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CHRISTINE BERNARDI† AND TONI SAYAH‡

Abstract. In this paper we study the time dependent Stokes problem with mixed boundary conditions. The problem is discretized by the backward Euler's scheme in time and finite elements in space. We establish an optimal a posteriori error with two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization.

Keywords. Stokes equations, mixed boundary conditions, finite element method, a posteriori analysis.

1. Introduction.

Let Ω be a bounded simply-connected open domain in $\mathbb{R}^3$, with a Lipschitz-continuous connected boundary $\partial\Omega$, and let $[0,T]$ denote an interval in $\mathbb{R}$ where T is a positive constant. We consider a partition without overlap of $\partial\Omega$ into two connected parts Γ_m and Γ. Let also $\mathbf{n}$ be the unit outward normal vector to Ω on its boundary $\partial\Omega$. We intend to work with the following time dependent Stokes system:

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t}(t,\mathbf{x}) - \nu\Delta\mathbf{u}(t,\mathbf{x}) + \nabla p(t,\mathbf{x}) = \mathbf{f}(t,\mathbf{x}) && \text{in }]0,T[\times\Omega,\\
&\text{div } \mathbf{u}(t,\mathbf{x}) = 0 && \text{in } [0,T]\times\Omega,\\
&\mathbf{u}(t,\mathbf{x}) = \mathbf{u}_D && \text{on } [0,T]\times\Gamma,\\
&\mathbf{u}(t,\mathbf{x})\,.\,\mathbf{n}(\mathbf{x}) = u_m && \text{on } [0,T]\times\Gamma_m,\\
&\mathbf{curl}\,\mathbf{u}(t,\mathbf{x})\times\mathbf{n}(\mathbf{x}) = \mathbf{0} && \text{on } [0,T]\times\Gamma_m,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0 && \text{in } \Omega,
\end{aligned} \tag{1.1}$$

where $\mathbf{f}$ represents a density of body forces and the viscosity ν is a positive constant. The unknowns are the velocity $\mathbf{u}$ and the pressure p of the fluid.

Indeed, the system of partial differential equations in (1.1) is provided with mixed boundary conditions which are standard Dirichler conditions on the velocity on Γ and conditions on the normal component of the velocity and the tangential components of the vorticity $\mathbf{curl}\,\mathbf{u}$ on Γ_m. A huge amount of work has been made concerning the discretization of the Stokes problem with Dirichlet boundary conditions on the velocity, see [14], [15], [16] and the references therein, and the a posteriori analysis of a finite element discretization for the time-dependent problem has been performed in several papers [13], [7]. Also a variational formulation with three unknowns (the vorticity, the velocity and the pressure) has been proposed in [10], [11] for handling the new boundary conditions, and a posteriori estimates have been proved in [1] and [12] for simple discretizations.

The aim of this work is to extend the a posteriori estimates to the more realistic case of mixed boundary conditions. We propose a very standard low cost discretization relying on the Euler's implicit scheme in time combined with finite elements in space, and prove optimal a posteriori error estimates for the discrete problem. To do this, we have rather follow the approach of [7] introduced in [3] which consists in uncoupling as much as possible the time and space errors in view of a simple adaptivity strategy.

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The outline of the paper is as follows:

- Section 2 is devoted to the study of the continuous problem.
- In section 3, we introduce the discrete problem and we recall its main properties.
- In section 4, we study the a posteriori errors and derive optimal estimates.

2. Analysis of the model

We suppose that $\partial\Gamma_m = \partial\Gamma$ is a Lipschitz-continuous submanifold of $\partial\Omega$. For simplicity, we work with zero boundary and initial conditions $\mathbf{u}_D = \mathbf{0}$, $u_m = 0$, $\mathbf{u}_0 = \mathbf{0}$; indeed the extension to the case of general conditions is rather obvious and only hinted in what follows. In view of the variational formulation of Problem (1.1), we recall the formula

$$-\Delta \mathbf{u} = \mathbf{curl}(\mathbf{curl}\,\mathbf{u}) - \nabla(\operatorname{div}\mathbf{u}).$$

Then Problem (1.1) can equivalently be written as (we suppress the variables $\mathbf{x}$ and t for brevity)

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + \nu\,\mathbf{curl}(\mathbf{curl}\,\mathbf{u}) + \nabla p = \mathbf{f} && \text{in }]0,T[\times\Omega,\\
&\operatorname{div}\,\mathbf{u} = 0 && \text{in } [0,T]\times\Omega,\\
&\mathbf{u}\times\mathbf{n} = \mathbf{0} && \text{on } [0,T]\times\Gamma,\\
&\mathbf{u}\,.\,\mathbf{n} = 0 && \text{on } [0,T]\times\partial\Omega,\\
&\mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{0} && \text{on } [0,T]\times\Gamma_m\\
&\mathbf{u} = \mathbf{0} && \text{in } \{0\}\times\Omega.
\end{aligned} \tag{2.1}$$

The reason for choosing this modified form is that the last boundary condition, namely $\mathbf{curl}\,\mathbf{u}\times\mathbf{n} = \mathbf{0}$ on Γ_m, can now be treated as a natural boundary condition.

In order to write the variational formulation of the previous problem, we introduce the Sobolev spaces:

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega),\, \partial^\alpha v \in L^p(\Omega), \quad \forall\,|\,\alpha\,|\leq m\}, \qquad H^m(\Omega) = W^{m,2}(\Omega),$$

equipped with the following semi-norm and norm :

$$|\,v\,|_{m,p,\Omega} = \left\{\sum_{|\alpha|=m}\int_\Omega |\,\partial^\alpha v(\mathbf{x})\,|^p\,d\mathbf{x}\right\}^{1/p} \quad \text{and} \quad \|\,v\,\|_{m,p,\Omega} = \left\{\sum_{k\leq m} |\,v\,|^p_{k,p,\Omega}\right\}^{1/p}.$$

As usual, we shall omit p when $p = 2$ and denote by $(\cdot,\cdot)$ the scalar product of $L^2(\Omega)$. We also consider the spaces

$$H(\operatorname{div},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3, \operatorname{div}\,\mathbf{v} \in L^2(\Omega)\}$$

and

$$H(\mathbf{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3, \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^3\}.$$

We recall [15, Chap. I, Section 2] that the normal trace operator $\mathbf{v} \mapsto \mathbf{v}\,.\,\mathbf{n}$ is defined from $H(\operatorname{div},\Omega)$ onto $H^{-1/2}(\partial\Omega)$ and the tangential trace operator $\mathbf{v} \mapsto \mathbf{v}\times\mathbf{n}$ is defined from $H(\mathbf{curl},\Omega)$ into $H^{-1/2}(\partial\Omega)^3$. In view of the boundary conditions in system (2.1), we thus consider the spaces

$$H_0(\operatorname{div},\Omega) = \{\mathbf{v} \in H(\operatorname{div},\Omega), \mathbf{v}\,.\,\mathbf{n} = 0 \text{ on } \partial\Omega\}$$

and

$$H_*(\mathbf{curl},\Omega) = \{\mathbf{v} \in H(\mathbf{curl},\Omega), \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \Gamma\}.$$

We set

$$X(\Omega) = H_0(\operatorname{div},\Omega) \cap H_*(\mathbf{curl},\Omega)$$

equipped with the semi-norm

$$||\mathbf{v}||_{X(\Omega)} = \left(||\operatorname{div}\mathbf{v}||^2_{L^2(\Omega)} + ||\,\mathbf{curl}\,\mathbf{v}||^2_{L^2(\Omega)^3}\right)^{1/2}.$$

Since Ω is simply-connected, we recall from [2, Cor. 3.16] that this quantity is a norm, which is equivalent to the graph norm of $H(\mathrm{div},\Omega)\cap H(\mathbf{curl},\Omega)$, i.e., that there exists a constant c only depending on Ω such that

$$\forall \mathbf{v}\in X(\Omega),\quad ||\mathbf{v}||_{L^2(\Omega)^3}\le c||\mathbf{v}||_{X(\Omega)}. \tag{2.2}$$

We denote by $L^2_\circ(\Omega)$ the space of functions in $L^2(\Omega)$ with a zero mean-value on Ω, and we introduce the kernel

$$V=\big\{\mathbf{v}\in X(\Omega);\ \forall q\in L^2_\circ(\Omega),\ \int_\Omega q(\mathbf{x})\ \mathrm{div}\,\mathbf{v}(\mathbf{x})\,d\mathbf{x}=0\big\},$$

which is a closed subspace of $X(\Omega)$ and coincides with

$$V=\big\{\mathbf{v}\in X(\Omega);\ \mathrm{div}\,\mathbf{v}=0 \text{ in } \Omega\big\}.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a,b[$ with values in a separable functional space, say Y. More precisely, let $\|\cdot\|_Y$ denote the norm of Y; then for any r, $1\le r\le\infty$, we define

$$L^r(a,b;Y)=\Big\{f \text{ mesurable in }]a,b[;\ \int_a^b \|f(\mathbf{t})\|_Y^r\,dt<\infty\Big\},$$

equipped with the norm

$$\|\ f\ \|_{L^r(a,b;Y)}=\Big(\int_a^b \|f(\mathbf{t})\|_Y^r\,dt\Big)^{1/r},$$

with the usual modifications if $r=\infty$. It is a Banach space if Y is a Banach space.

