Test vectors for trilinear forms when at least one representation is not supercuspidal

Mladen Dimitrov, Louise Nyssen

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Mladen Dimitrov * Louise Nyssen †

May 4, 2010

Abstract

Given three irreducible, admissible, infinite dimensional complex representations of $\mathrm{GL}_2(F)$, with F a local field, the space of trilinear functionals invariant by the group has dimension at most one. When it is one we provide an explicit vector on which the functional does not vanish assuming that not all three representations are supercuspidal.

1 Introduction

1.1 What is a test vector?

Let F be a local non-Archimedean field with ring of integers $\mathcal{O}$, uniformizing parameter π and finite residue field. Let V_1, V_2 and V_3 be three irreducible, admissible, infinite dimensional complex representations of $G = \mathrm{GL}_2(F)$ with central characters ω_1, ω_2 and ω_3 and conductors n_1, n_2 and n_3. Using the theory of Gelfand pairs, Dipendra Prasad proves in [P] that the space of G-invariant linear forms on $V_1 \otimes V_2 \otimes V_3$, with G acting diagonally, has dimension at most one and gives a precise criterion for this dimension to be one, that we will now explain.

Let $D^\times$ be the group of invertible elements of the unique quaternion division algebra D over F, and denote by R its unique maximal order. When V_i is a discrete series representation of G, denote by V_i^D the irreducible representation of $D^\times$ associated to V_i by the Jacquet-Langlands correspondence. Again, by the theory of Gelfand pairs, the space of $D^\times$-invariant linear forms on $V_1^D \otimes V_2^D \otimes V_3^D$ has dimension at most one.

A necessary condition for the existence of a non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$ or a non-zero $D^\times$-invariant linear form on $V_1^D \otimes V_2^D \otimes V_3^D$, that we will *always assume*, is that

$$\omega_1\omega_2\omega_3 = 1. \tag{1}$$

Theorem 1. *([P, Theorem 1.4],[P2, Theorem 2]) Let $\epsilon(V_1 \otimes V_2 \otimes V_3) = \pm 1$ denote the root number of the corresponding 8-dimensional symplectic representation of the Weil-Deligne group of F. When all the V_i's are supercuspidal, assume either that F has characteristic zero or that its residue characteristic is odd.*

*Institut de Mathématiques de Jussieu, Université Paris 7, UFR Mathématiques Site Chevaleret, Case 7012, 75205 Paris Cedex 13, France dimitrov@math.jussieu.fr

†Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2, CC 051, Place Eugène Bataillon, 34095 Montpellier Cedex lnyssen@math.univ-montp2.fr

0 *Mathematics Subject Classification (2000): 11F70*

Then $\epsilon(V_1 \otimes V_2 \otimes V_3) = 1$ if, and only if, there exists a non-zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, and $\epsilon(V_1 \otimes V_2 \otimes V_3) = -1$ if, and only if, all the V_i's are discrete series representations of G and there exists a non-zero $D^\times$-invariant linear form ℓ' on $V_1^D \otimes V_2^D \otimes V_3^D$.

Given a non-zero G-invariant linear form ℓ on $V_1 \otimes V_2 \otimes V_3$, our goal is to find a *pure tensor* in $V_1 \otimes V_2 \otimes V_3$ which is not in the kernel of ℓ. We call such a pure tensor a *test vector*.

Let v_i denote a new vector in V_i (see section 2.1). The following results are due to Dipendra Prasad and Benedict Gross. They show that tensor products of new vectors can sometimes be test vectors.

Theorem 2. *(i) ([P, Theorem 1.3]) If all the V_i's are unramified principal series, then $v_1 \otimes v_2 \otimes v_3$ is a test vector.*

(ii) ([GP, Proposition 6.3]) Suppose that for $1 \leq i \leq 3$, V_i is a twist of the Steinberg representation by an unramified character η_i. Then

- *either, $\eta_1\eta_2\eta_3(\pi) = -1$ and $v_1 \otimes v_2 \otimes v_3$ is a test vector.*
- *or, $\eta_1\eta_2\eta_3(\pi) = 1$ and the line in $V_1^D \otimes V_2^D \otimes V_3^D$ fixed by $R^\times \times R^\times \times R^\times$ is not in the kernel of ℓ'.*

However, as mentioned in [GP, Remark 7.5], new vectors do not always yield test vectors. Suppose, for example, that V_1 and V_2 are unramified, whereas V_3 is ramified, and denote by $K = \mathrm{GL}_2(\mathcal{O})$ the standard maximal compact subgroup of G. Since v_1 and v_2 are K-invariant and ℓ is G-equivariant, $v \mapsto \ell(v_1 \otimes v_2 \otimes v)$ defines a K-invariant linear form on V_3. In the meantime, V_3 and its contragredient are ramified, and therefore the above linear form has to be zero. In particular $\ell(v_1 \otimes v_2 \otimes v_3) = 0$. To go around this obstruction for new vectors to be test vectors, Gross and Prasad make a suggestion, which is the object of our first result:

Theorem 3. *If V_1 and V_2 are unramified and V_3 has conductor $n \geq 1$, then $\gamma^n \cdot v_1 \otimes v_2 \otimes v_3$ and $v_1 \otimes \gamma^n \cdot v_2 \otimes v_3$ are both test vectors, where $\gamma = \begin{pmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.*

In general we want to exhibit a test vector as an *explicit privileged* G-orbit inside the $G \times G \times G$-orbit of $v_1 \otimes v_2 \otimes v_3$, where G sits diagonally in $G \times G \times G$. Before stating our main result, let us explain a more general and systematic approach in the search for test vectors.

1.2 The tree for G

The vertices of the tree are in bijection with maximal open compact subgroups of G (or equivalently with lattices in F^2, up to homothetie) and its edges correspond to Iwahori subgroups of G, each Iwahori being the intersection of the two maximal compact subgroups sitting at the ends of the edge. Every Iwahori being endowed with two canonical $(\mathcal{O}/\pi)^\times$-valued characters, choosing one of those characters amounts to choosing an orientation on the corresponding edge. The standard Iwahori subgroup $I = I_1$ corresponds to the edge between K and $\gamma K \gamma^{-1}$, and changing the orientation on this edge amounts to replacing the character $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I \mapsto (d \mod \pi)$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I \mapsto (a \mod \pi)$.

More generally, for $n \geq 1$, the n-th standard Iwahori subgroup

$$I_n = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \pi^n \mathcal{O} & \mathcal{O}^\times \end{pmatrix}$$

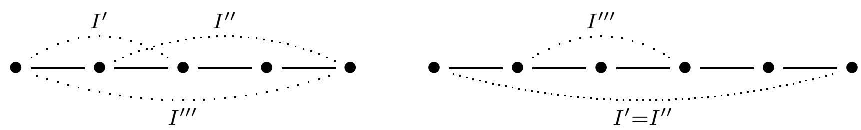
corresponds to the path between K and $\gamma^n K \gamma^{-n}$, the set of Iwahori subgroups of depth n is in bijection with the set of paths of length n on the tree, and choosing an orientation on such a path amounts to choosing one of the two $(\mathcal{O}/\pi^n)^\times$-valued characters of the corresponding Iwahori.

The new vector v_i is by definition a non-zero vector in the unique line of V_i on which I_{n_i} acts by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \omega_i(d)$. Clearly, for every $n \geq 1$, G acts transitively on the set of oriented paths of length n. Hence finding a G-orbit inside the $G\times G\times G$-orbit of $v_1 \otimes v_2 \otimes v_3$, amounts to finding a G-conjugacy class $I' \times I'' \times I'''$ inside the $G\times G\times G$-conjugacy class of $I_{n_1} \times I_{n_2} \times I_{n_3}$.

A most natural way of defining such a G-conjugacy class (almost uniquely) is by imposing the smallest of the three compact open subgroups to be the intersection of the two others.

For instance, the test vector $\gamma^n \cdot v_1 \otimes v_2 \otimes v_3$ in Theorem 3 corresponds to the G-conjugacy class of $\gamma^n K \gamma^{-n} \times K \times I_n$. The linear form on V_3 given by $v \mapsto \ell(\gamma^n \cdot v_1 \otimes v_2 \otimes v)$ is invariant by $\gamma^n K \gamma^{-n} \cap K = I_n$, hence belongs to the new line in the contragredient of V_3.

Visualized on the tree, the condition on the three compact open subgroups means that the longest path should be exactly covered by the two others, as shown on each of the following two pictures.



We would like to thank Dipendra Prasad for having shared this point of view with us.

1.3 Main result

Given an admissible representation V of G and a character η of $F^\times$, we let $V \otimes \eta$ denote the representation of G on the same space V with action multiplied by $\eta \circ \det$, called the twist of V by η.

If η_1, η_2 and η_3 are three characters of $F^\times$ such that $\eta_1\eta_2\eta_3 = 1$, then the G-representations $V_1 \otimes V_2 \otimes V_3$ and $(V_1 \otimes \eta_1) \otimes (V_2 \otimes \eta_2) \otimes (V_3 \otimes \eta_3)$ are identical, therefore

$$\mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) = \mathrm{Hom}_G((V_1 \otimes \eta_1) \otimes (V_2 \otimes \eta_2) \otimes (V_3 \otimes \eta_3), \mathbb{C}). \tag{2}$$

Hence finding a test vector in $V_1 \otimes V_2 \otimes V_3$ amounts to finding one in $(V_1 \otimes \eta_1) \otimes (V_2 \otimes \eta_2) \otimes (V_3 \otimes \eta_3)$ for some choice of characters η_1, η_2 and η_3 such that $\eta_1\eta_2\eta_3 = 1$. We would like to exhibit a test vector in the $G \times G \times G$-orbit of $v_1' \otimes v_2' \otimes v_3'$, where v_i' denotes a new vector in $V_i \otimes \eta_i$, and we want it to be fixed by an open compact subgroup as large as possible. Therefore the conductors of $V_i \otimes \eta_i$ should be as small as possible.

Denote by $n_i^{\min}$ the minimal possible value for the conductor of $V_i \otimes \eta$, when η varies. Finally, let $n^{\min}$ denote the minimal possible value of

$$\mathrm{cond}(V_1 \otimes \eta_1) + \mathrm{cond}(V_2 \otimes \eta_2) + \mathrm{cond}(V_3 \otimes \eta_3),$$

when (η_1, η_2, η_3) runs over all possible triples of characters such that $\eta_1\eta_2\eta_3 = 1$. Note that because of the latter condition, the inequality $n^{\min} \geq n_1^{\min} + n_2^{\min} + n_3^{\min}$ is strict in general.

Also note that the conductor of a representation is at least equal to the conductor of it's central character. Equality holds if, and only if, the representation is principal and has minimal conductor among it's twists.

