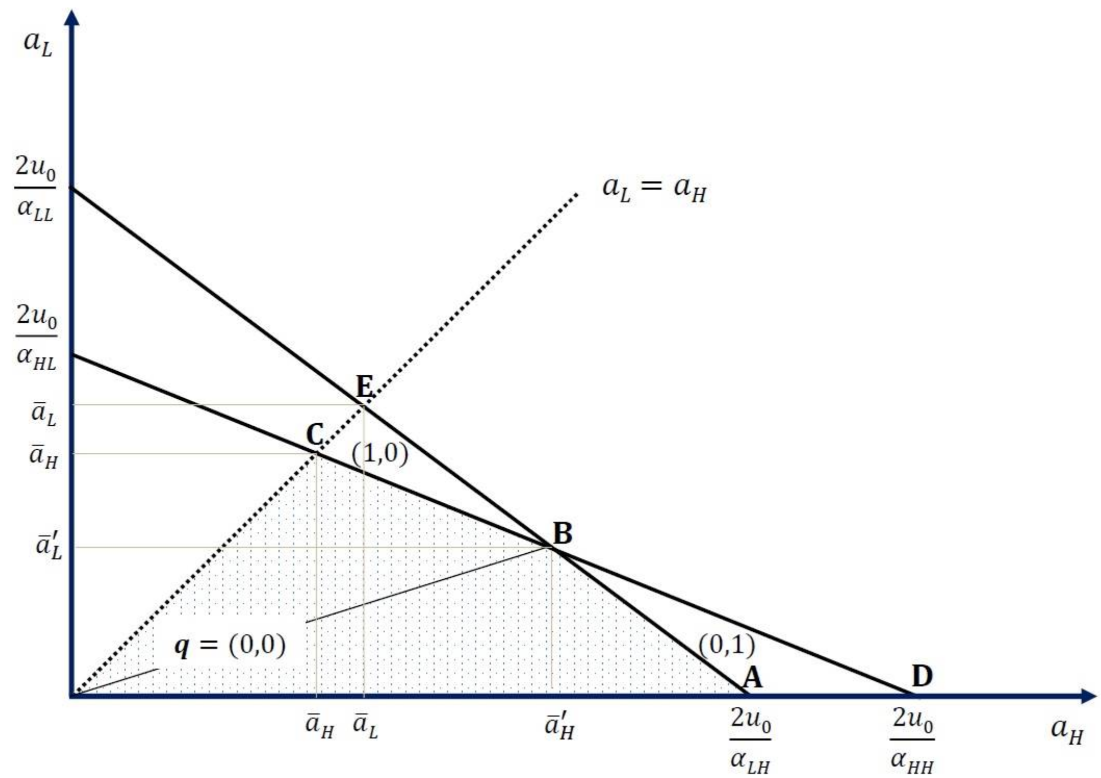


Figure 2: Optimal candidate contracts for the advertising platform

profit is computed as

$$\pi(0,0,a_H,a_L) \equiv u_0+\frac{1}{4}(\beta_{HH}+\beta_{HL})a_H+\frac{1}{4}(2\beta_{LH}+2\beta_{LL}-\beta_{HH}-\beta_{HL})a_L$$
$$-\frac{1}{2}\begin{cases}\alpha_{HH}a_H+\alpha_{HL}a_L & \text{if } a_H\geq a_L\geq \frac{\alpha_{LH}-\alpha_{HH}}{\alpha_{HL}-\alpha_{LL}}a_H\\ \alpha_{LH}a_H+\alpha_{LL}a_L & \text{if } a_L< \frac{\alpha_{LH}-\alpha_{HH}}{\alpha_{HL}-\alpha_{LL}}a_H\end{cases}$$

and it must be evaluated over the polygonal set with vertices $(0,0)$, A,B,C. So, our attention can be limited to the points A, B and C.

■ **Allocation** $(q_H,q_L)=(0,1)$

When $(q_H,q_L)=(0,1)$, the implementability condition (19) is simply given by $a_L\leq\rho a_H$ and IC_H^A and IR_L^A make IR_H^A redundant. Then we obtain $p_L^A=u_0$ and $p_H^A=u_0-\frac{1}{2}\alpha_{HH}a_H-\frac{1}{2}\alpha_{HL}a_L$. As explained, $p_H^A\geq 0$ requires (a_H,a_L) to belong to the triangle which has vertices $(0,0),B,D$ in the graph; our attention can be limited to the points B and D, and the platform's profit is given by

$$\pi(0,1,a_H,a_L)\equiv u_0+\frac{1}{4}\beta_{HH}a_H-\frac{1}{4}\alpha_{HH}a_H+(\frac{1}{2}\beta_{LH}-\frac{1}{4}\beta_{HH})a_L-\frac{1}{4}\alpha_{HL}a_L.$$

■ **Allocation** $(q_H,q_L)=(1,0)$

When $(q_H,q_L)=(1,0)$, (19) becomes $a_L\geq\rho a_H$ and IC_L^A and IR_H^A make IR_L^A redundant. Then we obtain $p_H^A=u_0$, $p_L^A=u_0-\frac{1}{2}\alpha_{LH}a_H-\frac{1}{2}\alpha_{LL}a_L$ and the profit is

$$\pi(1,0,a_H,a_L)=u_0+\frac{1}{4}\beta_{HL}a_H-\frac{1}{4}\alpha_{LH}a_H+(\frac{1}{2}\beta_{LL}-\frac{1}{4}\beta_{HL})a_L-\frac{1}{4}\alpha_{LL}a_L.$$

From $p_L^A\geq 0$, we need to restrict (a_H,a_L) to belong to the triangle which has vertices $(0,0),B,E$; our attention can be limited to the points B and E.

In summary, we have:

Lemma 2. *Consider the application with the linear specification and Assumption 2. The profit-maximizing mechanism is one among the following seven candidates:*

(a) Contracts A, B, or C with $(q_H,q_L)=(0,0)$;

(b) Contracts B or D with $(q_H,q_L)=(0,1)$;

(c) Contracts B or E with $(q_H,q_L)=(1,0)$.

6.3 No PD on side B

Now let us study when advertisers face a single menu of $a_H = a_L = a$. Since this case is a special case of the more general case in the previous subsection and the solution can be easily understood from Figure 2 following the diagonal, we relegate the proof to Appendix A and provide only the result:

Lemma 3. *Consider the application with the linear specification and Assumption 2. Conditional on no price discrimination on the advertising side, the optimal mechanism is either Contract C with* $(q_H, q_L) = (0, 0)$ *or Contract E with* $(q_H, q_L) = (1, 0)$.

6.4 Comparison

The previous analysis has identified all possible candidates for the optimal mechanism. Because we have many parameters, for clear comparison among them, we reduce the number of parameters to one. By doing so, we can gain further insight about under which condition a particular contract becomes optimal and when PD on one side complements or substitutes for the one on the other side.

Let $u_0 = 1$ without loss of generality and consider the following set of values which satisfy all assumptions made in this section and only one parameter δ is used:

$$\begin{cases} u_0 = 1, \quad \alpha_{HH} = 1 - \frac{1}{12}\delta, \qquad \alpha_{HL} = \frac{7}{9}, \qquad \alpha_{LH} = 1, \qquad \alpha_{LL} = \frac{1}{2} \\ \quad \beta_{HH} = 1 + \frac{1}{6}\delta, \qquad \beta_{HL} = 1.01, \qquad \beta_{LH} = 1.01, \qquad \beta_{LL} = 1 \end{cases} \tag{20}$$

where $\delta \in (0.06, \frac{10}{3})$ to satisfy the assumptions of $\alpha_{HL} - \alpha_{LL} > \alpha_{LH} - \alpha_{HH}$ and $\beta_{HH} > \beta_{HL}$. Then the points A, B, C, D, E have coordinates $(2, 0)$, $(\frac{40}{3\delta+20}, \frac{12\delta}{3\delta+20})$, $(\frac{72}{64-3\delta}, \frac{72}{64-3\delta})$, $(\frac{24}{12-\delta}, 0)$, $(\frac{4}{3}, \frac{4}{3})$, respectively. Remarkably, δ captures the intensity of type reversal in that as it increases, the net surplus generated by a H type's watching a H type's advertisement becomes larger.

