The Determinacy of Context-Free Games

Olivier Finkel

THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

THE DETERMINACY OF CONTEXT-FREE GAMES

OLIVIER FINKEL

Abstract. We prove that the determinacy of Gale-Stewart games whose winning sets are accepted by real-time 1-counter Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption. We show also that the determinacy of Wadge games between two players in charge of ω-languages accepted by 1-counter Büchi automata is equivalent to the (effective) analytic Wadge determinacy. Using some results of set theory we prove that one can effectively construct a 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that: (1) There exists a model of ZFC in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$; (2) There exists a model of ZFC in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined. Moreover these are the only two possibilities, i.e. there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

§**1. Introduction.** Two-players infinite games have been much studied in Set Theory and in Descriptive Set Theory, see [14, 13, 18]. In particular, if X is a (countable) alphabet having at least two letters and $A \subseteq X^\omega$, then the Gale-Stewart game $G(A)$ is an infinite game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed an infinite word $x = a_1b_1a_2b_2\ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play. The game $G(A)$ is said to be determined iff one of the two players has a winning strategy. A fundamental result of Descriptive Set Theory is Martin's Theorem which states that every Gale-Stewart game $G(A)$, where A is a Borel set, is determined [14].

On the other hand, in Computer Science, the conditions of a Gale Stewart game may be seen as a specification of a reactive system, where the two players are respectively a non terminating reactive program and the "environment". Then the problem of the synthesis of winning strategies is of great practical interest for the problem of program synthesis in reactive systems. In particular, if $A \subseteq X^\omega$, where X is here a finite alphabet, and A is effectively presented, i.e. accepted by a given finite machine or defined by a given logical formula, the following questions naturally arise, see [23, 15]: (1) Is the game $G(A)$ determined ? (2) If Player 1 has a winning strategy, is it effective, i.e. computable ? (3) What are the amounts of space and time necessary to compute such a winning strategy ? Büchi and Landweber gave a solution to the famous Church's Problem, posed in 1957, by stating that in a Gale Stewart game $G(A)$, where A is a regular

Key words and phrases. Automata and formal languages; logic in computer science; Gale-Stewart games; Wadge games; determinacy; effective analytic determinacy; context-free games; 1-counter automaton; models of set theory; independence from the axiomatic system ZFC.

0022-4812/00/0000-0000/$00.00

ω-language, one can decide who the winner is and compute a winning strategy given by a finite state transducer, see [24] for more information on this subject. In [23, 15] Thomas and Lescow asked for an extension of this result where A is no longer regular but deterministic context-free, i.e. accepted by some deterministic pushdown automaton. Walukiewicz extended Büchi and Landweber's Theorem to this case by showing first in [26] that that one can effectively construct winning strategies in parity games played on pushdown graphs and that these strategies can be computed by pushdown transducers. Notice that later some extensions to the case of higher-order pushdown automata have been established [1, 2].

In this paper, we first address the question (1) of the determinacy of Gale-Stewart games $G(A)$, where A is a context-free ω-language accepted by a (non-deterministic) pushdown automaton, or even by a 1-counter automaton. Notice that there are some context-free ω-languages which are (effective) analytic but non-Borel [6], and thus the determinacy of these games can not be deduced from Martin's Theorem of Borel determinacy. On the other hand, Martin's Theorem is provable in ZFC, the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developed. But the determinacy of Gale-Stewart games $G(A)$, where A is an (effective) analytic set, is not provable in ZFC; Martin and Harrington have proved that it is a large cardinal assumption equivalent to the existence of a particular real, called the real $0^\sharp$, see [13, page 637]. We prove here that the determinacy of Gale-Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is equivalent to the determinacy of (effective) analytic Gale-Stewart games and thus also equivalent to the existence of the real $0^\sharp$.

Next we consider Wadge games which were firstly studied by Wadge in [25] where he determined a great refinement of the Borel hierarchy defined via the notion of reduction by continuous functions, see Definition 4.1 below for a precise definition. These games are closely related to the notion of reducibility by continuous functions. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' iff there exists a continuous function $f : X^\omega \rightarrow Y^\omega$, such that $L = f^{-1}(L')$; this is then denoted by $L \leq_W L'$. On the other hand, the Wadge game $W(L, L')$ is an infinite game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. And it turned out that Player 2 has a winning strategy in the Wadge game $W(L, L')$ iff $L \leq_W L'$. It is easy to see that the determinacy of Borel Gale-Stewart games implies the determinacy of Borel Wadge games. On the other hand, Louveau and Saint-Raymond have proved that this latter one is weaker than the first one, since it is already provable in second-order arithmetic, while the first one is not. It is also known that the determinacy of (effective) analytic Gale-Stewart games is equivalent to the determinacy of (effective) analytic Wadge games, see [16]. We prove in this paper that the determinacy of Wadge games between two players in charge of ω-languages accepted by 1-counter Büchi automata is equivalent to the (effective) analytic Wadge determinacy, and thus also equivalent to the existence of the real $0^\sharp$.

Then, using some recent results from [8] and some results of Set Theory, we prove that, (assuming ZFC is consistent), one can effectively construct a 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that: (1) There exists a model of ZFC in which Player 2 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$; (2) There exists a model of ZFC in which the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.

Moreover these are the only two possibilities, i.e. there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

This paper is an extended version of a conference paper which appeared in the Proceedings of the 29 th International Symposium on Theoretical Aspects of Computer Science, STACS 2012, [10]. It contains the full proofs which could not be included in the conference paper due to lack of space.

Notice that as the results presented in this paper might be of interest to both set theorists and theoretical computer scientists, we shall recall in detail some notions of automata theory which are well known to computer scientists but not to set theorists. In a similar way we give a presentation of some results of set theory which are well known to set theorists but not to computer scientists.

The paper is organized as follows. We recall some known notions in Section 2. We study context-free Gale-Stewart games in Section 3 and context-free Wadge games in Section 4. Some concluding remarks are given in Section 5.

§**2. Recall of some known notions.** We assume the reader to be familiar with the theory of formal (ω-)languages [22, 20]. We recall the usual notations of formal language theory.

If Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots , k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ. A (finitary) *language* V over an alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An ω-*word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When $\sigma = a_1 \ldots a_n \ldots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:

$(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω-*words* over the alphabet Σ is denoted by Σ^ω. An ω-*language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega - V$, denoted V^-.

The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$.

If L is a finitary language (respectively, an ω-language) over the alphabet Σ then the set $\mathrm{Pref}(L)$ of prefixes of elements of L is defined by $\mathrm{Pref}(L) = \{u \in \Sigma^\star \mid \exists v \in L \; u \sqsubseteq v\}$.

We now recall the definition of k-counter Büchi automata which will be useful in the sequel.

Let k be an integer ≥ 1. A k-counter machine has k *counters*, each of which containing a non-negative integer. The machine can test whether the content of a given counter is zero or not. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counters. Notice that in this model transitions are allowed where the reading head of the machine does not move to the right. In other words, λ-transitions are allowed here.

Formally a k-counter machine is a 4-tuple $\mathcal{M}$=(K, Σ, Δ, q_0), where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0,1\}^k \times K \times \{0,1,-1\}^k$ is the transition relation. The k-counter machine $\mathcal{M}$ is said to be *real time* iff: $\Delta \subseteq K \times \Sigma \times \{0,1\}^k \times K \times \{0,1,-1\}^k$, i.e. iff there are no λ-transitions.

If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbf{N}$ is the content of the i^{th} counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k)$.

Thus the transition relation must obviously satisfy:
if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$ then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Let $\sigma = a_1 a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c_1^i, \ldots c_k^i)_{i \geq 1}$ is called a run of $\mathcal{M}$ on σ iff:

(1) $(q_1, c_1^1, \ldots c_k^1) = (q_0, 0, \ldots, 0)$

(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c_1^i, \ldots c_k^i) \mapsto_{\mathcal{M}} (q_{i+1}, c_1^{i+1}, \ldots c_k^{i+1})$ and such that $a_1 a_2 \ldots a_n \ldots = b_1 b_2 \ldots b_n \ldots$

For every such run r, $\mathrm{In}(r)$ is the set of all states entered infinitely often during r.

