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\(\lambda\mu\)-calculus and duality: call-by-name and call-by-value

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Abstract. Under the extension of Curry-Howard’s correspondence to classical logic, Gentzen’s NK and LK systems can be seen as syntax-directed systems of simple types respectively for Parigot’s \(\lambda\mu\)-calculus and Curien-Herbelin’s \(\bar{\lambda}\mu\bar{\mu}\)-calculus. We aim at showing their computational equivalence. We define translations between these calculi. We prove simulation theorems for an undirected evaluation as well as for call-by-name and call-by-value evaluations.

1 Introduction

Key systems for classical logic in proof theory are Gentzen’s NK and LK. The logical equivalence between the latter was proved in [Gentzen, 1934]. We deal with the extension of Curry-Howard’s correspondence between proofs and programs through the systems of simple types for the \(\lambda\mu\) and \(\bar{\lambda}\mu\bar{\mu}\)-calculi. This extension concerns some other calculi. It is initially Felleisen’s \(\lambda_c\)-calculus. Its type system is the intuitionistic natural deduction with the double negation axiom. Griffin proposed this axiom as the type for the \(c\)-operator in [Griffin, 1990]. However, we focus on calculi that correspond closer to Gentzen’s systems. The \(\lambda\mu\)-calculus was defined for NK in [Parigot, 1992]. The \(\lambda\mu\bar{\mu}\)-calculus was designed for LK in [Curien and Herbelin, 2000]. In the general case, these two calculi are not deterministic. There exists critical pairs. The \(\lambda\mu\bar{\mu}\)-calculus admits two deterministic projections depending on choosing one of the two possible symmetric orientations of a critical pair. They correspond to the call-by-name/call-by-value duality.

We aim at proving the computational equivalence between \(\lambda\mu\) and \(\bar{\lambda}\mu\bar{\mu}\)-calculi. A major step was reached with the proof of the simulation of the \(\lambda\mu\)-calculus by the \(\lambda\mu\bar{\mu}\)-calculus in [Curien and Herbelin, 2000]. It holds both for call-by-name and call-by-value evaluations. We present the call-by-name/call-by-value projections of the \(\lambda\mu\)-calculus in the same way as for the \(\lambda\mu\bar{\mu}\) in [Curien and Herbelin, 2000]. It consists of choosing one of the two possible orientations of a critical pair. We prove that the \(\lambda\mu\)-calculus simulates backwards the \(\lambda\mu\bar{\mu}\)-calculus in such a way that we obtain easily the same result for the call-by-name, for the call-by-value and for the simple type case. The
\(\lambda\mu\mu\)-calculus is composed of three syntactic categories: terms, contexts (or environments) and commands. The \(\lambda\mu\)-calculus is basically composed of terms and commands. We add contexts to the \(\lambda\mu\)-calculus. It eases mappings between the \(\lambda\mu\) and \(\lambda\mu\mu\)-calculi. We extend the translation from the \(\lambda\mu\)-calculus to the \(\lambda\mu\mu\)-calculus defined in [Curien and Herbelin, 2000] over the \(\lambda\mu\)-contexts. We define backwards a translation from the \(\lambda\mu\mu\)-calculus to the \(\lambda\mu\)-calculus. In section 2 we present the \(\lambda\mu\)-calculus. In section 3 we present the \(\lambda\mu\mu\)-calculus. In section 4 we define translations between these two calculi. In section 5 we prove simulation theorems that hold for call-by-name and call-by-value.

2 \(\lambda\mu\)-calculus

We follow the definition given in [Parigot, 1992]. Firstly, we present the grammar of terms and commands. Secondly, we present the system of simple types. Thirdly, we present generic reductions and their call-by-name and call-by-value projections. Fourthly, we extend both the grammar and the type system to the contexts. Basically, the \(\lambda\mu\)-calculus is composed of terms and commands. They are defined by mutual induction:

\[
 t ::= x \mid \lambda x.t \mid (t) t \mid \mu \alpha. c \quad c ::= [\alpha] t
\]

Symbols \(x\) range over \(\lambda\)-variables, symbols \(\alpha\) range over \(\mu\)-variables. We note \(x \in t\) or \(\alpha \in t\) the fact that \(x\) or \(\alpha\) has a free occurrence in \(t\). Symbols \(\lambda\) and \(\mu\) are binders. Two terms are equal modulo \(\alpha\)-equivalence.

The system of simple types for the \(\lambda\mu\)-calculus is based on two kinds of sequents. The first \(\Gamma \vdash t : T \mid \Delta\) concerns the terms and the second \(c : (\Gamma \vdash \Delta)\) concerns the commands in which \(T\) is a simple type obtained by the grammar \(T ::= X \mid T \to T\). \(\Gamma\) is a finite domain application from \(\lambda\)-variables to simple types and \(\Delta\) is a finite domain application from \(\mu\)-variables to simple types. \(\Gamma, \Gamma'\) denotes the union of the applications \(\Gamma\) and \(\Gamma'\). System rules are:

\[
\begin{array}{c}
\frac{x : A \vdash x : A}{\Gamma \vdash t : B \mid \Delta} \\
\frac{\Gamma \vdash t : A \mid \Delta}{\begin{array}{c}
[\alpha] t : (\Gamma \vdash \Delta \setminus \{\alpha : A\})
\end{array}} \\
\frac{\Gamma \vdash t : A \mid \Delta}{c : (\Gamma \vdash \Delta)} \\
\frac{\Gamma \vdash \Delta \mid \Delta'}{\Gamma, \Gamma' \vdash \Delta' \mid \Delta'}
\end{array} \quad (\ast)
\]

The restriction \((\ast)\) requires that \(\Gamma\) and \(\Gamma'\) match each other on the intersection of their domains. This holds for \(\Delta\) and \(\Delta'\) too.

The category of contexts is introduced in order to ease comparisons with the homonymous category of the \(\lambda\mu\mu\)-calculus. \(\lambda\mu\)-contexts are defined by mutual induction with the terms:

\[
 e ::= \alpha \mid \beta(t) \mid t \cdot e
\]

We can see contexts as commands with a hole to fill. The first construction \(\alpha\) expects a term \(t\) in order to provide the command \([\alpha] t\). The second \(\beta(t)\) expects
a term \( u \) in order to provide the command \([\beta](t)\) \( u \). The last \( t \cdot h \) puts the term \( t \) on a stack and expects another term to fill the hole.

