Realizability algebras III : some examples

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Introduction

The notion of *realizability algebra*, which was introduced in [17, 18], is a tool to study the proof-program correspondence and to build new models of set theory, which we call *realizability models of ZF*.
It is a variant of the well known notion of *combinatory algebra*, with a new instruction cc, and a new type for the *environments*.
The *sets of forcing conditions*, in common use in set theory, are (very) particular cases of realizability algebras ; and the forcing models of ZF are very particular cases of realizability models.

We show here how to extend an arbitrary realizability algebra, by means of a certain set of conditions, so that the axiom DC of dependent choice is realized.
In order to avoid introducing new instructions, we use an idea of A. Miquel [19].
This technique has applications of two kinds :

1. Construction of models of ZF + DC.
When the initial realizability algebra is not trivial (that is, if we are not in the case of forcing or equivalently, if the associated Boolean algebra $\gimel 2$ is $\neq \{0,1\}$), then we always obtain in this way a model of ZF *which satisfies DC + there is no well ordering of* $\mathbb{R}$.
By suitably choosing the realizability algebras, we can get, for instance, the relative consistency over ZF of the following two theories :

i) ZF + DC + there exists an increasing function $i \mapsto X_i$, from the countable atomless Boolean algebra $\mathcal{B}$ into $\mathcal{P}(\mathbb{R})$ such that :
$X_0 = \{0\}$; $i \neq 0 \Rightarrow X_i$ is uncountable ;
$X_i \cap X_j = X_{i \wedge j}$;
if $i \wedge j = 0$ then $X_{i \vee j}$ is equipotent with $X_i \times X_j$;
$X_i \times X_i$ is equipotent with X_i ;
there exists a surjection from X_1 onto $\mathbb{R}$;
if there exists a surjection from X_j onto X_i, then $i \leq j$;
if $i, j \neq 0, i \wedge j = 0$, there is no surjection from $X_i \oplus X_j$ onto $X_i \times X_j$;
more generally, if $A \subset \mathcal{B}$ and if there exists a surjection from $\bigcup_{j \in A} X_j$ onto X_i, then $i \leq j$ for some $j \in A$.

In particular, there exists a sequence of subsets of $\mathbb{R}$, the cardinals of which are not comparable, and also a sequence of subsets of $\mathbb{R}$, the cardinals of which are strictly decreasing.

ii) ZF + DC + there exists $X \subset \mathbb{R}$ such that :
X is uncountable and there is no surjection from X onto $\aleph_1$
(and therefore, every well orderable subset of X is countable) ;
$X \times X$ is equipotent with X ;
there exists a total order on X, every proper initial segment of which is countable ;
there exists a surjection from $X \times \aleph_1$ onto $\mathbb{R}$;
there exists an injection from $\aleph_1$ (thus also from $X \times \aleph_1$) into $\mathbb{R}$.

2. Curry-Howard correspondence.
With this technique of extension of realizability algebras, we can obtain a program from a proof, in ZF + DC, of an arithmetical formula F, which is a λ_c-term, that is, a λ-term containing cc, *but no other new instruction.*
This is a notable difference with the method given in [14, 15], where we use the instruction `quote` and which is, on the other hand, simpler and not limited to arithmetical formulas.

It is important to observe that the program we get in this way does not really depend on the given proof of $DC \to F$ in ZF, but only on the *program P extracted from this proof*, which is a closed λ_c-term. Indeed, we obtain this program by means of an operation of *compilation* applied to P (look at the remark at the end of the introduction of [17]).

Finally, apart from applications 1 and 2, we may notice theorem 26, which gives an interesting property of *every* realizability model : as soon as the Boolean algebra $\gimel 2$ is not trivial (i.e. if the model is not a forcing model), there exists a non well orderable individual.

1 Generalities

Realizability algebras

It is a first order structure, which is defined in [17]. We recall here briefly the definition and some essential properties :

A *realizability algebra* $\mathcal{A}$ is made up of three sets : Λ (the set of *terms*), Π (the set of *stacks*), $\Lambda \star \Pi$ (the set of *processes*) with the following operations :

$(\xi, \eta) \mapsto (\xi)\eta$ from Λ^2 into Λ (*application*) ;
$(\xi, \pi) \mapsto \xi \,.\, \pi$ from $\Lambda \times \Pi$ into Π (*push*) ;
$(\xi, \pi) \mapsto \xi \star \pi$ from $\Lambda \times \Pi$ into $\Lambda \star \Pi$ (*process*) ;
$\pi \mapsto \mathsf{k}_\pi$ from Π into Λ (*continuation*).

There are, in Λ, distinguished elements $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{cc}$, called *elementary combinators* or *instructions.*

Notation.
The term $(\ldots(((\xi)\eta_1)\eta_2)\ldots)\eta_n$ will be also written as $(\xi)\eta_1\eta_2\ldots\eta_n$ or $\xi\eta_1\eta_2\ldots\eta_n$.
For instance : $\xi\eta\zeta = (\xi)\eta\zeta = (\xi\eta)\zeta = ((\xi)\eta)\zeta$.

We define a preorder on $\Lambda \star \Pi$, denoted by $\succ$, which is called *execution* ;
$\xi \star \pi \succ \xi' \star \pi'$ is read as : *the process* $\xi \star \pi$ *reduces to* $\xi' \star \pi'$.
It is the smallest reflexive and transitive binary relation, such that, for any $\xi, \eta, \zeta \in \Lambda$ and $\pi, \varpi \in \Pi$, we have :

$(\xi)\eta \star \pi \succ \xi \star \eta \bullet \pi$.
$\mathsf{I} \star \xi \bullet \pi \succ \xi \star \pi$.
$\mathsf{K} \star \xi \bullet \eta \bullet \pi \succ \xi \star \pi$.
$\mathsf{W} \star \xi \bullet \eta \bullet \pi \succ \xi \star \eta \bullet \eta \bullet \pi$.
$\mathsf{C} \star \xi \bullet \eta \bullet \zeta \bullet \pi \succ \xi \star \zeta \bullet \eta \bullet \pi$.
$\mathsf{B} \star \xi \bullet \eta \bullet \zeta \bullet \pi \succ \xi \star (\eta)\zeta \bullet \pi$.
$\mathsf{cc} \star \xi \bullet \pi \succ \xi \star \mathsf{k}_\pi \bullet \pi$.
$\mathsf{k}_\pi \star \xi \bullet \varpi \succ \xi \star \pi$.

We are also given a subset $\perp\!\!\!\perp$ of $\Lambda \star \Pi$ such that :
$$\xi \star \pi \succ \xi' \star \pi',\ \xi' \star \pi' \in \perp\!\!\!\perp \ \Rightarrow\ \xi \star \pi \in \perp\!\!\!\perp.$$
Given two processes $\xi \star \pi, \xi' \star \pi'$, the notation $\xi \star \pi \succ\!\!\succ \xi' \star \pi'$ means :
$$\xi \star \pi \notin \perp\!\!\!\perp \Rightarrow \xi' \star \pi' \notin \perp\!\!\!\perp.$$
Therefore, obviously, $\xi \star \pi \succ \xi' \star \pi' \ \Rightarrow\ \xi \star \pi \succ\!\!\succ \xi' \star \pi'$.

Finally, we choose a set of terms $\mathrm{QP}_{\mathcal{A}} \subset \Lambda$, containing the elementary combinators : $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{cc}$ and closed by application. They are called the *proof-like terms of the algebra* $\mathcal{A}$. We write also QP instead of $\mathrm{QP}_{\mathcal{A}}$ if there is no ambiguity about $\mathcal{A}$.
The algebra $\mathcal{A}$ is called *coherent* if, for every proof-like term $\theta \in \mathrm{QP}_{\mathcal{A}}$, there exists a stack π such that $\theta \star \pi \notin \perp\!\!\!\perp$.

Remark. The *sets of forcing conditions* can be considered as degenerate cases of realizability algebras, if we present them in the following way : an inf-semi-lattice P, with a greatest element 1 and an initial segment $\perp\!\!\!\perp$ of P (the set of *false* conditions). Two conditions $p, q \in P$ are called *compatible* if their g.l.b. $p \wedge q$ is not in $\perp\!\!\!\perp$.
We get a realizability algebra if we set $\Lambda = \Pi = \Lambda \star \Pi = P$; $\mathsf{B} = \mathsf{C} = \mathsf{I} = \mathsf{K} = \mathsf{W} = \mathsf{cc} = 1$ and QP$= \{1\}$; $(p)q = p \bullet q = p \star q = p \wedge q$ and $\mathsf{k}_p = p$. The preorder $p \succ q$ is defined as $p \leq q$, i.e. $p \wedge q = p$. The condition of coherence is $1 \notin \perp\!\!\!\perp$.

c-terms and λ-terms

The terms of the language of combinatory algebra, which are built with variables, elementary combinators and the application (binary operation), will be called *combinatory terms* or c*-terms*, in order to distinguish them from the terms of the algebra $\mathcal{A}$, which are elements of Λ.
Each closed c-term (i.e. without variable) takes a value in the algebra $\mathcal{A}$, which is a proof-like term of $\mathcal{A}$.
Let us call *atom* a c-term of length 1, i.e. a constant symbol $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{cc}$ or a variable.

Lemma 1. *Every* c*-term* t *can be written, in a unique way, in the form* $t = (a)t_1 \ldots t_k$ *where* a *is an atom and* $t_1, \ldots, t_k$ *are* c*-terms.*

Immediate, by recurrence on the length of t.
Q.E.D.

The result of the *substitution of* $\xi_1, \ldots, \xi_n \in \Lambda$ *to the variables* $x_1, \ldots, x_n$ in a c-term t, is a *term* (i.e. an element of Λ) denoted by $t[\xi_1/x_1, \ldots, \xi_n/x_n]$ or, more briefly, $t[\vec{\xi}/\vec{x}]$. The inductive definition is :

$a[\vec{\xi}/\vec{x}] = \xi_i$ if $a = x_i (1 \le i \le n)$;
$a[\vec{\xi}/\vec{x}] = a$ if a is an atom $\neq x_1, \ldots, x_n$;
$(tu)[\vec{\xi}/\vec{x}] = (t[\vec{\xi}/\vec{x}])u[\vec{\xi}/\vec{x}]$.

Given a c-term t and a variable x, we define inductively on t, a new c-term denoted by $ʎx\,t$, which does not contain x. To this aim, we apply the first possible case in the following list :

1. $ʎx\,t = (\mathsf{K})t$ if t does not contain x.
2. $ʎx\,x = \mathsf{I}$.
3. $ʎx\,tu = (\mathsf{C}ʎx\,t)u$ if u does not contain x.
4. $ʎx\,tx = t$ if t does not contain x.
5. $ʎx\,tx = (\mathsf{W})ʎx\,t$ (if t contains x).
6. $ʎx(t)(u)v = ʎx(\mathsf{B})tuv$ (if uv contains x).

It is easy to see that this rewriting is finite, for any given c-term t : indeed, during the rewriting, no combinator is introduced inside t, but only in front of it. Moreover, the only changes in t are : moving parentheses and erasing occurrences of x. Now, rules 1 to 5 strictly decrease, and rule 6 does not increase, the part of t which remains under $ʎx$. Moreover, rule 6 can be applied consecutively only finitely many times.

Given a c-term t and a variable x, we now define the c-term $\lambda x\,t$ by setting :
$\lambda x\,t = ʎx\,(\mathsf{I})t$.

This enables us to translate every λ-term into a c-term. In the sequel, almost all c-terms will be written as λ-terms.

The fundamental property of this translation is given by theorem 2 :

Theorem 2. *Let t be a c-term with the only variables $x_1, \ldots, x_n$; let $\xi_1, \ldots, \xi_n \in \Lambda$ and $\pi \in \Pi$. Then $\lambda x_1 \ldots \lambda x_n\, t \star \xi_1 \,.\, \ldots \,.\, \xi_n \,.\, \pi \succ t[\xi_1/x_1, \ldots, \xi_n/x_n] \star \pi$.*

Lemma 3. *Let a be an atom, $t = (a)t_1 \ldots t_k$ a c-term with the only variables $x, y_1, \ldots, y_n$, and $\xi, \eta_1, \ldots, \eta_n \in \mathbf{\Lambda}$; then :*
$(ʎx\,t)[\vec{\eta}/\vec{y}] \star \xi \,.\, \pi \succ a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \,.\, \ldots \,.\, t_k[\xi/x, \vec{\eta}/\vec{y}] \,.\, \pi$.

The proof is done by induction on the number of rules 1 to 6 used to translate the term $ʎx\,t$. Consider the rule used first.

• Rule 1 : we have $(ʎx\,t)[\vec{\eta}/\vec{y}] \star \xi \,.\, \pi \equiv (\mathsf{K})t[\vec{\eta}/\vec{y}] \star \xi \,.\, \pi \succ \mathsf{K} \star t[\vec{\eta}/\vec{y}] \,.\, \xi \,.\, \pi \succ t[\vec{\eta}/\vec{y}] \star \pi$
$\equiv t[\xi/x, \vec{\eta}/\vec{y}] \star \pi$ because x is not in t. The result follows immediately.

• Rule 2 : we have $t = x, ʎx\,t = \mathsf{I}$ and the result is trivial.

In rules 3, 4, 5 or 6, we have $t = ut_k$ with $u = at_1 \ldots t_{k-1}$, by lemma 1.

• Rule 3 : $(ʎx\,t)[\vec{\eta}/\vec{y}] \star \xi \,.\, \pi \equiv ((\mathsf{C}ʎx\,u)t_k)[\vec{\eta}/\vec{y}] \star \xi \,.\, \pi \succ \mathsf{C} \star (ʎx\,u)[\vec{\eta}/\vec{y}] \,.\, t_k[\vec{\eta}/\vec{y}] \,.\, \xi \,.\, \pi$
$\succ (ʎx\,u)[\vec{\eta}/\vec{y}] \star \xi \,.\, t_k[\vec{\eta}/\vec{y}] \,.\, \pi$
$\succ a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \,.\, \ldots \,.\, t_{k-1}[\xi/x, \vec{\eta}/\vec{y}] \,.\, t_k[\vec{\eta}/\vec{y}]\pi$ by the induction hypothesis
$\equiv a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \,.\, \ldots \,.\, t_{k-1}[\xi/x, \vec{\eta}/\vec{y}] \,.\, t_k[\xi/x, \vec{\eta}/\vec{y}] \,.\, \pi$ since x is not in t_k.

In rules 4 and 5, we have $t_k = x$, i.e. $t = (u)x$.

• Rule 4 : we have $(\underline{\lambda} x\, t)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi \equiv u[\vec{\eta}/\vec{y}] \star \xi \cdot \pi \equiv u[\xi/x, \vec{\eta}/\vec{y}] \star \xi \cdot \pi$ because x is not in u. Since $u = at_1 \ldots t_{k-1}$ and $t_k = x$, the result follows immediately.

• Rule 5 : we have $t_k = x$ and $(\underline{\lambda} x\, t)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi \equiv (\mathsf{W}\underline{\lambda} x\, u)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi$
$\succ \mathsf{W} \star (\underline{\lambda} x\, u)[\vec{\eta}/\vec{y}] \cdot \xi \cdot \pi \succ (\underline{\lambda} x\, u)[\vec{\eta}/\vec{y}] \star \xi \cdot \xi \cdot \pi$
$\succ a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \cdot \ldots \cdot t_{k-1}[\xi/x, \vec{\eta}/\vec{y}] \cdot \xi \cdot \pi$ (by the induction hypothesis)
$\equiv a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \cdot \ldots \cdot t_k[\xi/x, \vec{\eta}/\vec{y}] \cdot \pi$.

• Rule 6 : we have $t_k = (v)w$ and $(\underline{\lambda} x\, t)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi \equiv (\underline{\lambda} x(\mathsf{B})uvw)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi$
$\succ \mathsf{B} \star u[\xi/x, \vec{\eta}/\vec{y}] \cdot v[\xi/x, \vec{\eta}/\vec{y}] \cdot w[\xi/x, \vec{\eta}/\vec{y}] \cdot \pi$ (by the induction hypothesis)
$\succ u[\xi/x, \vec{\eta}/\vec{y}] \star t_k[\xi/x, \vec{\eta}/\vec{y}] \cdot \pi$
$\succ a[\xi/x, \vec{\eta}/\vec{y}] \star t_1[\xi/x, \vec{\eta}/\vec{y}] \cdot \ldots \cdot t_{k-1}[\xi/x, \vec{\eta}/\vec{y}] \cdot t_k[\xi/x, \vec{\eta}/\vec{y}] \cdot \pi$.

Q.E.D.

Lemma 4. $(\lambda x\, t)[\vec{\eta}/\vec{y}] \star \xi \cdot \pi \succ t[\xi/x, \vec{\eta}/\vec{y}] \star \pi$.

Immediate by lemma 3 and the definition of $\lambda x\, t$ which is $\underline{\lambda} x(\mathsf{I})t$.

Q.E.D.

We can now prove theorem 2 by induction on n ; the case $n = 0$ is trivial.
We have $\lambda x_1 \ldots \lambda x_{n-1} \lambda x_n\, t \star \xi_1 \cdot \ldots \cdot \xi_{n-1} \cdot \xi_n \cdot \pi \succ (\lambda x_n t)[\xi_1/x_1, \ldots, \xi_{n-1}/x_{n-1}] \star \xi_n \cdot \pi$ (by induction hypothesis) $\succ t[\xi_1/x_1, \ldots, \xi_{n-1}/x_{n-1}, \xi_n/x_n] \star \pi$ by lemma 4.

Q.E.D.

The formal system

We write formulas and proofs in the language of first order logic. This formal language consists of :

• *individual variables* $x, y, \ldots$;
• *function symbols* $f, g, \ldots$ of various arities ; function symbols of arity 0 are called *constant symbols.*
• *relation symbols* ; there are three binary relation symbols : $\not\varepsilon, \notin, \subset$.

The terms of this first order language will be called *ℓ-terms* ; they are built in the usual way with individual variables and function symbols.

