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Scaling limit of the path leading to the leftmost particle in a branching random walk

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Summary. We consider a discrete-time branching random walk defined on the real line, which is assumed to be supercritical and in the boundary case. It is known that its leftmost position of the n -th generation behaves asymptotically like $\frac{3}{2} \ln n$, provided the non-extinction of the system. The main goal of this paper, is to prove that the path from the root to the leftmost particle, after a suitable normalization, converges weakly to a Brownian excursion in $D([0, 1], \mathbb{R})$.

Keywords. Branching random walk; spinal decomposition.

1 Introduction

We consider a branching random walk, which is constructed according to a point process \mathcal{L} on the line. Precisely speaking, the system is started with one initial particle at the origin. This particle is called the root, denoted by \emptyset . At time 1, the root dies and gives birth to some new particles, which form the first generation. Their positions constitute a point process distributed as \mathcal{L} . At time 2, each of these particles dies and gives birth to new particles whose positions – relative to that of their parent – constitute a new independent copy of \mathcal{L} . The system grows according to the same mechanism.

We denote by \mathbb{T} the genealogical tree of the system, which is clearly a Galton-Watson tree rooted at \emptyset . If a vertex $u \in \mathbb{T}$ is in the n -th generation, we write $|u| = n$ and denote its position by $V(u)$. Then $\{V(u), |u| = 1\}$ follows the same law as \mathcal{L} . The family of positions $(V(u); u \in \mathbb{T})$ is viewed as our branching random walk.

Throughout the paper, the branching random walk is assumed to be in the boundary case (Biggins and Kyprianou [5]):

$$(1.1) \quad \mathbf{E} \left[\sum_{|u|=1} 1 \right] > 1, \quad \mathbf{E} \left[\sum_{|x|=1} e^{-V(x)} \right] = 1, \quad \mathbf{E} \left[\sum_{|x|=1} V(x) e^{-V(x)} \right] = 0.$$

For any $y \in \mathbb{R}$, let $y_+ := \max\{y, 0\}$ and $\log_+ y := \log(\max\{y, 1\})$. We also assume the following integrability conditions:

$$(1.2) \quad \mathbf{E} \left[\sum_{|u|=1} V(u)^2 e^{-V(u)} \right] < \infty,$$

$$(1.3) \quad \mathbf{E}[X(\log_+ X)^2] < \infty, \quad \mathbf{E}[\tilde{X} \log_+ \tilde{X}] < \infty,$$

where

$$X := \sum_{|u|=1} e^{-V(u)}, \quad \tilde{X} := \sum_{|u|=1} V(u)_+ e^{-V(u)}.$$

We define I_n to be the leftmost position in the n -th generation, i.e.

$$(1.4) \quad I_n := \inf\{V(u), |u| = n\},$$

with $\inf \emptyset := \infty$. If $I_n < \infty$, we choose a vertex uniformly in the set $\{u : |u| = n, V(u) = I_n\}$ of leftmost particles at time n and denote it by $m^{(n)}$. We let $[\emptyset, m^{(n)}] = \{\emptyset =: m_0^{(n)}, m_1^{(n)}, \dots, m_n^{(n)} := m^{(n)}\}$ be the shortest path in \mathbb{T} relating the root \emptyset to $m^{(n)}$, and introduce the path from the root to $m^{(n)}$ as follows

$$(I_n(k); 0 \leq k \leq n) := (V(m_k^{(n)}); 0 \leq k \leq n).$$

In particular, $I_n(0) = 0$ and $I_n(n) = I_n$. Let σ be the positive real number such that $\sigma^2 = \mathbf{E} \left[\sum_{|u|=1} V(u)^2 e^{-V(u)} \right]$. Our main result is as follows.

Theorem 1.1 *The rescaled path $(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1)$ converges in law in $D([0, 1], \mathbb{R})$, to a normalized Brownian excursion $(e_s; 0 \leq s \leq 1)$.*

Remark 1.2 *It has been proved in [1], [11] and [2] that I_n is around $\frac{3}{2} \ln n$. In [3], the authors proved that, for the model of branching Brownian motion, the time reversed path followed by the leftmost particle converges in law to a certain stochastic process.*

Let us say a few words about the proof of Theorem 1.1. We first consider the path leading to $m^{(n)}$, by conditioning that its ending point I_n is located atypically below $\frac{3}{2} \ln n - z$

with large z . Then we apply the well-known spinal decomposition to show that this path, conditioned to $\{I_n \leq \frac{3}{2} \ln n - z\}$, behaves like a simple random walk staying positive but tied down at the end. Such a random walk, being rescaled, converges in law to the Brownian excursion (see [9]). We then prove our main result by removing the condition of I_n . The main strategy is borrowed from [2], but with appropriate refinements.

The rest of the paper is organized as follows. In Section 2, we recall the spinal decomposition by a change of measures, which implies the useful many-to-one lemma. We prove a conditioned version of Theorem 1.1 in Section 3. In Section 4, we remove the conditioning and prove the theorem.

Throughout the paper, we use $a_n \sim b_n$ ($n \rightarrow \infty$) to denote $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$; and let $(c_i)_{i \geq 0}$ denote finite and positive constants. We write $\mathbf{E}[f; A]$ for $\mathbf{E}[f \mathbf{1}_A]$. Moreover, $\sum_{\emptyset} := 0$ and $\prod_{\emptyset} := 1$.

2 Lyons' change of measures and spinal decomposition

For any $a \in \mathbb{R}$, let \mathbf{P}_a be the probability measure such that $\mathbf{P}_a((V(u), u \in \mathbb{T}) \in \cdot) = \mathbf{P}((V(u) + a, u \in \mathbb{T}) \in \cdot)$. The corresponding expectation is denoted by \mathbf{E}_a . Let $(\mathcal{F}_n, n \geq 0)$ be the natural filtration generated by the branching random walk and let $\mathcal{F}_{\infty} := \bigvee_{n \geq 0} \mathcal{F}_n$. We introduce the following random variables:

$$(2.1) \quad W_n := \sum_{|u|=n} e^{-V(u)}, \quad n \geq 0.$$

It follows immediately from (1.1) that $(W_n, n \geq 0)$ is a non-negative martingale with respect to (\mathcal{F}_n) . It is usually referred as the additive martingale. We define a probability measure \mathbf{Q}_a on \mathcal{F}_{∞} such that for any $n \geq 0$,

$$(2.2) \quad \left. \frac{d\mathbf{Q}_a}{d\mathbf{P}_a} \right|_{\mathcal{F}_n} := e^a W_n.$$

For convenience, we write \mathbf{Q} for \mathbf{Q}_0 .

Let us give the description of the branching random walk under \mathbf{Q}_a in an intuitive way, which is known as the spinal decomposition. We introduce another point process $\widehat{\mathcal{L}}$ with Radon-Nykodim derivative $\sum_{x \in \mathcal{L}} e^{-x}$ with respect to the law of \mathcal{L} . Under \mathbf{Q}_a , the branching random walk evolves as follows. Initially, there is one particle w_0 located at $V(w_0) = a$. At each step n , particles at generation n die and give birth to new particles independently according to the law of \mathcal{L} , except for the particle w_n which generates its children according

to the law of $\widehat{\mathcal{L}}$. The particle w_{n+1} is chosen proportionally to $e^{-V(u)}$ among the children u of w_n . We still call \mathbb{T} the genealogical tree of the process, so that $(w_n)_{n \geq 0}$ is a ray in \mathbb{T} , which is called the spine. This change of probabilities was presented in various forms; see, for example [15], [11] and [8].

It is convenient to use the following notation. For any $u \in \mathbb{T} \setminus \{\emptyset\}$, let \overleftarrow{u} be the parent of u , and

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

Let $\Omega(u)$ be the set of brothers of u , i.e. $\Omega(u) := \{v \in \mathbb{T} : \overleftarrow{v} = \overleftarrow{u}, v \neq u\}$. Let δ denote the Dirac measure. Then under \mathbf{Q}_a , $\sum_{|u|=1} \delta_{\Delta V(u)}$ follows the law of $\widehat{\mathcal{L}}$. Further, We recall the following proposition, from [11] and [15].

Proposition 2.1 (1) For any $|u| = n$, we have

$$(2.3) \quad \mathbf{Q}_a[w_n = u | \mathcal{F}_n] = \frac{e^{-V(u)}}{W_n}.$$

(2) Under \mathbf{Q}_a , the random variables $\left(\sum_{v \in \Omega(w_n)} \delta_{\Delta V(v)}, \Delta V(w_n) \right)$, $n \geq 1$ are i.i.d..

As a consequence of this proposition, we get the many-to-one lemma as follows:

Lemma 2.2 There exists a centered random walk $(S_n; n \geq 0)$ with $\mathbf{P}_a(S_0 = a) = 1$ such that for any $n \geq 1$ and any measurable function $g : \mathbb{R}^n \rightarrow [0, \infty)$, we have

$$(2.4) \quad \mathbf{E}_a \left[\sum_{|u|=n} g(V(u_1), \dots, V(u_n)) \right] = \mathbf{E}_a [e^{S_n - a} g(S_1, \dots, S_n)],$$

where we denote by $[[\emptyset, u]] = \{\emptyset =: u_0, u_1, \dots, u_{|u|} := u\}$ the ancestral line of u in \mathbb{T} .