Definition 1.1. (i) The representation V_i is *minimal* if $n_i = n_i^{\min}$.

(ii) The triple of representations (V_1, V_2, V_3) satisfying (1) is *minimal* if

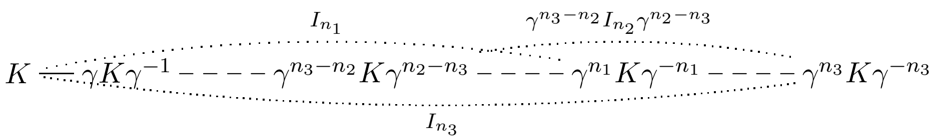
(a) either all non-supercuspidal V_i's are minimal,

(b) or none of the V_i's is supercuspidal and $n^{\min} = n_1 + n_2 + n_3$.

It is clear from the definition that for any V_1, V_2 and V_3, there exist characters η_1, η_2 and η_3 such that $\eta_1\eta_2\eta_3 = 1$ and $(V_1 \otimes \eta_1, V_2 \otimes \eta_2, V_3 \otimes \eta_3)$ is minimal. Our main result states:

Theorem 4. *Suppose that at least one of V_1, V_2 and V_3 is not supercuspidal, and that if two amongst them are supercuspidal with the same conductor then the third one is a ramified principal series. Assume that (V_1, V_2, V_3) is minimal and $\epsilon(V_1 \otimes V_2 \otimes V_3) = 1$. If $n_3 \geq n_1$ and $n_3 \geq n_2$, then $v_1 \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3$ and $\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3$ are both test vectors.*

Remark 1.2. The test vector $v_1 \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3$ can be visualized on the tree as follows:



Remark 1.3. Assume that (V_1, V_2, V_3) is minimal and that at least one of V_i's is not supercuspidal. Then $\epsilon(V_1 \otimes V_2 \otimes V_3) = -1$ if, and only if, one of the representations, say V_1, is a twist of the Steinberg representation by an unramified character η and V_2 is a discrete series whose contragredient is isomorphic to V_3 twisted by η (see [P, Propositions 8.4, 8.5, 8.6]).

Remark 1.4. Finding test vectors in the case when all the V_i's are supercuspidal remains an open question. Consider for example the case when the V_i's have trivial central characters and share the same conductor n. It is well known that the Atkin-Lehner involution $\left(\begin{smallmatrix} 0 & 1 \\ \pi^n & 0 \end{smallmatrix}\right)$ acts on v_i by the root number $\epsilon(V_i) = \pm 1$. It follows that if $\epsilon(V_1)\epsilon(V_2)\epsilon(V_3) = -1$, then $\ell(v_1 \otimes v_2 \otimes v_3) = 0$.

If V_1 is unramified and V_2, V_3 are supercuspidal of even conductor n, trivial central characters and $\epsilon(V_2)\epsilon(V_3) = -1$, then by applying the Atkin-Lehner involution one sees that $\ell(\gamma^{n/2} v_1 \otimes v_2 \otimes v_3) = 0$. Similarly, if V_1 is the Steinberg representation and V_2, V_3 are supercuspidal of odd conductor n, trivial central characters and $\epsilon(V_2)\epsilon(V_3) = 1$, then by applying the Atkin-Lehner involution one sees that $\ell(\gamma^{(n-1)/2} v_1 \otimes v_2 \otimes \cdot v_3) = 0$.

1.4 Application of test vectors to subconvexity

Test vectors for trilinear forms play an important role in various problems involving L-functions of triple products of automorphic representations of GL(2).

One such problem, studied by Bernstein-Reznikov in [BR1, BR2] and more recently by Michel-Venkatesh in [MV1, MV2], is about finding *subconvexity* bounds for the L-functions of automorphic representations of GL(2) along the critical line. More precisely, given a unitary automorphic representation Π of $\mathrm{GL}(N)$ over a number field E, the subconvexity bound asserts the existence of an absolute constant $\delta > 0$ such that :

$$L(\Pi, 1/2) \ll_{E,N} C(\Pi)^{1/4-\delta},$$

where $C(\Pi)$ denotes the analytic conductor of Π. We refer to [MV2] for the definition of $C(\Pi)$ and for various applications of subconvexity bounds to problems in number theory, such as Hilbert's eleventh problem. Let us just mention that the subconvexity bounds follow from the Lindelöff Conjecture, which is true under the Generalized Riemann Hypothesis.

In [MV2, 1.2] the authors establish the following subconvexity bound for $\mathrm{GL}(2) \times \mathrm{GL}(2)$:

$$L(\Pi_1 \otimes \Pi_2, 1/2) \ll_{E, C(\Pi_2)} C(\Pi_1)^{1/2-\delta},$$

and obtain as a corollary subconvexity bounds for GL(1) and GL(2). A key ingredient in their proof is to provide a test vector in the following setup: let F be the completion of E at a finite place and denote by V_i the local component of Π_i at F $(i = 1, 2)$. Let V_3 be a *minimal* principal series representation of $G = \mathrm{GL}_2(F)$ such that (1) is fulfilled, and denote by ℓ a normalized G-invariant trilinear form on $V_1 \otimes V_2 \otimes V_3$(the process of normalization is explained in [MV2, 3.4]). Then one needs to find a norm 1 test vector $v \otimes v' \otimes v'' \in V_1 \otimes V_2 \otimes V_3$ such that

$$\ell(v \otimes v' \otimes v'') \gg_{n_2} n_1^{-1/2},$$

which can be achieved either by using the test vectors from our main theorem, or by a direct computation in the Kirillov model as in [MV2, 3.6.1].

1.5 Organization of the paper

In section 2 we recall basic facts about induced admissible representations of G which are used in section 3 to prove Theorem 3 and a slightly more general version of Theorem 4 in the case when at most one of the representations is supercuspidal. Section 4 recalls some basic facts about Kirillov models and contains a proof of Theorem 4 in the case of two supercuspidal representations. Finally, in section 5 we study test vectors in reducible induced representation, as initiated in the work of Harris and Scholl [HS].

1.6 Acknowledgments

We would like to thank Philippe Michel for suggesting the study of this problem, and of course Benedict Gross and Dipendra Prasad for their articles full of inspiration. The second named author would like to thank also Paul Broussous and Nicolas Templier for many interesting conversations, and Wen-Ching Winnie Li for the opportunity to spend one semester at Penn State University, where the first draft of this paper was written. Finally, the first named author would like to thank Dipendra Prasad for several helpful discussions that took place during his visit of the Tata Institute of Fundamental Research in december 2008, as well as the institution for its hospitality and excellent conditions for work.

2 Background on induced admissible representations of G

2.1 New vectors and contragredient representation

Let V be an irreducible, admissible, infinite dimensional representation of G with central character ω. To the descending chain of open compact subgroups of G

$$K = I_0 \supset I = I_1 \supset \cdots \supset I_n \supset I_{n+1} \cdots$$

one can associate an ascending chain of vector spaces for $n \geq 1$:

$$V^{I_n,\omega} = \left\{ v \in V \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = \omega(d) v \text{ , for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_n \right\}.$$

Put $V^{I_0,\omega} = V^K$. There exists a minimal n such that the vector space $V^{I_n,\omega}$ is non-zero. It is necessarily one dimensional, called the new line, and any non-zero vector in it is called a *new vector* of V (see [C]). The integer n is the *conductor* of V. The representation V is said to be *unramified* if $n = 0$.

The contragredient representation $\widetilde{V}$ is the space of smooth linear forms φ on V, where G acts as follows:

$$\forall g \in G, \qquad \forall v \in V, \qquad (g \cdot \varphi)(v) = \varphi(g^{-1} \cdot v).$$

There is an isomorphism $\widetilde{V} \simeq V \otimes \omega^{-1}$, hence $\widetilde{V}$ and V have the same conductor n. Moreover, under this isomorphism the new line in $\widetilde{V}$ is sent to:

$$\left\{ v \in V \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v = \omega(a) v \text{ , for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_n \right\},$$

which is the image of the new line in V by the Atkin-Lehner involution $\left(\begin{smallmatrix} 0 & 1 \\ \pi^n & 0 \end{smallmatrix}\right)$.

2.2 Induced representations

Let (ρ, W) be a smooth representation of a closed subgroup H of G. Let Δ_H be the modular function on H. The induction of ρ from H to G, denoted $\mathrm{Ind}_H^G \rho$, is the space of functions f from G to W satisfying the following two conditions:

(i) for all $h \in H$ and $g \in G$ we have $f(hg) = \Delta_H(h)^{-\frac{1}{2}} \rho(h) f(g)$;

(ii) there exists an open compact subgroup K_f of G such that for all $k \in K_f$ and $g \in G$ we have $f(gk) = f(g)$.

The action of G is by right translation: for all $g, g' \in G, (g \cdot f)(g') = f(g'g)$. With the additional condition that f must be compactly supported modulo H, one gets the *compact induction* denoted by ind_H^G. When G/H is compact, there is no difference between Ind_H^G and ind_H^G.

Let B be the Borel subgroup of upper triangular matrices in G, and let T be the diagonal torus. The character Δ_T is trivial and we will use $\Delta_B = \delta^{-1}$ with $\delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |\frac{a}{d}|$ where $|\ |$ is the norm on F. The quotient $B \backslash G$ is compact and can be identified with $\mathbb{P}^1(F)$.

Let μ and μ' be two characters of $F^\times$ and χ be the character of B given by

$$\chi \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} = \mu(a) \mu'(d).$$

The next two sections are devoted to the study of new vectors in $V = \mathrm{Ind}_B^G(\chi)$.

2.3 New vectors in principal series representations

Assume that $V = \mathrm{Ind}_B^G(\chi)$ is a principal series representation of G, that is $\mu'\mu^{-1} \neq |\cdot|^{\pm 1}$. Then V has central character $\omega = \mu\mu'$ and conductor $n = \mathrm{cond}(\mu) + \mathrm{cond}(\mu')$. Let v denote a new vector in V.