Let $\pi(q_H, q_L; A)$ represent the profit at point A given (q_H, q_L). Then, we find that $\pi(0, 0; B) > \pi(0, 1; B)$, $\pi(0, 0; B) > \pi(1, 0; B)$, and $\pi(0, 0; B) > \pi(0, 0; A)$. In other words, we can eliminate the three contracts $\pi(0, 1; B)$, $\pi(1, 0; B)$, and $\pi(0, 0; A)$ from consideration as they are strictly dominated by $\pi(0, 0; B)$. Comparing the surviving four candidates leads to:

Lemma 4. *Consider the application with the linear specification with parameters given by (20). Then, the optimal mechanism is*

(a) Contract E with $(q_H, q_L) = (1, 0)$ *and* $a_L = a_H$ *if* $\delta \in S_1 \equiv (0.06, 0.659)$

(b) Contract C with $(q_H, q_L) = (0, 0)$ *and* $a_L = a_H$ *if* $\delta \in S_2 \equiv (0.659, 0.993)$

(c) Contract B with $(q_H, q_L) = (0, 0)$ *and* $a_H > a_L > 0$ *if* $\delta \in S_3 \equiv (0.993, 1.887)$

(d) Contract D with $(q_H, q_L) = (0, 1)$ *is* $a_H > a_L = 0$ *if* $\delta \in S_4 \equiv (1.887, \frac{10}{3})$.

where the neighboring contracts are tied at each border value of δ *.*

Proof. See Appendix A. □

For small enough $\delta \in S_1$, showing advertisements to H type consumers is not optimal as their nuisance cost is high relative to the advertising revenues generated from H type consumers. Conditional on advertising only to L type consumers, the platform ideally wants to implements $a_L > a_H$ on the advertising side as their nuisance from watching H type ads is much larger than the nuisance from L type ads. However, such a non-monotonic advertising schedule cannot be implemented on side B because of the implementability condition. Therefore, the platform chooses the uniform treatment of $a_L = a_H$, which leads to Contract E with $(q_H, q_L) = (1, 0)$.

As δ increases into S_2, it becomes optimal to show advertisements to both types of consumers. However, the platform still wants to choose $a_L > a_H$ and hence is constrained by the implementability condition on side B. This leads to pooling on both sides: $q_H = q_L = 0$ and $a_H = a_L = a_H$ which is Contract C.

As δ further increases and belongs to S_3, it is still optimal to show advertisements to both types of consumers but now δ is high enough that H type consumers generate much advertising revenue to H type advertisers while experiencing not much nuisance. Hence, the platform implements $a_H > a_L > 0$, which makes Contract B optimal.

Finally, for a high enough $\delta \in S_4$, H type consumers generates so much advertising revenue to H type advertisers while experiencing little nuisance that the platform wants to shutdown advertising to L type consumers and not to sell advertising service to L type advertisers, which leads to Contract D.

Now we turn to the original question: how the PD on both sides affects the profit of the platform compared to the PD on the consumer side only? First, for relatively small $\delta \in S_1 \cup S_2$, the PD on the advertising side substitutes the PD on the consumer side as the optimal contracts involves pooling on the advertising side: it can even involve pooling on the consumer side if Contract C is optimal. The platform wants to implement $a_L > a_H$ but it cannot due to the implementability condition, which makes $a_H = a_L$ second-best optimal. Forcing PD on the advertising side would reduce the platform's overall profit.

Second, for relatively large $\delta \in S_3 \cup S_4$, the PD on the advertising side complements the PD on the consumer side. As the platform wants to implement $a_H > a_H$, and given $(q_H, q_L) = (0, 0)$ adding the PD on the adverting side increases the platform's profit. This is a standard argument for introducing a second-degree PD in one-sided market. Furthermore, adding PD on the adverting side can increase the profit by allowing to implement a non-monotonic schedule on the consumer side, which is unique in a two-sided market.

Proposition 5. *Consider the application to the advertising platform with the linear specification and parameters given by (20).*

(i) The PD on the advertising side substitutes for the PD on the consumer side if $\delta \in S_1 \cup S_2$

(ii) The PD on the advertising side complements the PD on the consumer side if $\delta \in S_3 \cup S_4$.

(a) For $\delta \in S_3$*, it does so by implementing a strictly monotonic advertising schedule without affecting the allocation on the consumer side.*

(b) For $\delta \in S_4$*, it does so by implementing a non-monotonic schedule on the consumer side and a strictly monotonic advertising schedule.*

The above result can provide some insight about actual business practices by many online media platforms. For instance, YouTube recently launched its long-discussed paid subscription service, YouTube Red. This kind of PD on the consumer side corresponds to the monotonic schedule $(q_H, q_L) = (1, 0)$ in which H type consumers pay a certain fee to avoid the ads. Suppose for instance that YouTube Red means a change from Contract C with $(q_H, q_L) = (0, 0)$ (i.e., $\delta \in S_2$) to Contract E with $(q_H, q_L) = (1, 0)$ (i.e., $\delta \in S_1$). Then, this change involves an increase in advertising amount without changing its composition and an L type consumer gets worse off.

7 Concluding Remarks

In two-sided markets, introducing PD on one side may affect the incentive constraints and thus the set of implementable allocations on the other side so that PD on one side can complement or substitute for PD on the other side. Our model of a canonical two-sided PD can be applied to more specific environments as we demonstrated how our model can be adapted to online media advertising platforms in Section 6. Before we wrap up this paper, let us briefly discuss aforementioned two applications of network neutrality debate and privacy design.

In the ongoing debate of network neutrality, the key issue is whether it is desirable to introduce a tiered-Internet on the side of content providers; the network neutrality regulation implies that ISPs cannot apply price discrimination vis-à-vis content providers. Because it has not been controversial for ISPs to design menu pricing with different quality-price pairs on the side of end-users, we can rephrase the prime matter as "Do consumers and/or society can benefit from PD on both sides relative to PD on the consumer side only?" The answer to this inquiry requires deliberate investigation on welfare implications of the regulation. Our model provides a useful framework for such welfare analysis. Suppose that major content providers such as Netflix and YouTube (so-called "the haves") buy the prioritized delivery service and other non-major content providers ("the have nots") use non-prioritized lane. Consider the situation in which the enhanced quality from prioritized content delivery is more important to end-users subscribing to the premium Internet service. Then, the two qualities are complements. By contrast, if the faster-lane content is more important to basic users than to premium users, the qualities become substitutes. The neutrality regulation would affect ISPs' quality design not only on the content-providers' side but also on the consumer side via cross-side interactions. To our best knowledge, this two-sided interaction has not been studied; future research awaits.

Our model can also be applied to the context of online privacy and targeted advertising. Consumers care about the amount of personal information collected by data operators such as Google and Facebook as well as targeted advertising of which the precision may improve with more release of the personal information. Suppose that H type consumers are more reluctant to release personal information than L type consumers due to privacy concern while H type consumers' personal information is more profitable information source for targeted advertising. In this case a platform may end up choosing either pooling or a monotonic disclosure schedule on the consumer side – both types or H type consumers opt out of providing personal information – if the platform fails to commit to no data leakage or sales to a third party. If a private company's self-regulation on privacy protection does not resolve enough this kind of privacy concern by consumers, one can consider a public enforcement like European Commission's recent reform of data protection rules as a more effective data protection regime. If H type consumers agree to release useful personal information for better targeted advertising under a more strict public enforcement, both consumers and businesses may benefit from the external enforcement as it may facilitate better use of personal information for efficiency gains. We think our general framework can be suitably adapted to study the privacy issues related to targeted advertising from a two-sided market perspective.

It is to be hoped that our work will serve as a foundation from which studies of different specificity and greater depth may be undertaken for understanding second-degree price discrimination in two-sided markets.

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Appendix

A Mathematical Proofs

The proofs for other propositions and lemmas are discussed in the text. Thus, here we provide mathematical proofs for Lemma 1, Proposition 3, Lemma 3 and Lemma 4.

