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"Evaluating Assignment without Transfers: A Market Perspective"

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Evaluating Assignment without Transfers: A Market Perspective*

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Abstract

We show that every (random) assignment/allocation without transfers can be considered as a market outcome with personalized prices and an equal income. One can thus evaluate an assignment by investigating the prices and the induced opportunity sets. When prices are proportional across agents, the assignment is efficient; when prices are common, the assignment is both efficient and envy-free. Moreover, this market perspective reveals a weakness of envy-freeness.

Keywords: allocation without transfers, competitive equilibrium, equal incomes, market perspective, envy-free, Pareto efficient, coalitional-envy-free, random assignment

JEL Codes: C78, D61, D63

1 Introduction

In many settings, resources are allocated to agents without the use of monetary transfers, e.g., the allocation of office spaces or seats in college courses. Many of such problems are a assignment problem where multiple types of objects with n copies in total are to be allocated among n agents, and each agent is to receive exactly one object (Bogomolnaia and Moulin

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(2001)). Given an assignment/allocation, we are interested in evaluating it to see if it satisfies desirable properties such as efficiency and fairness. In the following, we focus on feasible assignments that can be either deterministic (i.e., individuals obtain either 0 or 1 unit of any object) or random (i.e., a lottery over deterministic assignments, or individuals are assigned with probability shares of objects).

In contrast to the Pareto efficiency being the leading concept of efficiency, there exist a variety of fairness criteria in the literature, such as symmetry (or equal treatment of equals) used in e.g., Zhou (1990) and Bogomolnaia and Moulin (2001) and envy-freeness in e.g., Varian (1974). Thomson (2011) discusses several other criteria.

Our analysis focuses on the case where objects do not rank agents while agents are endowed with cardinal preferences over objects. In such a setting of assignment without transfers, fairness is a natural requirement to impose. Given that every agent is viewed homogenous by objects, a "fair" assignment should ensure that every agent obtains her entitlement, which is no less than anyone else's. In other words, everyone should have equal claim to the objects.

One of the most commonly used fairness criteria is envy-freeness, which requires that there is no one preferring any other agent's assignment. Certainly, this implies symmetry. We begin our analysis by providing an example to show that envy-freeness is rather fragile and does not guarantee equal claim (Section 2). Specifically, given a non-envy-free assignment, a subset of agents may form a coalition, exchange (probability shares of) objects among themselves, and transform the assignment into an envy-free one without harming any of them. In other words, envy-freeness does not prevent some agents from obtaining preferential treatment.

Based this observation, we propose an alternative for evaluating assignments – a market perspective (Section 3). Supposing that every agent is endowed with an equal income, we are interested in the personalized prices that can rationalize her assignment as her optimal consumption bundle. We show that it is always feasible to find such personalized prices for any given assignment.

The benefits of such a market perspective is obvious: When objects are assigned to agent by price, it is straightforward to define fairness. Equal claim to the objects implies that the prices must be common across agents; when prices are common and incomes are equal, it is guaranteed that all agents face the same budget set or opportunity set (Varian (1976) and Thomson (2011)). We further illustrate the relationship among the properties in Section 4.

Such pseudo-market approaches have been considered in assignment problems since Hylland and Zeckhauser (1979) where a pseudo-market mechanism is introduced to assign objects to agents, and it has then been generalized in several ways, e.g., Budish (2011), Budish, Che, Kojima, and Milgrom (2013), and He, Miralles, and Yan (2012). This strand of existing literature focuses on what properties an assignment prescribed by such a pseudo-market mechanism can achieve, while being silent on evaluating other assignments in such a framework. In a similar sense, our paper also extends Miralles and Pycia (2014) who characterize the relationship between the pseudo-market mechanism and efficient assignments.

2 A Motivating Example

Let us consider an example with 3 objects, (A, B, C), and 4 agents, $(1, 2, 3, 4)$. There are one copy of each of objects A and B and 2 copies of object C. Each agent has a unit demand, i.e., she can be assigned exactly one unit of probability shares in all objects. Given the cardinal preferences as the following, we are interested in the properties of an assignment, Π.