We now assume that the data $\mathbf{f}$ belongs to $L^2(0,T;X(\Omega)')$ where $X(\Omega)'$ is the dual space of $X(\Omega)$, set $\mathbf{u}(t)=\mathbf{u}(t,.)$ and consider the following variational formulation in $]0,T[$: Find $\mathbf{u}(t)\in X(\Omega)$ such that,

$$\begin{gathered}\forall \mathbf{v}\in X(\Omega),\quad (\frac{\partial}{\partial t}\mathbf{u}(t),\mathbf{v})+\nu(\mathbf{curl}\,\mathbf{u}(t),\mathbf{curl}\,\mathbf{v})-(\mathrm{div}\,\mathbf{u}(t),p)=\langle\mathbf{f}(t),\mathbf{v}\rangle,\\ \forall q\in L^2_\circ(\Omega),\quad (\mathrm{div}\,\mathbf{u}(t),q)=0,\\ \mathbf{u}(0)=\mathbf{0}.\end{gathered} \tag{2.3}$$

Proposition 2.1. *Any solution of Problem* (2.3) *is a solution of Problem* (2.1) *where the first two equations are satisfied in the sense of distributions.*

Proof. Let $(\mathbf{u},p)$ be the solution of (2.3). Denoting by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with a compact support in Ω, we first take $\mathbf{v}$ in $\mathcal{D}(\Omega)^3$ in the first line of problem (2.3). This gives the first equation in problem (2.1). Next, it is readily checked from the Stokes formula that the second line of problem (2.3) is also satisfied when q is a constant, hence for all q in $L^2(\Omega)$. Thus, we take q in $\mathcal{D}(\Omega)$, which yields the second equation in problem (2.1). It also follows from the definition of $X(\Omega)$ that the first two boundary conditions in problem (2.1) hold. Finally, introducing an infinitely differentiable function $\boldsymbol{\varphi}$ with a compact support in Γ_m and choosing $\mathbf{v}$ as a lifting in $X(\Omega)\cap H^1(\Omega)^3$ of the extension of $\boldsymbol{\varphi}\times\mathbf{n}$ by zero to $\partial\Omega$ gives the last boundary condition of problem (2.1).

The spaces $L^2_\circ(\Omega)$ and $X(\Omega)$ verify a uniform inf-sup condition (see for instance [4] or [15, Chap. I, Cor. 2.4]): There exists a constant $\beta_*>0$ such that

$$\forall q\in L^2_\circ(\Omega),\qquad \sup_{\mathbf{v}\in X(\Omega)}\frac{\displaystyle\int_\Omega q(\mathbf{x})\ \mathrm{div}\,\mathbf{v}(\mathbf{x})\,d\mathbf{x}}{||\mathbf{v}||_{X(\Omega)}}\ge\beta_*||q||_{L^2(\Omega)}.$$

The arguments for the proof of the well-posedness of Problem 2.3 are exactly the same as [16, Chap. III, Thm 1.1], see also [14, Chap. V].

Proposition 2.2. *For any data* $\mathbf{f}$ *in* $L^2(0,T;X(\Omega)')$*, Problem* (2.3) *has a unique solution* $(\mathbf{u},p)$*.*

Remark 2.3. *This existence and uniqueness result easily extends to the case of non homogeneous boundary conditions as presented in system* (1.1)*, when these data stisfy*

$$\mathbf{u}_D \in L^2(0,T; H^{\frac{1}{2}}(\Gamma)^3), \qquad u_m \in L^2(0,T; H^{\frac{1}{2}}(\Gamma_m)), \qquad \mathbf{u}_0 \in L^2(\Omega)^3. \tag{2.4}$$

Remark 2.4. *In the case when Ω has a $C^{1,1}$ boundary or is convex, it is proved in* [2, Thm 2.17] *that the space $H_0(\mathrm{div}, \Omega) \cap H(\mathbf{curl}, \Omega)$ is contained in $H^1(\Omega)^3$. We recall also that when Ω is a polyhedron, the space of restrictions of functions of $X(\Omega)$ to $\Omega \setminus \bar{\Theta}$, where Θ is a neighborhood of the re-entrant corners of Ω inside Γ_m, is imbedded in $H^1(\Omega \setminus \bar{\Theta})$ (see the proof of* [4, Lemma 2.5] *for more details).*

3. The discrete problem

From now on, we assume that Ω is a polyhedron and that $\mathbf{f}$ belongs to $\mathcal{C}^0(0,T; X(\Omega)')$. In order to describe the time discretization with an adaptive choice of local time steps, we introduce a partition of the interval $[0,T]$ into subintervals $[t_{n-1}, t_n]$, $1 \le n \le N$, such that $0 = t_0 \le t_1 \le \cdots \le t_N = T$. We denote by τ_n the length of $[t_{n-1}, t_n]$, by τ the N-tuple $(\tau_1, \ldots, \tau_N)$, by $|\tau|$ the maximum of the τ_n, $1 \le n \le N$, and finally by σ_τ the regularity parameter

$$\sigma_\tau = \max_{2 \le n \le N} \frac{\tau_n}{\tau_{n-1}}.$$

From now on, we work with a regular family of partitions, i.e. we assume that σ_τ is bounded independently of τ.
We introduce the operator π_τ : For any Banach space X and any function g continuous from $]0,T]$ into X, $\pi_\tau g$ denotes the step function which is constant and equal to $g(t_n)$ on each interval $]t_{n-1}, t_n]$, $1 \le n \le N$. Similarly, with any sequence $(\phi_n)_{1 \le n \le N}$ in X, we associate the step function $\pi_\tau \phi_\tau$ which is constant and equal to ϕ_n on each interval $]t_{n-1}, t_n]$, $1 \le n \le N$.
Furthermore, for any Banach space X, with each family $(\mathbf{v}_n)_{0 \le n \le N}$ in X^{N+1}, we agree to associate the function $\mathbf{v}_\tau$ on $[0,T]$ which is affine on each interval $[t_{n-1}, t_n]$, $1 \le n \le N$, and equal to $\mathbf{v}_n$ at t_n, $0 \le n \le N$.

We now describe the space discretization. For each n, $0 \le n \le N$, let $(\mathcal{T}_{nh})_h$ be a regular family of triangulations of Ω by tetrahedra, in the usual sense that:

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_{nh}$;
- the intersection of two different elements of $\mathcal{T}_{nh}$, if not empty, is a vertex or a whole edge or a whole face of both of them;
- the ratio of the diameter of an element K in $\mathcal{T}_{nh}$ to the diameter of its inscribed sphere is bounded by a constant independent of n and h.

As usual, h denotes the maximal diameter of the elements of all $\mathcal{T}_{nh}$, $0 \le n \le N$, while for each n, h_n denotes the maximal diameter of the elements of $\mathcal{T}_{nh}$. For each κ in $\mathcal{T}_{nh}$ and each nonnegative integer k, we denote by $P_k(\kappa)$ the space of restrictions to κ of polynomials with 3 variables and total degree at most k.

In what follows, $c, c', C, C', c_1, \ldots$ stand for generic constants which may vary from line to line but are always independent of h and n. From now on, we call finite element space associated with $\mathcal{T}_{nh}$ a space of functions such that their restrictions to any element κ of $\mathcal{T}_{nh}$ belong to a space of polynomials of fixed degree.

For each n and h, we associate with $\mathcal{T}_{nh}$ two finite element spaces X_{nh} and M_{nh} which are contained in $X(\Omega)$ and $L^2_\circ(\Omega)$, respectively, and such that the following inf-sup condition holds for a constant $\beta > 0$:

$$\forall q_h \in M_{nh}, \quad \sup_{\mathbf{v}_h \in X_{nh}} \frac{\displaystyle\int_\Omega q_h(\mathbf{x})\, \mathrm{div}\, \mathbf{v}_h(\mathbf{x})\, d\mathbf{x}}{\|\mathbf{v}_h\|_{X(\Omega)}} \ge \beta \|q_h\|_{L^2(\Omega)}. \tag{3.1}$$

Indeed, there exist many examples of finite element spaces satisfying these conditions (the inf-sup condition being usually proved with X_{nh} replaced by $X_{nh} \cap H^1_0(\Omega)^3$), see [15, Chap. II]. We give one example of them dealing with continuous discrete pressures which is presented in [15, Chap. II, Section 4.1] for instance. The velocity is discretized with the "Mini-Element"

$$X_{nh} = \{\mathbf{v}_h \in X(\Omega);\ \forall \kappa \in \mathcal{T}_{nh},\ \mathbf{v}_h|_\kappa \in P_b(\kappa)^3\},$$

where the space $P_b(\kappa)$ is spanned by functions in $P_1(\kappa)$ and the bubble function on κ (for each element κ, the bubble function is equal to the product of the barycentric coordinates associated with the vertices of κ). The pressure is discretized with classical continuous finite element of order one

$$M_{nh} = \{q_h \in L^2_\circ(\Omega) \cap H^1(\Omega);\ \forall \kappa \in \mathcal{T}_{nh},\ q_h|_\kappa \in P_1(\kappa)\}.$$

As usual, we denote by V_{nh} the kernel

$$V_{nh} = \{\mathbf{v}_h \in X_{nh};\ \forall q_h \in M_{nh},\ \int_\Omega q_h(\mathbf{x})\ \mathrm{div}\ \mathbf{v}_h(\mathbf{x})\, d\mathbf{x} = 0\}.$$

The discrete problem associated with Problem (2.3) is: Knowing $\mathbf{u}_h^{n-1} \in X_{n-1h}$, find $(\mathbf{u}_h^n, p_h^n)$ with values in $X_{nh} \times M_{nh}$ solution of

$$\begin{aligned} \forall \mathbf{v}_h \in X_{nh}, \quad \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) \ +\ \nu(\mathbf{curl}\,\mathbf{u}_h^n, \mathbf{curl}\,\mathbf{v}_h) \ +\ \nu(\mathrm{div}\,\mathbf{u}_h^n, \mathrm{div}\,\mathbf{v}_h) \\ - (p_h^n, \mathrm{div}\,\mathbf{v}_h) = \langle \mathbf{f}^n, \mathbf{v}_h \rangle, \end{aligned} \tag{3.2}$$

$$\forall q_h \in M_{nh}, \quad (\mathrm{div}\,\mathbf{u}_h^n, q_h) = 0, \tag{3.3}$$

by assuming that $\mathbf{u}_h^0 = \mathbf{0}$ and taking

$$\mathbf{f}^n(\mathbf{x}) = \mathbf{f}(\mathbf{x}, t_n), \quad \text{for a.e} \quad \mathbf{x} \text{ in } \Omega. \tag{3.4}$$

We begin by showing a bound for the solution $\mathbf{u}_h^n$ of Problem (3.2) − (3.3).