A.1 Proof of Lemma 1

We prove it for side A. We can combine (IR_L^A) and (IC_H^A) to find that

$$p_L^A \leq \theta_L^A u_A^A(q_L^A) + \nu_H^B \theta_{LH}^A u_B^A(q_H^B) + \nu_L^B \theta_{LL}^A u_B^A(q_L^B).$$

Hence, we have

$$\begin{aligned}
& \theta_H^A u_A^A(q_H^A) + \nu_H^B \theta_{HH}^A u_B^A(q_H^B) + \nu_L^B \theta_{HL}^A u_B^A(q_L^B) - p_H^A \\
\geq \; & \theta_H^A u_A^A(q_L^A) + \nu_H^B \theta_{HH}^A u_B^A(q_H^B) + \nu_L^B \theta_{HL}^A u_B^A(q_L^B) - p_L^A \\
\geq \; & (\theta_H^A - \theta_L^A) u_A^A(q_L^A) + \nu_H^B(\theta_{HH}^A - \theta_{LH}^A) u_B^A(q_H^B) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A) u_B^A(q_L^B) \quad \text{(A.1)}
\end{aligned}$$

where $\theta_H^A - \theta_L^A > 0$ by assumption 1.

If there is no type reversal on side A, then we have $\theta_{HH}^A - \theta_{LH}^A > 0$ and $\theta_{HL}^A > \theta_{LL}^A$ and the R.H.S in (A.1) is non-negative, which allows to neglect (IR_H^A). If $\theta_{HH}^A - \theta_{LH}^A > 0$, then u_B^A increasing and $q_H^B \geq q_L^B$ imply that (A.1) is at least as large as

$$\begin{aligned}
& (\theta_H^A - \theta_L^A) u_A^A(q_L^A) + \nu_H^B(\theta_{HH}^A - \theta_{LH}^A) u_B^A(q_L^B) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A) u_B^A(q_L^B) \\
& \qquad = (\theta_H^A - \theta_L^A) u_A^A(q_L^A) + (\theta_H^A - \theta_L^A) u_B^A(q_L^B) \geq 0.
\end{aligned}$$

Then the standard arguments imply that (IR_L^A) and (IC_H^A) bind in the optimum, and that (IC_L^A) reduces to $q_H^A \geq q_L^A$.

A.2 Proof of Proposition 3

Here we prove a more detailed version of Proposition 3, and for that purpose we let

$$\phi(q_H^A) = \nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) u_1^A(q_H^A, q_H^B) + \nu_L^B(\theta_{HL} - \theta_{LL}) u_1^A(q_H^A, q_L^B)$$

denote the derivative of Φ^A with respect to q_H^A: notice that ϕ does not depend on q_L^A. As the case of negative sorting is symmetric to the case of positive sorting, we only provide the proof for the positive sorting of which the statement is refined as follows.

Refined version of Proposition 3(ii)

(i) Suppose that $u_{12} > 0$ and $u_{112} \geq 0$ (not needed for part (a)).

(a) When $q_H^B \geq q_L^B$, we have $F = M \cup D$.

(b) When $q_H^B < q_L^B$, we have that

(b1) If $\phi(0) \leq 0$, then $F = N \cup D$ if $\phi(0) \leq 0$.

(b2) If $\phi(0) > 0 > \lim_{q_H^A \to +\infty} \phi(q_H^A)$, then let $\overline{q}_H^A$ be uniquely defined by $\phi(\overline{q}_H^A) = 0$. The set F has the shape of a sandglass, such that it includes all points in M such that $q_L \leq \overline{q}_H^A$ and $q_H \leq \overline{q}_H^A$, and some points in N if $q_L^A > \overline{q}_H^A$.

(b3) If $\lim_{q_H^A \to +\infty} \phi(q_H^A) \geq 0$,then $F = M \cup D$.

(ii) Suppose that $u_{12} < 0$ and $u_{112} \leq 0$ (not needed for part (a)).

(a) when $q_H \leq q_L$, we have $F = M \cup D$.

(b) When $q_H > q_L$, we have that (b1-b3) from part (i) hold.

Proof of part (i): Complements: $u_{12}^A(q^A, q^B) > 0$ and $u_{112}^A(q^A, q^B) \geq 0$ for each q^A, q^B

1. If $q_H^B \geq q_L^B$, then $u_1^A(q_H^A, q_H^B) \geq u_1^A(q_H^A, q_L^B)$ and $\phi(q_H^A) \geq (\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A))u_1(q_H^A, q_L^B) > 0$. Therefore Φ^A is strictly increasing in q_H^A and $F = M \cup D$.

2. If $q_H^B < q_L^B$, then assume $u_{112}^A \geq 0$, that is u_{11}^A is increasing with respect to q^B, or equivalently u_{12}^A is increasing with respect to q^A. Then $\phi'(q_H^A) = \nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) u_{11}^A(q_H^A, q_H^B) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A) u_{11}^A(q_H^A, q_L^B) \leq (\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A))u_{11}^A(q_H^A, q_H^B) < 0$. Therefore ϕ is strictly decreasing.

   - If $\phi(0) \leq 0$, then $\phi(q_H^A) < 0$ for each $q_H^A > 0$. Therefore Φ^A is strictly decreasing in q_H^A and $F = N \cup D$.
   - If $\phi(0) > 0 > \lim_{q_H^A \to +\infty} \phi(q_H^A)$, then let $\overline{q}_H^A$ be uniquely defined by $\phi(\overline{q}_H^A) = 0$.

     Now fix q_L^A, and consider $q_L^A < \overline{q}_H^A$. Then $\phi(q_H^A) > 0$ for $q_H^A \in (0, q_L^A)$ and $\Phi^A(q_H^A, q_L^A) < 0$ for each $q_H^A < q_L^A$. Conversely, $\Phi^A(q_H^A, q_L^A) > 0$ at least for $q_H^A \in (q_L^A, \overline{q}_H^A]$, because Φ^A is increasing in q_H^A for $q_H^A \in (q_L^A, \overline{q}_H^A)$. Since $\phi(q_H^A) < 0$ for $q_H^A > \overline{q}_H^A$, it is possible that $\Phi^A(q_H^A, q_L^A) < 0$ for q_H^A sufficiently larger than $\overline{q}_H^A$.

Now consider $q_L^A > \overline{q}_H^A$. Then $\phi(q_H^A) < 0$ for each $q_H^A > q_L^A$, hence $\Phi^A(q_H^A, q_L^A) < 0$ for each $q_H^A > q_L^A$. Conversely, $\Phi^A(q_H^A, q_L^A) > 0$ at least for $q_H^A \in [\overline{q}_H^A, q_L^A)$ because Φ^A is decreasing in q_H^A for $q_H^A \in (\overline{q}_H^A, q_L^A)$. Since $\phi(q_H^A) > 0$ for $q_H^A < \overline{q}_H^A$, it is possible that $\Phi^A(q_H^A, q_L^A) < 0$ for q_H^A sufficiently smaller than $\overline{q}_H^A$.

In this case the feasible set is non convex, and has vaguely the shape of a sandglass.

- If $\lim_{q_H^A \to +\infty} \phi(q_H^A) \geq 0$, then Φ^A is strictly increasing in q_H^A, hence $(\mathbf{I}^A)$ is satisfied if and only if $(q_H^A, q_L^A) \in M \cup D$.

Proof of part (ii): Substitutes: $u_{12}^A(q^A, q^B) < 0$ and $u_{112}^A(q^A, q^B) \leq 0$ for each q^A, q^B

1. If $q_H^B \leq q_L^B$, then $u_1^A(q_H^A, q_H^B) \geq u_1^A(q_H^A, q_L^B)$ and $\phi(q_H^A) \geq (\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A))u_1^A(q_H^A, q_L^B) > 0$. Therefore $(\mathbf{I}^A)$ is equivalent to $q_H^A \geq q_L^A$.