DEFINITION 2.1. *A Büchi k-counter automaton is a 5-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'$=(K, Σ, Δ, q_0) is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is:*

$L(\mathcal{M})$= $\{\sigma \in \Sigma^\omega \mid$ *there exists a run r of $\mathcal{M}$ on σ such that* $\mathrm{In}(r) \cap F \neq \emptyset\}$

The class of ω-languages accepted by Büchi k-counter automata is denoted $\mathbf{BCL}(k)_\omega$. The class of ω-languages accepted by *real time* Büchi k-counter automata will be denoted **r-**$\mathbf{BCL}(k)_\omega$. The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by Büchi pushdown automata.

We assume the reader to be familiar with basic notions of topology which may be found in [14, 15, 22, 20]. There is a natural metric on the set Σ^ω of infinite words over a finite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual Cantor topology in which the *open subsets* of Σ^ω are of the form $W.\Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set.

For $V \subseteq \Sigma^\star$ we denote $\mathrm{Lim}(V) = \{x \in \Sigma^\omega \mid \exists^\infty n \geq 1 \ x[n] \in V\}$ the set of infinite words over Σ having infinitely many prefixes in V. Then the topological closure $\mathrm{Cl}(L)$ of a set $L \subseteq \Sigma^\omega$ is equal to $\mathrm{Lim}(\mathrm{Pref}(L))$. Thus we have also the following characterization of closed subsets of Σ^ω: a set $L \subseteq \Sigma^\omega$ is a closed subset of the Cantor space Σ^ω iff $L = \mathrm{Lim}(\mathrm{Pref}(L))$.

We now recall the definition of the *Borel Hierarchy* of subsets of X^ω.

DEFINITION 2.2. *For a non-null countable ordinal α, the classes Σ^0_α and Π^0_α of the Borel Hierarchy on the topological space X^ω are defined as follows: Σ^0_1 is the class of open subsets of X^ω, Π^0_1 is the class of closed subsets of X^ω, and for any countable*

ordinal $\alpha \geq 2$:
Σ^0_α *is the class of countable unions of subsets of X^ω in $\bigcup_{\gamma<\alpha} \Pi^0_\gamma$.*
Π^0_α *is the class of countable intersections of subsets of X^ω in $\bigcup_{\gamma<\alpha} \Sigma^0_\gamma$.*
A set $L \subseteq X^\omega$ is Borel iff it is in the union $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$, where ω_1 is the first uncountable ordinal.

There are also some subsets of X^ω which are not Borel. In particular the class of Borel subsets of X^ω is strictly included into the class Σ^1_1 of *analytic sets* which are obtained by projection of Borel sets. The *co-analytic sets* are the complements of analytic sets.

DEFINITION 2.3. *A subset A of X^ω is in the class Σ^1_1 of analytic sets iff there exist a finite alphabet Y and a Borel subset B of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where (x, y) is the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.*

We now recall the notion of completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq X^\omega$ is said to be a Σ^0_α (respectively, Π^0_α, Σ^1_1)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \Sigma^0_\alpha$ (respectively, $E \in \Pi^0_\alpha$, $E \in \Sigma^1_1$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.

We now recall the definition of classes of the arithmetical hierarchy of ω-languages, see [22]. Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that:
$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots , a_{n-1}, \sigma[a_n + 1]) \in R_L\}$,
where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The class Σ^1_1 is the class of *effective analytic sets* which are obtained by projection of arithmetical sets. An ω-language $L \subseteq X^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq \mathbb{N} \times \{0, 1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau (\tau \in \{0, 1\}^\omega \wedge \forall n \exists m((n, \tau[m], \sigma[m]) \in R_L))\}.$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is the projection of an ω-language over the alphabet $X \times \{0, 1\}$ which is in the class Π_2. The class Π^1_1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets.

Recall that the (lightface) class Σ^1_1 of effective analytic sets is strictly included into the (boldface) class $\boldsymbol{\Sigma}^1_1$ of analytic sets.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ^1_1 of effective analytic sets [4, 22]. On the other hand, one can construct, using a classical construction (see for instance [12]), from a Büchi Turing machine $\mathcal{T}$, a 2-counter Büchi automaton $\mathcal{A}$ accepting the same ω-language. Thus one can state the following proposition.

PROPOSITION 2.4. *An ω-language $L \subseteq X^\omega$ is in the class Σ^1_1 iff it is accepted by a non deterministic Büchi Turing machine, hence iff it is in the class $\mathbf{BCL}(2)_\omega$.*

§3. Context-free Gale-Stewart games.

We first recall the definition of Gale-Stewart games.

DEFINITION 3.1 ([13]). *Let $A \subseteq X^\omega$, where X is a finite alphabet. The Gale-Stewart game $G(A)$ is a game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed a word $x = a_1 b_1 a_2 b_2 \ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play.*

Let $A \subseteq X^\omega$ and $G(A)$ be the associated Gale-Stewart game. A strategy for Player 1 is a function $F_1 : (X^2)^\star \to X$ and a strategy for Player 2 is a function $F_2 : (X^2)^\star X \to X$. Player 1 follows the strategy F_1 in a play if for each integer $n \geq 1$ $a_n = F_1(a_1 b_1 a_2 b_2 \cdots a_{n-1} b_{n-1})$. If Player 1 wins every play in which she has followed the strategy F_1, then we say that the strategy F_1 is a winning strategy (w.s.) for Player 1. The notion of winning strategy for Player 2 is defined in a similar manner.

The game $G(A)$ is said to be determined if one of the two players has a winning strategy.

We shall denote **Det**($\mathcal{C}$)*, where $\mathcal{C}$ is a class of ω-languages, the sentence : "Every Gale-Stewart game $G(A)$, where $A \subseteq X^\omega$ is an ω-language in the class $\mathcal{C}$, is determined".*

Notice that, in the whole paper, we assume that ZFC is consistent, and all results, lemmas, propositions, theorems, are stated in ZFC unless we explicitely give another axiomatic framework.

We can now state our first result.

PROPOSITION 3.2. **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{r}$-$\mathbf{BCL}(8)_\omega$).

Proof. The implication **Det**(Σ_1^1) $\Longrightarrow$ **Det**($\mathbf{r}$-$\mathbf{BCL}(8)_\omega$) is obvious since $\mathbf{r}$-$\mathbf{BCL}(8)_\omega \subseteq \Sigma_1^1$.

To prove the reverse implication, we assume that **Det**($\mathbf{r}$-$\mathbf{BCL}(8)_\omega$) holds and we show that every Gale-Stewart game $G(A)$, where $A \subseteq X^\omega$ is an ω-language in the class Σ_1^1, or equivalently in the class $\mathbf{BCL}(2)_\omega$ by Proposition 2.4, is determined.

Let then $L \subseteq \Sigma^\omega$, where Σ is a finite alphabet, be an ω-language in the class $\mathbf{BCL}(2)_\omega$.

Let E be a new letter not in Σ, S be an integer ≥ 1, and $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ be the function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4) \ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}} \ldots$$

We proved in [7] that if $k = cardinal(\Sigma) + 2$, $S \geq (3k)^3$ is an integer, then one can effectively construct from a Büchi 2-counter automaton $\mathcal{A}_1$ accepting L a real time Büchi 8-counter automaton $\mathcal{A}_2$ such that $L(\mathcal{A}_2) = \theta_S(L)$. In the sequel we assume that we have fixed an integer $S \geq (3k)^3$ which is *even*.

Notice that the set $\theta_S(\Sigma^\omega)$ is a closed subset of the Cantor space $(\Sigma \cup \{E\})^\omega$. An ω-word $x \in (\Sigma \cup \{E\})^\omega$ is in $\theta_S(\Sigma^\omega)^-$ iff it has one prefix which is not in $\mathrm{Pref}(\theta_S(\Sigma^\omega))$. Let $L' \subseteq (\Sigma \cup \{E\})^\omega$ be the set of ω-words $y \in (\Sigma \cup \{E\})^\omega$ for which there is an integer $n \geq 1$ such that $y[2n-1] \in \mathrm{Pref}(\theta_S(\Sigma^\omega))$ and $y[2n] \notin \mathrm{Pref}(\theta_S(\Sigma^\omega))$. So if two players have alternatively written letters from the alphabet $\Sigma \cup \{E\}$ and have composed an infinite word in L', then it is Player 2 who has left the closed set $\theta_S(\Sigma^\omega)$. It is easy to see that L' is accepted by a real time Büchi 2-counter automaton.