Theorem 3.1. *At each time step, knowing $\mathbf{u}_h^{n-1} \in X_{n-1h}$, Problem* (3.2) − (3.3) *admits a unique solution $(\mathbf{u}_h^n, p_h^n)$ with values in $X_{nh} \times M_{nh}$. This solution satisfies, for $m = 1, \ldots, N$,*

$$\frac{1}{2}||\mathbf{u}_h^m||^2_{L^2(\Omega)^3} + \frac{\nu}{2}\sum_{n=1}^m \tau_n \|\mathbf{u}_h^n\|^2_{X(\Omega)} \le \frac{c^2}{\nu}||\pi_\tau \mathbf{f}||^2_{L^2(0,T;X(\Omega)')} \le \frac{c'^2}{\nu}||\mathbf{f}||^2_{L^\infty(0,T;X(\Omega)')}. \tag{3.5}$$

Proof. For $\mathbf{u}_h^{n-1} \in X_{n-1h}$, it is clear that Problem (3.2)–(3.3) has a unique solution $(\mathbf{u}_h^n, p_h^n)$ as a consequence of the coerciveness of the corresponding bilinear form on $X_{nh} \times X_{nh}$ and the inf-sup condition (3.1). Therefore, we take $\mathbf{v}_h = \mathbf{u}_h^n$ in (3.2) and we use the relation

$$a(a-b) = \frac{1}{2}a^2 + \frac{1}{2}(a-b)^2 - \frac{1}{2}b^2, \tag{3.6}$$

and inequality (2.2) to obtain the relation :

$$\frac{1}{2}||\mathbf{u}_h^n||^2_{L^2(\Omega)^3} - \frac{1}{2}||\mathbf{u}_h^{n-1}||^2_{L^2(\Omega)^3} + \nu\tau_n||\mathbf{u}_h^n||^2_{X(\Omega)} \le \frac{\tau_n\varepsilon}{2}||\mathbf{f}^n||^2_{X(\Omega)'} + \frac{\tau_n c^2}{2\varepsilon}||\mathbf{u}_h^n||^2_{X(\Omega)}.$$

We choose $\varepsilon = \dfrac{c^2}{\nu}$ and sum over $n = 1, \ldots m$. We obtain :

$$\frac{1}{2}||\mathbf{u}_h^m||^2_{L^2(\Omega)^3} + \frac{\nu}{2}\sum_{n=1}^m \tau_n\|\mathbf{u}_h^n\|^2_{X(\Omega)} \le \sum_{n=1}^m \frac{\tau_n c^2}{2\nu}||\mathbf{f}^n||^2_{X(\Omega)'}.$$

This implies the estimates.

Remark 3.2. *These results extend to the case of non homogeneous boundary conditions as presented in system* (1.1)*. Indeed, by assuming that these data satisfy*

$$\mathbf{u}_D \in \mathcal{C}^0([0,T] \times \Gamma)^3, \qquad u_m \in \mathcal{C}^0([0,T] \times \Gamma_m), \qquad \mathbf{u}_0 \in \mathcal{C}^0(\Omega)^3,$$

we can define a discrete problem where, with respect to (1.1)*, these data are replaced by appropriate interpolates. Then slightly more complex arguments (lifting of the discrete traces) lead to the existence and uniqueness result.*

4. A posteriori error analysis

We now intend to prove a posteriori error estimates between the exact solution $(\mathbf{u}, p)$ of Problem (2.3) and the numerical solution $(\mathbf{u}_h^n, p_h^n)$ of Problem (3.2) – (3.3). Several steps are needed for that.

4.1. Construction of the error indicators.

We first introduce the space

$$Z_{nh} = \{\mathbf{g}_h \in L^2(\Omega)^3;\ \forall \kappa \in \mathcal{T}_{nh},\ \mathbf{g}_h|_\kappa \in P_\ell(\kappa)^3\},$$

where ℓ is usually lower than the maximal degree of polynomials in X_{nh}, and, for $1 \le n \le N$, we fix an approximation $\mathbf{f}_h^n$ of the data $\mathbf{f}^n$ in Z_{nh}.

Next, for every element κ in $\mathcal{T}_{nh}$, we denote by
- ε_κ the set of faces of κ that are not contained in $\partial\Omega$,
- ε_κ^m the set of faces of κ which are contained in $\bar{\Gamma}_m$,
- Δ_κ the union of elements of $\mathcal{T}_{nh}$ that intersect κ,
- Δ_e the union of elements of $\mathcal{T}_{nh}$ that intersect the face e,
- h_κ the diameter of κ and h_e the diameter of the face e,
- and $[\cdot]_e$ the jump through e for each face e in an ε_κ (making its sign precise is not necessary).

Also, $\mathbf{n}_\kappa$ stands for the unit outward normal vector to κ on $\partial\kappa$.

For the demonstration of the next theorems, we introduce for an element κ of $\mathcal{T}_{nh}$, the bubble function ψ_κ (resp. ψ_e for the face e) which is equal to the product of the 4 barycentric coordinates associated with the vertices of κ (resp. of the 3 barycentric coordinates associated with the vertices of e). We also consider a lifting operator $\mathcal{L}_e$ defined on polynomials on e vanishing on ∂e into polynomials on the at most two elements κ containing e and vanishing on $\partial\kappa \setminus e$, which is constructed by affine transformation from a fixed operator on the reference element. We recall the next results from [17, Lemma 3.3].

Property 4.1. *Denoting by $P_r(\kappa)$ the space of polynomials of degree smaller than r on κ, we have*

$$\forall v \in P_r(\kappa), \qquad \begin{cases} c||v||_{0,\kappa} \le ||v\psi_\kappa^{1/2}||_{0,\kappa} \le c'||v||_{0,\kappa}, \\ |v|_{1,\kappa} \le c h_\kappa^{-1}||v||_{0,\kappa}. \end{cases} \tag{4.1}$$

Property 4.2. *Denoting by $P_r(e)$ the space of polynomials of degree smaller than r on e, we have*

$$\forall\, v \in P_r(e), \qquad c\|v\|_{0,e} \le \|v\psi_e^{1/2}\|_{0,e} \le c'\|v\|_{0,e},$$

and, for all polynomials v in $P_r(e)$ vanishing on ∂e, if κ is an element which contains e,

$$\|\mathcal{L}_e v\|_{0,\kappa} + h_e \mid \mathcal{L}_e v \mid_{1,\kappa} \le c h_e^{1/2}\|v\|_{0,e}.$$

We also introduce a Clément type regularization operator $\mathcal{C}_{nh}$ [8] which has the following properties, see [5, Section IX.3]: For any function $\mathbf{w}$ in $H^1(\Omega)^3$, $\mathcal{C}_{nh}\mathbf{w}$ belongs to the space of continuous affine finite elements and satisfies for any κ in $\mathcal{T}_{nh}$ and e in ε_κ,

$$||\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}||_{L^2(\kappa)^3} \le c h_\kappa ||\mathbf{w}||_{1,\Delta_\kappa} \qquad \text{and} \qquad ||\mathbf{w} - \mathcal{C}_{nh}\mathbf{w}||_{L^2(e)^3} \le c h_e^{1/2} ||\mathbf{w}||_{1,\Delta_e}. \tag{4.2}$$

For the a posteriori error studies, we consider the piecewise affine function $\mathbf{u}_h$ which take in the interval $[t_{n-1}, t_n]$ the values

$$\mathbf{u}_h(t) = \frac{t - t_{n-1}}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \mathbf{u}_h^{n-1},$$

and we prove optimal a posteriori error estimates by using the norm:

$$\begin{aligned}[][\mathbf{u} - \mathbf{u}_h]](t_n) = \Big(&||\mathbf{u}(t_n) - \mathbf{u}_h(t_n)||^2_{L^2(\Omega)^3} \\ &+ \nu \max\Big(\int_0^{t_n} ||\mathbf{u}(t) - \mathbf{u}_h(t)||^2_{X(\Omega)} dt, \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} ||\mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t)||^2_{X(\Omega)} dt\Big)\Big)^{1/2}.\end{aligned} \tag{4.3}$$

Since the solution of problem (2.3) is divergence-free, the solutions of Problems (2.3) and (3.2) – (3.3) verify for t in $]t_{n-1}, t_n]$ and for all $\mathbf{v}(t)$ in $X(\Omega)$:

$$\begin{gathered}(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t),\mathbf{v}(t))+\nu(\mathbf{curl}\,(\mathbf{u}(t)-\mathbf{u}_h(t)),\mathbf{curl}\,\mathbf{v}(t))+\nu(\mathrm{div}\,(\mathbf{u}(t)-\mathbf{u}_h(t)),\mathrm{div}\,\mathbf{v}(t))\\ -(\mathrm{div}\,\mathbf{v}(t),p(t)-\pi_\tau p_\tau(t))=(\mathbf{f}(t),\mathbf{v}(t))-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{v}(t))-\nu(\mathbf{curl}\,\mathbf{u}_h(t),\mathbf{curl}\,\mathbf{v}(t))\\ -\nu(\mathrm{div}\;\mathbf{u}_h(t),\mathrm{div}\,\mathbf{v}(t))+(\mathrm{div}\,\mathbf{v}(t),\pi_\tau p_\tau(t)),\end{gathered}\tag{4.4}$$

and for all $q(t)$ in $L^2_\circ(\Omega)$

$$\int_\Omega q(t,\mathbf{x})\;\mathrm{div}(\mathbf{u}(t,\mathbf{x})-\mathbf{u}_h(t,\mathbf{x}))\,d\mathbf{x}=-\int_\Omega q(t,\mathbf{x})\;\mathrm{div}\,\mathbf{u}_h(t,\mathbf{x})\,d\mathbf{x}\tag{4.5}$$

The residual $R(\mathbf{u}_h)$ is given in $L^2(0,T;X(\Omega)')$ by, for t in $]t_{n-1},t_n]$ and for all $\mathbf{v}(t)$ in $X(\Omega)$

$$\begin{aligned}\langle R(\mathbf{u}_h)(t),\mathbf{v}(t)\rangle &= (\mathbf{f}(t),\mathbf{v}(t))-(\frac{\partial\mathbf{u}_h}{\partial t}(t),\mathbf{v}(t))-\nu(\mathbf{curl}\,\mathbf{u}_h(t),\mathbf{curl}\,\mathbf{v}(t))\\ &\quad-\nu(\mathrm{div}\,\mathbf{u}_h(t),\mathrm{div}\,\mathbf{v}(t))+(\mathrm{div}\,\mathbf{v}(t),\pi_\tau p_\tau(t)).\end{aligned}\tag{4.6}$$