Preferences

	Object		
Agent	A	B	C
1	1	1/2	0
2	1	1/2	0
3	1	1/2	0
4	1	0	1/2

;

Assignment: Π

	Object		
Agent	A	B	C
1	1/3	1/3	1/3
2	1/3	1/3	1/3
3	1/3	1/3	1/3
4	0	0	1

.

Π is both envy-free and Pareto efficient (a formal proof is available upon request), and thus this assignment desirable based on traditional criteria.

Alternatively, let us consider a market perspective: Hypothetically, agents endowed with some incomes choose an optimal consumption bundle to maximize expected utility when facing a common vector of prices. Π is an outcome given a price vector $(p_A, p_B, p_C) = (3/2, 1, 1/2)$ and a budget vector $(b_1, b_2, b_3, b_4) = (1, 1, 1, 1/2)$.[1] More importantly, the assignment cannot be supported as an equilibrium where everyone has the same income, which will be revisited in the next section.

From this market perspective, is the assignment still desirable? It is clear that agents 1-3, who are endowed with higher incomes, are somehow treated more favorably and are allowed to choose from a larger opportunity set. As one would expect, this may cause certain kind of

[1]Equivalently, one may find personalized prices and an equal budget to support this assignment as a result of agents' expected utility maximization.

envy, although the assignment is envy-free. To see this potential lack of fairness, let us consider an exchange between agents 2 and 3, which leads to a new assignment $\hat{\Pi}$ as follows:

New Assignment: $\hat{\Pi}$

	Object		
Agent	A	B	C
1	1/3	1/3	1/3
2	1/2	0	1/2
3	1/6	1/3	1/6
4	0	0	1

Note that every agent is indifferent between her own assignment in Π and that in $\hat{\Pi}$, but agent 4 now envies agent 2 in $\hat{\Pi}$. This envy is consistent with what we have suspected when discovering that agent 4 is endowed with a lower income. **Although $\hat{\Pi}$ is certainly undesirable due to its lack of envy-freeness, agents 1-3 could form a coalition and change the assignment into Π while maintaining the same level of welfare**; judging Π by the commonly used envy-freeness, Π is desirable.

The above example thus illustrates the weakness of envy-freeness, and, more importantly, the usefulness of the market perspective in evaluating an assignment. We formalize this evaluation approach in the next section.

3 A Market Perspective to Evaluating Assignments

We consider the economy $\Gamma = \{\mathcal{I}, \mathcal{S}, Q, V\}$, where:

(i) $\mathcal{I} = \{i\}_{i=1}^{I}$ is a set of agents;

(ii) $\mathcal{S} = \{s\}_{s=1}^{S}$, $S \geq 3$, is a set of objects;

(iii) $Q = [q_s]_{s=1}^{S}$ is a capacity vector indicating the number of available copies of each object, and $q_s \in \mathbb{N}$, $\forall s$; $\sum_{s=1}^{S} q_s = I$, i.e., there are just enough objects to be allocated to all agents;[2]

(iv) $V = [\mathbf{v}_i]_{i\in\mathcal{I}}$, where $\mathbf{v}_i = [v_{i,s}]_{s\in\mathcal{S}}$ and $v_{i,s} \in [0,1]$ is agent i's von Neumann-Morgenstern utility associated with object s. We assume that $v_{i,s} \neq v_{i,s'}$ for all i and $s \neq s'$.

A feasible *random assignment* is a matrix $\Pi = [\boldsymbol{\pi}_i]_{i\in\mathcal{I}} \in \mathcal{A}$, where $\boldsymbol{\pi}_i = [\pi_{i,s}]_{s\in\mathcal{S}}$ and $\pi_{i,s} \in [0,1]$ is agent i's probability share at object s, or the probability that agent i is assigned s; $\mathcal{A}$ is the space of all feasible random assignments such that $\sum_{s\in\mathcal{S}} \pi_{i,s} = 1$ for all i and

[2] One may extend the analysis to the case $\sum_{s=1}^{S} q_s \neq I$.