Remark. Thus, we use four expressions with the word *term* : term, c-term, λ-term and ℓ-term.

The *atomic formulas* are the expressions $\top, \bot, t \not\varepsilon\, u, t \notin u, t \subset u$, where t, u are ℓ-terms.

Formulas are built as usual, from atomic formulas, *with the only logical symbols* $\to, \forall$:
• each atomic formula is a formula ;
• if A, B are formulas, then $A \to B$ is a formula ;
• if A is a formula and x an individual variable, then $\forall x\, A$ is a formula.

Notations. Let $A_1, \ldots, A_n, A, B$ be formulas. Then :
$A \to \bot$ is written $\neg A$;
$A_1 \to (A_2 \to \cdots \to (A_n \to B) \cdots)$ is written $A_1, A_2, \ldots, A_n \to B$;
$\neg A_1, \ldots, \neg A_n \to \bot$ is written $A_1 \vee \ldots \vee A_n$;
$(A_1, \ldots, A_n \to \bot) \to \bot$ is written $A_1 \wedge \ldots \wedge A_n$;
$\neg \forall x(A_1, \ldots, A_n \to \bot)$ is written $\exists x\{A_1, \ldots, A_n\}$.

The *rules of natural deduction* are the following (the A_i's are formulas, the x_i's are variables of c-term, t, u are c-terms, written as λ-terms) :

1. $x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i$.
2. $x_1 : A_1, \ldots, x_n : A_n \vdash t : A \to B, \quad x_1 : A_1, \ldots, x_n : A_n \vdash u : A \quad \Rightarrow \quad x_1 : A_1, \ldots, x_n : A_n \vdash tu : B$.
3. $x_1 : A_1, \ldots, x_n : A_n, x : A \vdash t : B \quad \Rightarrow \quad x_1 : A_1, \ldots, x_n : A_n \vdash \lambda x\, t : A \to B$.
4. $x_1 : A_1, \ldots, x_n : A_n \vdash t : A \quad \Rightarrow \quad x_1 : A_1, \ldots, x_n : A_n \vdash t : \forall x\, A$ where x is an individual variable which does not appear in $A_1, \ldots, A_n$.
5. $x_1 : A_1, \ldots, x_n : A_n \vdash t : \forall x\, A \quad \Rightarrow \quad x_1 : A_1, \ldots, x_n : A_n \vdash t : A[\tau/x]$ where x is an individual variable and τ is a ℓ-term.
6. $x_1 : A_1, \ldots, x_n : A_n \vdash \mathsf{cc} : ((A \to B) \to A) \to A$ (law of Peirce).
7. $x_1 : A_1, \ldots, x_n : A_n \vdash t : \bot \quad \Rightarrow \quad x_1 : A_1, \ldots, x_n : A_n \vdash t : A$ for every formula A.

Realizability models

We formalize set theory with the first order language described above. We write, in this language, the axioms of a theory named ZF_ε, which are given in [18].
The usual set theory ZF is supposed written with the only relation symbols $\notin, \subset$.
Then, ZF_ε is a *conservative extension* of ZF, which is proved in [18].

Let us consider a *coherent* realizability algebra $\mathcal{A}$, defined in a model $\mathcal{M}$ of ZFL, which is called the *ground model*. The elements of $\mathcal{M}$ will be called *individuals* (in order to avoid the word *set*, as far as possible).

We defined, in [18], a *realizability model*, denoted by $\mathcal{N}_\mathcal{A}$ (or even $\mathcal{N}$, if there is no ambiguity about the algebra $\mathcal{A}$).
It has the same domain (the same individuals) as $\mathcal{M}$ and the interpretation of the function symbols is the same as in $\mathcal{M}$.

Each closed formula F of ZF_ε with parameters in $\mathcal{M}$, has *two truth values* in $\mathcal{N}$, which are denoted by $\|F\|$ (which is a subset of Π) and $|F|$ (which is a subset of Λ).
Here are their definitions :

$|F|$ is defined immediately from $\|F\|$ as follows :

$$\xi \in |F| \;\Leftrightarrow\; (\forall \pi \in \|F\|)\, \xi \star \pi \in \perp\!\!\!\perp.$$

We shall write $\xi \Vdash F$ (read " ξ *realizes* F ") for $\xi \in |F|$.

$\|F\|$ is now defined by recurrence on the length of F :

- F is atomic ;

then F has one of the forms $\top, \bot, a \not\varepsilon b, a \subset b, a \notin b$ where a, b are parameters in $\mathcal{M}$.
We set :

$\|\top\| = \emptyset$; $\|\bot\| = \Pi$; $\|a \not\varepsilon b\| = \{\pi \in \Pi;\ (a, \pi) \in b\}$.

$\|a \subset b\|, \|a \notin b\|$ are defined simultaneously by induction on $(\mathrm{rk}(a) \cup \mathrm{rk}(b), \mathrm{rk}(a) \cap \mathrm{rk}(b))$ ($\mathrm{rk}(a)$ being the rank of a in $\mathcal{M}$).

$$\|a \subset b\| = \bigcup_c \{\xi \,.\, \pi;\ \xi \in \Lambda,\ \pi \in \Pi,\ (c, \pi) \in a,\ \xi \Vdash c \notin b\} \ ;$$

$$\|a \notin b\| = \bigcup_c \{\xi \,.\, \xi' \,.\, \pi;\ \xi, \xi' \in \Lambda,\ \pi \in \Pi,\ (c, \pi) \in b,\ \xi \Vdash a \subset c,\ \xi' \Vdash c \subset a\}.$$

- $F \equiv A \to B$; then $\|F\| = \{\xi \,.\, \pi \ ; \ \xi \Vdash A, \ \pi \in \|B\|\}$.
- $F \equiv \forall x\, A$: then $\|F\| = \bigcup_a \|A[a/x]\|$.

The following theorem, proved in [18], is an essential tool :

Theorem 5 (Adequacy lemma)**.**
Let $A_1, \ldots, A_n, A$ be closed formulas of ZF_ε, and suppose that $x_1 : A_1, \ldots, x_n : A_n \vdash t : A$.
If $\xi_1 \Vdash A_1, \ldots, \xi_n \Vdash A_n$ then $t[\xi_1/x_1, \ldots, \xi_n/x_n] \Vdash A$.
In particular, if $\vdash t : A$, then $t \Vdash A$.

Let F be a closed formula of ZF_ε, with parameters in $\mathcal{M}$. We say that $\mathcal{N}_\mathcal{A}$ *realizes* F or that F *is realized in* $\mathcal{N}_\mathcal{A}$ (which is written $\mathcal{N}_\mathcal{A} \Vdash F$ or even $\Vdash F$), if there exists a proof-like term θ such that $\theta \Vdash F$.

It is shown in [18] that *all the axioms of ZF_ε are realized in $\mathcal{N}_\mathcal{A}$,* and thus also all the axioms of ZF.

Definitions. Given a set of terms $X \subset \Lambda$ and a formula F, we shall use the notation $X \to F$ as an *extended formula* ; its truth value is $\|X \to F\| = \{\xi \,.\, \pi \ ; \ \xi \in X, \ \pi \in \|F\|\}$.

Two formulas $F[x_1, \ldots, x_n]$ and $G[x_1, \ldots, x_n]$ of ZF_ε will be called *interchangeable* if the formula $\forall x_1 \ldots \forall x_n (F[x_1, \ldots, x_n] \leftrightarrow G[x_1, \ldots, x_n])$ is realized.
That is, for instance, the case if $\|F[a_1, \ldots, a_n]\| = \|G[a_1, \ldots, a_n]\|$
or also if $\|F[a_1, \ldots, a_n]\| = \|\neg\neg G[a_1, \ldots, a_n]\|$
for every $a_1, \ldots, a_n \in \mathcal{M}$.

The following lemma gives a useful example :

Lemma 6. *For every formula A, define $\urcorner A \subset \Lambda$ by $\urcorner A = \{\mathsf{k}_\pi \ ; \ \pi \in \|A\|\}$.*
Then $\neg A \to B$ and $\urcorner A \to B$ are interchangeable, for every formula B.

We have immediately $\mathsf{k}_\pi \Vdash \neg A$ for every $\pi \in \|A\|$. Therefore, $\|\urcorner A \to B\| \subset \|\neg A \to B\|$ and it follows that $\mathsf{I} \Vdash (\neg A \to B) \to (\urcorner A \to B)$.
Conversely, let $\xi, \eta \in \Lambda$, $\xi \Vdash \urcorner A \to B$, $\eta \Vdash \neg B$ and let $\pi \in \|A\|$.
We have $\xi \mathsf{k}_\pi \Vdash B$, thus $(\eta)(\xi)\mathsf{k}_\pi \Vdash \bot$ and therefore $(\eta)(\xi)\mathsf{k}_\pi \star \pi \in \perp\!\!\!\perp$.
It follows that $\theta \star \xi \,.\, \eta \,.\, \pi \in \perp\!\!\!\perp$ with $\theta = \lambda x \lambda y (\mathsf{cc}) \lambda k (y)(x) k$.
Finally, we have shown that $\theta \Vdash (\urcorner A \to B) \to (\neg B \to A)$, from which the result follows.
Q.E.D.

Equality and type-like sets

The formula $x = y$ is, by definition, $\forall z (x \notin z \to y \notin z)$ *(Leibniz equality).*

If t, u are ℓ-terms and F is a formula of ZF_ε, with parameters in $\mathcal{M}$, we define the formula $t = u \hookrightarrow F$. When it is closed, its truth value is :
$\|t = u \hookrightarrow F\| = \|\top\| = \emptyset$ if $\mathcal{M} \models t \neq u$; $\|t = u \hookrightarrow F\| = \|F\|$ if $\mathcal{M} \models t = u$.
The formula $t = u \hookrightarrow \bot$ is written $t \neq u$.
The formula $t_1 = u_1 \hookrightarrow (t_2 = u_2 \hookrightarrow \cdots \hookrightarrow (t_n = u_n \hookrightarrow F) \cdots)$ is written :
$t_1 = u_1, t_2 = u_2, \ldots, t_n = u_n \hookrightarrow F$.

The formulas $t = u \to F$ and $t = u \hookrightarrow F$ are interchangeable, as is shown in the :

Lemma 7.
i) $\mathsf{CII} \Vdash \forall x \forall y\, ((x = y \to F) \to (x = y \hookrightarrow F))$ *;*
ii) $\mathsf{CI} \Vdash \forall x \forall y\, ((x = y \hookrightarrow F) \to (x = y \to F))$*.*

i) Trivial.
ii) Let a, b be individuals ; let $\xi \Vdash a = b \hookrightarrow F$, $\eta \Vdash a = b$ and $\pi \in \|F\|$.
We show that $\eta \star \xi \centerdot \pi \in \perp\!\!\!\perp$.
Let $c = \{(b, \pi)\}$; by hypothesis on η, we have $\eta \Vdash a \notin c \to b \notin c$. Since $\pi \in \|b \notin c\|$, it suffices to show that $\xi \Vdash a \notin c$. This is clear if $a \neq b$, since $\|a \notin c\| = \emptyset$ in this case.
If $a = b$, then $\xi \Vdash F$, by hypothesis on ξ, thus $\xi \star \pi \in \perp\!\!\!\perp$; but $\|a \notin c\| = \{\pi\}$ in this case, and therefore $\xi \Vdash a \notin c$.

Q.E.D.

We set $\daleth X = X \times \Pi$ for every individual X of $\mathcal{M}$; we define the quantifier $\forall x^{\daleth X}$ as follows :

$$\|\forall x^{\daleth X} F[x]\| = \bigcup_{a \in X} \|F[a]\|.$$

Of course, we set $\exists x^{\daleth X} F[x] \equiv \neg \forall x^{\daleth X} \neg F[x]$.
The quantifier $\forall x^{\daleth X}$ has the intended meaning, which is that the formulas $\forall x^{\daleth X} F[x]$ and $\forall x (x \,\varepsilon\, \daleth X \to F[x])$ are interchangeable. This is shown by the :

Lemma 8.
$\mathsf{CI} \Vdash \forall x^{\daleth X} F[x] \to \forall x^{\daleth X} \neg\neg F[x]$ *;*
$\mathsf{cc} \Vdash \forall x^{\daleth X} \neg\neg F[x] \to \forall x^{\daleth X} F[x]$ *;*
$\|\forall x^{\daleth X} \neg\neg F[x]\| = \|\forall x (\neg F[x] \to x \notin \daleth X)\|$*.*

Immediate.

Q.E.D.

Each *functional* $f : \mathcal{M}^n \to \mathcal{M}$, defined in $\mathcal{M}$ by a formula of ZF with parameters, gives a function symbol, that we denote also by f, and which has the same interpretation in the realizability model $\mathcal{N}_{\mathcal{A}}$.

Proposition 9.
Let $t, t_1, \ldots, t_n, u, u_1, \ldots, u_n$ be ℓ-terms, built with variables $x_1, \ldots, x_k$ and functional symbols of $\mathcal{M}$.
If $\mathcal{M} \models \forall x_1 \ldots \forall x_k (t_1 = u_1, \ldots, t_k = u_k \to t = u)$*, then :*
$\mathsf{I} \Vdash \forall x_1 \ldots \forall x_k (t_1 = u_1, \ldots, t_k = u_k \hookrightarrow t = u)$*.*
If $\mathcal{M} \models (\forall x_1 \in X_1) \ldots (\forall x_k \in X_k)(t_1 = u_1, \ldots, t_k = u_k \to t = u)$*, then :*
$\mathsf{I} \Vdash \forall x_1^{\daleth X_1} \ldots \forall x_k^{\daleth X_k} (t_1 = u_1, \ldots, t_k = u_k \hookrightarrow t = u)$*.*

Trivial.

Q.E.D.

Proposition 10. *If $f : X_1 \times \cdots \times X_n \to Y$ is a function in $\mathcal{M}$, its interpretation in $\mathcal{N}_{\mathcal{A}}$ is a function $f : \daleth X_1 \times \cdots \times \daleth X_n \to \daleth Y$.*

Indeed, let $f', f'' : \mathcal{M}^n \to \mathcal{M}$ be any two functionals which are extensions of the function f to the whole of $\mathcal{M}^n$. By proposition 9(ii), we have :
$\mathsf{I} \Vdash \forall x_1^{\daleth X_1} \ldots \forall x_k^{\daleth X_k} (f'(x_1, \ldots, x_k) = f''(x_1, \ldots, x_k))$.

Q.E.D.

An important example is the set $2 = \{0,1\}$ equipped with the trivial boolean functions, written $\wedge, \vee, \neg$. The extension to $\mathcal{N}_{\mathcal{A}}$ of these operations gives a structure of Boolean algebra on $\daleth 2$. It is called the *characteristic Boolean algebra* of the model $\mathcal{N}_{\mathcal{A}}$.

Conservation of well-foundedness

Theorem 11 says that every well founded relation in the ground model $\mathcal{M}$, gives a well founded relation in the realizability model $\mathcal{N}$.

Theorem 11. *Let $f : \mathcal{M}^2 \to 2$ be a function defined in the ground model $\mathcal{M}$ such that $f(x,y) = 1$ is a well founded relation on $\mathcal{M}$. Then, for every formula $F[x]$ of ZF_ε with parameters in $\mathcal{M}$:*
$\mathsf{Y} \Vdash \forall y\, (\forall x(f(x,y) = 1 \hookrightarrow F[x]) \to F[y]) \to \forall y\, F[y]$
with $\mathsf{Y} = AA$ *and* $A = \lambda a \lambda f (f)(a) a f$ *(or* $A = (\mathsf{W})(\mathsf{B})(\mathsf{BW})(\mathsf{C})\mathsf{B}$*).*

Let us fix $b \in X$ and let $\xi \Vdash \forall y\, (\forall x(f(x,y) = 1 \hookrightarrow F[x]) \to F[y])$.
We show, by induction on b, following the well founded relation $f(x,y) = 1$, that :
$\mathsf{Y} \star \xi \centerdot \pi \in \perp\!\!\!\perp$ for every $\pi \in \|F[b]\|$.
Thus, suppose that $\pi \in \|F[b]\|$; since $\mathsf{Y} \star \xi \centerdot \pi \succ \xi \star \mathsf{Y}\xi \centerdot \pi$, we need to show that $\xi \star \mathsf{Y}\xi \centerdot \pi \in \perp\!\!\!\perp$. By hypothesis, we have $\xi \Vdash \forall x(f(x,b) = 1 \hookrightarrow F[x]) \to F[b]$;
Thus, it suffices to show that $\mathsf{Y}\xi \Vdash f(a,b) = 1 \hookrightarrow F[a]$ for every $a \in X$.
This is clear if $f(a,b) \neq 1$, by definition of $\hookrightarrow$.
If $f(a,b) = 1$, we must show $\mathsf{Y}\xi \Vdash F[a]$, i.e. $\mathsf{Y} \star \xi \centerdot \varpi \in \perp\!\!\!\perp$ for every $\varpi \in \|F[a]\|$.
But this follows from the induction hypothesis.

Q.E.D.

Remarks.
i) If the function f is only defined on a set X in the ground model $\mathcal{M}$, we can apply theorem 11 to the extension f' of f defined by $f'(x,y) = 0$ if $(x,y) \notin X^2$.
This shows that, in the realizability model $\mathcal{N}$, the binay relation $f(x,y) = 1$ is well founded on $\daleth X$.
ii) We can use theorem 11 to show that the axiom of foundation of ZF_ε is realized in $\mathcal{N}_{\mathcal{A}}$.
Indeed, let us define $f : \mathcal{M}^2 \to 2$ by setting $f(x,y) = 1 \Leftrightarrow \exists z((x,z) \in y)$. The binary relation $f(x,y) = 1$ is obviously well founded in $\mathcal{M}$. Now, we have $\mathsf{I} \Vdash \forall x \forall y (f(x,y) \neq 1 \to x \notin y)$ because $\pi \in \|x \notin y\| \Rightarrow f(x,y) = 1$. Thus, the relation $x \varepsilon y$ is stronger than the relation $f(x,y) = 1$, which is well founded in $\mathcal{N}_{\mathcal{A}}$ by theorem 11.