Note that by (1.3), S_1 has the finite variance $\sigma^2 = \mathbf{E}[S_1^2] = \mathbf{E}[\sum_{|u|=1} V(u)^2 e^{-V(u)}]$.

2.1 Convergence in law for the one-dimensional random walk

Let us introduce some results about the centered random walk (S_n) with finite variance, which will be used later. For any $0 \leq m \leq n$, we define $\underline{S}_{[m, n]} := \min_{m \leq j \leq n} S_j$, and $\underline{S}_n = \underline{S}_{[0, n]}$. We denote by $R(x)$ the renewal function of (S_n) , which is defined as follows:

$$(2.5) \quad R(x) = \mathbf{1}_{\{x=0\}} + \mathbf{1}_{\{x>0\}} \sum_{k \geq 0} \mathbf{P}(-x \leq S_k < \underline{S}_{n-1}).$$

For the random walk $(-S_n)$, we define $\underline{S}_{[m,n]}^-$, \underline{S}_n^- and $R_-(x)$ similarly. It is known (see [10] p. 360) that there exists $c_0 > 0$ such that

$$(2.6) \quad \lim_{x \rightarrow \infty} \frac{R(x)}{x} = c_0.$$

Moreover, it is shown in [13] that there exist $C_+, C_- > 0$ such that for any $a \geq 0$,

$$(2.7) \quad \mathbf{P}_a(\underline{S}_n \geq 0) \sim \frac{C_+}{\sqrt{n}} R(a);$$

$$(2.8) \quad \mathbf{P}_a(\underline{S}_n^- \geq 0) \sim \frac{C_-}{\sqrt{n}} R_-(a).$$

We also state the following inequalities (see Lemmas 2.2 and 2.4 in [4], respectively).

Fact 2.3 (i) *There exists a constant $c_1 > 0$ such that for any $b \geq a \geq 0$, $x \geq 0$ and $n \geq 1$,*

$$(2.9) \quad \mathbf{P}(\underline{S}_n \geq -x; S_n \in [a-x, b-x]) \leq c_1(1+x)(1+b-a)(1+b)n^{-3/2}.$$

(ii) *Let $0 < \lambda < 1$. There exists a constant $c_2 > 0$ such that for any $b \geq a \geq 0$, $x, y \geq 0$ and $n \geq 1$,*

$$(2.10) \quad \mathbf{P}_x(S_n \in [y+a, y+b], \underline{S}_n \geq 0, \underline{S}_{[\lambda n, n]} \geq y) \leq c_2(1+x)(1+b-a)(1+b)n^{-3/2}.$$

Before we give the next lemma, we recall the definition of lattice distribution (see [10], p. 138). The distribution of a random variable X_1 is lattice, if it is concentrated on a set of points $\alpha + \beta\mathbb{Z}$, with α arbitrary. The largest β satisfying this property is called the span of X_1 . Otherwise, the distribution of X_1 is called non-lattice.

Lemma 2.4 *Let $(r_n)_{n \geq 0}$ be a sequence of real numbers such that $\lim_{n \rightarrow \infty} \frac{r_n}{\sqrt{n}} = 0$. Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a Riemann integrable function. We suppose that there exists a non-increasing function $\bar{f} : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $|f(x)| \leq \bar{f}(x)$ for any $x \geq 0$ and $\int_{x \geq 0} x \bar{f}(x) dx < \infty$. For $0 < \Delta < 1$, let $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$ be continuous. Let $a \geq 0$.*

(I) Non-lattice case. *If the distribution of $(S_1 - S_0)$ is non-lattice, then there exists a constant $C_1 > 0$ such that*

$$(2.11) \quad \lim_{n \rightarrow \infty} n^{3/2} \mathbf{E} \left[F \left(\frac{S_{[sn]}}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) f(S_n - y); \underline{S}_n \geq -a, \underline{S}_{[\Delta n, n]} \geq y \right] \\ = C_1 R(a) \int_{x \geq 0} f(x) R_-(x) dx \mathbf{E}[F(e_s; 0 \leq s \leq \Delta)],$$

uniformly in $y \in [0, r_n]$.

(II) Lattice case. If the distribution of $(S_1 - S_0)$ is supported in $(\alpha + \beta\mathbb{Z})$ with span β , then for any $d \in \mathbb{R}$,

$$(2.12) \quad \lim_{n \rightarrow \infty} n^{3/2} \mathbf{E} \left[F \left(\frac{S_{\lfloor sn \rfloor}}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) f(S_n - y + d); \underline{S}_n \geq -a, \underline{S}_{[\Delta n, n]} \geq y - d \right] \\ = C_1 R(a) \beta \sum_{j \geq \lceil -\frac{d}{\beta} \rceil} f(\beta j + d) R_-(\beta j + d) \mathbf{E}[F(e_s; 0 \leq s \leq \Delta)].$$

uniformly in $y \in [0, r_n] \cap \{\alpha n + \beta\mathbb{Z}\}$.

Proof of Lemma 2.4. The lemma is a refinement of Lemma 2.3 in [2], which proved the convergence in the non-lattice case when $a = 0$ and $F \equiv 1$. We consider the non-lattice case first. We denote the expectation on the left-hand side of (2.11) by $\chi(F, f)$. Observe that for any $K \in \mathbb{N}_+$,

$$\chi(F, f) = \chi(F, f(x)1_{(0 \leq x \leq K)}) + \chi(F, f(x)1_{(x > K)}).$$

Since $0 \leq F \leq 1$, we have $\chi(F, f(x)1_{(x > K)}) \leq \chi(1, f(x)1_{(x > K)})$, which is bounded by

$$\sum_{j \geq K} \mathbf{E}_a \left[f(S_n - y - a); \underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a, S_n \in [y + a + j, y + a + j + 1] \right].$$

Recall that $|f(x)| \leq \bar{f}(x)$ with \bar{f} non-increasing. We get that

$$\chi(1, f(x)1_{(x > K)}) \leq \sum_{j \geq K} \bar{f}(j) \mathbf{P}_a \left[\underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a, S_n \in [y + a + j, y + a + j + 1] \right].$$

It then follows from (2.10) that

$$(2.13) \quad \chi(1, f(x)1_{(x > K)}) \leq 2c_2(1 + a) \left(\sum_{j \geq K} \bar{f}(j)(2 + j) \right) n^{-3/2}.$$

Since $\int_0^\infty x \bar{f}(x) dx < \infty$, the sum $\sum_{j \geq K} \bar{f}(j)(2 + j)$ decreases to zero as $K \uparrow \infty$. We thus only need to estimate $\chi(F, f(x)1_{(0 \leq x \leq K)})$. Note that f is Riemann integrable. It suffices to consider $\chi(F, 1_{(0 \leq x \leq K)})$ with K a positive constant.

Applying the Markov property at time $[\Delta n]$ shows that

$$(2.14) \quad \chi(F, 1_{(0 \leq x \leq K)}) = \mathbf{E}_a \left[F \left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right); S_n \leq y + a + K, \underline{S}_n \geq 0, \underline{S}_{[\Delta n, n]} \geq y + a \right] \\ = \mathbf{E}_a \left[F \left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \Psi_K(S_{[\Delta n]}); \underline{S}_{[\Delta n]} \geq 0 \right],$$

where $\Psi_K(x) := \mathbf{P}_x \left[S_{n-\lfloor \Delta n \rfloor} \leq y + a + K, \underline{S}_{n-\lfloor \Delta n \rfloor} \geq y + a \right]$. By reversing time, we obtain that $\Psi_K(x) = \mathbf{P} \left[\underline{S}_m^- \geq (-S_m) + (y + a - x) \geq -K \right]$ with $m := n - \lfloor \Delta n \rfloor$.

We define τ_n as the first time when the random walk $(-S)$ hits the minimal level during $[0, n]$, namely, $\tau_n := \inf \{ k \in [0, n] : -S_k = \underline{S}_n^- \}$. Define also $\varkappa(z, \zeta; n) := \mathbf{P}(-S_n \in [z, z + \zeta], \underline{S}_n^- \geq 0)$ for any $z, \zeta \geq 0$. Then,

$$(2.15) \quad \begin{aligned} \Psi_K(x) &= \sum_{k=0}^m \mathbf{P} \left[\tau_m = k; \underline{S}_m^- \geq (-S_m) + (y + a - x) \geq -K \right] \\ &= \sum_{k=0}^m \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K; \varkappa(x - y - a, \underline{S}_k^- + K; m - k) \right], \end{aligned}$$

where the last equality follows from the Markov property.