$\sum_{i\in\mathcal{I}}\pi_{i,s}=q_s$ for all s. By the Birkhoff-von Neumann theorem, a feasible random assignment can be expressed as a lottery over deterministic assignments. A few more notations are needed:

Definition 1 *Given the economy Γ, a personalized price set $\mathcal{P}_\Gamma$ and the associated assignment Π_Γ, are defined as follows:*

i). Every agent is given an equal artificial income that is normalized to be 1.

ii). For a given $\mathbf{p}_i=[p_{i,s}]_{s\in\mathcal{S}}\in[0,+\infty]^S$ where $p_{i,s}$ is the price of object s for agent i, the demand correspondence of agent i, $\boldsymbol{\pi}_i(\mathbf{v}_i,\mathbf{p}_i)$, $\forall i$, is constructed as:[3]

$$\begin{aligned}\boldsymbol{\pi}_i^*(\mathbf{v}_i,\mathbf{p}_i) &= \arg\max_{[\pi_{i,s}]_{s\in\mathcal{S}}}\sum_{s\in\mathcal{S}}\pi_{i,s}v_{i,s},\\ s.t.\sum_{s\in\mathcal{S}}\pi_{i,s} &= 1;\ \pi_{i,s}\geq 0,\forall s\in\mathcal{S};\sum_{s\in\mathcal{S}}p_{i,s}\pi_{i,s}\leq 1.\end{aligned}$$

iii). $\mathcal{P}_\Gamma\equiv\left\{P^*=[\mathbf{p}_i^*]_{i\in\mathcal{I}}\in[0,+\infty]^{I\times S}\,|\exists\boldsymbol{\pi}_i\in\boldsymbol{\pi}_i^*(\mathbf{v}_i,\mathbf{p}_i^*),\ \sum_{i\in\mathcal{I}}\pi_{i,s}^*=q_s,\ \forall i\in\mathcal{I},\ \forall s\in\mathcal{S}.\right\}$;

iv) $\Pi_\Gamma(P^*)\equiv\left\{[\boldsymbol{\pi}_i]_{i\in\mathcal{I}}\in\mathcal{A}|\boldsymbol{\pi}_i\in\boldsymbol{\pi}_i^*(\mathbf{v}_i,\mathbf{p}_i^*),\ \forall i\in\mathcal{I}\right\}$ for $P^*\in\mathcal{P}_\Gamma$, and $\Pi_\Gamma\equiv\cup_{\forall P^*\in\mathcal{P}_\Gamma}\Pi_\Gamma(P^*)$.

In other words, $\mathcal{P}_\Gamma$ is the set of all possible personalized prices that can rationalize some assignment as a result of agents' utility maximization given equal incomes; $\Pi_\Gamma(P^*)$ is the set of assignments corresponding to P^*; and Π_Γ is the set of all possible assignments that can be supported as a result of agents' utility maximization given equal incomes. To make this market approach useful, we show that every feasible assignment can be represented in this way.

Lemma 1 *For any Γ, $\mathcal{P}_\Gamma\neq\varnothing$ and $\Pi_\Gamma=\mathcal{A}$.*

The validity of Lemma 1 is obvious, since for any $\boldsymbol{\pi}_i$ there exists at least one price vector $\mathbf{p}_i\in[0,+\infty]^S$ such that $\boldsymbol{\pi}_i\in\boldsymbol{\pi}_i^*(\mathbf{v}_i,\mathbf{p}_i)$, and there is no restriction on $\mathbf{p}_i$ across agents.

3.1 Efficiency

We are now ready to evaluate whether an assignment is Pareto efficient by investigating the properties of the prices.

Theorem 1 *An assignment $\Pi\in\mathcal{A}$ is efficient if and only if $\exists P^*=[\mathbf{p}_i^*]_{i\in\mathcal{I}}\in\mathcal{P}_\Gamma$, $\boldsymbol{\alpha}=[\alpha_i]_{i\in\mathcal{I}}\in(0,+\infty)^I$, and $\mathbf{p}^*\in[0,+\infty)^S$ such that $\mathbf{p}_i^*=\alpha_i\mathbf{p}^*$, $\forall i\in\mathcal{I}$ and $\Pi\in\Pi_\Gamma(P^*)$.*

[3] If $p_{i,s}=+\infty$, we define $+\infty\cdot 0=0$ and $+\infty\cdot\pi_{i,s}=+\infty$ if $\pi_{i,s}>0$.

Given that $\mathbf{p}_i^* = \alpha_i \mathbf{p}^*$, $\forall i \in \mathcal{I}$, and everyone has the same budget, it is equivalent to say that every agent faces a common price vector but are endowed with a personalized budget. Therefore, the "if" part is a result of the first welfare theorem, and the "only if" part follows Theorem 1 of Miralles and Pycia (2014).