2. If $q_H^B > q_L^B$, then assume $u_{112}^A \leq 0$, that is u_{11}^A is decreasing with respect to q^B, or equivalently u_{12}^A is decreasing with respect to q^A. Then $\phi'(q_H^A) = \nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) u_{11}^A(q_H^A, q_H^B) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A)u_{11}^A(q_H^A, q_L^B) \leq (\nu_H^B \left(\theta_{HH}^A - \theta_{LH}^A\right) + \nu_L^B(\theta_{HL}^A - \theta_{LL}^A))u_{11}^A(q_H^A, q_L^B) < 0$. Therefore ϕ is strictly decreasing and we obtain a feasible set similar to the case 2 above: (i) $N \cup D$ if $\phi(0) \leq 0$; (ii) a sandglass if $\phi(0) > 0 > \lim_{q_H^A \to +\infty} \phi(q_H^A)$; (iii) $M \cup D$ if $\lim_{q_H^A \to +\infty} \phi(q_H^A) \geq 0$. ■

A.3 Proof of Lemma 3

Then, the platform's price against advertisers is set to make L type advertisers earn zero net surplus:

$$p_L^B = p_H^B = \frac{1}{2} \left(\beta_{LH}(1 - q_H) + \beta_{LL}(1 - q_L)\right) a \tag{A.2}$$

A.3.1 Allocation $(q_H, q_L) = (0, 0)$

In this case IC_H^A and IC_L^A imply $p_L^A = p_H^A = p^A$, hence IR_H^A implies IR_L^A and $p^A = u_0 - \frac{1}{2}(\alpha_{HH} + \alpha_{HL})a$, with $a \leq \bar{a}_H \equiv \frac{2u_0}{\alpha_{HL} + \alpha_{HL}}$ in order to have $p^A \geq 0$. Since $p_L^B = p_H^B = \frac{1}{2}(\beta_{LH} + \beta_{LL})a$, the platform's profit is equal to

$$u_0 + \frac{1}{2}(\beta_{LH} + \beta_{LL})a - \frac{1}{2}(\alpha_{HH} + \alpha_{HL})a$$

and it should be maximized with respect to $a \in [0, \bar{a}_H]$. Assuming $\beta_{LH} + \beta_{LL} > \alpha_{HH} + \alpha_{HL}$, the optimal a is $\bar{a}_H$ (point C in Figure 2), hence the maximal value is

$$u_0 + (\frac{1}{2}\beta_{LH} + \frac{1}{2}\beta_{LL} - \frac{1}{2}\alpha_{HH} - \frac{1}{2}\alpha_{HL})\bar{a}_H.$$

A.3.2 Allocation $(q_H, q_L) = (0, 1)$

In this case IC_H^A and IC_L^A require $u_0 - \frac{1}{2}(\alpha_{HH} + \alpha_{HL})a - p_H^A \geq u_0 - p_L^A$ and $u_0 - p_L^A \geq u_0 - \frac{1}{2}(\alpha_{LH} + \alpha_{LL})a - p_H^A$. Combining the two inequality conditions, we obtain $(\alpha_{LH} + \alpha_{LL} - \alpha_{HH} - \alpha_{HL})a \geq 0$. Because of Assumption 2-(i), $(\alpha_{LH} + \alpha_{LL} - \alpha_{HH} - \alpha_{HL})a < 0$ and thus the derived condition holds only if $a = 0$. When $a = 0$, the platform extracts the full rent by charging $p_L^A = p_H^A = u_0$ and the profit is equal to u_0. As we assume that the platform is not viable without selling advertising, this situation is not optimal.

A.3.3 Allocation $(q_H, q_L) = (1, 0)$

In this case the constraints are given by

$$\begin{aligned}
&(\mathrm{IR}_H^A) \qquad && u_0 - p_H^A \geq 0 \\
&(\mathrm{IR}_L^A) \qquad && u_0 - \frac{1}{2}(\alpha_{LH} + \alpha_{LL})a - p_L^A \geq 0 \\
&(\mathrm{IC}_H^A) \qquad && u_0 - p_H^A \geq u_0 - \frac{1}{2}(\alpha_{HH} + \alpha_{HL})a - p_L^A \\
&(\mathrm{IC}_L^A) \qquad && u_0 - \frac{1}{2}(\alpha_{LH} + \alpha_{LL})a - p_L^A \geq u_0 - p_H^A
\end{aligned}$$

and the optimal tariffs are $p_H^A = u_0$, $p_L^A = u_0 - \frac{1}{2}(\alpha_{LH} + \alpha_{LL})a$, with $a_L \leq \bar{a}_L \equiv \frac{2u_0}{\alpha_{LH} + \alpha_{LL}}$ for $p_L^A \geq 0$. Since $p_L^B = p_H^B = \frac{1}{2}\beta_{LL}a$, the profit is

$$u_0 - \frac{1}{4}(\alpha_{LH} + \alpha_{LL})a + \frac{1}{2}\beta_{LL}a$$

and it should be maximized with respect to $a \in [0, \bar{a}_L]$. Assuming $2\beta_{LL} > \alpha_{LH} + \alpha_{LL}$, the optimal a is $\bar{a}_L$ (point E in Figure 2), hence the maximal profit given $(q_H, q_L) = (1, 0)$ is

$$u_0 + (\frac{1}{2}\beta_{LL} - \frac{1}{4}\alpha_{LH} - \frac{1}{4}\alpha_{LL})\bar{a}_L.$$

A.4 Proof of Lemma 4

Given $(q_H, q_L) = (0, 0)$, the profit function is

$$\begin{aligned}
\pi(0,0,a_H,a_L) &= 1+\frac{1}{4}\left(1+\frac{1}{6}\delta+\frac{101}{100}\right)a_H+\frac{1}{4}\left(2\cdot\frac{101}{100}+2-1-\frac{1}{6}\delta-\frac{101}{100}\right)a_L \\
&\quad -\frac{1}{2}\begin{cases} (1-\frac{1}{12}\delta)a_H+\frac{7}{9}a_L & \text{if } a_H \geq a_L \geq \frac{3\delta}{10}a_H \\ a_H+\frac{1}{2}a_L & \text{if } a_L < \frac{3\delta}{10}a_H \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
\pi(0,0;A) &= \frac{201}{200}+\frac{1}{12}\delta \\
\pi(0,0;B) &= \frac{2309\delta+6030-150\delta^2}{300(3\delta+20)} \\
\pi(0,0;C) &= \frac{1809}{25\,(64-3\delta)}
\end{aligned}$$

Given $(q_H, q_L) = (0, 1)$, the profit function is

$$\pi(0,1,a_H,a_L) = 1+\frac{1}{4}\cdot\left(1+\frac{1}{6}\delta\right)a_H-\frac{1}{4}\cdot\left(1-\frac{1}{12}\delta\right)a_H+\left(\frac{1}{2}\cdot\frac{101}{100}-\frac{1}{4}\cdot(1+\frac{1}{6}\delta)\right)a_L-\frac{1}{4}\cdot\frac{7}{9}a_L$$

$$\begin{aligned}
\pi(0,1;B) &= \frac{1}{150}\frac{934\delta-75\delta^2+3000}{3\delta+20} \\
\pi(0,1;D) &= \frac{1}{2}\frac{\delta+24}{12-\delta}
\end{aligned}$$

Given $(q_H, q_L) = (1, 0)$, the profit function is

$$\pi(1,0,a_H,a_L) = 1+\frac{1}{4}\left(\frac{101}{100}\right)a_H-\frac{1}{4}a_H+\left(\frac{1}{2}-\frac{1}{4}\cdot\frac{101}{100}\right)a_L-\frac{1}{4}\cdot\frac{1}{2}a_L$$

Hence,

$$\begin{aligned}
\pi(1,0;B) &= \frac{3}{100}\frac{149\delta+670}{3\delta+20} \\
\pi(1,0;E) &= \frac{7}{6}
\end{aligned}$$

Comparing these derived payoffs, we obtain the stated result. ■

Online Appendix

(Not for publication)

B Further analysis of the symmetric two-sided market

After identifying distortions in quality schedule, we solve for the optimal mechanism in a quadratic setting when qualities are complements.