**Definition 1.** Let \( t \) a term and \( e \) a context. The command \( e\{t\} \) is defined by induction on \( e \):

\[
e\{t\} = \begin{cases} [\alpha]t & \text{if } e = \alpha \\ [\beta](u)t & \text{if } e = \beta(u) \\ h\{(t)u\} & \text{if } e = u \cdot h \\
\end{cases}
\]

The type system is extended to another kind of sequents \( \Gamma \mid e : T \vdash \Delta \). The typing rules give the context \( e \) the type of the term \( t \) that fills the hole of \( e \):

\[
\frac{\Delta, \alpha \vdash \alpha \cdot A}{\Delta, \alpha \cdot A \vdash \Delta} \quad \frac{\Gamma, \beta : A \vdash \Delta, \beta : B}{\Gamma, \beta \vdash A \vdash \Delta, \beta : B} \quad \frac{\Gamma, e : A \vdash \Delta \mid e : A \vdash \Delta'} {\Gamma, e : A \vdash \Delta \mid e : A \vdash \Delta'}
\]

A sequent calculus like cut-rule can then be derived in this system as a term against context application.

**Lemma 1.** The rule \( \frac{\Gamma \vdash t : A \mid \Delta}{e\{t\} : \Gamma, \Gamma \vdash \Delta} \) holds in \( \lambda \mu \).

**Proof.** By induction on \( e \).

- if \( e = \alpha \) then \( e\{t\} = [\alpha]t \) and \( \frac{\Gamma, t : A \mid \Delta}{\Gamma, [\alpha]t \vdash \alpha \cdot A} \)
- if \( e = \beta(u) \) then \( e\{t\} = [\beta](u)t \) and
  \[
  \frac{\Gamma, u : A \vdash \Delta \mid \Delta'} {\Gamma, u : A \vdash \Delta, \Delta'} \quad \frac{\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'} {\Gamma, e' : A \vdash \Delta \mid e' : A \vdash \Delta'}
  \]
  \[
  \frac{\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'} {\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'}
  \]

- if \( e = u \cdot h \) then \( e\{t\} = h\{(t)u\} \) and
  \[
  \frac{\Gamma, t : A \vdash \Delta \mid \Delta'} {\Gamma, t : A \vdash \Delta, \Delta'} \quad \frac{\Gamma, t : A \vdash \Delta \mid \Delta'} {\Gamma, t : A \vdash \Delta, \Delta'}
  \]
  \[
  \frac{\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'} {\Gamma, e' : A \vdash \Delta \mid e' : A \vdash \Delta'}
  \]
  \[
  \frac{\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'} {\Gamma, e : A \vdash \Delta \mid e' : A \vdash \Delta'}
  \]

**Definition 2.** Let \( t \) a term, \( e \) a context and \( \alpha \) a \( \mu \)-variable, The term \( t[\alpha \leftarrow e] \)
- the substitution of \( \alpha \) by \( e \) in \( t \) – is defined by induction on \( t \):

\[
t[\alpha \leftarrow e] = \begin{cases} x & \text{if } t = x \\ \lambda x.u[\alpha \leftarrow e] & \text{if } t = \lambda x.u \\ v[\alpha \leftarrow e] & \text{if } t = (u)v \\ \mu \beta.e[\alpha \leftarrow e] & \text{if } t = \mu \beta.e \\
\end{cases}
\]

\[
e[\alpha \leftarrow e] = \begin{cases} e\{t[\alpha \leftarrow e]\} & \text{if } e = [\alpha]t \\ [\beta]t[\alpha \leftarrow e] & \text{if } e = [\beta]t \\
\end{cases}
\]
The computation notion is based on reductions. We remind one-step reduction rules:

\[(\lambda x. u) t \rightarrow_{\beta} u[x \leftarrow t] \quad \]  
\[(\mu \alpha. c) t \rightarrow_{\mu} \mu \alpha. c[\alpha \leftarrow t \cdot \alpha] \]

\[\mu \delta[\delta] t \rightarrow_{\theta} t \ (\text{if } \delta \notin t) \]

\[\beta[\mu \alpha. c \rightarrow_{\rho} \alpha[\alpha \leftarrow \beta] \]

The reduction \(\rightarrow_{\gamma}\) stands for the reflexive and transitive closure of \(\rightarrow\), and the reduction \(\rightarrow\) stands for the union of \(\rightarrow_{\gamma}\) for \(\gamma \in \{\beta, \mu, \mu', \rho, \theta\}\).

Some of these reductions are linear. Both of the \(\rho\) and \(\theta\)-reductions are linear because they correspond to the identity in NK. The \(\beta\)-reduction from the term \((\lambda x. t) y\) is linear because it consists of replacing a variable by another variable inside a term. It corresponds to a normalisation against an axiom rule in NK. The \(\beta\)-reduction from the term \((\lambda x. t) u\) where \(x\) has a single free occurrence in \(t\) is linear too because it consists either of substituting a single variable occurrence by any term. It corresponds either to a normalisation without a proof-tree branch duplication.

Reductions \(\sim_{\beta} \), \(\sim_{\gamma}\) and \(\sim\) have the same meanings as in the general case. The relation \(\approx\) is defined as the reflexive, transitive and symmetric closure of \(\sim\).

There exists a critical pair for computation determinism. Applicative terms \((\lambda x. t) \mu \beta. d\) and \((\mu \alpha. c) \mu \beta. d\) can be \(\beta\) or \(\mu'\)-rewritten in the first case and \(\mu\) or \(\mu'\)-rewritten in the second case. We can see the call-by-name and call-by-value disciplines as restrictions of the generic reductions.

The call-by-name evaluation consists of allowing every reduction but the \(\mu'\)-rule. The \(\beta\)-reduction holds in the first case and the \(\mu\)-reduction in the second. Formally the call-by-name reduction is \(\sim_{\beta} = \sim_{\mu} \setminus \sim_{\mu'}\).

The call-by-value evaluation consists of prohibiting \(\beta\) and \(\mu\)-reductions in which the argument is a \(\mu\)-abstraction. Formally we define a subset of terms called values by this grammar: \(v ::= x \mid \lambda x.t. \beta_v\) and \(\mu_v\)-reductions are defined instead of generic \(\beta\) and \(\mu\) ones:

\[(\lambda x. u) v \rightarrow_{\beta_v} u[x \leftarrow v] \quad \quad (\mu \alpha. c) v \rightarrow_{\mu_v} \mu \alpha. c[\alpha \leftarrow v \cdot \alpha] \]

The call-by-value reduction \(\sim_{\beta_v}\) is the union of \(\sim_{\gamma}\) for \(\gamma \in \{\beta_v, \mu_v, \mu', \rho, \theta\}\). Critical pairs are then \(\mu'\)-rewritten.