3.2 Fairness

We now formalize our discussion of fairness.

Definition 2 *A random assignment* Π *is* ***envy-free*** *if every agent prefers her own random assignment to the assignment of any other agent with respect to her expected utility, i.e.,*

$$\sum_{s \in \mathcal{S}} \pi_{i,s} v_{i,s} \geq \sum_{s \in \mathcal{S}} \pi_{j,s} v_{i,s}, \forall i, j.$$

Envy-freeness requires that no one prefers the assignment of someone else to her own. An agent's assignment is thus compared with all other observed individual assignments according to her preference, but there is no restriction on whether one desires other unobserved "fair" assignments. For example, the assignment in Section 2 satisfies envy-freeness; however, as shown earlier, a payoff-equivalent exchange between two agents creates envy. This is because envy-freeness does not guarantee that every agent has the equal access to the objects, which leads us to introduce a new concept.

Definition 3 *An assignment* Π *is* ***coalitional-envy-free*** *if any re-assignment from weakly Pareto-improving exchanges among any coalition of agents is still envy-free.*

Note that weakly Pareto-improving exchanges can lead to the same payoffs for the agents involved in the exchange. If an assignment is coalitional-envy-free, it is also envy-free; the reverse is not true. The example in Section 2 has shown that unequal incomes or proportional prices across agents may create coalitional-envy. We thus consider the idea of competitive equilibrium from equal incomes (Varian (1976) and Hylland and Zeckhauser (1979)).

Theorem 2 *For any economy* Γ*, there exists an assignment* $\Pi^* \in \mathcal{A}$ *that is the outcome of a competitive equilibrium from equal incomes (CEEI). That is,* $\exists P^* = [\mathbf{p}_i^*]_{i \in \mathcal{I}} \in \mathcal{P}_\Gamma$, $\mathbf{p}_i^* = \mathbf{p}_j^*$, $\forall i, j \in \mathcal{I}$ *and* $\Pi^* \in \Pi_\Gamma(P^*)$.

The proof is available in the working paper version of Hylland and Zeckhauser (1979) and He, Miralles, and Yan (2012) (Proposition 5). By the first welfare theorem, a CEEI outcome must be efficient. Besides, its outcome is envy-free as well.

The CEEI assignment of the "motivating example" in Section 2 is:

"Equal-claim" Assignment Π^*

	Object		
Agent	A	B	C
1	5/24	1/3	11/24
2	5/24	1/3	11/24
3	5/24	1/3	11/24
4	3/8	0	5/8

In fact this is an outcome of CEEI with the price vector $(p_A, p_B, p_C) = (8/3, 4/3, 0)$. We can verify that Π^* is coalitional-envy-free, i.e., there is no weakly Pareto-improving exchange among any subset of agents that leads to any envy. More generally, we have the following result (proof in Appendix):

Proposition 1 *A CEEI assignment is coalitional-envy-free.*

4 Relationship among the Criteria

In this last section, we summarize the relationship among Pareto efficiency, envy-freeness, coalitional-envy-freeness, and the CEEI assignments (Figure 1).

While many of the results are intuitive or have been proven in the previous section, we show additionally that there are assignments being both efficient and coalitional-envy-free but cannot be supported as an outcome of the CEEI.

Proposition 2 *Not every efficient and coalitional-envy-free assignment is a CEEI outcome.*

While leaving the proof in Appendix, we provide the following example where the assignment