When V is unramified the function $v : G \to \mathbb{C}$ is such that for all $b \in B$, $g \in G$ and $k \in K$

$$v(bgk) = \chi(b)\delta(b)^{\frac{1}{2}} v(g),$$

whereas, if V is ramified, then for all $b \in B$, $g \in G$ and $k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_n$,

$$v(bgk) = \chi(b)\delta(b)^{\frac{1}{2}} \omega(d) v(g).$$

We normalize v so that $v(1) = 1$ and put

$$\alpha^{-1} = \mu(\pi)|\pi|^{\frac{1}{2}} \qquad \text{and} \qquad \beta^{-1} = \mu'(\pi)|\pi|^{-\frac{1}{2}}.$$

Lemma 2.1. *If V is unramified then for all $r \geq 0$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} & \text{, if } k \in I_s \setminus I_{s+1} \text{ for } 0 \leq s \leq r-1, \\ \alpha^r & \text{, if } k \in I_r. \end{cases}$$

Similarly for $r \geq 1$,

$$(\gamma^r \cdot v - \alpha\gamma^{r-1} \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} - \alpha^{s+1}\beta^{r-1-s} & \text{, if } k \in I_s \setminus I_{s+1}, 0 \leq s \leq r-1, \\ 0 & \text{, if } k \in I_r, \end{cases}$$

$$\textit{and } (\gamma^r \cdot v - \beta\gamma^{r-1} \cdot v)(k) = \begin{cases} \alpha^r(1 - \frac{\beta}{\alpha}) & \text{, if } k \in I_r, \\ 0 & \text{, if } k \in K \setminus I_r. \end{cases}$$

Proof: If $k \in I_r$, then $\gamma^{-r}k\gamma^r \in K$, so $(\gamma^r \cdot v)(k) = \alpha^r v(\gamma^{-r}k\gamma^r) = \alpha^r$. Suppose that $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \setminus I_{s+1}$ for some $0 \leq s \leq r-1$. Then $\pi^{-s}c \in \mathcal{O}^\times$ and

$$(\gamma^r \cdot v)(k) = \alpha^r v \begin{pmatrix} a & \pi^r b \\ \pi^{-r}c & d \end{pmatrix} = \alpha^r v \begin{pmatrix} (ad-bc)\pi^{r-s} & a \\ 0 & \pi^{-r}c \end{pmatrix} = \alpha^s \beta^{r-s}.$$

The second part of the lemma follows by a direct computation. □

For the rest of this section we assume that V is ramified, that is $n \geq 1$. We put

$$m = \mathrm{cond}(\mu') \qquad \text{so that} \qquad n - m = \mathrm{cond}(\mu).$$

By [C, pp.305-306] the restriction to K of a new vector v is supported by the double coset of $\left(\begin{smallmatrix} 1 & 0 \\ \pi^m & 1 \end{smallmatrix}\right)$ modulo I_n. In particular if μ' is unramified ($m = 0$), then v is supported by $I_n \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) I_n = I_n \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) I_n = K \setminus I$.

If $1 \leq m \leq n-1$, then v is supported by $I_n \left(\begin{smallmatrix} 1 & 0 \\ \pi^m & 1 \end{smallmatrix}\right) I_n = I_m \setminus I_{m+1}$.

If μ is unramified, then v is supported by I_n. We normalize v so that $v\left(\begin{smallmatrix} 1 & 0 \\ \pi^m & 1 \end{smallmatrix}\right) = 1$.

Lemma 2.2. *Suppose that μ is unramified and μ' is ramified. Then, for all $r \geq 0$ and $k \in K$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r}, \\ 0 & , \text{ otherwise.} \end{cases}$$

$$\Big(\gamma^r \cdot v - \alpha^{-1}\gamma^{r+1} \cdot v\Big)(k) = \begin{cases} \alpha^r \mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in I_{n+r} \setminus I_{n+r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof: For $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ we have

$$\alpha^{-r}(\gamma^r \cdot v)(k) = v(\gamma^{-r} k \gamma^r) = v \begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix}.$$

It is easy to check that for every $s \geq 1$,

$$K \cap B\gamma^r I_s \gamma^{-r} = I_{s+r}. \tag{3}$$

It follows that $\gamma^r \cdot v$ has its support in I_{n+r}. If $k \in I_{n+r}$ then $c \in \pi^{m+r}\mathcal{O}^\times$ for some $m \geq n$, $d \in \mathcal{O}^\times$ and we have the following decomposition:

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} \det k & \pi^{-m} cb \\ 0 & \pi^{-m-r} cd \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^m & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & \pi^{m+r} c^{-1} \end{pmatrix}. \tag{4}$$

Hence

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu(\det k)\mu'(\pi^{-m-r} cd)(\mu\mu')(\pi^{m+r} c^{-1}) = \mu'(d).$$

□

Lemma 2.3. *Suppose that μ' is unramified and μ is ramified. Then for all $r \geq 0$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^s \beta^{r-s} \mu\left(\frac{\det k}{\pi^{-s} c}\right) & , \text{ if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_s \setminus I_{s+1}, \text{ with } 0 \leq s \leq r, \\ 0 & , \text{ if } k \in I_{r+1}. \end{cases}$$

Moreover, if $r \geq 1$, then

$$\Big(\gamma^r \cdot v - \beta\gamma^{r-1} \cdot v\Big)(k) = \begin{cases} \alpha^r \mu\left(\frac{\det k}{\pi^{-r} c}\right) & , \text{ if } k = \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in I_r \setminus I_{r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

Proof: We follow the pattern of proof of lemma 2.2. The restriction of $\gamma^r \cdot v$ to K is zero outside

$$K \cap B\gamma^r (K \setminus I)\gamma^{-r} = K \setminus I_{r+1}.$$

For $0 \leq s \leq r$ and $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_s \setminus I_{s+1}$ we use the following decomposition:

$$\begin{pmatrix} a & \pi^r b \\ \pi^{-r} c & d \end{pmatrix} = \begin{pmatrix} -\frac{\det k}{\pi^{-r} c} & a + \frac{\det k}{\pi^{-r} c} \\ 0 & \pi^{-r} c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 + \frac{d}{\pi^{-r} c} \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Since $d \in \mathcal{O}$ and $\pi^r c^{-1} \in \mathcal{O}$ we deduce that:

$$\alpha^{-r}(\gamma^r \cdot v)(k) = \mu\Big(\frac{\det k}{\pi^{-r}c}\Big)\mu'(-\pi^{-r}c)\left|\pi^r c^{-1}\right| = \mu\Big(\frac{\det k}{\pi^{-s}c}\Big)\alpha^{s-r}\beta^{r-s}.$$

$\square$

For the sake of completeness, we mention one more result. We omit the proof, since it will not be used in sequel of this paper.

Lemma 2.4. *If μ and μ' are both ramified ($0 < m < n$), then for all $r \geq 0$ and $k \in K$,*

$$(\gamma^r \cdot v)(k) = \begin{cases} \alpha^r \mu\Big(\frac{\det k}{\pi^{-(m+r)}c}\Big)\mu'(d) & , \text{ if } k = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in I_{m+r} \setminus I_{m+r+1}, \\ 0 & , \text{ otherwise.} \end{cases}$$

2.4 New vectors in special representations

In this section, we will assume that $\mathrm{Ind}_B^G(\chi)$ is reducible, that is $\frac{\mu'}{\mu} = |\cdot|^{\pm 1}$.

2.4.1 Case $\frac{\mu'}{\mu} = |\cdot|$

In this case, there exists a character η of $F^\times$ such that $\mu = \eta|\cdot|^{-\frac{1}{2}}$ and $\mu' = \eta|\cdot|^{\frac{1}{2}}$. The representation $\mathrm{Ind}_B^G((\eta \circ \det)\delta^{-\frac{1}{2}})$ has length 2 and has one irreducible one dimensional subspace, generated by the function $\eta \circ \det$. When η is trivial the quotient is called the Steinberg representation, denoted St. More generally, the quotient is isomorphic to $\eta \otimes \mathrm{St}$ and is called a special representation. There is a short exact sequence

$$0 \to \eta \otimes \mathbb{C} \to \mathrm{Ind}_B^G((\eta \circ \det)\delta^{-\frac{1}{2}}) \xrightarrow{\mathrm{proj}} \eta \otimes \mathrm{St} \to 0. \tag{6}$$

The representation $\eta \otimes \mathrm{St}$ is minimal if, and only if, η is unramified. Then the subspace of K-invariant vectors in $\mathrm{Ind}_B^G((\eta \circ \det)\delta^{-\frac{1}{2}})$ is the line $\eta \otimes \mathbb{C}$ with basis $\eta \circ \det$. Since

$$K = I \sqcup (B \cap K)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I$$

there exists v^I (resp. $v^{K\setminus I}$) in $\mathrm{Ind}_B^G((\eta \circ \det)\delta^{-\frac{1}{2}})$ taking value 1 (resp. 0) on I and 0 (resp. 1) on $K \setminus I$. Both v^I and $v^{K\setminus I}$ are I-invariant and $v^I + v^{K\setminus I}$ is K-invariant. Hence $\mathrm{proj}(v^I) = -\mathrm{proj}(v^{K\setminus I})$ is a new vector in $\eta \otimes \mathrm{St}$ whose conductor is 1.

Let us compute $\gamma^r \cdot v^I$ as a function on G. As in section 2.3, put

$$\alpha^{-1} = \mu(\pi)|\pi|^{\frac{1}{2}} = \eta(\pi) \qquad \text{and} \qquad \beta^{-1} = \mu'(\pi)|\pi|^{-\frac{1}{2}} = \eta(\pi).$$

Lemma 2.5. *For all $r \geq 0$, we have* $(\gamma^r \cdot v^I)(k) = \begin{cases} \alpha^r & , \text{ if } k \in I_{r+1}, \\ 0 & , \text{ if } k \in K \setminus I_{r+1}. \end{cases}$

Proof: By (3), we have $K \cap B\gamma^r I \gamma^{-r} = I_{r+1}$, hence $\gamma^r \cdot v^I$ vanishes on $K \setminus I_{r+1}$.

For $k \in I_{r+1}$, $\gamma^{-r}k\gamma^r \in I$, hence $\gamma^r \cdot v^I(k) = \alpha^r v^I(\gamma^{-r}k\gamma^r) = \alpha^r$. $\square$

2.4.2 Case $\frac{\mu'}{\mu} = |\cdot|^{-1}$

The notations and results from this section will only be used in section 5. There exists a character η of $F^\times$ such that $\mu = \eta|\cdot|^{\frac{1}{2}}$ and $\mu' = \eta|\cdot|^{-\frac{1}{2}}$. The representation $\mathrm{Ind}_B^G((\eta \circ \det)\delta^{\frac{1}{2}})$ has length 2 and the special representation $\eta \otimes \mathrm{St}$ is an irreducible subspace of codimension 1. There is a short exact sequence

$$0 \to \eta \otimes \mathrm{St} \to \mathrm{Ind}_B^G((\eta \circ \det)\delta^{\frac{1}{2}}) \xrightarrow{\mathrm{proj}^*} \eta \otimes \mathbb{C} \to 0. \tag{7}$$

When η is unramified, the space of K invariant vectors in $\mathrm{Ind}_B^G((\eta\circ\det)\delta^{\frac{1}{2}})$ is the line generated by the function v^K taking constant value 1 on K, that is for all b in B and k in K:

$$v^K(bk) = \eta\big(\det(b)\big)\delta(b).$$

We normalize the linear form proj^* by $\mathrm{proj}^*(v^K) = 1$. The function $\gamma \cdot v^K - \eta(\pi)^{-1}v^K$, whose image by proj^* is 0, is a new vector in $\eta \otimes \mathrm{St}$.