There is another way to define call-by-value into the \(\lambda \mu\)-calculus. The solution is detailed in [Ong and Stewart, 1997]. It consists of restricting the \(\mu'\)-rule to values instead of the \(\mu\):

\[(v) \mu \alpha. c \rightarrow_{\mu'} \mu \alpha. c[\alpha \leftarrow \alpha(v)] \]

Formally \(\sim_{\beta_v}\) becomes the union of \(\sim_{\gamma}\) for \(\gamma \in \{\beta_v, \mu, \mu', \rho, \theta\}\). In fact terms \((\lambda x.t) \mu \alpha. c\) and \((\mu \alpha. c) \mu \alpha'. c'\) are respectively \(\mu'\) and \(\mu\)-reduced because \(\mu \alpha. c\) is not a value in these cases. However, we follow Curien-Herbelin’s call-by-value definition.
We finish this section by a lemma. It is useful for the section 5 simulation theorems. Any command of the form \( e\{\mu \alpha .c\} \) is a redex. However, some cannot be reduced in call-by-name nor in call-by-value.

Lemma 2. \( e\{\mu \alpha .c\} \rightarrow c[\alpha \leftarrow e] \)

Proof. By induction on \( e \).

- if \( e = \beta \) then \( e\{\mu \alpha .c\} = [\beta]\,\mu \alpha .c \xrightarrow{\rho} c[\alpha \leftarrow \beta] \)
- if \( e = \beta (t) \) then \( e\{\mu \alpha .c\} = [\beta](t)\,\mu \alpha .c \xrightarrow{\mu'} [\beta]\,\mu \alpha .c[\alpha \leftarrow \alpha (t)] \xrightarrow{\rho} c[\alpha \leftarrow \alpha (t)][\alpha \leftarrow \beta] = c[\alpha \leftarrow \beta (t)] \)
- if \( e = t \cdot h \) then \( e\{\mu \alpha .c\} = h\{(\mu \alpha .c)\,t\} \xrightarrow{\mu} h\{(\mu \alpha .c)[\alpha \leftarrow t \cdot \alpha]\} \xrightarrow{\rho} c[\alpha \leftarrow t \cdot \alpha][\alpha \leftarrow h] = c[\alpha \leftarrow t \cdot h] \)

This lemma does not hold in call-by-name for the \( \beta (t) \) induction case because no \( \mu' \)-rule is allowed. It holds in call-by-value if \( t \) is a value for the \( h \cdot t \) induction case.

3 \( \lambda \mu \tilde{\mu} \)-calculus

The \( \lambda \mu \tilde{\mu} \)-calculus has the same relation against \( \text{LK} \) as the \( \lambda \mu \)-calculus against \( \text{NK} \). Reductions of \( \lambda \mu \tilde{\mu} \)-calculus correspond to the cut elimination steps in \( \text{LK} \) as well as the \( \lambda \mu \)-reductions correspond to the \( \text{NK} \)-normalisation. We follow the definition given in [Curien and Herbelin, 2000]. Firstly, we present the grammar of the \( \lambda \mu \tilde{\mu} \)-calculus. Secondly, we present the simple type system. Thirdly, we present generic reductions and their call-by-name and call-by-value projections.

The \( \lambda \mu \tilde{\mu} \)-calculus is basically composed of terms, commands and contexts\(^1\). They are defined by mutual induction:

\[
\begin{align*}
t & ::= x \mid \lambda x . t \mid \mu \alpha . c \\
c & ::= (t \mid e) \\
e & ::= \alpha \mid t \cdot e \mid \tilde{\mu} x . c
\end{align*}
\]

As in the \( \lambda \mu \), symbols \( x \) range over \( \lambda \)-variables, symbols \( \alpha \) range over \( \mu \)-variables and symbols \( \lambda, \mu \) and \( \tilde{\mu} \) are binders. Terms are equal modulo \( \alpha \)-equivalence.

This calculus symmetry looks like \( \text{LK} \)'s left/right symmetry. It is confirmed by its system of simple types. This system shares types with the \( \lambda \mu \)-calculus. It shares the same kinds of sequents too. Its rules are:

\(^1\) In [Dougherty et al., 2004] these are referred to respectively callers, callees and capsules. We kept the terminology in [Curien and Herbelin, 2000] that sounds closer to its meaning: terms are programs, contexts are environments and commands represent "a closed system containing both the program and its environment".
both consist of restricting the context construction. The first new grammar is µ
In fact µ is therefore a linear reduction. The contexts that are called stacks. ¯µ
reduction. It is a reduction. It is a \( \beta \)-calculus. The \( \beta \)-rule particular case in which \( e = \beta \).

We present one-step reduction rules. Substitutions inside the \( \lambda \mu \bar{\mu} \)-calculus are supposed to be known. Each rule concerns a command but the \( \theta \)-rule is restricted to the stacks:

\[
\langle \lambda x. u \mid t \cdot e \rangle \rightarrow_{\beta} \langle t \mid \bar{x}. (u \mid e) \rangle
\]

\[
\mu \bar{\delta}. (t \mid \delta) \rightarrow_{\theta} t (\delta \notin t)
\]

µ and \( \bar{\mu} \)-reductions are duals of each other. They correspond to the structural rules in LK. Reductions \( \rightarrow_{=\gamma} \) and \( \rightarrow_{\neq \gamma} \) have the same meanings as in the \( \lambda \mu \)-calculus. The \( \beta \)-rule is a mere term modification without term duplication. It is therefore a linear reduction. The \( \theta \)-reduction is linear too. There is no \( \rho \)-reduction. It is a µ-rule particular case in which \( e = \beta \).

This system is not deterministic. There is a single critical pair \( \langle \mu a.c \mid \bar{x}.x.d \rangle \).

It can be both µ or \( \bar{\mu} \)-rewritten so that Church-Rosser’s property does not hold.

In fact \( \langle \mu a .(x \mid y \cdot \alpha) \mid \bar{x}.x.(z \mid x \cdot \beta) \rangle \) is µ-rewritten as \( (x \mid y \cdot \bar{x}.x.(z \mid x \cdot \beta)) \) and is µ-rewritten as \( (x \mid y \cdot \bar{x}.x.(z \mid x \cdot \alpha \cdot \beta)) \). These are two different normal forms.