Using (3.2), we introduce the space residual R^h and the time residual R^τ:

$$R(\mathbf{u}_h)=(\mathbf{f}-\mathbf{f}^n)+(\mathbf{f}^n-\mathbf{f}_h^n)+R^h(\mathbf{u}_h)+R^\tau(\mathbf{u}_h),\tag{4.7}$$

such that, for t in $]t_{n-1},t_n]$, and for all $\mathbf{v}_h(t)$ in X_{nh}:

$$\langle R(\mathbf{u}_h)(t),\mathbf{v}(t)\rangle=\langle\mathbf{f}(t)-\mathbf{f}^n,\mathbf{v}(t)\rangle+\langle\mathbf{f}^n-\mathbf{f}_h^n+R^h(\mathbf{u}_h)(t),\mathbf{v}(t)-\mathbf{v}_h(t)\rangle+\langle R^\tau(\mathbf{u}_h)(t),\mathbf{v}(t)\rangle,\tag{4.8}$$

with

$$\begin{aligned}\langle R^h(\mathbf{u}_h)(t),\mathbf{v}(t)-\mathbf{v}_h(t)\rangle &=(\mathbf{f}_h^n-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1},\mathbf{v}(t)-\mathbf{v}_h(t))+(\mathrm{div}(\mathbf{v}(t)-\mathbf{v}_h(t)),p_h^n)\\ &\quad-\nu(\mathbf{curl}\;\mathbf{u}_h^n,\mathbf{curl}\,(\mathbf{v}(t)-\mathbf{v}_h(t)))-\nu(\mathrm{div}\,\mathbf{u}_h^n,\mathrm{div}\,(\mathbf{v}(t)-\mathbf{v}_h(t)))\\ &=\sum_{\kappa\in\mathcal{T}_{nh}}\Big\{\int_\kappa(\mathbf{f}_h^n-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})-\nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h^n+\nu\nabla\,\mathrm{div}\,\mathbf{u}_h^n-\nabla p_h^n)(\mathbf{x})\\ &\qquad\cdot(\mathbf{v}(t,\mathbf{x})-\mathbf{v}_h(t,\mathbf{x}))\,d\mathbf{x}\\ &\quad-\nu\sum_{e\in\varepsilon_\kappa}\int_e(\mathbf{curl}\;\mathbf{u}_h^n\times\mathbf{n}+\nu(\mathrm{div}\,\mathbf{u}_h^n)\mathbf{n}-p_h^n\mathbf{n})(\boldsymbol{\sigma})\cdot(\mathbf{v}(t,\boldsymbol{\sigma})-\mathbf{v}_h(t,\boldsymbol{\sigma}))\,d\boldsymbol{\sigma}\\ &\quad-\nu\sum_{e\in\varepsilon_\kappa^m}\int_e(\mathbf{curl}\;\mathbf{u}_h^n\times\mathbf{n})(\boldsymbol{\sigma})\cdot(\mathbf{v}(t,\boldsymbol{\sigma})-\mathbf{v}_h(t,\boldsymbol{\sigma}))\,d\boldsymbol{\sigma}\Big\}\end{aligned}\tag{4.9}$$

(where $\boldsymbol{\sigma}$ stands for the tangential coordinates on e) and

$$\begin{aligned}\langle R^\tau(\mathbf{u}_h)(t),\mathbf{v}(t)\rangle &=\nu(\mathbf{curl}\,(\mathbf{u}_h^n-\mathbf{u}_h(t)),\mathbf{curl}\;\mathbf{v}(t))+\nu(\mathrm{div}(\mathbf{u}_h^n-\mathbf{u}_h(t)),\mathrm{div}\,\mathbf{v}(t))\\ &=\frac{t_n-t}{\tau_n}\sum_{\kappa\in\mathcal{T}_{nh}}\Big\{\nu\int_\kappa\mathbf{curl}\,(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})(\mathbf{x})\cdot\;\mathbf{curl}\;\mathbf{v}(t,\mathbf{x})\,d\mathbf{x}\\ &\qquad+\nu\int_\kappa\mathrm{div}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})(\mathbf{x})\cdot\;\mathrm{div}\;\;\mathbf{v}(t,\mathbf{x})\,d\mathbf{x}\Big\}.\end{aligned}\tag{4.10}$$

All this leads to the following definition of the error indicators: For each κ in $\mathcal{T}_{nh}$,

$$\begin{aligned}(\eta_{n,\kappa}^h)^2 &= h_\kappa^2||\mathbf{f}_h^n-\frac{1}{\tau_n}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})-\nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h^n+\nu\nabla\,\mathrm{div}\,\mathbf{u}_h^n-\nabla p_h^n||_{0,\kappa}^2+||\,\mathrm{div}\,\mathbf{u}_h^n||_{0,\kappa}^2\\ &\quad+\sum_{e\in\varepsilon_\kappa}h_e||\,[\mathbf{curl}\,\mathbf{u}_h^n\times\mathbf{n}+\nu(\mathrm{div}\,\mathbf{u}_h^n)\mathbf{n}-p_h^n\mathbf{n}]_e||_{0,e}^2+\sum_{e\in\varepsilon_\kappa^m}h_e||\mathbf{curl}\,\mathbf{u}_h^n\times\mathbf{n}||_{0,e}^2,\\ (\eta_{n,\kappa}^\tau)^2 &= \tau_n\big(||\mathbf{curl}\,(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})||_{0,\kappa}^2+||\,\mathrm{div}(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})||_{0,\kappa}^2\big)=\tau_n||\mathbf{u}_h^n-\mathbf{u}_h^{n-1}||_{X(\Omega)}^2.\end{aligned}\tag{4.11}$$

Even if these indicators are a little complex, each term in them is easy to compute since it only depends on the discrete solution and involves (usually low degree) polynomials.

The following lemma justifies our choice of error indicators. We skip its proof which is nearly obvious (taking the definition of R^h in formula (4.9) with $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, using the Cauchy–Schwarz inequality and the continuity of the imbedding of $X(\Omega)$ in $H^1(\Omega)$, next by taking the definition of R^τ in formula (4.10) and using the Cauchy–Schwarz inequality).

Lemma 4.3. *The following estimates hold for* $1 \le n \le N$,

(1) *When* Ω *has no re-entrant corner inside* Γ_m*, for all* $\mathbf{v}$ *in* $X(\Omega)$ *and* $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$*:*

$$\langle R^h(\mathbf{u}_h), \mathbf{v} - \mathbf{v}_h \rangle \le C\Big(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta^h_{n,\kappa})^2 \Big)^{1/2} ||\mathbf{v}||_{X(\Omega)}. \tag{4.12}$$

(2) *For all* $\mathbf{v}$ *in* $X(\Omega)$ *and* t *in* $]t_{n-1}, t_n]$,

$$\langle R^\tau(\mathbf{u}_h)(t), \mathbf{v} \rangle \le C \frac{t_n - t}{\tau_n^{3/2}} \Big(\sum_{\kappa \in \mathcal{T}_{nh}} (\eta^\tau_{n,\kappa})^2 \Big)^{1/2} ||\mathbf{v}||_{X(\Omega)}. \tag{4.13}$$

4.2. **Upper bounds of the error.** To prove the upper bound, we follow the idea used by C. Bernardi and R. Verfürth in [7] in order to uncouple time and space errors. We introduce an auxiliary problem corresponding to the time discretization and calculate upper bounds for the errors between the solution of the last introduced problem and the exact solution firstly and the discrete solution secondly. Finally, we combine the obtained errors to derive the desired upper bound for the a posteriori error estimation.

We introduce the following time semi-discrete problem: Knowing $\mathbf{u}^{n-1}$ in $X(\Omega)$, find $(\mathbf{u}^n, p^n)$ with values in $X(\Omega) \times L^2_\circ(\Omega)$ solution of

$$\begin{aligned} \forall \mathbf{v} \in X(\Omega), \quad & \frac{1}{\tau_n}(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}) \; + \; \nu(\mathbf{curl}\,\mathbf{u}^n, \mathbf{curl}\,\mathbf{v}) \; + \; \nu(\operatorname{div}\mathbf{u}^n, \operatorname{div}\mathbf{v}) \\ & - (\operatorname{div}\mathbf{v}, p^n) = \langle \mathbf{f}(t_n), \mathbf{v} \rangle, \end{aligned} \tag{4.14}$$

$$\forall q \in L^2_\circ(\Omega), \quad (\operatorname{div}\mathbf{u}^n, q) = 0, \tag{4.15}$$

by assuming that $\mathbf{u}^0 = \mathbf{0}$. It is clear that Problem (4.14) – (4.15) has a unique solution owing to the ellipticity of the bilinear form and the inf-sup condition on the form for the divergence.