Let $\psi(x) := xe^{-x^2/2} \mathbf{1}_{(x \geq 0)}$. Combining Theorem 1 of [6] with (2.7) yields that

$$(2.16) \quad \varkappa(z, \zeta; n) = \mathbf{P}_0 \left[-S_n \in [z, z + \zeta]; \underline{S}_n^- \geq 0 \right] = \frac{C_- \zeta}{\sigma n} \psi \left(\frac{z}{\sigma \sqrt{n}} \right) + o(n^{-1}),$$

uniformly in $z \in \mathbb{R}_+$ and ζ in compact sets of \mathbb{R}_+ . Note that ψ is bounded on \mathbb{R}_+ . Therefore, there exists a constant $c_3 > 0$ such that for any $\zeta \in [0, K]$, $z \geq 0$ and $n \geq 0$,

$$(2.17) \quad \varkappa(z, \zeta; n) \leq c_3 \frac{(1 + K)}{n + 1}.$$

Let $k_n := \lfloor \sqrt{n} \rfloor$. We divide the sum on the right-hand side of (2.15) into two parts:

$$(2.18) \quad \Psi_K(x) = \sum_{k=0}^{k_n} + \sum_{k=k_n+1}^m \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K; \varkappa(x - y - a, \underline{S}_k^- + K; m - k) \right].$$

By (2.16), under the assumption that $y = o(\sqrt{n})$, the first part becomes that

$$(2.19) \quad \begin{aligned} &\frac{C_-}{\sigma m} \psi \left(\frac{x - a}{\sigma \sqrt{m}} \right) \sum_{k=0}^{k_n} \mathbf{E} \left[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \geq -K \right] + o(n^{-1}) \sum_{k=0}^{k_n} \mathbf{P} \left[-S_k = \underline{S}_k^- \geq -K \right] \\ &= \frac{C_-}{\sigma m} \psi \left(\frac{x - a}{\sigma \sqrt{m}} \right) \int_0^K R_-(u) du + o(n^{-1}), \end{aligned}$$

where the last equation comes from the fact that $\sum_{k \geq 0} \mathbf{E} \left[\underline{S}_k^- + K; -S_k = \underline{S}_k^- \geq -K \right] = \int_0^K R_-(u) du$. On the other hand, using (2.17) for $\varkappa(x - y - a, \underline{S}_k^- + K; m - k)$ and then applying (i) of Fact 2.3 imply that for n large enough, the second part of (2.18) is bounded

by

$$(2.20) \quad \begin{aligned} & \sum_{k=k_n+1}^m c_3 \frac{1+K}{m+1-k} \mathbf{P}(S_k^- \geq -K, -S_k \in [-K, 0]) \\ & \leq c_4 \sum_{k=k_n+1}^m \frac{(1+K)^3}{(m+1-k)k^{3/2}} = o(n^{-1}). \end{aligned}$$

By (2.19) and (2.20), we obtain that as n goes to infinity,

$$(2.21) \quad \Psi_K(x) = o(n^{-1}) + \frac{C_-}{\sigma(n - \lfloor \Delta n \rfloor)} \psi\left(\frac{x-a}{\sigma\sqrt{n - \lfloor \Delta n \rfloor}}\right) \int_0^K R_-(u) du,$$

uniformly in $x \geq 0$ and $y \in [0, r_n]$. Plugging it into (2.14) and then combining with (2.7) yield that

$$\begin{aligned} \chi(F, 1_{(0 \leq x \leq K)}) &= o(n^{-3/2}) + \frac{C_-}{\sigma(1-\Delta)n} \int_0^K R_-(u) du \\ &\quad \times \frac{C_+ R(a)}{\sqrt{\Delta n}} \mathbf{E}_a \left[F\left(\frac{S_{\lfloor sn \rfloor} - a}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) \psi\left(\frac{S_{\Delta n} - a}{\sigma\sqrt{(1-\Delta)n}}\right) \Big| S_{\Delta n} \geq 0 \right]. \end{aligned}$$

Theorem 1.1 of [7] says that under the conditioned probability $\mathbf{P}_a\left(\cdot \Big| S_{\Delta n} \geq 0\right), \left(\frac{S_{\lfloor r\Delta n \rfloor}}{\sigma\sqrt{\Delta n}}; 0 \leq r \leq 1\right)$ converges in law to a Brownian meander, denoted by $(\mathcal{M}_r; 0 \leq r \leq 1)$. Therefore,

$$\chi(F, 1_{(0 \leq x \leq K)}) \sim \frac{C_- C_+ R(a)}{\sigma n^{3/2} (1-\Delta) \sqrt{\Delta}} \int_0^K R_-(u) du \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right].$$

It remains to check that

$$(2.22) \quad \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right] = \sqrt{\frac{\pi}{2}} \mathbf{E} \left[F(e_s; 0 \leq s \leq \Delta) \right].$$

Let $(R_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel process. Then, as is shown in [12],

$$\begin{aligned} & \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[F\left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1-\Delta}}\right) \right] \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{(1-\Delta)\sqrt{\Delta}} \mathbf{E} \left[\frac{1}{R_1} F\left(\sqrt{\Delta} R_{s/\Delta}; 0 \leq s \leq \Delta\right) \psi\left(\frac{\sqrt{\Delta} R_1}{\sqrt{1-\Delta}}\right) \right], \\ &= \sqrt{\frac{\pi}{2}} \mathbf{E} \left[\frac{1}{(1-\Delta)^{3/2}} e^{-\frac{R_1^2}{2(1-\Delta)}} F\left(R_s; 0 \leq s \leq \Delta\right) \right], \end{aligned}$$

where the last equation follows from the scaling property of Bessel process. Let $(r_s; 0 \leq s \leq 1)$ be a standard three-dimensional Bessel bridge. Note that for any $\Delta < 1$, $(r_s; 0 \leq s \leq \Delta)$ is equivalent to $(R_s; 0 \leq s \leq \Delta)$, with density $(1 - \Delta)^{-3/2} \exp(-\frac{R_\Delta^2}{2(1-\Delta)})$ (see p. 468 (3.11) of [16]). Thus,

$$\frac{1}{(1 - \Delta)\sqrt{\Delta}} \mathbf{E} \left[F \left(\sqrt{\Delta} \mathcal{M}_{s/\Delta}; 0 \leq s \leq \Delta \right) \psi \left(\frac{\sqrt{\Delta} \mathcal{M}_1}{\sqrt{1 - \Delta}} \right) \right] = \sqrt{\frac{\pi}{2}} \mathbf{E} \left[F(r_s; 0 \leq s \leq \Delta) \right].$$

Since a normalized Brownian excursion is exactly a standard three-dimensional Bessel bridge, this yields (2.22). Therefore, (2.11) is proved by taking $C_1 = \sqrt{\frac{\pi}{2}} \frac{C_- C_+}{\sigma}$.

The proof of the lemma in the lattice case is along the same lines, except that we use Theorem 2 (instead of Theorem 1) of [6]. \square

3 Conditioning on the event $\{I_n \leq \frac{3}{2} \ln n - z\}$

On the event $\{I_n \leq \frac{3}{2} \ln n - z\}$, we analyze the sample path leading to a particle located at the leftmost position at the n th generation. For $z \geq 0$ and $n \geq 1$, let $a_n(z) := \frac{3}{2} \ln n - z$ if the distribution of \mathcal{L} is non-lattice and let $a_n(z) := \alpha n + \beta \lfloor \frac{\frac{3}{2} \ln n - \alpha n}{\beta} \rfloor - z$ if the distribution of \mathcal{L} is supported by $\alpha + \beta\mathbb{Z}$. This section is devoted to the proof of the following proposition.

Proposition 3.1 *For any $\Delta \in (0, 1]$ and any continuous functional $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$,*

$$(3.1) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E} \left[F(e_s; 0 \leq s \leq \Delta) \right] \right| = 0.$$

We begin with some preliminary results.

For any $0 < \Delta < 1$ and $L, K \geq 0$, we denote by $J_{z,K,L}^\Delta(n)$ the following collection of particles:

$$(3.2) \quad \left\{ u \in \mathbb{T} : |u| = n, V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \geq a_n(z + L) \right\}.$$

Lemma 3.2 *For any $\varepsilon > 0$, there exists $L_\varepsilon > 0$ such that for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,*

$$(3.3) \quad \mathbf{P} \left(m^{(n)} \notin J_{z,K,L}^\Delta(n), I_n \leq a_n(z) \right) \leq \left(e^K + \varepsilon(1 + z - K) \right) e^{-z}.$$

Proof. It suffices to show that for any $\varepsilon \in (0, 1)$, there exists $L_\varepsilon \geq 1$ such that for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.4) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \left(e^K + \varepsilon(1 + z - K)\right)e^{-z}.$$

We observe that

$$(3.5) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \mathbf{P}\left(\exists u \in \mathbb{T} : V(u) \leq -z + K\right) \\ + \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z + L)\right).$$

On the one hand, by (2.4),

$$(3.6) \quad \mathbf{P}\left(\exists u \in \mathbb{T} : V(u) \leq -z + k\right) \leq \sum_{n \geq 0} \mathbf{E}\left[\sum_{|u|=n} \mathbf{1}_{\{V(u) \leq -z + K < \min_{k < n} V(u_k)\}}\right] \\ = \sum_{n \geq 0} \mathbf{E}[e^{S_n}; S_n \leq -z + K < \underline{S}_{n-1}] \leq e^{-z+K}.$$