Call-by-name and call-by-value disciplines still deal with this problem. They both consist of restricting the context construction. The first new grammar is called \( \lambda \mu \bar{\mu}_T \) and the second is called \( \lambda \mu \bar{\mu}_Q \).

The call-by-name evaluation consists of restricting the µ-rule to a subset of contexts that are called stacks. \( \lambda \mu \bar{\mu}_T \)-grammar is:

\[
t ::= x \mid \lambda x. t \mid \mu a.c \quad e ::= \langle t \mid e \rangle
\]

The \( \mu_n \)-rule is restricted to the stacks:

\[
\langle \mu a.c \mid s \rangle \rightarrow_{\mu_n} c[\alpha \leftarrow s]
\]

Call-by-name reduction \( \rightarrow_{\gamma} \) is the union of \( \rightarrow_{=\gamma} \) for \( \gamma \in \{\beta, \mu_n, \bar{\mu}, \theta\} \). The critical pair can then only be µ-rewritten. This reduction was proved confluent and stable in the \( \lambda \mu \bar{\mu}_T \)-calculus in [Curien and Herbelin, 2000].

The call-by-value oriented grammar consists of allowing the \( t \cdot e \) context construction only for values. \( \lambda \mu \bar{\mu}_Q \)-grammar is:

\[
t ::= x \mid \lambda x. t \mid \mu a.c \quad v ::= x \mid \lambda x. t \quad c ::= \langle t \mid e \rangle
\]

The \( \bar{\mu}_n \)-rule is restricted to values:

\[
\langle v \mid \bar{x}.x.c \rangle \rightarrow_{\bar{\mu}_n} c[x \leftarrow v]
\]

Call-by-value reduction \( \rightarrow_{\gamma} \) is the union of \( \rightarrow_{=\gamma} \) for \( \gamma \in \{\beta, \mu, \bar{\mu}_n, \theta\} \). The command \( \langle \mu a.c \mid \mu a'.c' \rangle \) can then only be µ-rewritten. This reduction was proved confluent and stable in the \( \lambda \mu \bar{\mu}_Q \)-calculus in [Curien and Herbelin, 2000].
The $\beta'$-rule contracts as shortcut for both a linear $\beta$-rule and a $\tilde{\mu}$-rule:

$$\langle \lambda x. u \mid t \cdot e \rangle \rightarrow_{\beta'} \langle u[x \leftarrow t] \mid e \rangle$$

This $\beta'$-rule is obviously compatible with the call-by-name evaluation. It is also compatible with the call-by-value because $t$ is a value by definition of $\overline{\lambda\mu}\tilde{\mu}Q$.

4 Translations between $\lambda\mu$ and $\overline{\lambda\mu}\tilde{\mu}$-calculi

We define a translation $(\cdot)^\dagger$ from $\lambda\mu$ to $\overline{\lambda\mu}\tilde{\mu}$. It extends that of Curien-Herbelin to the $\lambda\mu$-contexts. We define backwards a translation $(\cdot)^\flat$ from $\overline{\lambda\mu}\tilde{\mu}$ to $\lambda\mu$. We prove properties about their compatibilities with the simple type system and about their compositions.

Definition 3. Application $(\cdot)^\dagger$ maps any $\lambda\mu$-term $t$, command $c$ and context $e$ respectively to a $\overline{\lambda\mu}\tilde{\mu}$-term, command and context. $(\cdot)^\dagger$ is defined by induction on $t$, $c$ and $e$:

$$t^\dagger = \begin{cases} 
  x & \text{if } t = x \\
  \lambda x. u^\dagger & \text{if } t = \lambda x. u \\
  \mu\beta.(v^\dagger \mid \tilde{\mu}y.(u^\dagger \mid y \cdot \beta)) & \text{if } t = (u)v \quad (\ast) \\
  \mu\alpha.c^\dagger & \text{if } t = \mu\alpha.c
\end{cases}$$

$$c^\dagger = [\alpha]t^\dagger = \langle t^\dagger \mid [\alpha] \rangle$$

$$e^\dagger = \begin{cases} 
  \alpha & \text{if } e = \alpha \\
  \tilde{\mu}y.(t^\dagger \mid y \cdot \beta) & \text{if } e = \beta(t) \quad (\ast\ast) \\
  t^\dagger \cdot h^\dagger & \text{if } e = t \cdot h
\end{cases}$$

Condition $(\ast)$ requires that variables $y$ and $\beta$ have no free occurrence in $u$ neither in $v$. Condition $(\ast\ast)$ requires that $y \notin t$. A straightforward induction leads us to state that $t$ and $t^\dagger$ have the same free variables set.

It seems more natural to translate $(u)v$ by $\mu\beta.(v^\dagger \mid u^\dagger \cdot \beta)$. This shorter term corresponds in LK to the arrow elimination rule in NK too. But it would not be compatible with the call-by-value evaluation. For example, $(x)\mu\alpha.c$ would be translated as $\mu\beta.(x \mid \mu\alpha.c^\dagger \cdot \beta)$ in this case. It can not be reduced by any rule in the $\overline{\lambda\mu}\tilde{\mu}$-calculus. However, $(x)\mu\alpha.c$ can be $\mu'$-reduced in the $\lambda\mu$-calculus.

$(u)v$ should be translated as $\mu\beta.(u^\dagger \mid \tilde{\mu}y.(v^\dagger \mid \tilde{\mu}x.(y \mid x \cdot \beta)))$ with Ong and Stewart’s call-by-value definition in [Ong and Stewart, 1997].

We show that translation $(\cdot)^\dagger$ is compatible with the type system. If a typing environment for a term $t$ exists, it holds for $t^\dagger$.