The class $\mathbf{r}$-$\mathbf{BCL}(8)_\omega \supseteq \mathbf{r}$-$\mathbf{BCL}(2)_\omega$ is closed under finite union in an effective way, so $\theta_S(L) \cup L'$ is accepted by a real time Büchi 8-counter automaton $\mathcal{A}_3$ which can be effectively constructed from $\mathcal{A}_2$.

As we have assumed that **Det**($\mathbf{r}$-$\mathbf{BCL}(8)_\omega$) holds, the game $G(\theta_S(L) \cup L')$ is determined, i.e. one of the two players has a w.s. in the game $G(\theta_S(L) \cup L')$. We now show that the game $G(L)$ is itself determined.

We shall say that, during an infinite play, Player 1 "goes out" of the *closed* set $\theta_S(\Sigma^\omega)$ if the final play y composed by the two players has a prefix $y[2n] \in \mathrm{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n+1] \notin \mathrm{Pref}(\theta_S(\Sigma^\omega))$. We define in a similar way the sentence "Player 2 goes out of the *closed* set $\theta_S(\Sigma^\omega)$".

Assume first that Player 1 has a w.s. F_1 in the game $G(\theta_S(L) \cup L')$. Then Player 1 never "goes out" of the set $\theta_S(\Sigma^\omega)$ when she follows this w.s. because otherwise the final play y composed by the two players has a prefix $y[2n] \in \mathrm{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n+1] \notin \mathrm{Pref}(\theta_S(\Sigma^\omega))$ and thus $y \notin \theta_S(L) \cup L'$. Consider now a play in which Player 2 does not go out of $\theta_S(\Sigma^\omega)$. If player 1 follows her w.s. F_1 then the two players remain in the set $\theta_S(\Sigma^\omega)$. But we have fixed S to be an **even** integer. So the two players compose an ω-word

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

and the letters $x(k)$ are written by player 1 for k an odd integer and by Player 2 for k an even integer because S is even. Moreover Player 1 wins the play iff the ω-word $x(1)x(2)x(3)\ldots x(n)\ldots$ is in L. This implies that Player 1 has also a w.s. in the game $G(L)$.

Assume now that Player 2 has a w.s. F_2 in the game $G(\theta_S(L) \cup L')$. Then Player 2 never "goes out" of the set $\theta_S(\Sigma^\omega)$ when he follows this w.s. because otherwise the final play y composed by the two players has a prefix $y[2n-1] \in \mathrm{Pref}(\theta_S(\Sigma^\omega))$ such that $y[2n] \notin \mathrm{Pref}(\theta_S(\Sigma^\omega))$ and thus $y \in L'$ hence also $y \in \theta_S(L) \cup L'$. Consider now a play in which Player 1 does not go out of $\theta_S(\Sigma^\omega)$. If player 2 follows his w.s. F_2 then the two players remain in the set $\theta_S(\Sigma^\omega)$. So the two players compose an ω-word

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

where the letters $x(k)$ are written by player 1 for k an odd integer and by Player 2 for k an even integer. Moreover Player 2 wins the play iff the ω-word $x(1)x(2)x(3)\ldots x(n)\ldots$ is not in L. This implies that Player 2 has also a w.s. in the game $G(L)$. □

THEOREM 3.3. *Det*(Σ^1_1) $\Longleftrightarrow$ *Det*($\mathbf{CFL}_\omega$) $\Longleftrightarrow$ *Det*($\mathbf{BCL}(1)_\omega$).

Proof. The implications **Det**(Σ^1_1) $\Longrightarrow$ **Det**($\mathbf{CFL}_\omega$) $\Longrightarrow$ **Det**($\mathbf{BCL}(1)_\omega$) are obvious since $\mathbf{BCL}(1)_\omega \subseteq \mathbf{CFL}_\omega \subseteq \Sigma^1_1$.

To prove the reverse implication **Det**($\mathbf{BCL}(1)_\omega$) $\Longrightarrow$ **Det**(Σ^1_1), we assume that **Det**($\mathbf{BCL}(1)_\omega$) holds and we show that every Gale-Stewart game $G(L)$, where $L \subseteq X^\omega$ is an ω-language in the class $\mathbf{r}$-$\mathbf{BCL}(8)_\omega$ is determined. Then Proposition 3.2 will imply that **Det**(Σ^1_1) also holds.

Let then $L(\mathcal{A}) \subseteq \Gamma^\omega$, where Γ is a finite alphabet and $\mathcal{A}$ is a real time Büchi 8-counter automaton.

We now recall the following coding which was used in the paper [7].

Let K be the product of the eight first prime numbers. An ω-word $x \in \Gamma^\omega$ was coded by the ω-word

$$h_K(x) = A.C^K.x(1).B.C^{K^2}.A.C^{K^2}.x(2).B.C^{K^3}.A.C^{K^3}.x(3).B\ldots$$

$$\ldots B.C^{K^n}.A.C^{K^n}.x(n).B\ldots$$

over the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$, where A, B, C are new letters not in Γ. We are going to use here a slightly different coding which we now define. Let then

$$h(x) = C^K.C.A.x(1).C^{K^2}.A.C^{K^2}.C.x(2).B.C^{K^3}.A.C^{K^3}.C.A.x(3)\ldots$$

$$\ldots C^{K^{2n}}.A.C^{K^{2n}}.C.x(2n).B.C^{K^{2n+1}}.A.C^{K^{2n+1}}.C.A.x(2n+1)\ldots$$

We now explain the rules used to obtain the ω-word $h(x)$ from the ω-word $h_K(x)$.

(1) The first letter A of the word $h_K(x)$ has been suppressed.

(2) The letters B following a letter $x(2n+1)$, for $n \geq 1$, have been suppressed.

(3) A letter C has been added before each letter $x(2n)$, for $n \geq 1$.

(4) A block of two letters $C.A$ has been added before each letter $x(2n+1)$, for $n \geq 1$. The reasons behind this changes are the following ones. Assume that two players alternatively write letters from the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$ and that they finally produce an ω-word in the form $h(x)$. Due to the above changes we have now the two following properties which will be useful in the sequel.

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters C, the first letter which is not a C has always been written by Player 2.

We proved in [7] that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting $L(\mathcal{A}) \subseteq \Gamma^\omega$, one can effectively construct a Büchi 1-counter automaton $\mathcal{A}_1$ accepting the ω-language $h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-$. We can easily check that the changes in $h_K(x)$ leading to the coding $h(x)$ have no influence with regard to the proof of this result in [7] and thus one can also effectively construct a Büchi 1-counter automaton $\mathcal{A}_2$ accepting the ω-language $h(L(\mathcal{A})) \cup h(\Gamma^\omega)^-$.

On the other hand we can remark that all ω-words in the form $h(x)$ belong to the ω-language $H \subseteq (\Gamma_1)^\omega$ of ω-words y of the following form:

$$y = C^{n_1}.C.A.x(1).C^{n_2}.A.C^{n'_2}.C.x(2).B.C^{n_3}.A.C^{n'_3}.C.A.x(3)\ldots$$

$$\ldots C^{n_{2n}}.A.C^{n'_{2n}}.C.x(2n).B.C^{n_{2n+1}}.A.C^{n'_{2n+1}}.C.A.x(2n+1)\ldots$$

where for all integers $i \geq 1$ the letters $x(i)$ belong to Γ and the n_i, n'_i, are even non-null integers. Notice that it is crucial to allow here for arbitrary n_i, n'_i and not just $n_i = n'_i = K^i$ because we obtain this way a *regular* ω-language H.

An important fact is the following property of H which extends the same property of the set $h(\Gamma^\omega)$. Assume that two players alternatively write letters from the alphabet $\Gamma_1 = \Gamma \cup \{A, B, C\}$ and that they finally produce an ω-word y in H in the above form. Then we have the two following facts:

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters C, the first letter which is not a C has always been written by Player 2.

Let now $V = \text{Pref}(H) \cap (\Gamma_1)^\star.C$. So a finite word over the alphabet Γ_1 is in V iff it is a prefix of some word in H and its last letter is a C. It is easy to see that the topological closure of H is

$$\text{Cl}(H) = H \ \cup \ V.C^\omega.$$

Notice that an ω-word in $\text{Cl}(H)$ is not in $h(\Gamma^\omega)$ iff a sequence of consecutive letters C has not the good length. Thus if two players alternatively write letters from the alphabet Γ_1 and produce an ω-word $y \in \text{Cl}(H) - h(\Gamma^\omega)$ then it is Player 2 who has gone out of the set $h(\Gamma^\omega)$ at some step of the play. This will be important in the sequel.