B.1 Quality distortions and non-responsiveness

As a benchmark let us write the profit function under complete information:

$$\begin{aligned}\pi^{FB}(q_H, q_L) &= 2\nu_H^2\theta_{HH}u(q_H, q_H) + 2\nu_H\nu_L\left[\theta_{HL}u(q_H, q_L) + \theta_{LH}u(q_L, q_H)\right] \\ &\quad +2{\nu_L}^2\theta_{LL}u(q_L, q_L) - 2\nu_H C(q_H) - 2\nu_L C(q_L).\end{aligned}$$

With private information consider now the standard approach in which we assume that only IR_L and IC_H bind. Substituting the transfers obtained from the binding constraints into the platform's objective gives the following profit function:

$$\begin{aligned}\widehat{\pi}(q_H, q_L) &\equiv \pi^{FB}(q_H, q_L) - 2\nu_H\left[\nu_H(\theta_{HH} - \theta_{LH})u(q_L, q_H) + \nu_L(\theta_{HL} - \theta_{LL})u(q_L, q_L)\right] \\ &= 2\nu_H^2\theta_{HH}u(q_H, q_H) + 2\nu_H\nu_L\left[\theta_{HL}u(q_H, q_L) + \theta^v_{LH}u(q_L, q_H)\right] \qquad \text{(B.1)} \\ &\quad +2{\nu_L}^2\theta^v_{LL}u(q_L, q_L) - 2\nu_H C(q_H) - 2\nu_L C(q_L)\end{aligned}$$

where $\theta^v_{LH} = \theta_{LH} - \frac{\nu_H}{\nu_L}(\theta_{HH} - \theta_{LH})$ and $\theta^v_{LL} = \theta_{LL} - \frac{\nu_H}{\nu_L}(\theta_{HL} - \theta_{LL})$. Hence, we can see that $\widehat{\pi}$ differs from π^{FB} only because θ^v_{LH} and θ^v_{LL} respectively replace θ_{LH} and θ_{LL}, where the two are related such that

$$\theta_{LH} \gtreqless \theta^v_{LH} \Leftrightarrow (\theta_{HH} - \theta_{LH}) \gtreqless 0; \qquad \theta_{LL} \gtreqless \theta^v_{LL} \Leftrightarrow (\theta_{HL} - \theta_{LL}) \gtreqless 0.$$

When we neglect the cross-group interactions and consider a one-sided market, we can prove that there is only a downward distortion for q_L as $\nu_H(\theta_{HH}-\theta_{LH})+\nu_L(\theta_{HL}-\theta_{LL}) > 0$ from Assumption 1. However, we show that the cross-group interaction can induce an upward distortion even at the top. To see this, see the FOC with respect to q_H:

$$\nu_H\theta_{HH}\left[u_1(q_H, q_H) + u_2(q_H, q_H)\right] + \nu_L\left[\theta_{HL}u_1(q_H, q_L) + \theta^v_{LH}u_2(q_L, q_H)\right] = C'(q_H), \quad \text{(B.2)}$$

If $(\theta_{HH} - \theta_{LH}) > 0$, then we have $\theta_{LH} > \theta^v_{LH}$. Additionally if $u_2 > 0$ is assumed so that u is strictly increasing in both its arguments, the q_H satisfying (B.2) is smaller than the first-best quality q_H^{FB} for a given $q_L = q_L^{FB}$. Note that this downward distortion at q_H arises through the *cross-group interactions* that would have been absent in a one-sided market. For the same reasoning, if $(\theta_{HH} - \theta_{LH}) < 0$ and $u_2 > 0$, the q_H will be distorted upward compared to q_H^{FB}.

Next, the FOC with respect to q_L is given by

$$\nu_H \left[\theta_{HL} u_2(q_H, q_L) + \theta^v_{LH} u_1(q_L, q_H)\right] + \nu_L \theta^v_{LL} \left[u_1(q_L, q_L) + u_2(q_L, q_L)\right] = C'(q_L). \quad \text{(B.3)}$$

Consider no type reversal $((\theta_{HH} - \theta_{LH}) > 0,\ (\theta_{HL} - \theta_{LL}) > 0)$ and assume $u_2 > 0$. Then, given $q_H = q_H^{FB}$, the q_L that satisfies (B.3) is smaller than q_L^{FB}. In this case, the well-known downward distortion in one-sided market is reinforced by the downward distortion due to cross-group interactions in a two-sided market. However, if $(\theta_{HH} - \theta_{LH}) > 0 > (\theta_{HL} - \theta_{LL})$ (type reversal with a positive sorting), as $\theta_{LL} < \theta^v_{LL}$, the q_L that satisfies (B.3) given $q_H = q_H^{FB}$ can be higher or lower than q_L^{FB}.

B.2 Symmetric quadratic setting with complements

When qualities are complements and there is type-reversal of a positive sorting, the implementable set is composed of all monotonic schedule and possibly some non-monotonic schedule with relatively large gap $q_L - q_H$. Thus, the implementable set itself may not be a convex set and thus finding the optimal mechanism can be challenging. We here analyze the optimal mechanism for a symmetric quadratic setting with the complementarity in the qualities. This analysis confirms the general insight in a more visible manner through explicit solutions. Let us begin by specifying the utility function:

$$\tilde{u}(q^A, q^B) = q^A - \frac{1}{2}(q^A)^2 + q^B - \frac{1}{2}(q^B)^2 + \alpha q^A q^B$$

with $\alpha \in [0, 1)$. We assume that $(\theta_{HL} - \theta_{LL}) \in (-(\theta_{HH} - \theta_{LH}), 0)$, $\nu_H = \frac{1}{2}$ and $C(q) = q^2/2$.

Unfortunately, $\tilde{u}$ is not monotone increasing in q^A, q^B, as it has a global max point at $(q^A, q^B) = (\frac{1}{1-\alpha}, \frac{1}{1-\alpha})$; this suggests to consider $\tilde{u}$ as defined in the square $S = [0, \frac{1}{1-\alpha}] \times [0, \frac{1}{1-\alpha}]$. Even in this refined domain, $\tilde{u}$ is not monotone increasing in q^A, q^B: for instance, it is decreasing with respect to q^B for $q^B > 1 + \alpha q^A$. For this reason, we consider the function u defined below, after introducing a suitable partition of the set

S:

$$
\begin{aligned}
R_1 &= \{(q^A, q^B) : q^B \in [0, \frac{1}{1-\alpha}),\ q^A \in (1 + \alpha q^B, \frac{1}{1-\alpha}]\}, \\
R_2 &= \{(q^A, q^B) : q^B \in [0, \frac{1}{1-\alpha}),\ q^A \in [q^B, 1 + \alpha q^B]\}, \\
R_3 &= \{(q^A, q^B) : q^A \in [0, \frac{1}{1-\alpha}),\ q^B \in (q^A, 1 + \alpha q^A]\}, \\
R_4 &= \{(q^A, q^B) : q^A \in [0, \frac{1}{1-\alpha}],\ q^B \in [1 + \alpha q^A, \frac{1}{1-\alpha}]\}
\end{aligned}
$$

The Figure B.1 illustrates the partitions of the domain set S.