Integers

Let $\phi, \alpha \in \Lambda$ and $n \in \mathbb{N}$; we define $(\phi)^n \alpha \in \Lambda$ by setting $(\phi)^0 \alpha = \alpha$; $(\phi)^{n+1}\alpha = (\phi)(\phi)^n \alpha$.
For $n \in \mathbb{N}$, we define $\underline{n} = (\sigma)^n \underline{0}$ with $\underline{0} = \mathsf{KI}$ and $\sigma = (\mathsf{BW})(\mathsf{B})\mathsf{B}$;
$\underline{n}$ is "the integer n" and σ the "successor" in combinatory logic.
The essential property of $\underline{0}$ and σ is : $\underline{0} \star \phi \centerdot \alpha \centerdot \pi \succ \alpha \star \pi$; $\sigma \star \nu \centerdot \phi \centerdot \alpha \centerdot \pi \succ \nu \star \phi \centerdot \phi\alpha \centerdot \pi$.
The following lemmas 12 and 13 will be used in section 3.

Lemma 12.
Let $O, \varsigma \in \Lambda$ *be such that :* $O \star \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ \alpha \star \pi$ *and* $\varsigma \star \nu \centerdot \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ \nu \star \phi \centerdot \phi\alpha \centerdot \pi$

for every $\alpha, \zeta, \nu, \phi \in \Lambda$ *and* $\pi \in \Pi$.
Then, for every $n \in \mathbb{N}, \alpha, \zeta, \phi \in \Lambda$ *and* $\pi \in \Pi$ *:*
i) $(\varsigma)^n O \star \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ (\phi)^n \alpha \star \pi$ *; in particular,* $\underline{n} \star \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ (\phi)^n \alpha \star \pi$
ii) $(\varsigma)^n O \star \mathsf{CB}\phi \centerdot \zeta \centerdot \alpha \centerdot \pi \succ\!\!\succ \zeta \star (\phi)^n \alpha \centerdot \pi$.

i) Proof by recurrence on n ; this is clear if $n = 0$; if $n = m + 1$, we have :
$\varsigma \star (\varsigma)^m O \centerdot \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ (\varsigma)^m O \star \phi \centerdot \phi\alpha \centerdot \pi \succ\!\!\succ (\phi)^m (\phi)\alpha \star \pi$ by the recurrence hypothesis.
The particular case is $O = \underline{0}, \varsigma = \sigma$.
ii) By (i), we have $(\varsigma)^n O \star \mathsf{CB}\phi \centerdot \zeta \centerdot \alpha \centerdot \pi \succ\!\!\succ (\mathsf{CB}\phi)^n \zeta \star \alpha \centerdot \pi$.
We now show, by recurrence on n, that $(\mathsf{CB}\phi)^n \zeta \star \alpha \centerdot \pi \succ\!\!\succ \zeta \star (\phi)^n \alpha \centerdot \pi$.
This is clear if $n = 0$; if $n = m + 1$, we have :
$(\mathsf{CB}\phi)^n \zeta \star \alpha \centerdot \pi \succ\!\!\succ \mathsf{CB}\phi \star (\mathsf{CB}\phi)^m \zeta \centerdot \alpha \centerdot \pi \succ\!\!\succ \mathsf{C} \star \mathsf{B} \centerdot \phi \centerdot (\mathsf{CB}\phi)^m \zeta \centerdot \alpha \centerdot \pi \succ\!\!\succ$
$\mathsf{B} \star (\mathsf{CB}\phi)^m \zeta \centerdot \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ (\mathsf{CB}\phi)^m \zeta \star \phi\alpha \centerdot \pi \succ\!\!\succ \zeta \star (\phi)^m (\phi)\alpha \centerdot \pi$ (by the recurrence hypothesis).

Q.E.D.

Lemma 13.
Let $\Omega, \Sigma \in \Lambda$ *be such that :* $\Omega \star \delta \centerdot \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ \alpha \star \pi$ *and* $\Sigma \star \nu \centerdot \delta \centerdot \phi \centerdot \alpha \centerdot \pi \succ\!\!\succ \nu \star \delta \centerdot \phi \centerdot \phi\alpha \centerdot \pi$
for every $\alpha, \delta, \nu, \phi \in \Lambda$ *and* $\pi \in \Pi$. *For instance :* $\Omega = (K)(K)I$ *;* $\Sigma = (B)(BW)(B)B$.
Then, for every $n \in \mathbb{N}, \alpha, \delta, \zeta, \phi \in \Lambda$ *and* $\pi \in \Pi$ *:*
i) $(\Sigma)^n \Omega \star \delta \centerdot \phi \centerdot \alpha \centerdot \pi \succ (\phi)^n \alpha \star \pi$.
ii) $(\Sigma)^n \Omega \star \delta \centerdot \mathsf{CB}\phi \centerdot \zeta \centerdot \alpha \centerdot \pi \succ \zeta \star (\phi)^n \alpha \centerdot \pi$.

Same proof as lemma 12.

Q.E.D.

We set $\mathbb{N}_{\mathcal{A}} = \{(n, \underline{n} \centerdot \pi) \; ; \; n \in \mathbb{N}, \pi \in \Pi\}$; it is shown below that $\mathbb{N}_{\mathcal{A}}$ is the set of integers of the realizability model $\mathcal{N}_{\mathcal{A}}$.
We define the quantifier $\forall x^{\text{int}}$ as follows :

$$\|\forall x^{\text{int}} F[x]\| = \{\underline{n} \centerdot \pi \; ; \; n \in \mathbb{N}, \; \pi \in \|F[n]\|\}.$$

that is also :

$$\|\forall x^{\text{int}} F[x]\| = \|\forall n^{\daleth\mathbb{N}} (\{\underline{n}\} \to F[n])\|.$$

The formulas $\forall x^{\text{int}} F[x]$ and $\forall x (x \,\varepsilon\, \mathbb{N}_{\mathcal{A}} \to F[x])$ are interchangeable, as is shown in the :

Lemma 14.
$\lambda x \lambda n \lambda y (y)(x) n \Vdash \forall x^{int} F[x] \to \forall x^{int} \neg\neg F[x]$ *;*
$\lambda x \lambda n (\mathsf{cc})(x) n \Vdash \forall x^{int} \neg\neg F[x] \to \forall x^{int} F[x]$ *;*
$\|\forall x^{int} \neg\neg F[x]\| = \|\forall x (\neg F[x] \to x \notin \mathbb{N}_{\mathcal{A}})\|$.

Immediate

Q.E.D.

Lemma 15.
i) $K \Vdash \forall x (x \notin \daleth\mathbb{N} \to x \notin \mathbb{N}_{\mathcal{A}})$.
ii) $\lambda x (x) \underline{0} \Vdash 0 \notin \mathbb{N}_{\mathcal{A}} \to \bot$ *;* $\lambda f \lambda x (f)(\sigma) x \Vdash \forall y^{\daleth\mathbb{N}} ((y+1) \notin \mathbb{N}_{\mathcal{A}} \to y \notin \mathbb{N}_{\mathcal{A}})$.
iii) $I \Vdash \forall x^{int} \big(\forall y^{\daleth\mathbb{N}} (F[y] \to F[y+1]), F[0] \to F[x]\big)$ *for every formula* $F[x]$ *of* ZF_ε.

i) and ii) Immediate.
iii) Let $n \in \mathbb{N}$, $\phi \Vdash \forall y^{\beth\mathbb{N}}(F[y] \to F[y+1]), \alpha \Vdash F[0]$ et $\pi \in \|F[n]\|$. We must show : $\underline{n} \star \phi \centerdot \alpha \centerdot \pi \in \perp\!\!\!\perp$ i.e., by lemma 12, $(\phi)^n\alpha \star \pi \in \perp\!\!\!\perp$.
But it is clear, by recurrence on n, that $(\phi)^n\alpha \Vdash F[n]$ for every $n \in \mathbb{N}$.

Q.E.D.

Lemma 15(i) shows that $\mathbb{N}_{\mathcal{A}}$ is a subset of $\beth\mathbb{N}$.
But it is clear that $\beth\mathbb{N}$ contains 0 and is closed by the function $n \mapsto n+1$.
Now, by lemma 15(ii) and (iii), $\mathbb{N}_{\mathcal{A}}$ is the smallest subset of $\beth\mathbb{N}$ which contains 0 and is closed by the function $n \mapsto n+1$. Therefore :
$\mathbb{N}_{\mathcal{A}}$ is the set of integers of the model $\mathcal{N}_{\mathcal{A}}$.

2 The characteristic Boolean algebra $\beth 2$

Function symbols

Let us now define the principal function symbols commonly used in the sequel :
• The projections $pr_0 : X \times Y \to X$ and $pr_1 : X \times Y \to Y$ defined by :
$pr_0(x,y) = x,\ pr_1(x,y) = y$
give, in $\mathcal{N}_{\mathcal{A}}$, a *bijection* from $\beth(X \times Y)$ onto $\beth X \times \beth Y$.
• We define, in $\mathcal{M}$, the function $\text{app} : Y^X \times X \to Y$ (read *application*) by setting :
$\text{app}(f,x) = f(x)$ for $f \in Y^X$ and $x \in X$.
This gives, in $\mathcal{N}_{\mathcal{A}}$, an application $\text{app} : \beth(Y^X) \times \beth X \to \beth Y$.
We shall write $f(x)$ for $\text{app}(f,x)$.

Theorem 16.
If $X \neq \emptyset$, the function app *gives an injection from $\beth(Y^X)$ into $(\beth Y)^{\beth X}$. Indeed, we have :*
$\mathsf{I} \Vdash \forall f^{\beth(Y^X)} \forall g^{\beth(Y^X)} \big(\forall x^{\beth X}(app(f,x) = app(g,x)) \to f = g\big)$.

Let $f,g \in Y^X$, $\xi \Vdash \forall x^{\beth X}(\text{app}(f,x) \neq \text{app}(g,x) \to \bot)$ and $\pi \in \|f \neq g \to \bot\|$.
We must show $\xi \star \pi \in \perp\!\!\!\perp$. We choose $a \in X$; then $\xi \Vdash (f(a) \neq g(a) \to \bot)$.
If $f = g$, we have $\|f(a) \neq g(a) \to \bot\| = \|f \neq g \to \bot\| = \|\bot \to \bot\|$. Hence the result.
If $f \neq g$, we could choose a such that $f(a) \neq g(a)$.
Then, $\|f(a) \neq g(a) \to \bot\| = \|f \neq g \to \bot\| = \|\top \to \bot\|$. Hence the result.

Q.E.D.

• Let $\text{sp} : \mathcal{M} \to \{0,1\}$ (read *support*) the unary function symbol defined by :
$\text{sp}(\emptyset) = 0$; $\text{sp}(x) = 1$ if $x \neq \emptyset$.
In the realizability model $\mathcal{N}_{\mathcal{A}}$, we have $\text{sp} : \mathcal{N} \to \beth 2$.
• Let $\mathsf{P} : \{0,1\} \times \mathcal{M} \to \mathcal{M}$ (read *projection*) the binary function symbol defined by :
$\mathsf{P}(0,x) = \emptyset$; $\mathsf{P}(1,x) = x$.
In the realizability model $\mathcal{N}_{\mathcal{A}}$, we have $\mathsf{P} : \beth 2 \times \mathcal{N} \to \mathcal{N}$.
In the following, we shall write ix instead of $\mathsf{P}(i,x)$.
When t,u are ℓ-terms with values in $\beth 2$, we write $t \leq u$ for $t \wedge u = t$.

Proposition 17.
i) $\mathsf{I} \Vdash \forall i^{\beth 2} \forall x(i(jx) = (i \wedge j)x)$.

ii) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x(ix = x \rightleftharpoons sp(x) \le i)$.
iii) *If* $\emptyset \in E$, *then* $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x^{\daleth E}(ix \,\varepsilon\, \daleth E)$.
iv) *If* $f : \mathcal{M}^n \to \mathcal{M}$ *is a function symbol such that* $f(\emptyset, \ldots, \emptyset) = \emptyset$, *then :*
$\mathsf{I} \Vdash \forall j^{\daleth 2}\forall x_1 \ldots \forall x_n\,(jf(x_1, \ldots, x_n) = f(jx_1, \ldots, jx_n))$.
v) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x\,(i \ne 1 \to \forall y(y \not\varepsilon\, ix))$ *and therefore* $\mathsf{K}^2\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x\,(i \ne 1 \to \forall y(y \notin ix))$.

Trivial.

Q.E.D.

Remark. Proposition 17(v) shows that, in the realizability model $\mathcal{N}$, every non empty individual has support 1.

Because of property (iv), we shall define, as far as possible, each function symbol f in $\mathcal{M}$, so that to have $f(\emptyset, \ldots, \emptyset) = \emptyset$.

• Thus, let us change the ordered pair (x, y) by setting $(\emptyset, \emptyset) = \emptyset$. Then, we have :
$\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x\forall y\,(i(x, y) = (ix, iy))$.

• We define the binary function symbol $\sqcup : \mathcal{M}^2 \to \mathcal{M}$ by setting : $a \sqcup b = a \cup b$.

Remark. The extension to $\mathcal{N}$ of this operation *is not* the union $\cup$.

The operation $\daleth_i$

Let $E \in \mathcal{M}$ be such that $\emptyset \in E$. In $\mathcal{M}$, we define $\daleth_i E$ for $i \in 2$ by setting :
$\daleth_0 E = \daleth\{\emptyset\} = \{\emptyset\} \times \Pi$; $\daleth_1 E = \daleth E = E \times \Pi$.
In this way, we have now defined $\daleth_i E$ in $\mathcal{N}$, for every $i \,\varepsilon\, \daleth 2$.

Proposition 18.
i) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x\forall y(i(x \sqcup y) = ix \sqcup iy)$.
ii) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall j^{\daleth 2}\forall x((i \vee j)x = ix \sqcup jx)$.
iii) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall j^{\daleth 2}\forall x\forall y\forall z(i \wedge j = 0, z = ix \sqcup jy \hookrightarrow iz = ix)$.
$\mathsf{I} \Vdash \forall i^{\daleth 2}\forall j^{\daleth 2}\forall x\forall y\forall z(i \wedge j = 0, z = ix \sqcup jy \hookrightarrow jz = jy)$.
iv) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall j^{\daleth 2}\forall x^{\daleth E}\forall y^{\daleth E}\forall z\,(i \wedge j = 0, z = ix \sqcup jy \hookrightarrow z \,\varepsilon\, \daleth_{i \vee j} E)$.

Trivial.

Q.E.D.

Proposition 19.
If $\emptyset \in E, E'$, *the following formulas are realized :*
i) $\daleth_i E$ *increases with* i. *In particular,* $\daleth_i E \subset \daleth E$.
ii) The ε*-elements of* $\daleth_i E$ *are the* ix *for* $x \,\varepsilon\, \daleth E$.
iii) The ε*-elements of* $\daleth_i E$ *are those of* $\daleth E$ *such that* $sp(x) \le i$.
iv) The only ε*-element common to* $\daleth_i E$ *and* $\daleth_{1-i} E$ *is* $\emptyset$.
v) If $i \wedge j = 0$, *then the application* $x \mapsto (ix, jx)$ *is a bijection from* $\daleth_{i \vee j} E$ *onto* $\daleth_i E \times \daleth_j E$. *The inverse function is* $(x, y) \mapsto x \sqcup y$.
vi) $\daleth_i(E \times E') = \daleth_i E \times \daleth_i E'$.

We check immediately i), ii), iii), iv) below :
i) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall j^{\daleth 2}\forall x(i \wedge j = i \hookrightarrow (x \not\varepsilon\, \daleth_j E \to x \not\varepsilon\, \daleth_i E))$.
ii) $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x^{\daleth E}(ix \,\varepsilon\, \daleth_i E)$; $\mathsf{I} \Vdash \forall i^{\daleth 2}\forall x^{\daleth E}(ix \ne x \to x \not\varepsilon\, \daleth_i E)$.

iii) $\mathsf{I} \Vdash \forall i^{\beth 2}\forall x^{\beth E}(x \notin\!\!\varepsilon\, \beth_i E \to \mathrm{sp}(x)i \neq \mathrm{sp}(x))$; $\mathsf{I} \Vdash \forall i^{\beth 2}\forall x^{\beth E}(\mathrm{sp}(x)i \neq \mathrm{sp}(x) \to x \notin\!\!\varepsilon\, \beth_i E)$;
iv) $\mathsf{I} \Vdash \forall i^{\beth 2}\forall x^{\beth E}\forall y^{\beth E}(ix = (1-i)y \hookrightarrow ix = \emptyset)$.

v) By proposition 18(ii), we have $ix \sqcup jx = (i \vee j)x = x$ if $x \,\varepsilon\, \beth_{i\vee j} E$.
By proposition 18(iii,iv), if $x, y \,\varepsilon\, \beth E$, there exists $z \,\varepsilon\, \beth_{i\vee j} E$ such that $iz = ix, jz = jy$, namely $z = ix \sqcup jy$.

vi) By proposition 17(iv), we have $\mathsf{I} \Vdash \forall i^{\beth 2}\forall x\forall y(i(x,y) = (ix, iy))$.

Q.E.D.

Proposition 20. *Let $E, E' \in \mathcal{M}$ be such that $\emptyset \in E, E'$ and E is equipotent with E'. Then : $\Vdash \forall i^{\beth 2}(\beth_i E$ is equipotent with $\beth_i E')$.*

Let ϕ be, in $\mathcal{M}$, a bijection from E onto E', such that $\phi(\emptyset) = \emptyset$. Then ϕ is, in $\mathcal{N}$, a bijection from $\beth E$ onto $\beth E'$. But we have immediately : $\mathsf{I} \Vdash \forall i^{\beth 2}\forall x^{\beth E}(\phi(ix) = i\phi(x))$. This shows that ϕ is a bijection from $\beth_i E$ onto $\beth_i E'$.