It is very easy to see that the ω-language H is regular and to construct a Büchi automaton $\mathcal{H}$ accepting it. Moreover it is known that the class $\mathbf{BCL}(1)_\omega$ is effectively closed under intersection with regular ω-languages (this can be seen using a classical construction of a product automaton, see [3, 20]). Thus one can also construct a Büchi 1-counter automaton $\mathcal{A}_3$ accepting the ω-language $h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H]$.

We denote also U the set of finite words u over Γ_1 such that $|u| = 2n$ for some integer $n \geq 1$ and $u[2n-1] \in \text{Pref}(H)$ and $u = u[2n] \notin \text{Pref}(H)$.

Now we set:

$$\mathcal{L} \ = \ h(L(\mathcal{A})) \ \cup \ [h(\Gamma^\omega)^- \cap H] \ \cup \ V.C^\omega \ \cup \ U.(\Gamma_1)^\omega$$

Notice that $\mathcal{L}$ is obtained as the union of the image of $L(\mathcal{A})$ by h and of three sets which are at the end only accessible through Player 2.

We have already seen that the ω-language $h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H]$ is accepted by a Büchi 1-counter automaton $\mathcal{A}_3$. On the other hand the ω-language H is regular and it is accepted by a Büchi automaton $\mathcal{H}$. Thus the finitary language $\text{Pref}(H)$ is also regular, the languages U and V are also regular, and the ω-languages $V.C^\omega$ and $U.(\Gamma_1)^\omega$ are regular. This implies that one can construct a Büchi 1-counter automaton $\mathcal{A}_4$ accepting the language $\mathcal{L}$.

By hypothesis we assume that **Det**$(\mathbf{BCL}(1)_\omega)$ holds and thus the game $G(\mathcal{L})$ is determined. We are going to show that this implies that the game $G(L(\mathcal{A}))$ itself is determined.

Assume firstly that Player 1 has a winning strategy F_1 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word z, and Player 2 "does not go out of the set $h(\Gamma^\omega)$" then we claim that also Player 1, following her strategy F_1, "does not go out of the set $h(\Gamma^\omega)$". Indeed if Player 1 goes out of the set $h(\Gamma^\omega)$ then due to the above remark this would imply that Player 1 also goes out of the set $\text{Cl}(H)$: there is an integer $n \geq 0$ such that $z[2n] \in \text{Pref}(H)$ but $z[2n+1] \notin \text{Pref}(H)$. So $z \notin h(L(\mathcal{A})) \ \cup \ [h(\Gamma^\omega)^- \cap H] \cup \ V.C^\omega$. Moreover it follows from the definition of U that $z \notin U.(\Gamma_1)^\omega$. Thus If Player 1 goes out of the set $h(\Gamma^\omega)$ then she looses the game.

Consider now an infinite play in which Player 2 "does not go out of the set $h(\Gamma^\omega)$". Then Player 1, following her strategy F_1, "does not go out of the set $h(\Gamma^\omega)$". Thus the two players write an infinite word $z = h(x)$ for some infinite word $x \in \Gamma^\omega$. But the letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 1 wins the play iff $x \in L(\mathcal{A})$ and Player 1 wins always the play when she uses her strategy F_1. This implies that Player 1 has also a w.s. in the game $G(L(\mathcal{A}))$.

Assume now that Player 2 has a winning strategy F_2 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word z, and Player 1 "does not go out of the set $h(\Gamma^\omega)$" then we claim that also Player 2, following his strategy F_2, "does not go out of the set $h(\Gamma^\omega)$". Indeed if Player 2 goes out of the set $h(\Gamma^\omega)$ and the final play z remains in $\mathrm{Cl}(H)$ then $z \in [h(\Gamma^\omega)^- \cap H] \cup V.C^\omega \subseteq \mathcal{L}$ and Player 2 looses. If Player 1 does not go out of the set $\mathrm{Cl}(H)$ and at some step of the play, Player 2 goes out of $\mathrm{Pref}(H)$, i.e. there is an integer $n \geq 1$ such that $z[2n-1] \in \mathrm{Pref}(H)$ and $z[2n] \notin \mathrm{Pref}(H)$, then $z \in U.(\Gamma_1)^\omega \subseteq \mathcal{L}$ and Player 2 looses.

Assume now that Player 1 "does not go out of the set $h(\Gamma^\omega)$". Then Player 2 follows his w. s. F_2, and then "never goes out of the set $h(\Gamma^\omega)$". Thus the two players write an infinite word $z = h(x)$ for some infinite word $x \in \Gamma^\omega$. But the letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 2 wins the play iff $x \notin L(\mathcal{A})$ and Player 2 wins always the play when he uses his strategy F_2. This implies that Player 2 has also a w.s. in the game $G(L(\mathcal{A}))$. □

Looking carefully at the above proof, we can obtain the following stronger result:

THEOREM 3.4. **Det**(Σ^1_1) $\Longleftrightarrow$ **Det**$(\mathbf{CFL}_\omega)$ $\Longleftrightarrow$ **Det**$(\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega)$.

Proof. We return to the above proof of Theorem 3.3, with the same notations.

We proved in [7] that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting $L(\mathcal{A}) \subseteq \Gamma^\omega$, one can effectively construct a Büchi 1-counter automaton $\mathcal{A}_1$ accepting the ω-language $h_K(L(\mathcal{A})) \cup h_K(\Gamma^\omega)^-$ having the additional property: during any run of $\mathcal{A}_1$ there are at most K consecutive λ-transitions, where K is the product of the eight first prime numbers.

Then the Büchi 1-counter automaton $\mathcal{A}_3$, accepting the ω-language

$$h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H],$$

has the same property because the ω-language H is regular and any regular ω-language is accepted by a real-time Büchi or Muller automaton, so the result follows from a classical construction of a product automaton, see [20]. Finally the Büchi 1-counter automaton $\mathcal{A}_4$ accepting the language

$$\mathcal{L} = h(L(\mathcal{A})) \cup [h(\Gamma^\omega)^- \cap H] \cup V.C^\omega \cup U.(\Gamma_1)^\omega$$

has also the same property.

Thus we have actually proved that **Det**(Σ^1_1) is equivalent to the determinacy of all games $G(L(\mathcal{B}))$, where $\mathcal{B}$ is a Büchi 1-counter automaton having also this property: during any run at most K consecutive λ-transitions may occur.

We now prove that **Det**$(\mathbf{r}\text{-}\mathbf{BCL}(1)_\omega)$ implies the determinacy of such games.

We now assume that **Det**(**r**-$\mathbf{BCL}(1)_\omega$) holds and we consider a Büchi 1-counter automaton $\mathcal{B}$ reading words over an alphabet Γ having the property: during any run at most K consecutive λ-transitions may occur.

Consider now the mapping $\phi_K : \Gamma^\omega \rightarrow (\Gamma \cup \{F\})^\omega$ which is simply defined by: for all $x \in \Gamma^\omega$,

$$\phi_K(x) = F^K.x(1).F^K.x(2)\ldots F^K.x(n).F^K.x(n+1).F^K \ldots$$

Then the ω-language $\phi_K(L(\mathcal{B})$ is accepted by a real time Büchi 1-counter automaton $\mathcal{B}'$ which can be effectively constructed from the Büchi 1-counter automaton $\mathcal{B}$, see [5]. Notice that the set $\phi_K(\Gamma^\omega)$ is a regular closed subset of $(\Gamma \cup \{F\})^\omega$. Let now L'' be the set of ω-words $y \in (\Gamma \cup \{F\})^\omega$ such that there is an integer $n \geq 0$ with $y[2n-1] \in \mathrm{Pref}(\phi_K(\Gamma^\omega))$ and $y[2n] \notin \mathrm{Pref}(\phi_K(\Gamma^\omega))$. The ω-language L'' is regular since $\phi_K(\Gamma^\omega)$ is regular and so $\mathrm{Pref}(\phi_K(\Gamma^\omega))$ is regular. Thus the ω-language $\phi_K(L(\mathcal{B})) \cup L''$ is accepted by a real time Büchi 1-counter automaton $\mathcal{B}''$. Therefore the game $G(\phi_K(L(\mathcal{B})) \cup L'')$ is determined.