Let us compute v^K as functions on G. As in section 2.3, put

$$\alpha^{-1} = \mu(\pi)|\pi|^{\frac{1}{2}} = \eta(\pi)|\pi| \qquad \text{and} \qquad \beta^{-1} = \mu'(\pi)|\pi|^{-\frac{1}{2}} = \eta(\pi)|\pi|^{-1}$$

Lemma 2.6. *For all $r \geq 0$,*

$$(\gamma^r \cdot v^K)(k) = \begin{cases} \alpha^s\beta^{r-s} & \text{, if } k \in I_s \setminus I_{s+1} \text{ for } 0 \leq s \leq r-1, \\ \alpha^r & \text{, if } k \in I_r. \end{cases}$$

Similarly for $r \geq 1$,

$$(\gamma^r \cdot v^K - \alpha\gamma^{r-1} \cdot v^K)(k) = \begin{cases} \alpha^s\beta^{r-s} - \alpha^{s+1}\beta^{r-1-s} & \text{, if } k \in I_s \setminus I_{s+1}, 0 \leq s \leq r-1, \\ 0 & \text{, if } k \in I_r, \end{cases}$$

$$\text{and } (\gamma^r \cdot v^K - \beta\gamma^{r-1} \cdot v^K)(k) = \begin{cases} \alpha^r(1 - \frac{\beta}{\alpha}) & \text{, if } k \in I_r, \\ 0 & \text{, if } k \in K \setminus I_r. \end{cases}$$

It is worth noting that v^K behaves as the new vector in an unramified representation (see Lemma 2.1). The proof is the same.

3 The case when at most one representation is supercuspidal

In this section we prove the following result.

Theorem 5. *Assume that (V_1, V_2, V_3) is minimal, $\epsilon(V_1 \otimes V_2 \otimes V_3) = 1$ and that at most one representation is supercuspidal. Then, up to a permutation of the V_i's, exactly one of the following holds:*

(a) $n_3 > n_1$, $n_3 > n_2$, and $\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3$ and $v_1 \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3$ are both test vectors;

(b) $n_1 = n_2 \geq n_3$, and $v_1 \otimes v_2 \otimes \gamma^i v_3$ is a test vector, for all $0 \leq i \leq n_1 - n_3$.

By symmetry, it is enough to prove in case (a) that $\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.

Lemma 3.1. *Under the assumptions in theorem 5, if n_1, n_2 and n_3 are not all equal, then V_1 and V_2 are non-supercuspidal and minimal.*

Proof: Assume first that we are in case (a), that is $n_3 > n_1$ and $n_3 > n_2$. Since all representations of conductor at most 1 are non-supercuspidal and minimal, we may assume that $n_3 \geq 2$. Moreover by (1):

$$\mathrm{cond}(\omega_3) \leq \max(\mathrm{cond}(\omega_1), \mathrm{cond}(\omega_2)) \leq \max(n_1, n_2) < n_3,$$

hence V_3 is either supercuspidal or non-minimal. Since (V_1, V_2, V_3) is minimal, this proves our claim in this case.

Assume next that we are in case (b), that is $n_1 = n_2 > n_3$. As in previous case, we may assume that $n_1 = n_2 \geq 2$. Then if only one amongst V_1 and V_2 is non-supercuspidal and minimal, say V_1, one would obtain

$$\mathrm{cond}(\omega_1) = n_1 > \max(n_2 - 1, n_3) \geq \max(\mathrm{cond}(\omega_2), \mathrm{cond}(\omega_3)),$$

which is false by (1). Hence the claim. □

If $n_1 = n_2 = n_3$ then we can assume without loss of generality that V_1 and V_2 are non-supercuspidal and minimal. Furthermore, by Theorem 2 one can assume that the V_i's are not all three unramified, nor are all three twists of the Steinberg representation by unramified characters. Finally, if all the three representations have conductor one and if exactly one among them is special, we can assume without loss of generality that this is V_3.

3.1 Choice of models

If V_i is a principal series for $i = 1$ or 2, then by minimality there exist characters μ_i and μ'_i of $F^\times$, at least one of which is unramified, such that $\mu'_i\mu_i^{-1} \neq |\cdot|^{\pm 1}$ and

$$V_i = \mathrm{Ind}_B^G \chi_i \quad \text{, where} \quad \chi_i \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu_i(a)\mu'_i(d).$$

Using the natural isomorphism

$$\mathrm{Ind}_B^G \chi_i \cong \mathrm{Ind}_B^G \chi'_i \text{ , where } \chi'_i \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu'_i(a)\mu_i(d)$$

one can assume that μ_1 and μ'_2 are unramified.

If V_i is a special representation, then by minimality there exists an unramified character η_i such that $V_i = \eta_i \otimes \mathrm{St}$. We put then

$$\mu_i = \eta_i\,|\cdot|^{-\frac{1}{2}}, \quad \mu'_i = \eta_i\,|\cdot|^{\frac{1}{2}} \text{ and } \quad \chi_i = (\eta_i \circ \det)\delta^{-\frac{1}{2}}$$

and choose as model for V_i the exact sequence (6):

$$0 \to \eta_i \otimes \mathbb{C} \to \mathrm{Ind}_B^G(\chi_i) \xrightarrow{\mathrm{proj}_i} V_i \to 0.$$

As new vectors, we choose $v_1 = \mathrm{proj}_1(v_1^I)$ in V_1 and $v_2 = \mathrm{proj}_2(v_2^{K\setminus I})$ in V_2.

3.2 Going down using Prasad's exact sequence

We will now explain how Prasad constructs a non-zero G-invariant linear form on $V_1 \otimes V_2 \otimes V_3$. First, there is a canonical isomorphism:

$$\mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_G(V_1 \otimes V_2, \widetilde{V_3}). \tag{8}$$

Lemma 3.2. *We have*

$$\mathrm{Hom}_G(V_1 \otimes V_2, \widetilde{V_3}) \xrightarrow{\sim} \mathrm{Hom}_G\Big(\mathrm{Res}_G \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2), \widetilde{V_3}\Big),$$

where the restriction is taken with respect to the diagonal embedding of G in $G \times G$.

Proof: This is clear when V_1 and V_2 are principal series. Suppose $V_2 = \eta_2 \otimes \mathrm{St}$. Tensoring the exact sequence (6) for V_2 with the projective G-module V_1 and taking $\mathrm{Hom}_G(\cdot, \widetilde{V_3})$ yields a long exact sequence:

$$0 \to \mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \to \mathrm{Hom}_G\Big(V_1 \otimes \mathrm{Ind}_B^G(\chi_2), \widetilde{V_3}\Big) \to \mathrm{Hom}_G\Big(V_1 \otimes \eta_2, \widetilde{V_3}\Big).$$

By minimality and by the assumption made in the beginning of section 3, we have

$$\mathrm{Hom}_G(V_1 \otimes \eta_2, \widetilde{V_3}) = 0. \tag{9}$$

Hence there is a canonical isomorphism:

$$\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_G\Big(V_1 \otimes \mathrm{Ind}_B^G(\chi_2), \widetilde{V_3}\Big).$$

This proves the lemma when V_1 is principal series. Finally, if $V_1 = \eta_1 \otimes \mathrm{St}$ for some unramified character η_1, then analogously there is a canonical isomorphism:

$$\mathrm{Hom}_G\Big(V_1 \otimes \mathrm{Ind}_B^G(\chi_2), \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_G\Big(\mathrm{Ind}_B^G(\chi_1) \otimes \mathrm{Ind}_B^G(\chi_2), \widetilde{V_3}\Big).$$

□

The action of G on $(B \times B)\backslash(G \times G) \cong \mathbb{P}^1(F) \times \mathbb{P}^1(F)$ has precisely two orbits. The first is the diagonal $\Delta_{B\backslash G}$, which is closed and can be identified with $B\backslash G$. The second is its complement which is open and can be identified with $T\backslash G$ via the bijection:

$$\begin{array}{rcl} T\backslash G & \longrightarrow & \Big(B\backslash G \times B\backslash G\Big) \setminus \Delta_{B\backslash G} \\ Tg & \longmapsto & \left(Bg, B \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) g\right) \end{array}$$

Hence, there is a short exact sequence of G-modules:

$$0 \to \mathrm{ind}_T^G(\chi_1\chi_2') \xrightarrow{\mathrm{ext}} \mathrm{Res}_G \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2) \xrightarrow{\mathrm{res}} \mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}) \to 0. \tag{10}$$

The surjection res is given by the restriction to the diagonal. The injection ext takes a function $h \in \mathrm{ind}_T^G(\chi_1\chi_2')$ to a function $H \in \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2)$ vanishing on $\Delta_{B\backslash G}$, such that for all $g \in G$

$$H\Big(g, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g\Big) = h(g).$$

Applying the functor $\mathrm{Hom}_G\Big(\bullet, \widetilde{V_3}\Big)$ yields a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_G\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) \to \mathrm{Hom}_G\Big(\mathrm{Res}_G \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2), \widetilde{V_3}\Big) \to \\ \to \mathrm{Hom}_G\Big(\mathrm{ind}_T^G\Big(\chi_1\chi_2'\Big), \widetilde{V_3}\Big) \to \mathrm{Ext}_G^1\Big(\mathrm{Ind}_B^G\Big(\chi_1\chi_2\delta^{\frac{1}{2}}\Big), \widetilde{V_3}\Big) \to \cdots \end{aligned} \tag{11}$$

Lemma 3.3. $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}), \widetilde{V_3}) = 0.$

Proof: If, say V_2 is special, then the claim is exactly (9), so we can assume that V_1 and V_2 are both principal series.

Suppose that $\mathrm{Hom}_G(\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}), \widetilde{V_3}) \neq 0$, in particular, V_3 is not supercuspidal.

If V_1 and V_2 are both ramified, this contradicts the minimality assumption, namely that $n^{\min} = n_1 + n_2 + n_3$, since $n_2 = \mathrm{cond}(V_2 \otimes \mu_2^{-1})$ whereas $n_3 > \mathrm{cond}(V_3 \otimes \mu_2)$.