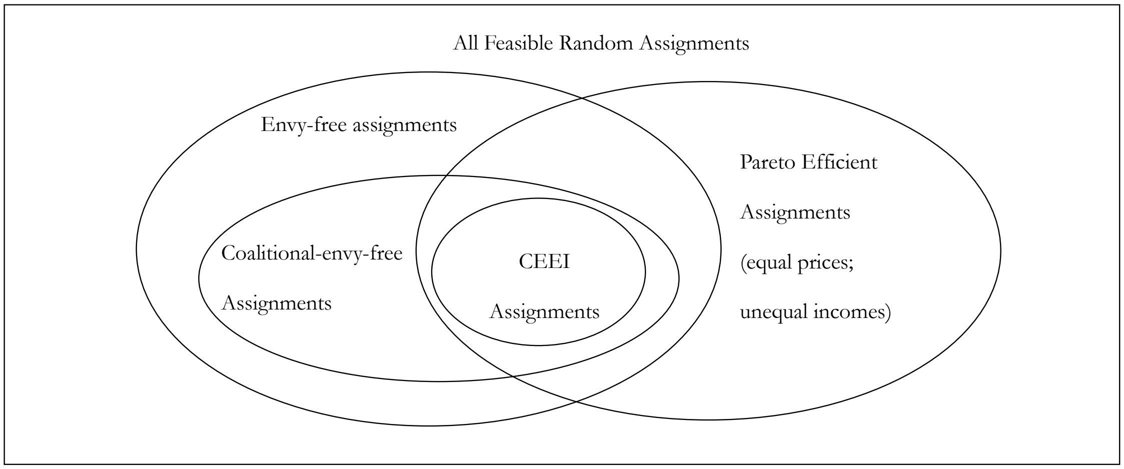

Figure 1: Properties of Random Assignments

Notes: CEEI stands for competitive equilibrium from equal incomes.

is both efficient and coalitional-envy-free but cannot be a CEEI outcome:

Preferences

Agent	Object A	B	C
1	1	1/2	0
2	0	1	1/2
3	1	0	1/2

;

Assignment: Π

Agent	Object A	B	C
1	1/2	1/2	0
2	0	1/2	1/2
3	1/2	0	1/2

.

Appendix: Proofs

Proof of Proposition 1. We first show that if the optimal bundle of an agent is not obtaining one unit of her most preferred object, the agent must exhaust her total budget.

Denote Π as a CEEI assignment and $(\pi_1, \cdots, \pi_S)$ as the agent's optimal bundle facing price $(p_1, \cdots, p_S)$. Without loss of generality, let the first object be her most preferred object. Suppose $\pi_1 < 1$ and the budget spent by the agent is $b = \sum_{\forall s} p_s \pi_s < 1$.

There must exist $s' \in \{2, \cdots, S\}$ such that $\pi_{s'} > 0$. Besides, $(\pi_1 + \varepsilon, \cdots, \pi_s - \varepsilon, \cdots, \pi_S)$ is more preferred to $(\pi_1, \cdots, \pi_S)$ and is affordable as long as $0 < \varepsilon < \min\left\{\pi_s, \frac{1-b}{p_1}\right\}$, which is a contradiction with $(\pi_1, \cdots, \pi_S)$ being the agent's optimal choice facing price $(p_1, \cdots, p_S)$.

Second, any coalition of weakly Pareto-improving exchanges cannot involve a member who obtains one unit of her most preferred object. Any change to the assignments of such agents would make them worse off, given everyone's most-preferred object is unique by assumption.

Therefore, all the members in any potential coalition of weakly Pareto-improving exchanges must exhaust their total budget in Π when facing a common price vector $(p_1, \cdots, p_S)$.

Suppose there exists a coalition of weakly Pareto-improving exchanges, $\widetilde{\mathcal{I}} \subset \mathcal{I}$. Then

$$\sum_{s\in\mathcal{S}} p_s \sum_{i\in\widetilde{\mathcal{I}}} \pi_{i,s} = \sum_{s\in\mathcal{S}} p_s \sum_{i\in\widetilde{\mathcal{I}}} \widetilde{\pi}_{i,s} = \sum_{i\in\widetilde{\mathcal{I}}} \sum_{s\in\mathcal{S}} p_s \widetilde{\pi}_{i,s} = \left|\widetilde{\mathcal{I}}\right|,$$

where $\widetilde{\pi}_{i,s}$ is the probability share after the exchange for $i \in \widetilde{\mathcal{I}}$ and $s \in \mathcal{S}$. Furthermore, this implies that $\forall i \in \widetilde{\mathcal{I}}$, $\sum_{s\in\mathcal{S}} p_s \widetilde{\pi}_{i,s} = 1$. Otherwise, there must exist $\widetilde{i} \in \widetilde{\mathcal{I}}$ such that $\sum_{s\in\mathcal{S}} p_s \widetilde{\pi}_{\widetilde{i},s} < 1$, which implies $\sum_{s\in\mathcal{S}} \widetilde{\pi}_{\widetilde{i},s} v_{\widetilde{i},s} < \sum_{s\in\mathcal{S}} \pi_{\widetilde{i},s} v_{\widetilde{i},s}$ because $\pi_{\widetilde{i}}$ is optimal given p_s while $\widetilde{\pi}_{\widetilde{i}}$ is also affordable. This contradicts the exchange being weakly Pareto-improving.