Lemma 3. $\Gamma \vdash t : A \mid \Delta \implies \Gamma \vdash t^\dagger : A \mid \Delta$

Proof. By a straightforward induction on $t$. We show the less than obvious cases.
- if \( t = (u \cdot v) \) then \( t^\circ = \mu \beta. \langle u^\circ \mid y \cdot \beta \rangle \) and

\[
\Gamma \vdash u^\circ : A \rightarrow B \mid \Delta \quad \frac{\cfrac{\Gamma \vdash v : A \mid \Delta \quad \cfrac{\Gamma \vdash \mu y. v : A \mid \Delta} \quad (u^\circ \mid y \cdot \beta) \circ (\Gamma)\{\Delta,\beta \circ B \mid \beta \circ B \}}}{(u^\circ \mid y \cdot \beta) \circ (\Gamma)\{\Delta,\beta \circ B \mid \beta \circ B \}} \]

\[
\Gamma, \Gamma' \vdash \mu \beta. \langle u^\circ \mid y \cdot \beta \rangle ; B \mid \Delta, \Delta'
\]

- if \( e = \beta(t) \) then \( e^\circ = \tilde{\mu} y. \langle t^\circ \mid y \cdot \beta \rangle \) and

\[
\Gamma \vdash t^\circ : (A \rightarrow B) \mid \Delta \quad \frac{\cfrac{\Gamma \vdash y : B \mid \Delta \quad \cfrac{\Gamma \vdash \mu \delta. c : A \mid \Delta} \quad (\Gamma)\{\Delta,\beta \circ B \mid \beta \circ B \}}}{(t^\circ \mid y \cdot \beta) \circ (\Gamma)\{\Delta,\beta \circ B \mid \beta \circ B \}} \]

\[
\Gamma, \Gamma' \vdash \mu \delta. c : B \mid \Delta, \Delta'
\]

**Definition 4.** Application \((\cdot)^\circ\) maps backwards any \( \lambda \mu \tilde{\mu} \)-term \( t \) to a \( \lambda \mu \)-term. **Definition 4** is used to translate any \( \lambda \mu \tilde{\mu} \)-command \( e \). Definition of the \( \lambda \mu \)-contexts is used to map the \( \lambda \mu \tilde{\mu} \)-contexts \( e \) as well. \((\cdot)^\circ\) is built by induction on \( t \), \( c \) and \( e \):

\[
t^\circ = \begin{cases} x & \text{if } t = x \\
\lambda x. u^\circ & \text{if } t = \lambda x. u \\
\mu \alpha. e^\circ & \text{if } t = \mu \alpha. e \end{cases}
\]

\[
e^\circ = \begin{cases} e & \text{if } e = \alpha \\
\e^\circ \cdot h^\circ & \text{if } e = t \cdot h \\
\beta(\lambda x. \mu \delta. e^\circ) & \text{if } e = \tilde{\mu} x. c \quad (\ast) \end{cases}
\]

Condition \((\ast)\) requires that \( \delta \notin \text{free}(c) \). \( t \) and \( t^\circ \) have the same free variables set. Application \((\cdot)^\circ\) is compatible with the type system too.

**Lemma 4.** \( \Gamma \vdash t : A \mid \Delta \implies \Gamma \vdash t^\circ : A \mid \Delta \)

**Proof.** By a straightforward induction on \( t \). We give two cases.

- if \( e = \langle t \mid e \rangle \) then \( e^\circ = e^\circ \{t^\circ\} \)

\[
\Gamma \vdash t : A \mid \Delta \quad \frac{\cfrac{\Gamma \vdash t = e\cdot \alpha^\circ : A \mid \Delta \quad \cfrac{\Gamma \vdash \mu \delta. e : A \mid \Delta \quad \cfrac{\Gamma \vdash \mu \delta. e \circ A : A \mid \Delta} \quad \cfrac{\Gamma \vdash \mu \delta. e} \quad \text{lem. 3}}}{\Gamma \vdash e^\circ = \beta(\lambda x. \mu \delta. e^\circ) \mid \beta(\lambda x. \mu \delta. e^\circ) \circ A \mid \Delta, \beta \circ B \mid \beta \circ B}}}{\Gamma \vdash e^\circ : (\Gamma)\{\Delta, \beta \circ B \mid \beta \circ B \}} \]

- if \( e = \tilde{\mu} x. c \) then \( e^\circ = \beta(\lambda x. \mu \delta. e^\circ) \) and

\[
\Gamma \vdash \mu \delta. e \circ A : A \mid \Delta \quad \frac{\cfrac{\Gamma \vdash e \circ A : A \mid \Delta \quad \cfrac{\Gamma \vdash \mu \delta. e : A \mid \Delta} \quad \cfrac{\Gamma \vdash \mu \delta. e \circ A} \quad \text{lem. 3}}}{\Gamma \vdash e^\circ : (\Gamma)\{\Delta, \beta \circ B \mid \beta \circ B \}} \]

\[
\Gamma \vdash \mu \delta. e \circ A : A \mid \Delta \quad \frac{\cfrac{\Gamma \vdash \mu \delta. e : A \mid \Delta} \quad \cfrac{\Gamma \vdash \mu \delta. e \circ A} \quad \text{lem. 3}}}{\Gamma \vdash e^\circ : (\Gamma)\{\Delta, \beta \circ B \mid \beta \circ B \}} \]
We focus on properties about the composition of $\top$ and $\circ$. We want to state that $t^\top \circ t = t$ and that $t \circ t^\top = t$ for any term. But it is not the case, these results hold modulo linear reductions.

**Theorem 1.** $t^\top \sim t$

**Proof.** By a straightforward induction on $t$. Every case is obtained successively by expanding definitions 3, 1, 4 and by applying the induction hypothesis. We give the case which uses linear reductions additionally.

- if $t = (u)v$ then

\[
(u)v^\top = \mu \beta.\langle \mu y.\langle u^\top | y \cdot \beta \rangle \rangle^\top
\]

\[
= \mu \beta.\langle \lambda y.\mu \delta.\langle \beta \rangle (u^\top) y \rangle v^\top
\]

\[
\sim \mu \beta.\langle \lambda y.\mu \delta.\langle \beta \rangle (u) y \rangle v
\]

\[
\sim_{\beta} \mu \beta.\langle \beta \rangle (u) v
\]

\[
\sim_{\theta} \mu \beta.\langle \beta \rangle (u) v
\]

We prove two lemmas before stating backwards that $\circ t^\top$ is the identity modulo linear reductions. The first lemma is useful to prove the second.

**Lemma 5.** $\langle t_0 \top t_1 \ldots t_n \top | e \rangle \sim \langle t_0 \top | t_1 \top \ldots t_n \top | e \rangle$

**Proof.** By induction on $n$.

- if $n = 0$ then it is obvious

- if $n = m + 1$ then

\[
\langle t_0 \top t_1 \ldots t_m t_{m+1} \top | e \rangle = \langle \mu \beta.\langle t_m+1 \top | \mu y.\langle t_0 \top t_1 \ldots t_m \top | y \cdot \beta \rangle | e \rangle
\]

\[
\sim_{\mu} \langle t_m+1 \top | \mu y.\langle t_0 \top t_1 \ldots t_m \top | y \cdot e \rangle \rangle
\]

\[
\sim_{\mu} \langle t_0 \top t_1 \ldots t_m \top | t_{m+1} \top | e \rangle
\]

\[
\sim \langle t_0 \top | t_1 \top \ldots t_m \top \cdot t_{m+1} \top | e \rangle
\]

The second lemma shows how to map a definition command.