It is now easy to prove that the game $G(L(\mathcal{B}))$ itself is determined, reasoning as in the proof of Proposition 3.2. Details are here left to the reader. □

REMARK 3.5. *The proofs of Proposition 3.2 and Theorems 3.3 and 3.4 provide actually the following effective result. Let $L \subseteq X^\omega$ be an ω-language in the class Σ^1_1, or equivalently in the class $\mathbf{BCL}(2)_\omega$, which is accepted by a Büchi 2-counter automaton $\mathcal{A}$. Then one can effectively construct from $\mathcal{A}$ a real time Büchi 1-counter automaton $\mathcal{B}$ such that the game $G(L)$ is determined if and only if the game $G(L(\mathcal{B}))$ is determined. Moreover Player 1 (respectively, Player 2) has a w.s. in the game $G(L)$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{B}))$.*

§4. Context-free Wadge games.

We first recall the notion of Wadge games.

DEFINITION 4.1 (Wadge [25]). *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in Y$, then Player 1 writes a letter $a_2 \in X$, and so on. The two players alternatively write letters a_n of X for Player 1 and b_n of Y for Player 2. After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written an ω-word $b \in Y^\omega$. Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps. Player 2 wins the play iff [$a \in L \leftrightarrow b \in L'$], i.e. iff: [($a \in L$ and $b \in L'$)* or *($a \notin L$ and $b \notin L'$ and b is infinite)].*

Recall that a strategy for Player 1 is a function $\sigma : (Y \cup \{s\})^\star \rightarrow X$. And a strategy for Player 2 is a function $f : X^+ \rightarrow Y \cup \{s\}$. The strategy σ is a winning strategy for Player 1 iff she always wins a play when she uses the strategy σ, i.e. when the n^{th} letter she writes is given by $a_n = \sigma(b_1 \ldots b_{n-1})$, where b_i is the letter written by Player 2 at step i and $b_i = s$ if Player 2 skips at step i. A winning strategy for Player 2 is defined in a similar manner.

The game $W(L, L')$ is said to be determined if one of the two players has a winning strategy. In the sequel we shall denote **W-Det**($\mathcal{C}$), where $\mathcal{C}$ is a class of ω-languages, the sentence: "All Wadge games $W(L, L')$, where $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ are ω-languages in the class $\mathcal{C}$, are determined".

There is a close relationship between Wadge reducibility and games.

DEFINITION 4.2 (Wadge [25]). *Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$. L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.*
The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The equivalence classes of $\equiv_W$ are called Wadge degrees.

THEOREM 4.3 (Wadge). *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ where X and Y are finite alphabets. Then $L \leq_W L'$ if and only if Player 2 has a winning strategy in the Wadge game $W(L, L')$.*

The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$. Using Wadge games, Wadge proved that, up to the complement and $\equiv_W$, it is a well ordered hierarchy which provides a great refinement of the Borel hierarchy.

THEOREM 4.4 (Wadge). *The class of Borel subsets of X^ω, for a finite alphabet X, equipped with $\leq_W$, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d_W^0 from WH onto $|WH| - \{0\}$, such that for all $L, L' \subseteq X^\omega$:*
$d_W^0 L < d_W^0 L' \leftrightarrow L <_W L'$ and
$d_W^0 L = d_W^0 L' \leftrightarrow [L \equiv_W L'$ or $L \equiv_W L'^{-}]$.

We can now state the following result on determinacy of context-free Wadge games.

THEOREM 4.5.
Det(Σ_1^1) $\Longleftrightarrow$ **W-Det**($\mathbf{CFL}_\omega$) $\Longleftrightarrow$ **W-Det**($\mathbf{BCL}(1)_\omega$) $\Longleftrightarrow$ **W-Det**($\mathbf{r}$-$\mathbf{BCL}(1)_\omega$).

In order to prove this theorem, we first recall the notion of operation of sum of sets of infinite words which has as counterpart the ordinal addition over Wadge degrees, and which will be used later.

DEFINITION 4.6 (Wadge). *Assume that $X \subseteq Y$ are two finite alphabets, $Y - X$ containing at least two elements, and that $\{X_+, X_-\}$ is a partition of $Y - X$ in two non empty sets. Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, then*

$$L' + L =_{df} L \cup \{u.a.\beta \mid u \in X^\star, \ (a \in X_+ \text{ and } \beta \in L') \text{ or } (a \in X_- \text{ and } \beta \in L'^{-})\}$$

Notice that a player in charge of a set $L' + L$ in a Wadge game is like a player in charge of the set L but who can, at any step of the play, erase his previous play and choose to be this time in charge of L' or of L'^{-}. Notice that he can do this only one time during a play. We shall use this property below.

We now prove the following lemmas.

LEMMA 4.7. *Let $L \subseteq \Sigma^\omega$ be an analytic but non Borel set. Then it holds that $L \equiv_W \emptyset + L$.*

Notice that in the above lemma, $\emptyset$ is viewed as the empty set over an alphabet Γ such that $\Sigma \subseteq \Gamma$ and cardinal $(\Gamma - \Sigma) \geq 2$. Recall also that the emptyset and the whole set Γ^ω are located at the first level of the Wadge hierarchy and that their Wadge degree is equal to 1.

Proof. Firstly, it is easy to see that $L \leq_W \emptyset + L$: Player 2 has clearly a winning strategy in the Wadge game $W(L, \emptyset + L)$ which consists in copying the play of Player 1.

Secondly, we now assume that $L \subseteq \Sigma^\omega$ is an analytic but non Borel set and we show that Player 2 has a winning strategy in the Wadge game $W(\emptyset + L, L)$. Recall that we can infer from Hurewicz's Theorem, see [14, page 160], that an analytic subset of Σ^ω is either Π^0_2-hard or a Σ^0_2-set. Consider now the Wadge game $W(\emptyset + L, L)$. The successive letters written by Player 1 will be denoted $x(1), x(2), \ldots x(n), \ldots$ We now describe a winning strategy for Player 2.

We first assume that Player 1 remains in charge of the set L. As long as $[x[n].\Sigma^\omega \cap L]$ is Π^0_2-hard, Player 2 copies the letters written by Player 1. If for some integer $n \geq 1$, $[x[n-1].\Sigma^\omega \cap L]$ is Π^0_2-hard but $[x[n].\Sigma^\omega \cap L]$ is not Π^0_2-hard then $[x[n].\Sigma^\omega \cap L]$ is a Σ^0_2-set. If $[x[n].\Sigma^\omega \cap L]$ is Σ^0_2-complete then Player 2 writes the same letter $x(n)$ and as long as $[x[k].\Sigma^\omega \cap L]$ is Σ^0_2-complete, for $k \geq n$, Player 2 continues to copy the letters written by Player 1. If for some integer $k \geq n$, $[x[k].\Sigma^\omega \cap L]$ is not Σ^0_2-complete, then it is a Σ^0_2-set which is not complete and it follows from the study of the Wadge hierarchy that $[x[k].\Sigma^\omega \cap L]$ is a Δ^0_2-set. Let p be the first such integer $k \geq n$. Player 2 may skip at step p of the play. And now the Wadge game is reduced to the Wadge game $W(\emptyset + [x[p].\Sigma^\omega \cap L], [x[p-1].\Sigma^\omega \cap L])$. Player 2 has a winning strategy in this game because $\emptyset + [x[p].\Sigma^\omega \cap L]$ is still a Δ^0_2-set while $[x[p-1].\Sigma^\omega \cap L]$ is Π^0_2-hard or Σ^0_2-hard. Thus Player 2 follows the winning strategy in this game and he wins the Wadge game $W(\emptyset + L, L)$.

If at some step of a play as described above there is an integer $k \geq n$ such that $[x[k].\Sigma^\omega \cap L]$ is Π^0_2-hard or Σ^0_2-hard and $x(k+1) \in \Gamma - \Sigma$, then this means that Player 1 is now like a player in charge of the empty set or of the whole set Γ^ω which are located at the first level of the Wadge hierarchy. But after the k first steps of the play, Player 2 has also written $x[k]$ and he is like a player in charge of a set which is Π^0_2-hard or Σ^0_2-hard. Thus Player 2 has a w.s. to win the play from this step. □

LEMMA 4.8. **W-Det**(Σ^1_1) $\Longleftrightarrow$ **W-Det**(**r**-$\mathbf{BCL}(8)_\omega$).

