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CANONICAL EQUIVALENCE RELATIONS ON NETS OF $PS_{c_0}$

J. LOPEZ-ABAD

ABSTRACT. We give a list of canonical equivalence relations on discrete nets of the positive unit sphere of $c_0$. This generalizes results of W. T. Gowers and A. D. Taylor.

1. Introduction

Let $\text{FIN}$ be the family of nonempty finite sets of positive integers. A block sequence is an infinite sequence $(x_n)_n$ of elements of $\text{FIN}$ such that for every $n$ one has $\max x_n < \min x_{n+1}$ (usually written as $x_n < x_{n+1}$). The combinatorial subspace $\langle (x_n)_n \rangle$ given by $(x_n)_n$ is the set of finite unions $x_{n_0} \cup \cdots \cup x_{n_m}$. Using this terminology, the Hindman’s pigeonhole principle of $\text{FIN}$ states that every finite coloring of $\text{FIN}$ is constant in some combinatorial subspace, or, equivalently, every equivalence relation on $\text{FIN}$ with finitely many classes has a restriction to some combinatorial subspace with only one class. It is easy to see, for example by considering the equivalent relation defined by $s \sim t$ iff $\min s = \min t$, that this is no longer the case for equivalence relations with an arbitrary number of classes. Nevertheless, it is still possible to classify them, much in the spirit of the original motivation of F. P. Ramsey for discovering his famous Theorem. A result of Taylor states that every equivalence relation on $\text{FIN}$ can be reduced, by restriction to a combinatorial subspace, to one of the following five canonical relations:

$$\min, \max, (\min, \max), =, \text{FIN}^2,$$

naturally defined by $s \min t$ iff the minimum of $s$ is equal to the minimum of $t$, $s \max t$ iff the maximum of $s$ is equal to the maximum of $t$, $s(\min, \max)t$ iff both minimum and maximum are the same.

Following some geometric ideas exposed in Section 2, one can generalize $\text{FIN}$ as follows: Given a positive integer $k$, let $\text{FIN}_k$ be the set of mappings $x : \mathbb{N} \to \{0, 1, \ldots, k\}$, called $k$-vectors, whose support $\text{supp} x = \{n : x(n) \neq 0\}$ is finite and with $k$ in their range. One can naturally extend the union operation on $\text{FIN}$ to the join operation $\lor$ on $\text{FIN}_k$ by $(x \lor y)(n) = \max\{x(n), y(n)\}$. Let $T : \text{FIN}_k \to \text{FIN}_{k-1}$ be the mapping defined by $T(x)(n) = \max\{x(n) - 1, 0\}$. A $k$-block sequence $(x_n)_n$ is an infinite sequence of members of $\text{FIN}_k$ such that $\max\text{supp} x_n < \min\text{supp} x_{n+1}$ for every $n$. The $k$-combinatorial subspace $\langle (x_n)_n \rangle$ defined by a $k$-block sequence $(x_n)_n$ is the set of combinations of the form $T^{i_0}x_{n_0} \lor \cdots \lor T^{i_m}x_{n_m}$ with the condition that $i_j = 0$ for some $j$, and where $T^i x$ is defined by $T^i x(n) = \max\{x(n) - i, 0\}$ for $i > 0$ and $T^0 = \text{Id}$. Gowers has proved in that $\text{FIN}_k$ possesses the exact analogue of the pigeonhole principle of $\text{FIN}$: Every equivalence relation on $\text{FIN}_k$ with finitely many classes has a restriction to some

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combinatorial subspace with only one class. The aim of this paper is to characterize equivalence relations on \( \text{FIN}_k \) with arbitrary number of classes. More precisely, we are going to give a non redundant finite list \( \mathcal{T}_k \) of equivalence relations such that any other equivalence relation on \( \text{FIN}_k \) can be reduced, modulo restriction to some \( k \)-combinatorial subspace, to one in the list \( \mathcal{T}_k \).

Indeed, the elements of \( \mathcal{T}_k \) are determined by characteristics of a typical \( k \)-vector. Easy examples of these are the minimum and maximum of a finite set, that determine the Taylor’s list for \( \text{FIN}_2 \). Generalizing this, given an integer \( i \) with \( 1 \leq i \leq k \) let \( \min_i \) be the least integer \( n \) such that \( s(n) = i \). Another more complex example is the following. Given two integers \( i \) and \( l \) such that \( 1 \leq l \leq i - 1 \leq k \), let us assign to a given vector \( s \) of \( \text{FIN}_k \) the set of integers \( n \) such that \( \min_{i-1} s \leq n \leq \min_i s \) and \( s(n) = l \). We illustrate this with the following picture.

**Figure 1.** an example of invariant

So, our first task will be to guess all the natural characteristics of a \( k \)-vector. Although these characteristics are not well defined for an arbitrary \( k \)-vector, we will show that every \( k \)-block sequence will have a \( k \)-block subsequence, called here a system of staircases for which all the vectors have all natural characteristics well defined. The precise definitions are given in Section 3.

In order to show that every equivalence relation is, when restricted to some \( k \)-combinatorial subspace, in \( \mathcal{T}_k \) we follow the ideas of Taylor’s proof [5]. Let us explain this. Given an equivalence relation \( \sim \) on \( \text{FIN} \) one defines the coloring \( c : [\text{FIN}]^3 \to \{0,1\}^4 \) by

\[
a = (a_0, a_1, a_2) \mapsto \begin{cases} 
c(a)(0) = 1 & \text{iff } a_0 \sim a_1 \\
c(a)(1) = 1 & \text{iff } a_0 \cup a_1 \sim a_0 \\
c(a)(2) = 1 & \text{iff } a_0 \cup a_1 \sim a_1 \\
c(a)(3) = 1 & \text{iff } a_0 \cup a_1 \cup a_2 \sim a_0 \cup a_2,
\end{cases}
\]

where \( [\text{FIN}]^3 \) is the set of 3-sequences of finite sets \( (a_0, a_1, a_2) \) such that \( a_0 < a_1 < a_2 \). Since \( [\text{FIN}]^3 \) has a pigeonhole principle (this is a simple extension of Gowers’ result), one can find a block sequence \( X = (x_n)_n \) such that \( c \) is constant on \( [X]^3 \) with value \( s_0 \in \{0,1\}^4 \). An analysis of the value \( s_0 \) identifies the restriction of the equivalence relation \( \sim \) to \( X \) as one of the five relations \( \min, \max, (\min, \max), =, \text{FIN}^2 \). Let us re-write the coloring \( c \) in a way that will be easy to generalize to \( \text{FIN}_k \). Fix an alphabet of countably many variables \( \{x_n\}_n \). An \( \sim \)-equation \( e \) is a pair \( ((x_{i_0}, \ldots, x_{i_l}), (x_{j_0}, \ldots, x_{j_m})) \), written as \( x_{i_0} \cup \cdots \cup x_{i_l} \sim x_{j_0} \cup \cdots \cup x_{j_m} \), such that
of the positive sphere of $c_0$, extending some standard concepts coming from Banach space theory to $\text{FIN}_k$. We also state the W. T. Gowers Pigeonhole principle of $\text{FIN}_k$. The notion of equation is introduced in Section 3, together with the natural characteristics of a vector of $\text{FIN}_k$. We describe the vectors for which these invariants are well defined, and we show that they appear “everywhere”. We also define the family $T_k$. In Section 4 our main theorem is proved, and in Section 5 we give an explicit formula to compute the cardinality of $T_k$. Sections 6 and 7 deal with the finite version of our main result, and with some consequences for equivalence relations on the positive sphere of $c_0$.

2. First Definitions and Results

Recall that $c_0 = c_0(\mathbb{R})$ is the Banach space of sequences of real numbers converging to 0, with the sup-norm defined for a vector $\vec{x} = (x_n)_n$ of $c_0$ by $\|\vec{x}\| = \sup_n |x_n|$. Let $(e_n)_n$ be its natural Schauder basis, i.e., $e_n(m) = \delta_{n,m}$. The support of a vector $\vec{x} = (x_n)_n$, is defined
as supp $\bar{x} = \{ n : x_n \neq 0 \}$ and let $c_{00}$ be the linear subspace of $c_0$ consisting of the vectors $\bar{x} = (x_n)_n$ with finite support, i.e., only finitely many of the coordinates of $\bar{x}$ are not zero. Given two vectors $\bar{x}$ and $\bar{y}$ of $c_{00}$ we write $\bar{x} < \bar{y}$ to denote that $\max\text{supp} \bar{x} < \min\text{supp} \bar{y}$.

Let $PS_{c_0}$ be the set of norm one positive vectors of $c_0$, i.e., the set of all vectors $\bar{x} = (x_n)_n$ such that $\|\bar{x}\| = 1$, and such that $x_n \geq 0$, for every $n$, and let $PB_{c_0}$ be the set of positive vectors of the unit ball of $c_0$. Observe that $PB_{c_0}$ is a lattice with respect to $(x_n)_n \vee (y_n)_n = (\max \{x_n, y_n\})_n$ and $(x_n)_n \wedge (y_n)_n = (\min \{x_n, y_n\})_n$, with $0 = (0)_n$, and $1 = (1)_n$. Notice also that $PS_{c_0}$ is closed under the operation $\vee$, and that $x \vee y = x + y$ if $x$ and $y$ have disjoint support. In general, given two subsets $N \subseteq A$ of $c_0$, and a positive number $\delta$ we say that $N$ is a $\delta$-net of $A$ iff for every $\bar{a} \in A$ there is some $\bar{x} \in N$ such that $\|\bar{a} - \bar{x}\| \leq \delta$.

For a given $\delta$ with $0 < \delta < 1$, let $k$ be the least integer such that $1/(1 + \delta)^{k-1} \leq \delta$, and let $\varepsilon = 1/(1 + \delta)$. Let

$$\mathcal{N}_\delta = \{ x \in PB_{c_{00}} : x(i) \in \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{k-1}, 0\} \}$$

$$\mathcal{M}_\delta = \{ x \in PS_{c_{00}} : x(i) \in \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{k-1}, 0\} \}.$$ 

Since $\varepsilon^i - \varepsilon^{i+1} = \varepsilon^i(1 - \varepsilon) = \varepsilon^i(\delta/(1 + \delta)) < \delta$ and $\varepsilon^{k-1} \leq \delta$, it follows that $\mathcal{N}_\delta$ and $\mathcal{M}_\delta$ are $\delta$-nets of $PB_{c_0}$ and of $PS_{c_0}$, respectively. The set $\mathcal{N}_\delta$ is a sub-lattice of $PB_{c_0}$ with respect to $\vee$ and $\wedge$, and it is closed under scalar multiplication by $\varepsilon$, identifying $\varepsilon^i = 0$ for $l \geq k$ (which means that we identify the coordinates less than $\varepsilon^k$ with 0). Also, for two $\bar{x}, \bar{y} \in \mathcal{M}_\delta$, we have that $\bar{x} \vee \varepsilon^i \bar{y}, \varepsilon^i \bar{x} \vee \bar{y} \in \mathcal{M}_\delta$, for every $0 \leq i \leq k - 1$. Finally, note that $\mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \varepsilon^i \mathcal{M}_\delta$, a disjoint union.

We define the mapping $\Theta = \Theta_\delta : \mathcal{N}_\delta \rightarrow \{0, 1, \ldots, k\}^\mathbb{N}$ by

$$\Theta((x_m)_m)(n) = \begin{cases} k - \log_\varepsilon(x_n) & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0. \end{cases}$$

We can now give an equivalent definition of $FIN_k$ using the mapping $\Theta$, and therefore giving a geometrical interpretation of it.

**Definition 2.1.** Fix, for a given integer $k$, a positive real number $\delta = \delta(k)$ such that $1/(1 + \delta)^{k-1} = \delta$. Let $FIN_k = \Theta(\mathcal{M}_\delta)$, i.e., the set of functions $s : \mathbb{N} \rightarrow \{0, 1, \ldots, k\}$ eventually 0, and with $k$ in the range. The elements of $FIN_k$ are called $k$-vectors.

Observe that $\Theta^r \mathcal{N}_\delta = \bigcup_{i=0}^{k-1} \Theta^r \varepsilon^i \mathcal{M}_\delta$, and that $\Theta^r \varepsilon^i \mathcal{M}_\delta$ is the set of all functions $s : \mathbb{N} \rightarrow \{0, 1, \ldots, k\}$ eventually 0, and with $k - i$ in the range. So, $\Theta^r \varepsilon^i \mathcal{M}_\delta = FIN_{k-i}$. Hence, $\Theta^r \mathcal{N}_\delta = \bigcup_{i=1}^k \text{FIN}_i =: \text{FIN}_{\leq k}$, whose members are called $(\leq k)$-vectors.

We can transfer the algebraic structure of $N_k \subseteq c_{00}$ to $\text{FIN}_{\leq k}$ via $\Theta$. In particular, for $s, t \in \text{FIN}_{\leq k}$, let the support of $s$ be supp $s = \{ n : s(n) \neq 0 \}$; we write $s < t$ to denote that max supp $s < \min$ supp $t$, define $s \vee t$ and $s \wedge t$ by

$$(s \vee t)(n) = \max \{s(n), t(n)\} \quad \text{and} \quad (s \wedge t)(n) = \min \{s(n), t(n)\},$$

and let $T$ be the transfer of the multiplication by $\varepsilon$, i.e., for a $(\leq k)$-vector $s$, let

$$T(s) = \Theta(\varepsilon \Theta^{-1}(s)) = (s - 1) \vee 0.$$
Let \( S : \text{FIN}_{k-1} \to \text{FIN}_k \) be an inverse map for \( T \), defined for a \((k-1)\)-vector \( a \) by

\[
S(a)(n) = \begin{cases} 
  a(n) + 1 & \text{if } n \in \text{supp } a \\
  0 & \text{if not.}
\end{cases}
\]

It turns out that \( \text{FIN}_{\leq k} \) is a lattice with operations \( \vee \) and \( \wedge \), and it is closed under \( T \). We will use the order \( \leq_L \) to denote the lattice-order of \( \text{FIN}_{\leq k} \), i.e., for \( s, t \in \text{FIN}_{\leq k} \), we write \( s \leq_L t \) iff \( s \wedge t = s \). Note that \( \text{FIN}_i \vee \text{FIN}_j = \text{FIN}_{\max(i,j)} \) and \( \text{FIN}_i \wedge \text{FIN}_j = \text{FIN}_{\min(i,j)} \). We will use \( s + t \) for \( s \vee t \) whenever \( s < t \).

We now pass to introduce some combinatorial notions. A sequence of \( k \)-vectors \( (s_n) \) is called a finite \( k \)-block sequence if \( (s_n) \) is finite and if \( s_n < s_{n+1} \) for every \( n \); if such sequence is infinite, then we call it a (infinite) \( k \)-block sequence. We write \( \text{FIN}_k^{[\infty]} \), \( \text{FIN}_k^{[n]} \) and \( \text{FIN}_k^{[<\infty]} \) to denote respectively the set of finite \( k \)-block sequences, finite \( k \)-block sequences of length \( n \), and the set of finite \( k \)-block sequences.

The \( k \)-combinatorial subspace \( \langle \alpha \rangle \) defined by a finite or infinite \( k \)-block sequence \( \alpha = (s_n)_n \) is the set of all \( k \)-vectors of \( \alpha \) defined by

\[
\langle \alpha \rangle = \Theta((\text{LinSpan } \Theta^{-1}\{s_n\}_n) \cap \mathcal{N}_d),
\]

where \( \text{LinSpan } A \) denotes the linear span of a given subset \( A \) of \( c_0 \). Using this one has that \( \text{FIN}_k = \langle (\Theta e_n)_n \rangle \). Similarly, we define for a given integer \( i \leq k \) the set \( \langle \alpha \rangle_i \) of \( i \)-vectors of \( \alpha \). A main property of the \( k \)-block sequences \( (a_n)_n \) is that \( e_n \mapsto a_n \) naturally extends to a lattice isomorphism between \( \text{FIN}_k \) and \( \langle (a_n)_n \rangle \) that preserves the operation \( T \).

For \( M \leq N \leq \infty \), and \( \alpha = (s_n)_n \) let \( [\alpha]^{[M]} \) be the set of \( k \)-block subsequences of \( \alpha \), defined as \( [\alpha]^{[M]} = \{ (s_n)_{n \leq M} \in \text{FIN}_k^{[M]} : s_n \in \langle \alpha \rangle (0 \leq n < M) \} \). Without loss of generality we will identify \( [\alpha]^{[1]} \) with \( \langle \alpha \rangle \).

Given two finite block sequences \( \alpha \) and \( \beta \), and two infinite ones \( A \) and \( B \), we define \( \alpha \leq \beta \) if and only if \( \alpha \in [\beta]^{[|\alpha|]} \), \( \alpha \leq A \) if and only if \( \alpha \in [A]^{[|\alpha|]} \), and \( B \leq A \) if and only if \( B \in [A]^{[\infty]} \). Notice that all these definitions come from the notion of subspace. For example, \( A \in \langle B \rangle \) if and only if the space generated by \( \Theta^{-1}A \) is a subspace of the space generated by \( \Theta^{-1}B \).

For a \( k \)-block sequence \( A = (a_i)_i \) and \( a \in \langle A \rangle \), since \( \langle A \rangle = \Theta(\text{LinSpan } \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_d) \), we have that \( \Theta^{-1}a \in \Theta^{-1}\{a_i\}_i \cap \mathcal{N}_d \). Therefore, \( \Theta^{-1}a = \sum_{i=0}^{m} \varepsilon_i^d \Theta^{-1}a_i \), for some \( m \), and with possibly some \( d_i = 0 \). This implies that \( a = \Theta(\sum_{i=0}^{m} \varepsilon_i^d \Theta^{-1}a_i) = \sum_{i=0}^{m} \Theta(\varepsilon_i^d \Theta^{-1}a_i) = \sum_{i=0}^{m} T^{d_i}a_i \).

Finally, an infinite sequence \( \langle A_r \rangle_{r \in \mathbb{N}} \) of infinite \( k \)-block sequences \( A_r = (a^*_n)_n \) is called a fusion sequence of \( A \in \text{FIN}_k^{[\infty]} \) if for all \( r \in \mathbb{N} \):

- (a) \( A_{r+1} \leq A_r \leq A \),
- (b) \( a^*_0 < a^*_{0+1} \).

The infinite \( k \)-block sequence \( A_{\infty} = (a^*_0)_r \in \mathbb{N} \) is called the fusion \( k \)-block sequence of the sequence \( \langle M_r \rangle_{r \in \mathbb{N}} \).

**Definition 2.2.** Given a \( k \)-block sequence \( A = (a_n)_n \), let \( C_A : \langle A \rangle \to \text{FIN}_k \) be the mapping satisfying

\[
a = \sum_{n=0}^{\infty} T^{k-C_A(\langle A \rangle)(a)} a_n,
\]

(2)
for every $k$-block vector $a$ of $A$. Since $\Theta^{-1}a = \sum_{n \geq 0} \varepsilon^{k-C_A(a)(n)}\Theta^{-1}a_n$, for every $a$, the mapping $C_A$ is well defined. We call the sum in (2) the canonical decomposition of $a$ in $A$. Notice that $C_A(a) \in \text{FIN}_k$ for every $a$.

For two $(\leq k)$-vectors $s$ and $t$,

(a) we write $s \subseteq t$ when $t\supp s = s$, i.e., if $t$ restricted to the support of $s$ is equal to $s$, and

(b) we write $s \perp t$ when there is no $u \in \text{FIN}_{\leq k}$ such that $u \subseteq s, t$, i.e., if $s(n) \neq t(n)$ for every $n \in \text{dom } s \cap \text{dom } t$.

Using this, if $s = \sum_{n=0}^\infty T^{k-l_n}a_n$, then $T^{k-l_n}a_n \subseteq a$, for every $n$, while $T^{k-l_n}a_n \perp T^{k-l_n'}a_n$, for every $n \neq n'$. It follows that:

**Proposition 2.3.** Fix $A = (a_n)_n$, $a \in \langle A \rangle$ and an integer $n$. If there are some $r \leq k$ and $m$ such that $T^{k-r}a_n(m) = a(m) \neq 0$, then necessarily $C_A(a)(n) = r$ (i.e., $T^{k-r}a_n \subseteq a$).

The following is Gowers’ pigeonhole principle for $\text{FIN}_k$.

**Theorem 2.4.** [6] If $\text{FIN}_k$ is partitioned into finitely many pieces, then there is $A \in \text{FIN}_k[\infty]$ such that $\langle A \rangle$ is in only one of the pieces.