Q.E.D.

Some general theorems

Theorems 21 to 29, which are shown in this section, are valid *in every realizability model.*
In the ground model $\mathcal{M}$, which satisfies ZFL, we denote by κ the cardinal of $\Lambda \cup \Pi \cup \mathbb{N}$ (which we shall also call the *cardinal of the algebra* $\mathcal{A}$) and by $\kappa_+ = \mathcal{P}(\kappa)$ the power set of κ.

Theorem 21.
Let $\forall\vec{x}\forall y\, F[\vec{x}, y]$ be a closed formula of ZF_ε with parameters in $\mathcal{M}$ (where $\vec{x} = (x_1, \ldots, x_n)$). Then, there exists in $\mathcal{M}$, a functional $f_F : \kappa \times \mathcal{M}^n \to \mathcal{M}$ such that :
i) If $\vec{a}, b \in \mathcal{M}$ and $\xi \Vdash F[\vec{a}, b]$, then there exists $\alpha \in \kappa$ such that $\xi \Vdash F[\vec{a}, f_F(\alpha, \vec{a})]$.
ii) $\mathsf{CI} \Vdash \forall\vec{x}\forall y\,\big(F[\vec{x}, y] \to \exists\nu^{\beth\kappa} F[\vec{x}, f_F(\nu, \vec{x})]\big)$.

i) Let $\xi \mapsto \alpha_\xi$ be an injection from Λ into κ. Using the principle of choice in $\mathcal{M}$ (which satisfies $V = L$), we can define a functional $f_F : \kappa \times \mathcal{M}^n \to \mathcal{M}$ such that, in $\mathcal{M}$, we have : $\forall\vec{x}\forall y (\forall\xi \in \Lambda)\,(\xi \Vdash F[\vec{x}, y] \Rightarrow \xi \Vdash F[\vec{x}, f_F(\alpha_\xi, \vec{x})])$.
ii) Let $\xi \Vdash F[\vec{a}, b]$, $\eta \Vdash \forall\nu^{\beth\kappa}\neg F[\vec{a}, f_F(\nu, \vec{a})]$ and $\pi \in \Pi$.
Thus, we have $\eta \Vdash \neg F[\vec{a}, f_F(\alpha_\xi, \vec{a})]$; by definition of f_F, we have $\xi \Vdash F[\vec{a}, f_F(\alpha_\xi, \vec{a})]$. Therefore $\eta \star \xi \centerdot \pi \in \perp\!\!\!\perp$, and $\mathsf{CI} \star \xi \centerdot \eta \centerdot \pi \in \perp\!\!\!\perp$.

Q.E.D.

Subsets of $\beth\kappa_+$

Theorem 22. *Let $\forall x\forall y\forall z\, F[x, y, z]$ be a closed formula of ZF_ε, with parameters in $\mathcal{M}$. Then, there exists, in $\mathcal{M}$, a functional $\beta_F : \mathcal{M} \to \kappa_+$ such that :*
$\mathsf{W} \Vdash \forall z\,\big(\forall x\forall y\forall y'(F[x, y, z], F[x, y', z] \to y = y') \to \forall i^{\beth 2}\forall x(F[x, i\beta_F(z), z] \to sp(x) \geq i)\big)$.

By theorem 21(i), there exists, in $\mathcal{M}$, a functional $g : \kappa \times \mathcal{M}^2 \to \mathcal{M}$ such that :
$(*)$ For $a, b, c \in \mathcal{M}$ and $\xi \Vdash F[a, b, c]$, there exists $\alpha \in \kappa$ such that $\xi \Vdash F[a, g(\alpha, a, c), c]$.

Using the principle of choice in $\mathcal{M}$, we define a functional $\beta_F : \mathcal{M} \to \kappa_+$ such that :
for every $\alpha \in \kappa$ and $c \in \mathcal{M}$, we have $\beta_F(c) \neq g(\alpha, \emptyset, c)$.
This is possible since κ_+ is of cardinal $> \kappa$.

Now let : $a, c \in \mathcal{M}$, $i \in \{0,1\}$, $\phi \Vdash \forall x \forall y \forall y'(F[x,y,c], F[x,y',c], y \neq y' \to \bot)$,
$\xi \Vdash F[a, i\beta_F(c), c]$, $\eta \Vdash \mathrm{sp}(a)i \neq i$ and $\pi \in \Pi$.

We must show that $\mathsf{W} \star \phi \cdot \xi \cdot \eta \cdot \pi \in \perp\!\!\!\perp$, that is $\phi \star \xi \cdot \xi \cdot \eta \cdot \pi \in \perp\!\!\!\perp$.
We set $b = i\beta_F(c)$ and therefore, we have $\xi \Vdash F[a,b,c]$.
Thus, by $(*)$, we have $\xi \Vdash F[a, g(\alpha,a,c), c]$ for some $\alpha \in \kappa$.

Let us show that $\|b \neq g(\alpha,a,c)\| \subset \|\mathrm{sp}(a)i \neq i\|$; there are three possible cases :
If $i = 0$, then $\|\mathrm{sp}(a)i \neq i\| = \|0 \neq 0\| = \Pi$, hence the result.
If $i = 1$ and $a \neq \emptyset$, then $\|\mathrm{sp}(a)i \neq i\| = \|1 \neq 1\| = \Pi$, hence the result.
If $i = 1$ and $a = \emptyset$, then :
$\|b \neq g(\alpha,a,c)\| = \|i\beta_F(c) \neq g(\alpha,a,c)\| = \|\beta_F(c) \neq g(\alpha,\emptyset,c)\| = \|\top\| = \emptyset$, by definition of β_F, hence the result.

It follows that $\eta \Vdash b \neq g(\alpha,a,c)$. Now, we have seen that :
$\xi \Vdash F[a,b,c]$ and $\xi \Vdash F[a, g(\alpha,a,c), c]$.
Therefore, by hypothesis on ϕ, we have $\phi \star \xi \cdot \xi \cdot \eta \cdot \pi \in \perp\!\!\!\perp$.

Q.E.D.

Corollary 23. *The following formulas are realized :*
i) $\forall E \forall i^{\gimel 2}$(*there is no surjection from* $\bigcup\{\gimel_j E \; ; \; j \,\varepsilon\, \gimel 2, j \not\geq i\}$ *onto* $\gimel_i \kappa_+$).
ii) $\forall E \forall i^{\gimel 2} \forall j^{\gimel 2}$(*if there exists a surjection from* $\gimel_j E$ *onto* $\gimel_i \kappa_+$ *then* $j \geq i$).
iii) $\forall i^{\gimel 2} \forall j^{\gimel 2}(i, j \neq 0, i \wedge j = 0 \to$
(*there is no surjection from* $\gimel_i \kappa_+ \oplus \gimel_j \kappa_+$ *onto* $\gimel_i \kappa_+ \times \gimel_j \kappa_+$)).

Remark. The notation $\bigcup\{\gimel_j E \; ; \; j \,\varepsilon\, \gimel 2, j \not\geq i\}$ denotes any individual X of $\mathcal{N}$ such that :
$\mathcal{N} \models \forall x (x \,\varepsilon\, X \leftrightarrow \exists j^{\gimel 2}(j \not\geq i \wedge x \,\varepsilon\, \gimel_j E))$.

i) We apply theorem 22, with the formula $F[x,y,z] \equiv (x,y) \,\varepsilon\, z$.
In the realizabiblity model $\mathcal{N}$, we have $\beta_F : \mathcal{N} \to \gimel \kappa_+$.
Let z_0 be, in $\mathcal{N}$, a surjective function onto $\gimel_i \kappa_+$.
We have $\beta_F(z_0) \,\varepsilon\, \gimel \kappa_+$, and therefore $i\beta_F(z_0) \,\varepsilon\, \gimel_i \kappa_+$.
If x_0 is such that $(x_0, i\beta_F(z_0)) \,\varepsilon\, z_0$, then $\mathrm{sp}(x_0) \geq i$ by theorem 22. Therefore, for any individual E, we have $x_0 \,\varepsilon\, \gimel_j E \Rightarrow j \geq i$, by proposition 19(iii).

ii) It is a trivial consequence of (i).

iii) We take $E = \kappa_+$; since $i, j \neq 0, i \wedge j = 0$, we have $i, j \not\geq i \vee j$; by (i), there is no surjection from $\gimel_i \kappa_+ \cup \gimel_j \kappa_+$ onto $\gimel_{i \vee j} \kappa_+$.
Now, since $i \wedge j = 0$, $\gimel_{i \vee j} \kappa_+$ is equipotent with $\gimel_i \kappa_+ \times \gimel_j \kappa_+$ by proposition 19(v).
Moreover, $\emptyset$ is the only ε-element common to $\gimel_i \kappa_+$ and $\gimel_j \kappa_+$ by proposition19(iv).
But these sets contain a countable subset by theorem 25. It follows that $\gimel_i \kappa_+ \cup \gimel_j \kappa_+$ is equipotent with $\gimel_i \kappa_+ \oplus \gimel_j \kappa_+$.

Q.E.D.

Theorem 24. *The formula :* (*there exists a surjection from* $\gimel \kappa_+$ *onto* $2^{\gimel \kappa}$) *is realized.*

In the ground model $\mathcal{M}$, there exists a bijection from $\kappa_+ = 2^\kappa$ onto $2^{\kappa \times \Pi}$. Therefore, in $\mathcal{N}$, there exists a bijection from $\gimel \kappa_+$ onto $\gimel 2^{\kappa \times \Pi}$.

We now need a surjection from $\gimel 2^{\kappa\times\Pi}$ onto $2^{\gimel\kappa}$.
Let $\phi : \mathcal{M} \to 2^{\kappa\times\Pi}$ be the unary function symbol defined by $\phi(x) = x \cap (\kappa\times\Pi)$.
In $\mathcal{N}$, we have $\phi : \mathcal{N} \to \gimel 2^{\kappa\times\Pi}$. Now, we check immediately that :
i) $\mathsf{I} \Vdash \forall\nu\forall x^{\gimel 2^{\kappa\times\Pi}}(\nu \notin \gimel\kappa \to \nu \notin x)$ because $\|\nu \notin a\| \subset \|\nu \notin \gimel\kappa\|$ for all $a \in \mathcal{P}(\kappa\times\Pi)$.
ii) $\mathsf{I} \Vdash \forall x\forall\nu^{\gimel\kappa}(\nu \notin x \rightleftharpoons \nu \notin \phi(x))$ because $\|\nu \notin a\| = \|\nu \notin \phi(a)\|$ for all $\nu \in \kappa$.
From (i), it follows that $\gimel 2^{\kappa\times\Pi}$ is, in $\mathcal{N}$, a set of subsets of $\gimel\kappa$;
from (ii), it follows that it contains at least one representative for each equivalence class of extensionality.
Thus, the desired surjection simply associates, with each ε-element of $\gimel 2^{\kappa\times\Pi}$, its equivalence class of extensionality.

Q.E.D.

Theorem 25. *Let $E \in \mathcal{M}$ be infinite and such that $\emptyset \in E$. Then we have :*
$\Vdash \forall i^{\gimel 2}(i \neq 0 \to$ *there exists an injection from* $\mathbb{N}$ *into* $\gimel_i E)$.

In $\mathcal{M}$, let $\phi : \mathbb{N} \to (E \setminus \{\emptyset\})$ be injective. In $\mathcal{N}$, we have $\phi : \gimel\mathbb{N} \to \gimel E$.
The desired function is $n \mapsto i\phi(n)$. Indeed, we have :
$\mathsf{I} \Vdash \forall i^{\gimel 2}\forall m^{\gimel\mathbb{N}}\forall n^{\gimel\mathbb{N}}(i \neq 0 \to i\phi(m+n+1) \neq i\phi(m))$.
This shows that the restriction of this function to $\mathbb{N}_{\mathcal{A}}$ (the set of integers of $\mathcal{N}_{\mathcal{A}}$) is injective.

Q.E.D.

Theorem 26. $\Vdash \forall i^{\gimel 2}\,(i \neq 0, i \neq 1 \to (\gimel\kappa_+$ *cannot be well ordered*$))$.

Let $i \in \gimel 2, i \neq 0, 1$; then, $\gimel_i\kappa_+$ and $\gimel_{1-i}\kappa_+$ are infinite (theorem 25) and $\subset \gimel\kappa_+$ by proposition 19(i). But there exists no surjection from $\gimel_i\kappa_+$ onto $\gimel_{1-i}\kappa_+$, neither from $\gimel_{1-i}\kappa_+$ onto $\gimel_i\kappa_+$, by corollary 23.

Q.E.D.

Remark. By theorem 26, if the Boolean algebra $\gimel 2$ is not trivial, then $\gimel\kappa_+$ is not well orderable. On the other hand, it can be shown that, if this Boolean algebra is trivial, then the realizability model $\mathcal{N}$ is an extension by forcing of the ground model $\mathcal{M}$. In this case, $\mathcal{N}$ itself can be well ordered, since we suppose that the ground model $\mathcal{M}$ satisfies ZFL.

A strict order on $\gimel\kappa_+$

A binary relation $<$ on X is a *strict order* if it is transitive ($x < y, y < z \Rightarrow x < z$) and antireflexive ($x \not< x$). This strict order is called *total* if we have : $x < y$ or $y < x$ or $x = y$.
If $(X_0, <_0), (X_1, <_1)$ are two strictly ordered sets, then the *strict order product* $<$ on $X_0 \times X_1$ is defined by : $(x_0, x_1) < (y_0, y_1) \Leftrightarrow x_0 < y_0$ and $x_1 < y_1$.

Lemma 27. *The strict order product of $<_0, <_1$ is well founded if and only if one of the strict orders $<_0, <_1$ is well founded.*

Proof of $\Rightarrow$: by contradiction ; if $<_0$ and $<_1$ are not well founded, we have :
$\forall y_0\,(\forall x_0(x_0 <_0 y_0 \to F_0[x_0]) \to F_0[y_0])$; $\neg F_0[b_0]$;
$\forall y_1\,(\forall x_1(x_1 <_1 y_1 \to F_1[x_1]) \to F_1[y_1])$; $\neg F_1[b_1]$;
for some formulas F_0, F_1 and some individuals b_0, b_1. It follows that :

$\forall y_0 \forall y_1 (\forall x_0 \forall x_1 (x_0 <_0 y_0, x_1 <_1 y_1 \to G[x_0, x_1]) \to G[y_0, y_1])$; $\neg G[b_0, b_1]$
where $G[x_0, x_1] \equiv F_0[x_0] \vee F_1[x_1]$.

Proof of $\Leftarrow$: suppose that $<_0$ is well founded and let $G[x_0, x_1]$ be any formula.
Let $F[x_0] \equiv \forall x_1 G[x_0, x_1]$. We have to prove $\forall y_0 \forall y_1 G[y_0, y_1]$, i.e. $\forall y_0 F[y_0]$ with the hypothesis $\forall y_0 \forall y_1 (\forall x_0 \forall x_1 (x_0 <_0 y_0, x_1 <_1 y_1 \to G[x_0, x_1]) \to G[y_0, y_1])$. But this implies : $\forall y_0 (\forall x_0 (x_0 <_0 y_0 \to F[x_0]) \to F[y_0])$ and the result follows, because $<_0$ is well founded.

Q.E.D.

We denote by $\triangleleft$ a strict well ordering on κ_+, in $\mathcal{M}$; we suppose that its least element is $\emptyset$ and that the cardinal of each proper initial segment is $\leq \kappa$.

This gives a binary function from κ_+^2 into $\{0, 1\}$, denoted by $(x \triangleleft y)$, which is defined as follows : $(x \triangleleft y) = 1 \Leftrightarrow x \triangleleft y$.
We can extend it to the realizability model $\mathcal{N}_\mathcal{A}$, which gives a function from $(\daleth\kappa_+)^2$ into $\daleth 2$.

Lemma 28. *The following propositions are realized :*
If $i \,\varepsilon\, \daleth 2, i \neq 0$, then $(x \triangleleft y) = i$ is a strict ordering of $\daleth_i \kappa_+$, which we denote by $\triangleleft_i$.
If i is an atom of the Boolean algebra $\daleth 2$, then this ordering is total.

We have immediately :
i) $\mathsf{I} \Vdash \forall x^{\daleth\kappa_+} \forall y^{\daleth\kappa_+} \forall z^{\daleth\kappa_+} ((x \triangleleft y) \wedge (y \triangleleft z) \leq (x \triangleleft z))$; $\mathsf{I} \Vdash \forall x^{\daleth\kappa_+} ((x \triangleleft x) = 0)$.
ii) $\mathsf{I} \Vdash \forall i^{\daleth 2} \forall x^{\daleth\kappa_+} \forall y^{\daleth\kappa_+} ((ix \triangleleft iy) \leq i)$.
iii) $\mathsf{I} \Vdash \forall x^{\daleth\kappa_+} \forall y^{\daleth\kappa_+} ((x \triangleleft y) = 0, (y \triangleleft x) = 0 \hookrightarrow x = y)$.
It follows from (i) that, if $i \neq 0$, then $(x \triangleleft y) \geq i$ is a strict ordering relation on $\daleth\kappa_+$.
It follows from (ii), that this relation, restricted to $\daleth_i \kappa_+$, is equivalent to $(x \triangleleft y) = i$.
Finally, it follows from (iii), that the relation $(x \triangleleft y) = i$, restricted to $\daleth_i \kappa_+$, is total when i is an atom of $\daleth 2$.

Q.E.D.