On the other hand, denoting $A_n(z) := [a_n(z) - 1, a_n(z)]$ for any $z \geq 0$,

$$\mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), \min_{0 \leq k \leq n} V(u_k) \geq -z + K, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z + L)\right) \\ = \mathbf{P}_{z-K}\left(\exists |u| = n : V(u) \leq a_n(K), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(K + L)\right) \\ \leq \sum_{\ell \geq L+K} \sum_{j=K}^{j=K+\ell} \mathbf{P}_{z-K}\left(\exists |u| = n : V(u) \in A_n(j), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \in A_n(\ell)\right).$$

According to Lemma 3.3 in [2], there exist constants $1 > c_5 > 0$ and $c_6 > 0$ such that for any $n \geq 1$, $L \geq 0$ and $x, z \geq 0$,

$$(3.7) \quad \mathbf{P}_x\left(\exists u \in \mathbb{T} : |u| = n, V(u) \in A_n(z), \min_{0 \leq k \leq n} V(u_k) \geq 0, \min_{\Delta n \leq k \leq n} V(u_k) \in A_n(z + L)\right) \\ \leq c_6(1 + x)e^{-c_5 L}e^{-x-z}.$$

Hence, combining (3.6) with (3.5) yields that

$$\mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \\ \leq e^{-z+K} + \sum_{\ell \geq L} \sum_{0 \leq j \leq \ell} c_6(1 + z - K)e^{-c_5(\ell-j)}e^{-z-j} \\ \leq \left(e^K + c_7 \sum_{\ell \geq L} e^{-c_5 \ell}(1 + z - K)\right)e^{-z},$$

where the last inequality comes from the fact that $\sum_{j \geq 0} e^{-(1-c_5)j} < \infty$. We take $L_\varepsilon = -c_8 \ln \varepsilon$ so that $c_7 \sum_{\ell \geq L} e^{-c_5 \ell} \leq \varepsilon$ for all $L \geq L_\varepsilon$. Therefore, for any $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.8) \quad \mathbf{P}\left(\exists |u| = n : V(u) \leq a_n(z), u \notin J_{z,K,L}^\Delta(n)\right) \leq \left(e^K + \varepsilon(1+z-K)\right)e^{-z},$$

which completes the proof. \square

For $b \in \mathbb{Z}_+$, we define

$$(3.9) \quad \mathcal{E}_n = \mathcal{E}_n(z, b) := \{\forall k \leq n - b, \min_{u \geq w_k, |u|=n} V(u) > a_n(z)\}.$$

We note that on the event $\mathcal{E}_n \cap \{I_n \leq a_n(z)\}$, any particle located at the leftmost position must be separated from the spine after time $n - b$.

Lemma 3.3 *For any $\eta > 0$ and $L > 0$, there exist $K(\eta) > 0$, $B(L, \eta) \geq 1$ and $N(\eta) \geq 1$ such that for any $b \geq B(L, \eta)$, $n \geq N(\eta)$ and $z \geq K \geq K(\eta)$,*

$$(3.10) \quad \mathbf{Q}\left(\mathcal{E}_n^c, w_n \in J_{z,K,L}^\Delta(n)\right) \leq \eta(1+L)^2(1+z-K)n^{-3/2}.$$

We feel free to omit the proof of Lemma 3.3 since it is just a slightly stronger version of Lemma 3.8 in [2]. It follows from the same arguments.

Let us turn to the proof of Proposition 3.1. We break it up into 3 steps.

Step (I) (The conditioned convergence of $(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta)$ for $\Delta < 1$ in the non-lattice case)

Assume that the distribution of \mathcal{L} is non-lattice in this step. Recall that $a_n(z) = \frac{3}{2} \ln n - z$. The tail distribution of I_n has been given in Propositions 1.3 and 4.1 of [2], recalled as follows.

Fact 3.4 ([2]) *There exists a constant $C > 0$ such that*

$$(3.11) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - C \right| = 0.$$

Furthermore, for any $\varepsilon > 0$, there exist $N_\varepsilon \geq 1$ and $\Lambda_\varepsilon > 0$ such that for any $n \geq N_\varepsilon$ and $\Lambda_\varepsilon \leq z \leq \frac{3}{2} \ln n - \Lambda_\varepsilon$,

$$(3.12) \quad \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - C \right| \leq \varepsilon. \quad \square$$

For any continuous functional $F : D([0, \Delta], \mathbb{R}) \rightarrow [0, 1]$, it is convenient to write that

$$(3.13) \quad \Sigma_n(F, z) := \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \mathbf{1}_{\{I_n \leq a_n(z)\}} \right].$$

In particular, if $F \equiv 1$, $\Sigma_n(1, z) = \mathbf{P}(I_n \leq a_n(z))$. Thus,

$$(3.14) \quad \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} = \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \middle| I_n \leq a_n(z) \right].$$

Let us prove the following convergence for $0 < \Delta < 1$,

$$(3.15) \quad \lim_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| = 0.$$

Proof of (3.15). For any $n \geq 1$, $L \geq 0$ and $z \geq K \geq 0$, let

$$(3.16) \quad \Pi_n(F) = \Pi_n(F, z, K, L) := \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \mathbf{1}_{\{m^{(n)} \in J_{z, K, L}^\Delta(n)\}} \right].$$

By Lemma 3.2, we obtain that for $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq K \geq 0$,

$$(3.17) \quad \left| \Sigma_n(F, z) - \Pi_n(F) \right| \leq \left(e^K + \varepsilon(1 + z - K) \right) e^{-z}.$$

Note that $m^{(n)}$ is chosen uniformly among the particles located at the leftmost position. Thus,

$$\begin{aligned} \Pi_n(F) &= \mathbf{E} \left[\sum_{|u|=n} \mathbf{1}_{(u=m^{(n)}, u \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(u_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right] \\ &= \mathbf{E} \left[\frac{1}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \sum_{|u|=n} \mathbf{1}_{(V(u)=I_n, u \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(u_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right]. \end{aligned}$$

Applying the change of measures given in (2.2), it follows from Proposition 2.1 that

$$(3.18) \quad \Pi_n(F) = \mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \mathbf{1}_{(V(w_n)=I_n, w_n \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \right].$$

In order to estimate Π_n , we restrict ourselves to the event \mathcal{E}_n . Define

$$\Lambda_n(F) := \mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{|u|=n} \mathbf{1}_{(V(u)=I_n)}} \mathbf{1}_{(V(w_n)=I_n, w_n \in J_{z, K, L}^\Delta(n))} F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right); \mathcal{E}_n \right].$$

In view of Lemma 3.3, for any $b \geq B(L, \eta)$, $n \geq N(\eta)$ and $z \geq K \geq K(\eta)$,

$$(3.19) \quad \begin{aligned} \left| \Pi_n(F) - \Lambda_n(F) \right| &\leq \mathbf{E}_{\mathbf{Q}} \left[e^{V(w_n)}; w_n \in J_{z, K, L}^\Delta(n), \mathcal{E}_n^c \right] \\ &\leq e^{-z} n^{-3/2} \mathbf{Q} \left(\mathcal{E}_n^c, w_n \in J_{z, K, L}^\Delta(n) \right) \\ &\leq \eta(1 + L)^2(1 + z - K)e^{-z}. \end{aligned}$$

On the event $\mathcal{E}_n \cap \{I_n \leq a_n(z)\}$, $\Lambda_n(F)$ equals

$$\mathbf{E}_{\mathbf{Q}} \left[\frac{e^{V(w_n)}}{\sum_{u > w_{n-b}, |u|=n} \mathbf{1}_{\{V(u)=I_n\}}} \mathbf{1}_{\{V(w_n)=I_n, w_n \in J_{z,K,L}^\Delta(n)\}} F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right); \mathcal{E}_n \right].$$

Let, for $x \geq 0$, $L > 0$, and $b \geq 1$,

$$\begin{aligned} f_{L,b}(x) &:= \mathbf{E}_{\mathbf{Q}_x} \left[\frac{e^{V(w_b)-L} \mathbf{1}_{\{V(w_b)=I_b\}}}{\sum_{|u|=b} \mathbf{1}_{\{V(u)=I_b\}}}, \min_{0 \leq k \leq b} V(w_k) \geq 0, V(w_b) \leq L \right] \\ (3.20) \quad &\leq \mathbf{Q}_x \left(\min_{0 \leq k \leq b} V(w_k) \geq 0, V(w_b) \leq L \right). \end{aligned}$$

We choose n large enough so that $\Delta n \leq n - b$. Thus, applying the Markov property at time $n - b$ yields that

$$\begin{aligned} (3.21) \quad \Lambda_n(F) &= n^{3/2} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) f_{L,b}(V(w_{n-b}) - a_n(z+L)); \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z + K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L), \mathcal{E}_n \right]. \end{aligned}$$