This naturally extends to higher dimensions.

**Lemma 2.5.** [7] Suppose that $f : \text{FIN}_k[\infty] \to \{0, \ldots, l - 1\}$. Then there is a block sequence $X$ such that $f$ is constant on $[X][n]$.

**Proof.** The proof is done by induction on $n$. Suppose it is true for $n - 1$. We can find, by a repeated use of Theorem 2.4, a fusion sequence $(X_r)_r$, $X_r = (x^r_i)_i$, such that for every $r$ and every $(b_0, \ldots, b_{n-2}) \in ([x^r_i]_{i \leq r})[n-1]$ the coloring $f$ is constant on the set $\{(b_0, \ldots, b_{n-2}, x) : x \in X_r\}$ with value $\varepsilon((b_0, \ldots, b_{n-2}), r)$. By construction one has that $X_r \leq X_s$ if $r \leq s$. So it follows that $\varepsilon((b_0, \ldots, b_{n-2}), r) = \varepsilon((b_0, \ldots, b_{n-2}), s)$ for every $(b_0, \ldots, b_{n-2}) \in [b_r][n-1]$ and every $r < s$.

This allows us to define $\varepsilon : [X_\infty][n-1] \to \{0, 1, \ldots, l - 1\}$ by $\varepsilon((b_0, \ldots, b_{n-2}) = \varepsilon((b_0, \ldots, b_{n-2}), r)$, for some (any) integer $r$, where $X_\infty = (x^r_i)_i$ is the fusion $k$-block sequence of $(X_r)_r$. This coloring $\varepsilon$ can be easily interpreted as a coloring of $\text{FIN}_k[\infty-1]$, so by the inductive hypothesis there is some $X \leq X_\infty$ such that $\varepsilon$ is constant on $[X][n-1]$, and therefore $f$ is also constant on $[X][n]$.

3. EQUATIONS, STAIRCASES AND CANONICAL EQUIVALENCE RELATIONS

Roughly speaking, terms are natural mappings that assign $k$-vectors to finite block sequences of $k$-vectors of a fixed length $n$, and which are defined from the operations $+$ and $T^i$ of $\text{FIN}_k$.

For example, the mapping that assigns to a block sequence $(a_1, a_2)$ of $k$-vectors the $k$-vector $a_1 + Ta_2$ is a $k$-term which can be understood as the mapping with two variables $x_1, x_2$ defined by $f(x_1, x_2) = x_1 + T x_2$.

From two fixed $k$-terms $f$ and $g$ of $n$ variables and one equivalence relation $\sim$ on $\text{FIN}_k$ we can define the natural coloring $c_{f,g} : [\text{FIN}_k][n] \to \{0, 1\}$ via $c_{f,g}(a_1, \ldots, a_n) = 1$ if and only if $f(a_1, \ldots, a_n) \sim g(a_1, \ldots, a_n)$. A $k$-equation will be $f \sim g$. The pigeonhole principle in Lemma 2.3 gives that for every equation $f \sim g$ ($f$ and $g$ with $n$ variables) there is some infinite block sequence $A$ such that, either for every $(a_1, \ldots, a_n)$ in $[A][n]$, $f(a_1, \ldots, a_n) \sim g(a_1, \ldots, a_n)$, or for all $(a_1, \ldots, a_n)$ in $[A][n]$, $f(a_1, \ldots, a_n) \not\sim g(a_1, \ldots, a_n)$, i.e., in $A$ the equation $f \sim g$ is either true
or false. As we explained in the introduction, Taylor proves that an equivalence relation \( \sim \) on FIN is determined by a list of 4 equations (precisely, \( x_0 \sim x_1, x_0 \sim x_0 + x_1, x_1 \sim x_0 + x_1 \) and \( x_0 + x_1 + x_2 \sim x_0 + x_2 \)). This is going to be also the case for arbitrary \( k \), of course with a more complex list of equations.

### 3.1. Terms and equations.

**Definition 3.1.** Let \( X = \{x_n\}_{n \geq 1} \) be a countable infinite alphabet of variables. Consider the trivial map \( x : X \to \mathbb{N} \) defined by \( x_n \mapsto x(x_n) = n \). A free \( k \)-term \( p \) is a map of the form \( s \circ x \) where \( s \) is a \( k \)-vector, i.e., it is a map \( p : X \to \{0, \ldots, k\} \) such that \( \text{supp} \, p \) is finite, and \( k \) is in the range of \( p \). A natural representation of \( p \) is

\[
p = p(x_0, \ldots, x_l) = \sum_{i=0}^{l} T^{k-m_i} x_i,
\]

where \( 0 \leq m_i \leq k \), and at least one \( m_i = k \). For example \( T^2 x_1 + T x_2 + x_4 \), and \( x_1 + x_3 \) are both free 3-terms. Notice that, if \( p \) is a free \( k \)-term, then \( p \circ x^{-1} \) is a \( k \)-vector. A free \((\leq k)\)-term is \( s \circ x \), where \( s \) is a \((\leq k)\)-vector. It follows that the set of free \((\leq k)\)-terms is a lattice. For example

\[
p(x_0, \ldots, x_n) \vee q(x_0, \ldots, x_m) = (p \circ x^{-1} \vee q \circ x^{-1}) \circ x.
\]

We also have defined the operator \( T \) for a \( k \)-term \( p(x_0, \ldots, x_n) \) by

\[
T(p(x_0, \ldots, x_n)) = (T(p \circ x^{-1}) \circ x).
\]

For every \((\leq k)\)-term \( p(x_0, \ldots, x_n) = \sum_{i=0}^{n} T^{k-m_i} x_i \) we consider the following kind of substitutions:

(a) Given a sequence of free \((\leq k)\)-terms \( t_0, \ldots, t_n \), consider the substitution of each \( x_i \) by \( t_i \)

\[
p(t_0, \ldots, t_n) = \bigvee_{i=0}^{n} T^{k-m_i} t_i.
\]

In the case that \( p \) and \( t_0, \ldots, t_n \) are free \( k \)-terms, then \( p(t_0, \ldots, t_n) \) is also a free \( k \)-term.

(b) For a block sequence \((a_0, \ldots, a_n)\) of \((\leq k)\)-vectors, replace each \( x_i \) by \( a_i \)

\[
p(a_0, \ldots, a_n) = \sum_{i=0}^{n} T^{k-m_i} a_i.
\]

If \( p \) is a free \( k \)-term, and \( a_0, \ldots, a_n \) are \( k \)-vectors, then the result of the substitution \( p(a_0, \ldots, a_n) \) is a \( k \)-vector. The main reason to introduce free \( k \)-terms is the following notion of equations.

**Definition 3.2.** A free \( k \)-equation (free equation in short) is a pair \( \{p(x_0, \ldots, x_n), q(x_0, \ldots, x_n')\} \) of free \( k \)-terms. Given a fixed equivalence relation \( \sim \) on \( \text{FIN}_k \), we will write the previous free equation as

\[
p(x_0, \ldots, x_n) \sim q(x_0, \ldots, x_n').
\]

Given \( s, t, i_0 \) and \( i_1 \)-vectors respectively, a free \( j_0 \)-term \( p \), and a free \( j_1 \)-term \( q \) such that \( \max\{i_l, j_l\} = k \) for \( l = 0, 1 \), we consider the equations of the form \( s + p \sim t + q \) and \( p + s \sim q + t \), called \( k \)-equations (or equations, if there is no possible confusion). The substitutions of \((b_0, \ldots, b_n)\) in the equation \( s + p \sim t + q \) will be allowed only when \( b_0 > s, t \), and for an
equation \( p + s \sim q + t \), provided that \( b_n < s, t \). This last condition implies that only finitely many substitutions are allowed for this latter equations, in contrast with the equations of the form \( s + p \sim t + q \).

**Definition 3.3.** We say that a \( k \)-equation \( s + p(x_0, \ldots, x_n) \sim t + q(x_0, \ldots, x_n) \) (or \( p(x_0, \ldots, x_n) + s \sim q(x_0, \ldots, x_n) + t \)) holds (or is true) in \( A \) if for every \( (a_0, \ldots, a_n) \) in \([A]^{[n+1]}\) with \( a_0 > s, t \) (resp. \( a_n < s, t \)), \( s + p(a_0, \ldots, a_n) \sim s + q(a_0, \ldots, a_n) \) (resp. \( p(a_0, \ldots, a_n) + s \sim q(a_0, \ldots, a_n) + s \)). The equation \( s + p(x_0, \ldots, x_n) \sim t + q(x_0, \ldots, x_n) \) (or \( p(x_0, \ldots, x_n) + s \sim q(x_0, \ldots, x_n) + t \)) is false in \( A \) if for every \( (a_0, \ldots, a_n) \) in \([A]^{[n+1]}\) with \( a_0 > s, t \) (resp. \( a_n < s, t \)), \( s + p(a_0, \ldots, a_n) \not\sim s + q(a_0, \ldots, a_n) \) (resp. \( p(a_0, \ldots, a_n) + s \not\sim q(a_0, \ldots, a_n) + s \)). The equation is decided in \( A \) if it is either true in \( A \) or false in \( A \).

It is clear that, given a \( k \)-equation \( p(x_0, \ldots, x_n) \sim q(x_0, \ldots, x_n') \), we can assume that \( n = n' \), since we can extend the terms of the equation adding summands of the form \( T^k x \) and not changing the “meaning” of the \( k \)-equation.

Some properties of equations that will be useful are given in the following.

**Proposition 3.4.** Suppose that all free \( k \)-equations with at most five variables are decided in a given \( k \)-block sequence \( A \). Then:

(i) If \( x_0 + T^{k-i}x_1 + x_2 \sim x_0 + x_2 \) is true in \( A \), then \( x_0 + T^{k-j}x_1 + x_2 \sim x_0 + x_2 \) is true in \( A \) for every \( j \leq i \).

(ii) If \( x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \) or \( Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \) are true in \( A \), then \( x_0 + x_1 + x_2 \sim x_0 + x_2 \) is also true in \( A \).

(iii) If the equation \( x_0 + x_1 + T^i x_2 \sim x_0 + T^i x_2 \) is true in \( A \), then the equation \( x_0 + x_1 + T^j x_2 \sim x_0 + T^j x_2 \) is also true in \( A \) for every \( j \leq i \).

(iv) If the equation \( T^i x_0 + x_1 + x_2 \sim T^i x_0 + x_2 \) is true in \( A \), then the equation \( T^j x_0 + x_1 + x_2 \sim T^j x_0 + x_2 \) is also true in \( A \) for every \( j \leq i \).

(v) If the equation \( x_0 + T^{k-r_1}x_1 + T^{k-r_2}x_2 \sim x_0 + T^{k-r_2}x_2 \) holds, then also the equation \( x_0 + T^{k-r_1}x_1 + T^{k-r_2}x_2 \sim x_0 + T^{k-r_2}x_2 \) for every \( r_1 > r_2 \) and \( r_0 \).

**Proof.** Suppose that the \( k \)-block sequence \( A \) decides all the equations with at most five variables.

(i): Fix \( j < i \). Then,

\[
x_0 + T^{k-i}x_1 + T^{k-j}x_2 + x_3 \sim x_0 + T^{k-i}(x_1 + T^{i-j}x_2) + x_3 \sim x_0 + x_3 \text{ hold in } A.
\]

Hence,

\[
x_0 + T^{k-i}x_1 + (T^{k-j}x_2 + x_3) \sim x_0 + (T^{k-j}x_2 + x_3) \text{ holds in } A,
\]

and we are done.

(ii): Suppose now that \( x_0 + x_1 + T x_2 \sim x_0 + T x_2 \) is true in \( A \). Then

\[
x_0 + x_2 + T x_3 \sim x_0 + T x_3 \text{ and } x_0 + x_1 + x_2 + T x_3 \sim x_0 + T x_3 \text{ are true in } A.
\]

Hence, \( x_0 + x_1 + x_2 + T x_3 \sim x_0 + x_2 + T x_3 \) holds in \( A \), and therefore, \( x_0 + x_1 + x_2 \sim x_0 + x_2 \) is true in \( A \).
(iii): Suppose that \( x_0 + x_1 + T^i x_2 \sim x_0 + T^i x_2 \) is true in \( A \), and fix \( j \geq i \). Then, \( x_0 + x_1 + x_2 + T^j(x_3 + T^{i-j} x_4) \sim x_0 + x_1 + x_2 + T^j x_3 + T^i x_4 \sim x_0 + T^i x_4 \) hold in \( A \), and
\[
x_0 + x_1 + T^j(x_2 + T^{i-j} x_3) \sim x_0 + x_1 + T^j x_2 + T^i x_3 \sim x_0 + T^i x_3 \text{ hold in } A,
\]
which implies what we wanted.

(iv): This is showed in a similar manner that (iii).

(v): Fix \( r_1 > r_2 \) and \( r_0 \) and suppose that the equation \( x_0 + T^{k-r_1} x_1 + T^{k-r_0} x_2 \sim x_0 + T^{k-r_0} x_2 \) holds in \( A \). Then, \( x_0 + T^{k-r_2} x_1 + T^{k-r_1} x_2 + T^{k-r_0} x_3 \sim x_0 + T^{k-r_1}(T^{r_1-r_2} x_1 + x_2) + T^{k-r_0} x_3 \sim x_0 + T^{k-r_0} x_3 \) and \( (x_0 + T^{k-r_2} x_1) + T^{k-r_1} x_2 + T^{k-r_0} x_3 \sim x_0 + T^{k-r_2} x_1 + T^{k-r_0} x_3 \) holds in \( A \). Therefore, \( x_0 + T^{k-r_2} x_1 \sim T^{k-r_0} x_3 \sim x_0 + T^{k-r_0} x_3 \) is true in \( A \).

3.2. Systems of staircases, canonical and staircase equivalence relations. Classifying equivalence relations of \( \text{FIN}_k \) is roughly the same as finding properties of a typical \( k \)-vector. One of these properties can be the cardinality, or, for example, the minimum or maximum of its support. Indeed Taylor’s result on \( \text{FIN} \) tells that these are the relevant properties of \( 1 \)-vectors.

For an arbitrary \( k > 1 \), one expects a longer list of properties. One example is obtained by considering for a given \( k \)-vector \( a \) the least integer \( n \) of the support of \( a \) such that \( a(n) = k \); another one is obtained by fixing \( i \) with \( 1 \leq i \leq k \) and considering the least \( n \) such that \( a(n) = i \). This is not always well defined, since for \( i < k \) there are \( k \)-vectors where \( i \) does not appear in their range. Nevertheless, this last property seems very natural to consider. Indeed we are going introduce a type of \( k \)-block sequences, called systems of staircases, where these properties, and some others, are well defined for every \( k \)-vector of their combinatorial subspaces.

**Definition 3.5.** Given an integer \( i \in [1, k] \) let \( \min_i, \max_i : \text{FIN}_k \to \mathbb{N} \) be the mappings \( \min_i(s) = \min s^{-1}\{i\} \), \( \max_i(s) = \max s^{-1}\{i\} \), if defined, and 0 otherwise. A \( k \)-vector \( a \) is a **system of staircases** (sos in short) if and only if

(i) Range \( s = \{0, 1, \ldots, k\} \),
(ii) \( \min_i a < \min_j a < \max_j a < \max_i a \), for \( i < j \leq k \),
(iii) for every \( 1 \leq i \leq k \),
\[
\text{Range } a[\min_{i-1} a, \min_i a] = \{0, \ldots, i\},
\]
\[
\text{Range } a[\max_i a, \max_{i-1} a] = \{0, \ldots, i\},
\]
\[
\text{Range } a[\min_k a, \max_k a] = \{0, \ldots, k\}.
\]

The following figure illustrates the previous definition.

![Figure 2. A typical sos.](image-url)
A block subspace \( A = (a_n)_n \) is a system of staircases iff every \( k \)-vector in \( \langle A \rangle \) is an sos. In the next proposition we show, among other properties, that for every \( k \)-block sequence \( A \) there is sos \( B \in [A[^\infty].\)

**Proposition 3.6.**

(i) \( T \) preserves sos, i.e., if \( a \) is an sos \( k \)-vector, then \( Ta \) is an sos \( (k - 1) \)-vector.

(ii) \( T^{k-j}a+b, a+T^{k-j}b \) are sos’s, provided that \( a < b \) are sos’s. Therefore, for every \( k \)-term \( p(x_0, \ldots, x_n) \) and every block sequence of sos \( (a_0, \ldots, a_n) \in [\text{FIN}_k][n+1] \), the substitution \( p(a_0, \ldots, a_n) \) is also an sos.

(iii) A \( k \)-block sequence \( A = (a_n)_n \) is an sos if and only if every \( B \) is an sos for every \( n \).

(iv) If \( A \) is an sos, then any other \( B \leq A \) is also an sos.

(v) For every \( A \) there is some \( B \leq A \) which is an sos.

**Proof.** It is not difficult to prove (i) and (ii) (for the last part of (ii), one can use induction on the complexity of the \( k \)-term \( p \)). To show (iii), let us suppose that \( a_n \) is an sos for every \( n \), and let us fix \( a \in \langle (a_n)_n \rangle \). Then there is a \( k \)-term \( p(x_0, \ldots, x_n) \) such that \( p(a_0, \ldots, a_n) = a \). Therefore, by (ii), \( a \) is an sos. Assertion (iv) easily follows from (ii). Finally, Let us prove (v):

Fix \( A = (a_n)_n \). For each \( n \), let

\[
c_n = \sum_{j=1}^k T^{k-j}a(2k-1)n+1-j+1 \sum_{j=1}^{k-1} T^{k-(k-j)}a(2k-1)n+k-1+j.
\]

Notice that for every \( n \) one has that

\[
\text{Range } c_n[0, \min_k(c_n)] = \text{Range } c_n[\max_k(c_n), \infty) = \{0, \ldots, k\}.
\]

Therefore, \( \text{Range } T^{k-j}c_n[0, \min_j T^{k-j}(c_n)] = \text{Range } T^{k-j}c_n[\max_j T^{k-j}(c_n), \infty) = \{0, \ldots, j\} \)

for each \( j \leq k \). For \( n \geq 0 \), let

\[
b_n = \sum_{j=1}^{k} T^{k-j}c_n(3k-1)+j+1 \sum_{j=1}^{k} T^{k-j}c_n(3k-1)+k-1+j+1 \sum_{j=1}^{k-1} T^{k-(k-j)}c_n(3k-1)+2k-1+j.
\]

Now it is not difficult to prove that every \( b_n \) is an sos.

**Definition 3.7.** An equivalence relation \( \sim \) on \( \text{FIN}_k \) is canonical \(^1\) in \( A \) if and only if every \( k \)-equation are decided in every sos \( B \in [A] \) in the same way, i.e., iff for every \( k \)-equation \( p \sim q \), either for every sos \( B \in [A] \) one has that \( p \sim q \) is true in \( B \), or for every sos \( B \in [A] \) one has that \( p \sim q \) is false in \( B \). We will say that \( \sim \) is canonical if it is canonical in \( \text{FIN}_k \).

Canonical equivalence relations are those for which all the equations \( p \sim q \) are decided in every sos in the same way. It is not difficult to see that all the equivalence relations of the list \( \{\min, \max, (\min, \max), \text{FIN}^2\} \) are canonical in \( \text{FIN} \). Taylor’s result for \( \text{FIN} \) says that there are no more canonical equivalence relations than the ones in this list. It will be shown later that

\(^1\)this name is not arbitrary chosen: We will show that every equivalence relation is, when restricted to some combinatorial subspace, canonical.
for every \( k \) there is also a finite list of canonical equivalence relations. Indeed we will give an explicit description of how canonical equivalence relations look like.

In order to do the same to the equivalence relations in \( \text{FIN}_k \) we have to give a list of relations naturally defined for a typical sos.