Theorem 4.4. *The following a posteriori error estimate holds between the velocity* $\mathbf{u}$ *of Problem* (2.3) *and the velocity* $\mathbf{u}_\tau$ *associated with the solutions* $(\mathbf{u}^n)_{0 \le n \le N}$ *of Problem* (4.14) – (4.15)*: For* $1 \le m \le N$,

$$\begin{aligned} & ||\mathbf{u}(t_m) - \mathbf{u}_\tau(t_m)||^2_{L^2(\Omega)} + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||^2_{X(\Omega)} ds \\ & \quad \le C\Big(\sum_{n=1}^m \sum_{\kappa \in \mathcal{T}_{nh}} (\eta^\tau_{n,\kappa})^2 + \sum_{n=1}^m \tau_n ||\mathbf{u}^n - \mathbf{u}^n_h||^2_{X(\Omega)} + ||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X(\Omega)')} \Big). \end{aligned} \tag{4.16}$$

Proof. By combining Problems (2.3) and (4.14)-(4.15), we observe that the pair $(\mathbf{u} - \mathbf{u}_\tau, p - \pi_\tau p_\tau)$ satisfies

$$(\mathbf{u} - \mathbf{u}_\tau)(0) = \mathbf{0} \qquad a.e. \text{ in } \Omega,$$

and that, for $1 \le n \le N$, for *a.e.* t in $]t_{n-1}, t_n]$ and for all $(\mathbf{v}, q) \in X(\Omega) \times L^2_\circ(\Omega)$,

$$\begin{aligned} & (\partial_t(\mathbf{u} - \mathbf{u}_\tau), \mathbf{v}) + \nu(\mathbf{curl}(\mathbf{u} - \mathbf{u}_\tau), \mathbf{curl}\,\mathbf{v}) + \nu(\operatorname{div}(\mathbf{u} - \mathbf{u}_\tau), \operatorname{div}\mathbf{v}) - (\operatorname{div}\mathbf{v}, p - \pi_\tau p_\tau) \\ & \qquad = \langle \mathbf{f} - \pi_\tau \mathbf{f}, \mathbf{v} \rangle + \nu(\mathbf{curl}(\mathbf{u}^n - \mathbf{u}_\tau), \mathbf{curl}\,\mathbf{v}) + \nu(\operatorname{div}(\mathbf{u}^n - \mathbf{u}_\tau), \operatorname{div}\mathbf{v}), \\ & -(\operatorname{div}(\mathbf{u} - \mathbf{u}_\tau), q) = 0. \end{aligned} \tag{4.17}$$

By taking in the last system $\mathbf{v} = \mathbf{u} - \mathbf{u}_\tau$ and $q = p - \pi_\tau p_\tau$ and subtracting the second line from the first one, we obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt} ||\mathbf{u} - \mathbf{u}_\tau||^2_{0,\Omega} + \nu ||\mathbf{u} - \mathbf{u}_\tau||^2_{X(\Omega)} \\ & \qquad \le \Big(\frac{1}{\nu^{1/2}} ||\mathbf{f} - \pi_\tau \mathbf{f}||_{X(\Omega)'} + \nu^{1/2} ||\mathbf{u}^n - \mathbf{u}_\tau||_{X(\Omega)} \Big) \nu^{1/2} ||\mathbf{u} - \mathbf{u}_\tau||_{X(\Omega)}, \end{aligned}$$

whence

$$\frac{d}{dt}||\mathbf{u}-\mathbf{u}_\tau||^2_{0,\Omega}+\nu||\mathbf{u}-\mathbf{u}_\tau||^2_{X(\Omega)}\leq 2\Big(\frac{1}{\nu}||\mathbf{f}-\pi_\tau\mathbf{f}||^2_{X(\Omega)'}+\nu||\mathbf{u}^n-\mathbf{u}_\tau||^2_{X(\Omega)}\Big).$$

We remark that, for all t in $[t_{n-1},t_n]$,

$$(\mathbf{u}^n-\mathbf{u}_\tau)(t)=\frac{t_n-t}{\tau_n}(\mathbf{u}^n-\mathbf{u}^{n-1}),$$

integrate the last inequality between t_{n-1} and t_n and sum over n to obtain

$$\begin{aligned}&||\mathbf{u}(t_m)-\mathbf{u}_\tau(t_m)||^2_{0,\Omega}+\nu\int_0^{t_m}||\mathbf{u}(s)-\mathbf{u}_\tau(s)||^2_{X(\Omega)}ds\\&\qquad\leq C\Big(||\mathbf{f}-\pi_\tau\mathbf{f}||^2_{L^2(0,t_m;X(\Omega)')}+\sum_{n=1}^m\tau_n||\mathbf{u}^n-\mathbf{u}^{n-1}||^2_{X(\Omega)}\Big).\end{aligned}$$

By using a triangle inequality, we have

$$||\mathbf{u}^n-\mathbf{u}^{n-1}||_{X(\Omega)}\quad\leq\quad||\mathbf{u}^n-\mathbf{u}_h^n||_{X(\Omega)}+||\mathbf{u}_h^n-\mathbf{u}_h^{n-1}||_{X(\Omega)}+||\mathbf{u}^{n-1}-\mathbf{u}_h^{n-1}||_{X(\Omega)},$$

whence the desired result follows due to the regularity of the family of partitions $[t_{n-1},t_n]$.

To derive an a posteriori estimate between the solution $\mathbf{u}$ of Problem (2.3) and the solution $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of (3.2)–(3.3), it suffices to obtain an a posteriori estimate between the solution $\mathbf{u}_\tau$ of Problem (4.14) – (4.15) and the solution $\mathbf{u}_h$, and to apply the triangle inequality using the previous theorem.

We observe that, for any $\mathbf{v}$ in $X(\Omega)$ and $\mathbf{v}_h$ in $X_{nh}(\Omega)$,

$$\begin{aligned}\frac{1}{\tau_n}((\mathbf{u}^n-\mathbf{u}^{n-1})-(\mathbf{u}_h^n-\mathbf{u}_h^{n-1}),\mathbf{v})\ +\ \nu(\mathbf{curl}(\mathbf{u}^n-\mathbf{u}_h^n),\mathbf{curl}\,\mathbf{v})\ +\ \nu(\operatorname{div}(\mathbf{u}^n-\mathbf{u}_h^n),\operatorname{div}\mathbf{v})\\-\ (\operatorname{div}\mathbf{v},p^n-p_h^n)=\langle\mathbf{f}^n-\mathbf{f}_h^n+R^h\mathbf{u}_h^n,\mathbf{v}-\mathbf{v}_h\rangle,\end{aligned}\tag{4.18}$$

and

$$\int_\Omega q(t,\mathbf{x})\ \operatorname{div}(\mathbf{u}^n-\mathbf{u}_h^n)(\mathbf{x})\,d\mathbf{x}=-\int_\Omega q(t,\mathbf{x})\ \operatorname{div}\mathbf{u}_h^n(\mathbf{x})\,d\mathbf{x}.\tag{4.19}$$

A further lemma is needed to handle the non-zero right-hand side of equation (4.19).

Let now Π denotes the operator defined from $X(\Omega)$ into itself as follows: For each $\mathbf{v}$ in $X(\Omega)$, $\Pi\mathbf{v}$ denotes the velocity $\mathbf{w}$ of the unique weak solution $(\mathbf{w},r)$ in $X(\Omega)\times L^2_\circ(\Omega)$ of the Stokes problem

$$\begin{aligned}&\forall\mathbf{t}\in X(\Omega),\quad(\mathbf{curl}\,\mathbf{w},\mathbf{curl}\,\mathbf{t})+(\operatorname{div}\mathbf{w},\operatorname{div}\mathbf{t})-(\operatorname{div}\mathbf{t},r)=0,\\&\forall q\in L^2_\circ(\Omega),\quad(\operatorname{div}\mathbf{w},q)=(\operatorname{div}\mathbf{v},q).\end{aligned}\tag{4.20}$$

The next lemma states some properties of the operator Π.

Lemma 4.5. *The operator Π has the following properties:*

(1) *For all $\mathbf{v}$ in V, $\Pi\mathbf{v}$ is zero.*
(2) *The following estimates hold for all $\mathbf{v}$ in $X(\Omega)$,*

$$||\mathbf{v}-\Pi\mathbf{v}||_{X(\Omega)}\leq||\mathbf{v}||_{X(\Omega)}\qquad\text{and}\qquad||\Pi\mathbf{v}||_{X(\Omega)}\leq\frac{1}{\beta_*}||\operatorname{div}\mathbf{v}||_{L^2(\Omega)}.$$

Proof. Part (1) of the lemma is obvious. Moreover, since $\mathbf{v}-\Pi\mathbf{v}$ has vanishing divergence, we conclude from the weak form of the Stokes problem that

$$\nu(\mathbf{curl}\,\Pi\mathbf{v},\mathbf{curl}(\mathbf{v}-\Pi\mathbf{v}))+\nu(\operatorname{div}\Pi\mathbf{v},\operatorname{div}(\mathbf{v}-\Pi\mathbf{v}))=(\operatorname{div}(\mathbf{v}-\Pi\mathbf{v}),r)=0.$$

This proves the first estimate in part (2) of the lemma. Similarly, we obtain

$$||\Pi\mathbf{v}||^2_{X(\Omega)}=(\mathbf{curl}\,\Pi\mathbf{v},\mathbf{curl}\,\Pi\mathbf{v})+(\operatorname{div}\Pi\mathbf{v},\operatorname{div}\Pi\mathbf{v})=(\operatorname{div}\Pi\mathbf{v},r)=(\operatorname{div}\mathbf{v},r)\leq||\operatorname{div}\mathbf{v}||_{L^2(\Omega)}||r||_{L^2(\Omega)}$$

and

$$\begin{aligned}\beta_*||r||_{L^2(\Omega)} &\leq \sup_{\mathbf{z}\in X(\Omega)} \frac{(\operatorname{div}\mathbf{z}, r)}{||\mathbf{z}||_{X(\Omega)}} = \sup_{\mathbf{z}\in X(\Omega)} \frac{(\mathbf{curl}\,\Pi\mathbf{v}, \mathbf{curl}\,\mathbf{z}) + (\operatorname{div}\Pi\mathbf{v}, \operatorname{div}\mathbf{z})}{||\mathbf{z}||_{X(\Omega)}} \\ &\leq ||\Pi\mathbf{v}||_{X(\Omega)}.\end{aligned}$$

This proves the second estimate in part (2) of the lemma.

We are now in a position to prove a posteriori estimate corresponding to the problem (4.18).