The above results implies that there is no agent who will envy any $\widetilde{i} \in \widetilde{\mathcal{I}}$ or any $i \in \mathcal{I} \diagdown \widetilde{\mathcal{I}}$, because $\forall i \in \mathcal{I}$, $[\pi_{i,s}]_{s\in\mathcal{S}}$ in Π is already the optimal consumption bundle among all possible $[\widehat{\pi}_{i,s}]_{s\in\mathcal{S}}$ that satisfies $\sum_{s\in\mathcal{S}} p_s \widehat{\pi}_{i,s} \leq 1$. ■

Proof of Proposition 2. We first show the above assignment is efficient. If a Pareto improvement Π' involves assignment change to $i \in \{1,3\}$, then it requires that $\pi'_{i,A} > \frac{1}{2}$. If $\pi'_{i,A} = \frac{1}{2}$, then any other assignment with the constraint $\pi'_{i,B} + \pi'_{i,C} = \frac{1}{2}$ and $\left(\pi'_{i,B}, \pi'_{i,C}\right) \neq \left(\pi_{i,B}, \pi_{i,C}\right)$ leads to a payoff strictly less than $\frac{3}{4}$, which is i's payoff in Π. If instead $\pi'_{1,A} < \frac{1}{2}$, then agent 1's payoff is:

$$\pi'_{1,A} + \frac{1}{2}\pi'_{1,B} = \pi'_{1,A} + \frac{1}{2}\left(1 - \pi'_{1,A} - \pi'_{1,C}\right) = \frac{1}{2}\left(1 + \pi'_{1,A} - \pi'_{1,C}\right) < \frac{1}{2}\left(\frac{3}{2} - \pi'_{1,C}\right) \leq \frac{3}{4}.$$

If $\pi'_{3,A} < \frac{1}{2}$, then agent 3's payoff is:

$$\pi'_{3,A} + \frac{1}{2}\pi'_{3,C} = \pi'_{3,A} + \frac{1}{2}\left(1 - \pi'_{3,A} - \pi'_{3,B}\right) = \frac{1}{2}\left(1 + \pi'_{3,A} - \pi'_{3,B}\right) < \frac{1}{2}\left(\frac{3}{2} - \pi'_{3,B}\right) \leq \frac{3}{4}.$$

Hence, any Pareto improvement Π' cannot involve assignment change to $i \in \{1,3\}$. Otherwise $\pi'_{1,A} + \pi'_{3,A} > 1$, which is not feasible. This further implies that Π is efficient since agent 2's assignment cannot be changed without changing the assignment of $\{1,3\}$ in Π.

We then show that Π is coalitional-envy-free. Note that Π is envy-free and thus the potential coalition to be considered are $\{1,3\}$, $\{1,2\}$, or $\{2,3\}$. The argument of efficiency directly rules out coalition $\{1,3\}$ since any weakly improved changes to $\{1,3\}$ require that $\pi'_{i,A} > \frac{1}{2} = \pi_{i,A}$, $i \in \{1,3\}$. This also rules out $\{1,2\}$ and $\{2,3\}$ since $\pi_{2,A} = 0$.

Finally, we show this efficient and coalitional-envy-free assignment cannot be a CEEI outcome. Suppose instead that $P^* = (p_A^*, p_B^*, p_C^*)$ is a vector of prices that leads to a CEEI outcome. We must have $p_A^* > 1$ and $p_B^* > 1$, otherwise Π is not optimal. But $[\pi_{1,s}]_{s\in\{A,B,C\}}$ cannot be affordable when $p_A^* > 1$ and $p_B^* > 1$. A contradiction.

In fact, a CEEI outcome of this example is a price vector $(p_A, p_B, p_C) = (2, 1, 0)$ and:

Assignment: Π^*

	Object		
Agent	A	B	C
1	1/2	0	1/2
2	0	1	0
3	1/2	0	1/2

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■

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