**Definition 3.8.** For a set \( X \), a \( k \)-block sequence \( A \), and an arbitrary map \( f : \langle A \rangle \to X \) we define the relation \( R_f \) on \( \langle A \rangle \) by \( sR_f t \) if and only if \( f(s) = f(t) \). Whenever there is no possible confusion, we are going to use the notation \( sft \) instead of \( sR_f t \). Now fix an sos \( A \). Recall that \( \min_i(s) = \min \{ n : s(n) = i \} \) for a given integer \( i \in \{1, k\} \) and \( s \in \langle A \rangle \). This mapping can be interpreted as \( \min_i : \langle A \rangle \to \text{FIN}_i \) in the following way

\[
\min_i(s)(n) = \begin{cases} 
  i & \text{if } n = \min_i(s) \\
  0 & \text{otherwise}.
\end{cases}
\]

Extending this, define, for \( I \subseteq \{1, \ldots, k\} \), the mapping \( \min_I : \langle A \rangle \to \text{FIN}_{\max I} \subseteq \text{FIN}_{\leq k} \) by \( \min_I(s)(n) = i \) if \( n = \min_i(s) \), for \( i \in I \) and 0 otherwise, i.e., \( \min_I(s) = \{ (\min_i(s), i) : i \in I \} \) and extended by 0 in the rest. Similarly, let

\[
\max_i(s)(n) = \begin{cases} 
  i & \text{if } n = \max_i(s) \\
  0 & \text{otherwise},
\end{cases}
\]

and let \( \max_I : \text{FIN}_k \to \text{FIN}_{\max I} \) be defined by \( \max_I(s) = \{ (\max_i(s), i) : i \in I \} \), again extended by 0. Clearly \( \min_I = \bigvee_{i \in I} \min_i \) and \( \max_I = \bigvee_{i \in I} \max_i \), where for two mappings \( f, g : \langle A \rangle \to \text{FIN}_{\leq k} \) we define \( (f \vee g)(s) = f(s) \vee g(s) \).

We now introduce a more sophisticated class of functions. For \( l \leq i - 1 \), let \( \theta_{i,l}^0, \theta_{i,l}^1 : \langle A \rangle \to \text{FIN}_l \) be the mappings defined by

\[
\theta_{i,l}^0(s) = \{ (n, l) : n \in (\min_{i-1}(s), \min_i(s)) \& s(n) = l \}, \text{ extended by 0, and}
\]

\[
\theta_{i,l}^1(s) = \{ (n, l) : n \in (\max_i(s), \max_{i-1}(s)) \& s(n) = l \}, \text{ extended by 0.}
\]

In other words, for a given integer \( n \)

\[
\theta_{i,l}^0(s)(n) = \begin{cases} 
  l & \text{if } n \in (\min_{i-1}(s), \min_i(s)) \text{ and } s(n) = l \\
  0 & \text{otherwise, and}
\end{cases}
\]

\[
\theta_{i,l}^1(s)(n) = \begin{cases} 
  l & \text{if } n \in (\max_i(s), \max_{i-1}(s)) \text{ and } s(n) = l \\
  0 & \text{otherwise.}
\end{cases}
\]

For example, for \( k = 4, i = 3, l = 2 \) and a given sos 4-vector \( s \), \( \theta_{3,2}^2(s) \) is the 2-vector such that \( (\theta_{3,2}^2(s))(n) = 2 \) for every \( n \) such that

(a) \( s(n) = 2 \), and

(b) \( n \) is in the interval between \( \min_2(s) \) (i.e., the first \( m \) such that \( s(m) = 2 \)) and \( \min_3(s) \) (i.e., the first \( m \) such that \( s(m) = 3 \)), and it is zero otherwise.

For \( 1 \leq l \leq k \), let

\[
\theta_{l}^2(s) = \{ (n, l) : n \in (\min_k(s), \max_k(s)) \& s(n) = l \}, \text{ extended by zero.}
\]

We illustrate this with another example: For \( k = 4, l = 3 \) and an sos 4-vector \( s \), \( \theta_{4,3}^3(s) \) is the 3-vector with value \( l = 3 \) in every element \( n \) of the support of \( s \) such that
Remark 3.9. (i) Sometimes we will use $\min_i$ or $\max_i$ as a integers instead of $i$-vectors, i.e., for example $\min_i(s)$ will denote the unique integer $n$ such that $\min_i(s)(n) = i$. 
(ii) Also, we can extend the mappings $f$ defined before for $\FIN_k$ to all $\FIN_{\leq k}$ by setting $\bar{f}(s) = f(s)$, if it is well defined, and $\bar{f}(s) = 0$, if not. For example, for a $(\leq k)$-vector $s$, $\min_{i-1}(s)(n) = i$ if $i \in \text{Range } s$ and $n$ is the minimum $m$ such that $s(m) = i$, and $\min_i(s) = 0$ otherwise; and $\theta^0_{i,t}(s)$ will have the same definition, provided that the mappings $\min_{i-1}$ and $\min_i$ are well defined for $s$, and so on.

Proposition 3.10. Suppose that $l$ is such that $-1 < l \leq i - 1$. Then, 
(i) $\sim_{\theta^0_{i,t}} \subseteq \sim_{\min_{i-1}} \cap \sim_{\min_i} \subseteq \sim_{\max_i} \cap \sim_{\max_{i-1}}$, and $\sim_{\theta^2_{i,t}} \subseteq \sim_{\min_k} \cap \sim_{\max_k}$.
(ii) $\sim_{\theta^2_{i,t}} \subseteq \sim_{\theta^2_{i,t+1}}$ and $\sim_{\theta^0_{i,t+1}} \subseteq \sim_{\theta^0_{i,t+1}}$.

Proof. We prove the result in (i) for $\theta^0_{i,t}$. The other cases can be shown in a similar way. 
Suppose that $\theta^0_{i,t}(s) = \theta^0_{i,t}(t)$; we show that $\min_{i-1}(s) = \min_{i-1}(t)$. Let $n$ be such that $\min_{i-1}(t)(n) = i - 1$. By symmetry, it suffices to prove that $s(n) = i - 1$. So, let $r$ be the unique integer such that $T^{k-C_A(t)}(r)a_r(n) = i - 1$. Note that $C_A(t)(r) \geq i - 1$. There are two cases to consider:
(a) $C_A(t)(r) = i - 1$. Since $a_r$ is an sos, there is some $m \geq n$ such that $T^{k-C_A(t)(r)}a_r(m) = l$, and hence $\theta^0_{i,t}(t)(m) = l$ and $\theta^0_{i,t}(s)(m) = l$. This implies that $C_A(s)(r) = C_A(t)(r)$, and hence $T^{k-C_A(t)(r)}a_r \subseteq s$. Hence, $s(n) = T^{k-C_A(t)(r)}a_r(n) = i - 1$.
(b) $C_A(t)(r) > i - 1$. Then, $\theta^0_{i,t}$ is well defined for $T^{k-C_A(t)(r)}a_r$, and $\theta^0_{i,t}(T^{k-C_A(t)(r)}a_r) \subseteq \theta^0_{i,t}(t)$, which implies that $T^{k-C_A(t)(r)}a_r \subseteq s$, and again we are done.

Let us now prove the result for $\theta^2_{i,t}$ in (ii). Suppose that $\theta^2_{i,t}(s) = \theta^2_{i,t}(t)$, i.e., 
\[ \{n \in [\min_k(s), \max_k(s)] : s(n) = l\} = \{n \in [\min_k(s), \max_k(s)] : t(n) = l\}. \]

Let $n \in (\min_k(s), \max_k(s))$ be such that $s(n) = l + 1$. We show that $t(n) = l + 1$. Let $r$ be the unique integer such that $T^{k-C_A(s)}(r)a_r(n) = l + 1$. Then, $C_A(s)(r) \geq l + 1$, and since $a_r$ is an sos, $T^{k-C_A(s)(r)}a_r^{-1}\{l\} \neq \emptyset$.

Claim. $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap (\min_k(s), \max_k(s)) \neq \emptyset$.

Proof of Claim: Let $r_0, r_1$ be the unique integers such that $a_{r_0}(\min_k(s)) = a_{r_1}(\max_k(s)) = k$. Observe that $r_0 \leq r \leq r_1$. There are two cases: If $r_0 < r < r_1$, then we are done since $(T^{k-C_A(s)(r)}a_r)^{-1}\{l\} \cap [\min_k(s), \max_k(s)] = (T^{k-C_A(s)(r)}a_r)^{-1}\{l\}$ is non empty.

Suppose that $r_0 = r$ (the case $r_1 = r$ is similar). Then, $C_A(s)(r) = k$, and $\min_k s = \min_k a_r$. So, $(a_r)^{-1}\{l\} \cap (\min_k a_r, \max_k a_r) \neq \emptyset$, since $a_r$ is an sos, and therefore $\text{Range } a_r$ $(\min_k a_r, \max_k a_r) = \{0, \ldots, k\}$. \hfill $\square$
Now that for every \( m \in (T^{k-C_A(s(r))}a_r)^{-1}\{l\} \cap (\text{min}_k s, \text{max}_k s) \) one has that \( t(m) = l \), since \((T^{k-C_A(s(r))}a_r)^{-1}\{l\} \cap (\text{min}_k s, \text{max}_k s) \subseteq \theta^t_l(t)\). By Proposition 2.3, \( C_A(t)(r) = C_A(s(r)) \), and hence \( T^{k-C_A(s(r))}a_r \subseteq t \), which implies that \( t(n) = T^{k-C_A(s(r))}a_r(n) = s(n) = l \).

The second inclusion in (ii) is shown in a similar manner. The details are left to the reader. \( \square \)

The collection of mappings introduced in Definition 3.8 can be divided into pieces as follows.

**Definition 3.11.** Let \( F_{\text{min}} = \{\text{min}_1, \ldots, \text{min}_k\} \), \( F_{\text{max}} = \{\text{max}_1, \ldots, \text{max}_k\} \), \( F_{\text{mid}} = \{\theta^t_l : i \in \{1, \ldots, k\}, l \in \{1, \ldots, i - 1\}\} \), for \( \varepsilon = 0, 1 \), and \( F_{\text{mid}} = \{\theta^2_l : l \in \{1, \ldots, k\}\} \cup \{0\} \). Set

\[
F = F_{\text{min}} \cup F_{\text{max}} \cup F_{\text{mid}^0} \cup F_{\text{mid}} \cup F_{\text{mid}^1}.
\]

Given a \( k \)-block sequence \( A \) we say that a function \( f : \langle A \rangle \to \text{FIN}_{\leq k} \) is a *staircase* function (in \( A \)) if it is in the lattice closure of \( F \). An equivalence relation \( \sim \) in \( A \) is a *staircase* (in \( A \)) iff \( \sim = \sim_f \) for some staircase mapping \( f \).

**Definition 3.12.** Let \( f, g : \langle A \rangle \to \text{FIN}_k \) be two functions defined on the \( k \)-combinatorial subspace defined by \( A \).

(i) We say that \( f \) and \( g \) are *incompatible*, and we write \( f \perp g \), when \( f(s) \perp f(s) \) for every \( s \in \{A\} \).

(ii) We write \( f < g \) to denote that \( f(s) < g(s) \) for every \( s \in \{A\} \).

(iii) We say that \( f \) and \( g \) are *equivalent* (in \( A \)), and we write \( f \equiv g \), when \( \sim_f \equiv \sim_g \), i.e., if \( f \) and \( g \) define the same equivalence relation in \( A \).

**Remark 3.13.** The family \( F \) is pairwise incompatible, i.e. if \( f \neq g \) in \( F \) then \( f \perp g \). Also, if \( f < g \) then \( f \perp g \).

The following makes the notion of staircase relation more explicit.

**Proposition 3.14.** Suppose that \( A \) is an sos, and suppose that \( f : \langle A \rangle \to \text{FIN}_{\leq k} \). Then the following are equivalent: (i) \( f \) is staircase.

(ii) There are \( I_0 \subseteq \{1, \ldots, k\} \), \( J_\varepsilon \subseteq \{j \in I_\varepsilon : j - 1 \in I_\varepsilon\} \), \( (l_j^{(\varepsilon)})_{j \in J_\varepsilon} \) with \( l_j^{(\varepsilon)} \leq j - 1 \) (for \( \varepsilon = 0, 1 \)) and \( l_k^{(2)} \) such that

\[
f = \text{min}_{I_0} \lor \bigvee_{j \in J_0} \theta^{0}_{l_j^{(0)}(0)} \lor \theta^{2}_{l_k^{(2)}} \lor \text{max}_{I_1} \lor \bigvee_{j \in J_1} \theta^{1}_{l_j^{(1)}}.
\]

We say that \( (I_0, J_0, (l_j^{(0)})_{j \in J_0}, I_1, (l_j^{(1)})_{j \in J_1}, l_k^{(2)}) \) are the values of \( f \).

(iii) Either \( f = 0 \) or there is a unique sequence \( f_0 < f_1 < \cdots < f_n \), \( f_0 \neq 0 \) such that \( f \equiv \bigvee_{i=0}^n f_i \) in \( A \).

**Proof.** This decomposition is a direct consequence of the fact that \( F \) is a pairwise incompatible family and the inclusions exposed in Proposition 3.10. \( \square \)

**Proposition 3.15.** Fix a staircase mapping \( f \) with decomposition \( f = f_0 \cup \cdots \cup f_n \) with \( f_0 < \cdots < f_n \) in \( F \), an sos \( A = (a_n)_n \) and \( k \)-vectors \( s \) and \( t \) of \( A \). Then

(i) \( f(s) = f(t) \) if and only if \( f_i(s) = f_i(t) \) for every \( 0 \leq i \leq n \).

(ii) \( f(s) = f(t) \) iff \( f(s|\text{supp } t) = f(t) \) and \( f(t|\text{supp } s) = f(s) \). \( \square \)

\( ^2 \)Notice that \( s|\text{supp } t \) is not necessarily a \( k \)-vector, but we can still apply \( f \) to it; see Remark 3.7.
(iii) Suppose that \( s_0, s_1 < t_0, t_1 \) are \((\leq k)\)-vectors of \( A \) such that \( s_0 + t_0, s_1 + t_1 \) and \( s_0 + t_1 \) are \( k \)-vectors. If \( f(s_0 + t_0) = f(s_1 + t_1) \), then \( f(s_0 + t_0) = f(s_0 + t_1) \).

**Proof.** (ii) follows from the fact that \( f_i < f_j \) for \( i < j \). Let us check (ii) using (i). We may assume that \( f \in \mathcal{F} \). There are several cases to consider.

(a) \( f = \min_i \). Suppose that \( \min_i(s) = \min_i(t) \). Then, \( i \in \text{Range } s|\supp t \) and hence \( \min_i(s|\supp t) = \min_i s = \min_i t = \min_i(t|\supp s) \). Suppose now that \( \min_i s < \min_i t \). Then, \( \min_i s < \min_i t \leq \min_i(t|\supp s) \). So, \( \min_i(t|\supp s) \neq \min_i s \).

(b) \( f = \max_i \) is shown in the same way.

(c) \( f = \theta^0_{i,l} \). Suppose that \( \theta^0_{i,l}(s) = \theta^0_{i,l}(t) \). Then, by (a), \( \min_j s = \min_j t|\supp s \) and \( \min_j t = \min_j s|\supp t \), where \( j = i - 1 \) or \( j = i \). Fix \( n \in (\min_{i-1}(s), \min_i(s)) \) such that \( s(n) = l \). Then, \( t(n) = l \), and hence \( \theta^0_{i,l}(t(s))(n) = l \). Now suppose that \( \theta^0_{i,l}(t(s))(n) = l \). Then, \( t(n) = l \), and hence \( s(n) = l \).

Suppose that \( \theta^0_{i,l}(s) = \theta^0_{i,l}(t|\supp s) \) and \( \theta^0_{i,l}(t) = \theta^0_{i,l}(s|\supp t) \). Then, \( \min_j(s) = \min_j(t) \) for \( j = i - 1, i \). Fix \( n \) such that \( \theta^0_{i,l}(s)(n) = l \). Then, \( \theta^0_{i,l}(t|\supp s)(n) = l \), which implies that \( t(n) = l \).

(d) The cases of \( f = \theta^1_{i,l} \) and \( f = \theta^2_{i,l} \) have a similar proof that (c).

Let us prove (iii). To this do, fix \( s_0, s_1, t_0, t_1 \) as in the statement, and suppose that \( f(s_0 + t_0) = f(s_1 + t_1) \). Suppose that \( f = \min_i \). If \( \min_i(s_0 + t_0) = \min_i(s_0) \), then clearly \( \min_i(s_0 + t_0) = \min_i(s_0 + t_1) \). If not we have that \( \min_i(s_0 + t_0) = \min_i(t_0) \), hence by our assumptions \( \min_i(s_1 + t_1) = \min_i(t_0) \). Since \( s_1 < t_0 \), it follows that \( \min_i(s_1 + t_1) = \min_i(t_1) \) and we are done. Suppose now that \( f = \max_i \). If \( \max_i(s_0 + t_0) = \max_i(s_0) \), then \( \max_i(s_1 + t_1) = \max_i(s_1) \) (now using the fact that \( t_1 > s_1 \)), and therefore, \( t_1 \) is a \((< i)\)-vector. So, \( \max_i(s_0 + t_0) = \max_i(s_0 + t_1) \). If \( \max_i(s_0 + t_0) = \max_i(t_0) \), then \( \max_i(s_1 + t_1) = \max_i(t_1) \) and we are done. Suppose now that \( f = \theta^0_{i,l} \) and suppose that \( \theta^0_{i,l}(s_0 + t_0)(n) = \theta^0_{i,l}(s_1 + t_1)(n) = l \). If \( s_1(n) = l \), then \( s_0(l) = l \), and hence \( (s_0 + t_1)(n) = l \). If \( t_1(n) = l \), then clearly \( (s_0 + t_1)(n) = l \). By symmetry, we are done in this case. The cases \( f = \theta^1_{i,l} \) and \( f = \theta^2_{i,l} \) have a similar proof. We leave the details to the reader. \( \square \)

**Proposition 3.16.** Any staircase equivalence relation is canonical.

**Proof.** By Proposition 3.15, it suffices to prove the result only for staircases functions \( f \in \mathcal{F} \). So, we fix \( f \in \mathcal{F} \), set \( \sim = \sim_f \) and consider an equation \( p(x_0, \ldots, x_n) \sim q(x_0, \ldots, x_n) \) where \( p(x_0, \ldots, x_n) = \sum_{d=0}^{n} T^{k-m_d} x_d \) and \( q(x_0, \ldots, x_n) = \sum_{d=0}^{n} T^{k-n_d} x_d \). Set \( p^* = p \circ \pi^{-1} \) and \( q^* = q \circ \pi^{-1} \). So \( p^*(d) = m_d \) and \( q^*(d) = u_d \) for \( d \leq n \) and 0 for the rest. Fix two sos’s \( A \) and \( B \) (\( B \) can be equal to \( A \), and suppose that \( p(a_0, \ldots, a_n) \sim q(a_0, \ldots, a_n) \) for some \( (a_0, \ldots, a_n) \in [A]^{n+1} \). We show that \( p(b_0, \ldots, b_n) \sim q(b_0, \ldots, b_n) \) for every \( (b_0, \ldots, b_n) \in [B]^{n+1} \). There are several cases to consider depending on \( f \).

(a) \( f = \min_i \). Let \( d_0 \) be the first \( d \) such that \( m_d \geq i \), and \( d_1 \) be the first \( d \) such that \( u_d \geq i \). Then \( \min_i(p(a_0, \ldots, a_n)) = \min_i(T^{-m_d} a_d) \) and \( \min_i(q(a_0, \ldots, a_n)) = \min_i(T^{-u_d} a_d) \). Since \( \min_i(T^{-m_d} a_d) = \min_i(T^{-n_d} a_d) \), we have that \( d_0 = d_1 \) (otherwise, \( a_d \perp a_d \)). Hence \( m_d_0 = u_d_1 \) (because \( T^r a \perp T^r a \) if \( r \neq s \)). So \( p \) and \( q \) satisfy that for every \( d < d_0 \), both \( m_d \) and \( u_d \) are less than \( i \) and \( m_d_0 = u_d_0 = i \). This implies that \( \min_i p(b_0, \ldots, b_n) = T^{-m_d_0} b_d_0 = \min_i q(b_0, \ldots, b_n) \).
(b) $f = \max_i$ has a similar proof.