Theorem 4.6. *If the domain* Ω *has no re-entrant corner inside* Γ_m*, the following a posteriori error estimate holds between the solutions* u^m *and* u_h^m *of Problem* (4.14) – (4.15) *and* (3.2) – (3.3)

$$||\mathbf{u}^m - \mathbf{u}_h^m||_{0,\Omega}^2 + \sum_{n=1}^{m} \tau_n ||\mathbf{u}^n - \mathbf{u}_h^n||_{X(\Omega)}^2 \leq c \sum_{n=1}^{m} \tau_n \big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||_{0,\kappa}^2 + (\eta_{n,\kappa}^h)^2 \big). \tag{4.21}$$

Proof. For abbreviation we set

$$\mathbf{e}^n = \mathbf{u}^n - \mathbf{u}_h^n, \;\; 0 \leq n \leq N \qquad \text{and} \qquad \varepsilon^n = p^n - p_h^n, \;\; 0 \leq n \leq N.$$

For any n, $1 \leq n \leq N$, we then have

$$\begin{aligned}&\frac{1}{2}||\mathbf{e}^n||_{L^2(\Omega)^3}^2 - \frac{1}{2}||\mathbf{e}^{n-1}||_{L^2(\Omega)^3}^2 + \frac{1}{2}||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)}^2 + \nu\tau_n||\mathbf{e}^n||_{X(\Omega)}^2 \\ &\qquad = (\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\mathbf{e}^n).\end{aligned} \tag{4.22}$$

We obtain

$$\begin{aligned}&(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\mathbf{e}^n) \\ &\quad = (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\Pi\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\Pi\mathbf{e}^n) \\ &\quad +(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n - \Pi\mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}(\mathbf{e}^n - \Pi\mathbf{e}^n)) \\ &\quad +\nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}(\mathbf{e}^n - \Pi\mathbf{e}^n)) - \tau_n(\operatorname{div}(\mathbf{e}^n - \Pi\mathbf{e}^n), \varepsilon^n).\end{aligned} \tag{4.23}$$

By observing that $\operatorname{div}(\mathbf{e}^n - \Pi\mathbf{e}^n) = 0$ and inserting $\mathbf{v} = \mathbf{e}^n - \Pi\mathbf{e}_n$ in equation (4.18), this yields for every $\mathbf{v}_h \in X_{nh}(\Omega)$

$$\begin{aligned}&(\mathbf{e}^n - \mathbf{e}^{n-1}, \mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\mathbf{e}^n) \\ &\quad = (\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) + \nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\Pi\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\Pi\mathbf{e}^n) \\ &\quad +\tau_n\langle \mathbf{f}^n - \mathbf{f}_h^n, \mathbf{v} - \mathbf{v}_h\rangle + \tau_n\langle R^h\mathbf{u}_h^n, \mathbf{v} - \mathbf{v}_h\rangle.\end{aligned} \tag{4.24}$$

Next, we evaluate all the terms on the right-hand side separately. Taking into account that $\Pi\mathbf{e}_n = -\Pi\mathbf{u}_h^n$ and using Lemma 4.5, the first term can be bounded as:

$$\begin{aligned}(\mathbf{e}^n - \mathbf{e}^{n-1}, \Pi\mathbf{e}^n) &\leq \frac{1}{2}||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)^3}^2 + \frac{1}{2}||\Pi\mathbf{e}^n||_{L^2(\Omega)^3}^2 \\ &\leq \frac{1}{2}||\mathbf{e}^n - \mathbf{e}^{n-1}||_{L^2(\Omega)^3}^2 + c||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)^3}^2\end{aligned}$$

Similarly, we derive from Lemma 4.5 the estimate for the second and third terms

$$\begin{aligned}\nu\tau_n(\mathbf{curl}\,\mathbf{e}^n, \mathbf{curl}\,\Pi\mathbf{e}^n) + \nu\tau_n(\operatorname{div}\mathbf{e}^n, \operatorname{div}\Pi\mathbf{e}^n) &\leq \frac{\nu\tau_n}{4}||\mathbf{e}^n||_{X(\Omega)}^2 + \nu\tau_n||\Pi\mathbf{e}^n||_{X(\Omega)}^2 \\ &\leq \frac{\nu\tau_n}{4}||\mathbf{e}^n||_{X(\Omega)}^2 + c\tau_n||\operatorname{div}\mathbf{u}_h^n||_{L^2(\Omega)^3}^2\end{aligned}$$

To estimate the other terms in the right-hand side of (4.24), we take $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, and use the continuity of $X(\Omega)$ into $H^1(\Omega)$ (see Remark 2.4), Lemma 4.5, and the relation $ab \le \frac{a^2}{4} + b^2$, to derive

$$\begin{aligned}
\tau_n\langle \mathbf{f}^n - \mathbf{f}_h^n, \mathbf{v} - \mathcal{C}_{nh}\mathbf{v}\rangle &\le c_1\tau_n \sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa ||\mathbf{f}^n - \mathbf{f}_h^n||_{L^2(\kappa)^3} ||\mathbf{v}||^2_{H^1(\Delta_\kappa)^3}\\
&\le c_2\tau_n \Big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||^2_{L^2(\kappa)^3}\Big)^{1/2} ||\mathbf{v}||_{H^1(\Omega)^3}\\
&\le c_3\tau_n \Big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||^2_{L^2(\kappa)^3}\Big)^{1/2} ||\mathbf{v}||_{X(\Omega)}\\
&\le \frac{2c_3^2\tau_n}{\nu}\Big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||^2_{L^2(\kappa)^3}\Big) + \frac{\nu\tau_n}{8}||\mathbf{e}^n||^2_{X(\Omega)}.
\end{aligned}$$

To conclude, we bound the last term in the right-hand side of (4.24). We obtain by using the definition of R^h, Lemma 4.5 and the relation $ab \le \frac{a^2}{4} + b^2$:

$$\begin{aligned}
\tau_n\langle R^h\mathbf{u}_h^n, \mathbf{v} - \mathbf{v}_h\rangle &\le C\tau_n\Big(\sum_{\kappa\in\mathcal{T}_{nh}} (\eta_{n,\kappa}^h)^2\Big)^{1/2} ||\mathbf{v}||_{X(\Omega)}.\\
&\le \frac{C^2}{\nu}\sum_{\kappa\in\mathcal{T}_{nh}} \tau_n(\eta_{n,\kappa}^h)^2 + \frac{\nu\tau_n}{4}||\mathbf{e}^n||^2_{X(\Omega)}
\end{aligned}$$

Equation (4.24), the relation $||\operatorname{div}\mathbf{u}_h^n||^2_{0,\kappa} \le (\eta_{n,\kappa}^h)^2$ and the above bounds give

$$\frac{1}{2}||\mathbf{e}^n||^2_{L^2(\Omega)^3} - \frac{1}{2}||\mathbf{e}^{n-1}||^2_{L^2(\Omega)^3} + \frac{3}{8}\nu\tau_n||\mathbf{e}^n||^2_{X(\Omega)} \le C\tau_n \sum_{\kappa\in\mathcal{T}_{nh}} \big(h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n)||^2_{L^2(\kappa)^3} + (\eta_{n,\kappa}^h)^2\big) \quad (4.25)$$

Summing with respect to n yields the desired estimate.

Remark 4.7. *When Ω has at least a re-entrant corner inside Γ_m, it is only known* [9] *that $X(\Omega)$ is contained in $H^{\frac{1}{2}}(\Omega)$. So the previous estimate has to be replaced by*

$$||\mathbf{u}^m - \mathbf{u}_h^m||^2_{0,\Omega} + \sum_{n=1}^m \tau_n||\mathbf{u}^n - \mathbf{u}_h^n||^2_{X(\Omega)} \le c\sum_{n=1}^m \tau_n\big(\sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + h_\kappa^{-1}\,(\eta_{n,\kappa}^h)^2\big).$$

So, a lack of optimality of $\max_{\kappa\in\mathcal{T}_{nh}} h_\kappa^{-\frac{1}{2}}$ occurs in the upper bound.

Corollary 4.8. *If the domain Ω has no re-entrant corners inside Γ_m, the following a posteriori error estimate holds between the velocity $\mathbf{u}$ solution of Problem* (2.3) *and the velocity $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of Problem* (3.2) – (3.3)

$$\begin{aligned}
||\mathbf{u}(t_m) - \mathbf{u}_h^m||^2_{L^2(\Omega)} + \int_0^{t_m} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_{X(\Omega)}ds \le C\Big(\sum_{n=1}^m\sum_{\kappa\in\mathcal{T}_{nh}} \big(\tau_n(\eta_{n,\kappa}^h)^2 + (\eta_{n,\kappa}^\tau)^2\big)\\
+ \sum_{n=1}^m \tau_n \sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + ||\mathbf{f} - \pi_\tau\mathbf{f}||^2_{L^2(0,t_m;X(\Omega)')}\Big).
\end{aligned} \quad (4.26)$$

Proof. The proof is a direct consequence of Theorems 4.4 and 4.6. First, we use the triangle inequality

$$\begin{aligned}
||\mathbf{u}(t_n) - \mathbf{u}_h^n||^2_{L^2(\Omega)} + \int_0^{t_n} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_{X(\Omega)}ds \le 2||\mathbf{u}(t_n) - \mathbf{u}_\tau(t_n)||^2_{L^2(\Omega)}\\
+2\int_0^{t_n} ||\mathbf{u}(s) - \mathbf{u}_\tau(s)||^2_{X(\Omega)}ds + 2||\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n)||^2_{L^2(\Omega)} + 2\int_0^{t_n} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_{X(\Omega)}ds.
\end{aligned} \quad (4.27)$$

Second, the fact that $\mathbf{u}_\tau - \mathbf{u}_h$ is piecewise affine, equal to $\mathbf{u}^n - \mathbf{u}_h^n$ at t_n, gives, by using Simpson formula,

$$\int_{t_{n-1}}^{t_n} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_{X(\Omega)} ds = \frac{\tau_n}{3}\Big(||\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n)||^2_{X(\Omega)} + ||\mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1})||^2_{X(\Omega)}$$
$$+ \big(\mathbf{curl}(\mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1})), \mathbf{curl}(\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n))\big)$$
$$+ \big(\mathrm{div}(\mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1})), \mathrm{div}(\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n))\big)\Big)$$

and the inequalites $ab \geq -\frac{1}{4}a^2 - b^2$ and $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ yield the bound

$$\frac{\tau_n}{4}||\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n)||^2_{X(\Omega)} \leq \int_{t_{n-1}}^{t_n} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_{X(\Omega)} ds$$
$$\leq \frac{\tau_n}{2}(||\mathbf{u}_\tau(t_n) - \mathbf{u}_h(t_n)||^2_{X(\Omega)} + ||\mathbf{u}_\tau(t_{n-1}) - \mathbf{u}_h(t_{n-1})||^2_{X(\Omega)}).$$

By using the relation $\tau_n \leq \sigma_\tau \tau_{n-1}$, the last inequality yields

$$\frac{1}{4}\sum_{n=1}^m \tau_n ||\mathbf{u}^n - \mathbf{u}_h^n||^2_{X(\Omega)} \leq \sum_{n=1}^m \int_{t_{n-1}}^{t_n} ||\mathbf{u}_\tau(s) - \mathbf{u}_h(s)||^2_{X(\Omega)} ds \quad \leq \quad \frac{1+\sigma_\tau}{2}\sum_{n=1}^m \tau_n ||\mathbf{u}^n - \mathbf{u}_h^n||^2_{X(\Omega)}. \tag{4.28}$$

Theorems 4.4 and 4.6 conclude the result.