Let us introduce the following quantity by removing the restriction to \mathcal{E}_n :

$$\begin{aligned} (3.22) \quad \Lambda_n^I(F) &:= n^{3/2} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F\left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta\right) f_{L,b}(V(w_{n-b}) - a_n(z+L)); \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z + K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L) \right]. \end{aligned}$$

We immediately observe that

$$\begin{aligned} (3.23) \quad \left| \Lambda_n(F) - \Lambda_n^I(F) \right| &\leq n^{3/2} e^{-z} \mathbf{Q} \left(f_{L,b}(V(w_{n-b}) - a_n(z+L)), \right. \\ &\quad \left. \min_{0 \leq k \leq n-b} V(w_k) \geq -z + K, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L); (\mathcal{E}_n)^c \right). \end{aligned}$$

By (3.20), we check that $\left| \Lambda_n(F) - \Lambda_n^I(F) \right| \leq n^{3/2} e^{-z} \mathbf{Q}(w_n \in J_{z,K,L}^\Delta(n), (\mathcal{E}_n)^c)$. Applying Lemma 3.3 again implies that

$$(3.24) \quad \left| \Lambda_n(F) - \Lambda_n^I(F) \right| \leq \eta(1+L)^2(1+z-K)e^{-z}.$$

Combining with (3.19), we obtain that for any $b \geq B(L, \eta)$, $z \geq K \geq K(\eta)$ and n large enough,

$$(3.25) \quad \left| \Pi_n(F) - \Lambda_n^I(F) \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z}.$$

Note that $(V(w_k); k \geq 1)$ is a centered random walk under \mathbf{Q} and that it is proved in [2] that $f_{L,b}$ satisfies the conditions of Lemma 2.4. By (I) of Lemma 2.4, we get that

$$(3.26) \quad \lim_{n \rightarrow \infty} \Lambda_n^I(F) = \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \delta)],$$

where $\alpha_{L,b}^I := C_1 \int_{x \geq 0} f_{L,b}(x) R_-(x) dx \in [0, \infty)$. Thus, by (3.25), one sees that for any $b \geq B(L, \eta)$ and $z \geq K \geq K(\eta)$,

$$(3.27) \quad \limsup_{n \rightarrow \infty} \left| \Pi_n(F) - \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z}.$$

Going back to (3.17), we deduce that for any $L \geq L_\varepsilon$, $b \geq B(L, \eta)$ and $z \geq K \geq K(\eta)$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| \Sigma_n(F, z) - \alpha_{L,b}^I R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \\ & \leq 2\eta(1+L)^2(1+z-K)e^{-z} + \left(e^K + \varepsilon(1+z-K) \right) e^{-z}. \end{aligned}$$

Recall that $\lim_{z \rightarrow \infty} \frac{R(z)}{z} = c_0$. We multiply each term by $\frac{e^z}{z}$, and then let z go to infinity to conclude that

$$(3.28) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \Sigma_n(F, z) - \alpha_{L,b}^I c_0 \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq 2\eta(1+L)^2 + \varepsilon.$$

In particular, taking $F \equiv 1$ gives that

$$(3.29) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) - \alpha_{L,b}^I c_0 \right| \leq 2\eta(1+L)^2 + \varepsilon.$$

It follows from Fact 3.4 that $|C - \alpha_{L,b}^I c_0| \leq 2\eta(1+L)^2 + \varepsilon$. We thus choose $0 < \varepsilon < C/10$ and $0 < \eta \leq \frac{\varepsilon}{2(1+L_\varepsilon)^2}$ so that $2C > \alpha_{L_\varepsilon, b}^I c_0 > C/2 > 0$.

Therefore, for any $\varepsilon \in (0, C/10)$, $0 < \eta \leq \frac{\varepsilon}{2(1+L_\varepsilon)^2}$, $L = L_\varepsilon$ and $b \geq B(L_\varepsilon, \eta)$,

$$(3.30) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq \frac{4\varepsilon}{C/2 - 2\varepsilon},$$

which completes the proof of (3.15) in the non-lattice case.

Step (II) (The conditioned convergence of $(\frac{I_n(s)}{\sigma\sqrt{n}}; 0 \leq s \leq \Delta)$ for $\Delta < 1$ in the lattice case) Assume that the law of \mathcal{L} is supported by $\alpha + \beta\mathbb{Z}$ with span β . Recall that $a_n(0) = \alpha n + \beta \lfloor \frac{3}{2} \frac{\ln n - \alpha n}{\beta} \rfloor$ and that $a_n(z) = a_n(0) - z$. We use the same notation of Step (I). Let us prove

$$(3.31) \quad \lim_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| = 0.$$

Suppose that $z \in \beta\mathbb{Z}$. Whereas the arguments of Step (I), we obtain that for any $L \geq L_\varepsilon$, $b \geq B(L, \eta)$, $z \geq K \geq K(\eta)$ and n sufficiently large,

$$(3.32) \quad \left| \Sigma_n(F, z) - \Lambda_n^{II}(F) \right| \leq 2\eta(1+L)^2(1+z-K)e^{-z} + \left(e^K + \varepsilon(1+z-K) \right) e^{-z},$$

where

$$\Lambda_n^{II}(F) = \Lambda^{II}(F, z, K, L, b) := e^{a_n(0)} e^{-z} \mathbf{E}_{\mathbf{Q}} \left[F \left(\frac{V(w_{\lfloor sn \rfloor})}{\sigma \sqrt{n}}; 0 \leq s \leq \Delta \right) \times \right. \\ \left. f_{L,b}(V(w_{n-b} - a_n(z+L))) \right]; \quad \min_{0 \leq k \leq n-b} V(w_k) \geq -z + K, \quad \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L).$$

Under \mathbf{Q} , the distribution of $V(w_1) - V(w_0)$ is also supported by $\alpha + \beta\mathbb{Z}$. Let $d = d(L, b) := \beta \lceil \frac{\alpha b - L}{\beta} \rceil - \alpha b + L$ and $\lambda_n := n^{3/2} e^{-a_n(0)}$. Recall that $f_{L,b}$ is well defined in (3.20), it follows from (II) of Lemma 2.4 that

$$(3.33) \quad \lim_{n \rightarrow \infty} \lambda_n \Lambda_n^{II}(F) = \alpha_{L,b}^{II} R(z-K) e^{-z} \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)].$$

where $\alpha_{L,b}^{II} := C_1 \beta \sum_{j \geq 0} f_{L,b}(\beta j + d) R_-(\beta j + d) \in [0, \infty)$. Observe that $1 \leq \lambda_n \leq e^\beta$. Combining with (3.32), we conclude that

$$(3.34) \quad \limsup_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{e^z}{z} \lambda_n \Sigma_n(F, z) - \alpha_{L,b}^{II} c_0 \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq e^\beta (2\eta(1+L)^2 + \varepsilon).$$

We admit for the moment that there exist $0 < c_9 < c_{10} < \infty$ such that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all L, b large enough. Then take $\varepsilon < \frac{c_9 c_0}{4e^\beta}$, $L = L_\varepsilon$, $\eta = \frac{\varepsilon}{2(1+L_\varepsilon)^2}$ and $b \geq B(L_\varepsilon, \eta)$ so that $e^\beta (2\eta(1+L)^2 + \varepsilon) < c_9 c_0 / 2 \leq \alpha_{L_\varepsilon, b}^{II} c_0 / 2 \leq 2c_{10} c_0$. Note that $\frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} = \frac{e^{-z} \lambda_n \Sigma_n(F, z)}{e^{-z} \lambda_n \Sigma_n(1, z)}$. We thus deduce from (3.34) that

$$(3.35) \quad \limsup_{\beta\mathbb{Z} \ni z \rightarrow \infty} \limsup_{n \rightarrow \infty} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \mathbf{E}[F(e_s, 0 \leq s \leq \Delta)] \right| \leq \frac{4\varepsilon}{c_9 c_0 / e^\beta - 2\varepsilon},$$

which tends to zero as $\varepsilon \downarrow 0$.

It remains to prove that $\alpha_{L,b}^{II} \in [c_9, c_{10}]$ for all L, b large enough. Instead of investigating the entire system, we consider the branching random walk killed at 0. Define

$$(3.36) \quad I_n^{kill} := \inf \{ V(u) : |u| = n, V(u_k) \geq 0, \forall 0 \leq k \leq n \},$$

and we get the following fact from Corollary 3.4 and Lemma 3.6 of [2].

Fact 3.5 ([2]) *There exists a constant $c_{11} > 0$ such that for any $n \geq 1$ and $x, z \geq 0$,*

$$(3.37) \quad \mathbf{P}_x(I_n^{kill} \leq a_n(z)) \leq c_{11}(1+x)e^{-x-z}.$$

Moreover, there exists $c_{12} > 0$ such that for any $n \geq 1$ and $z \in [0, a_n(1)]$,

$$(3.38) \quad \mathbf{P}(I_n^{kill} \leq a_n(z)) \geq c_{12}e^{-z}.$$

Even though Fact 3.5 is proved in [2] under the assumption that the distribution of \mathcal{L} is non-lattice, the lattice case is actually recovered from that proof.