(c) $f = \theta_{i,l}^0$. By Proposition 3.11, $\sim_{i,l}^0 \subseteq \sim_{\min_i-1} \cap \sim_{\min_i}$. Hence $\min_{i-\varepsilon} p(a_0, \ldots, a_n) = \min_{i-\varepsilon} q(a_0, \ldots, a_n)$ for $\varepsilon = 0, 1$. Define, for $\varepsilon = 0, 1$, $d_\varepsilon$ as the least integer $d$ such that $p^*(d_j) = q^*(d_j) \geq i - 1 + \varepsilon$. So, $d_0 \leq d_1$ and

$$\theta_{i,l}^0 p(a_0, \ldots, a_n) = \theta_{i,l}^0 \sum_{d=d_0}^{d_1} T^{k-m_d} a_d$$

(7)

$$\theta_{i,l}^0 q(a_0, \ldots, a_n) = \theta_{i,l}^0 \sum_{j=d_0}^{d_1} T^{k-u_d} a_d.$$  (8)

We see now that for every $d \in [d_0, d_1]$ either $m_d$ and $u_d$ are both less than $l$ or $m_d = u_d$. To do this, suppose that $d \in [d_0, d_1]$ is such that $m_d \geq l$. Then $\theta_{i,l}^0 T^{k-m_d} a_d \subseteq \theta_{i,l}^0 p(a_0, \ldots, a_n) = \theta_{i,l}^0 q(a_0, \ldots, a_n)$. Since for $d \neq d^0$ in $[d_0, d_1]$ one has that $T^{k-u_d} a_d' \perp T^{k-m_d} a_d$, it follows that $T^{k-m_d} a_d \subseteq T^{k-u_d} a_d$, and hence $u_d = m_d$.

(d) The cases $f = \theta_{i,l}^1$ and $f = \theta_{i,l}^2$ have a similar proof. \hfill $\square$

Let us now give some other properties of equations for staircase equivalence relations.

**Proposition 3.17.** Suppose that $\sim$ is a staircase equivalence relation with values $I_0$, $J_0$, $I_1$, $J_1$, $(l_{j,0})_{j \in J_0}$, $(l_{j,1})_{j \in J_1}$ and $l_{k,2}$, and suppose that $A$ is an sos.

(i) Let $0 \leq r_0 < r_1 \leq r_2$. If $T^{k-r_0} x_0 + T^{k-r_2} x_1 + x_2 \sim T^{k-r_0} x_0 + x_2$ is true in $A$, then $r_1 \notin I_0$.

(ii) If $l_{k,0}^2 = -1$, then the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true in $A$. If $l_{k,0}^2 \neq -1$, then for every $0 < l < l_{k,0}^2$, the equation $x_0 + T^{k-l} x_1 + x_2 \sim x_0 + x_2$ holds in $A$.

(iii) Suppose that $i \notin I_0$, and let $j = \max I_0 \cap [1, i]$. Then the equation $T^{k-j} x_0 + T^{k-i} x_1 + x_2 \sim T^{k-j} x_0 + x_2$ is true in $A$.

(iv) If $l_{j,0}^0 = -1$, then the equation $T^{k-(j-1)} x_0 + T^{k-(j-1)} x_1 + x_2 \sim T^{k-(j-1)} x_0 + x_2$ is true in $A$.

(v) Suppose that $l_{j,0}^0 \neq -1$, and let $h < l_{j,0}^0$. Then the equation $T^{k-(j-1)} x_0 + T^{k-h} x_1 + x_2 \sim T^{k-(j-1)} x_0 + x_2$ is true in $A$.

(vi) Suppose that $p(x_0, \ldots, x_n)$ is a $(\leq k)$-term, and suppose that $p(x_0, \ldots, x_n) + T^{k-l} x_{n+1} + x_{n+3} \sim p(x_0, \ldots, x_n) + T^{k-l} x_{n+2} + x_{n+3}$ holds in $A$. Then $p(x_0, \ldots, x_n) + T^{k-l} x_{n+1} + x_{n+2} \sim p(x_0, \ldots, x_n) + x_{n+2}$ also holds.

The analogous symmetric results are also true.

**Proof.** We give some of the proofs. The rest are quite similar, and the details are left to the reader. The main idea is to use the decomposition of $f = \bigvee_{i=0}^n f_i$ be the decomposition of $f$ into elements of $\mathcal{F}$ with $f_0 < \cdots < f_n$.

(i): Fix $(a_0, a_1, a_2) \in [A]^{[3]}$. Then $\min_{r_1}(T^{k-r_0} a_0 + T^{k-r_2} a_1 + a_2) = \min_{r} T^{k-r_2} a_1$, while $\min_{r_1}(T^{k-r_0} a_0 + a_2) = \min_{r_1}(a_2)$. Hence $\min_{r_1}(T^{k-r_0} a_0 + T^{k-r_2} a_1 + a_2) = \min_{r_1}(T^{k-r_0} a_0 + a_2)$.

For the rest of the points (ii) to (vi) one shows that in each case the corresponding equations for $\sim_{f_i}$ hold for every $0 \leq i \leq r$, and then use Proposition 3.11 to conclude that the desired equation also holds. \hfill $\square$
Theorem 4.1. If \( \sim \) holds, a contradiction. Notice that this proves that if \( \sim \) is a min-relation, then \( \sim \) is a max-relation.

4. The main Theorem

The next theorem is the main result of this paper.

Theorem 4.1. For every \( k \) and every equivalence relation \( \sim \) on \( \text{FIN}_k \) there is an sos \( B \) such that \( \sim \) restricted to \( (B) \) is a staircase equivalence relation.

Again we use Taylor’s result, now to expose the role of equations. Fix an equivalence relation \( \sim \) on \( \text{FIN} \). A diagonal procedure shows that we can find a block sequence \( A = (a_n)_n \) such that for every \( i_0, i_1, i_2, i_3, j_0, j_1, j_2, j_3 \in \{0, 1\} \) and every \( s, t \in \langle A \rangle \), the equation

\[
s + T^{i_0}x_0 + T^{i_1}x_1 + T^{i_2}x_2 + T^{i_3}x_3 \sim t + T^{j_0}x_0 + T^{j_1}x_1 + T^{j_2}x_2 + T^{j_3}x_3 \text{ is decided in } A. \quad (9)
\]

For arbitrary \( k \), the corresponding result is stated in Lemma 3.12. We consider the same cases considered in original Taylor’s proof:

(a) \( x_0 \sim x_1 \) holds. Then \( \sim \) is \( \langle A \rangle^2 \) on \( \langle A \rangle \): Let \( s, t \in \langle A \rangle \), pick \( u > s, t \), and hence \( s, t \sim u \).

(b) \( x_0 \sim x_1 \) is false, \( x_0 + x_1 \sim x_0 \) is true, and \( x_0 + x_1 \sim x_1 \) is false. Let us check that \( \sim \) is \( \sim_{\min} \) on \( \langle A \rangle \). Fix \( s, t \in \langle A \rangle \). Suppose that \( s \sim_{\min} t \), and let \( n \) be the least integer such that \( C_A(s)(n) = 1 \). Then \( s = a_n + s', t = a_n + t' \), and using the fact that \( x_0 + x_1 \sim x_0 \) holds, \( s, t \sim a_n \). Suppose now that \( s \neq_{\min} t \), and suppose that \( \min(s) < \min(t) \), and pick \( n \) as before. Then \( s \sim a_n \), \( a_n < t \), and \( a_n \sim t \), a contradiction.

(c) \( x_0 \sim x_1 \) is false, \( x_0 + x_1 \sim x_0 \) is false, and \( x_0 + x_1 \sim x_1 \) is true. Similar proof to 2. shows that \( \sim \) is \( \sim_{\min} \) on \( \langle A \rangle \).

(d) \( x_0 \sim x_1 \) is false, \( x_0 + x_1 \sim x_0 \) and \( x_0 + x_1 \sim x_1 \) are false, and \( x_0 + x_1 + x_2 \sim x_0 + x_2 \) is true. We show that \( \sim \) is \( \sim_{\min} \cap \sim_{\max} \) on \( \langle A \rangle \). It is rather easy to prove that \( \sim_{\min} \cap \sim_{\max} \subseteq \sim \) on \( \langle A \rangle \). For the converse, suppose that \( \max s \neq \max t \) and \( s \sim t \). We may assume that \( \max s < \max t \). Let \( n \) be the maximal integer \( m \) such that \( C_A(t)(m) = 1 \). Then, \( t = t' + a_n \), and hence the equation \( s \sim t' + x_0 \) holds and hence \( t' + x_0 + x_1 \sim t' + x_0 \) also holds which implies that \( x_0 + x_1 \sim x_0 \) holds, a contradiction. Notice that this proves that if \( x_0 + x_1 \sim x_0 \) is false, then \( \sim \subseteq \sim_{\max} \). We assume that \( \max s = \max t \) but \( \min s \neq \min t \). Suppose that \( \min s \sim t \) and work for a contradiction. Let \( n_0, n_1 \) be the minimum and the maximum of the support of \( s \) in \( A \) resp., and let \( m_0 \) be the minimum of the support of \( t \) in \( A \). Then \( s = a_{n_0} + s' + a_{n_1} \), \( t = a_{m_0} + t' + a_{n_1} \). Using that the equation \( x_0 + x_1 + x_2 \sim x_0 + x_2 \) is true, we may assume that \( s' = t' = 0 \). Since \( n_0 < m_0 \leq n_1 \), either the equation \( x_0 + x_2 \sim x_1 + x_2 \) is true or the equation \( x_0 + x_1 \sim x_1 \) is true. But the first case implies that the equations \( x_0 + x_3 \sim x_1 + x_2 + x_3 \) and \( x_0 + x_3 \sim x_2 + x_3 \) hold and hence \( x_0 \sim x_0 + x_1 \) holds, a contradiction.
(e) \(x_0 \sim x_1, x_0 + x_1 \sim x_0, x_0 + x_1 \sim x_1, x_0 + x_1 + x_2 \sim x_0 + x_2\) are false. Then \(\sim\) is on \((A)\).

Suppose that \(s \sim t\), and suppose that \(s \neq t\). Since \(x_0 + x_1 \sim x_0\) is false, then \(s = \max t\) (see 4. above). Let \(n\) be the maximal integer \(m < \max s\) such that \(C_A(s)(m) \neq C_A(t)(m)\), and without loss of generality we assume that \(C_A(s)(n) = 1\) and \(C_A(s)(n) = 0\). Then, \(s = s' + a_n + s''\), and \(t = t' + s''\), with \(t' < a_n\). Therefore the equation \(s' + x_0 + x_1 \sim t'' + x_1\) holds, which implies that \(s' + x_0 + x_1 + x_2, s' + x_0 + x_2 \sim t'' + x_2\) holds, and hence the equation \(x_0 + x_1 + x_2 \sim x_0 + x_2\) is true, a contradiction.

For arbitrary \(k\) the proof is done by induction on \(k\), making use of several lemmas. From now on we fix an equivalence relation \(\sim\) on \(\text{FIN}_k\). Our approach is the following. By the pigeonhole principle Theorem \[2.4\] there is always an sos \(A\) who decides a finite class of equations. It turns out that two kind of equations we are interested in are of the form \(x_0 + s \sim x_0 + t, s + x_0 \sim t + x_0\) where \(s\) and \(t\) are \((k - 1)\)-vectors. The reason is that if they are decided, then we can define naturally the \((k - 1)\)-equivalence relations

\[
s_0 \sim t \text{ iff } s + x_0 \sim t + x_0 \text{ holds,}
\]

\[
s_1 \sim t \text{ iff } x_0 + s \sim x_0 + t \text{ holds.}
\]

and then use the inductive hypothesis to detect both \(\sim_0\) and \(\sim_1\) as \((k - 1)\)-staircase equivalence relations. The next thing to do is to interpret \(\sim_0\) and \(\sim_1\) as \(k\)-relations \(\sim_0\) and \(\sim_1\), and then prove that in a suitable restriction \(\sim_0 \cap \sim_1\). Finally, a few more equations decided in some sos will force the decomposition \(\sim = \sim_0 \cap \sim_1 \cap R\) for a suitable staircase relation \(R\).

**Lemma 4.2.** There is some sos \(A = (a_n)_n\) such that for every 5-tuples \(i, j \in \{0, \ldots, k\}\), and every \((\leq k)\)-vectors \(s\) and \(t\) of \((A)\), the \(k\)-equation

\[
s + \sum_{l=0}^{4} T^{i(l)} x_l \sim t + \sum_{l=0}^{4} T^{j(l)} x_l
\]

is decided in \(A\).

**Proof.** We find a fusion sequence \((A_r)_r\) of \(k\)-block sequences, \(A_r = (a^n_r)_n\) such that for every integer \(r\) the equations \(s + \sum_{l=0}^{4} T^{i(l)} x_l \sim t + \sum_{l=0}^{4} T^{j(l)} x_l\) are decided in \(A_r\) for every \((\leq k)\)-vectors \(s, t\) of \((a^i_r)_{i < r}\) and \((a^j_r)_{j < r}\). Once we have done this, the fusion sequence \(A = (a^r_r)_r\) works for our purposes: Fix an equation \(e, s + \sum_{l=0}^{4} T^{i(l)} x_l \sim t + \sum_{l=0}^{4} T^{j(l)} x_l,\) and let \(r\) be the least integer such that \(s, t\) are \((\leq k)\)-vectors of \((a^i_r)_{i < r}\). Then \(e\) is decided in \(A_r\), hence it is also decided in \(A\).

We justify the existence of the demanded fusion sequence. Suppose we have already defined \(A_r = (a^n_r)_n\). Let \(L\) be the set of all the \(k\)-equations of the form

\[
s + \sum_{l=0}^{4} T^{i(l)} x_l \sim t + \sum_{l=0}^{4} T^{j(l)} x_l
\]

where \(s\) and \(t\) are \((\leq k)\)-vectors in \((a^i_r)_{i < r}) \leq k\) and \(i, j \in \{0, \ldots, k\}\). Let

\[
\Lambda : \left[\left(a^n_n\right)_{n \geq 1}\right]^{[5]} \to \{0, 1\}^L
\]
be the finite coloring defined for each \((c_0, \ldots, c_4) \in \{[a^{(r)}_n]_{n \geq 1}\}^{[5]}\) and each equation \(e\) of the form
\[s + \sum_{i=0}^4 T^{(i)} x_i \sim t + \sum_{i=0}^4 T^{(i)} x_i \in \mathcal{L}\] by \(\Lambda(c_0, \ldots, c_4)(e) = 0\) iff
\[s + \sum_{i=0}^4 T^{(i)} c_i \sim t + \sum_{i=0}^4 T^{(i)} c_i.\]

By Lemma 23, there is \(A_{r+1} \in \{[a^{(r)}_n]_{n \geq 1}\}^{[\infty]}\) such that \(A\) is constant on \([A_{r+1}]^{[5]}\), which is equivalent to all the equations considered above being decided in \(A_{r+1}\). \(\Box\)

4.1. The inductive step. The relations \(\sim_0\) and \(\sim_1\). Suppose that Theorem 4.1 holds for \(k - 1\). Our intention is, of course, to prove the case for \(k\). To do this we first associate two \(k - 1\)-relations to our fixed \(k\)-relation \(\sim\) as follows.

**Lemma 4.3.** There is an sos \(A\) and two staircase \(k - 1\)-equivalence relations \(\sim_0\) and \(\sim_1\) on \(\langle A \rangle_{k-1}\) such that for every \(s, t \in \langle A \rangle_{k-1}\),

- the \(k\)-equation \(s + x_0 \sim t + x_0\) is true in \(A\) if and only if \(s \sim_0 t\), and
- the \(k\)-equation \(x_0 + s \sim x_0 + t\) is true in \(A\) if and only if \(s \sim_1 t\).

Moreover \(\sim_0\) and \(\sim_1\) are such that for any two \((k - 1)\)-vectors \(s\) and \(t\) of \(A\),

- \(s \sim_0 t\) iff the \((k - 1)\)-equation \(s + x \sim_0 t + x\) holds in \(A\), and
- \(s \sim_1 t\) iff the \((k - 1)\)-equation \(x + s \sim_0 x + t\) holds in \(A\).

**Proof.** Let \(B = (b_n)_n\) be an sos satisfying Lemma 4.2. Then for \((k - 1)\)-vectors \(s\) and \(t\) of \(B\) the \((k - 1)\)-equations \(s + x_0 \sim t + x_0\) are decided in \(B\). Now define the relation \(\sim'\) on \(\langle B \rangle_{k-1}\) as follows. For \(s, t \in \langle B \rangle_{k-1}\),

\[s \sim' t\text{ iff } s + x_0 \sim t + x_0\text{ holds in } B.\]

It is not difficult to see that \(\sim'\) is an equivalence relation. By the inductive hypothesis there is some \((k - 1)\)-block sequence \(B' = (b'_n)_n \in \{[Tb_n]_4\}^{[\infty]}\) and some canonical equivalence relation \(\sim_0\) such that \(\sim'\) coincides with \(\sim_0\) on \(B'\) (since, by Proposition 3.10, all staircase equivalence relations are canonical). The \(k\)-block sequence \(A = (Sb'_n)_{n \geq 1}\) and the \(k\)-equivalence relation \(\sim_0\) clearly satisfy [14]. We prove assertion [12] for \(\sim_0\). To do this, suppose that \(s \sim_0 t\). Then the \(k\)-equation \(s + x_0 \sim t + x_0\) holds. Since the equation \(s + Tx_0 + x_1 \sim t + Tx_0 + x_1\) is decided, it must be true. It follows that for every \(k\)-vector \(b > s, t\) we have that \(s + Tb \sim_0 t + Tb\). Since \(\sim_0\) is canonical, we obtain that the \((k - 1)\)-equation

\[s + x_0 \sim t + x_0\text{ holds in } A,\]

as desired. Now assume that [14] is true. Fix a \((k - 1)\)-vector \(u > s, t\). Then \(s + u \sim_0 t + u\), i.e.,
the \(k\)-equation \(s + u + x_0 \sim t + u + x_0\) holds. Hence \(s + x_0 \sim t + x_0\), that is \(s \sim_0 t\).

We justify now the existence of a staircase \(k - 1\)-equivalence relation \(\sim_1\) and an sos \(A\) such that the statements [1] and [3] hold. We can find a fusion sequence \((A_r)_r\), \(A_r = (a^{(r)}_n)_n\), of \(k\)-block sequences of \(A\), and a list \(\{a^{(r)}_n\}_{n \in \langle a^{(r)}_n \leq r \rangle}_k\) defined on \(\langle A_r \rangle_{k-1}\) such that for every \(s, t \in \langle A_r \rangle_{k-1}\),

\[a + s \sim a + t\text{ if and only if } s \sim_0 t.\]
Let $A_{\infty} = (a^i_n)_n$ be the fusion sequence of $(A_r)_r$. Now for every $a \in A_{\infty}$ let $n(a)$ be unique integer unique $n$ such that $a \in (\langle a^i_n \rangle_{i<n}) \setminus (\langle a^i_n \rangle_{i<n-1})$. Define the finite coloring

$$c : (A_{\infty}) \to \text{canonical equivalence relations on FIN}_{k-1}$$

by $c(a) = \sim^{n(a)}_a$. By Lemma 2.3 there is some $A \in [A_{\infty}]^{[\infty]}$ in which $c$ is constant, with value $\sim_1$. We check that $A$ and $\sim_1$ satisfy what we want. Fix $a \in (A)$ and two $k-1$-vectors $s, t$ of $A$ with $a < s, t$; then $a \in \theta(n(a))$ and $s, t$ are $k-1$-block sequences of $A_{n(a)}$. So, $a + s \sim a + t$ if and only if $s \sim_{n(a)} t$ if and only if $s \sim_1 t$. Notice that in particular all equations $x_0 + s \sim x_0 + t$ are decided in $A$.

Let us prove now the assertion (13). To do this, fix two $(k-1)$-vectors $s, t$ of $A$. If $s \sim_1 t$, then $x_0 + s \sim x_0 + t$. Given a $(k - 1)$-vector $u < s, t$, choose a $k$-vector $a < u$ in $\langle (Sb'_n)_{n \geq 0} \rangle$. Then $a + u + s \sim a + u + t$, and this implies that $u + s \sim u + t$; in other words, the $(k - 1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Suppose now that the $(k - 1)$-equation $x_0 + s \sim_1 x_0 + t$ holds. Pick $(k - 1)$-vector $u < s, t$. Then the $k$-equation $x_0 + u + s \sim x_0 + u + t$ is true, and hence also $x_0 + s \sim x_0 + t$ holds (since this equation is decided).