**Lemma 6.** $\langle t^\top | e^\top \rangle \sim \langle t^\top | e^\top \rangle$

**Proof.** By induction on $e$.

- if $e = \alpha$ then it is obvious by definitions 1 and 2

- if $e = \beta(u)$ then

\[
\beta(u)\langle t^\top \rangle = \langle \beta \rangle (u)^\top
\]

\[
= \langle \mu \gamma.\langle t^\top | \mu y.\langle u^\top | y \cdot \gamma \rangle \rangle \rangle | \beta
\]

\[
\sim_{\mu} \langle t^\top | \mu y.\langle u^\top | y \cdot \beta \rangle \rangle
\]

\[
= \langle t^\top | \beta(u)^\top \rangle
\]
Theorem 2. \( t^\dagger \sim^* t \)

Proof. By induction on \( t \). We apply definitions 3, 4 successively and the induction hypothesis. We give a typical case and another which needs either the previous lemma or linear reductions.

- if \( e = u \cdot h \) then
  \[
  u \cdot h(t)^t = h((t) u)^\dagger \\
  \sim^* (t) (u^t | h^t) \\
  \sim^* (t^t | u^t, h^t) \\
  = (t^t | u \cdot h^t)
  \]

5 Simulations between \( \lambda \mu \) and \( \bar{\lambda} \mu \bar{\mu} \)-calculi

We want to prove that the \( \lambda \mu \)-calculus simulates and is simulated backwards by the \( \bar{\lambda} \mu \bar{\mu} \)-calculus. We focus on the undirected evaluation. Call-by-name and call-by-value are drawn from this.

We begin with the simulation of the \( \lambda \mu \) by the \( \bar{\lambda} \mu \bar{\mu} \). The next four lemmas show results of a \( \lambda \mu \)-substitution after a \( \beta \), \( \mu \), \( \mu' \) and \( \rho \)-reduction. Each proof consists successively of

- expanding the \( \lambda \mu \)-substitution
- expanding the definition of \( ( )^t \)
- applying the induction hypothesis if necessary
- factorising the \( \bar{\lambda} \mu \bar{\mu} \)-substitution
- factorising the definition of \( ( )^t \)

We give basic cases and those which use lemmas additionally for any proof.

Lemma 7. \( t[x \leftarrow u]^t = t^t[x \leftarrow u^t] \)

Proof. By induction on \( t \).

- if \( t = x \) then \( x[x \leftarrow u]^t = u^t = x^t[x \leftarrow u^t] \)
- if $t = y$ then $y[x ← u]^\dagger = y = y^\dagger[x ← u^\dagger]$
- if $t = (v) w$ then
  \[(v) w[x ← u]^\dagger = (v[x ← u]) w[x ← u]^\dagger = \mu \beta, (w[x ← u]^\dagger \mid \mu \gamma, (v[x ← u] \mid y \cdot \beta)) = \mu \beta, (w^\dagger \mid \mu \gamma, (v^\dagger \mid y \cdot \beta) [x ← u^\dagger]) = (v) w^\dagger[x ← u^\dagger]
  \]

**Lemma 8.** $t[\alpha ← u \cdot \alpha]^\dagger \rightarrow^* t[\alpha ← u^\dagger \cdot \alpha]$  

**Proof.** By induction on $t$.
- if $t = (a) b$ then
  \[(a) b[\alpha ← u \cdot \alpha]^\dagger = (a[\alpha ← u \cdot \alpha]) b[\alpha ← u \cdot \alpha]^\dagger = \mu \beta, (b[\alpha ← u \cdot \alpha]^\dagger \mid \mu \gamma, (a[\alpha ← u \cdot \alpha]^\dagger \mid y \cdot \beta)) \rightarrow^* \mu \beta, (b^\dagger [\alpha ← u^\dagger \cdot \alpha] \mid \mu \gamma, (a^\dagger [\alpha ← u^\dagger \cdot \alpha] \mid y \cdot \beta)) = \mu \beta, (b^\dagger \mid \mu \gamma, (a^\dagger \mid y \cdot \beta)) [\alpha ← u^\dagger \cdot \alpha] = (a) b^\dagger [\alpha ← u^\dagger \cdot \alpha]
  \]
- if $c = [\alpha] w$ then
  \[[\alpha] w[\alpha ← u \cdot \alpha]^\dagger = [\alpha] w[\alpha ← u \cdot \alpha]^\dagger = u \cdot \alpha \{w[\alpha ← u \cdot \alpha]^\dagger\}^\dagger = \{w[\alpha ← u \cdot \alpha]^\dagger\}^\dagger [u^\dagger \cdot \alpha] \rightarrow^* \{w^\dagger [\alpha ← u^\dagger \cdot \alpha] \mid u^\dagger \cdot \alpha\} = \{w^\dagger \mid \alpha ← u^\dagger \cdot \alpha\} = [\alpha] w^\dagger [\alpha ← u^\dagger \cdot \alpha]
  \]

**Lemma 9.** $t[\alpha ← u(\alpha)]^\dagger \rightarrow^* t[\alpha ← \mu \gamma, (y \mid y \cdot \alpha)] u^\dagger$

**Proof.** By induction on $t$.
- if $t = (a) b$ then
  \[(a) b[\alpha ← u(\alpha)]^\dagger = (a[\alpha ← u(\alpha)]) b[\alpha ← u(\alpha)]^\dagger = \mu \beta, (b[\alpha ← u(\alpha)]^\dagger \mid \mu \gamma, (a[\alpha ← u(\alpha)]^\dagger \mid y \cdot \beta)) \rightarrow^* \mu \beta, (b^\dagger [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)] \mid \mu \gamma, (a^\dagger [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)] \mid y \cdot \beta)) = \mu \beta, (b^\dagger \mu \gamma, (a^\dagger \mu \gamma, (u^\dagger \mid y \cdot \alpha) \mid y \cdot \beta)) [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)] = (a) b^\dagger [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)]
  \]
- if $c = [\alpha] w$ then
  \[[\alpha] w[\alpha ← u(\alpha)]^\dagger = [\alpha] (w[\alpha ← u(\alpha)]) u^\dagger = \{w[\alpha ← u(\alpha)] \mid \alpha\} = \{w[\alpha ← u(\alpha)] \mid u^\dagger \cdot \alpha\} \rightarrow^* \{w^\dagger [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)] \mid u^\dagger \cdot \alpha\} = \{w^\dagger \mid \alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)\} = [\alpha] w^\dagger [\alpha ← \mu \gamma, (u^\dagger \mid y \cdot \alpha)]
  \]
Lemma 10. \( t[\alpha \leftarrow \beta]^\dagger = t^\dagger[\alpha \leftarrow \beta] \)

Proof. By induction on \( t \).