Lemma 29. *The following propositions are realized :*
i) $\forall i^{\daleth 2}$(the application $x \mapsto (ix, (1-i)x)$ is an isomorphism of strictly ordered sets from $(\daleth\kappa_+, \triangleleft)$ onto $(\daleth_i \kappa_+, \triangleleft_i) \times (\daleth_{1-i} \kappa_+, \triangleleft_{1-i})$).
ii) $\forall i^{\daleth 2}$(either $\daleth_i \kappa_+$ or $\daleth_{1-i} \kappa_+$ is a well founded ordered set).

i) It follows from proposition 19(v), that the application $x \mapsto (ix, (1-i)x)$ is a bijection from $\daleth\kappa_+$ onto $\daleth_i \kappa_+ \times \daleth_{1-i} \kappa_+$. In fact, it is an *isomorphism* of ordered sets, since we have :
$\mathsf{I} \Vdash \forall i^{\daleth 2} \forall x^{\daleth\kappa_+} \forall y^{\daleth\kappa_+} ((x \triangleleft y) = (ix \triangleleft iy) \vee ((1-i)x \triangleleft (1-i)y))$ and therefore :
$\Vdash \forall i^{\daleth 2} \forall x^{\daleth\kappa_+} \forall y^{\daleth\kappa_+} ((x \triangleleft y) = 1 \leftrightarrow (ix \triangleleft iy) = i \wedge ((1-i)x \triangleleft (1-i)y) = 1 - i)$.
ii) By theorem 11, the relation $(x \triangleleft y) = 1$ is well founded on $\daleth\kappa_+$. Thus, the result follows immediately from (i) and lemma 27.

Q.E.D.

$\daleth\kappa$ countable

In this section, we consider some consequences of the hypothesis : ($\daleth\kappa$ is countable).

Non extensional and dependent choice

The formula $\forall z(z \notin y \to z \notin x)$ will be written $x \subseteq y$.
The formula $\forall x \forall y \forall y'((x, y) \,\varepsilon\, f, (x, y') \,\varepsilon\, f \to y = y')$ will be written $\mathrm{Func}(f)$ (read : f *is a function*).

The formula $\forall z \exists f\, (f \subseteq z \wedge \mathrm{Func}(f) \wedge \forall x \forall y \exists y'((x, y) \,\varepsilon\, z \to (x, y') \,\varepsilon\, f))$ is called the *non extensional axiom of choice* and denoted by NEAC.

It is easily shown [18] that ZF_ε+ NEAC $\vdash$ DC (axiom of dependent choice). On the other hand, we have built, in [18], a model of ZF_ε + NEAC + $\neg$AC ; and other such models will be given in the present paper. In all these models, $\mathbb{R}$ is not well orderable.

Theorem 30.
There exists a closed c*-term* H *such that* $H \Vdash$ ($\gimel\kappa$ *is countable*) $\to$ *NEAC.*

We apply theorem 21(ii) to the formula $(x, y) \,\varepsilon\, z$. We get a function symbol g such that $\mathsf{C\,I} \Vdash \forall x \forall y \forall z((x, y) \,\varepsilon\, z \to \exists \nu^{\gimel\kappa}(x, g(\nu, x, z)) \,\varepsilon\, z)$.
Therefore, it suffices to prove NEAC in ZF_ε, by means of this formula and the additional hypothesis : ($\gimel\kappa$ is countable). Now, from this hypothesis, it follows that there exists a strict well ordering $<$ on $\gimel\kappa$. Then, we can define the desired function f by means of the comprehension scheme :
$(x, y) \,\varepsilon\, f \leftrightarrow (x, y) \,\varepsilon\, z \wedge \exists \nu^{\gimel\kappa} \big(y = g(\nu, x, z) \wedge \forall \alpha^{\gimel\kappa}(\alpha < \nu \to (x, g(\alpha, x, z)) \notin z\big)$.
Intuitively, $f(x) = g(\nu, x, z)$ for the least $\nu \,\varepsilon\, \gimel\kappa$ such that $(x, g(\nu, x, z)) \,\varepsilon\, z$.

Q.E.D.

Subsets of $\mathbb{R}$

Theorem 31. $\Vdash$ ($\gimel\kappa$ *is countable*) $\to$
every bounded above subset of the ordered set $(\gimel\kappa_+, \triangleleft)$ *is countable.*

Every proper initial segment of the well ordering $\triangleleft$ on κ_+ is of cardinal κ. Thus, there exists a function $\phi : \kappa \times \kappa_+ \to \kappa_+$ such that, for each $x \in \kappa_+$, $x \neq \emptyset$, the function $\alpha \mapsto \phi(\alpha, x)$ is a surjection from κ onto $\{y \in \kappa_+ \;;\; y \triangleleft x\}$. Then, we have immediately :
$\mathsf{I} \Vdash \forall x^{\gimel\kappa_+} \forall y^{\gimel\kappa_+} \big((y \triangleleft x) = 1 \hookrightarrow (\forall \alpha^{\gimel\kappa}(y \neq \phi(\alpha, x)) \to \bot)\big)$.
This shows that, in $\mathcal{N}$, there exists a surjection from $\gimel\kappa$, onto every subset of $\gimel\kappa_+$ which is bounded from above for the strict ordering $\triangleleft$.
Thus, all these subsets of $\gimel\kappa_+$ are countable, since $\gimel\kappa$ is.

Q.E.D.

Theorem 32. $\Vdash$ ($\gimel\kappa$ *is countable*) $\to$ *there exists an injection from* $\gimel\kappa_+$ *into* $\mathbb{R}$.

We have obviously $\Vdash$ ($\gimel\kappa$ is countable $\to \gimel 2$ is countable), and therefore :
$\Vdash$ ($\gimel\kappa$ is countable $\to (\gimel 2)^{\gimel\kappa}$ is equipotent to $\mathbb{R}$).
Now, by theorem 16, we have : $\Vdash$ (there is an injection from $\gimel\kappa_+ = \gimel(2^\kappa)$ into $(\gimel 2)^{\gimel\kappa}$).

Q.E.D.

Theorem 33. *The following formula is realized :*
($\gimel\kappa$ *is countable*) $\to$ *there exists an application* $i \mapsto X_i$ *from the countable Boolean algebra*

$\beth 2$ *into* $\mathcal{P}(\mathbb{R})$ *such that :*
i) $X_0 = \{\emptyset\}$ *;* $i \neq 0 \to X_i$ *is uncountably infinite ;*
ii) $X_i \times X_i$ *is equipotent with* X_i *;*
iii) $X_i \cap X_j = X_{i \wedge j}$ *and therefore* $i \leq j \to X_i \subset X_j$ *;*
iv) $i \wedge j = 0 \to X_{i \vee j}$ *is equipotent with* $X_i \times X_j$ *;*
v) there exists a surjection from X_1 *onto* $\mathbb{R}$.
vi) if A *is a subset of* $\beth 2$ *and if there is a surjection from* $\bigcup_{j \varepsilon A} X_j$ *onto* X_i*, then* $i \leq j$ *for some* $j \,\varepsilon\, A$.
vii) if there is a surjection from X_j *onto* X_i*, then* $i \leq j$ *;*
viii) if $i, j \neq 0, i \wedge j = 0$*, then there is no surjection from* $X_i \oplus X_j$ *onto* $X_i \times X_j$.

For each $i \,\varepsilon\, \beth 2$, let us denote by X_i the image of $\beth_i \kappa_+$ by the injection from $\beth \kappa_+$ into $\mathbb{R}$, given by theorem 32.
i) The fact that X_i is infinite for $i \neq 0$ is a consequence of theorem 25.
If $i = 1$, X_i is uncountable by (vi). If $i \neq 0, 1$ and X_i is countable, then X_{1-i} is infinite and thus, there exists a surjection from X_{1-i} onto X_i. This contradicts corollary 23.
ii) by proposition 19(vi), $\beth_i \kappa_+ \times \beth_i \kappa_+$ is equipotent with $\beth_i(\kappa_+^2)$, thus also with $\beth_i \kappa_+$ by proposition 20.
iii) If $a \,\varepsilon\, \beth_i \kappa_+$ and $a \,\varepsilon\, \beth_j \kappa_+$, then $ia = a$, and therefore $(i \wedge j)a = ja = a$.
iv) This is proposition 19(v).
v) Application of theorem 24.
vi), vii), viii) Applications of corollary 23.
Q.E.D.

Theorem 33 is interesting only if the countable Boolean algebra $\beth 2$ is not trivial. In this case, $\mathbb{R}$ cannot be well ordered, by theorems 26 and 32.

In section 3 below, given an *arbitrary* realizability algebra $\mathcal{A}$, we build a new algebra $\mathcal{B}$ such that :
• $\mathcal{N}_\mathcal{B}$ realizes the formula : ($\beth \kappa$ is countable).
• The (countable) Boolean algebra $\beth 2$ of the model $\mathcal{N}_\mathcal{B}$ is elementarily equivalent to the algebra $\beth 2$ of $\mathcal{N}_\mathcal{A}$.

In the sequel, we shall consider two interesting cases :
$\beth 2$ is atomless ; $\beth 2$ has four ε-elements.

3 Collapsing $\beth \kappa$

Extending a realizability algebra

In the ground model $\mathcal{M}$, we consider a realizability algebra $\mathcal{A}$, the elementary combinators of which are denoted by $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{cc}$ and the continuations k_π for $\pi \in \Pi$.

We define the combinators $\mathsf{B}^*, \mathsf{C}^*, \mathsf{I}^*, \mathsf{K}^*, \mathsf{W}^*, \mathsf{cc}^*$, and the continuations k_π^* as follows :
$\mathsf{B}^* = \lambda n \lambda x \lambda y \lambda z (xn)(\mathsf{C})yz = \big((\mathsf{C})(\mathsf{BC})(\mathsf{C})(\mathsf{B})(\mathsf{BB})\mathsf{B}\big)\mathsf{C}$;
$\mathsf{C}^* = \lambda n \lambda x \lambda y \lambda z (x) nzy = (\mathsf{C})(\mathsf{B})\mathsf{C}$;
$\mathsf{I}^* = \lambda n \lambda x (x) n = \mathsf{C}\,\mathsf{I}$;
$\mathsf{K}^* = \lambda n \lambda x \lambda y (x) n = (\mathsf{C})(\mathsf{B})\mathsf{K}$;

$\mathsf{W}^* = \lambda n\lambda x\lambda y(x)nyy = (\mathsf{C})(\mathsf{B})\mathsf{W}$;
$\mathsf{k}_\pi^* = \lambda n\lambda x(\mathsf{k}_\pi)(x)n = (\mathsf{C})(\mathsf{B})\mathsf{k}_\pi$;
$\mathsf{cc}^* = \lambda n\lambda x(\mathsf{cc})\lambda k(xn)\lambda n\lambda x(k)(x)n$
$= \Big((\mathsf{C})\Big((\mathsf{C})\Big((\mathsf{B})((\mathsf{B})(\mathsf{B})\mathsf{C})\mathsf{C}\Big)(\mathsf{C})(\mathsf{B})\Big((\mathsf{B})(\mathsf{B})((\mathsf{B})(\mathsf{B})\mathsf{cc})\mathsf{B}\Big)\mathsf{B}\Big)\mathsf{C}\Big)\mathsf{B}.$

Therefore, we have :

$\mathsf{B}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \mathsf{C}\eta\zeta \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{C}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{I}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{K}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{W}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{k}_\pi^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \varpi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \pi$;
$\mathsf{cc}^* \star \nu \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \nu \mathbin{\scriptstyle\bullet} \mathsf{k}_\pi^* \mathbin{\scriptstyle\bullet} \pi.$

(reminder : the notation $\xi \star \pi \succ\!\!\succ \xi' \star \pi'$ means $\xi \star \pi \notin \perp\!\!\!\perp \Rightarrow \xi' \star \pi' \notin \perp\!\!\!\perp$).

Let κ be an infinite cardinal of $\mathcal{M}$, $\kappa \geq \mathrm{card}(\Lambda \cup \Pi)$; we consider the tree (usually called $\kappa^{<\omega}$) of functions, the domain of which is an integer, with values in κ.
Let P be the ordered set obtained by adding a least element $\mathbb{O}$ to this tree.
P is an inf-semi-lattice, the greatest element $\mathbf{1}$ of which is the function $\emptyset$.
The greatest lower bound of $p, q \in P$, denoted by pq, is p (resp. q) if $p, q \neq \mathbb{O}$ and $q \subset p$ (resp. $p \subset q$). It is $\mathbb{O}$ in every other case.

Remark. $P \setminus \{\mathbb{O}\} = \kappa^{<\omega}$ is the ordered set used, in the method of forcing, to *collapse* (i.e. make countable) the cardinal κ.

We define a new realizability algebra $\mathcal{B}$ by setting :
$\boldsymbol{\Lambda} = \Lambda \times P$; $\boldsymbol{\Pi} = \Pi \times P$; $\boldsymbol{\Lambda} \star \boldsymbol{\Pi} = (\Lambda \star \Pi) \times P$;
$(\xi, p) \mathbin{\scriptstyle\bullet} (\pi, q) = (\xi \mathbin{\scriptstyle\bullet} \pi, pq)$; $(\xi, p) \star (\pi, q) = (\xi \star \pi, pq)$; $(\xi, p)(\eta, q) = (\mathsf{C}\xi\eta, pq)$.
$\mathbf{B} = (\mathsf{B}^*, \mathbf{1})$; $\mathbf{C} = (\mathsf{C}^*, \mathbf{1})$; $\mathbf{I} = (\mathsf{I}^*, \mathbf{1})$; $\mathbf{K} = (\mathsf{K}^*, \mathbf{1})$; $\mathbf{W} = (\mathsf{W}^*, \mathbf{1})$;
$\mathbf{cc} = (\mathsf{cc}^*, \mathbf{1})$; $\mathbf{k}_{(\pi,p)} = (\mathsf{k}_\pi^*, p)$.

We define, in $\mathcal{M}$, a function symbol from $P \times \mathbb{N}$ into $\{0, 1\}$, denoted by $(p \ll n)$, by setting :
$(p \ll n) = 1 \Leftrightarrow p \neq \mathbb{O}$ and the domain of p is an integer $\leq n$.

We define $\perp\!\!\!\perp_{\mathcal{B}}$, that we shall denote also by $\perp\!\!\!\perp\!\!\!\perp$, as follows :
$(\xi \star \pi, p) \in \perp\!\!\!\perp\!\!\!\perp \Leftrightarrow (\forall n \in \mathbb{N})((p \ll n) = 1 \Rightarrow \xi \star \underline{n} \mathbin{\scriptstyle\bullet} \pi \in \perp\!\!\!\perp)$ for $p \in P$, $\xi \in \Lambda$ and $\pi \in \Pi$.
In particular, we have $(\xi \star \pi, \mathbb{O}) \in \perp\!\!\!\perp\!\!\!\perp$ for any $\xi \in \Lambda, \pi \in \Pi$.

We check now that $\mathcal{B}$ is a realizability algebra.

- $(\xi, p)(\eta, q) \star (\pi, r) \notin \perp\!\!\!\perp\!\!\!\perp \Rightarrow (\xi, p) \star (\eta, q) \mathbin{\scriptstyle\bullet} (\pi, r) \notin \perp\!\!\!\perp\!\!\!\perp$:

Suppose that $(\xi \star \eta \mathbin{\scriptstyle\bullet} \pi, pqr) \in \perp\!\!\!\perp\!\!\!\perp$; we must show $(\mathsf{C}\xi\eta \star \pi, pqr) \in \perp\!\!\!\perp\!\!\!\perp$ i.e. $\mathsf{C}\xi\eta \star \underline{n} \mathbin{\scriptstyle\bullet} \pi \in \perp\!\!\!\perp$ for $(pqr \ll n) = 1$. Now, we have $\mathsf{C}\xi\eta \star \underline{n} \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \underline{n} \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{B}^*, \mathbf{1}) \star (\xi, p) \mathbin{\scriptstyle\bullet} (\eta, q) \mathbin{\scriptstyle\bullet} (\zeta, r) \mathbin{\scriptstyle\bullet} (\pi, s) \notin \perp\!\!\!\perp\!\!\!\perp \Rightarrow (\xi, p) \star (\eta, q)(\zeta, r) \mathbin{\scriptstyle\bullet} (\pi, s) \notin \perp\!\!\!\perp\!\!\!\perp$:

Suppose that $(\xi, p) \star (\eta, q)(\zeta, r) \mathbin{\scriptstyle\bullet} (\pi, s) \in \perp\!\!\!\perp\!\!\!\perp$ i.e. $(\xi \star \mathsf{C}\eta\zeta \mathbin{\scriptstyle\bullet} \pi, pqrs) \in \perp\!\!\!\perp\!\!\!\perp$.
We must show :
$(\mathsf{B}^* \star \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \pi, pqrs) \in \perp\!\!\!\perp\!\!\!\perp$ i.e. $\mathsf{B}^* \star \underline{n} \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \pi \in \perp\!\!\!\perp$ for $(pqrs \ll n) = 1$.
Now, we have $\mathsf{B}^* \star \underline{n} \mathbin{\scriptstyle\bullet} \xi \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \zeta \mathbin{\scriptstyle\bullet} \pi \succ\!\!\succ \xi \star \underline{n} \mathbin{\scriptstyle\bullet} \mathsf{C}\eta\zeta \mathbin{\scriptstyle\bullet} \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{C}^*, \mathbf{1}) \star (\xi, p) \mathbin{\scriptstyle\bullet} (\eta, q) \mathbin{\scriptstyle\bullet} (\zeta, r) \mathbin{\scriptstyle\bullet} (\pi, s) \notin \perp\!\!\!\perp\!\!\!\perp \Rightarrow (\xi, p) \star (\zeta, r) \mathbin{\scriptstyle\bullet} (\eta, q) \mathbin{\scriptstyle\bullet} (\pi, s) \notin \perp\!\!\!\perp\!\!\!\perp$:

Suppose that $(\xi \star \zeta \mathbin{\scriptstyle\bullet} \eta \mathbin{\scriptstyle\bullet} \pi, pqrs) \in \perp\!\!\!\perp\!\!\!\perp$; we must show :

$(\mathsf{C}^* \star \xi \centerdot \eta \centerdot \zeta \centerdot \pi, pqrs) \in \perp\!\!\!\perp$ i.e. $\mathsf{C}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi \in \perp\!\!\!\perp$ for $(pqrs \ll n) = 1$.
Now, we have $\mathsf{C}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi \succ\!\!\succ \xi \star \underline{n} \centerdot \zeta \centerdot \eta \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{I}^*, \mathbf{1}) \star (\xi, p) \centerdot (\pi, q) \notin \perp\!\!\!\perp \Rightarrow (\xi, p) \star (\pi, q) \notin \perp\!\!\!\perp$:

Suppose that $(\xi \star \pi, pq) \in \perp\!\!\!\perp$; we must show :
$(\mathsf{I}^* \star \xi \centerdot \pi, pq) \in \perp\!\!\!\perp$ i.e. $\mathsf{I}^* \star \underline{n} \centerdot \xi \centerdot \pi \in \perp\!\!\!\perp$ for $(pq \ll n) = 1$. Now, we have :
$\mathsf{I}^* \star \underline{n} \centerdot \xi \centerdot \pi \succ\!\!\succ \xi \star \underline{n} \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{K}^*, \mathbf{1}) \star (\xi, p) \centerdot (\eta, q) \centerdot (\pi, r) \notin \perp\!\!\!\perp \Rightarrow (\xi, p) \star (\pi, r) \notin \perp\!\!\!\perp$:

Suppose that $(\xi \star \pi, pr) \in \perp\!\!\!\perp$; we must show :
$(\mathsf{K}^* \star \xi \centerdot \eta \centerdot \pi, pqr) \in \perp\!\!\!\perp$ i.e. $\mathsf{K}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \pi \in \perp\!\!\!\perp$ for $(pqr \ll n) = 1$. Now, we have :
$\mathsf{K}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \pi \succ\!\!\succ \xi \star \underline{n} \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{W}^*, \mathbf{1}) \star (\xi, p) \centerdot (\eta, q) \centerdot (\pi, r) \notin \perp\!\!\!\perp \Rightarrow (\xi, p) \star (\eta, q) \centerdot (\eta, q) \centerdot (\pi, r) \notin \perp\!\!\!\perp$:

Suppose that $(\xi \star \eta \centerdot \eta \centerdot \pi, pqr) \in \perp\!\!\!\perp$; we must show :
$(\mathsf{W}^* \star \xi \centerdot \eta \centerdot \pi, pqr) \in \perp\!\!\!\perp$ i.e. $\mathsf{W}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \pi \in \perp\!\!\!\perp$ for $(pqr \ll n) = 1$. Now, we have :
$\mathsf{W}^* \star \underline{n} \centerdot \xi \centerdot \eta \centerdot \pi \succ\!\!\succ \xi \star \underline{n} \centerdot \eta \centerdot \eta \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{cc}^*, \mathbf{1}) \star (\xi, p) \centerdot (\pi, q) \notin \perp\!\!\!\perp \Rightarrow (\xi, p) \star (\mathsf{k}_\pi^*, q) \centerdot (\pi, q) \notin \perp\!\!\!\perp$:

Suppose that $(\xi \star \mathsf{k}_\pi^* \centerdot \pi, pq) \in \perp\!\!\!\perp$; we must show :
$(\mathsf{cc}^* \star \xi \centerdot \pi, pq) \in \perp\!\!\!\perp$ i.e. $\mathsf{cc}^* \star \underline{n} \centerdot \xi \centerdot \pi \in \perp\!\!\!\perp$ for $(pq \ll n) = 1$.
Now, we have $\mathsf{cc}^* \star \underline{n} \centerdot \xi \centerdot \pi \succ\!\!\succ \xi \star \underline{n} \centerdot \mathsf{k}_\pi^* \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

- $(\mathsf{k}_\pi^*, p) \star (\xi, q) \centerdot (\varpi, r) \notin \perp\!\!\!\perp \Rightarrow (\xi, q) \star (\pi, p) \notin \perp\!\!\!\perp$:

Suppose that $(\xi \star \pi, pq) \in \perp\!\!\!\perp$; we must show :
$(\mathsf{k}_\pi^* \star \xi \centerdot \varpi, pqr) \in \perp\!\!\!\perp$ i.e. $\mathsf{k}_\pi^* \star \underline{n} \centerdot \xi \centerdot \varpi \in \perp\!\!\!\perp$ for $(pqr \ll n) = 1$.
Now, we have $\mathsf{k}_\pi^* \star \underline{n} \centerdot \xi \centerdot \varpi \succ\!\!\succ \xi \star \underline{n} \centerdot \pi$ which is in $\perp\!\!\!\perp$ by hypothesis.

For each closed c-term τ (built with the elementary combinators and the application), we define τ^* by recurrence, as follows :
if τ is an elementary combinator, τ^* is already defined ;
we set $(tu)^* = \mathsf{C}t^*u^*$.

In the algebra $\mathcal{B}$, the value of the combinator τ is $\tau_\mathcal{B} = (\tau_\mathcal{A}^*, \mathbf{1})$.
In particular, the integer n of the algebra $\mathcal{B}$ is $\underline{n}_\mathcal{B} = (\underline{n}^*, \mathbf{1})$.
We have $\underline{0}_\mathcal{B} = (\underline{0}^*, \mathbf{1}) = (\mathsf{K}^*, \mathbf{1})(\mathsf{I}^*, \mathbf{1})$; therefore : $\underline{0}^* = \mathsf{C}\mathsf{K}^*\mathsf{I}^*$.
We have $(\underline{n+1})_\mathcal{B} = ((\underline{n+1})^*, \mathbf{1}) = (\sigma^*, \mathbf{1})(\underline{n}^*, \mathbf{1})$; therefore : $(\underline{n+1})^* = \mathsf{C}\sigma^*\underline{n}^*$.
Thus, we have, for every $n \in \mathbb{N}$: $\underline{n}^* = (\mathsf{C}\sigma^*)^n\underline{0}^*$.

We define the proof-like terms of the algebra $\mathcal{B}$ as terms of the form $(\theta, \mathbf{1})$ where θ is a proof-like term of the algebra $\mathcal{A}$. The condition of coherence for $\mathcal{B}$ is therefore :
If θ is a proof-like term of $\mathcal{A}$, there exist $n \in \mathbb{N}$ and $\pi \in \Pi$ such that $\theta \star \underline{n} \centerdot \pi \notin \perp\!\!\!\perp$.
If $\mathcal{A}$ is coherent, then so is $\mathcal{B}$: indeed, if θ is a proof-like term of $\mathcal{A}$, then so is $\theta\underline{0}$; this gives a stack π such that $\theta\underline{0} \star \pi \notin \perp\!\!\!\perp$.

Notations.
The realizability models associated with the algebras $\mathcal{A}$ and $\mathcal{B}$ are denoted respectively by $\mathcal{N}_\mathcal{A}$ and $\mathcal{N}_\mathcal{B}$.
The truth value of a formula F in the realizability model $\mathcal{N}_\mathcal{B}$ will be denoted by $\|F\|_\mathcal{B}$ or also $|\!|\!|F|\!|\!|$.
We write $(\xi, p) \Vdash_\mathcal{B} F$ or $(\xi, p) |\!|\!\!\vdash F$ to say that (ξ, p) realizes the formula F in the realizability model $\mathcal{N}_\mathcal{B}$.

The collapsing function

We now define $\mathcal{G} \in \mathcal{M}$ in the following way :
$\mathcal{G} = \{((m,\alpha),(\pi,p))\,;\ m \in \mathbb{N},\ \alpha \in \kappa,\ \pi \in \Pi,\ p \in P\setminus\{\mathbb{O}\},\ p(m)$ is defined and $p(m) = \alpha\}$.

Theorem 34.
The formula $(\mathcal{G}$ *is a surjection from* $\mathbb{N}$ *onto* $\beth\kappa)$ *is realized in the model* $\mathcal{N}_{\mathcal{B}}$.
More precisely, we have :
i) $(\theta_0, \mathbf{1}) \Vdash \forall x \forall y \forall y'\,((x,y)\,\varepsilon\,\mathcal{G}, y \neq y' \to (x,y') \notin \mathcal{G})$ *with* $\theta_0 = \lambda n \lambda k \lambda x (x)n$ *;*
ii) $(\theta_1, \mathbf{1}) \Vdash \forall y^{\beth\kappa}[\forall x^{int}((x,y) \notin \mathcal{G}) \to \bot]$ *with* $\theta_1 = \lambda n \lambda x((((n)(\mathsf{C}B)(\mathsf{C})\sigma^*)(\mathsf{C})x)\underline{0}^*)(\sigma)n$,
and $\sigma = (B\mathsf{W})(B)B$ *(successor).*

i) Let $m \in \mathbb{N}, \alpha, \alpha' \in \kappa$, $(\pi,p) \in |||(m,\alpha) \notin \mathcal{G}|||$, $(\pi',p') \in |||(m,\alpha') \notin \mathcal{G}|||$
and $(\xi,q) \Vdash \alpha \neq \alpha'$.
Thus, we have $m \in \mathrm{dom}(p), m \in \mathrm{dom}(p'), p(m) = \alpha$ and $p'(m) = \alpha'$.
By lemma 6, we can replace the formula $(m,\alpha)\,\varepsilon\,\mathcal{G}$, which is $\neg((m,\alpha) \notin \mathcal{G})$, with the set of terms $\urcorner((m,\alpha) \notin \mathcal{G})$ which is $\{\mathsf{k}_{(\pi,p)}\ ;\ (\pi,p) \in |||(m,\alpha) \notin \mathcal{G}|||\}$.
Therefore, we have to show that :
$(\theta_0, \mathbf{1}) \star \mathsf{k}_{(\pi,p)} \centerdot (\xi,q) \centerdot (\pi',p') \in \mathbb{\perp\!\!\!\perp}$ that is $(\theta_0 \star \mathsf{k}_\pi^* \centerdot \xi \centerdot \pi', pp'q) \in \mathbb{\perp\!\!\!\perp}$.
This is obvious if $pp'q = \mathbb{O}$. Otherwise, p and p' are compatible, thus $\alpha = \alpha'$.
Let n be such that $(pp'q \ll n) = 1$; we must show that $\theta_0 \star \underline{n} \centerdot \mathsf{k}_\pi^* \centerdot \xi \centerdot \pi' \in \perp\!\!\!\perp$ i.e. $\xi \star \underline{n} \centerdot \pi' \in \perp\!\!\!\perp$.
Now, we have $(\xi,q) \Vdash \bot$ by hypothesis on (ξ,q), thus $(\xi,q) \star (\pi',\mathbf{1}) \in \mathbb{\perp\!\!\!\perp}$.
Since $(q \ll n) = 1$, it follows that $\xi \star \underline{n} \centerdot \pi' \in \perp\!\!\!\perp$.

ii) Let us first show that $\theta_1 \star \underline{n} \centerdot \eta \centerdot \varpi \succ\!\!\succ \eta \star \underline{n+1} \centerdot \underline{n}^* \centerdot \varpi$ for each $n \in \mathbb{N}, \eta \in \Lambda$ and $\varpi \in \Pi$. We have $\theta_1 \star \underline{n} \centerdot \eta \centerdot \varpi \succ\!\!\succ \underline{n} \star (\mathsf{CB})(\mathsf{C})\sigma^* \centerdot \mathsf{C}\eta \centerdot \underline{0}^* \centerdot \underline{n+1} \centerdot \varpi$.
By lemma 12(ii), in which we set $\zeta = \mathsf{C}\eta, \phi = \mathsf{C}\sigma^*, \alpha = \underline{0}^*, \varsigma = \sigma, O = \underline{0}$ and $\pi = \underline{n+1} \centerdot \varpi$, we obtain : $\theta_1 \star \underline{n} \centerdot \eta \centerdot \varpi \succ\!\!\succ \mathsf{C}\eta \star \underline{n}^* \centerdot \underline{n+1} \centerdot \varpi$ (since $\underline{n}^* = (\mathsf{C}\sigma^*)^n \underline{0}^*$) $\succ\!\!\succ \eta \star \underline{n+1} \centerdot \underline{n}^* \centerdot \varpi$.

We prove now that $(\theta_1, \mathbf{1}) \Vdash \forall y^{\beth\kappa}[\forall x^{\mathrm{int}}((x,y) \notin \mathcal{G}) \to \bot]$.
Let $\alpha \in \kappa$, $(\eta, p_0) \Vdash \forall x^{\mathrm{int}}((x,\alpha) \notin \mathcal{G})$ and $(\varpi, q_0) \in \Pi \times P$;
we show that $(\theta_1, \mathbf{1}) \star (\eta, p_0) \centerdot (\varpi, q_0) \in \mathbb{\perp\!\!\!\perp}$.
This is trivial if $p_0 q_0 = \mathbb{O}$; otherwise, let $n \in \mathbb{N}$ be such that $(p_0 q_0 \ll n) = 1$.
We must show that $\theta_1 \star \underline{n} \centerdot \eta \centerdot \varpi \in \perp\!\!\!\perp$, that is $\eta \star \underline{n+1} \centerdot \underline{n}^* \centerdot \varpi \in \perp\!\!\!\perp$.
But we have $(\eta, p_0) \Vdash \{(\underline{n}^*, \mathbf{1})\} \to (n,\alpha) \notin \mathcal{G}$ by hypothesis on η.
Since $(p_0 q_0 \ll n) = 1$, we can define $q \in P$ with domain $n+1$ such that $q \supset p_0 q_0$ and $q(n) = \alpha$. Then, we have $(\varpi, q) \in |||(n,\alpha) \notin \mathcal{G}|||$ by definition of $\mathcal{G}$.
We have thus $(\eta, p_0) \star (\underline{n}^*, \mathbf{1}) \centerdot (\varpi, q) \in \mathbb{\perp\!\!\!\perp}$ that is $(\eta \star \underline{n}^* \centerdot \varpi, p_0 q) \in \mathbb{\perp\!\!\!\perp}$.
But we have $p_0 q = q$, and therefore $(\eta \star \underline{n}^* \centerdot \varpi, q) \in \mathbb{\perp\!\!\!\perp}$.
Since $(q \ll n+1) = 1$, it follows that $\eta \star \underline{n+1} \centerdot \underline{n}^* \centerdot \varpi \in \perp\!\!\!\perp$.

Q.E.D.

Corollary 35. $\mathcal{N}_{\mathcal{B}}$ *realizes the non extensional axiom of choice and thus also DC.*

Indeed, by theorem 34, the model $\mathcal{N}_{\mathcal{B}}$ realizes the formula : ($\beth\kappa$ is countable).
But we have $\kappa = \mathrm{card}(\boldsymbol{\Lambda} \cup \boldsymbol{\Pi} \cup \mathbb{N})$, since $\kappa \geq \mathrm{card}(\Lambda \cup \Pi \cup \mathbb{N})$ and $\kappa = \mathrm{card}(P)$.
Therefore $\mathcal{N}_{\mathcal{B}}$ realizes NEAC, by theorem 30.

Q.E.D.

Remark. Intuitively, the model $\mathcal{N}_\mathcal{B}$ is an extension of the model $\mathcal{N}_\mathcal{A}$ obtained by forcing, by collapsing $\beth\kappa$. We cannot apply directly the usual theory of forcing, because $\beth\kappa$ is not defined in ZF.

Elementary formulas

Elementary formulas are defined as follows, where t, u are ℓ-terms, i.e. terms built with variables, individuals, and function symbols defined in $\mathcal{M}$:

- $\top, \bot$ are elementary formulas ;
- if U is an elementary formula, then $t = u \hookrightarrow U$ and $\forall x\, U$ are too ;
- if U, V are elementary formulas, then $U \to V$ too ;
- if U is an elementary formula, then $\forall n^{\text{int}} U$ too.

Remark. $t \neq u$ is an elementary formula, and also $t \notin \beth u$ (which can be written $f(t,u) \neq 1$ where f is the function symbol defined in $\mathcal{M}$ by : $f(a,b) = 1$ iff $a \in b$).
If U is an elementary formula, then $\forall x^{\beth t} U$ is too : indeed, it is written $\forall x(f(x,t) = 1 \hookrightarrow U)$.

For each elementary formula U, we define two formulas U_p and U^p, with one additional free variable p, by the conditions below.
Condition 1 defines U^p by means of U_p ; conditions 2 to 5 define U_p by recurrence :

1. $U^p \equiv \forall q^{\beth P} \forall n^{\text{int}}((pq \ll n) = 1 \hookrightarrow U_q)$;
2. $\bot_p \equiv \bot$ and $\top_p \equiv \top$;
3. $(t = u \hookrightarrow U)_p \equiv (t = u \hookrightarrow U_p)$; $(\forall x\, U[x])_p \equiv \forall x\, U_p[x]$;
4. $(U \to V)_p \equiv \forall q^{\beth P} \forall r^{\beth P}\, (p = qr \hookrightarrow (U^q \to V_r))$;
5. $(\forall n^{\text{int}} U[n])_p \equiv \forall n^{\beth \mathbb{N}}(\{\underline{n}^*\} \to U_p[n])$, in other words :
   $\|(\forall n^{\text{int}} U[n])_p\| = \{\underline{n}^* \centerdot \pi \,;\, n \in \mathbb{N}, \pi \in \|U_p[n]\|\}$.

Lemma 36. *For each closed elementary formula U, we have :*
$(\pi, p) \in |||U||| \Leftrightarrow \pi \in \|U_p\|$; $(\xi, p) \Vdash\!\!\vdash U \Leftrightarrow \xi \Vdash U^p$.