Analogously, let $m^{kill,(n)}$ be the particle chosen uniformly in the set $\{u : |u| = n, V(u) = I_n^{kill}, \min_{0 \leq k \leq n} V(u_k) \geq 0\}$. Moreover, let $\Sigma_n^{kill}(1, z) := \mathbf{P}[I_n^{kill} \leq a_n(z)]$ and $\Pi_n^{kill}(1, z, z, L) := \mathbf{P}[I_n^{kill} \leq a_n(z), m^{kill,(n)} \in J_{z,z,L}^\Delta(n)]$. By (3.7) again, we check that for all $L \geq L_\varepsilon$,

$$(3.39) \quad \begin{aligned} & \left| \Sigma_n^{kill}(1, z) - \Pi_n^{kill}(1, z, z, L) \right| \\ & \leq \mathbf{P}\left[\exists |u| = n : V(u) \leq a_n(z); \min_{0 \leq k \leq n} V(u_k) \geq 0; \min_{\Delta n \leq k \leq n} V(u_k) \leq a_n(z+L)\right] \\ & \leq \varepsilon e^{-z}. \end{aligned}$$

Recounting the arguments of Step (1), one sees that for any $L \geq L_\varepsilon, b \geq B(L, \eta), z \geq K(\eta)$ and n sufficiently large,

$$(3.40) \quad \left| \Pi_n^{kill}(1, z, z, L) - \Lambda_n^{kill} \right| \leq 2\eta(1+L)^2 e^{-z},$$

where

$$(3.41) \quad \Lambda_n^{kill} := \mathbf{E}_{\mathbf{Q}} \left[f^{kill}(V(w_{n-b})); \min_{0 \leq k \leq n-b} V(w_k) \geq 0, \min_{\Delta n \leq k \leq n-b} V(w_k) \geq a_n(z+L) \right],$$

with $f^{kill}(x) := \mathbf{E}_{\mathbf{Q}_x} \left[\frac{e^{V(w_b)} \mathbf{1}_{\{V(w_b) = I_b^{kill}\}}}{\sum_{|u|=b} \mathbf{1}_{\{V(u) = I_b^{kill}, \min_{0 \leq j \leq b} V(u_j) \geq 0\}}}; \min_{0 \leq k \leq b} V(w_k) \geq a_n(z+L), V(w_b) \leq a_n(z) \right]$. For $\varepsilon > 0$ and n sufficiently large, it has been proved in [2] that

$$(3.42) \quad \left| e^z \Lambda_n^{II}(1, z, z, L, b) - \Lambda_n^{kill} \right| \leq \varepsilon.$$

Recalling the convergence (3.33) with $K = z$ and $F \equiv 1$, we deduce from (3.39), (3.40) and (3.42) that for any $L \geq L_\varepsilon, b \geq B(L, \eta)$ and $z \geq K(\eta)$,

$$(3.43) \quad \limsup_{n \rightarrow \infty} \left| \lambda_n \Sigma_n^{kill}(1, z) - \alpha_{L,b}^{II} e^{-z} \right| \leq e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right) e^{-z},$$

since $R(0) = 1$ and $1 \leq \lambda_n \leq e^\beta$. Fact 3.5 implies that $c_{12} \leq e^z \lambda_n \mathbf{P}(I_n^{kill} \leq a_n(z)) \leq c_{11} e^\beta$. Hence, we obtain that

$$(3.44) \quad c_{12} - e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right) \leq \alpha_{L,b}^{II} \leq e^\beta c_{11} + e^\beta \left(2\eta(1+L)^2 + 2\varepsilon \right).$$

Let $c_{10} := c_{11} e^\beta + c_{12}$ and $c_9 := 3c_{12}/4 > 0$. For any $\varepsilon < e^{-\beta} c_{12}/12$, we take $L = L_\varepsilon$ and $\eta \leq \varepsilon/2(1+L_\varepsilon)^2$. Then $c_{10} > \alpha_{L,b}^{II} \geq c_9 > 0$ for $b \geq B(L_\varepsilon, \eta)$. This completes the second step.

Step (III)(The tightness) Actually, it suffices to prove the following proposition.

Proposition 3.6 *For any $\eta > 0$,*

$$(3.45) \quad \lim_{\delta \rightarrow 0} \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \mid I_n \leq a_n(z) \right) = 0.$$

The first two steps allow us to obtain the following fact whether the distribution is lattice or non-lattice.

Fact 3.7 *There exist constants $c_{13}, c_{14} \in (0, \infty)$ such that*

$$(3.46) \quad c_{13} \leq \liminf_{z \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) \leq \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{e^z}{z} \mathbf{P}(I_n \leq a_n(z)) \leq c_{14}.$$

Proof of Proposition 3.6. First, we observe that for any $M \geq 1$ and $\delta \in (0, 1/2)$,

$$\begin{aligned} & \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \delta \sigma \sqrt{n}, I_n \leq a_n(z) \right) \\ & \leq \mathbf{P} \left(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n), I_n \leq a_n(z) \right) + \mathbf{P} \left(I_n(n - \lfloor \delta n \rfloor) \geq M \sigma \sqrt{\delta n}, I_n \leq a_n(z) \right) + \chi(\delta, z, n). \end{aligned}$$

where $\chi(\delta, z, n) := \mathbf{P} \left(m_n^{(n)} \in J_{z,0,L}^{1/2}(n), I_n(n - \lfloor \delta n \rfloor) \leq M \sigma \sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \right)$.

It follows from Lemma 3.2 that for any $\varepsilon > 0$, if $L \geq L_\varepsilon$, $n \geq 1$ and $z \geq 0$,

$$(3.47) \quad \mathbf{P} \left(m_n^{(n)} \notin J_{z,0,L}^{1/2}(n), I_n \leq a_n(z) \right) \leq (1 + \varepsilon(1+z))e^{-z}.$$

Then dividing each term of (3.47) by $\mathbf{P}(I_n \leq a_n(z))$ yields that

$$(3.48) \quad \begin{aligned} & \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta \sigma \sqrt{n} \mid I_n \leq a_n(z) \right) \\ & \leq \frac{(1 + \varepsilon(1+z))e^{-z}}{\mathbf{P}(I_n \leq a_n(z))} + \mathbf{P} \left(I_n(n - \lfloor \delta n \rfloor) \geq M \sigma \sqrt{\delta n} \mid I_n \leq a_n(z) \right) + \frac{\chi(\delta, z, n)}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

On the one hand, by Fact 3.7,

$$(3.49) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{(1 + \varepsilon(1 + z))e^{-z}}{\mathbf{P}(I_n \leq a_n(z))} \leq \frac{\varepsilon}{c_{13}}.$$

On the other hand, Steps (I) and (II) tell us that for any $1 > \delta > 0$ and $M \geq 1$,

$$(3.50) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left[I_n(n - \lfloor \delta n \rfloor) \geq M\sigma\sqrt{\delta n} \mid I_n \leq a_n(z) \right] = \mathbf{P}[e_{1-\delta} \geq M\sqrt{\delta}],$$

which, by Chebyshev's inequality, is bounded by $\frac{\mathbf{E}[e_{1-\delta}]}{M\sqrt{\delta}} = \frac{4\sqrt{1-\delta}}{M\sqrt{2\pi}}$. Consequently,

$$(3.51) \quad \begin{aligned} & \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n - k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \\ & \leq \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{\chi(\delta, z, n)}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

Let us estimate $\chi(\delta, z, n)$. One sees that

$$\chi(\delta, z, n) \leq \mathbf{E} \left[\sum_{|u|=n} \mathbf{1}_{\{u \in J_{z,L}^{1/2}(n); \sup_{0 \leq k \leq \delta n} |V(u_{n-k}) - V(u)| \geq \eta\sigma\sqrt{n}; V(u_{n-\lfloor \delta n \rfloor}) \leq M\sigma\sqrt{\delta n}\} \right].$$

By Lemma 2.4, it becomes that

$$\begin{aligned} \chi(\delta, z, n) & \leq \mathbf{E} \left[e^{S_n}; S_n \leq a_n(z), \underline{S}_n \geq -z, \underline{S}_{[n/2, n]} \geq a_n(z + L), \right. \\ & \quad \left. S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |S_{n-k} - S_n| \geq \eta\sigma\sqrt{n} \right] \\ & \leq n^{3/2} e^{-z} \Upsilon(\delta, z, n), \end{aligned}$$

where $\Upsilon(\delta, z, n) := \mathbf{P} \left(S_n \leq a_n(z), \underline{S}_n \geq -z, \underline{S}_{[n/2, n]} \geq a_n(z + L), S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n}, \sup_{0 \leq k \leq \delta n} |S_{n-k} - S_n| \geq \eta\sigma\sqrt{n}, S_{n-\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n} \right)$.