Next, we bound the function $\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))$.

Theorem 4.9. *If the domain Ω has no re-entrant corner inside Γ_m, the following a posteriori error estimate holds between the solution $(\mathbf{u}, p)$ of problem (2.3) and the pair $(\mathbf{u}_h, \pi_\tau p_\tau)$ associated with the solutions of Problem (3.2) – (3.3): For $1 \leq m \leq N$,*

$$||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))||_{L^2(0,t_m;X(\Omega)')} \leq C\Big(\sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} \big(\tau_n(\eta^h_{n,\kappa})^2 + (\eta^\tau_{n,\kappa})^2\big)$$
$$+ \sum_{n=0}^m \tau_n \sum_{\kappa\in\mathcal{T}_{nh}} h_\kappa^2 ||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,\kappa} + ||\mathbf{f} - \pi_\tau \mathbf{f}||^2_{L^2(0,t_m;X(\Omega)')}\Big). \tag{4.29}$$

Proof. We derive from (4.4) that

$$||\frac{\partial}{\partial t}(\mathbf{u} - \mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))||_{X(\Omega)'}$$
$$= \sup_{\mathbf{v}\in X(\Omega)} \frac{-\nu(\mathbf{curl}\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathbf{curl}\,\mathbf{v}) - \nu(\mathrm{div}\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathrm{div}\,\mathbf{v}) + \langle R(\mathbf{u}_h)(t), \mathbf{v}\rangle}{||\mathbf{v}||_{X(\Omega)}}. \tag{4.30}$$

For the first term of the right - side, we have

$$\nu(\mathbf{curl}\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathbf{curl}\,\mathbf{v}) + \nu(\mathrm{div}\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathrm{div}\,\mathbf{v}) \leq \nu||\mathbf{u}(t) - \mathbf{u}_h(t)||_{X(\Omega)}||\mathbf{v}||_{X(\Omega)}.$$

We use equation (4.8) with $\mathbf{v}_h = R_h\mathbf{v}$ and Lemma 4.3 to bound the second term of the right-hand side. Finally, by integrating over t from t_{n-1} to t_n, summing over n and using Corollary 4.8, we obtain the results.

To conclude the upper bound, we bound the quantity $\sum_{n=1}^m \int_{t_{m-1}}^{t_m} ||\mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t)||^2_{X(\Omega)} dt$.

Theorem 4.10. *The following a posteriori error estimate holds between the velocity $\mathbf{u}$ solution of Problem (2.3) and the velocity $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of Problem (3.2) – (3.3)*

$$\sum_{n=1}^m \int_{t_{n-1}}^{t_n} ||\mathbf{u}(s) - \pi_\tau \mathbf{u}_h(s)||^2_{X(\Omega)} ds \leq C\Big(\int_0^{t_n} ||\mathbf{u}(s) - \mathbf{u}_h(s)||^2_{X(\Omega)} ds + \sum_{n=1}^m \sum_{\kappa\in\mathcal{T}_{nh}} (\eta^\tau_{n,\kappa})^2\Big). \tag{4.31}$$

Proof. We consider the velocity $\mathbf{u}$ solution of Problem (2.3) and the velocity $\mathbf{u}_h$ corresponding to the solutions $\mathbf{u}_h^n$ of Problem (3.2) – (3.3). We have for t in $]t_{n-1}, t_n]$:

$$\begin{aligned}||\mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t)||^2_{X(\Omega)} &\leq \big(||\mathbf{u}(t) - \mathbf{u}_h(t)||_{X(\Omega)} + ||\mathbf{u}_h(t) - \mathbf{u}_h^n||_{X(\Omega)}\big)^2 \\ &\leq \big(||\mathbf{u}(t) - \mathbf{u}_h(t)||_{X(\Omega)} + \frac{t-t_n}{\tau_n}||\mathbf{u}_h^n - \mathbf{u}_h^{n-1}||_{X(\Omega)}\big)^2 \\ &\leq 2\big(||\mathbf{u}(t) - \mathbf{u}_h(t)||^2_{X(\Omega)} + \big(\frac{t-t_n}{\tau_n}\big)^2||\mathbf{u}_h^n - \mathbf{u}_h^{n-1}||^2_{X(\Omega)}\big).\end{aligned}$$

When integrating between t_{n-1} and t_n and summing over n, the estimate follows from the definition of the $\eta^\tau_{n,\kappa}$.

The bounds (4.26), (4.29) and (4.31) constitute our upper bounds.

Remark 4.11. *In the case of non-homogeneous boundary and initial conditions, the error indicators remain exactly the same. However, the estimates involve some further terms, due to the approximations of $\mathbf{u}_0$ (which is rather easy to treat) and $\mathbf{u}_D$ and u_m (which requires appropriate liftings of the traces on the boundary).*

4.3. **Upper bounds of the indicators.** We now prove upper bounds of the indicators (or equivalently lower bounds of the error) and we begin with the term $\eta^h_{n,\kappa}$.

Theorem 4.12. *The following estimate holds:*

$$\begin{aligned}\tau_n\,(\eta^h_{n,\kappa})^2 \quad \leq \quad & c\Big(||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))||^2_{L^2(t_{n-1},t_n;X(w_\kappa)')} + \nu||\mathbf{u}-\mathbf{u}_h^n||^2_{L^2(t_{n-1},t_n;X(w_\kappa))} \\ & + ||\mathbf{f} - \mathbf{f}^n||^2_{L^2(t_{n-1},t_n;X(w_\kappa)')} + \tau_n\,h_\kappa^2||\mathbf{f}^n - \mathbf{f}_h^n||^2_{0,w_\kappa}\Big),\end{aligned} \tag{4.32}$$

where w_κ denotes the union of the elements of $\mathcal{T}_{nh}$ that share at least a face with κ.

Proof. We denote by $L(W)$ the right-hand side of (4.32) for the domain w_κ replaced by a domain W. The solution $\mathbf{u}$ of Problem (2.3) and the solution velocity $\mathbf{u}_h$ associated with the solution $(\mathbf{u}_h^n)_{0\leq n\leq N}$ of Problem (3.2) – (3.3) verify : For all $\mathbf{v}$ in $X(\Omega)$, $\mathbf{v}_h$ in X_{nh} and t in $]t_{n-1}, t_n]$ $(1 \leq n \leq N)$,

$$\begin{aligned}&(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t), \mathbf{v}(t)) + \nu(\mathbf{curl}\,(\mathbf{u}(t) - \mathbf{u}_h^n), \mathbf{curl}\,\mathbf{v}(t)) + \nu(\mathrm{div}\,(\mathbf{u}(t)-\mathbf{u}_h^n), \mathrm{div}\,\mathbf{v}(t)) \\ &-(\mathrm{div}\,\mathbf{v}(t), (p(t) - \pi_\tau p_\tau(t))) = \langle \mathbf{f}(t) - \mathbf{f}^n, \mathbf{v}(t)\rangle + \langle \mathbf{f}^n - \mathbf{f}_h^n + R^h(\mathbf{u}_h), \mathbf{v}(t) - \mathbf{v}_h(t)\rangle.\end{aligned} \tag{4.33}$$

Next, we estimate successively every term of $\eta^h_{n,\kappa}$.
(1) First of all, we take $\mathbf{v}_h = \mathbf{0}$ and

$$\mathbf{v} = \mathbf{v}_\kappa = \begin{cases} (\mathbf{f}_h^n - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) - \nabla p_h^n - \nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h^n + \nu\nabla\,\mathrm{div}\,\mathbf{u}_h^n)\psi_\kappa & \text{on } \kappa \\ 0 & \text{on } \Omega\setminus\kappa\end{cases}$$

and we integrate between t_{n-1} and t_n to obtain:

$$\begin{aligned}&\tau_n||(\mathbf{f}_h^n - \tfrac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) - \nabla p_h^n - \nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h^n + \nu\nabla\,\mathrm{div}\,\mathbf{u}_h^n)\psi_\kappa^{1/2}||^2_{L^2(\kappa)^3} \\ &\leq c\Big(\tau_n^{1/2}||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))||_{L^2(t_{n-1},t_n;X(\kappa)')}|\mathbf{v}_\kappa|_{1,\kappa} \\ &+\nu\tau_n^{1/2}||\mathbf{u}-\mathbf{u}_h^n||_{L^2(t_{n-1},t_n;X(\kappa))}|\mathbf{v}_\kappa|_{1,\kappa} + \tau_n^{1/2}||\mathbf{f}-\mathbf{f}^n||_{L^2(t_{n-1},t_n;X(\kappa)')}||\mathbf{v}_\kappa||_{X(\kappa)} \\ &+\tau_n||\mathbf{f}^n - \mathbf{f}_h^n||_{0,\kappa}||\mathbf{v}_\kappa||_{0,\kappa}\Big).\end{aligned}$$

By using the inequality $ab \leq 2a^2 + \frac{1}{8}b^2$ for all the terms of the right-hand side, multiplying by h_κ^2, remarking that $||\mathbf{v}_\kappa||_{X(\kappa)} \leq c|\mathbf{v}_\kappa|_{1,\kappa}$ and thanks to Property 4.1, we obtain

$$\tau_n\,h_\kappa^2||(\mathbf{f}_h^n - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) - \nabla p_h^n - \nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_h^n + \nu\nabla\,\mathrm{div}\,\mathbf{u}_h^n)\psi_\kappa^{1/2}||^2_{L^2(\kappa)^3} \leq L(\kappa). \tag{4.34}$$