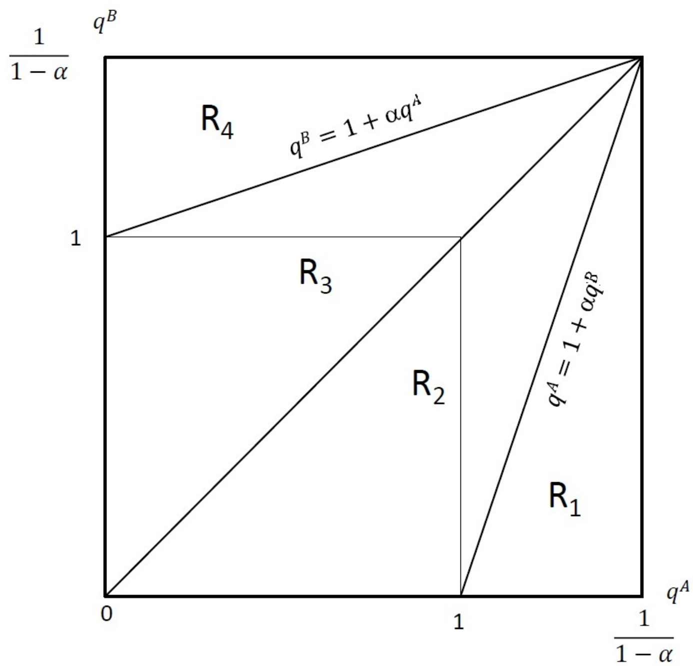


Figure B.1: The domain with partitions for the symmetric quadratic setting

Then we define u in S as follows:

$$
u(q^A, q^B) = \begin{cases} \tilde{u}(1 + \alpha q^B, q^B) & \text{if } (q^A, q^B) \in R_1 \\ \tilde{u}(q^A, q^B) & \text{if } (q^A, q^B) \in R_2 \cup R_3 \\ \tilde{u}(q^A, 1 + \alpha q^A) & \text{if } (q^A, q^B) \in R_4 \end{cases}
$$

In order to understand this definition, consider for instance $q^A \in [0, \frac{1}{1-\alpha})$, and recall that $\tilde{u}$ is strictly decreasing with respect to q^B if $q^B > 1+\alpha q^A$. Then, for $q^B > 1+\alpha q^A$, $u(q^A, q^B)$ is defined as $\tilde{u}(q^A, 1+\alpha q^A)$, such that u is constant with respect to q^B in the set R_4.

For this setting, it is interesting to notice the following:

$$q_H^{FB} \geq q_L^{FB} \quad \text{if} \quad \theta_{HH} \geq \theta_{LL}, \qquad q_H^{FB} < q_L^{FB} \quad \text{if} \quad \theta_{HH} < \theta_{LL}$$

Under incomplete information, if we assume that IR_L and IC_H bind, and neglect IR_H and IC_L, then we find $\hat{\pi}$ in (B.1) and $\hat{q}_H, \hat{q}_L$ is such that

$$\widehat{q}_H \geq \widehat{q}_L \quad \text{if} \quad \theta_{HH} \geq \theta_{LL}^v, \qquad \widehat{q}_H < \widehat{q}_L \quad \text{if} \quad \theta_{HH} < \theta_{LL}^v$$

Since $(\theta_{HL} - \theta_{LL}) < 0$, we have $\theta_{LL}^v > \theta_{LL}$. Therefore, if the first-best schedule is non-monotonic, then $(\widehat{q}_H, \widehat{q}_L)$ is non-monotonic as well. Moreover, $(\widehat{q}_H, \widehat{q}_L)$ can be non-monotonic even if the first-best schedule is monotonic. As we explained previously, this is because $\theta_{LL}^v > \theta_{LL}$ can create an upward distortion in $\widehat{q}_L$.

Under incomplete information, $\left(q_H^{SB}, q_L^{SB}\right)$ does not necessarily coincide with $(\widehat{q}_H, \widehat{q}_L)$ because $(\widehat{q}_H, \widehat{q}_L)$ may fail to satisfy IC_L and/or IR_H. Precisely, given that IR_L and IC_H bind, IC_L and IR_H reduce to

$$[u(q_H, q_H) - u(q_L, q_H)] + r[u(q_H, q_L) - u(q_L, q_L)] \geq 0 \tag{B.4}$$

$$u(q_L, q_H) + ru(q_L, q_L) \geq 0 \tag{B.5}$$

with $r = \frac{(\theta_{HL} - \theta_{LL})}{(\theta_{HH} - \theta_{LH})} \in (-1, 0)$. Next lemma identifies the subset of S in which (B.4) is satisfied, as a function of $\alpha \in [0, 1)$.

Lemma 5.

Let $\alpha_1 = \dfrac{1 - |r|}{1 + |r|}, \qquad \alpha_2 = \dfrac{1 - |r|}{2|r|}$ *and*

$$b = \sqrt{\frac{1}{2}(1 - \frac{1}{|r|} + \frac{1 + |r|}{|r|}\alpha)}, \qquad c = \frac{2(1 - |r|)}{1 + |r|(2\alpha - 1)} > 0, \qquad d = \frac{(2\alpha - 1) + |r|}{1 + |r|(2\alpha - 1)}.$$

(i) If $r < -\frac{1}{3}$, then $\alpha_2 < 1$ and the set of (q_H, q_L) which satisfy (B.4) depends on α as follows:

$$\begin{cases} R_1 \cup R_2 & \text{if } \alpha \in [0, \alpha_1] \\ R_1 \cup R_2 \cup \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha}) \text{ and } q_L \in [\frac{1-b}{1-\alpha} + bq_H, \frac{1}{1-\alpha}]\} & \text{if } \alpha \in (\alpha_1, \alpha_2) \\ R_1 \cup R_2 \cup \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha}) \text{ and } q_L \in [c + dq_H, \frac{1}{1-\alpha}]\} & \text{if } \alpha \in [\alpha_2, 1) \end{cases}$$

(ii) If $r \geq -\frac{1}{3}$, then $\alpha_2 \geq 1$ and the set of (q_H, q_L) which satisfy (B.4) depends on α as

follows:

$$\begin{cases} R_1 \cup R_2 & \textit{if } \alpha \in [0, \alpha_1] \\ R_1 \cup R_2 \cup \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha}) \textit{ and } q_L \in [\frac{1-b}{1-\alpha} + bq_H, \frac{1}{1-\alpha}]\} & \textit{if } \alpha \in (\alpha_1, 1) \end{cases}$$

The inequality (B.4) has a different expression depending on whether we consider (q_H, q_L) in R_1, or in $R_2 \cup R_3$, or in R_4. Precisely, it is equivalent to

$$\tilde{u}(q_H, q_H) - \tilde{u}(q_L, 1 + \alpha q_L) + r[\tilde{u}(1 + \alpha q_L, q_L) - \tilde{u}(q_L, q_L)] \geq 0 \text{ if } (q_H, q_L) \in R_1 \tag{B.6}$$

$$(q_L - q_H)(q_L - c - dq_H) \geq 0 \text{ if } (q_H, q_L) \in R_2 \cup R_3 \tag{B.7}$$

$$\tilde{u}(q_H, q_H) - \tilde{u}(1 + \alpha q_H, q_H) + r[\tilde{u}(q_H, 1 + \alpha q_H) - \tilde{u}(q_L, q_L)] \geq 0 \text{ if } (q_H, q_L) \in R_4 \tag{B.8}$$

Figure B.2 represents this set in the three cases of $\alpha \in [0, \alpha_1]$, $\alpha \in (\alpha_1, \alpha_2)$, and $\alpha \in [\alpha_2, 1)$. Notice that for $\alpha \in (\alpha_1, \alpha_2)$, the line $q_L = \frac{1-b}{1-\alpha} + bq_H$ lies above the line $q_L = 1 + \alpha q_H$, that is it is entirely in R_4 and the feasible set consists of the points in S which are on or below the diagonal, plus a subset of R_4. For $\alpha = \alpha_2$, the two lines $q_L = \frac{1-b}{1-\alpha} + bq_H$ and $q_L = c + dq_H$ both coincide with $q_L = 1 + \alpha q_H$, and for $\alpha > \alpha_2$, the line $q_L = c + dq_H$ is included in R_3, but is bounded away from the line $q_L = q_H$ even as $\alpha \to 1$: when α tends to 1, the line $q_L = c + dq_H$ tends to the line $q_L = 2\alpha_1 + q_H$. Thus the set $R_1 \cup R_2 \cup \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha})$ and $q_L \in [c + dq_H, \frac{1}{1-\alpha}]\}$ is a strict subset of S for each $\alpha \in [\alpha_2, 1)$.

The proof for Lemma 4 is in what follows.