Proof. The implication **W-Det**(Σ^1_1) $\Longrightarrow$**W-Det**(**r**-$\mathbf{BCL}(8)_\omega$) is obvious since **r**-$\mathbf{BCL}(8)_\omega$ $\subseteq \Sigma^1_1$.

To prove the reverse implication, we assume that **W-Det**(**r**-$\mathbf{BCL}(8)_\omega$) holds and we are going to show that every Wadge game $W(L, L')$, where $L \subseteq (\Sigma_1)^\omega$ and $L' \subseteq (\Sigma_2)^\omega$ are ω-languages in the class Σ^1_1, or equivalently in the class $\mathbf{BCL}(2)_\omega$ by Proposition 2.4, is determined. Notice that if the two ω-languages are Borel we already know that the game $W(L, L')$ is determined; thus we have only to consider the case where at least one of these languages is non-Borel. Let then $k_1 = \mathrm{cardinal}(\Sigma_1) + 2$, $k_2 = \mathrm{cardinal}(\Sigma_2) + 2$, and $S \geq \max[(3\mathrm{k}_1)^3, (3\mathrm{k}_2)^3]$ be an integer. We now use the mapping $\theta_S : (\Sigma_1)^\omega \rightarrow (\Sigma_1 \cup \{E\})^\omega$, defined in [7] and recalled in the proof of Proposition 3.2, and the similar one $\theta'_S : (\Sigma_2)^\omega \rightarrow (\Sigma_2 \cup \{E\})^\omega$. It is proved in [7] that one can effectively construct, from Büchi 2-counter automata $\mathcal{A}_1$ and $\mathcal{A}_2$ accepting L and L', some real time Büchi 8-counter automata accepting the ω-lannguages $\theta_S(L)$ and $\theta'_S(L')$. Then the Wadge game $W(\theta_S(L), \theta'_S(L'))$ is determined. We consider now the two following cases:

First case. Player 2 has a w.s. in the game $W(\theta_S(L), \theta'_S(L'))$.

If L' is Borel then $\theta'_S(L')$ is easily seen to be Borel (see [7]) and then $\theta_S(L)$ is also Borel because $\theta_S(L) \leq_W \theta'_S(L')$ and thus L is also Borel and thus the game $W(L, L')$ is determined. Assume now that L' is not Borel. Consider the Wadge game $W(L, \emptyset + L')$. We claim that Player 2 has a w.s. in that game which is easily deduced from a w.s. of Player 2 in the Wadge game $W(\theta_S(L), \theta'_S(L'))$. Consider a play in this latter game where Player 1 remains in the closed set $\theta_S((\Sigma_1)^\omega)$: she writes a beginning of a word in the form

$$x(1).E^S.x(2).E^{S^2}.x(3)\ldots x(n).E^{S^n}\ldots$$

Then player 2 writes a beginning of a word in the form

$$x'(1).E^S.x'(2).E^{S^2}.x'(3)\ldots x'(p).E^{S^p}\ldots$$

where $p \leq n$. Then the strategy for Player 2 in $W(L, \emptyset + L')$ consists to write $x'(1).x'(2)\ldots x'(p)$. when Player 1 writes $x(1).x(2)\ldots x(n)$.. If the strategy for Player 2 in $W(\theta_S(L), \theta'_S(L'))$ was at some step to go out of the set $\theta'_S((\Sigma_2)^\omega)$ then this means that his final word is surely outside $\theta'_S((\Sigma_2)^\omega)$, and that the final word of Player 1 is also surely outside $\theta_S(L)$, because Player 2 wins the play. Then Player 2 in the Wadge game $W(L, \emptyset + L')$ can make as he is now in charge of the emptyset and play anything (without skipping anymore) so that his final ω-word is also outside $\emptyset + L'$. So we have proved that Player 2 has a w.s. in the Wadge game $W(L, \emptyset + L')$ or equivalently that $L \leq_W \emptyset + L'$. But by Lemma 4.7 we know that $L' \equiv_W \emptyset + L'$ and thus $L \leq_W L'$ which means that Player 2 has a w.s. in the Wadge game $W(L, L')$.

Second case. Player 1 has a w.s. in the game $W(\theta_S(L), \theta'_S(L'))$.

Notice that this implies that $\theta'_S(L') \leq_W \theta_S(L)^-$. Thus if L is Borel then $\theta_S(L)$ is Borel (see [7]), $\theta_S(L)^-$ is also Borel, and $\theta'_S(L')$ is Borel as the inverse image of a Borel set by a continuous function, and L' is also Borel, so the Wadge game $W(L, L')$ is determined. We assume now that L is not Borel and we consider the Wadge game $W(\emptyset + L, L')$. Player 1 has a w.s. in this game which is easily constructed from a w.s. of the same player in the game $W(\theta_S(L), \theta'_S(L'))$ as follows. For this consider a play in this latter game where Player 2 does not go out of the closed set $\theta_S((\Sigma_2)^\omega)$. Then player 2 writes a beginning of a word in the form

$$x'(1).E^S.x'(2).E^{S^2}.x'(3)\ldots x'(p).E^{S^p}\ldots$$

Player 1, following her w.s. composes a beginning of a word in the form

$$x(1).E^S.x(2).E^{S^2}.x(3)\ldots x(n).E^{S^n}\ldots$$

where $p \leq n$. Then the strategy for Player 1 in $W(\emptyset + L, L')$ consists to write $x(1).x(2)\ldots x(n)$ when Player 2 writes $x'(1).x'(2)\ldots x'(p)$. If the strategy for Player 1 in $W(\theta_S(L), \theta'_S(L'))$ was at some step to go out of the set $\theta_S((\Sigma_1)^\omega)$ then this means that her final word is surely outside $\theta_S((\Sigma_1)^\omega)$, and that the final word of Player 2 is also surely in the set $\theta'_S(L')$ (at least if he produces really an infinite word in ω steps). In that case Player 1 in the game $W(\emptyset + L, L')$ can decide to be now in charge of the emptyset and play anything so that her final ω-word is outside $\emptyset + L$. So we have proved that Player 1 has a w.s. in the Wadge game $W(\emptyset + L, L')$. Using a very similar reasoning as in Lemma 4.7 where it is proved that $L \equiv_W \emptyset + L$ we can see that Player 1 has also a w.s. in the Wadge game $W(L, L')$. □

LEMMA 4.9. **W-Det(BCL$(1)_\omega$) $\Longrightarrow$ W-Det(r-BCL$(8)_\omega$).**

Proof. We assume that **W-Det**($\mathbf{BCL}(1)_\omega$) holds. Let then $L \subseteq (\Sigma_1)^\omega$ and $L' \subseteq (\Sigma_2)^\omega$ be ω-languages in the class **r-**$\mathbf{BCL}(8)_\omega$. We are going to show that the Wadge game $W(L, L')$ is determined. We now use the mapping $h_K : (\Sigma_1)^\omega \to (\Sigma_1 \cup \{A, B, C\})^\omega$ defined in [7] and recalled in the proof of the above Theorem 3.3. Similarly we have the mapping $h'_K : (\Sigma_2)^\omega \to (\Sigma_2 \cup \{A, B, C\})^\omega$ where we replace the alphabet Σ_1 by the alphabet Σ_2. It is proved in [7] that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting $L \subseteq (\Sigma_1)^\omega$, (respectively, $\mathcal{A}'$ accepting $L' \subseteq (\Sigma_2)^\omega$) one can effectively construct a Büchi 1-counter automaton $\mathcal{A}_1$ accepting the ω-language $h_K(L) \cup h_K((\Sigma_1)^\omega)^-$ (respectively, $\mathcal{A}'_1$ accepting $h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-$). Thus the Wadge game $W(h_K(L) \cup h_K((\Sigma_1)^\omega)^-, h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-)$ is determined.