(2) Second, we take $\mathbf{v}_h = \mathbf{0}$ and for any $e \in \varepsilon_\kappa$, we denote by κ' the other element containing e. We introduce the function

$$R^h_{n,e} = \begin{cases} [\mathbf{curl}\,\mathbf{u}^n_h \times \mathbf{n} + \nu(\operatorname{div}\mathbf{u}^n_h)\mathbf{n} - p^n_h\mathbf{n}]_e & \text{if } e \in \varepsilon_\kappa \\ \mathbf{curl}\,\mathbf{u}^n_h \times \mathbf{n} \quad \text{if } e \in \varepsilon^m_\kappa, \end{cases} \tag{4.35}$$

and we take $\mathbf{v} = \mathbf{v}_e = \mathcal{L}_e(R^h_{n,e}\psi_e)$ extended by zero to Ω. Then, we integrate between t_{n-1} and t_n to obtain:

$$\begin{aligned}
&\tau_n||R^h_{n,e}\psi_e^{1/2}||^2_{L^2(e)^3} \\
&\quad \le c\Big(\tau_n||(\mathbf{f}^n_h - \frac{1}{\tau_n}(\mathbf{u}^n_h - \mathbf{u}^{n-1}_h) - \nabla p^n_h - \nu\,\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}^n_h + \nu\nabla\operatorname{div}\mathbf{u}^n_h)||_{L^2(\kappa\cup\kappa')^3}||\mathbf{v}_e||_{L^2(\kappa\cup\kappa')^3} \\
&\qquad +\tau_n^{1/2}||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t)\nabla(p(t) - \pi_\tau p_\tau(t))||_{L^2(t_{n-1},t_n;X(\kappa\cup\kappa')')}|\mathbf{v}_e|_{1,\kappa\cup\kappa'} \\
&\qquad +\nu\tau_n^{1/2}||\mathbf{u}-\mathbf{u}^n_h||_{L^2(t_{n-1},t_n;X(\kappa\cup\kappa'))}|\mathbf{v}_e|_{1,\kappa\cup\kappa'} + \tau_n^{1/2}||\mathbf{f}-\mathbf{f}^n||_{L^2(t_{n-1},t_n;X(\kappa\cup\kappa')')}||\mathbf{v}_e||_{X(\kappa\cup\kappa')} \\
&\qquad +\tau_n||\mathbf{f}^n - \mathbf{f}^n_h||_{0,\kappa\cup\kappa'}||\mathbf{v}_e||_{0,\kappa\cup\kappa'}\Big).
\end{aligned} \tag{4.36}$$

By using the inequality $ab \le 2a^2 + \frac{1}{8}b^2$ for all the terms of the right-hand side, multiplying by h_e, remarking that $||\mathbf{v}_\kappa||_{X(\kappa)} \le c|\mathbf{v}_\kappa|_{1,\kappa}$, using Property 4.1 and summing over $\partial\kappa$, we obtain:

$$\tau_n\Big(\sum_{e\in\varepsilon_\kappa} h_e||\mathbf{curl}\,\mathbf{u}^n_h \times \mathbf{n} + \nu(\operatorname{div}\mathbf{u}^n_h)\mathbf{n} - p^n_h\mathbf{n}||^2_{0,e} + \sum_{e\in\varepsilon^m_\kappa} h_e||\mathbf{curl}\,\mathbf{u}^n_h \times \mathbf{n}||^2_{0,e}\Big) \le L(w_\kappa) \tag{4.37}$$

(3) Finally, we take in the equation (4.5)

$$q(t) = q_\kappa(t) = \operatorname{div}\mathbf{u}_h(t)\xi_\kappa,$$

where ξ_κ denotes the characteristic function of κ, and we integrate between t_{n-1} and t_n to get

$$||\operatorname{div}\mathbf{u}_h||_{L^2(t_{n-1},t_n;L^2(\kappa)^3)} \le ||\operatorname{div}(\mathbf{u}-\mathbf{u}_h)||_{L^2(t_{n-1},t_n;L^2(\kappa)^3)}.$$

The same argument which gives the inequality (4.2), can be applied here and gives

$$\begin{aligned}
\frac{1}{4}\tau_n||\operatorname{div}\mathbf{u}^n_h||_{0,\kappa} &\le ||\operatorname{div}\mathbf{u}_h||_{L^2(t_{n-1},t_n;L^2(\kappa)^3)} \\
&\le ||\mathbf{u}-\mathbf{u}_h||_{L^2(t_{n-1},t_n;X(\kappa))}.
\end{aligned} \tag{4.38}$$

Combining estimates (4.34), (4.37) and (4.38) leads to desired results.

To finish the lower bound, we have to bound the term $\eta^\tau_{n,\kappa}$.

Theorem 4.13. *The following estimate holds*

$$\begin{aligned}
(\eta^\tau_{n,\kappa})^2 \le\ & c\big(||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \nabla(p(t) - \pi_\tau p_\tau(t))||^2_{L^2(t_{n-1},t_n;X(\kappa)')} + ||\mathbf{u}-\mathbf{u}_h||^2_{L^2(t_{n-1},t_n;X(\kappa))} \\
& +||\mathbf{f}-\mathbf{f}^n||^2_{L^2(t_{n-1},t_n;X(\kappa)')} + \tau_n h^2_\kappa||\mathbf{f}^n - \mathbf{f}^n_h||^2_{0,\kappa} + \tau_n(\eta^h_{n,\kappa})^2\Big).
\end{aligned} \tag{4.39}$$

Proof. We consider Equation (4.4) and the definition of the operators R, R^h and R^τ to obtain : For all $\mathbf{v} \in X(\Omega)$, $\mathbf{v}_h \in X_{nh}$ and $t \in]t_{n-1}, t_n]$ $(1 \le n \le N)$,

$$\begin{aligned}
&(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t), \mathbf{v}(t)) + \nu(\mathbf{curl}\,(\mathbf{u}(t) - \mathbf{u}_h(t)), \mathbf{curl}\,\mathbf{v}(t)) + \nu(\operatorname{div}(\mathbf{u}(t) - \mathbf{u}_h(t)), \operatorname{div}\mathbf{v}(t)) \\
&\qquad -(\operatorname{div}\mathbf{v}(t), (p(t) - \pi_\tau p_\tau(t))) = \langle(\mathbf{f}(t) - \mathbf{f}^n), \mathbf{v}(t)\rangle \\
&\qquad\qquad +\langle(\mathbf{f}^n - \mathbf{f}^n_h) + R^h(\mathbf{u}_h), \mathbf{v}(t) - \mathbf{v}_h(t)\rangle + \langle(R^\tau(\mathbf{u}_h), \mathbf{v}(t)\rangle.
\end{aligned} \tag{4.40}$$

By taking

$$\mathbf{v} = \mathbf{v}_\kappa = \begin{cases} (\mathbf{u}^n_h - \mathbf{u}^{n-1}_h)\psi_\kappa & \text{on } \kappa \\ 0 & \text{on } \Omega \setminus \kappa, \end{cases}$$

and $\mathbf{v}_h = \mathcal{C}_{nh}\mathbf{v}$, and by integrating between t_{n-1} and t_n and by using the Cauchy-Schartz inequality, we obtain:

$$\begin{aligned}\nu\frac{1}{2}\tau_n||(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\psi_\kappa^{1/2}||^2_{X(\kappa)} \quad \leq \quad & c\big(\tau_n^{1/2}||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \nabla(p(t)-\pi_\tau p_\tau(t))||_{L^2(t_{n-1},t_n;X(\kappa)')}|\mathbf{v}_\kappa|_{1,\kappa}\\ &+\nu\tau_n^{1/2}||\mathbf{u}-\mathbf{u}_h||_{L^2(t_{n-1},t_n;X(\kappa))}|\mathbf{v}_\kappa|_{1,\kappa}\\ &+\tau_n^{1/2}||\mathbf{f}-\mathbf{f}^n||_{L^2(t_{n-1},t_n;X(\kappa)')}||\mathbf{v}_\kappa||_{X(\kappa)}\\ &+\tau_n h_\kappa||\mathbf{f}^n - \mathbf{f}_h^n||_{0,\kappa}|\mathbf{v}_\kappa|_{1,\kappa} + \tau_n\gamma_{n,\kappa}|\mathbf{v}_\kappa|_{1,\kappa}\big).\end{aligned}$$

By using the inequality $ab \leq 2a^2 + \frac{1}{8}b^2$ for all the right hand side and Property 4.1, we obtain the result.

4.4. **Conclusions.** We have proved that, when the domain Ω has no re-entrant corner inside Γ_m, the pressure and the velocity verify the upper bound:

$$\begin{aligned}&[[\mathbf{u}-\mathbf{u}_h]]^2(t_m) + ||\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h)(t) + \nabla(p(t)-\pi_\tau p_\tau(t))||^2_{L^2(0,t_m;X(\Omega)')}\\ &\quad\leq C\Big(\sum_{n=1}^m\sum_{\kappa\in\mathcal{T}_{nh}}\big(\tau_n(\eta_{n,\kappa}^h)^2 + (\eta_{n,\kappa}^\tau)^2\big) + \sum_{n=1}^m\tau_n\sum_{\kappa\in\mathcal{T}_{nh}}h_\kappa^2||\mathbf{f}^n-\mathbf{f}_h^n||^2_{0,\kappa} + ||\mathbf{f}-\pi_\tau\mathbf{f}||^2_{L^2(0,t_m;X(\Omega)')}\Big),\end{aligned} \tag{4.41}$$

while the lower bounds follow from (4.39) and (4.32).

We observe that estimate (4.41) is optimal: Up to the terms involving the data, the full error is bounded by a constant times the sum of all indicators. Estimates (4.39) and (4.32) are local in space and local in time. The indicator $\eta_{n,\kappa}^\tau$ can be interpreted as a measure for the error of the time-discretization. Correspondingly, it can be used for controlling the step-size in time. On the other hand, the other indicator $\eta_{n,\kappa}^h$ can be viewed as a measure for the error of the space discretization and can be used to adapt the mesh-size in space. We refer to [6, Section 6] for the detailed description of a simple adaptivity strategy relying on similar estimates.

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