Otherwise, if for example V_1 is unramified, then $n_2 = n_3 > n_1 = 0$ which is impossible by the assumptions in theorem 5. □

By [P, Corollary 5.9] it follows that $\mathrm{Ext}^1_G(\mathrm{Ind}_B^G(\chi_1\chi_2\delta^{\frac{1}{2}}), \widetilde{V_3}) = 0$, hence (11) yields:

$$\mathrm{Hom}_G\Big(\mathrm{Res}_G \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2), \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big). \tag{12}$$

Finally, by Frobenius reciprocity

$$\mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_T\Big(\chi_1\chi_2', \widetilde{V_{3|T}}\Big). \tag{13}$$

Since by (1) the restriction of $\chi_1\chi_2'$ to the center equals ω_3^{-1}, it follows from [W, Lemmes 8-9] that the latter space is one dimensional. Thus, we have five canonically isomorphic lines with corresponding bases:

$$\begin{array}{rcc} 0 \neq \ell & \in & \mathrm{Hom}_G\Big(V_1 \otimes V_2 \otimes V_3, \mathbb{C}\Big) \\ & & \downarrow \wr \\ 0 \neq \psi & \in & \mathrm{Hom}_G\Big(\mathrm{Ind}_B^G(\chi_1) \otimes \mathrm{Ind}_B^G(\chi_2) \otimes V_3, \mathbb{C}\Big) \\ & & \downarrow \wr \\ 0 \neq \Psi & \in & \mathrm{Hom}_G\Big(\mathrm{Res}_G \mathrm{Ind}_{B\times B}^{G\times G}(\chi_1 \times \chi_2), \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ 0 \neq \Phi & \in & \mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\chi_1\chi_2'), \widetilde{V_3}\Big) \\ & & \downarrow \wr \\ 0 \neq \varphi & \in & \mathrm{Hom}_T\Big(\chi_1\chi_2', \widetilde{V_{3|T}}\Big) \end{array} \tag{14}$$

Observe that φ is a linear form on V_3 satisfying:

$$\forall t \in T, \qquad \forall v \in V_3, \qquad \varphi(t\cdot v) = (\chi_1\chi_2')(t)^{-1}\varphi(v). \tag{15}$$

Moreover, for all $v \in \mathrm{Ind}_B^G(\chi_1)$, $v' \in \mathrm{Ind}_B^G(\chi_2)$ and $v'' \in V_3$, we have the formula:

$$\ell(\mathrm{proj}_1(v) \otimes \mathrm{proj}_2(v') \otimes v'') = \psi(v \otimes v' \otimes v'') = \int_{T\backslash G} v(g)v'\Big(\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) g\Big)\varphi(g \cdot v'')dg, \tag{16}$$

where for $i = 1, 2$, proj_i is the map defined in (6), if V_i is special, and identity otherwise.

3.3 Going up

Lemma 3.4. *For all $i \in \mathbb{Z}$, $\varphi(\gamma^i \cdot v_3) \neq 0$.*

Proof: Take any $v_0 \in V_3$ such that $\varphi(v_0) \neq 0$. By smoothness v_0 is fixed by the principal congruence subgroup $\ker(K \to \mathrm{GL}_2(\mathcal{O}/\pi^{s_0}))$, for some $s_0 \geq 0$. Then $\varphi(\gamma^{s_0} \cdot v_0) = (\mu_1\mu_2')(\pi^{s_0})\varphi(v_0) \neq 0$ and $\gamma^{s_0}\cdot v_0$ is fixed by the congruence subgroup

$$I^1_{2s_0} := \left\{ k \in K \middle| k \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\pi^{2s_0}} \right\}.$$

Hence $\varphi(V_3^{I_s^1}) \neq \{0\}$, for all $s \geq 2s_0$. Since I_s/I_s^1 is a finite abelian group, $V_3^{I_s^1}$ decomposes as a direct sum of spaces indexed by the characters of I_s/I_s^1. By (15) and by the fact that $\mu_1\mu_2'$ is unramified, φ vanishes on all summands of $V_3^{I_s^1}$ other than $V_3^{I_s,\omega_3}$ (defined in section 2.1). Hence $\varphi(V_3^{I_s,\omega_3}) \neq \{0\}$. By [C, p.306] the space $V_3^{I_s,\omega_3}$ has the following basis:

$$\left(v_3 \quad , \quad \gamma\cdot v_3 \quad , \ldots , \quad \gamma^{s-n_3}\cdot v_3\right).$$

It follows that $\varphi(\gamma^i\cdot v_3) \neq 0$ for some $i \in \mathbb{Z}$, hence by (15), $\varphi(\gamma^i\cdot v_3) \neq 0$ for all $i \in \mathbb{Z}$.

Note that the claim also follows from the first case in [GP, Proposition 2.6] applied to the split torus T of G. □

Let $n = \max(n_1, n_2, n_3) \geq 1$ and put

$$J_n = \begin{pmatrix} 1 & \mathcal{O} \\ \pi^n \mathcal{O} & 1 \end{pmatrix}.$$

Consider the unique function $h \in \mathrm{ind}_T^G(\chi_1\chi_2')$ which is zero outside the open compact subset TJ_n of $T\backslash G$ and such that for all $b_0 \in \mathcal{O}$ and $c_0 \in \pi^n \mathcal{O}$ we have $h\left(\begin{smallmatrix} 1 & b_0 \\ c_0 & 1 \end{smallmatrix}\right) = 1$.

For every $0 \leq i \leq n - n_3$, J_n fixes $\gamma^i\cdot v_3$.

By definition, the function $g \mapsto h(g)\varphi(g\cdot v_3)$ factors through $G \to T\backslash G$ and by lemma 3.4:

$$\Big(\Phi(h)\Big)(\gamma^i\cdot v_3) = \int_{T\backslash G} h(g)\, \varphi(g\gamma^i\cdot v_3)dg = \varphi(\gamma^i\cdot v_3)\int_{J_n} dk_0 \neq 0. \tag{17}$$

Now, we will compute $H = \mathrm{ext}(h)$ as a function on $G \times G$. Recall that $H : G \times G \to \mathbb{C}$ is the unique function satisfying:

(i) for all $b_1, b_2 \in B$, $g_1, g_2 \in G$, $H(b_1g_1, b_2g_2) = \chi_1(b_1)\chi_2(b_2)\delta^{\frac{1}{2}}(b_1b_2)H(g_1, g_2)$,

(ii) for all $g \in G$, $H(g, g) = 0$ and $H(g, \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) g) = h(g)$.

Since $G = BK$, H is uniquely determined by its restriction to $K \times K$. Following the notations of section 2.3 put

$$\alpha_i^{-1} = \mu_i(\pi)|\pi|^{\frac{1}{2}} \quad \text{and} \quad \beta_i^{-1} = \mu_i'(\pi)|\pi|^{-\frac{1}{2}}.$$

Lemma 3.5. *For all* $k_1 = \begin{pmatrix} * & * \\ c_1 & d_1 \end{pmatrix}$ *and* $k_2 = \begin{pmatrix} * & * \\ c_2 & d_2 \end{pmatrix}$ *in* K *we have*

$$H(k_1, k_2) = \begin{cases} \omega_1(d_1)\omega_2\left(\frac{-\det k_2}{c_2}\right) & , \text{ if } k_1 \in I_n \text{ and } k_2 \in K \setminus I, \\ 0 & , \text{ otherwise.} \end{cases} \tag{18}$$

Proof: By definition $H(k_1, k_2) = 0$ unless there exist $k_0 = \begin{pmatrix} 1 & b_0 \\ c_0 & 1 \end{pmatrix} \in J_n$ such that

$$k_1 k_0^{-1} \in B \qquad \text{and} \qquad k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B,$$

in which case

$$H(k_1, k_2) = \chi_1(k_1 k_0^{-1}) \chi_2\Big(k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) \delta^{\frac{1}{2}}\Big(k_1 k_0^{-1} k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big) h(k_0). \tag{19}$$

From $k_1 k_0^{-1} \in B$, we deduce that $\frac{c_1}{d_1} = c_0 \in \pi^n \mathcal{O}$. From $k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in B$ we deduce that $\frac{d_2}{c_2} = b_0 \in \mathcal{O}$. Since, for $i \in \{1, 2\}$, both c_i and d_i are in $\mathcal{O}$, and at least one is in $\mathcal{O}^\times$ it follows that

$$d_1, c_2 \in \mathcal{O}^\times, \qquad d_2 \in \mathcal{O} \qquad \text{and} \qquad c_1 \in \pi^n \mathcal{O}. \tag{20}$$

Hence $k_1 \in I_n$ and $k_2 \in K \setminus I$. Moreover

$$k_1 k_0^{-1} = \begin{pmatrix} \frac{\det k_1}{d_1 \det k_0} & * \\ 0 & d_1 \end{pmatrix} \text{ and } k_2 k_0^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{-\det k_2}{c_2 \det k_0} & * \\ 0 & c_2 \end{pmatrix}.$$

Since $n \geq n_2$ and $n \geq 1$ we have $\mu_2(\det k_0) = 1$, hence

$$H(k_1, k_2) = \mu_1'(d_1) \mu_2\left(\frac{-\det k_2}{c_2}\right) = \omega_1(d_1) \omega_2\left(\frac{-\det k_2}{c_2}\right). \tag{21}$$

Conversely, if $k_1 \in I_n$ and $k_2 \in K \setminus I$ one can take $k_0 = \begin{pmatrix} 1 & d_2 c_2^{-1} \\ c_1 d_1^{-1} & 1 \end{pmatrix}$. □

Remark 3.6. One can define h and compute the corresponding H for values of n smaller than $\max(n_1, n_2, n_3)$. However, H does not need to decompose as a product of functions of one variable as in the above lemma, and the corresponding element in $V_1 \otimes V_2$ will not be a pure tensor. For example, if $n_3 = 0$ and $n_1 = n_2 \geq 0$, we can take $n = 0$ and put $J_0 = \left(\begin{smallmatrix} 1 & \mathcal{O} \\ \mathcal{O} & 1 \end{smallmatrix}\right) \cap \mathrm{GL}_2(F)$. Then by (19) and (20) one finds that for all $k_1 \in K$ and $k_2 \in K$

$$H(k_1, k_2) = \begin{cases} \frac{\omega_2(-\det k_2)}{\mu_1 \mu_2 |\cdot|(d_1 c_2 - c_1 d_2)} & \text{, if } d_1 \in \mathcal{O}^\times,\ c_2 \in \mathcal{O}^\times \text{ and } d_1 c_2 \neq c_1 d_2; \\ 0 & \text{, otherwise.} \end{cases}$$

Now, we want to express $H \in V_1 \otimes V_2$ in terms of the new vectors v_1 and v_2. Put

$$v_1^* = \begin{cases} \gamma^n \cdot v_1 - \beta_1 \gamma^{n-1} \cdot v_1 & \text{, if } V_1 \text{ is unramified,} \\ \gamma^{n-1} \cdot v_1^I & \text{, if } V_1 \text{ is special,} \\ \gamma^{n-n_1} \cdot v_1 & \text{, otherwise,} \end{cases}$$
$$\text{and } v_2^* = \begin{cases} v_2 - \alpha_2^{-1} \gamma \cdot v_2 & \text{, if } V_2 \text{ is unramified,} \\ v_2^{K \setminus I} & \text{, if } V_2 \text{ is special,} \\ v_2 & \text{, otherwise.} \end{cases} \tag{22}$$

Lemma 3.7. *With the notations of (22), H is a non-zero multiple of $v_1^* \otimes v_2^*$.*

Proof: Both H and $v_1^* \otimes v_2^*$ are elements in $\mathrm{Ind}_{B\times B}^{G\times G}\Big(\chi_1 \times \chi_2\Big)$, hence it is enough to compare their restrictions to $K \times K$. By Lemmas 2.1, 2.2, 2.3, 2.5 and 3.5 both restrictions are supported by $I_n \times (K \setminus I)$.