Proof by recurrence on the length of the formula U.
1. We have $(\xi, p) \Vdash\!\!\vdash U \Leftrightarrow (\xi, p) \star (\pi, q) \in \perp\!\!\!\perp\!\!\!\perp$ for $(\pi, q) \in |||U|||$, that is :
$(\xi \star \pi, pq) \in \perp\!\!\!\perp\!\!\!\perp$ for every $\pi \in \|U_q\|$, by the recurrence hypothesis, or also :
$(\forall q \in P)(\forall \pi \in \|U_q\|)(\forall n \in \mathbb{N})((pq \ll n) = 1 \Rightarrow \xi \star \underline{n} \centerdot \pi \in \perp\!\!\!\perp)$ which is equivalent to :
$\xi \Vdash \forall q^{\beth P} \forall n^{\text{int}}((pq \ll n) = 1 \hookrightarrow U_q)$ that is $\xi \Vdash U^p$.
2 and 3. Obvious.
4. Any element of $|||U \to V|||$ has the form $(\xi, q) \centerdot (\pi, r)$, i.e. $(\xi \centerdot \pi, p)$, with $p = qr$, $(\xi, q) \Vdash\!\!\vdash U$ and $(\pi, r) \in |||V|||$;
by the recurrence hypothesis, this is equivalent to $\xi \centerdot \pi \in \|U^q \to V_r\|$.
5. We have $|||\forall n^{\text{int}} U[n]||| = |||\forall n^{\beth \mathbb{N}}(\{(\underline{n}^*, \mathbf{1})\} \to U[n])|||$
$= \{(\underline{n}^*, \mathbf{1}) \centerdot (\pi, p) \,;\, n \in \mathbb{N}, (\pi, p) \in |||U[n]|||\} = \{(\underline{n}^* \centerdot \pi, p) \,;\, n \in \mathbb{N}, (\pi, p) \in |||U[n]|||\}$.
Thus, by the recurrence hypothesis, it is $\{(\underline{n}^* \centerdot \pi, p) \,;\, n \in \mathbb{N}, \pi \in \|U_p[n]\|\}$.
Q.E.D.

Lemma 37.
For each elementary formula U, there exist two proof-like terms θ^0_U, θ^1_U, such that :

i) $\theta^0_U \Vdash \forall p^{\daleth P}\forall n^{int}((p \ll n) = 1 \hookrightarrow (U \to U_p))$;
ii) $\theta^1_U \Vdash \forall p^{\daleth P}\forall n^{int}((p \ll n) = 1 \hookrightarrow (U_p \to U))$;
iii) $\tau^0_U \Vdash \forall p^{\daleth P}\forall n^{int}((p \ll n) = 1 \hookrightarrow (U \to U^p))$;
iv) $\tau^1_U \Vdash \forall p^{\daleth P}\forall n^{int}((p \ll n) = 1 \hookrightarrow (U^p \to U))$;
with $\tau^0_U = \lambda n\lambda x\lambda m(\theta^0_U)mx$ *and* $\tau^1_U = \lambda n\lambda x(\theta^1_U n)(x)n$.

We first show (iii) and (iv) from (i) and (ii).
(i)⇒(iii)
Let $p \in P$ and $n \in \mathbb{N}$ be such that $(p \ll n) = 1$; let $\xi \Vdash U$ and $\pi \in \|U^p\|$.
We have to show : $\lambda n\lambda x\lambda m(\theta^0_U)mx \star \underline{n} \cdot \xi \cdot \pi \in \perp\!\!\!\perp$.
Now, by the definition (1) of U^p, there exist $q \in P$, $m \in \mathbb{N}$ and $\varpi \in \|U_q\|$ such that $(pq \ll m) = 1$ and $\pi = \underline{m} \cdot \varpi$. Therefore, we have $(q \ll m) = 1$ and, by (i) :
$\theta^0_U \star \underline{m} \cdot \xi \cdot \varpi \in \perp\!\!\!\perp$, hence $\lambda n\lambda x\lambda m(\theta^0_U)mx \star \underline{n} \cdot \xi \cdot \underline{m} \cdot \varpi \in \perp\!\!\!\perp$.
(ii)⇒(iv)
Let $p \in P$, $n \in \mathbb{N}, \xi \in \Lambda$ and $\pi \in \|U\|$ such that $(p \ll n) = 1$ and $\xi \Vdash U^p$.
We have to show : $\lambda n\lambda x(\theta^1_U n)(x)n \star \underline{n} \cdot \xi \cdot \pi \in \perp\!\!\!\perp$ i.e. $\theta^1_U \star \underline{n} \cdot \xi\underline{n} \cdot \pi \in \perp\!\!\!\perp$.
But, by the definition (1) of U^p, in which we set $q = p$, we have $\xi\underline{n} \Vdash U_p$; therefore, the desired result follows from (ii).

We now show (i) and (ii) by recurrence on the length of U.

- If U is $\perp$ or $\top$, we take $\theta^0_U = \theta^1_U = \lambda n\lambda x\, x$.
- If $U \equiv (t = u \hookrightarrow V)$ or $U \equiv \forall x\, V$, then $\theta^0_U = \theta^0_V$ and $\theta^1_U = \theta^1_V$ by (3).
- If $U \equiv V \to W$, let $q, r \in \mathbb{N}$ and $p = qr$; let $n \in \mathbb{N}$ such that $(p \ll n) = 1$. We have :

$\tau^0_V\underline{n} \Vdash V \to V^q$; $\tau^1_V\underline{n} \Vdash V^q \to V$; $\theta^0_W\underline{n} \Vdash W \to W_r$; $\theta^1_W\underline{n} \Vdash W_r \to W$.
Let $\xi \Vdash V \to W$; then, by the recurrence hypothesis, we have :
$(\theta^0_W\underline{n})\circ\xi \Vdash V \to W_r$ and $(\theta^0_W\underline{n})\circ\xi\circ(\tau^1_V\underline{n}) \Vdash V^q \to W_r$.
Thus, by (4), we obtain $\theta^0_U = \lambda n\lambda x\lambda y(\theta^0_W n)(x)(\tau^1_V n)y$.
Now, let $\xi \Vdash V^q \to W_r$; then, by the recurrence hypothesis, we have :
$(\theta^1_W\underline{n})\circ\xi \Vdash V^q \to W$ and $(\theta^1_W\underline{n})\circ\xi\circ(\tau^0_V\underline{n}) \Vdash V \to W$.
Thus, by (4), we obtain $\theta^1_U = \lambda n\lambda x\lambda y(\theta^1_W n)(x)(\tau^0_V n)y$.

- If $U \equiv \forall n^{\text{int}}V[n]$, we first prove :

Lemma 38.
There exist two proof-like terms T_0, T_1 such that, for every closed formula F of ZF_ε :
i) $T_0 \Vdash \forall n^{\daleth\mathbb{N}}((\{\underline{n}^*\} \to F) \to (\{\underline{n}\} \to F))$.
ii) $T_1 \Vdash \forall n^{\daleth\mathbb{N}}((\{\underline{n}\} \to F) \to (\{\underline{n}^*\} \to F))$.
iii) For every elementary formula $V[n]$, we have :
$T_0 \Vdash (\forall n^{int}V[n])_p \to \forall n^{int}V_p[n]$ *and* $T_1 \Vdash \forall n^{int}V_p[n] \to (\forall n^{int}V[n])_p$.

i) We apply lemma 12(ii) to the realizability algebra $\mathcal{A}$, with :
$\varsigma = \sigma$, $O = \underline{0}$, $\phi = \mathsf{C}\sigma^*$ and $\alpha = \underline{0}^*$. For every $n \in \mathbb{N}, \zeta \in \Lambda$ and $\pi \in \Pi$, we obtain :
$\underline{n} \star (\mathsf{CB})(\mathsf{C})\sigma^* \cdot \zeta \cdot \underline{0}^* \cdot \pi \succ\!\!\succ \zeta \star \underline{n}^* \cdot \pi$, since $\underline{n}^* = (\mathsf{C}\sigma^*)^n\underline{0}^*$.
Therefore, if we set $T_0 = \lambda f\lambda n((n)(\mathsf{CB})(\mathsf{C})\sigma^*)f\underline{0}^*$, we have $T_0 \star \zeta \cdot \underline{n} \cdot \pi \succ\!\!\succ \zeta \star \underline{n}^* \cdot \pi$.
Thus, we have $T_0 \Vdash \forall n^{\daleth\mathbb{N}}((\{\underline{n}^*\} \to F) \to (\{\underline{n}\} \to F))$.

ii) We apply now lemma 12(i) to the realizability algebra $\mathcal{B}$, with :
$\varsigma = \sigma_{\mathcal{B}}$, $O = \underline{0}_{\mathcal{B}}$, $\phi = (\mathsf{C}\Sigma, \mathbf{1})$ $\alpha = (\Omega, \mathbf{1})$ and $\Omega = \lambda d\lambda f\lambda a\, a$; $\Sigma = \lambda n\lambda d\lambda f\lambda a(ndf)(f)a$.
Since $\underline{n}_{\mathcal{B}} = (\sigma_{\mathcal{B}})^n\underline{0}_{\mathcal{B}} = (\underline{n}^*, \mathbf{1})$, we get, by setting $\Sigma_2 = (\mathsf{C})^2\Sigma$:
$(\underline{n}^*, \mathbf{1}) \star (\mathsf{C}\Sigma, \mathbf{1}) \,\bullet\, (\Omega, \mathbf{1}) \,\bullet\, (\varpi, \mathbf{1}) \succ\!\!\succ ((\Sigma_2)^n\Omega, \mathbf{1}) \star (\varpi, \mathbf{1})$
because $((\mathsf{C}\Sigma, \mathbf{1}))^n(\Omega, \mathbf{1}) = ((\Sigma_2)^n\Omega, \mathbf{1})$. We write this as :
$(\underline{n}^* \star \mathsf{C}\Sigma \,\bullet\, \Omega \,\bullet\, \varpi, \mathbf{1}) \succ\!\!\succ ((\Sigma_2)^n\Omega \star \varpi, \mathbf{1})$.
It follows that $\underline{n}^* \star \underline{0} \,\bullet\, \mathsf{C}\Sigma \,\bullet\, \Omega \,\bullet\, \varpi \succ\!\!\succ (\Sigma_2)^n\Omega \star \underline{d} \,\bullet\, \varpi$ for some $d \in \mathbb{N}$.
Let us take $\varpi = \mathsf{CB}\sigma \,\bullet\, \zeta \,\bullet\, \underline{0} \,\bullet\, \pi$. We obtain :
$\underline{n}^* \star \underline{0} \,\bullet\, \mathsf{C}\Sigma \,\bullet\, \Omega \,\bullet\, \mathsf{CB}\sigma \,\bullet\, \zeta \,\bullet\, \underline{0} \,\bullet\, \pi \succ\!\!\succ (\Sigma_2)^n\Omega \star \underline{d} \,\bullet\, \mathsf{CB}\sigma \,\bullet\, \zeta \,\bullet\, \underline{0} \,\bullet\, \pi$.
Now, we apply lemma 13(ii), with $\phi = \sigma$ and $\alpha = \underline{0}$ (note that $\Sigma_2 = (\mathsf{C})^2\Sigma$ satisfies the hypothesis of lemma 13).
We obtain $(\Sigma_2)^n\Omega \star \underline{d} \,\bullet\, \mathsf{CB}\sigma \,\bullet\, \zeta \,\bullet\, \underline{0} \,\bullet\, \pi \succ \zeta \star (\sigma)^n\underline{0} \,\bullet\, \pi$ and therefore :
$\underline{n}^* \star \underline{0} \,\bullet\, \mathsf{C}\Sigma \,\bullet\, \Omega \,\bullet\, \mathsf{CB}\sigma \,\bullet\, \zeta \,\bullet\, \underline{0} \,\bullet\, \pi \succ\!\!\succ \zeta \star \underline{n} \,\bullet\, \pi$.
Finally, if we set $T_1 = \lambda f\lambda n((((n\underline{0})(\mathsf{C})\Sigma)\Omega)(\mathsf{C})\mathsf{B}\sigma)f\underline{0}$, we have :
$T_1 \star \zeta \,\bullet\, \underline{n}^* \,\bullet\, \pi \succ\!\!\succ \zeta \star \underline{n} \,\bullet\, \pi$ and therefore $T_1 \Vdash \forall n^{\gimel\mathbb{N}}((\{\underline{n}\} \to F) \to (\{\underline{n}^*\} \to F))$.
iii) This follows immediately from (i) and (ii), by definition of $(\forall n^{\mathrm{int}}V[n])_p$.

Q.E.D.

We can now finish the proof of lemma 37, considering the last case which is :

- $U \equiv \forall m^{\mathrm{int}}V[m]$.

We show that $\theta^0_U = \lambda n\lambda x(T_1)\lambda m(\theta^0_V n)(x)m$.
By the recurrence hypothesis, we have $\theta^0_V \Vdash \forall p^{\gimel P}\forall n^{\mathrm{int}}((p \ll n) = 1 \hookrightarrow (V[m] \to V_p[m]))$.
Let $p \in P, n \in \mathbb{N}, \xi \in \Lambda$ be such that $(p \ll n) = 1$ and $\xi \Vdash \forall m^{\mathrm{int}}V[m]$.
Then, for every $m \in \mathbb{N}$, we have $\xi\underline{m} \Vdash V[m]$; thus $(\theta^0_V\underline{n})(\xi)m \Vdash V_p[m]$ and therefore $\lambda m(\theta^0_V\underline{n})(\xi)m \Vdash \forall m^{\mathrm{int}}V_p[m]$. By lemma 38(iii), we get $(T_1)\lambda m(\theta^0_V\underline{n})(\xi)m \Vdash (\forall m^{\mathrm{int}}V[m])_p$ and therefore : $\lambda x(T_1)\lambda m(\theta^0_V\underline{n})(x)m \Vdash \forall m^{\mathrm{int}}V[m] \to (\forall m^{\mathrm{int}}V[m])_p$. Finally : $\lambda n\lambda x(T_1)\lambda m(\theta^0_V n)(x)m \Vdash \forall p^{\gimel P}\forall n^{\mathrm{int}}((p \ll n) = 1 \hookrightarrow (\forall m^{\mathrm{int}}V[m] \to (\forall m^{\mathrm{int}}V[m])_p))$.
We show now that $\theta^1_U = \lambda n\lambda x\lambda m(\theta^1_V n)(T_0)xm$.
By the recurrence hypothesis, we have $\theta^1_V \Vdash \forall p^{\gimel P}\forall n^{\mathrm{int}}((p \ll n) = 1 \hookrightarrow (V_p[m] \to V[m]))$;
Let $p \in P, n \in \mathbb{N}, \xi \in \Lambda$ be such that $(p \ll n) = 1$ and $\xi \Vdash (\forall m^{\mathrm{int}}V[m])_p$.
By lemma 38(iii), we have $T_0\xi \Vdash \forall m^{\mathrm{int}}V_p[m]$, thus $T_0\xi\underline{m} \Vdash V_p[m]$.
Therefore $(\theta^1_V\underline{n})(T_0)\xi\underline{m} \Vdash V[m]$, and $\lambda m(\theta^1_V\underline{n})(T_0)\xi m \Vdash \forall m^{\mathrm{int}}V[m]$, hence the result.

Q.E.D.

Theorem 39.
The same closed elementary formulas, with parameters in $\mathcal{M}$, are realized in the models $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$.

Let U be a closed elementary formula, which is realized in $\mathcal{N}_{\mathcal{A}}$ and let θ be a proof-like term such that $\theta \Vdash U$. Then, we have $(\tau^0_U)\underline{n}\theta \Vdash U^p$ for $(p \ll n) = 1$, by lemma 37(iii) ; therefore, setting $p = \emptyset = \mathbf{1}$, we have $((\tau^0_U)\underline{0}\theta, \mathbf{1}) \Vvdash U$ by lemma 36.
Therefore, the formula U is also realized in the model $\mathcal{N}_{\mathcal{B}}$.
Conversely, if $(\theta, \mathbf{1}) \Vvdash U$ with $\theta \in \mathrm{QP}$, we have $\theta \Vdash U^{\mathbf{1}}$, by lemma 36. Thus $\tau^1_U\underline{0}\,\theta \Vdash U$ by lemma 37(iv).

Q.E.D.

Remark. For instance :

- If the Boolean algebra $\daleth 2$ has four ε-elements or if it is atomless, in the model $\mathcal{N}_{\mathcal{A}}$, the same goes for the model $\mathcal{N}_{\mathcal{B}}$.
- Arithmetical formulas are elementary. Therefore, by theorem 39, the models $\mathcal{N}_{\mathcal{A}}$ and $\mathcal{N}_{\mathcal{B}}$ realize the same arithmetical formulas. In fact, this was already known, because they are the same as the arithmetical formulas which are true in $\mathcal{M}$ [15, 16].

Arithmetical formulas and dependent choice

In this section, we obtain, by means of the previous results, a technique to transform into a program, a given proof, in ZF + DC, of an arithmetical formula F.
We notice that this program is a closed c-term, written with the elementary combinators $\mathsf{B}, \mathsf{C}, \mathsf{I}, \mathsf{K}, \mathsf{W}, \mathsf{cc}$ *without any other instruction.*

Thus, let us consider a proof of $\mathrm{ZF}_\varepsilon \vdash$ NEAC $\to F$. It gives us a closed c-term Φ_0 such that $\Phi_0 \Vdash$ NEAC $\to F$, in every realizability algebra.
We now describe a rewriting on closed c-terms, which will transform Φ_0 into a closed c-term Φ such that $\Phi \Vdash F$ in every realizability algebra $\mathcal{A}$.
By theorem 30, we have $\Phi_1 \Vdash (\daleth\kappa$ is countable$) \to F$ with $\Phi_1 = \lambda x(\Phi_0)(\mathsf{H})x$.
We apply this result in the algebra $\mathcal{B}$, which gives :
$(\Phi_1^*, \mathbf{1}) \mathrel{|\!|\!|\!-} (\daleth\kappa$ is countable$) \to F$.
Now, theorem 34 gives a closed c-term Δ such that $(\Delta, \mathbf{1}) \mathrel{|\!|\!|\!-} (\daleth\kappa$ is countable$)$.
It follows that $(\Phi_1^*, \mathbf{1})(\Delta, \mathbf{1}) \mathrel{|\!|\!|\!-} F$, i.e. $(\Psi, \mathbf{1}) \mathrel{|\!|\!|\!-} F$, with $\Psi = \mathsf{C}\Phi_1^*\Delta$.
Since F is an arithmetical formula, it is an elementary formula.
Therefore, by lemma 36, we have $\Psi \Vdash F^{\mathbf{1}}$. Now, by lemma 37(iv), we have :
$\tau_F^1 \Vdash \forall p^{\daleth P} \forall n^{\mathrm{int}}((p \ll n) = 1 \hookrightarrow (F^p \to F))$.
We set $p = \mathbf{1}$ and $n = 0$, and we obtain $\tau_F^1 \underline{0} \Vdash F^{\mathbf{1}} \to F$.
Finally, by setting $\Phi = (\tau_F^1)\underline{0}\Psi$, we have $\Phi \Vdash F$.