- if \( c = [\alpha] u \) then
  \[ [\alpha] u[\alpha \leftarrow \beta]^\dagger = [\beta] u[\alpha \leftarrow \beta]^\dagger \]
  \[ = \langle u[\alpha \leftarrow \beta]^\dagger \mid \beta \rangle \]
  \[ = \langle u^\dagger[\alpha \leftarrow \beta] \mid \beta \rangle \]
  \[ = \langle u^\dagger \mid \alpha \rangle[\alpha \leftarrow \beta] \]
  \[ = [\alpha] u^\dagger[\alpha \leftarrow \beta] \]

Theorem 3 (simulation of the \( \lambda\mu \)-calculus by the \( \lambda\mu\nu \)-calculus).

\[ t \rightarrow_\gamma v \implies \exists u \ t^\dagger \xrightarrow{\star} u \land v^\dagger \xrightarrow{\star} u \]

Proof. By cases on \( \gamma \).

- if \( \gamma = \beta \) then
  \[ (\lambda x.u)^v^\dagger = [\beta] \langle v^\dagger \mid \tilde{\mu} y.\langle \lambda x.u^\dagger \mid y \cdot \beta \rangle \rangle \]
  \[ \xrightarrow{\beta} [\beta] \langle v^\dagger \mid \tilde{\mu} y.\langle \lambda x.u^\dagger \mid y \cdot \beta \rangle \rangle \]
  \[ \xrightarrow{\tilde{\mu}} [\beta] \langle v^\dagger \mid \tilde{\mu} x.\langle u^\dagger \mid \beta \rangle \rangle \]
  \[ \xrightarrow{\theta} u^\dagger[x \leftarrow v^\dagger] \]
  \[ = u[x \leftarrow v^\dagger] \]

- if \( \gamma = \mu \) then
  \[ (\mu x.c)^v^\dagger = [\mu] \langle v^\dagger \mid \tilde{\mu} y.\langle \mu x.c^\dagger \mid y \cdot \beta \rangle \rangle \]
  \[ \xrightarrow{\mu} [\mu] \langle v^\dagger \mid \tilde{\mu} x.\langle c^\dagger \mid x \cdot \alpha \rangle \rangle \]
  \[ \xrightarrow{\mu} [\mu] \langle v^\dagger \mid \tilde{\mu} y.\langle c^\dagger \mid y \cdot \alpha \rangle \rangle \]
  \[ \approx [\mu] \langle v^\dagger \mid \alpha \rangle[\alpha \leftarrow v^\dagger \cdot \alpha]^\dagger \]

- if \( \gamma = \mu' \) then
  \[ (v) \mu x.c^v^\dagger = [\mu] \langle v^\dagger \mid \tilde{\mu} y.\langle v^\dagger \mid y \cdot \beta \rangle \rangle \]
  \[ \xrightarrow{\mu} [\mu] \langle v^\dagger \mid \tilde{\mu} y.\langle v^\dagger \mid y \cdot \alpha \rangle \rangle \]
  \[ \approx [\mu] \langle v^\dagger \mid \alpha \rangle[\alpha \leftarrow v^\dagger \cdot \alpha]^\dagger \]

- if \( \gamma = \rho \) then
  \[ [\beta] \mu x.c^v^\dagger = [\beta] \mu x.c^\dagger \]
  \[ \xrightarrow{\rho} [\beta] \mu x.c^\dagger \]
  \[ \approx [\beta] \mu x.c^\dagger \]

- if \( \gamma = \theta \) then \( \mu \delta.\langle t^\dagger \mid \delta \rangle \xrightarrow{\theta} t^\dagger \)

Corollary 1 (call-by-name case). \( t \rightarrow_n v \implies \exists u \ t^\dagger \xrightarrow{\star_n} u \land v^\dagger \xrightarrow{\star_n} u \)
Proof. By cases on $\beta$ and $\mu$-rules.

$(\lambda x.u) v$ is $\beta$-reduced in call-by-name without any restriction. It is simulated in the $\lambda\mu\tilde{\mu}$-calculus by a $\tilde{\mu}$-reduction. The latter is in call-by-name without any restriction too.

$(\mu\alpha.c) v$ is $\mu$-reduced in call-by-name without any restriction. It is simulated in the $\lambda\mu\tilde{\mu}$-calculus by a $\mu$-reduction. The latter is in call-by-name if $v^\dagger \cdot \alpha$ is a stack. It is the case by definition.  

Corollary 2 (call-by-value case). $t \rightarrow v \iff \exists u . t^\dagger \* u \land v^\dagger \sim\!_v u$

Proof. By cases on $\beta$, $\mu$ and $\mu'$-rules.

$(\lambda x.u) v$ is $\beta$-reduced in call-by-value if $v$ is a value. It is simulated in the $\lambda\mu\tilde{\mu}$-calculus by a $\mu$-reduction. The latter is in call-by-value if $v^\dagger$ is a value. It is the case by the definition of $\lambda\mu\tilde{\mu}_Q$.

$(\mu\alpha.c) v$ is $\mu$-reduced in call-by-value if $v$ is a value. It is simulated in the $\lambda\mu\tilde{\mu}$-calculus by a $\mu$-reduction. The latter is in call-by-value without any restriction.

$(v) \mu\alpha.c$ is $\mu'$-reduced in call-by-value without any restriction. It is simulated in the $\lambda\mu\tilde{\mu}$-calculus by a $\mu$-reduction. The latter is in call-by-value without any restriction as well.

The $\lambda\mu\tilde{\mu}$-simulation by the $\lambda\mu$-calculus requires preliminary lemmas showing that $(\cdot)^\circ$ commutes over $\lambda\mu$ and $\lambda\mu\tilde{\mu}$-substitutions. Each proof consists of

- expanding the $\lambda\mu\tilde{\mu}$-substitution
- expanding the definition of $(\cdot)^\circ$
- applying the induction hypothesis if necessary
- factorising the $\lambda\mu$-substitution
- factorising the definition of $(\cdot)^\circ$

Lemma 11. $t[x \leftarrow u]^\circ = t^\circ[x \leftarrow u^\circ]$

Proof. By induction on $t$.