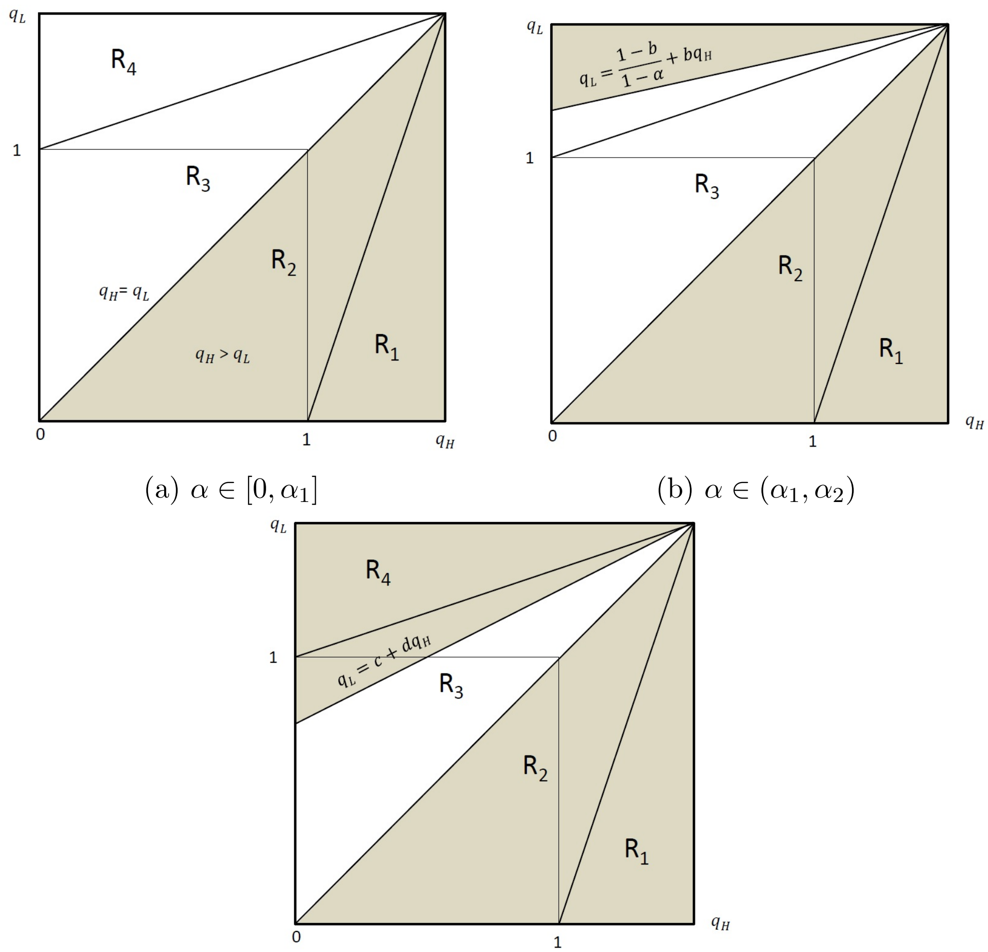
Step 1 (B.4) holds for each point in $R_1 \cup R_2$. For each $(q_H, q_L) \in R_2$, we find that (B.7) holds because $q_L - q_H \leq 0$ and $q_L - c - dq_H \leq q_H - c - dq_H < 0$, given that $q_H \in (0, \frac{1}{1-\alpha})$. Hence, each $(q_H, q_L) \in R_2$ satisfies (B.4). Regarding R_1, the term $\tilde{u}(q_H, q_H)$ in the left hand side in (B.6) is at least as large as $\tilde{u}(1 + \alpha q_L, 1 + \alpha q_L)$, therefore the left hand side in (B.6) is at least as large as $\frac{1}{2}(1+r+2\alpha)(1-q_L+\alpha q_L)^2 > 0$. Hence IC$_L$ holds at each point in R_1.

Step 2 The subset of $R_3 \cup R_4$ in which (B.4) is satisfied depends on α as follows

$$\begin{cases} \varnothing & \text{if } \alpha \in [0, \alpha_1] \\ \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha}) \text{ and } q_L \in [\frac{1-b}{1-\alpha} + bq_H, \frac{1}{1-\alpha}]\} & \text{if } \alpha \in (\alpha_1, \alpha_2) \\ \{(q_H, q_L) : q_H \in [0, \frac{1}{1-\alpha}) \text{ and } q_L \in [c + dq_H, \frac{1}{1-\alpha}]\} & \text{if } \alpha \in [\alpha_2, 1) \end{cases}$$

Step 2.1 $\alpha \in [0, \alpha_1]$**.**

For each $(q_H, q_L) \in R_3$, (B.4) is equivalent to $q_L - c - dq_H \geq 0$, but $q_L - c - dq_H \leq -(1 - q_H + \alpha q_H)\frac{1+r+2\alpha r}{1+r-2\alpha r} < 0$, in which the first inequality follows from $q_L \leq 1 + \alpha q_H$,



(a) $\alpha \in [0, \alpha_1]$ (b) $\alpha \in (\alpha_1, \alpha_2)$

(c) $\alpha \in [\alpha_2, 1)$

Figure B.2: The set of feasible allocations for the symmetric quadratic setting: complements

and the second inequality follows from $\alpha \leq \alpha_1$. Regarding R_4, if $\alpha \leq \alpha_1$ then the left hand side in (B.8) has a unique maximizer at $q_H = \frac{1}{1-\alpha}$, $q_L = \frac{1}{1-\alpha}$, and the maximum value is 0. Hence (B.4) is violated in $R_3 \cup R_4$ for each $\alpha \in [0, \alpha_1]$.

Step 2.2 $\alpha \in (\alpha_1, \alpha_2)$**.**

For $\alpha \in (\alpha_1, \alpha_2)$, we can argue as in the proof of Step 2.1 to establish that (B.4) is violated in R_3. Regarding R_4, the left hand side in (B.8) is non negative at $(q_H, q_L) \in R_4$ if and only if $\frac{1-b}{1-\alpha} + bq_H \leq q_L \leq \frac{1}{1-\alpha}$.

Step 2.3 $\alpha \in [\alpha_2, 1)$**.**

Regarding R_3, for each $q_H \in [0, \frac{1}{1-\alpha}]$ the inequality $c + dq_H \leq 1 + \alpha q_H$ holds given that $\alpha > \alpha_2$,[16] hence (B.4) is satisfied in R_3 if and only if $q_L \geq c + dq_H$. Re-

[16]The inequality holds at $x = 0$ and at $x = \frac{1}{1-\alpha}$, hence it holds for each $x \in (0, \frac{1}{1-\alpha})$.

garding R_4, the term $\tilde{u}(q_L, q_L)$ in the left hand side in (B.8) is at least as large as $\tilde{u}(1+\alpha q_H, 1+\alpha q_H)$, therefore the left hand side in (B.8) is at least as large as $-\frac{1}{2}(1-q_H+\alpha q_H)^2(1+r+2\alpha r)$, which is non-negative because $\alpha \geq \alpha_2$. Hence (B.4) holds at each point in R_4. ■

Proposition 6. *Consider the symmetric quadratic setting with type reversal. Suppose that the qualities are complements.*

(i) If $\theta_{HH} \geq \theta_{LL}^v$, then $q_H^{SB} = \hat{q}_H, q_L^{SB} = \hat{q}_L$.

(ii) Assume $\theta_{HH} < \theta_{LL}^v$. Then,

(a) If $\alpha \in [0, \alpha_1]$, then $q_H^{SB} = q^p$, $q_L^{SB} = q^p$.

(b) If $\theta_{HH} \geq \frac{3}{4}\theta_{HL} - \frac{3}{4}$, then $q_H^{SB} = q^p$, $q_L^{SB} = q^p$ for each $\alpha \in (\alpha_1, \alpha_2)$. If $\theta_{HH} < \frac{3}{4}\theta_{HL} - \frac{3}{4}$, then there exist parameter values (with α close to α_2) such that the q_H^{SB}, q_L^{SB} belong to region R_4, implying $q_H^{SB} < q_L^{SB}$.

The result in this proposition is immediate, as $\theta_{HH} \geq \theta_{LL}^v$ implies $\hat{q}_H > \hat{q}_L$, which satisfies (B.4) because $R_1 \cup R_2$ is the set of points in S such that $q_H \geq q_L$. Moreover, from (B.5) it is immediate that $\hat{q}_H \geq \hat{q}_L$ makes IR$_H$ satisfied, given that $r \in (-1, 0)$. Hence $q_H^{SB} = \hat{q}_H, q_L^{SB} = \hat{q}_L$.