In order to avoid repetitions or cumbersome notations, we will only give the final result:

$$\begin{aligned} &H = \lambda_1\lambda_2\mu_2(-1)\alpha_1^{n_1-n}(v_1^* \otimes v_2^*) \text{ , where} \\ &\lambda_i = \begin{cases} \left(1 - \frac{\beta_i}{\alpha_i}\right)^{-1} & \text{, if } V_i \text{ is unramified,} \\ 1 & \text{, if } V_i \text{ is ramified.} \end{cases} \end{aligned} \tag{23}$$

If V_i is unramified ($i = 1, 2$), then $\beta_i \neq \alpha_i$ and $\lambda_i \neq 0$. □

Since by definition, for any $v \in V_3$, we have $\psi(H \otimes v) = \Psi(H)(v) = \Phi(h)(v)$, it follows from Lemma 3.7 and (17) that for every i, $0 \leq i \leq n - n_3$:

$$\psi(v_1^* \otimes v_2^* \otimes \gamma^i \cdot v_3) \neq 0. \tag{24}$$

At this stage, we do have an explicit test vector, which is $\mathrm{proj}_1(v_1^*) \otimes \mathrm{proj}_2(v_2^*) \otimes v_3 \in V_1 \otimes V_2 \otimes V_3$. By section 2.4.1 we have :

$$\begin{aligned} \mathrm{proj}_1(v_1^*) &= \begin{cases} \gamma^n \cdot v_1 - \beta_1\gamma^{n-1} \cdot v_1 & \text{, if } V_1 \text{ is unramified,} \\ \gamma^{n-n_1} \cdot v_1 & \text{, otherwise,} \end{cases} \\ \text{and } \mathrm{proj}_2(v_2^*) &= \begin{cases} v_2 - \alpha_2^{-1}\gamma \cdot v_2 & \text{, if } V_2 \text{ is unramified,} \\ v_2 & \text{, otherwise.} \end{cases} \end{aligned} \tag{25}$$

In the next two sections we will simplify it and deduce Theorems 3 and 5.

3.4 Proof of Theorem 3

Suppose that $n_1 = n_2 = 0$, so that $n = \max(n_1, n_2, n_3) = n_3 \geq 1$. Then (24) yields:

$$\ell\Big((\gamma^n \cdot v_1 - \beta_1\gamma^{n-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \neq 0.$$

This expression can be simplified as follows. Consider for $m \geq 0$ the linear form:

$$\psi_m(\bullet) = \ell(\gamma^m \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}.$$

As observed in the introduction, ψ_m is invariant by $\gamma^m K \gamma^{-m} \cap K = I_m$, hence vanishes for $m < n = \mathrm{cond}(\widetilde{V_3})$. Therefore, for $n \geq 2$:

$$\begin{aligned} &\ell\Big((\gamma^n \cdot v_1 - \beta_1\gamma^{n-1} \cdot v_1) \otimes (\gamma \cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \\ &\quad = -\alpha_2\psi_n(v_3) + \beta_1\alpha_2\psi_{n-1}(v_3) + \psi_{n-1}(\gamma^{-1} \cdot v_3) - \beta_1\psi_{n-2}(\gamma^{-1} \cdot v_3) \\ &\quad = -\alpha_2\psi_n(v_3) \\ &\quad = -\alpha_2\ell(\gamma^n \cdot v_1 \otimes v_2 \otimes v_3) \neq 0. \end{aligned}$$

If $n = 1$, only the two terms in the middle vanish and we obtain

$$\alpha_2\ell(\gamma \cdot v_1 \otimes v_2 \otimes v_3) + \beta_1\ell(v_1 \otimes \gamma \cdot v_2 \otimes v_3) \neq 0.$$

Put $w = \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. Then $w\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K$ and $\gamma^{-1}w = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \in \pi K$. Hence:

$$\begin{aligned} \beta_1 \ell(v_1 \otimes \gamma\cdot v_2 \otimes v_3) &= \beta_1 \ell(\gamma\gamma^{-1}w\cdot v_1 \otimes w\gamma\cdot v_2 \otimes w\cdot v_3) \\ &= \beta_1 \omega_1(\pi)\ell\big(\gamma\cdot v_1 \otimes v_2 \otimes w\cdot v_3\big) \\ &= \alpha_1^{-1}\ell(\gamma\cdot v_1 \otimes v_2 \otimes w\cdot v_3). \end{aligned}$$

Therefore
$$\ell\Big(\gamma\cdot v_1 \otimes v_2 \otimes (w\cdot v_3 + \alpha_1\alpha_2 v_3)\Big) \neq 0.$$

In particular
$$\Psi(\gamma\cdot v_1 \otimes v_2) \neq 0.$$

Since $\gamma K \gamma \cap K = I$, $\Psi(\gamma\cdot v_1 \otimes v_2) \in \widetilde{V_3}^{I,\omega_3{}^{-1}}$, cannot vanish on the line ${V_3}^{I,\omega_3}$, which is generated by v_3, and therefore

$$\ell(\gamma\cdot v_1 \otimes v_2 \otimes v_3) = \Psi(\gamma\cdot v_1 \otimes v_2)(v_3) \neq 0.$$

Hence, if $n \geq 1$, $\gamma^n\cdot v_1 \otimes v_2 \otimes v_3$ is a test vector. By symmetry $v_1 \otimes \gamma^n\cdot v_2 \otimes v_3$ is a test vector too. This completes the proof of Theorem 3. □

3.5 End of the proof of Theorem 5

By Theorem 3 we may assume that V_1 or V_2 is ramified.

If V_1 and V_2 are both ramified then Theorem 5 follows directly from (24) and (25).

If V_1 is unramified (24) yields:

$$\ell\Big((\gamma^n\cdot v_1 - \beta_1\gamma^{n-1}\cdot v_1) \otimes v_2 \otimes v_3\Big) \neq 0.$$

Since $n_1 = 0 < n_2$, we are in case (a) of Theorem 5, hence $n_2 < n_3 = n$, which implies $\gamma^{n_3-1}K\gamma^{1-n_3} \cap I_{n_2} = I_{n_3-1}$ and

$$\ell(\gamma^{n_3-1}\cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}^{I_{n_3-1},\omega_3^{-1}} = \{0\}.$$

Therefore $\ell(\gamma^{n_3}\cdot v_1 \otimes v_2 \otimes v_3) \neq 0$, that is $\gamma^{n_3}\cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.

Finally, if V_2 is unramified (24) yields:

$$\ell\Big(\gamma^{n_3-n_1}\cdot v_1 \otimes (\gamma\cdot v_2 - \alpha_2 v_2) \otimes v_3\Big) \neq 0.$$

Since $n_2 = 0 < n_1$, we are in case (a) of Theorem 5, hence $n_1 < n_3 = n$, which implies

$$\ell(\gamma^{n_3-n_1-1}\cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}^{I_{n_3-1},\omega_3^{-1}} = \{0\}.$$

It follows that $\ell(\gamma^{n_3-n_1}\cdot v_1 \otimes \gamma\cdot v_2 \otimes v_3) = \ell(\gamma^{n_3-n_1-1}\cdot v_1 \otimes v_2 \otimes \gamma^{-1}\cdot v_3) = 0$.
Therefore $\ell(\gamma^{n_3-n_1}\cdot v_1 \otimes v_2 \otimes v_3) \neq 0$, that is $\gamma^{n_3-n_1}\cdot v_1 \otimes v_2 \otimes v_3$ is a test vector.
The proof of Theorem 5 is now complete. □

4 Proof of Theorem 4 when two of the representations are supercuspidal

The proof in this case follows the original approach of Prasad [P, page 18]. We are indebted to Paul Broussous who has first obtained and shared with us some of the results described here.

Suppose given V_1, V_2 and V_3 as in theorem 4 and such that exactly two of the V_i's are supercuspidal. The condition (1) forces the representation with the largest conductor V_3 to be supercuspidal and we may assume that V_2 is supercuspidal too, whereas V_1 is minimal.

4.1 Kirillov model for supercuspidal representations

Suppose given an irreducible supercuspidal representation V of G with central character ω. Fix a non-trivial additive character ψ on F of conductor 0. We identify F with the unipotent subgroup N of B and denote by $\psi \boxtimes \omega$ the corresponding character of $NF^\times$. Then the compactly induced representation $\mathrm{ind}_{NF^\times}^B(\psi \boxtimes \omega)$ is naturally isomorphic to the space $\mathcal{C}_c^\infty(F^\times)$ of compactly supported locally constant functions on $F^\times$ on which B acts as follows:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot f(x) = \omega(d)\psi\left(\frac{b}{d}\right) f\left(\frac{ax}{d}\right). \tag{26}$$

It is well known (see [B, §4.7] that the restriction of V to B is irreducible and isomorphic to $\mathrm{ind}_{NF^\times}^B(\psi \boxtimes \omega)$. In other terms there is an unique way to endow the latter with a G-action making it isomorphic to V. Hence the action of B on $\mathcal{C}_c^\infty(F^\times)$ defined in (26) can be uniquely extended to G so that the resulting representation is isomorphic to V. It is called the *Kirillov model* of V, with respect to ψ.

The characteristic function of $\mathcal{O}^\times$ is a new vector in the Kirillov model.

4.2 Choice of models

We first choose a model for V_1. Consider the character χ_1 of B defined by $\chi_1\left(\begin{smallmatrix} a & * \\ 0 & d \end{smallmatrix}\right) = |\frac{a}{d}|^{-\frac{1}{2}}\omega_1(d)$. The claim of the theorem is invariant by unramified twists. By the minimality assumption, after twisting V_1 by an appropriate unramified character (and V_2 by its inverse), we can assume either that $V_1 = \mathrm{Ind}_B^G\chi_1$, or that V_1 is the Steinberg representation. In both cases V_1 is the unique irreducible quotient of $\mathrm{Ind}_B^G\chi_1$.