Reversing time yields that

$$(3.52) \quad \begin{aligned} \Upsilon(\delta, z, n) & \leq \mathbf{P} \left(\underline{S}_n^- \geq -a_n(0), \underline{S}_{n/2}^- \geq -L, -S_n \in [-a_n(z), -a_n(z + L)], \right. \\ & \quad \left. \sup_{0 \leq k \leq \delta n} | -S_k | \geq \eta\sigma\sqrt{n}, -S_{\lfloor \delta n \rfloor} \leq M\sigma\sqrt{\delta n} - a_n(z + L) \right). \end{aligned}$$

Applying the Markov property at time $\lfloor \delta n \rfloor$, we obtain that

$$(3.53) \quad \Upsilon(\delta, z, n) = \mathbf{E} \left[\Theta(-S_{\lfloor \delta n \rfloor}); \underline{S}_{\delta n}^- \geq -L, \sup_{0 \leq k \leq \delta n} | -S_k | \geq \eta\sigma\sqrt{n} \right],$$

where $\Theta(x) := \mathbf{1}_{\{x \leq M\sigma\sqrt{\delta n} - a_n(z+L)\}} \mathbf{P}_x \left(\underline{S}_{(1/2-\delta)n}^- \geq -L, \underline{S}_{(1-\delta)n}^- \geq -a_n(0), -S_{n-[\delta n]} \in [-a_n(z), -a_n(z+L)] \right)$. Reversing time again implies that

$$\Theta(x) \leq \mathbf{1}_{\{x \leq M\sigma\sqrt{\delta n}\}} \mathbf{P} \left(\underline{S}_{(1-\delta)n} \geq -z - L, \underline{S}_{[n/2, (1-\delta)n]} \geq a_n(z+2L), S_{n-[\delta n]} \in [x + a_n(z+L), x + a_n(z)] \right).$$

By (2.10), $\Theta(x) \leq c_{15}(1+z+L)(1+L)(1+M\sigma\sqrt{\delta n}+2L)n^{-3/2}$. Plugging it into (3.53) and taking n large enough so that $1+2L < \eta\sigma\sqrt{\delta n}$, we get that

$$\Upsilon(\delta, z, n) \leq c_{15}(1+z)(1+L)^2 n^{-3/2} (M+\eta)\sigma\sqrt{\delta n} \mathbf{E} \left[\underline{S}_{\delta n}^- \geq -L, \sup_{0 \leq k \leq \delta n} |-S_k| \geq \eta\sigma\sqrt{n} \right].$$

Recall that $\chi(\delta, z, n) \leq e^{-z} n^{3/2} \Upsilon(\delta, z, n)$. We check that

$$(3.54) \quad \chi(\delta, z, n) \leq c_{15} e^{-z} (1+z)(1+L)^2 (M+\eta)\sigma \times \mathbf{E}_L \left[\sup_{0 \leq k \leq \delta n} (-S_k) \geq \eta\sigma\sqrt{n} \mid \underline{S}_{\delta n}^- \geq 0 \right] \left(\sqrt{\delta n} \mathbf{P}_L \left[\underline{S}_{\delta n}^- \geq 0 \right] \right).$$

On the one hand, by Theorem 1.1 of [7], $\mathbf{E}_L \left[\sup_{0 \leq k \leq \delta n} (-S_k) \geq \eta\sigma\sqrt{n} \mid \underline{S}_{\delta n}^- \geq 0 \right]$ converges to $\mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})$ as $n \rightarrow \infty$. On the other hand, (2.7) shows that $\sqrt{\delta n} \mathbf{P}_L \left[\underline{S}_{\delta n}^- \geq 0 \right]$ converges to $C_- R_-(L)$ as $n \rightarrow \infty$. Therefore,

$$\limsup_{n \rightarrow \infty} \chi(\delta, z, n) \leq c_{15} e^{-z} (1+z)(1+L)^2 (M+\eta)\sigma C_- R_-(L) \times \mathbf{P} \left(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta} \right).$$

Going back to (3.51) and letting $z \rightarrow \infty$, we deduce from Fact 3.7 that

$$(3.55) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \leq \frac{\varepsilon}{c_{13}} + \frac{2}{M} + \frac{c_{15}(1+L)^2 (M+\eta)\sigma C_- R_-(L) \times \mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})}{c_{13}}.$$

Notice that $\mathbf{P}(\sup_{0 \leq s \leq 1} \mathcal{M}_s \geq \eta/\sqrt{\delta})$ decreases to 0 as $\delta \downarrow 0$. Take $M \geq 2/\varepsilon$. We conclude that for any $0 < \varepsilon < c_{13}$,

$$(3.56) \quad \limsup_{\delta \rightarrow 0} \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P} \left(\sup_{0 \leq k \leq \delta n} |I_n(n-k) - I_n| \geq \eta\sigma\sqrt{n} \mid I_n \leq a_n(z) \right) \leq \frac{\varepsilon}{c_{13}} + \varepsilon,$$

which completes the proof of Proposition 3.6. And Proposition 3.1 is thus proved. \square

4 Proof of Theorem 1.1

Let us prove the main theorem now. It suffices to prove that for any continuous functional $F : D([0, 1], \mathbb{R}) \rightarrow [0, 1]$, we have

$$(4.1) \quad \lim_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E} \left[F(e_s, 0 \leq s \leq 1) \right] \right| = 0.$$

Proof of (4.1). Define for $A \geq 0$,

$$(4.2) \quad \mathcal{Z}[A] := \{u \in \mathbb{T} : V(u) \geq A > \max_{k < |u|} V(u_k)\}.$$

For any particle $u \in \mathcal{Z}[A]$, there is a subtree rooted at u . If $|u| \leq n$, let

$$I_n(u) := \min_{v \geq u, |v|=n} V(v).$$

Moreover, assume m_n^u is the particle uniformly chosen in the set $\{|v| = n : v \geq u, V(v) = I_n(u)\}$. Similarly, we write $[\emptyset, m_n^u] := \{\emptyset =: m_0^u, m_1^u, \dots, m_n^u\}$. The trajectory leading to m_n^u is denoted by $\{V(m_k^u); 0 \leq k \leq n\}$. Let ω_A be the particle uniformly chosen in $\{u \in \mathcal{Z}[A] : |u| \leq n, I_n(u) = I_n\}$.

Let $\mathcal{Y}_A := \{\max_{u \in \mathcal{Z}[A]} |u| \leq M, \max_{u \in \mathcal{Z}[A]} V(u) \leq M\}$. Then for any $\varepsilon > 0$, there exist $M := M(A, \varepsilon)$ large enough such that $\mathbf{P}(\mathcal{Y}_A^c) \leq \varepsilon$. It follows that

$$(4.3) \quad \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \right| \\ \leq \varepsilon + \mathbf{P}[|I_n - a_n(0)| \geq A/2].$$

We then check that for $n \geq M$,

$$(4.4) \quad \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\ = \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{V(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right].$$

Define another trajectory $\{\tilde{V}(m_k^u); 0 \leq k \leq n\}$ as follows.

$$(4.5) \quad \tilde{V}(m_k^u) := \begin{cases} V(u) & \text{if } k < |u|; \\ V(m_k^u) & \text{if } |u| \leq k \leq n. \end{cases}$$

It follows that

$$\begin{aligned}
(4.6) \quad & \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] + o_n(1),
\end{aligned}$$

where $o_n(1) \rightarrow 0$ as n goes to infinity.

Define the sigma-field $\mathcal{G}_A := \sigma\{(u, V(u), I_n(u)); u \in \mathcal{Z}[A]\}$. Note that on \mathcal{Y}_A , $I_n = \min_{u \in \mathcal{Z}[A]} I_n(u)$ as long as $n \geq M$. One sees that $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$ is \mathcal{G}_A -measurable for all n large enough. Thus,

$$\begin{aligned}
(4.7) \quad & \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| \mathcal{G}_A, u = \omega_A \right]; \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right].
\end{aligned}$$

Further, we notice by the branching property that conditioned on $\{(u, V(u)); u \in \mathcal{Z}[A]\}$, the subtrees generated by $u \in \mathcal{Z}[A]$ are independent copies of the original one, started from $V(u)$, respectively. Therefore, given $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$,

$$\begin{aligned}
& \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{\tilde{V}(m_{\lfloor sn \rfloor}^u)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| \mathcal{G}_A, u = \omega_A \right] \\
&= \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{I(\lfloor s(n - |u|) \rfloor)}{\sigma \sqrt{n - |u|}}; 0 \leq s \leq 1 \right) \middle| I_{n-|u|} \leq a_n(-r_u) \right] + o_n(1),
\end{aligned}$$

where $r_u := \min\{\min_{v \in \mathcal{Z}[A] \setminus \{u\}} I_n(v) - a_n(0), A/2\} - V(u)$ is independent of $I_{n-|u|}$. Thus, (4.6) becomes that

$$\begin{aligned}
(4.8) \quad & \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] \\
&= \mathbf{E} \left[\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} \mathbf{E} \left[F \left(\frac{I(\lfloor s(n - |u|) \rfloor)}{\sigma \sqrt{n - |u|}}; 0 \leq s \leq 1 \right) \middle| I_{n-|u|} \leq a_n(-r_u) \right]; \right. \\
& \quad \left. \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] + o_n(1).
\end{aligned}$$

The event $\mathcal{Y}_A \cap \{|I_n - a_n(0)| \leq A/2\}$ ensures that $A/2 + M \geq -r_u \geq A/2$. The conditioned convergence has been given in Proposition 3.1. We need a slightly stronger version here.