- if $t = x$ then $x[x \leftarrow u]^\circ = u^\circ = x^\circ[x \leftarrow u^\circ]$
- if $t = y$ then $y[x \leftarrow u]^\circ = y^\circ[x \leftarrow u^\circ]$
- if $t = \langle t \mid e \rangle$ then

$$
\langle t \mid e \rangle[x \leftarrow u]^\circ = \langle t[x \leftarrow u] \mid e[x \leftarrow u] \rangle^\circ = e[x \leftarrow u]^\circ \{ t[x \leftarrow u]^\circ \} = e^\circ[x \leftarrow u^\circ] \{ t^\circ[x \leftarrow u^\circ] \} = e^\circ[t^\circ][x \leftarrow u^\circ] = (t \mid e)^\circ[x \leftarrow u^\circ]
$$

Lemma 12. $t[\alpha \leftarrow h]^\circ = t^\circ[\alpha \leftarrow h^\circ]$

Proof. By induction on $t$. 
\[
\begin{align*}
\text{if } c = \langle t \mid e \rangle \text{ then } \\
\langle t \mid e \rangle[\alpha \leftarrow h] = \langle t[\alpha \leftarrow h] \mid e[\alpha \leftarrow h] \rangle \\
= e[\alpha \leftarrow h] \cup \{t[\alpha \leftarrow h] \}
\end{align*}
\]

\[
\begin{align*}
\text{if } c = \alpha \text{ then } \alpha \cup \{t \rangle[\alpha \leftarrow h] \}
\end{align*}
\]

\[
\begin{align*}
\text{if } c = \beta \text{ then } \beta \cup \{t \rangle[\alpha \leftarrow h] \}
\end{align*}
\]

Theorem 4 (simulation of the \(\bar{\lambda}\mu\bar{\mu}\)-calculus by the \(\lambda\mu\)-calculus).

\[
t \rightarrow_{\gamma} v \implies \exists u \ t \overset{*}{\rightarrow} \ u \overset{*}{\rightarrow} v
\]

Proof. By cases on \(\gamma\).

- if \(\gamma = \beta'\) then

\[
\langle \lambda x. u \mid v \cdot e \rangle = v \cdot e \cdot \{\lambda x. u\}
\]

- if \(\gamma = \mu\) then

\[
\langle \mu \alpha. c \mid e \rangle = \mu \cdot \{\mu \alpha. c\}
\]

- if \(\gamma = \tilde{\mu}\) then

\[
\langle t \mid \tilde{\mu} x. c \rangle = \langle \beta \rangle (\lambda x. \mu \delta. c\rangle) \ t
\]

Corollary 3 (call-by-name case). \(t \rightarrow_{\gamma} v \implies \exists u \ t \overset{*}{\rightarrow} \ u \overset{*}{\rightarrow} v\)

Proof. By cases on \(\beta', \mu\) and \(\tilde{\mu}\)-rules.

\(\langle \lambda x. u \mid v \cdot e \rangle\) is \(\beta'\)-reduced in call-by-name without any restriction. It is simulated in the \(\lambda\mu\)-calculus by a \(\beta\)-reduction. The latter is in call-by-name without any restriction too.

\(\langle \lambda x. u \mid v \cdot e \rangle\) is \(\beta'\)-reduced in call-by-name if \(e \neq \tilde{\mu} x. c\) else it were \(\tilde{\mu}\)-reduced. It is simulated in the \(\lambda\mu\)-calculus with the help of lemma 2. The latter is in call-by-name if \(e \neq \beta(t)\) i.e. if \(e \neq \tilde{\mu} x. c\). It is the case by definition 2.

\(\langle t \mid \tilde{\mu} x. c \rangle\) is \(\tilde{\mu}\)-reduced in call-by-name without any restriction. It is simulated in the \(\lambda\mu\)-calculus by a \(\beta\)-reduction. The latter is in call-by-name without any restriction as well.
Corollary 4 (call-by-value case). \( t \rightarrow v \quad \Rightarrow \quad \exists u \quad t \overset{\beta'}{\rightarrow} v \quad u \overset{\mu}{\rightarrow} v^o \)

Proof. By cases on \( \beta' \), \( \mu \) and \( \tilde{\mu} \)-rules.

\( \langle \lambda x. u | v \cdot e \rangle \) is \( \beta' \)-reduced in call-by-value if \( v \) is a value. It is simulated in the \( \lambda \mu \)-calculus by a \( \beta \)-reduction. The latter is in call-by-value if \( v^o \) is a value. It is the case by definition 4.

\( \langle \mu \alpha. c | e \rangle \) is \( \mu \)-reduced in call-by-value if \( e \) is either a \( \mu \)-variable or a context of the form \( v \cdot h \) where \( v \) is a value or a \( \mu \)-abstraction by the definition of \( \tilde{\lambda} \mu \tilde{\mu} Q \). It is simulated in the \( \lambda \mu \)-calculus with the help of lemma 2. The latter is in call-by-value if \( v^o \) is a value in a context of the form \( h^o \cdot v^o \) i.e. if \( v \) is a value in a \( v \cdot h \) context. It is the case by definition 4.

\( \langle t | \tilde{\mu} x. c \rangle \) is \( \tilde{\mu} \)-reduced in call-by-value if \( t \) is a value. It is simulated in the \( \lambda \mu \)-calculus by a \( \beta \)-reduction. The latter is in call-by-value if \( t^o \) is a value. It is the case by definition 4.

6 Conclusion

Analysis of the \( \lambda \mu \) and \( \tilde{\lambda} \mu \tilde{\mu} \)-calculi has shown their computational equivalence. It holds for undirected evaluations of pure calculi (see theorems 3 and 4). This result is then easily obtained for call-by-name and call-by-value evaluations (see corollaries 1, 2, 3 and 4). It concerns the simple type system too (see lemmas 3 and 4).

The simulation of the \( \tilde{\lambda} \mu \tilde{\mu} \)-calculus by the \( \lambda \mu \)-calculus is smoother than the simulation of the \( \lambda \mu \)-calculus by the \( \tilde{\lambda} \mu \tilde{\mu} \)-calculus. The first is obtained with the help of linear reductions whereas the second is obtained with the help of linear expansions.

This work can be extended in three ways. The first consists of proving the same results for the call-by-value evaluation of the \( \lambda \mu \)-calculus defined in [Ong and Stewart, 1997]. The second consists of defining CPS translations to \( \lambda \)-calculus in order to complete [Curien and Herbelin, 2000]. The third consists of extending the type system to the other logical constants.

References