Assuming again that L or L' is non-Borel, we can now easily show that the Wadge game $W(L, L')$ is determined: Player 1 (resp., Player 2) has a w.s. in the Wadge game $W(L, L')$ iff she (resp., he) has a w.s in the Wadge game

$$W(h_K(L) \cup h_K((\Sigma_1)^\omega)^-, h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-).$$

We can use a very similar reasoning as in the proof of the preceding lemma. A key argument is that if Player 1, who is in charge of the set $h_K(L) \cup h_K((\Sigma_1)^\omega)^-$ in the Wadge game $W(h_K(L) \cup h_K((\Sigma_1)^\omega)^-, h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-)$, goes out of the closed set $h_K((\Sigma_1)^\omega)$, then at the end of the play she has written an ω-word which is *surely* in her set. A similar argument holds for Player 2. Details are here left to the reader. □

LEMMA 4.10. **W-Det(r-**$\mathbf{BCL}(1)_\omega$**)** $\Longrightarrow$ **W-Det(r-**$\mathbf{BCL}(8)_\omega$**)**.

Proof. We return to the proof of the preceding lemma. Notice that we needed only the determinacy of Wadge games of the form

$$W(h_K(L) \cup h_K((\Sigma_1)^\omega)^-, h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-),$$

where $L \subseteq (\Sigma_1)^\omega$ and $L' \subseteq (\Sigma_2)^\omega$ are ω-languages in the class **r-**$\mathbf{BCL}(8)_\omega$, to prove that **W-Det(r-**$\mathbf{BCL}(8)_\omega$ holds. On the other hand, as noticed in the proof of Theorem 3.4, the ω-languages $h_K(L) \cup h_K((\Sigma_1)^\omega)^-$ and $h'_K(L') \cup h'_K((\Sigma_2)^\omega)^-$ are actually accepted by Büchi 1-counter automata $\mathcal{A}_1$ and $\mathcal{A}'_1$ having the following additional property: during any run of $\mathcal{A}_1$ (respectively, $\mathcal{A}'_1$) there are at most K consecutive λ-transitions. Thus it suffices now to show that **W-Det(r-**$\mathbf{BCL}(1)_\omega$) implies the determinacy of Wadge games $W(L(\mathcal{A}_1), L(\mathcal{A}'_1))$, where $\mathcal{A}_1$ and $\mathcal{A}'_1$ are Büchi 1-counter automata having this additional property.

We now assume that **W-Det(r-**$\mathbf{BCL}(1)_\omega$) holds and we consider such a Wadge game $W(L(\mathcal{A}_1), L(\mathcal{A}'_1))$. where $L(\mathcal{A}_1) \subseteq (\Sigma_1)^\omega$ and $L(\mathcal{A}'_1) \subseteq (\Sigma_2)^\omega$. Consider the mapping $\phi_K : (\Sigma_1)^\omega \to (\Sigma_1 \cup \{F\})^\omega$ which is simply defined by: for all $x \in (\Sigma_1)^\omega$,

$$\phi_K(x) = F^K.x(1).F^K.x(2) \ldots F^K.x(n).F^K.x(n+1).F^K \ldots$$

and the mapping $\phi'_K : (\Sigma_2)^\omega \to (\Sigma_2 \cup \{F\})^\omega$ which is defined in the same way.

Then the ω-languages $\phi_K(L(\mathcal{A}_1))$ and $\phi'_K(L(\mathcal{A}'_1))$ are accepted by real time Büchi 1-counter automata. Thus the Wadge game $W(\phi_K(L(\mathcal{A}_1)), \phi'_K(L(\mathcal{A}'_1)))$ is determined.

Assuming again that at least $L(\mathcal{A}_1)$ or $L(\mathcal{A}'_1)$ is non-Borel, it is now easy to show that the Wadge game $W(L(\mathcal{A}_1), L(\mathcal{A}'_1))$ is determined: Player 1 (respectively, Player 2) has a w.s. in the Wadge game $W(L(\mathcal{A}_1), L(\mathcal{A}'_1))$ iff she (respectively, he) has a w.s in the Wadge game $W(\phi_K(L(\mathcal{A}_1)), \phi'_K(L(\mathcal{A}'_1)))$. We can use a very similar reasoning as in the proof of the Lemma 4.8. A key argument is that if Player 1,who is in charge of the set $\phi_K(L(\mathcal{A}_1))$ in the Wadge game $W(\phi_K(L(\mathcal{A}_1)), \phi'_K(L(\mathcal{A}'_1)))$, goes out of the closed set $\phi_K((\Sigma_1)^\omega)$, then at the end of the play she has written an ω-word which is *surely* out of her set. A similar argument holds for Player 2. Details are here left to the reader. $\square$

Finally Theorem 4.5 follows from Lemmas 4.7, 4.8, 4.9, 4.10, and from the known equivalence **Det**$(\Sigma^1_1) \Longleftrightarrow$ **W-Det**(Σ^1_1). $\square$

Recall that, assuming that ZFC is consistent, there are some models of ZFC in which **Det**(Σ^1_1) does not hold. Therefore there are some models of ZFC in which some Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$, where $\mathcal{A}$ and $\mathcal{B}$ are Büchi 1-counter automata, are not determined. We are going to prove that this may be also the case when $\mathcal{B}$ is a Büchi automaton (without counter). To prove this, we use a recent result of [8] and some results of set theory, so we now briefly recall some notions of set theory and refer the reader to [8] and to a textbook like [13] for more background on set theory.

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. The axioms of ZFC express some natural facts that we consider to hold in the universe of sets. A model (**V**, $\in$) of an arbitrary set of axioms $\mathbb{A}$ is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of $\mathbb{A}$. We often say " the model **V**" instead of "the model (**V**, $\in$)".

We say that two sets A and B have same cardinality iff there is a bijection from A onto B and we denote this by $A \approx B$. The relation $\approx$ is an equivalence relation. Using the axiom of choice AC, one can prove that any set A can be well-ordered so there is an ordinal γ such that $A \approx \gamma$. In set theory the cardinal of the set A is then formally defined as the smallest such ordinal γ. The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The continuum hypothesis CH says that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$ which is the cardinal of the continuum.

If **V** is a model of ZF and **L** is the class of *constructible sets* of **V**, then the class **L** is a model of ZFC + CH. Notice that the axiom V=L, which means "every set is constructible", is consistent with ZFC because **L** is a model of ZFC + V=L.

Consider now a model **V** of ZFC and the class of its constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which is another model of ZFC. It is known that the ordinals of **L** are also the ordinals of **V**, but the cardinals in **V** may be different from the cardinals in **L**. In particular, the first uncountable cardinal in **L** is denoted $\aleph_1^{\mathbf{L}}$, and it is in fact an ordinal of **V** which is denoted $\omega_1^{\mathbf{L}}$. It is well-known that in general this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model **V** of the axiomatic system ZFC + V=L the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds, but in some other models of ZFC the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$.

The following result was proved in [8].

THEOREM 4.11. *There exists a real-time* 1*-counter Büchi automaton $\mathcal{A}$, which can be effectively constructed, such that the topological complexity of the ω-language $L(\mathcal{A})$ is not determined by the axiomatic system* ZFC. *Indeed it holds that :*

(1) *(*ZFC + V=L*). The ω-language $L(\mathcal{A})$ is an analytic but non-Borel set.*
(2) *(*ZFC + $\omega_1^{\mathbf{L}} < \omega_1$*). The ω-language $L(\mathcal{A})$ is a Π^0_2-set.*

We now state the following new result. To prove it we use in particular the above Theorem 4.11, the link between Wadge games and Wadge reducibility, the Π^0_2-completeness of the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$, the Shoenfield's Absoluteness Theorem, and the notion of extensions of a model of ZFC.

THEOREM 4.12. *Let $\mathcal{B}$ be a Büchi automaton accepting the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$. Then one can effectively construct a real-time 1-counter Büchi automaton $\mathcal{A}$ such that:*

(1) *(ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). Player 2 has a winning strategy F in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. But F cannot be recursive and not even hyperarithmetical.*
(2) *(ZFC + $\omega_1^{\mathbf{L}} = \omega_1$). The Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.*

Proof. Let $\mathcal{A}$ be a real-time 1-counter Büchi automaton, which can be effectively constructed by Theorem 4.11 and satisfying the properties given by this theorem. The automaton $\mathcal{A}$ reads ω-words over a finite alphabet Σ and we can assume, without loss of generality, that $\Sigma = \{0,1\}$. On the other hand the ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$ is regular and there is a (deterministic) Büchi automaton $\mathcal{B}$ accepting it. Moreover it is well known that this language $L(\mathcal{B})$ is Π^0_2-complete (in every model of ZFC), see [20, 22].