According to Proposition 3.1, for any $\varepsilon > 0$, there exists $z_\varepsilon > 0$ such that for all $z \geq z_\varepsilon$,

$$(4.9) \quad \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| < \varepsilon.$$

Thus, for any $z \geq z_\varepsilon$, there exists $N_z \geq 1$ such that for any $n \geq N_z$,

$$(4.10) \quad \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| < 2\varepsilon.$$

Take $A = 2z_\varepsilon$ and $K = M$. We say that for n sufficiently large,

$$(4.11) \quad \sup_{z \in [z_\varepsilon, z_\varepsilon + K]} \left| \mathbf{E} \left[F \left(\frac{I(\lfloor s(n) \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \leq 3\varepsilon.$$

In the lattice case, (4.11) follows immediately. We only need to prove it in the non-lattice case.

Recall that $\Sigma_n(F, z) = \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); I_n \leq a_n(z) \right]$ with $0 \leq F \leq 1$. Then, for any $\ell > 0$ and $z \geq 0$,

$$(4.12) \quad \begin{aligned} & \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \\ & \leq \left| \frac{\Sigma_n(F, z) - \Sigma_n(F, z + \ell)}{\Sigma_n(1, z)} \right| + \left| \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \\ & = \frac{1}{\Sigma_n(1, z)} \left(\left| \Sigma_n(F, z) - \Sigma_n(F, z + \ell) \right| + \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \left| \Sigma_n(1, z + \ell) - \Sigma_n(1, z) \right| \right). \end{aligned}$$

Since $0 \leq F \leq 1$, the two following inequalities

$$\begin{aligned} \left| \Sigma_n(F, z) - \Sigma_n(F, z + \ell) \right| &= \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); a_n(z + \ell) < I_n \leq a_n(z) \right] \\ &\leq \mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z)), \end{aligned}$$

and $\frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \leq 1$ hold. Note also that $|\Sigma_n(1, z + \ell) - \Sigma_n(1, z)| = \mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z))$.

It follows that

$$(4.13) \quad \begin{aligned} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| &\leq 2 \frac{\mathbf{P}(a_n(z + \ell) < I_n \leq a_n(z))}{\mathbf{P}(I_n \leq a_n(z))} \\ &= 2 - 2 \frac{\mathbf{P}(I_n \leq a_n(z + \ell))}{\mathbf{P}(I_n \leq a_n(z))}. \end{aligned}$$

In view of Fact 3.4, we take $\frac{3}{2} \ln n - \Lambda_{\varepsilon'} \geq \ell + z > z \geq \Lambda_{\varepsilon'}$ so that for any $n \geq N_{\varepsilon'}$,

$$(4.14) \quad \frac{\mathbf{P}(I_n \leq a_n(z + \ell))}{\mathbf{P}(I_n \leq a_n(z))} \geq \frac{(C - \varepsilon')(z + \ell)e^{-z - \ell}}{(C + \varepsilon')ze^{-z}} \geq \frac{C - \varepsilon'}{C + \varepsilon'} e^{-\ell}.$$

For $\varepsilon' = C\varepsilon/8 > 0$, we choose $\zeta = \frac{\varepsilon}{4}$ so that $\frac{C-\varepsilon'}{C+\varepsilon'}e^{-\zeta} \geq 1 - \frac{\varepsilon}{2}$. As a consequence, for any $\Lambda_{\varepsilon'} \leq z \leq \frac{3}{2} \ln n - \Lambda_{\varepsilon'} - \zeta$, $0 \leq \ell \leq \zeta$ and $n \geq N_{\varepsilon'}$,

$$(4.15) \quad \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z + \ell)}{\Sigma_n(1, z + \ell)} \right| \leq 2 \left(1 - \frac{C - \varepsilon'}{C + \varepsilon'} e^{-\ell} \right) \leq \varepsilon.$$

For $\varepsilon > 0$, z_ε can be chosen so that $[z_\varepsilon, z_\varepsilon + K] \subset [\Lambda_{\varepsilon'}, \frac{3}{2} \ln n - \Lambda_{\varepsilon'}]$ for $n \geq e^K N_{\varepsilon'}$. For any integer $0 \leq j \leq \lceil K/\zeta \rceil$, let $z_j := z_\varepsilon + j\zeta$. Then $[z_\varepsilon, z_\varepsilon + K] \subset \cup_{0 \leq j \leq \lceil K/\zeta \rceil} [z_j, z_{j+1}]$. Take $N'_\varepsilon = \max_{0 \leq j \leq \lceil K/\zeta \rceil} \{N_{z_j}, e^K N_{\varepsilon'}\}$. By (4.10) and (4.15), we conclude that for any $n \geq N'_\varepsilon$,

$$\begin{aligned} & \sup_{z \in [z_\varepsilon, z_\varepsilon + K]} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \middle| I_n \leq a_n(z) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq \sup_{0 \leq j \leq \lceil K/\zeta \rceil} \left| \frac{\Sigma_n(F, z_j)}{\Sigma_n(1, z_j)} - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| + \sup_{0 \leq j < \lceil K/\zeta \rceil} \sup_{z_j \leq z \leq z_{j+1}} \left| \frac{\Sigma_n(F, z)}{\Sigma_n(1, z)} - \frac{\Sigma_n(F, z_j)}{\Sigma_n(1, z_j)} \right| \\ & \leq 3\varepsilon. \end{aligned}$$

We continue to prove the main theorem. Since $\sum_{u \in \mathcal{Z}[A]} \mathbf{1}_{(u=\omega_A)} = 1$, we deduce from (4.8) and (4.11) that for n sufficiently large,

$$\begin{aligned} & \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right); \mathcal{Y}_A, |I_n - a_n(0)| \leq A/2 \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq 3\varepsilon \mathbf{P}(\mathcal{Y}_A; |I_n - a_n(0)| \leq A/2) + o_n(1) + \mathbf{P}(\mathcal{Y}_A^c) + \mathbf{P}(|I_n - a_n(0)| \geq A/2) \\ & \leq 4\varepsilon + o_n(1) + \mathbf{P}(|I_n - a_n(0)| \geq A/2). \end{aligned}$$

Going back to (4.3), we conclude that for n large enough,

$$\left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \leq 5\varepsilon + 2\mathbf{P}(|I_n - a_n(0)| \geq A/2) + o_n(1).$$

Let n go to infinity and then make $\varepsilon \downarrow 0$. Therefore,

$$(4.16) \quad \begin{aligned} & \limsup_{n \rightarrow \infty} \left| \mathbf{E} \left[F \left(\frac{I_n(\lfloor sn \rfloor)}{\sigma \sqrt{n}}; 0 \leq s \leq 1 \right) \right] - \mathbf{E}[F(e_s, 0 \leq s \leq 1)] \right| \\ & \leq \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} 2\mathbf{P}(|I_n - a_n(0)| \geq z). \end{aligned}$$

It remains to show that $\limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(|I_n - a_n(0)| \geq z) = 0$. Because of Fact (3.7), it suffices to prove that

$$(4.17) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) = 0.$$

In the non-lattice case, Theorem 1.1 of [2] implies it directly. In the lattice case, we see that for n large enough,

$$(4.18) \quad \mathbf{P}(I_n \geq a_n(0) + z) \leq \mathbf{E} \left[\prod_{u \in \mathcal{Z}[A]} (1 - \Phi_u(z, n)); \mathcal{Y}_A \right] + \varepsilon,$$

with $\Phi_u(z, n) := \mathbf{P}(I_{n-|u|} \leq a_n(V(u) - z))$. Take $A = 2z$ here. Then it follows from Fact 3.7 that for n large enough and for any particle $u \in \mathcal{Z}[A]$,

$$(4.19) \quad \Phi_u(z, n) \geq c_{13}/2(V(u) - z)e^{z-V(u)} \geq \frac{c_{13}}{4}V(u)e^{z-V(u)}.$$

(4.18) hence becomes that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) &\leq \mathbf{E} \left[\prod_{u \in \mathcal{Z}[A]} (1 - \frac{c_{13}}{4}V(u)e^{z-V(u)}); \mathcal{Y}_A \right] + \varepsilon \\ &\leq \mathbf{E} \left[\exp \left(- \frac{c_{13}}{4}e^z \sum_{u \in \mathcal{Z}[A]} V(u)e^{-V(u)} \right) \right] + \varepsilon. \end{aligned}$$

It has been proved that as A goes to infinity, $\sum_{u \in \mathcal{Z}[A]} V(u)e^{-V(u)}$ converges almost surely to some limit D_∞ , which is strictly positive on the set of non-extinction of \mathbb{T} , (see (5.2) in [2]). We end up with

$$(4.20) \quad \limsup_{z \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{P}(I_n \geq a_n(0) + z) \leq \varepsilon,$$

which completes the proof of Theorem 1.1. \square

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