Finally, we justify the existence of the fusion sequence $(A_r)_r$. Suppose we have already defined $A_r = (a^r_n)_n$ fulfilling its corresponding requirements. For every $a \in (\langle a^i_n \rangle_{i<r})$, put $\sim^{n+1}_a = \sim^a_n$. For every $a \in (\langle a^i_n \rangle_{i \leq r}) \setminus (\langle a^i_n \rangle_{i<r})$, let $R_a$ be the relation on $(\langle a^r_n \rangle_{n \geq 1})_{k-1}$ defined by

$$sR_at \text{ if and only if } a + s \sim a + t.$$ 

By the inductive hypothesis, we can find some $B \leq (a^r_n)_{n \geq 1}$ such that for every $a \in (\langle a^i_n \rangle_{i \leq r}) \setminus (\langle a^i_n \rangle_{i<r})$ the relation $R_a$ is staircase when restricted to $B$. Then $A_{r+1} = B$ satisfies the requirements.

Roughly speaking, the assertions (12) and (13) tell that the $(k - 1)$-relation $\sim_0$ does not depend on the part of a $(k - 1)$-vector before $\min_{k-1}$ and that $\sim_1$ does not depend on the part of a $(k - 1)$-vector after $\max_{k-1}$. Indeed (12) and (13) determine the form of $\sim_0$ and $\sim_1$. To express this mathematically we introduce the following useful notation.

**Definition 4.4.** For $l \leq k$, let $\max^l_k : \text{FIN}_k \to \text{FIN}_k$ be defined by

$$\max^l_k(s)(n) = \begin{cases} s(n) & \text{if } n \leq \max_k(s) \text{ and } s(n) \geq l, \\ 0 & \text{otherwise.} \end{cases}$$

In other words $\max^l_k$ is the staircase function with values $I_0 = \{l, \ldots, k\}$, $J_0 = \{l + 1, \ldots, k\}$, for every $j \in J_0$, $l^{(0)}_j = l$, $l^{(2)}_j = l$ and $I_1 = \{k\}$. Symmetrically, we can define $\min^l_k$ by $\min^l_k(s)(n) = s(n)$ if $n \leq \min_k(s)$ and $s(n) \geq l$, and 0 otherwise.

**Proposition 4.5.** Suppose that $R$ is a staircase relation, and suppose that $A$ is an sos. The following are equivalent:

(i) For every $k$-vectors $s, t$ of $A$, one has that $sRt$ iff $x + sR x + t$ holds in $A$.

(ii) Either $R$ is a max-relation or there is some max-relation $R'$ and some $l \in \{1, \ldots, k\}$ such that $R = R' \cap \max^l_k$.

The analogous result for $s + xR t + x$ is also true.
Proof. Fix a staircase relation $R$ with values $I_x, J_x, (u^{(e)}_{j})_{j \in J_x}$ ($e = 0, 1$) and $l^{(2)}_k$ such that for every $k$-vectors $s, t$ one has that $s R t$ if $x + s R x + t$ holds. Suppose that $I_0 \neq \emptyset$, since otherwise $R$ is a max-relation. Let $l = \min I_0$. We show that $I_0 = \{l, l + 1, \ldots, k\}$, $J_0 = \{l + 1, \ldots, k\}$, for every $j \in J_0$, $l^{(0)}_j = l$, $l^{(2)}_j = l$ and $k \in I_1$. First we show that $l^{(2)}_k \neq -1$. If not, the equation $x_0 + x_1 + x_2 R x_0 + x_2$ is true and hence the equation $x_1 + x_2 R x_2$ is true, which implies that $l \notin I_0$, a contradiction. If $l^{(2)}_k > l$, then the equation $x_0 + T^{k-l} x_1 + x_2 R x_0 + x_2$ is true and hence the equation $T^{k-l} x_1 + x_2 R x_2$ is true, which implies again that $l \notin I_0$. If $l^{(2)}_k < l$, then the equation $T^{k-l} x_0 + x_1 R x_1$ holds and hence the equation

$$x_0 + T^{k-l} x_1 + x_2 R x_0 + x_2 \text{ holds,}$$

which contradicts the definition of $l^{(2)}_k$.

We now show that $I_0 = \{l, \ldots, k\}$. It is clear that $I_0 \subseteq \{l, \ldots, k\}$ since $l$ is the minimum of $I_0$. We prove the reverse inclusion $\{l, \ldots, k\} \subseteq I_0$. Suppose not, and set $j = \min\{l, \ldots, k\} \setminus I_0$.

Then the equation $T^{k-j-1} x_0 + T^{k-j} x_1 + x_2 R T^{k-j-1} x_0 + x_2$ is true and hence the equation $x_0 + T^{k-j} x_1 + x_2 R T^{k-j-1} x_0 + x_2$ is true, which implies that the equation $x_0 + T^{k-j} x_1 + x_2 R x_0 + x_2$ also holds. This contradicts the fact that $j > l$ and that $R \subseteq R^{(2)}_I$.

Notice that $I_0 = \{l, \ldots, k\}$ implies that $J_1 = \{l + 1, \ldots, k\}$.

We show that $l^{(0)}_j = l$ for all $j \geq l + 1$. Suppose that $l^{(0)}_j = -1$. This implies that the equation $T^{k-(j-1)} x_0 + T^{k-(j-1)} x_1 + x_2 R T^{k-(j-1)} x_0 + x_2$ holds. Again by adding one variable at the beginning of both terms and using the fact that $j - 1 \geq l$ we can arrive at a contradiction to the fact that $l^{(2)}_k = l$. Suppose now that $l^{(0)}_j < l$. Then the equation $T^{k-l} x_0 + x_1 R x_1$ is true, and adding a variable we arrive at a contradiction. Suppose that $l^{(0)}_j > l$, then the equation

$$T^{k-(j-1)} x_0 + T^{k-j} x_1 + x_2 R T^{k-(j-1)} x_0 + x_2 \text{ is true},$$

which yields a contradiction in the same way as before. It is not difficult to check that the converse and the analogous situation for $\min$ are also true.

Proposition 4.4 and (12) and (13) determine the relations $\sim_0$ and $\sim_1$ as follows.

**Corollary 4.6.** The relation $\sim_0$ is either a min-relation or there is some $l \leq k - 1$ and some min-relation $R$ such that $\sim_0 = R \cap \min^I_{k-1}$ and $\sim_1$ is either a max-relation or there is some $l \leq k - 1$ and some max-relation $R$ such that $\sim_1 = R \cap \max^I_{k-1}$. □

Recall that $\sim_0$ and $\sim_1$ are both staircase equivalence relations of $\text{FIN}_{k-1}$. We now give the proper interpretation of both as $k$-relations. Suppose that $k > 1$. We know that either $\sim_1$ is a max-relation, or $\sim_1 = \max^I_{k-1} \cap R$, with $R$ a max-relation. Let

$$\sim_1' = \begin{cases} \sim_1 & \text{if } \sim_1 \text{ is a max-relation} \\ \theta^1_{k,l} \cap R & \text{if } I_0 \neq \emptyset. \end{cases}$$
Notice that in the second case we have that \( \max_k \subseteq \sim'_0 \). We do the same for \( \sim_0 \): It is either a min-relation or \( \sim_0 = R \cap \min_{k-1}^{l} \), being \( R \) a min-relation. Let

\[
\sim'_0 = \begin{cases} 
\sim_0 & \text{if } \sim_0 \text{ is a min-relation} \\
R \cap \theta_{k,l}^0 & \text{if } I_1 \neq \emptyset.
\end{cases}
\]

In this second case we have that \( \min_k \subseteq \sim'_0 \). For \( k = 1 \), let \( \sim'_0 = \sim'_1 = \text{FIN}_2^1 \).

So, although \( \sim_0 \) is not a min-relation and \( \sim_1 \) is not a max-relation, their corresponding interpretations \( \sim'_0 \) and \( \sim'_1 \) as \( k \)-relations are a min-relation and a max-relation, respectively.

The relations \( \sim'_0 \) and \( \sim'_1 \) have similar properties than \( \sim_0 \) and \( \sim_1 \).

**Proposition 4.7.** Let \( s \) and \( t \) be \((k-1)\)-vectors. Then

(i) \( s + x \sim'_0 t + x \) holds iff \( s \sim_0 t \) iff \( s + x \sim t + x \) holds.
(ii) \( x + s \sim'_1 x + t \) holds iff \( s \sim_1 t \) iff \( x + s \sim x + t \) holds.
(iii) \( s + x_0 + x_1 \sim'_0 t + x_0 + x_2 \) holds iff \( s \sim_0 t \).
(iv) \( x_0 + x_2 + s \sim'_1 x_1 + x_2 + t \) holds iff \( s \sim_1 t \).

**Proof.** We show the result for \( \sim_1 \); for \( \sim_0 \) the proof is similar, and we leave the details to the reader. If \( \sim_1 \) is a max relation, then there is nothing to prove. Suppose that \( \sim_1 = \max_{k-1}^{l} \cap R \), for some \( l \leq k-1 \), where \( R \) is a max relation. So, \( \sim'_1 = \theta_{k,l}^{(1)} \cap R \) and we only have to show that

\[
s \max_{k-1}^{l} t \text{ iff the equation } x + s \theta_{k,l}^{(1)} x + t \text{ holds,} \tag{17}
\]

which is not difficult to check (see Figure below).

![Figure 3. The relation between \( \sim_1 \) and \( \sim'_1 \)]
Definition 4.8. Let $D = (d_n)_n$ be a $k$-block sequence and a $k$-vector $s = \sum_{n \geq 0} T^{k-C_D(s)(n)}d_n$ of $D$, and let $n_0 = n_0(s)$ and $n_1 = n_1(s)$ be respectively the minimal and the maximal elements of the set of integers $n$ such that $C_D(s)(n) = k$. We define the first part of $s$ in $D$ as the $(\leq k-1)$-vector $f_DS = \sum_{n < n_0} T^{k-C_D(s)(n)}d_n$, the middle part of $s$ in $D$ as the $(\leq k)$-vector $m_DS = \sum_{n \in (n_0, n_1)} T^{k-C_D(s)(n)}d_n$ and the last part of $s$ in $D$, as the $(\leq k-1)$-vector $l_DS = \sum_{n > n_1} T^{k-C_D(s)(n)}d_n$. Using this, we have the decomposition

$$s = f_DS + b_n + m_DS + b_n + l_DS.$$ 

So $f_DS$ is the part of $s$ before the occurrence of $\text{min}_k s$, $m_DS$ is the part of $s$ between $\text{min}_k s$ and $\text{max}_k s$, and $l_DS$ is the part of $s$ after $\text{max}_k s$. All these definitions are local, depending on a fixed sos $D$.

Let $A = (a_n)_n$ satisfy both Lemmas 4.2 and 4.3, and let $B = (b_n)_n$ be defined for every $n$ by $b_n = Ta_{3n} + a_{3n+1} + Ta_{3n+2}$. The role of $B$ is to guarantee that for every $k$-vector $s$ of $B$ the first part $f_BS$ and the last part $l_BS$ are both $(k-1)$-vectors. We need this because $\sim_\epsilon (\epsilon = 0, 1)$ gives information only about $(k-1)$-vectors, since it is a $k-1$-relation.

From now on we work in $B$, unless we explicitly say the contrary. The following proposition tells us that many equations are decided in $B$.

Proposition 4.9. Let $p(x_1, \ldots, x_{n-1})$ and $q(x_1, \ldots, x_{n-1})$ be $(\leq k-1)$-terms. Then:

(i) The equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ is decided in $B$.

(ii) The equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ holds in $B$ iff the equation $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ holds in $B$.

The analogous results for $\sim_0'$ are also true.

Proof. Fix two $(\leq k-1)$-terms $p = p(x_1, \ldots, x_{n-1})$, $q = q(x_1, \ldots, x_{n-1})$.

(i) Fix a finite block sequence $(c_0, \ldots, c_{n-1})$ in $B$. Suppose that $c_0 + p(c_1, \ldots, c_{n-1}) \sim_1 c_0 + q(c_1, \ldots, c_{n-1})$. By definition of $B$, $c_0 = c'_0 + c''_0$, where $c'_0$ is a $k$-vector of $A$ and $c''_0$ is a $(k-1)$-vector of $A$. Hence,

$$c''_0 + p(c_1, \ldots, c_{n-1}) \sim_1 c''_0 + q(c_1, \ldots, c_{n-1}).$$

Since the relation $\sim_1$ is $(k-1)$-canonical in $A$, the $(k-1)$-equation

$$x_0 + p(x_1, \ldots, x_{n-1}) \sim_1 x_0 + q(x_1, \ldots, x_{n-1})$$

is true in $A$.

Fix $(d_0, \ldots, d_{n-1})$ in $B$, and set $d_0 = d''_0 + d''_0$. Then,

$$d''_0 + p(d_1, \ldots, d_{n-1}) \sim_1 d''_0 + q(d_1, \ldots, d_{n-1}),$$

and hence, the equation

$$x_0 + p(d_1, \ldots, d_{n-1}) \sim_1 x_0 + q(d_1, \ldots, d_{n-1})$$

holds in $A$, which implies that $d_0 + p(d_1, \ldots, d_{n-1}) \sim_1 d_0 + q(d_1, \ldots, d_{n-1})$, as desired.

(ii) Suppose that $x_0 + p(x_1, \ldots, x_{n-1}) \sim x_0 + q(x_1, \ldots, x_{n-1})$ holds in $B$. Then for a given block sequence $(c_0, c_1, \ldots, c_{n-1})$ in $B$, the equation

$$x_0 + Tc_0 + p(c_1, \ldots, c_{n-1}) \sim x_0 + Tc_0 + q(c_1, \ldots, c_{n-1})$$

holds in $B$. 

By Proposition 4.7, the assertion (22) implies that
\[ x_0 + Tc_0 + p(c_1, \ldots, c_{n-1}) \sim'_1 x_0 + Tc_0 + q(c_1, \ldots, c_{n-1}) \] holds in \( \mathbb{B} \).
(23)
\[ \sim \]
Since \( \sim'_1 \) is canonical, the equation
\[ x_0 + Tx_1 + p(x_2, \ldots, x_n) \sim'_1 x_0 + Tx_1 + q(x_2, \ldots, x_n) \] holds in \( \mathbb{B} \).
(24)
Therefore the equation \( x_0 + p(x_1, \ldots, x_{n-1}) \sim'_1 x_0 + q(x_1, \ldots, x_{n-1}) \) holds in \( \mathbb{B} \), as desired. \( \square \)

**Proposition 4.10.** Suppose that \( a, b \) are \( k \)-vectors of \( \mathbb{B} \), \( s, t \) are \( \leq (k-1) \)-vectors of \( \mathbb{B} \) such that \( a < s, b < t \) and suppose that \( a + s \sim'_1 b + t \).

(i) If \( a, b < s, t \), then \( a + s \sim a + t \).

(ii) If \( l_A a = l_B b = 0 \), and \( \max_k(a) > \max_k(b) \), then \( b + s \sim b + t \).

The corresponding analogous results for \( \sim'_0 \) are also true.

**Proof.** Let us check (i): By point (iv) of Proposition 3.17, we have that \( a + s \sim'_1 a + t \). By construction of \( \mathbb{B} \), \( a = a' + a'' \) where \( a' \) is a \( k \)-vector and \( a'' \) is a \( (k-1) \)-vector, both of \( \mathbb{A} \). But since the relation is \( \sim'_1 \) is staircase, it is canonical, and hence the \( k \)-equation
\[ x_0 + a'' + s \sim'_1 x_0 + a'' + t \] holds in \( \mathbb{A} \).
(25)
It follows from Proposition 4.9 that \( a'' + s \sim a'' + t \), and hence, by definition of \( \sim_1 \), the \( k \)-equation
\[ x_0 + a'' + s \sim x_0 + a'' + t \] holds in \( \mathbb{A} \).
(26)
Replacing in (24) \( x_0 \) by \( a' \), we obtain that \( a + s \sim a + t \).

(ii): Since \( l_A a = l_B b = 0 \), we have that \( a + s = a' + a_{no} + s \) and \( b + t = b' + a_{mo} + t \). Since \( \max_k(a) > \max_k(b) \), it follows that \( n_0 > m_0 \). This together with the fact that \( a + s \sim'_1 b + t \) implies that \( \max_k \not\subset \sim'_1 \) and hence, by definition, \( \sim'_1 \) has to be max-relation. Set \( i = \max I_1(\sim'_1) < k \). Then the equation
\[ p(x_0, \ldots, x_r) + T^{k-i'} x_{r+1} \sim'_1 q(x_0, \ldots, x_r) + T^{k-i'} x_{r+1} \] is true,
(27)
for every terms \( p \) and \( q \), and every \( i' \geq i \). Now set
\[ t = t' + T^{k-j} a_{no} + t'' \]
Notice that \( t'' \) is an \( i \)-vector, and \( s \) is an \( i' \)-vector for some \( i' \geq i \). By (27),
\[ a + s = a' + a_{mo} + s \sim'_1 b' + a_{no} + t' + T^{k-j} a_{mo} + s \sim'_1 b' + a_{no} + s = b + s \]
(28)
Hence, \( b + t \sim'_1 b + s \), and since \( b < s, t \), 1. implies that \( b + s \sim b + t \). \( \square \)

Our intention is to show that \( \sim \subset \sim'_1 \). To do this, we decompose the relation \( \sim'_1 \) as the final step of a chain \( \sim'_1(1) \subset \cdots \subset \sim'_1(k) = \sim'_1 \) and we prove by induction on \( j \) that \( \sim \subset \sim'_1(1) \).

**Definition 4.11.** Suppose that \( R \) is a max-relation with values \( I_0 = \emptyset, I_1, J_1 \) and \((I_j^{(1)})_{j \in J_1}\). For every \( i \leq k - 1 \) we define \( I_1(i) = I_1 \cap [0, i], J_1(i) = J_1 \cap [0, i] \), and let \( R(i) \) be the staircase
Proposition 4.13. Suppose that \( s \) is nothing to prove. Suppose that \( k > 0 \) and \( l > 0 \). Observe that each \( R(i) \) is also a staircase equivalence relation on every sos of \( \text{FIN}_i \).

Roughly speaking, \( R(i) \) is the staircase equivalence relation whose values are the ones from \( R \) which are smaller than \( i \).

Remark 4.12. One has that for a given \( i \leq k - 1 \), \( sR(i)t \) iff the equation with variable \( x \)

\[
x + s\max_i(s), \max_1(s)]R(i)x + t[\max_i(s), \max_1(s)] \quad \text{holds.} \tag{29}
\]

Proposition 4.13. Suppose that \( R \) is a max-relation of \( \text{FIN}_k \). Fix \( j' < j < j'' \), and suppose that \( s \) is a \( j' \)-vector, \( t \) is a \((<j)\)-vector, and \( a \) is a \( j'' \)-vector such that \( a + sR(j)T^ia + t \) for some \( l > 0 \). Then, \( R(j) = R(j') \), and hence \( s''R(j)\) holds.

Proof. Set \( s' = a + s \) and \( t' = T^ia + t \), and suppose that \( s'R(j)t' \). We are going to show that \( I_2(j) = I_2(j') \), which will imply that \( R(j) = R(j') \), as desired. We know that \( s'[\max_j(s'), \max_1(s')]R(j)t'[\max_j(s'), \max_1(s')] \). Notice that for every \( r \in [j, j') \), \( \max_r(s') = \max_r(a) \), hence \( \max_r(s') = \max_r(t') \), since \( a \) and \( T^ia \) have nothing in common except 0's. This implies that \( I_2(j) \subseteq [j', 1] \) and hence \( I_2(j) = I_2(j') \).

Lemma 4.14. \( \sim \leq \sim_1(j) \), for every \( j \leq k \). In particular, \( \sim \leq \sim_1 \).

Proof. The proof is by induction on \( j \). Notice that if \( k = 1 \), then \( \sim_1 = \text{FIN}_1^2 \) and hence there is nothing to prove. Suppose that \( k > 1 \). Let \( I_1, J \) and \((l^{(1)})_{j \in J} \) be the values of \( \sim_1 \).