Lemma 4.1. *The natural inclusion of* $\widetilde{V_1}$ *in* $\mathrm{Ind}_B^G(\chi_1^{-1})$ *induces an isomorphism:*

$$\mathrm{Hom}_G(V_2 \otimes V_3, \widetilde{V_1}) \xrightarrow{\sim} \mathrm{Hom}_G(V_2 \otimes V_3, \mathrm{Ind}_B^G(\chi_1^{-1})).$$

Proof: The lemma is clear if V_1 is a principal series. If V_1 is the Steinberg representation, the condition $\epsilon(V_1 \otimes V_2 \otimes V_3) = 1$ implies that $\mathrm{Hom}_G(V_2 \otimes V_3, \mathbb{C}) = \mathrm{Hom}_G(V_2, \widetilde{V_3}) = 0$. The lemma then follows from the long exact sequence obtained by applying the functor $\mathrm{Hom}_G(V_2 \otimes V_3, \bullet)$ to the short exact sequence (7). □

By Frobenius reciprocity:

$$\mathrm{Hom}_G(V_2 \otimes V_3, \mathrm{Ind}_B^G(\chi_1^{-1})) \xrightarrow{\sim} \mathrm{Hom}_B(V_2 \otimes V_3, \chi_1^{-1}\delta^{\frac{1}{2}}).$$

Let us choose Kirillov models for V_2 (resp. V_3) with respect to ψ (resp. $\overline{\psi}$), so that vectors in V_2 and V_3 are elements in $\mathcal{C}_c^\infty(F^\times)$. For $v' \in V_2$ and $v'' \in V_3$ we define:

$$\Phi(v', v'') = \int_{F^\times} v'(x)v''(x)|x|^{-1}d^\times x. \tag{27}$$

Lemma 4.2. *We have* $0 \neq \Phi \in \mathrm{Hom}_B(V_2 \otimes V_3, \chi_1^{-1}\delta^{\frac{1}{2}})$.

Proof: Since v_2 and v_3 are given by the characteristic function of $\mathcal{O}^\times$, $\Phi(v_2, v_3) = 1 \neq 0$. By (1), Φ respects the central action. Since $\psi\overline{\psi} = 1$, Φ is also equivariant with respect to the action of N. Finally, for any $a \in F^\times$,

$$\Phi((\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \cdot v', (\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \cdot v'') = \int_{F^\times} v'(ax)v''(ax)|x|^{-1}d^\times x = |a|\Phi(v', v'') =$$

$$= (\chi_1^{-1}\delta^{\frac{1}{2}})\,(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix})\,\Phi(v', v'').$$

□

It follows then from [B, Proposition 4.5.5] that for any $v \otimes v' \otimes v'' \in V_1 \otimes V_2 \otimes V_3$ we have

$$\ell(v \otimes v' \otimes v'') = \int_K v(k)\Phi(k \cdot v', k \cdot v'')dk. \tag{28}$$

4.3 The case of unequal conductors

In this subsection we assume that $n_2 \neq n_3$, so $n_2 < n_3$. Since V_1 is minimal, it follows then from (1) that $n_1 < n_3$.

We first show that $\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3$ is a test vector. Since $\Phi(v_2, v_3) \neq 0$ by lemma 4.2, it follows that $0 \neq \ell(\bullet \otimes v_2 \otimes v_3) \in \widetilde{V_1}^{I_{n_3}, \omega_1^{-1}}$, hence there exists $0 \leq i \leq n_3 - n_1$ such that $\ell(\gamma^i \cdot v_1 \otimes v_2 \otimes v_3) \neq 0$. Now, for every $0 \leq i < n_3 - n_1$, we have

$$I_{n_3-1} \subset \gamma^i I_{n_1} \gamma^{-i} \cap I_{n_2}$$

hence

$$\ell(\gamma^i \cdot v_1 \otimes v_2 \otimes \bullet) \in \widetilde{V_3}^{I_{n_3-1}, \omega_3^{-1}} = \{0\}.$$

Therefore $\ell(\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3) \neq 0$ as wanted.

Next, we show that $v_1 \otimes \gamma^{n_3-n_2} v_2 \otimes v_3$ is a test vector, assuming that $\ell(\bullet \otimes \gamma^{n_3-n_2} v_2 \otimes v_3) \neq 0$. As in the previous paragraph, there exists $0 \leq i \leq n_3 - n_1$ such that $\ell(\gamma^i v_1 \otimes \gamma^{n_3-n_2} v_2 \otimes v_3) \neq 0$. Moreover, for every $0 < i \leq n_3 - n_1$, we have

$$\gamma I_{n_3-1} \gamma^{-1} \subset \gamma^i I_{n_1} \gamma^{-i} \cap \gamma^{n_3-n_2} I_{n_2} \gamma^{n_2-n_3}$$

hence

$$\ell(\gamma^i \cdot v_1 \otimes \gamma^{n_2-n_3} v_2 \otimes \bullet) \in \widetilde{V_3}^{\gamma I_{n_3-1}\gamma^{-1}, \omega_3^{-1}} = \{0\}.$$

Therefore $\ell(v_1 \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3) \neq 0$ as wanted.

Finally, we prove the above assumption that $\ell(\bullet \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3) \neq 0$.

Recall that $(\begin{smallmatrix} 0 & 1 \\ \pi^{n_i} & 0 \end{smallmatrix}) \cdot v_i$ is sent by the isomorphism $V_i \otimes \omega_i^{-1} \cong \widetilde{V_i}$ to a new vector in $\widetilde{V_i}$. Moreover by (2) any test vector in $\widetilde{V_1} \otimes \widetilde{V_2} \otimes \widetilde{V_3}$ yields a test vector in $V_1 \otimes V_2 \otimes V_3$. By applying lemma 4.2 to $\widetilde{V_1} \otimes \widetilde{V_2} \otimes \widetilde{V_3}$ one gets

$$\ell\,(\bullet \otimes (\begin{smallmatrix} 0 & 1 \\ \pi^{n_2} & 0 \end{smallmatrix}) \cdot v_2 \otimes (\begin{smallmatrix} 0 & 1 \\ \pi^{n_3} & 0 \end{smallmatrix}) \cdot v_3) \neq 0,$$

hence $\ell(\bullet \otimes \gamma^{n_3-n_2} \cdot v_2 \otimes v_3) \neq 0$. This completes the proof of theorem 4 in this case.

4.4 The case of equal conductors

In this subsection we assume that $n_2 = n_3$, hence V_1 is a ramified principal series. Since V_1 is minimal, it follows then from (1) that $n_1 < n_3$. By (28) and lemma 2.2 we have

$$\ell(\gamma^{n_3-n_1} \cdot v_1 \otimes v_2 \otimes v_3) = \int_{I_{n_3}} v_1(k)\Phi(k \cdot v_2, k \cdot v_3)dk = {\alpha_1}^{n_3-n_1} \int_{I_{n_3}} (\omega_1\omega_2\omega_3)(d)dk \neq 0.$$

where d is the lower right coefficient of k.

Recall again that $\left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_i} & 0 \end{smallmatrix}\right) \cdot v_i$ is sent by the isomorphism $V_i \otimes \omega_i^{-1} \cong \widetilde{V_i}$ to a new vector in $\widetilde{V_i}$. Moreover by (2) any test vector in $\widetilde{V_1} \otimes \widetilde{V_2} \otimes \widetilde{V_3}$ yields a test vector in $V_1 \otimes V_2 \otimes V_3$, hence

$$\ell\left(\omega_1^{-1}(\det(\gamma^{n_3-n_1}))\gamma^{n_3-n_1} \cdot \left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_1} & 0 \end{smallmatrix}\right) \cdot v_1 \otimes \left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_2} & 0 \end{smallmatrix}\right) \cdot v_2 \otimes \left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_3} & 0 \end{smallmatrix}\right) \cdot v_3\right) \neq 0,$$

$$\ell(v_1 \otimes v_2 \otimes v_3) = \omega_1(\pi^{n_3-n_1})\ell\left(\left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_3} & 0 \end{smallmatrix}\right)^{-1} \gamma^{n_3-n_1} \left(\begin{smallmatrix} 0 & 1 \\ \pi^{n_1} & 0 \end{smallmatrix}\right) \cdot v_1 \otimes v_2 \otimes v_3\right) \neq 0.$$

This completes the proof of Theorem 4. □

5 Test vectors in reducible induced representation

In this section, we generalize the local part of the paper [HS] by Michael Harris and Anthony Scholl on trilinear forms and test vectors when some of the V_i's are reducible principal series of G. The results of Harris and Scholl have as a global application the fact that a certain subspace, constructed by Beilinson, in the motivic cohomology of the product of two modular curves is a line. However, we are not going to follow them in this direction.

As in [HS], we will only consider reducible principal series having infinite dimensional subspaces (see section 2.4.2), since for those having infinite dimensional quotients (see section 2.4.1) test vector can be obtained by preimage of test vectors in the quotient. It follows then from [HS, Propositions 1.5, 1.6 and 1.7] that under the assumption (1):

$$\dim \mathrm{Hom}_G(V_1 \otimes V_2 \otimes V_3, \mathbb{C}) = 1. \tag{29}$$

This is particularly interesting for $V_1 = V_2 = V_3 = \mathrm{Ind}_B^G(\delta^{\frac{1}{2}})$ since, according to Theorem 2(ii), the space $\mathrm{Hom}_G(\mathrm{St} \otimes \mathrm{St} \otimes \mathrm{St}, \mathbb{C})$ vanishes.

Remark 5.1. The case when for $1 \leq i \leq 3$, $V_i = \mathrm{Ind}_B^G((\eta_i \circ \det)\delta^{\frac{1}{2}})$, with $\eta_1\eta_2\eta_3$ non-trivial (quadratic), is not contained explicitly in [HS], but can be handled as follows. Since

$$\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G((\eta_1\eta_2 \circ \det)\delta^{\frac{1}{2}}), \mathrm{Ind}_B^G(({\eta_3}^{-1} \circ \det)\delta^{-\frac{1}{2}})\Big) = 0,$$

it follows easily from the short exact sequence (7) for V_3 that there is an isomorphism

$$\mathrm{Hom}_G(V_1 \otimes V_2, \widetilde{V_3}) \xrightarrow{\sim} \mathrm{Hom}_G(V_1 \otimes V_2, \mathrm{St} \otimes \eta_3^{-1}),$$

and the latter space is one dimensional by [HS, Proposition 1.6].

In [HS], Harris and Scholl also exhibit test vectors when the three representations involved have a line of K-invariant vectors. The following proposition generalizes their results.