The case of $\theta_{HH} < \theta_{LL}^v$ is more difficult to deal with, since we have $\hat{q}_H < \hat{q}_L$, and precisely $(\hat{q}_H, \hat{q}_L) \in R_3$, and we know from Lemma 1 that IC$_L$ is violated at some points in $R_3 \cup R_4$.

Part (ii)-(a) holds since when $\alpha \in [0, \alpha_1]$, the feasible set is $R_1 \cup R_2$ hence $(\hat{q}_H, \hat{q}_L)$ is infeasible. Then we maximize $\hat{\pi}(q, q) = (\theta_{LH} + \theta_{LL})\, u(q, q) - 2C(q)$ with respect to q, and find the maximizer $q^p = \frac{\theta_{LH}+\theta_{LL}}{(1-\alpha)(\theta_{LH}+\theta_{LL})+1}$ (with $\hat{\pi}(q^p, q^p) = (\theta_{LH}+\theta_{LL})q^p$). Since also IR$_H$ is satisfied when $q_L = q_H$, it follows that $q_H^{SB} = q^p$, $q_L^{SB} = q^p$.

Part (ii)-(b) is about the case in which some non monotonic allocation is feasible. Precisely, if $\alpha \in (\alpha_1, \alpha_2)$, then the feasible set consists of $R_1 \cup R_2$, and a subset of R_4. Yet, it is still the case that $(\hat{q}_H, \hat{q}_L)$ is infeasible, since our assumptions (included $\theta_{LL}^v > \theta_{HH}$) imply $(\hat{q}_H, \hat{q}_L) \in R_3$. In order to find q_H^{SB}, q_L^{SB} we need to evaluate $\max_{q_H} \hat{\pi}(q_H, \frac{1-b}{1-\alpha} + bq_H) \equiv \hat{\pi}_{R_4}$, and compare it with $\hat{\pi}(q^p, q^p)$. If $\hat{\pi}(q^p, q^p) \geq \hat{\pi}_{R_4}$, then $q_H^{SB} = q^p, q_L^{SB} = q^p$; if $\hat{\pi}(q^p, q^p) < \hat{\pi}_{R_4}$, then (q_H^{SB}, q_L^{SB}) belongs to R_4, as it is possible to prove that IR$_H$ is satisfied. Characterizing exactly when $\hat{\pi}(q^p, q^p) \geq \hat{\pi}_{R_4}$ as α varies in (α_1, α_2) is possible in principle, as we can always obtain closed form solutions, but those closed forms are quite complicated. Part (ii)-(b) establishes that if $\theta_{HL} - \theta_{HH}$ is negative, or not too positive, then $\hat{\pi}(q^p, q^p) > \hat{\pi}_{R_4}$ for each $\alpha \in (\alpha_1, \alpha_2)$, whereas if θ_{HL}

is sufficiently larger than θ_{HH}, then for some parameters $\hat{\pi}(q^p, q^p) < \hat{\pi}_{R_4}$ if α is close to α_2.[17]

We now move to consider $\alpha \in [\alpha_2, 1)$, and we find that dealing with this case is quite difficult. In detail, it is possible that $(\hat{q}_H, \hat{q}_L)$ is infeasible, and then we need to compare the optimal pooling contract with the optimal (q_H, q_L) in $R_3 \cup R_4$, which is found by maximizing $\hat{\pi}(q_H, c + dq_H)$ with respect to q_H. Precisely, let $\tilde{q}_H = \arg\max_{q_H} \hat{\pi}(q_H, c + dq_H)$, and $\tilde{q}_L = c + d\tilde{q}_H$. If $\hat{\pi}(q^p, q^p) \geq \hat{\pi}(\tilde{q}_H, \tilde{q}_L)$, then $q_H^{SB} = q^p, q_L^{SB} = q^p$, but if $\hat{\pi}(q^p, q^p) < \hat{\pi}(\tilde{q}_H, \tilde{q}_L)$, then $q_H^{SB} = \tilde{q}_H, q_L^{SB} = \tilde{q}_L$, provided that $\tilde{q}_H, \tilde{q}_L$ satisfies IR$_H$. However, it is also possible that $(\hat{q}_H, \hat{q}_L) \in R_3$, and thus it is feasible. In this case $q_H^{SB} = \hat{q}_H, q_L^{SB} = \hat{q}_L$ if IR$_H$ is satisfied. We are no longer able to cover these cases for general parameter values; instead, we offer a particular numeric example with full characterization for every possible $\alpha \in [0, 1)$ below.[18]

Consider parameter values such that $\theta_{HH} = 0.8$, $\theta_{HL} = 0.81$, $\theta_{LH} = 0.6$, $\theta_{LL} = 1$. Then, we can compute $\theta_{LH}^v = 0.4$, $\theta_{LL}^v = 1.19$ and $r = -\frac{19}{20}$, $\alpha_1 = \frac{1}{39}$, $\alpha_2 = \frac{1}{38}$, $b = \sqrt{\frac{39}{38}\alpha - \frac{1}{38}}$, $c = \frac{2}{38\alpha+1}$, $d = \frac{40\alpha-1}{38\alpha+1}$.

(i) If $\alpha \in [0, \frac{1}{38}]$, then $q_H^{SB} = q^p$, $q_L^{SB} = q^p$;

(ii) If $\alpha \in (\frac{1}{38}, \frac{1}{6}]$, then $q_H^{SB} = q^p$, $q_L^{SB} = q^p$;

(iii) If $\alpha \in (\frac{1}{6}, 0.1913]$, then q_H^{SB}, q_L^{SB} is such that $q_L^{SB} = c + dq_H^{SB}$ and such that IR$_H$ binds;

(iv) If $\alpha \in (0.1913, \frac{40}{123}]$, then $q_H^{SB} = \tilde{q}_H$, $q_L^{SB} = \tilde{q}_L$;

(v) If $\alpha \in (\frac{40}{123}, 0.8671]$, then $q_H^{SB} = \hat{q}_H$, $q_L^{SB} = \hat{q}_L$;

(vi) If $\alpha \in (0.8671, 1)$, then q_H^{SB}, q_L^{SB} is obtained by maximizing $\hat{\pi}$ subject to IR$_H$ binding.

Part (i) is a corollary of Proposition 6 to the case of $\alpha \in (0, \frac{1}{38})$, since $\theta_{HH} \geq \frac{3}{4}\theta_{HL} - \frac{3}{4}$ is satisfied. The remaining parts can be distinguished between (ii)-(iv), which refer to the case in which $(\hat{q}_H, \hat{q}_L)$ is infeasible, and (v)-(vi), which refers to the case in which $\hat{q}_H, \hat{q}_L$ belongs to R_3.

When $\alpha \in (\frac{1}{38}, \frac{40}{123}]$, $(\hat{q}_H, \hat{q}_L)$ is infeasible. Therefore we need to identify the best (q_H, q_L) on the line $q_L = c + dq_H$, denoted $(\tilde{q}_H, \tilde{q}_L)$, and to compare it with the pooling

[17] This is the case, for instance, if $\theta_{HH} = 3, \theta_{HL} = 5.1, \theta_{LH} = 1.6, \theta_{LL} = 5.7$, and $\alpha = \frac{2}{3}$.

[18] Detailed mathematical derivations are available upon request.

contract. It turns out that the pooling contract is superior for $\alpha \in (\frac{1}{38}, \frac{1}{6}]$, whereas $(\tilde{q}_H, \tilde{q}_L)$ is superior for $\alpha > \frac{1}{6}$. However, $\tilde{q}_H, \tilde{q}_L$ satisfies IR_H only for $\alpha \in (0.1913, \frac{40}{123}]$, but violates IR_H for $\alpha \in (\frac{1}{6}, 0.1913]$; in such a case the optimal contract is such that all the four constraints bind.

For $\alpha > \frac{40}{123}$, $(\hat{q}_H, \hat{q}_L)$ is feasible (i.e., it satisfies IC_L), therefore it is the optimal contract if it satisfies IR_H, which occurs if $\alpha \in (\frac{40}{123}, 0.8671)$. For greater values of α, we need to take into account also IR_H to find the optimal contracts.