\( j = 1 \): Suppose that \( I \in I_1 \) (otherwise there is nothing to prove), i.e., \( \sim_1(1) = \sim_{max_1} \). Suppose that \( s \sim t \) but \( \max_1(s) < \max_1(t) \), and let \( n \) and \( i \) be the unique integers such that

\[
\max_1 T^{k-i}a_n = \max_1 t \text{ and } t = t' + T^{k-i}a_n. \tag{30}
\]

So, \( s = s' + T^{k-i}a_n \), for some \( i' < i \) and some \( k \)-vector \( s' \). The fact that \( s \sim t \) implies that the equation \( s' + T^{k-i}x_0 \sim t' + T^{k-i}x_0 \) holds in \( \mathbb{B} \), which implies that the equation \( s' + T^{k-i}(x_0 + T^{i'}x_1) \sim t' + T^{k-i}(x_0 + T^{i'}x_1) \) holds. Therefore

\[
s' + T^{k-i}x_0 + \sim t' + T^{k-i}x_0 + T^{k-i+i'}x_1 \text{ holds,} \tag{31}
\]

implies that the equation

\[
t' + T^{k-i}x_0 \sim t' + T^{k-i}x_0 + T^{k-i+i'}x_1 \text{ is true,} \tag{32}
\]

and hence, also

\[
x_0 + T^{k-i+i'}x_1 \sim x_0 \text{ is true in } \mathbb{B}. \tag{33}
\]

But since \( j - i + i' < k \), we have that

\[
x_0 + Tx_1 \sim x_0 + Tx_1 + T^{k-i+i'}x_2 \text{ is true,} \tag{34}
\]
and by Proposition 1.9, we have that
\[ x_0 + T x_1 \sim_1 x_0 + T x_1 + T^{k-\ell_1} x_2 \] holds, (35)
which contradicts the fact that \( 1 \in I_1 \).

Notice that \( j \sim j + 1 \). Assume that \( \sim \subseteq \sim_1 (j) \) and let us conclude that \( \sim \subseteq \sim_1 (j + 1) \). There are two cases:

(a) \( j \notin I_1 \): Suppose that \( j + 1 \in I_1 \) (otherwise, there is nothing to prove), and set
\[ \beta = \max I_1 \cap [0, j]. \] (36)

Since \( \beta \) can be 0. By definition of \( \sim_1 \), we know that if \( j + 1 = k \) belongs to \( I_1 \), then \( j = k - 1 \) also belongs to \( I_1 \). So, \( j + 1 < k \). We only need to show that \( \sim \subseteq \max j + 1 \): Suppose that \( s \sim t \), and \( \max j + 1 s < \max j + 1 t \); set \( s = s' + T^{k-l} a_n + s'' \), \( t = t' + T^{k-l} a_n + t'' \), with \( l < l' \), \( l' \geq j + 1 \), and \( (< j + 1)\)-vectors \( s'' \) and \( t'' \). Observe that in the previous decomposition of \( s \), \( s' \) needs to be a \( k \)-vector. By the inductive hypothesis,
\[ s' + T^{k-l} a_n + s'' \sim_1 (j) t' + T^{k-l_1} a_n + t''. \] (37)
Since \( \sim_1 \) is a staircase equivalence relation, (iv) of Proposition 3.15 gives that
\[ s' + T^{k-l} a_n + s'' \sim (j) s' + T^{k-l} a_n + t'' \] (\( t'' \) can be 0), (38)
which implies that \( s' + T^{k-l} a_n + s'' \sim_1 s' + T^{k-l} a_n + t'' \), and hence, by Proposition 4.10, \( s' + T^{k-l} a_n + s'' \sim s' + T^{k-l} a_n + t'' \). Resuming, we have that
\[ s' + T^{k-l} a_n + t'' \sim t' + T^{k-l} a_n + t'', \] (39)
and hence, the equation
\[ s' + T^{k-l} x_0 + T^{k-a} x_1 \sim t' + T^{k-l} x_0 + T^{k-a} x_1 \] holds, (40)
where \( j \geq \alpha \geq \beta \) is such that \( t'' \in \text{FIN}_\alpha \). Notice that since \( j \notin I_1 \), and \( j \geq \alpha \geq \beta = \max I_1 \cap [0, \ldots, j] \), the equation
\[ x_0 + T^{k-\alpha} x_1 + T^{k-\alpha} x_2 \sim_1 x_0 + T^{k-\alpha} x_2 \] is true, (41)
for all \( r \leq j \). Hence,
\[ x_0 + T^{k-\alpha} x_1 + T^{k-\alpha} x_2 \sim_1 x_0 + T^{k-\alpha} x_2 \] is true. (42)

There are two new two subcases to consider:

(a.1) \( l \leq j \). Then
\[ s' + T^{k-\alpha} x_2 \sim s' + T^{k-l} x_1 + T^{k-\alpha} x_2 \sim t' + T^{k-l} x_1 + T^{k-\alpha} x_2 \] is true, (43)
and hence,
\[ x_0 + T^{k-l} x_1 + T^{k-l} x_2 + T^{k-\alpha} x_3 \sim x_0 + T^{k-l} x_1 + T^{k-\alpha} x_3 \] is true, (44)
which implies that
\[ x_0 + T^{k-l} x_1 + T^{k-l} x_2 + T^{k-\alpha} x_3 \sim x_0 + T^{k-l} x_1 + T^{k-\alpha} x_3 \] is true, (45)

By Proposition 4.9,
\[ x_0 + T^{k-l} x_1 + T^{k-l} x_2 \sim_1 x_0 + T^{k-\alpha} x_2 \] holds. (46)
which contradicts the fact that \( j + 1 \in I_1 \).

(a.2) \( j + 1 \leq l < l' \). Then, the equation

\[ s' + T^{k-l}(x_0 + T^{l-j}x_1) + T^{k-\alpha}x_2 \sim s' + T^{k-l}x_0 + T^{k-\alpha}x_2 \text{ holds,} \]

and hence,

\[ t' + T^{k-l'}x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim t' + T^{k-l'}x_0 + T^{k-\alpha}x_2 \text{ holds,} \]

which implies that

\[ x_0 + T^{k-(j+l'-l)}x_1 + T^{k-\alpha}x_2 \sim x_0 + T^{k-\alpha}x_2 \text{ holds.} \]

Since \( l' - i > 0 \) the assertion \([\square]\) contradicts the fact that \( j + 1 \in I_1 \).

(b) \( j \in I_1 \). We assume that \( j + 1 \in I_1 \) because otherwise there is nothing to prove. Then

\[ \sim_1(j + 1) = \sim_1(j) \cap \delta_{j+1,l} \cap \max, \]

where \( l = j_{j+1}^{(1)} \). Suppose that \( s \sim t \). By the inductive hypothesis, \( s \sim_1(j)t \), and in particular \( \max_j(s) = \max_j(t) \). Let \( m_0 = \max\{\max_{j+1}s, \max_{j+1}t\} \). First we show that

\[ (s'|m_0, \max_j(s))]^{-1}(l) = (l|m_0, \max_j(s))]^{-1}(l), \]

i.e., for all \( n \in [m_0, \max_j(s)] \), \( s(n) = l \) iff \( t(n) = l \). Suppose not, and let

\[ m_1 = \max\{m \in [m_0, \max_j(s)] : (s(m) = l \text{ or } t(m) = l) \text{ and } s(m) \neq t(m)\}. \]

Suppose that \( s(m_1) = l \), and that \( t(m_1) \neq 0 \). Let \( n_1 \) be the unique integer \( n \) such that \( T^{k-C_2(n)}a_n(m_1) = s(m_1) = l \), and let \( h = C_2(n_1) \geq l \). So, \( h' = C_2(n_1) \neq h \), \( h = s' + T^{-h}a_n + s'' \), and \( t = t' + T^{-h'}a_n + t'' \), with \( s'', t'' \) both \( j \)-vectors. By definition of \( m_1 \), the equation

\[ x + s'' \sim_1(j + 1)x + t'' \text{ holds,} \]

and hence,

\[ x + s'' \sim_1 x + t'' \text{ and } x + s'' \sim x + t'' \text{ also both hold.} \]

So, \( s' + T^{-h}a_n + s'' \sim t'' + T^{-h'}a_n + s'' \), and hence, the equation

\[ s' + T^{-h}x_0 + T^{-j}x_1 \sim t' + T^{-h'}x_0 + T^{-j}x_1 \text{ holds.} \]

There are two subcases to consider:

(b.1) \( h > h' \). Since \( x_0 + T^{-r}x_1 + T^{-j}x_2 \sim_1 x_0 + T^{-j}x_2 \) is true, the equation \( x_0 + T^{-r}x_1 + T^{-j}x_2 \sim x_0 + T^{-j}x_2 \) holds for every \( r < l \). Since \( l + h' - h < l \),

\[ s' + T^{-h}x_0 + T^{-l}x_1 + T^{-j}x_2 \sim s' + T^{-h}(x_0 + T^{h-l}x_1) + T^{-j}x_2 \sim \]

\[ t' + T^{-h'}(x_0 + T^{h-l}x_1) + T^{-j}x_2 \sim t' + T^{-h'}x_0 + T^{h-l}x_1 + T^{-j}x_2 \sim \]

\[ \sim t' + T^{-h'}x_0 + T^{-j}x_2 \sim s' + T^{-h}x_0 + T^{-j}x_2 \text{ hold.} \]

Notice that we have used that \( h \geq l \), and so \( T^{-l} \) makes sense. Summarizing, the equation

\[ s' + T^{-h}x_0 + T^{-l}x_1 + T^{-j}x_2 \sim s' + T^{-h}x_0 + T^{-j}x_2 \]

and hence, the equation

\[ x_0 + T^{-l}x_1 + T^{-j}x_2 \sim_1 x_0 + T^{-j}x_2 \text{ holds,} \]
which is a contradiction with the fact that $\sim'_1 \subseteq \theta^1_{j+1,l}$.

(b.2) $h < h'$. Then $h' > l$, and repeating the previous argument used for the case $h > h'$, we conclude that the equation

$$t' + T^{k-h'}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-j}x_2$$

holds, (59)

and hence,

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim x_0 + T^{k-j}x_2$$

which is a contradiction.

The proof will be finished once we show that $\max_j s = \max_j t$. So suppose otherwise, without loss of generality, that $\max_j s > \max_j t$. Let $n_1 \in \mathbb{N}$ be such that $\max_j(s) = \max_j(T^{k-h}b_n)$, where $h = C_2(n_1) \geq j + 1$. Then one has the decomposition $s = s' + T^{k-h}a_n + s''$, $t = t' + T^{k-h'}a_n + t''$, where $h' < h$ and $s'', t''$ are $j$-vectors. From (61), it follows that

$$x_0 + s'' \sim (j+1)x_0 + t''$$

holds, (62)

This implies that the equation

$$s' + T^{k-h}x_0 + T^{k-j}x_1 \sim t' + T^{k-h'}x_0 + T^{k-j}x_1$$

is true. (63)

Using a similar argument to the above, we arrive at the equation

$$s' + T^{k-h}(x_0 + T^{l}x_1) + T^{k-j}x_2 \sim t' + T^{k-h'}(x_0 + T^{l}x_1) + T^{k-j}x_2$$

is true, (64)

and hence,

$$s' + T^{k-h}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim t' + T^{k-h'}x_0 + T^{k-j}x_2 \sim s' + T^{k-h}x_0 + T^{k-j}x_2$$

which is again a contradiction, since it implies that

$$x_0 + T^{k-l}x_1 + T^{k-j}x_2 \sim x_0 + T^{k-j}x_2$$

holds. (67)

\[ \square \]

**Proposition 4.15.** Suppose that $a, b$ are $k$-vectors of $\mathbb{B}$, $s, t$ are $(\leq (k-1))$-vectors of $\mathbb{B}$ such that $a < s$ and $b < t$, and suppose that $a + s \sim b + t$.

(i) If $a < t$ and $b < s$, then $a + s \sim a + t$ and hence $a + t \sim b + t$.

(ii) If $a < t$ and $\max_k a < \max_k b$, then $a + s \sim a + t$ and hence $a + t \sim b + t$.

**Proof.** (i) is a consequence of Proposition 4.4(1) and Lemma 4.14. Let us prove (ii). To do this, suppose that $a, b, s, t$ are as in the statement. By Lemma 4.14 one has that $a + s \sim_1 b + t$. Since $\max_k(a + s) < \max_k(b + t)$, we have that $\sim'_1 = \sim_1$, where $\sim_1$ is a max-relation of $\text{FIN}_k$. This implies that $s \sim_1 t$, from which the desired result easily follows. \[ \square \]
4.2. Determining the relation $\sim$. We already know that $\sim \subseteq \sim'$. The following identifies the staircase equivalence relation that will be equal to $\sim$ on $\mathbb{B}$ in terms of which equations hold or not in $\mathbb{B}$. This will conclude the proof of Theorem [1.1).

Theorem 4.16.

(i) Suppose that $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ is false, and $x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2$ is false. Then $\sim = \sim_0 \cap \sim_1 \cap \sim'_1$. 

(ii) Suppose that $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true.
   (a) If $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ are both false, then $\sim = \sim_0 \cap \min_k \cap \max_k \cap \sim_1'$.
   (b) If $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is true, and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is false, then $\sim = \sim_0 \cap \max_k \cap \sim_1'$.
   (c) If $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is false, and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is true, then $\sim = \sim_0 \cap \min_k \cap \sim_1'$.
   (d) If $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ are both true, then $\sim = \sim_0 \cap \sim_1'$.

The proof is done in various steps. (i) is in Corollary [4.20] and (ii.a), (ii.b), (ii.c) and (ii.d) in Corollary [4.23], and Lemmas [4.21], [4.23] and [4.26] respectively. We start with the following proposition that gives one of the inclusions.

Proposition 4.17.

(i) If the equation $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is true, then $\sim_0 \cap \max_k \cap \sim_1' \subseteq \sim$.

(ii) If the equation $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is true, then $\sim_0 \cap \min_k \cap \sim_1' \subseteq \sim$.

(iii) If the equation $x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2$ is true, then $\sim_0 \cap \min_k \cap \sim_1' \subseteq \sim$, for every $l \leq k$.

(iv) If the equation $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true, then $\sim_0 \cap \min_k \cap \max_k \cap \sim_1' \subseteq \sim$.

(v) If the equations $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ are both true, then $\sim_0 \cap \sim_1' \subseteq \sim$.

Proof. (i): Suppose that the equation $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ holds. Then, $Tx_0 + Tx_1 + x_2 + x_3 \sim Tx_0 + x_3$ holds, and the equations $Tx_0 + Tx_1 + x_2 + x_3 \sim Tx_0 + x_3 \sim Tx_0 + x_2 + x_3$ also hold. This implies that the equation

$$Tx_0 + T x_1 + x_2 \sim Tx_0 + x_2$$

Hence the relation $\sim_0$ is a min-relation, which implies that $\sim'_0$ so is a min-relation. Set $R = \sim'_0 \cap \max_k \cap \sim'_1$ and suppose that $sRt$. Then $\max_k s = \max_k t$. Let $n$ be such that $\max_k s = \max_k b_n$. Therefore, $s = s' + a_{3n+1} + s''$ and $t = t' + a_{3n+1} + t''$. It is not difficult to show that the equation

$$Tx_0 + x_1 + x_2 \sim Rx_0 + x_2$$

So, we may assume that $s'$ and $t'$ are $(k-1)$-vectors of $\mathbb{A}$. Since $s \sim'_0 t$, we have that $s' + a_{3n+1} + s'' \sim s' + a_{3n+1} + t''$. Since $s \sim'_0 t$, we have that $s' + a_{3n+1} + t'' \sim'_0 t' + a_{3n+1} + t''$, and hence $s' \sim'_0 t'$, which implies that $s' + x \sim t' + x$ is true. In particular $s' + a_{3n+1} + t'' \sim t' + a_{3n+1} + t''$, i.e., $s \sim t$. 


The proofs of (ii), (iii) and (iv) are similar. We leave the details to the reader. Let us check point (v): Fix $s = f_ks + a_n + m_is + a_n^1 + l_is$, $t = f_k t + a_m + m_i t + a_m^1 + l_it$ such that $s R t$, where $R = \sim^0_0 \cap \sim^1_1$. If $m_0 = n_0$, then $s R \cap \min t$, and hence we are done by 2. So, suppose that $n_0 < m_0$. Since $\sim^0_0$ is a min-relation and $\sim^1_1$ is a max-relation, the equations $Tx_0 + x_1 + x_2 R Tx_0 + x_2$ and $x_0 + x_1 + Tx_2 R x_0 + Tx_2$ are true. Therefore, $s R f_ks + a_n + l_is$ and $t R f_ks + a_m + l_it$. Since $s \sim f_ks + a_n + l_is$ and $t \sim f_k t + a_m + l_it$ the proof will be finished if we show that

$$f_ks + a_n + l_is \sim f_k t + a_m + l_it. \tag{70}$$

Since $f_ks + a_n + l_is \sim f_k t + a_m + l_it$ and $f_ks + a_n + l_is \sim f_k t + a_m + l_it$, by the last point of Proposition 4.10 (for both $\sim^0_0$ and $\sim^1_1$), we have that

$$f_ks + a_n + l_is \sim f_k t + a_m + l_it \text{ and } f_ks + a_n + l_is \sim f_ks + a_n + l_is. \tag{71}$$

But $f_ks + a_n + l_is \sim f_ks + a_n + l_is$, and we are done.

**Lemma 4.18.** If the equation $x_0 + x_1 + Tx_2 \sim x_0 + Tx_2$ is false, then $\sim \subseteq \max_k$.

**Proof.** Suppose that $s \sim t$ but $\max_k s > \max_k t$. Set

$$s = f_ks + a_n + m_is + a_n^1 + l_is$$

$$t = f_k t + a_m + m_i t + a_m^1 + l_it,$$

where $n_1 > m_1$. Let $l_is = t' + T^{k_i}a_n^1 + t''$, where $t' < T^{k_i}a_n^1 < t''$, and $i < k$. By Proposition 4.13

$$f_k t + a_m + m_it + a_m^1 + t' + T^{k_i}a_n^1 + l_is \sim f_ks + a_n + m_is + a_n^1 + l_is, \tag{72}$$

and therefore, the equation

$$f_k t + a_m + m_it + a_m^1 + t' + T^{k_i}a_n^1 + l_is \sim f_ks + a_n + m_is + x_0 + Tx_1 \text{ holds.} \tag{73}$$

Since $\sim \subseteq \sim^1_1$ and $\sim^1_1$ is a canonical relation, the $\sim^1_1$-equation

$$f_k t + a_m + m_it + a_m^1 + t' + T^{k_i}a_n^1 + l_is \sim f_ks + a_n + m_is + x_0 + Tx_1 \text{ holds.} \tag{74}$$

Since $\sim^1_1$ is a staircase relation, the truth of the last equation implies that $k \notin I_1(\sim^1_1)$, and hence $\sim^1_1$ is a max-relation with $\max(I_1(\sim^1_1))$ at most $k - 1$. Therefore,

$$f_k t + a_m + m_it + a_m^1 + t' + T^{k_i}a_n^1 + l_is \sim f_ks + a_n + m_is + x_0 + Tx_1 \text{ is true,} \tag{75}$$

which implies that

$$f_k t + a_m + m_it + a_m^1 + t' + T^{k_i}a_n^1 + l_is \sim f_ks + a_m + m_is \sim f_k t + a_m + m_it + a_m^1 + t' + Tx_1 \text{ is true.} \tag{76}$$

Hence, the equation

$$f_k t + a_m + m_it + a_m^1 + t' + Tx_1 \sim f_ks + a_n + m_is + x_0 + Tx_1 \text{ holds,} \tag{77}$$

from which we conclude that

$$x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \text{ is true,} \tag{78}$$

a contradiction.
**Lemma 4.19.** Suppose that \( x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2 \) is true but \( x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2 \) is false. Then \( \sim \subseteq \sim_{a_1^2} \). In particular, \( \sim \subseteq \min_k \cap \max_k \).