Proposition 6. *(i) Suppose that for* $1 \le i \le 3$, $V_i = \mathrm{Ind}_B^G((\eta_i \circ \det)\delta^{\frac{1}{2}})$, *with* η_i *unramified character such that* $\eta_1^2\eta_2^2\eta_3^2 = 1$. *Then* $v_1^K \otimes v_2^K \otimes v_3^K$ *is a test vector.*

(ii) Suppose that for $1 \le i \le 2$, $V_i = \mathrm{Ind}_B^G((\eta_i \circ \det)\delta^{\frac{1}{2}})$, *with* η_i *unramified, and* V_3 *is irreducible such that* $\eta_1^2\eta_2^2\omega_3 = 1$. *Then* $\gamma^{n_3}\cdot v_1^K \otimes v_2^K \otimes v_3$ *and* $v_1^K \otimes \gamma^{n_3}\cdot v_2^K \otimes v_3$ *are test vectors.*

(iii) Suppose that $V_1 = \mathrm{Ind}_B^G((\eta_1 \circ \det)\delta^{\frac{1}{2}})$ *with* η_1 *unramified, and that* V_2 *and* V_3 *are irreducible with* $\eta_1^2\omega_2\omega_3 = 1$. *Suppose that either* V_2 *is non-supercuspidal and minimal, or* V_2 *and* V_3 *are both supercuspidal with distinct conductors. Then exactly one of the following holds:*

(a) $n_3 > n_2$ *and* $v_1^K \otimes \gamma^{n_3-n_2}\cdot v_2 \otimes v_3$ *and* $\gamma^{n_3}\cdot v_1^K \otimes v_2 \otimes v_3$ *are both test vectors;*

(b) $n_3 = n_2$ *and , for every* i, $0 \le i \le n_3$, $\gamma^i\cdot v_1^K \otimes v_2 \otimes v_3$ *is a test vector;*

(c) V_2 *is special,* $n_3 = 0$, *and* $v_1^K \otimes v_2 \otimes \gamma\cdot v_3$ *and* $\gamma\cdot v_1^K \otimes v_2 \otimes v_3$ *are both test vectors.*

Remark 5.2. One should observe that the test vectors in Proposition 6 :

– do not belong to *any proper subrepresentation* of $V_1 \otimes V_2 \otimes V_3$;

– are fixed by larger open compact subgroups of $G \times G \times G$, than those fixing the test vectors in the irreducible subrepresentation of $V_1 \otimes V_2 \otimes V_3$ given by Theorem 4.

Proof: As explained in the introduction, twisting allows us to assume that $\eta_1 = \eta_2 = 1$.
(i) If $\eta_3 = 1$ this is [HS, Proposition 1.7]. Otherwise η_3 is the unramified quadratic character and we consider Prasad's short exact sequence (10):

$$0 \to \mathrm{ind}_T^G 1 \xrightarrow{\mathrm{ext}} V_1 \otimes V_2 \xrightarrow{\mathrm{res}} \mathrm{Ind}_B^G \delta^{\frac{3}{2}} \to 0. \tag{30}$$

Since $\mathrm{Hom}_G(\mathrm{Ind}_B^G\delta^{\frac{3}{2}}, \mathrm{Ind}_B^G((\eta_3{}^{-1} \circ \det)\delta^{-\frac{1}{2}})) = 0$, one has :

$$\mathrm{Hom}_G(V_1 \otimes V_2, \widetilde{V_3}) \xrightarrow{\sim} \mathrm{Hom}_G(\mathrm{ind}_T^G 1, \widetilde{V_3}) \xrightarrow{\sim} \mathrm{Hom}_T(1, \widetilde{V_{3|T}}).$$

Denote by φ a generator of the latter. It follows from the proof of Lemma 3.4, where the irreducibility of V_3 is not used, only it's smoothness, that $\varphi(v_3^K) \ne 0$ (the point is that by (7), a basis of the I_s-invariants in V_3 is given by $\gamma^i\cdot v_3^K$ for $0 \le i \le s$).

It follows then by exactly the same argument as in the proof of [P, Theorem 5.10], that $v_1^K \otimes v_2^K \otimes v_3^K$ is a test vector. The only point to check is that the denominator in the formula displayed in the middle of [P, page 20] does not vanish.

(ii) For $n_3 = 0$, this is [HS, Proposition 1.6].

For $n_3 \ge 1$, again by Lemma 3.4 we have $\varphi(v_3^K) \ne 0$ and the usual process, as in the proof of Theorem 3, allows to prove that $\gamma^{n_3}\cdot v_1^K \otimes v_2^K \otimes v_3$ and $v_1^K \otimes \gamma^{n_3}\cdot v_2^K \otimes v_3$ are test vector.

(iii)(a) If V_2 and V_3 are both supercuspidal the claim follows from lemma 4.1 by exactly same arguments that allowed to prove theorem 4 in this case. So we can assume that V_2 is non-supercuspidal and minimal.

First we choose a model of V_2 such that μ_2 is unramified and consider the exact sequence (10):

$$0 \to \mathrm{ind}_T^G(\delta^{-\frac{1}{2}}\chi_2) \xrightarrow{\mathrm{ext}} V_1 \otimes V_2 \xrightarrow{\mathrm{res}} \mathrm{Ind}_B^G(\delta\chi_2) \to 0.$$

If $\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G(\delta\chi_2), \widetilde{V_3}\Big) = 0$, then we obtain isomorphisms

$$\mathrm{Hom}_G\Big(V_1 \otimes V_2, \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_G\Big(\mathrm{ind}_T^G(\delta^{-\frac{1}{2}}\chi_2), \widetilde{V_3}\Big) \xrightarrow{\sim} \mathrm{Hom}_T\Big(\delta^{\frac{1}{2}}\chi_2, \widetilde{V_{3|T}}\Big)$$

and as in section 3 we obtain that $v_1^K \otimes \gamma^{n_3-n_2}\cdot v_2 \otimes v_3$ is a test vector.

If $\mathrm{Hom}_G\Big(\mathrm{Ind}_B^G(\delta\chi_2), \widetilde{V_3}\Big) \neq 0$, then $n_3 > n_2$ implies that there exists an unramified character η such that $\delta\chi_2 = (\eta \circ \det)\delta^{-\frac{1}{2}}$ and $\widetilde{V_3} = \eta \otimes \mathrm{St}$. So $n_2 = 0$ and $n_3 = 1$. It is easy then to check that the image of $v_1^K \otimes \gamma\cdot v_2 \otimes \in V_1 \otimes V_2$ by res is not a multiple of $\eta \circ \det$, hence it yields a non zero element of $\widetilde{V_3}$. Since $\gamma^{-1}K\gamma \cap K = I$, it is actually a non zero element of $\widetilde{V_3}^{I,\omega_3^{-1}}$, hence $v_1^K \otimes \gamma\cdot v_2 \otimes v_3$ is a test vector.

By choosing a model of V_2 with μ_2' unramified, and applying the above arguments to $V_2 \otimes \mathrm{Ind}_B^G(\delta^{\frac{1}{2}})$ one can prove that $\gamma^{n_3}\cdot v_1^K \otimes v_2 \otimes v_3$ is a test vector.

(iii)(b) For $n_3 = 0$, this is [HS, Proposition 1.5].

For $n_3 \geq 1$, assume first that $\mathrm{Hom}_G\Big(V_2, \widetilde{V_3}\Big) \neq 0$. Then the G-invariant trilinear form on $V_1 \otimes V_2 \otimes V_3$ is obtained by composing $\mathrm{proj}_1^* \otimes \mathrm{id} \otimes \mathrm{id}$ with the natural pairing between $V_2 \simeq \widetilde{V_3}$ and V_3. Since the natural pairing between $\widetilde{V_3}$ and V_3 is non-zero on a couple of new vectors, it follows that for all i, $\gamma^i\cdot v_1^K \otimes v_2 \otimes v_3$ is a test vector.

If $\mathrm{Hom}_G\Big(V_2, \widetilde{V_3}\Big) = 0$, we apply the techniques of section 3 to $V_2 \otimes V_3 \otimes \mathrm{Ind}_B^G(\delta^{\frac{1}{2}})$. There are isomorphisms

$$\mathrm{Hom}_G(V_2 \otimes V_3 \otimes V_1, \mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}_T\Big(\chi_3\chi_2', \mathrm{Ind}_B^G(\delta^{-\frac{1}{2}})_{|T}\Big) \xrightarrow{\sim} \mathrm{Hom}_T\Big(\chi_3\chi_2', \widetilde{V_{1|T}}\Big).$$

Taking a generator φ of the latter space, one has $\varphi(\gamma^i\cdot v_1^K) \neq 0$ for all i, by adapting the proof of Lemma 3.4 as above. Then exactly the same computations as in the proof of Theorem 4(b) show that $\gamma^i v_1^K \otimes v_2 \otimes v_3$ is a test vector, for all $0 \leq i \leq n_3$.

(iii)(c) This case follows from (iii)(a) applied to $V_1 \otimes V_3 \otimes V_2$. □

References

[B] D. Bump *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics. 55. Cambridge University Press. xiv, 574 p.

[BR1] J. Bernstein and A. Reznikov, *Estimates of automorphic functions*, Mosc. Math. J., 4 (2004), pp. 19–37, 310.

[BR2] ——, *Periods, subconvexity of L-functions and representation theory*, J. Differential Geom., 70 (2005), pp. 129–141.

[BH] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), vol. 335 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2006.

[C] W. Casselman, *On some results of Atkin and Lehner*, Math. Ann., 201 (1973), pp. 301–314.

[GP] B. H. Gross and D. Prasad, *Test vectors for linear forms*, Math. Ann., 291 (1991), pp. 343–355.

[HS] M. Harris and A. J. Scholl, *A note on trilinear forms for reducible representations and Beilinson's conjectures*, J. Eur. Math. Soc. (JEMS), 3 (2001), pp. 93–104.

[MV1] P. Michel and A. Venkatesh, *Equidistribution, L-functions and ergodic theory: on some problems of Yu. Linnik*, in International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 421–457.

[MV2] ——, *The subconvexity problem for* GL(2), to appear in Publ. Math IHES. arXiv:0903.3591 (2009).

[P] D. Prasad, *Trilinear forms for representations of* GL(2) *and local* ϵ*-factors*, Compositio Math., 75 (1990), pp. 1–46.

[P2] ——, *Relating invariant linear form and local epsilon factors via global methods*, Duke Math. J., 138 (2007), pp. 233–261. With an appendix by Hiroshi Saito.

[W] J.-L. Waldspurger, *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie*, Compositio Math., 54 (1985), pp. 173–242.