A relative consistency result

In [18], we have defined a countable realizability algebra $\mathcal{A}$ such that the characteristic Boolean algebra $\daleth 2$ of the model $\mathcal{N}_{\mathcal{A}}$ is atomless (in this example, we have $\kappa = \mathbb{N}$).
If we apply the technique of section 3, in order to collapse $\daleth\kappa$, we obtain a realizability algebra $\mathcal{B}$ and a model $\mathcal{N}_{\mathcal{B}}$, the characteristic Boolean algebra of which is also atomless.
Indeed, the property : ($\daleth 2$ is atomless) is expressed by an elementary formula.
But now $\daleth 2$ is *the countable atomless Boolean algebra* (they are all isomorphic). Therefore, by applying theorems 30 and 33, we obtain the relative consistency result (i) announced in the introduction.

Remark. We note that this method applies to every realizability algebra such that we have :
$\Vdash$ ($\daleth 2$ is an atomless Boolean algebra).

4 A two threads model ($\beth 2$ with four elements)

In this section, we suppose that $\mathcal{A}$ is a *standard realisability algebra* [18].
This means, by definition, that the terms and the stacks are finite sequences, built with :
the alphabet B, C, I, K, W, cc, k, $\,\bullet\,$, (,), [,]
a countable set of *term constants* (also called *instructions*),
a countable set of *stack constants*
and that they are defined by the following rules :
B, C, I, K, W, cc and all the term constants are terms ;
if t, u are terms, the sequence $(t)u$ is a term ;
if π is a stack, the sequence $\mathsf{k}[\pi]$ is a term (denoted by k_π) ;
each stack constant is a stack ;
if t is a term and π is a stack, then $t\,\bullet\,\pi$ is a stack.
If t is a term and π is a stack, then the ordered pair (t, π) is a *process*, denoted by $t \star \pi$.
A proof-like term of $\mathcal{A}$ is a term which does not contain the symbol k ; or, which is the same, a term which does not contain any stack constant.
We now build a realizability model in which $\beth 2$ has exactly 4 elements.
We suppose that there are exactly two stack constants π^0, π^1 and one term constant d.
For $i \in \{0, 1\}$, let Λ^i (resp. Π^i) be the set of terms (resp. stacks)
which contain the only stack constant π^i.
For $i, j \in \{0, 1\}$, define $\perp\!\!\!\perp^i_j$ as the least set $P \subset \Lambda^i \star \Pi^i$ of processes such that :
1. $\mathsf{d} \star \underline{j}\,\bullet\,\pi \in P$ for every $\pi \in \Pi^i$.
2. $\xi \star \pi \in \Lambda^i \star \Pi^i$, $\xi' \star \pi' \in P$, $\xi \star \pi \succ \xi' \star \pi' \Rightarrow \xi \star \pi \in P$
3. If at least two out of three processes $\xi\star\pi$, $\eta\star\pi$, $\zeta\star\pi$ are in P, then $\mathsf{d}\star\underline{2}\,\bullet\,\xi\,\bullet\,\eta\,\bullet\,\zeta\,\bullet\,\pi \in P$.

Remarks.
The preorder $\succ$ on $\Lambda \star \Pi$ was defined at the beginning of section 1.
We express condition 2 by saying that P *is saturated in* $\Lambda^i \star \Pi^i$.
Following this definition of $\succ$, the constant d is a *halting instruction.* Indeed, we have :

$$\mathsf{d} \star \pi \succ \xi \star \varpi \;\Leftrightarrow\; \xi \star \varpi = \mathsf{d} \star \pi.$$

We define $\perp\!\!\!\perp$ by : $\quad \Lambda \star \Pi \setminus \perp\!\!\!\perp = (\Lambda^0 \star \Pi^0 \setminus \perp\!\!\!\perp^0_0) \cup (\Lambda^1 \star \Pi^1 \setminus \perp\!\!\!\perp^1_1)$
In other words, a process is in $\perp\!\!\!\perp$ if and only if
either it is in $\perp\!\!\!\perp^0_0 \cup \perp\!\!\!\perp^1_1$ or it contains both stack constants π^0, π^1.

Lemma 40. *If* $\xi \star \pi \in \perp\!\!\!\perp^i_j$ *and* $\xi \star \pi \succ \xi' \star \pi'$ *then* $\xi' \star \pi' \in \perp\!\!\!\perp^i_j$ *(closure by reduction).*

Suppose that $\xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$, $\xi_0 \star \pi_0 \in \perp\!\!\!\perp^i_j$ and $\xi'_0 \star \pi'_0 \notin \perp\!\!\!\perp^i_j$. We may suppose that :
$(*)$ $\qquad \xi_0 \star \pi_0 \succ \xi'_0 \star \pi'_0$ in exactly one step of reduction.
Let us show that $\perp\!\!\!\perp^i_j \setminus \{\xi_0 \star \pi_0\}$ has properties 1,2,3 defining $\perp\!\!\!\perp^i_j$, which will contradict the definition of $\perp\!\!\!\perp^i_j$:
1. If $\xi_0 \star \pi_0 = \mathsf{d} \star \underline{j}\,\bullet\,\pi$, with $\pi \in \Pi^i$, then $\mathsf{d} \star \underline{j}\,\bullet\,\pi \succ \xi'_0 \star \pi'_0$, thus $\xi'_0 \star \pi'_0 = \mathsf{d} \star \underline{j}\,\bullet\,\pi$.
Therefore $\xi'_0 \star \pi'_0 \in \perp\!\!\!\perp^i_j$, which is false.
2. Suppose $\xi\star\pi \in \Lambda^i\star\Pi^i$, $\xi\star\pi \succ \xi'\star\pi' \in \perp\!\!\!\perp^i_j$, $\xi'\star\pi' \neq \xi_0\star\pi_0$. Then $\xi\star\pi \in \perp\!\!\!\perp^i_j$, by (2).
If $\xi \star \pi = \xi_0 \star \pi_0$, then $\xi_0 \star \pi_0 \succ \xi' \star \pi'$; since $\xi' \star \pi' \neq \xi_0 \star \pi_0$, it follows from $(*)$ that $\xi'_0 \star \pi'_0 \succ \xi' \star \pi'$ and therefore $\xi'_0 \star \pi'_0 \in \perp\!\!\!\perp^i_j$, which is false.

3. Suppose that two out of the processes $\xi \star \pi$, $\eta \star \pi$, $\zeta \star \pi$ are in $\perp\!\!\!\perp_j^i \setminus \{\xi_0 \star \pi_0\}$, but $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi$ is not. From (3), it follows that $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi = \xi_0 \star \pi_0$.
Thus, $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi \succ \xi_0' \star \pi_0'$, and therefore $\xi_0' \star \pi_0' = \mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi$.
Therefore $\xi_0' \star \pi_0' \in \perp\!\!\!\perp_j^i$, which is false.

Q.E.D.

Lemma 41. $\perp\!\!\!\perp_0^i \cap \perp\!\!\!\perp_1^i = \emptyset$.

We prove that $(\Lambda^i \star \Pi^i \setminus \perp\!\!\!\perp_1^i) \supset \perp\!\!\!\perp_0^i$ by showing that $\Lambda^i \star \Pi^i \setminus \perp\!\!\!\perp_1^i$ has properties 1, 2, 3 which define $\perp\!\!\!\perp_0^i$.
1. $\mathsf{d} \star \underline{0} \centerdot \pi \notin \perp\!\!\!\perp_1^i$ because $\perp\!\!\!\perp_1^i \setminus \{\mathsf{d} \star \underline{0} \centerdot \pi\}$ has properties 1, 2, 3 defining $\perp\!\!\!\perp_1^i$.
2. Follows from lemma 40.
3. Suppose $\xi_0 \star \pi_0$, $\eta_0 \star \pi_0 \notin \perp\!\!\!\perp_1^i$; we show that $\mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0 \notin \perp\!\!\!\perp_1^i$ by showing that $\perp\!\!\!\perp_1^i \setminus \{\mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0\}$ has properties 1, 2, 3 defining $\perp\!\!\!\perp_1^i$.

1. Clearly, $\mathsf{d} \star \underline{1} \centerdot \pi' \in (\perp\!\!\!\perp_1^i \setminus \{\mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0\})$ for every $\pi' \in \Pi^i$.
2. Suppose that $\xi \star \pi \in \Lambda^i \star \Pi^i$, $\xi \star \pi \succ \xi' \star \pi' \in \perp\!\!\!\perp_1^i$, $\xi' \star \pi' \neq \mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0$ and that $\xi \star \pi \notin (\perp\!\!\!\perp_1^i \setminus \{\mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0\})$.
   From (2), it follows that $\xi \star \pi = \mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0$ which contradicts $\xi \star \pi \succ \xi' \star \pi'$.
3. Suppose that two out of the processes $\xi \star \pi$, $\eta \star \pi$, $\zeta \star \pi$ are in $\perp\!\!\!\perp_1^i \setminus \{\mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0\}$ but that $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi$ is not.
   It follows from (3) that $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi = \mathsf{d} \star \underline{2} \centerdot \xi_0 \centerdot \eta_0 \centerdot \zeta_0 \centerdot \pi_0$, i.e.
   $\xi = \xi_0, \eta = \eta_0, \zeta = \zeta_0$ and $\pi = \pi_0$. But this contradicts the hypothesis :
   $\xi_0 \star \pi_0$, $\eta_0 \star \pi_0 \notin \perp\!\!\!\perp_1^i$.

Q.E.D.

Theorem 42. *This realizability algebra is coherent.*

Let $\theta \in \mathrm{QP}$ be such that $\theta \star \pi^0 \in \perp\!\!\!\perp_0^0$ and $\theta \star \pi^1 \in \perp\!\!\!\perp_1^1$. Then $\theta \star \pi^0 \in \perp\!\!\!\perp_0^0 \cap \perp\!\!\!\perp_1^0$ which contradicts lemma 41.

Q.E.D.

Lemma 43. *$\mathsf{d}\underline{2} \Vdash$ (the boolean algebra $\daleth 2$ has at most four ε-elements).*

We show that $\mathsf{d}\,\underline{2} \Vdash \forall x^{\daleth 2} \forall y^{\daleth 2}(x \neq 0, y \neq 1, x \neq y \to x \wedge y \neq x)$.
Let $i, j \in \{0, 1\}$, $\xi \Vdash i \neq 0, \eta \Vdash j \neq 1, \zeta \Vdash i \neq j$ and $\pi \in \|i \wedge j \neq i\|$.
Since $\|i \wedge j \neq i\| \neq \emptyset$, we have $i \leq j$. Thus, there are three possibilities for (i, j) :
$i = j = 0$; $i = j = 1$; $i = 0, j = 1$.
In each case, two out of the terms ξ, η, ζ realize $\perp$. Thus, we have $\mathsf{d} \star \underline{2} \centerdot \xi \centerdot \eta \centerdot \zeta \centerdot \pi \in \perp\!\!\!\perp$.

Q.E.D.

Remark. If $\pi \in \Pi \setminus (\Pi^0 \cup \Pi^1)$, then $\xi \star \pi \in \perp\!\!\!\perp$ for every term ξ. Thus, we can remove these stacks and consider only $\Pi^0 \cup \Pi^1$.

We define two individuals in this realizability model :
$\gamma_0 = (\{0\} \times \Pi^0) \cup (\{1\} \times \Pi^1)$; $\gamma_1 = (\{1\} \times \Pi^0) \cup (\{0\} \times \Pi^1)$.
Obviously, $\gamma_0, \gamma_1 \subset \daleth 2 = \{0, 1\} \times \Pi$. Now we have :
$\|\forall x^{\daleth 2}(x \notin \gamma_0)\| = \Pi^0 \cup \Pi^1 = \|\perp\|$ and therefore $\mathsf{I} \Vdash \neg \forall x^{\daleth 2}(x \notin \gamma_0)$.
$\mathsf{d}\underline{0} \Vdash 0 \notin \gamma_0$ and $\mathsf{d}\underline{1} \Vdash 1 \notin \gamma_0$.

It follows that γ_0, γ_1 are not ε-empty and that every ε-element of γ_0, γ_1 is $\neq 0, 1$.
Therefore :

The Boolean algebra $\gimel 2$ *has exactly four* ε*-elements.*

We have $\xi \Vdash \forall x^{\gimel 2}(x \,\varepsilon\, \gamma_0, x \,\varepsilon\, \gamma_1 \to \bot)$ for *every* term ξ :
Indeed, let $i, j \in \{0, 1\}$; using lemma 6, we replace the formula $i \,\varepsilon\, \gamma_j$, i.e. $\neg(i \notin \gamma_j)$, with $\urcorner(i \notin \gamma_j)$ which is $\{\mathsf{k}_\pi \,;\, \pi \in \Pi^{i+j}\}$. Therefore, we have to check :
$\rho_0 \in \Pi^0, \rho_1 \in \Pi^1 \Rightarrow \xi \star \mathsf{k}_{\rho_0} \,.\, \mathsf{k}_{\rho_1} \,.\, \pi \in \perp\!\!\!\perp$ which is clear.

In the same way, we get :
$\lambda x \lambda y \lambda z\, z \Vdash \forall x \forall y (x \,\varepsilon\, \gamma_i, y \,\varepsilon\, \gamma_i, x \neq y \to \bot)$.
It follows that γ_0, γ_1 are singletons and that their ε-elements are the two atoms of $\gimel 2$.

$\gimel 2$ has four ε-elements and $\gimel \kappa$ is countable

We now apply to the algebra $\mathcal{A}$ the technique expounded in section 3, in order to make $\gimel \kappa$ countable ; this gives a realizability algebra $\mathcal{B}$.
In this case, we have $\kappa = \mathbb{N}$, and therefore $\kappa_+ = \mathcal{P}(\kappa) = \mathbb{R}$.

Now, there is an elementary formula which express that the Boolean algebra $\gimel 2$ has four ε-elements, for instance : $\exists x^{\gimel 2}\{x \neq 0, x \neq 1\} \wedge \forall x^{\gimel 2} \forall y^{\gimel 2}(x \neq 1, y \neq 1, x \neq y \to xy = 0)$.
Therefore, the realizability model $\mathcal{N}_\mathcal{B}$ realizes the following two formulas :

($\gimel 2$ has four ε-elements) ; ($\gimel \kappa$ is countable) ;
and therefore also NEAC by theorem 30.

Let us denote by i_0, i_1 the two atoms of $\gimel 2$; thus, we have $i_1 = 1 - i_0$.

We suppose that $\mathcal{M} \models V = L$; thus, there exists on $\kappa_+ = \mathcal{P}(\mathbb{N}) = \mathbb{R}$ a strict well ordering $\lhd$ of type $\aleph_1$. This gives a function from $\mathbb{R}^2$ into $\{0, 1\}$, denoted by $(x \lhd y)$, which is defined as follows : $(x \lhd y) = 1 \Leftrightarrow x \lhd y$.
We can extend it to $\mathcal{N}_\mathcal{A}$ and $\mathcal{N}_\mathcal{B}$, which gives a function from $(\gimel \mathbb{R})^2$ into $\gimel 2$.
From lemmas 28 and 29, we get :

For $i = i_0$ or i_1, the relation $(x \lhd y) = i$ is a strict total ordering on $\gimel_i \mathbb{R}$ and one of these two relations is a well ordering ;
in order to fix the ideas, we shall suppose that it is for $i = i_0$.
The relation $(x \lhd y) = 1$ is a strict order relation on $\gimel \mathbb{R}$, which is well founded.
The application $x \mapsto (i_0 x, i_1 x)$ from $\gimel \mathbb{R}$ onto $\gimel_{i_0} \mathbb{R} \times \gimel_{i_1} \mathbb{R}$ is an isomorphism of strictly ordered sets.

It follows from theorem 25, that each of the sets $\gimel_{i_0} \mathbb{R}$, $\gimel_{i_1} \mathbb{R}$ contain a countable subset.
By corollary 23, there is no surjection from each one of the sets $\gimel_{i_0} \mathbb{R}, \gimel_{i_1} \mathbb{R}$ onto the other.
Thus, there is no surjection from $\mathbb{N}$ onto $\gimel_{i_0} \mathbb{R}$ or onto $\gimel_{i_1} \mathbb{R}$.
Therefore, the well ordering on $\gimel_{i_0} \mathbb{R}$ has, at least, the order type $\aleph_1$ in $\mathcal{N}_\mathcal{B}$.

Now, by theorem 31, every subset of $\gimel \mathbb{R}$, which is bounded from above for the ordering $\lhd$, is countable ; thus, the same goes for the proper initial segments of $\gimel_{i_0} \mathbb{R}$ and $\gimel_{i_1} \mathbb{R}$, since these sets are totally ordered and $\gimel \mathbb{R}$ is isomorphic to $\gimel_{i_0} \mathbb{R} \times \gimel_{i_1} \mathbb{R}$.

It follows that the well ordering on $\gimel_{i_0} \mathbb{R}$ is at most $\aleph_1$, and therefore exactly $\aleph_1$.
Moreover, *there exists, on* $\gimel_{i_1} \mathbb{R}$*, a total ordering, every proper initial segment of which is countable.*

Then, we can apply theorem 33, to the sets X_{i_0}, X_{i_1} which are the images of $\beth_{i_0}\mathbb{R}, \beth_{i_1}\mathbb{R}$ by the injection from $\beth\kappa_+$ into $\mathbb{R}$, which is given by theorem 32. By setting $X = X_{i_1}$, we obtain exactly the result (ii) of relative consistency announced in the introduction.

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