**Proof.** Fix \( l \) as in the statement. Since we assume that the equation

\[
x_0 + T^{k-l}x_1 + x_2 \sim x_0 + x_2
\]

is false, then by Proposition 3.4(1,2), we know that

\[
x_0 + x_1 + Tx_2 \sim x_0 + Tx_2
\]

is false. Then by Lemma 4.18, we obtain that \( \sim \subseteq \max_k \). Suppose that \( s \sim t \). Take the decomposition

\[
s = f_\beta s + b_0 + m_\beta s + b_m + l_\beta s
\]

\[
t = f_\beta t + b_{1n} + m_\beta t + b_m + l_\beta s
\]

so, by Lemma 4.19. Let \( \sim \subseteq \max_k \). Suppose that \( s \sim t \). Take the decomposition

\[
s = f_\beta s + b_0 + m_\beta s + b_m + l_\beta s
\]

\[
t = f_\beta t + b_{1n} + m_\beta t + b_m + l_\beta s
\]

where we implicitly assume that \( l_\beta s \sim l_\beta t \), since \( s \sim t \). Observe that showing that \( s\theta_2^\beta t \) is the same that proving that

\[
\text{for all } n \in [\min\{n_0, n_1\}, m], \text{ either } C_\beta(s)(n), C_\beta(t)(n) < l, \text{ or } C_\beta(s)(n) = C_\beta(t)(n).
\]

Assume on the contrary that (81) is false, and let \( \alpha \) be the last \( n \in [\min\{n_0, n_1\}, m] \) for which

\[
\max\{C_\beta(s)(n), C_\beta(t)(n)\} \geq l \text{ and } C_\beta(s)(n) \neq C_\beta(t)(n).
\]

Set \( l_0 = C_\beta(s)(\alpha) \), and \( l_1 = C_\beta(s)(\alpha) \). Notice that \( \alpha < m \). Without loss of generality, we assume that \( l_1 < l_0 \) (the other case has a similar proof). Set

\[
s' = \sum_{n < \alpha} T^{k-C_\beta(s)(n)}b_n
\]

\[
t' = \sum_{n < \alpha} T^{k-C_\beta(t)(n)}b_n.
\]

Using this notation, we have that the equation

\[
s' + T^{k-l_0}x_0 + x_1 \sim t' + T^{k-l_1}x_0 + x_1 \text{ holds.}
\]

There are two cases:

- \( n_0 \leq n_1 \). We first show that in this case \( s' + T^{k-l_0}x_0 \) is a \( k \)-term. If \( n_0 = n_1 \), then \( \alpha > n_0 \), and hence \( s' \) is a \( k \)-vector. Suppose that \( n_0 < n_1 \). If \( \alpha > n_0 \), then \( s' \) is a \( k \)-term. If \( \alpha = n_0 \), then \( l_0 = k \), and clearly \( s' + T^{k-k}x_0 = s' + x_0 \) is a \( k \)-term. We consider two subcases:
  - (a) \( l_1 < l \leq l_0 \). Then, by our assumption that \( x_0 + T^{k-(l-1)}x_1 + x_2 \sim x_0 + x_2 \) holds, we have that
    \[
s' + T^{k-l_0}x_0 + T^{k-l_1}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds.}
    \]
    By (84),
    \[
s' + T^{k-l_0}x_0 + T^{k-l_1}x_1 + x_2 \sim t' + T^{k-l_1}x_0 + T^{k-l_1}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds,}
    \]
    which implies that the equation
    \[
s' + T^{k-l_0}x_0 + T^{k-l_0}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds.}
    \]
  This contradicts the fact that \( l_0 \geq l \).
(b) \( l \leq l_1 < l_0 \). Then,
\[
s' + T^{k-l_0}x_0 + T^{k-l_1}(T^{l_0-1})x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds},
\]
and by (83),
\[
s' + T^{k-l_0}x_0 + T^{k-l_1}(T^{l_0-1})x_1 + x_2 \sim t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-1})x_1 + x_2 \sim
\]
\[
\sim s' + T^{k-l_0}x_0 + T^{k-l_1}x_1 + x_2 \text{ holds}.
\]
Again, this yields a contradiction.

\( n_1 < n_0 \). It can be shown that \( t' + T^{k-l_1}x_0 \) is a \( k \)-term. We consider the same two subcases as above:

(a) \( l_1 < l \leq l_0 \). Then
\[
t' + T^{k-l_1}x_0 + T^{k-l_1}x_1 + x_2 \sim s' + T^{k-l_1}x_0 + x_2 \text{ holds},
\]
and hence,
\[
s' + T^{k-l_0}x_0 + T^{k-l_0}x_1 + x_2 \sim s' + T^{k-l_0}x_0 + x_2 \text{ holds},
\]
which, by (83), implies that
\[
t' + T^{k-l_1}x_0 + T^{k-l_0}x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds},
\]
a contradiction, since \( l_0 \geq l \).

(b) \( l \leq l_1 < l_0 \). Then
\[
t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-1})x_1 + x_2 \sim t' + T^{k-l_1}x_0 + x_2 \text{ holds}.
\]
Using that
\[
t' + T^{k-l_1}x_0 + T^{k-l_1}(T^{l_0-1})x_1 + x_2 \sim s' + T^{k-l_0}x_0 + T^{k-l_1}x_1 + x_2 \sim
\]
\[
\sim t' + T^{k-l_1}x_0 + T^{k-l_1}x_1 + x_2 \text{ holds},
\]
we arrive at a contradiction. \( \square \)

**Corollary 4.20.** Suppose that \( x_0 + T^{k-l_1}x_1 + x_2 \sim x_0 + x_2 \) is a \( k \)-term. Then, \( \sim = x_0' \cap \sim_{\delta_1} \cap \sim_1 \).

**PROOF.** By Proposition 4.17, \( \sim_0 \cap \sim_{\delta_1} \cap \sim_1 \subseteq \sim \). We only need to show that \( \sim \subseteq \sim_0 \). Suppose that \( s \sim t \), and consider the decomposition
\[
s = f_\delta s + a_{n_0} + m_\delta s + a_{m_0} + l_\delta s
\]
\[
t = f_\delta t + a_{n_1} + m_\delta t + a_{m_1} + l_\delta t.
\]
Since \( \max_\delta s = \max_\delta t \), we have that \( m_0 = m_1 \), and since \( s \sim_1 t \), by Proposition 4.17(4), we may assume that \( l_\delta s \sim_1 l_\delta t \). By Lemma 4.19, \( s \sim_{\delta_2} t \), and using the fact that the equations \( x_0 + T^{k-j}x_1 + x_2 \sim x_0 + x_2 \) are true for all \( j < l \), we may also assume that \( n_0 = n_1 \) and \( m_\delta s = m_\delta t \). Therefore, the equation \( f_\delta s + x_0 \sim f_\delta t + x_0 \) holds. By definition of \( \sim_0 \), we have that \( f_\delta s \sim_0 f_\delta t \), and by Proposition 4.17(3), \( s \sim_0 t \), as desired. \( \square \)

**Lemma 4.21.** Suppose that \( Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \) is a \( k \)-term. Then, \( x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \) is false. Then, \( \sim = \sim_0 \cap \max_\delta k \cap \sim_1 \).
We only need to show that $\sim \subseteq \sim_0$. Suppose that $s \sim t$. Consider the following decompositions of $s$ and $t$

$$s = f_A s + a_{n_0} + m_A s + a_m + l_A s$$
$$t = f_A t + a_{n_1} + m_A t + a_m + l_A t.$$  

Notice that, since $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is true, we have that $x_0 + x_1 + x_2 \sim x_0 + x_2$ is true. Hence, we may assume that $m_A s = m_A t = 0$. Notice also that, since $Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$ is true,

$$s \sim f_A s + l_A s$$

and since $f_A s$ and $f_A t$ are $(k-1)$-vectors (this is why we use the decompositions of vectors of $\mathbb{B}$ in $\mathbb{A}$), we have that $s \sim f_A s$. By Proposition 4.15(2), we have that

$$s \sim f_A s + a_{n_2} + l_A t$$

and hence (since $l_A t$ is a $(k-1)$-vector), the equation

$$f_A s + x_0 + x_2 \sim f_A t + x_1 + x_2$$

holds, (103)

which implies that the equation

$$f_A s + x_0 + x_2 \sim f_A s + x_2$$

is true. (104)

Since $f_A s$ is a $(k-1)$-vector, we have that

$$Tx_0 + x_1 + x_2 \sim Tx_0 + x_2$$

is true, (105)

a contradiction. □
Lemma 4.23. Suppose that \( x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \) is true, and \( Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \) is false. Then, \( \sim = \sim'_0 \cap \min_k \cap \sim'_1 \).

Proof. By Proposition 4.17, we have that \( \sim'_0 \cap \min_k \cap \sim'_1 \subseteq \sim \). Let us show that \( \sim \subseteq \sim'_0 \cap \min_k \cap \sim'_1 \). By Proposition 4.22 and Lemma 4.14, we have that \( \sim \subseteq \min_k \cap \sim'_1 \). So, we only need to show that \( \sim \subseteq \sim'_0 \). Suppose that \( s \sim t \) with

\[
 s = f_A s + a_n + m_k s + a_m + l_k s \\
 t = f_A t + a_n + m_k t + a_m + l_k t.
\]

Since the equation \( x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \) is true, we have that

\[
 f_A s + a_n + m_k s + a_m + l_k s \sim f_A t + a_n + l_k t,
\]

and, by Proposition 4.14,

\[
 f_A s + a_n + l_k s \sim f_A t + a_n + l_k s,
\]

which easily leads to that \( s \sim'_0 t \). □

Lemma 4.24. Suppose that \( x_0 + x_1 + x_2 \sim x_0 + x_2 \) is true, and that \( x_0 + x_1 + Tx_2 \sim x_0 + Tx_2 \) and \( Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \) are both false. Then, \( \sim \subseteq \min_k \cap \max_k \).

Proof. By Lemma 4.18, we know that \( \sim \subseteq \max_k \) and by Lemma 4.14, \( \sim \subseteq \sim'_1 \). So, we only need to show that \( \sim \subseteq \min_k \). Suppose that \( s \sim t \), set

\[
 s = f_A s + a_n + m_k s + a_m + l_k s \\
 t = f_A t + a_n + m_k t + a_m + l_k t.
\]

Suppose on the contrary that \( n_0 < n_1 \). There are two cases to consider:

1. \( n_1 = m \). Hence, \( n_0 < m \) and

\[
 s \sim f_A s + a_n + m + l_A s \text{ and } t = f_A t + a_m + l_A t.
\]

By Proposition 4.13,

\[
 f_A s + a_n + m + l_A s \sim f_A t + a_m + l_A s,
\]

which implies that the equation

\[
 f_A s + x_0 + x_1 \sim f_A t + x_1 \text{ is true,}
\]

a contradiction, since \( f_A s \) is a \((k - 1)\)-vector.

2. \( n_1 < m \). Then, by our assumptions, and Proposition 4.13,

\[
 f_A s + a_n + m + l_A s \sim f_A t + a_n + a_m + l_A s.
\]

Hence, the equation

\[
 f_A s + x_0 + x_2 \sim f_A t + x_1 + x_2 \text{ is true,}
\]

which readily implies that \( Tx_0 + x_1 + x_2 \sim Tx_0 + x_2 \) must be true, a contradiction. □

Corollary 4.25. Suppose that \( x_1 + x_2 + x_3 \sim x_1 + x_3 \) is true, and that \( x_1 + x_2 + Tx_3 \sim x_1 + Tx_3 \) and \( Tx_1 + x_2 + x_3 \sim Tx_1 + x_3 \) are both false. Then, \( \sim = \sim'_0 \cap \min_k \cap \max_k \cap \sim'_1 \).

PROOF. By Proposition 4.17, \( \sim_0 \cap \min_k \cap \max_k \cap \sim'_0 \subseteq \sim \). Let us show the opposite inclusion. By Lemma 4.24, we have that \( \sim \subseteq \min_k \cap \max_k \). It remains to show that \( \sim \subseteq \sim'_0 \). Suppose that \( s \sim t \), where \( s = f_h s + a_n + m_h s + a_n + l_h s \) and \( t = f_h t + a_n + m_h t + a_m + l_h s \) (we may assume that \( l_h s = l_h t \), since \( \max_k(s) = \max_k(t) \)). There are two cases: \( n < m \). Then, \( f_h s + a_n + a_m + l_h s \sim f_h t + a_n + m_h t + a_m + l_h s \) which directly implies that \( s \sim'_0 t \). The proof for \( n_0 = m \) is quite similar. \( \square \\

**Lemma 4.26.** Suppose that \( T x_0 + x_1 + x_2 \sim T x_0 + x_2 \) and \( x_0 + x_1 + T x_2 \sim x_0 + T x_2 \) are both true. Then, \( \sim = \sim'_0 \cap \sim'_1 \).

PROOF. It is enough to show that \( \sim \subseteq \sim'_0 \). Suppose that \( s \sim t \), with \( s = f_h s + a_{n_0} + m_h s + a_{n_0} + l_h s \) and \( t = f_h t + a_{n_1} + m_h t + a_{n_1} + l_h t \). We may assume that \( s = f_h s + a_{n_0} + l_h s \), and \( t = f_h t + a_{n_1} + l_h t \). W.l.o.g. we assume that \( n_0 \leq n_1 \), and hence, by Proposition 4.13,

\[
f_h s + a_{n_0} + l_h t \sim f_h t + a_{n_1} + l_h t. \tag{113}
\]

Case \( n_0 = n_1 \). By definition of \( \sim'_0 \), (113) implies that

\[
f_h t + a_{n_0} + l_h t \sim f_h t + a_{n_0} + l_h t, \tag{114}
\]

but trivially \( f_h s + a_{n_0} + l_h t \sim f_h s + a_{n_0} + l_h s \), and we are done.

Case \( n_0 < n_1 \). Then,

\[
f_h s + x_0 + T x_2 \sim f_h t + x_1 + T x_2 \text{ is true}, \tag{115}
\]

which easily yields

\[
f_h s + x_1 + T x_3 \sim f_h t + x_1 + T x_3 \text{ is true}. \tag{116}
\]

This implies that \( s \sim'_0 t \). \( \square \\

**Corollary 4.27.** Every equivalence relation on \( \FIN_k \) is canonical in some sos. \( \square 

This corollary has the following local version.

**Corollary 4.28.** For every block sequence \( A \) and every equivalence relation \( \sim \) on \( \langle A \rangle \) there is an sos \( B \in [A]^\infty \) on which \( \sim \) is canonical.

PROOF. Fix the canonical isomorphism \( \Lambda : \FIN_k \rightarrow \langle A \rangle \) (i.e., the extension of \( \Theta e_n \mapsto a_n \)). It is not difficult to show the following facts:

\( i \) \( B = (b_n)_n \) is an sos iff \( FB = (F b_n)_n \) is an sos.

\( ii \) For every canonical equivalence relation \( \sim_{\text{can}} \), every sos \( B \), and \( s, t \in \langle B \rangle \), \( s \sim_{\text{can}} t \) iff \( F^{-1} s \sim_{\text{can}} F^{-1} t \).

We define \( \sim' \) on \( \FIN_k \) by \( s \sim' t \) iff \( Fs \sim Ft \). Find a canonical equivalence relation \( \sim_{\text{can}} \) and an sos \( B \) such that \( \sim \) and \( \sim_{\text{can}} \) are the same on \( \langle B \rangle \). Let \( C = FB \), which is an sos. Then \( \sim \) and \( \sim_{\text{can}} \) are the same in \( \langle C \rangle \): \( s \sim_{\text{can}} t \) iff \( F^{-1} s \sim_{\text{can}} F^{-1} t \) iff \( F^{-1} s \sim' F^{-1} t \) iff \( s \sim t \). \( \square \\

**Corollary 4.29.** Every canonical equivalence relation is a staircase equivalence relation.
By standard methods, we conclude that \( R \) and that \( \Gamma(a, x) \). Hence, \( \sim' \) is a staircase equivalence relation in \( \mathbb{B} = (Ta_{3n} + a_{3n+1} + Ta_{3n+2}) \). Let \( \sim' \) be this staircase relation, which is equal to \( \sim \) when restricted to \( \mathbb{B} \). We show that \( \sim \) and \( \sim' \) are not only equal in \( \mathbb{B} \), but also in \( A \). Fix \( s \) and \( t \) in \( A \), and take their canonical decompositions in \( A \):

\[
s = \sum_{n \geq 0} T^{k-C_A(s)(n)} a_n \quad \text{and} \quad t = \sum_{n \geq 0} T^{k-C_A(t)(n)} a_n.
\]

Suppose first that \( s \sim t \). Since \( \sim \) is canonical, the equation

\[
\sum_{n \geq 0} T^{k-C_A(s)(n)} x_n \sim \sum_{n \geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } A,
\]

and hence, also in \( \mathbb{B} \), i.e.,

\[
\sum_{n \geq 0} T^{k-C_A(s)(n)} x_n \sim' \sum_{n \geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } \mathbb{B}.
\]

But since \( \sim' \) is staircase, it is canonical (Proposition 3.10), and hence, equation (118) also holds in \( A \), and in particular, \( s \sim' t \).

Suppose now that \( s \sim t \). Since \( \sim' \) is canonical in any sos, the equation

\[
\sum_{n \geq 0} T^{k-C_A(s)(n)} x_n \sim' \sum_{n \geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } A,
\]

hence, also in \( \mathbb{B} \). By definition, \( \sim' \) is equal to \( \sim \) restricted to \( \mathbb{B} \), and hence

\[
\sum_{n \geq 0} T^{k-C_A(s)(n)} x_n \sim \sum_{n \geq 0} T^{k-C_A(t)(n)} x_n \text{ holds in } \mathbb{B}.
\]

Since \( \sim \) is canonical, the equation (120) holds in \( A \), and in particular, \( s \sim t \). \( \square \)

5. Counting

The purpose now is to give an explicit formula for the number \( t_k \) of staircase equivalence relations on \( \text{FIN}_k \). To do this, recall that \( e_n(t) = \sum_{j=0}^{n} \frac{1}{j!} \) is the exponential sum-function and that \( \Gamma(a, x) = \int_{x}^{\infty} e^{-t} \, dt \) is the incomplete Gamma function. Recall also that \( \Gamma(n, 1) = (n - 1)! e^{-1} e_{n-1}(1) \) for every integer \( n \).

Let \( \mathcal{A}_k, \mathcal{B}_k \) be the set of min-relations and max-relations respectively, and set \( a_k = |\mathcal{A}_k| \) and \( b_k = |\mathcal{B}_k| \). Let \( \mathcal{C}_k \subseteq \mathcal{A}_k \) be the set of min-relations \( R \) such that \( k \not\in I_0(R) \), and let \( \mathcal{D}_k \subseteq \mathcal{B}_k \) be the set of max-relations \( R \) such that \( k \not\in I_1(R) \). Set \( c_k = |\mathcal{C}_k| \) and \( d_k = |\mathcal{D}_k| \). Notice that

(i) \( c_k = a_k - 1 \),

(ii) \( \mathcal{A}_k = \mathcal{A}_{k-1} \cup \{ R \cap \sim_{\min_k} : R \in \mathcal{C}_{k-1} \} \cup \{ R \cap \sim_{\max_k \cap \sim_{\theta_k}} : l = -1 \text{ or } l = 1, \ldots, k - 1, R \in \mathcal{A}_{k-1} \setminus \mathcal{C}_{k-1} \} \). So, \( a_k = a_{k-1} + c_{k-1} + k(a_{k-1} - c_{k-1}) \).

Hence,

\[
a_k = (k + 1)a_{k-1} - (k - 1)a_{k-2}, \quad a_0 = 1, \quad a_1 = 2.
\]

By standard methods, we conclude that

\[
a_k = \frac{e (1 + k) k! \Gamma(1 + k, 1)}{\Gamma(2 + k)} = k! e_k(1).
\]
Now let $T_k$ be the set of staircase equivalence relations of $FIN_k$ and $t_k = |T_k|$. Then,

\[ T_k = \left\{ (R \cap S : R \in \mathcal{A}_k, S \in B_k) \setminus \{ R \cap S : R \in \mathcal{A}_k \setminus C_k, S \in B_k \setminus D_k \} \cup \{ R \cap S \cap \sim_{R}^{2} : R \in \mathcal{A}_k \setminus C_k, S \in B_k \setminus D_k, l = -1 \text{ or } l = 1, \ldots, k \right\}. \]  

(123)

(124)

Hence,

\[ t_k = a_k^2 - (a_k - c_k)^2 + (k + 1)(a_k - c_k)^2 = k(a_k - a_{k-1})^2 + a_k^2 \]  

(125)

and from (122) and (127), we obtain that

\[ t_k = (k e_k(1))^2 + k (k! e_k(1) - (k-1)! e_{k-1}(1))^2, \]  

(126)

or, equivalently,

\[ t_k = e^2 \left[ k \left[ \Gamma(k,1) - \Gamma(k+1,1) \right]^2 + \Gamma(k+1,1)^2 \right]. \]  

(127)

This is a table with the first few values of $t_k$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_k$</td>
<td>1</td>
<td>5</td>
<td>43</td>
<td>619</td>
<td>13829</td>
<td>446881</td>
<td>19790815</td>
</tr>
</tbody>
</table>

**Remark 5.1.** Let us say that a canonical equivalence relation $R$ is **linked free** iff $I_0(R)$ and $I_1(R)$ have no consecutive members and $k \notin I_0(R) \cap I_1(R)$. The number $t_k$ of linked free canonical equivalence relations of $FIN_k$ is the Fibonacci number $F_{2k+2}$ for $2k+2$, since $F_{i+2}$ is the number of subsets of $\{1,2,\ldots,l\}$ with no consecutive elements, and since $R$ is linked free iff the set $I_0(R) \cup \{2k+1-i : i \in I_1(R)\} \subseteq \{1,2,\ldots,2k\}$ has no consecutive numbers.

6. The finite version

**Theorem 6.1.** For every $m$ there is some $n = n(m)$ such that for every equivalence relation $\sim$ on $\langle e_0, \ldots, e_n \rangle$ there is some sos $(a_0, \ldots, a_{m-1})$ such that $\sim$ is a staircase equivalence relation in $(a_0, \ldots, a_{m-1})$.

**Proof.** Suppose not. Then, there is some $m$ such that for every $n$ there is some equivalence relation $\sim_n$ on $\langle e_0, \ldots, e_n \rangle$ which is not a staircase relation when restricted to any sos $(a_0, \ldots, a_{m-1})$ of $(e_i)_{i=0}^n$. Let $U$ be a non-principal ultrafilter on $\mathbb{N}$, and define the equivalence relation $\sim$ on $FIN_k$ by

\[ s \sim t \text{ if and only if } \{ n : sR_n t \} \in U, \]

where $R_n = \sim_n \cup (FIN_k)^2$ is an equivalence relation on $FIN_k$. It is easy to see that $\sim$ is an equivalence relation. By Theorem 5.1 there is some sos $A = (a_n)_n$ on which $\sim$ is a staircase equivalence relation, say $\sim_{can}$. Choose $n$ large enough such that:

(i) $(a_0, \ldots, a_{m-1}) \preceq (e_i)_{i=0}^n$

(ii) For $s, t \in (a_0, \ldots, a_{m-1})$ one has that $s \sim t$ iff $s \sim_n t$.

This can be done as follows: For every pair $s, t \in (a_0, \ldots, a_{m-1})$, let

\[ A_{s,t} = \begin{cases} \{ n : s \sim_n t \} \in U & \text{if } s \sim t \\ \{ n : s \not\sim_n t \} \in U & \text{if } s \not\sim t. \end{cases} \]  

(128)

Let $n = \min \bigcap_{s,t \in (a_0, \ldots, a_{m-1})} A_{s,t}$. Then $\sim_n$ is $\sim$ restricted to $(a_0, \ldots, a_{m-1})$, and hence is a staircase equivalence relation, a contradiction. \[\square\]
Corollary 6.2. For every \( m \) there is some \( n = n(m) \) such that for every equivalence relation \( \sim \) on \( \langle e_0, \ldots, e_n \rangle \) there is some sos \( (a_0, \ldots, a_{m-1}) \preceq (e_0, \ldots, e_n) \) such that \( \sim \) is a canonical equivalence relation on \( \langle a_0, \ldots, a_{m-1} \rangle \).

Proof. Let \( n = n(m) \) be given by Theorem 5.1. Fix \( b_0, \ldots, b_n \), and an equivalence relation \( \sim \). Let \( F \) be the canonical isomorphism between \( \langle e_0, \ldots, e_n \rangle \) and \( \langle b_0, \ldots, b_n \rangle \). Define \( \sim' \) on \( \langle e_i \rangle_{i=1}^n \) via \( F \), i.e., \( s \sim' t \) if and only if \( F(s) \sim F(t) \). Fix an sos \( (c_i)_{i=0}^{m-1} \preceq (e_i)_{i=1}^n \) and a staircase equivalence relation \( \sim_{\text{can}} \) such that \( s \sim' t \) if and only if \( s \sim_{\text{can}} t \), for every \( s, t \in \langle e_i \rangle_{i=0}^{m-1} \). Let \( b_i = Fc_i \), for every \( i = 0, \ldots, m-1 \). Then \( \langle b_0, \ldots, b_m \rangle \) is an sos since sos are preserved under isomorphisms, and \( \sim_{\text{can}} \) is well defined on \( \langle b_0, \ldots, b_{m-1} \rangle \). Since \( \sim_{\text{can}} \) is staircase one has that
\[
s \sim_{\text{can}} t \text{ if and only if } F^{-1}s \sim_{\text{can}} F^{-1}t
\]
for every \( s, t \in \langle b_0, \ldots, b_{m-1} \rangle \). Hence,
\[
s \sim_{\text{can}} t \text{ iff } F^{-1}s \sim_{\text{can}} F^{-1}t \text{ iff } F^{-1}s \sim' F^{-1}t \text{ iff } s \sim t.
\]
Therefore \( \sim_{\text{can}} \) and \( \sim \) coincide on \( \langle b_0, \ldots, b_{m-1} \rangle \). \( \square \)

Definition 6.4. We say that a staircase relation \( \sim \) is symmetric iff \( I_1(\sim) = I_0(\sim) = I \), \( J_1(\sim) = J_0(\sim) = J \) and \( l_i^{(0)} = l_i^{(1)} \) for every \( j \in J \).

Corollary 6.5. For every \( m \) there is some \( n = n(m) \) such that for every equivalence relation \( \sim \) on \( \langle e_0, \ldots, e_n \rangle \) there are disjointly supported sos’s \( a_0, \ldots, a_{m-1} \in \langle (e_i)_{i=0}^n \rangle \) such that \( \sim \) is a symmetric staircase relation in \( \langle a_0, \ldots, a_{m-1} \rangle \).

Before we give the proof of this, let us observe that if \( a_0, \ldots, a_{m-1} \) are disjointly supported \( k \)-vectors then the mapping \( c_i \rightarrow a_i \) extends to a lattice-isomorphism from \( \langle (e_i)_{i=0}^n \rangle \rightarrow \langle (a_i)_{i=0}^{m-1} \rangle \) that preserves the operation \( T \).

Proof. Fix an integer \( m \). Let \( n \) be given by Theorem 5.1 when applied to \( 2m \). Suppose that \( \sim \) is an equivalence relation on \( \langle (e_i)_{i=0}^n \rangle \). Then there is some sos \( (b_i)_{i=0}^{2m-1} \) such that \( \sim \) is a staircase relation when restricted to \( \langle (b_i)_{i=0}^{2m-1} \rangle \). Let \( a_i = b_i + b_{2m-i-1} \) for every \( 0 \leq i \leq m-1 \). A typical vector \( b \in \langle (a_i)_{i=0}^{m-1} \rangle \) is of the form
\[
b = \sum_{i=0}^{m-1} T^{k-r_i}b_i + \sum_{i=0}^{m-1} T^{k-r_i}b_{2m-i-1}.
\]
Let \( s, t \in \langle (a_i)_{i=0}^{m-1} \rangle \). Then one has that
\[
\min(s) = \min(t) \text{ iff } \max(s) = \max(t), \text{ and } \quad (129)\]
\[
\theta_{i,l}^0(s) = \theta_{i,l}^0(t) \text{ iff } \theta_{i,l}^1(s) = \theta_{i,l}^1(t). \quad (130)\]
Let \((I_0, J_0, (I_j^{(0)})_{j \in J_0}, I_1, J_1, (I_j^{(1)})_{j \in J_1}, I_k^{(2)})\) be the values of \(\sim\) when restricted to \(\langle b_i \rangle_{i=0}^{2m-1}\). Using Proposition 7.1, it follows that our fixed relation \(\sim\) is when restricted to \(\langle a_i \rangle_{i=0}^{n-1}\) a symmetric staircase relation with values

\[(I_0 \cup I_1, J_0 \cup J_1, (I_j)_{j \in J_0 \cup J_1}, I_0 \cup I_1, J_0 \cup J_1, (l_j)_{j \in J_0 \cup J_1}, l_k^{(2)})\]

and where for each \(j \in J_0 \cup J_1\)

\[l_j = \begin{cases} \min \{l_j^{(0)}, l_j^{(1)}\} & \text{if } j \in J_0 \cap J_1 \\ l_j^{(0)} & \text{if } j \in J_0 \setminus J_1 \\ l_j^{(1)} & \text{if } j \in J_1 \setminus J_0. \end{cases}\]

Remark 6.6. (1) Pr"{o}mel and Voigt were the firsts to observe in 129 the Corollary 6.5 for FIN. We thank the referee for pointing us out this.

(2) Let \(S_k\) be the set of symmetric staircase relations of \(FIN_k\), and set \(s_k = |S_k|\). Using the notation from the Section 3 one has that

\[S_k = C_k \cup \{\sim \cap \sim_{q_2} : \sim \in A_k \setminus C_k, \text{ and } l = -1, 1, \ldots, k\}.

Hence

\[s_k = c_k + (a_k - c_k)(k + 1) = a_{k-1} + (k + 1)(a_k - a_{k-1}) = (k + 1)!e_k(1) - k!e_{k-1}(1).

7. CANONICAL RELATIONS AND CONTINUOUS MAPS ON \(PS_{c_0}\)

Our result on equivalence relations on \(FIN_k\) gives some consequences about equivalence relations on \(PS_{c_0}\). Let us start with some natural definitions.

For a fixed \(\delta > 0\), let \(k\) be the first integer such that \(1/(1 + \delta)^{k-1} < \delta\), and set \(\delta_i = (1 + \delta)^{i-k}\), for \(0 \leq i \leq k\). For \(0 \leq i \leq k + 1\), let

\[\gamma_i(\delta) = \begin{cases} \frac{\delta_{i-1} + \delta_i}{2} & \text{if } 1 \leq i \leq k \\ 0 & \text{if } i = 0 \\ \delta_k = 1 & \text{if } i = k + 1 \end{cases}\]

and for \(0 \leq i \leq k\), let

\[I_i^{(\delta)} = \begin{cases} \gamma_i(\delta), \gamma_{i+1}(\delta) & \text{if } 0 \leq i < k \\ \gamma_k(\delta), \gamma_{k+1}(\delta) = 1 & \text{if } i = k. \end{cases}\]

We have then that \(\delta_i \in I_i^{(\delta)}\) for every \(0 \leq i \leq k\), and that \([0, 1] = \bigcup_{i=0}^{k} I_i^{(\delta)}\), a disjoint union.

For \(x = (x_m)_m \in PB_{c_0}\) and \(n \in \mathbb{N}\), let \(\Gamma_n^{(\delta)}(x)\) be the unique \(0 \leq i \leq k\) such that \(x_n \in I_i^{(\delta)}\), and define \(\Gamma_\delta : PB_{c_0} \to FIN_{<k}\) by \(\Gamma_\delta(x) = (\Gamma_n^{(\delta)}(x))_n\). Notice that \(\Gamma_\delta(PS_{c_0}) \subseteq FIN_k\). A vector \(x \in PS_{c_0}\) is called a \(\delta\)-sos iff \(\Gamma_\delta x\) is an sos. A block sequence \((x_n)_n\) of vectors of \(PS_{c_0}\) is called a \(\delta\)-sos iff every \(x \in PS_X\) is a \(\delta\)-sos. The next proposition is not difficult to prove.

Proposition 7.1. Fix \( \rho \in [0, 1] \), \( x, y \in PB_{c_0}\), and a \(k\)-vector \( s \) of \( FIN_k \). Let \( i \) be the unique integer such that \( \rho \in I_i^{(\delta)} \). Then,

(i) \( \Gamma_\delta(x + y) = \Gamma_\delta(x) + \Gamma_\delta(y) \) and \( \Gamma_\delta(\rho e_n) = T_{\delta}^{-i} \Gamma_\delta(e_n) = T_{\delta}^{-i}(\Theta_\delta^{-1} e_n) \).
\(\Gamma_\delta(\rho\Theta_{\delta}^{-1}x) = T_{k-1}^k\Gamma_\delta(\Theta_{\delta}^{-1}x)\). It follows that if \((a_n)_n\) is an sos k-block sequence, then \((\Theta_{\delta}^{-1}a_n)_n\) is a \(\delta\)-sos.

\(\square\)

**Definition 7.2.** Given a staircase mapping \(f\) of \(\text{FIN}_k\), we consider the following two extensions to an arbitrary \(\delta\)-sos \(X = (x_n)_n\). The first one is \(f^{(0)} : P_S X \to \text{FIN}_{\leq k}\), closing the following diagram:

\[
\begin{array}{ccc}
P_S X & \xrightarrow{\Gamma_\delta} & (\Gamma_\delta x_n)_n \\
\downarrow{f^{(0)}} & & \downarrow{f} \\
\text{FIN}_{\leq k} & & \\
\end{array}
\]

The second one is \(f^{(1)} : P_S X \to PB_{c_0}\), defined by \(f^{(1)}(x)(n) = x(n)\) iff \(f^{(0)}x(n) \neq 0\).

**Proposition 7.3.** Fix a staircase \(f\), and some \(\delta\)-sos \(X\).

(i) \((f \circ g)^{(i)} = f^{(i)} \circ g^{(i)}\), for \(i = 0, 1\) and \(\circ\) equal to \(\lor\) or \(\land\).

(ii) \(f^{(1)}\) is a Baire class 1 function.

(iii) If \(f^{(1)}x = f^{(1)}y\), then \(f^{(0)}x = f^{(0)}y\) for every \(x, y \in P_S X\).

(iv) \(\|\Theta_{\delta}^{-1}f^{(0)}x - f^{(1)}x\| \leq \delta\) for every \(x \in P_S X\).

(v) For every k-vector \(a \in (\langle \Gamma_\delta x_n \rangle)_n\), \(f^{(1)}\Theta_{\delta}^{-1}a = f^{(0)}\Theta_{\delta}^{-1}a = fa\). Therefore \(f^{(1)}\Theta_{\delta}^{-1}a = f^{(1)}\Theta_{\delta}^{-1}b\) iff \(f^{(0)}\Theta_{\delta}^{-1}a = f^{(0)}\Theta_{\delta}^{-1}b\), for every k-vectors \(a, b \in \langle \Gamma_\delta x_n \rangle_n\).

(vi) For every \(x \in P_S X\) there is some k-vector \(\bar{x}\) such that \(\|x - f^{(0)}\Theta_{\delta}^{-1}\bar{x}\| \leq \delta\) and \(f^{(0)}x = f^{(0)}\Theta_{\delta}^{-1}\bar{x}\).

**Proof.** (i) is not difficult to check. Let us show (ii). To do this, suppose that \(f\) is a staircase mapping. Then \(f\) is in the algebraic closure of \(\mathcal{F}\) (see Definition 3.11), i.e., there is a finite list \(f_0, \ldots, f_n \in \mathcal{F}\) such that \(f = f_0 \circ_0 f_1 \circ_1 f_2 \circ_2 \cdots \circ_{n-1} f_n\), where \(\circ_i\) is either \(\lor\) or \(\land\) for every \(i = 0, \ldots, n - 1\). By point (i) one has that \(f^{(1)} = f_0^{(1)} \circ_0 f_1^{(1)} \circ_1 f_2^{(1)} \circ_2 \cdots \circ_{n-1} f_n^{(1)}\). Since for every point \(x \in P_S X\) the support of \(f^{(1)}(x)\) is finite, we may assume that \(f \in \mathcal{F}\). We give the proof for the case \(f = \text{min}_i\). The other cases can be shown in a similar way. For \(l > 0\) we define the following perturbations of the intervals \(I_i^{(\delta)}\), let

\[
I_{i,l}^{(\delta)} = \begin{cases} 
(\gamma_i(\delta) - \frac{1}{l}, \gamma_{i+1}(\delta)) & \text{if } i < k \\
(\gamma_k(\delta) - \frac{1}{l}, 1] & \text{if } i = k.
\end{cases}
\]

These are open intervals of \(P_S c_0\). For each \(l\), let \(f_l : P_S X \to PB_X\) be defined for \(n \in \mathbb{N}\) as follows,

\[
f_l(x)(n) = \begin{cases} 
x(n) & \text{if } x(n) \in I_{i,l}^{(\delta)} \text{ and for all } m < n \ x(n) \in [0, \gamma_i(\delta)) \\
0 & \text{if not.}
\end{cases}
\]

Let us see that \(f_l\) is continuous, and that \(f_l \to_l f\). Suppose that \(x_r \to x\), with \(x_r, x \in P_S X\). Let \(n\) be the unique integer such that \(f_l(x)(n) = x(n) > 0\), i.e., \(x(n) \in I_{i,l}^{(\delta)}\) and \(x(m) \in [0, \gamma_i(\delta))\) for every \(m < n\). Since both sets are open, there must be some \(r'\) such that \(x_{r'}(n) \in I_{i,l}^{(\delta)}\) and
Proposition 7.4. Fix $\delta > 0$, a staircase equivalence relation $R_f$, and a $k$-block sequence $A = (a_n)_n$, where $k = \delta(k)$. Set $X = (x_n = \Theta_\delta^{-1}a_n)_n$ and $R = R_{f(1)}$.

(i) For every $x \in PS_X$ there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$ and hence $y \in ([x']_R)$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$ and hence $y \in ([x']_R)$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$ and hence $y \in ([x']_R)$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$. We show that $[x]_R \subseteq ([x']_R)$. For every $z \in [x]_R$, there is a $k$-vector $\bar{x}$ of $A$ such that $\|x - \Theta_\delta^{-1}\bar{x}\| \leq \delta$ and $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}])_R$. Suppose that $y \in [x]_R \cap PS_X$. Then $f(1)x = f(1)y$, and hence $f(0)y = f(0)x'$. Let $\bar{y}$ be a $k$-vector of $A$ such that $\|y - \Theta_\delta^{-1}\bar{y}\| \leq \delta$ and $f(0)y = f(0)\Theta_\delta^{-1}\bar{y}$, and set $y' = \Theta_\delta^{-1}\bar{y}$. Then, $f(0)x' = f(0)y'$, which implies that $f(1)x' = f(1)y'$, i.e., $y' \in [x']_R$.
In the case of equivalence relations with some additional properties, we have the following stronger result.

**Proposition 7.7.** Fix $\delta, \gamma > 0$, set $k = k(\delta)$, and suppose that $R$ is an equivalence relation on $PS_{c_0}$ such that

(i) for every $x, y \in PS_{c_0}$ and every $z \in PS_{c_0}$ with $x \wedge y \leq_L z \leq_L x \vee y$, if $(x, y) \in R$, then $(x, z) \in R$, and

(ii) for every sos $k$-block sequence $B = (b_n)_n$ and every $x \in PS_{(\Theta_\delta^{-1}b_n)_n}$ there is some $k$-vector $\bar{x}$ of $B$ such that $[x]_R \subseteq ([\Theta_\delta^{-1}\bar{x}]_R)_\gamma$.

Then, there is some $\delta$-sos $X$ and some $\delta$-staircase equivalence relation $\tilde{R}$ such that

(a) for every $R$-equivalent classes $\alpha$ in $PS_X$, there is a $\tilde{R}$-equivalent class $\beta$ in $PS_X$ such that $\alpha \subseteq \beta_{k+\gamma}$, and

(b) for every $\tilde{R}$-equivalence class $\beta$ there is a $R$-equivalence class $\alpha$ such that $\beta \subseteq (\alpha)_\delta$.

**Proof.** Define $\tilde{R}$ on $\text{FIN}_k$ via $\Theta_\delta$. Then, there is some sos $A = (a_n)$ and some staircase equivalence relation $R_f$ such that $\tilde{R}$ is $R_f$ on $\langle A \rangle$. Let $\tilde{R} = R_{f(1)}$, and $X = (x_n)_n$, where $x_n = \Theta_\delta^{-1}a_n$ for every $n$. (b) is shown in Proposition 7.6. Let us show (a). Fix $x \in PS_X$, and choose a $k$-vector $\bar{x}$ of $A$ such that $[x]_R \subseteq ([x']_R)_\gamma$ where $x' = \Theta_\delta^{-1}\bar{x}$. Let us show that $[x']_R \subseteq ([x']_R)_\delta$ on $PS_X$. Fix $y \in [x']_R$. Then, there is some $k$-vector $\bar{y}$ of $A$ such that $x' \wedge y \leq_L y' \leq_L x' \vee y$ and $\|y - y'\| \leq \delta$, where $y' = \Theta_\delta^{-1}\bar{y}$. Hence, $y' \in [x']_R$, and therefore, $y' \in [x']_R$. \qed

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**References**


**EQUIPE DE LOGIQUE MATHEMATIQUE, UNIVERSITÉ PARIS 7- DENIS DIDEROT, C.N.R.S. -UMR 7056, 2, PLACE JUSSIEU- CASE 7012, 75251 PARIS CEDEX 05, FRANCE**

**E-mail address:** abad@logique.jussieu.fr