Consider now a model V_1 of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). By Theorem 4.11, in this model the ω-language $L(\mathcal{A})$ is a Π^0_2-set. Thus $L(\mathcal{A}) \leq_W L(\mathcal{B})$ because the ω-language $L(\mathcal{B})$ is Π^0_2-complete. This implies that Player 2 has a winning strategy F in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. This strategy is a mapping $F : \{0,1\}^+ \to \{0,1\} \cup \{s\}$ hence it can be coded in a recursive manner by an infinite word $X_F \in \{0,1\}^\omega$ which may be identified with a subset of the set $\mathbb{N}$ of natural numbers. We now claim that this strategy is not constructible, or equivalently that the set $X_F \subseteq \mathbb{N}$ does not belong to the class $\mathbf{L}^{V_1}$ of constructible sets in the model V_1. Recall that a real-time 1-counter Büchi automaton $\mathcal{A}$ has a finite description to which can be associated, in an effective way, a unique natural number called its index, so we have a Gödel numbering of real-time 1-counter Büchi automata, see [12, page 369] for such a coding of Turing machines, and [13, page 162] about Gödel numberings of formulae. We denote $\mathcal{A}_z$ the real time Büchi 1-counter automaton of index z reading words over $\{0,1\}$. In a similar way we denote $\mathcal{B}_z$ the Büchi automaton of index z reading words over $\{0,1\}$. Then there exist integers z_0 and z'_0 such that $\mathcal{A} = \mathcal{A}_{z_0}$ and $\mathcal{B} = \mathcal{B}_{z'_0}$. If $x \in \{0,1\}^\omega$ is the ω-word written by Player 1 during a play of a Wadge game $W(L(\mathcal{A}_z), L(\mathcal{B}_{z'}))$ and Player 2 follows a strategy G, the ω-word $(x \star G) \in (\{0,1,s\})^\omega$ is defined by $(x \star G)(n) = G(x[n])$ for all integers $n \geq 1$ and $(x \star G)(/s)$ is obtained from $(x \star G)$ by deleting the letters s, so that $(x \star G)(/s)$ is the word written by Player 2 at the end of the play. We can now easily see that the sentence: "G is a winning strategy for Player 2 in the Wadge game $W(L(\mathcal{A}_z), L(\mathcal{B}_{z'}))$" can be expressed by a Π^1_2-formula $P(z, z', G)$ (we assume here that the reader has some familiarity with this notion which can be found in [19]):

$$\forall x \in \Sigma^\omega [\ (x \in L(\mathcal{A}_z) \text{ and } (x \star G)(/s) \in L(\mathcal{B}_{z'})) \text{ or}$$
$$(x \notin L(\mathcal{A}_z) \wedge (x \star G)(/s) \text{ is infinite } \wedge (x \star G)(/s) \notin L(\mathcal{B}_{z'}))]$$

Recall that $x \in L(\mathcal{A}_z)$ can be expressed by a Σ^1_1-formula (see [9]). And $(x \star G)(/s) \in L(\mathcal{B}_{z'})$ can be expressed by $\exists y \in \Sigma^\omega (y = (x \star G)(/s) \text{ and } y \in L(\mathcal{B}_{z'}))$, which is also a

Σ_1^1-formula since $(x\star G)(/s)$ is recursive in x and G. Moreover "$(x\star G)(/s)$ is infinite" means that $(x \star G)$ contains infinitely many letters in $\{0,1\}$; this is an arithmetical statement in x and G. Finally the formula $P(z,z',G)$ is a Π_2^1-formula.

Towards a contradiction, assume now that the winning strategy F for Player 2 in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$ belongs to the class $\mathbf{L}^{V_1}$ of constructible sets in the model V_1. The relation $P_F \subseteq \mathbb{N}\times\mathbb{N}$ defined by $P_F(z,z')$ iff $P(z,z',F)$ is a $\Pi_2^1(F)$-relation, i.e. a relation with is Π_2^1 with parameter F. By Shoenfield's Absoluteness Theorem (see [13, page 490]), the relation $P_F \subseteq \mathbb{N}\times\mathbb{N}$ would be absolute for the models $\mathbf{L}^{V_1}$ and V_1 of ZFC. This means that the set $\{(z,z')\in\mathbb{N}\times\mathbb{N} \mid P_F(z,z')\}$ would be the same set in the two models $\mathbf{L}^{V_1}$ and V_1. In particular, the pair (z_0,z_0') belongs to P_F in the model V_1 since F is a w.s. for Player 2 in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$. This would imply that F is also a w.s. for Player 2 in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$ in the model $\mathbf{L}^{V_1}$. But $\mathbf{L}^{V_1}$ is a model of ZFC + V=L so in this model the ω-language $L(\mathcal{A})$ is an analytic but non-Borel set and $L(\mathcal{A}) \leq_W L(\mathcal{B})$ does not hold. This contradiction shows that the w.s. F is not constructible in V_1. On the other hand every set $A \subseteq \mathbb{N}$ which is Π_2^1 or Σ_2^1 is constructible, see [13, page 491]. Thus X_F is neither a Π_2^1-set nor a Σ_2^1-set; in particular, the strategy F is not recursive and not even hyperarithmetical, i.e. not Δ_1^1.

Consider now a model V_2 of (**ZFC** + $\omega_1^{\mathbf{L}} = \omega_1$).

Notice first that Theorem 4.11 (1) is easily extended to models of (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$) since [8, Corollary 4.8] is easily seen to be true if we replace (ZFC + V=L) by (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$): in a model of (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$) the largest thin Π_1^1-set in Σ^ω is uncountable and has no perfect subset hence it can not be a Borel set because the class of Borel sets has the perfect set property. And thus [8, Theorem 5.1] is also true if we replace (ZFC + V=L) by (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$), because this follows from the fact that the largest thin Π_1^1-set in Σ^ω is not Borel.

Then in the model V_2 the ω-language $L(\mathcal{A})$ is an analytic but non-Borel set. Thus $L(\mathcal{A}) \leq_W L(\mathcal{B})$ does not hold because the ω-language $L(\mathcal{B})$ is Π_2^0-complete. This implies that Player 2 has no winning strategy in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$. We now claim that Player 1 too has no winning strategy in this Wadge game. Towards a contradiction assume that Player 1 has a w.s. F' in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$. Using Cohen's method of forcing developed in 1963, we can show that there exists an extension $V_3 \supset V_2$ such that V_3 is a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). The construction of such a model is due to Levy and presented in [13, page 202]: one can start from the model V_2 of (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$) and construct by forcing a generic extension $V_3 \supset V_2$ in which $\omega_1^{V_2}$ is collapsed to ω; in this extension the inequality $\omega_1^{\mathbf{L}} < \omega_1$ holds. We can show, as above, that the sentence "G is a winning strategy for Player 1 in the Wadge game $W(L(\mathcal{A}_z),L(\mathcal{B}_{z'}))$" can be expressed by a Π_2^1-formula $Q(z,z',G)$. We denote $Q_{F'}(z,z') \leftrightarrow Q(z,z',F')$. By Shoenfield's Absoluteness Theorem, the relation $Q_{F'} \subseteq \mathbb{N}\times\mathbb{N}$ would be absolute for the models V_2 and V_3 of ZFC. Thus (z_0,z_0') would belong to $Q_{F'}$ in V_3 and this means that Player 1 would have a w.s. in the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$ in the model V_3. But V_3 is a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). Thus in this model the ω-language $L(\mathcal{A})$ is a Π_2^0-set, the relation $L(\mathcal{A}) \leq_W L(\mathcal{B})$ holds, and Player 2 has a w.s. in the $W(L(\mathcal{A}),L(\mathcal{B}))$. This is a contradiction because it is impossible that both players have a w.s. in the same Wadge game. Finally we have proved that in V_2 none of the players has a winning strategy and thus the Wadge game $W(L(\mathcal{A}),L(\mathcal{B}))$ is not determined. □

REMARK 4.13. *Every model of* ZFC *is either a model of* (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$) *or a model of* (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$). *Thus there are no models of* ZFC *in which Player 1 has a winning strategy in the Wadge game* $W(L(\mathcal{A}), L(\mathcal{B}))$.

REMARK 4.14. *In order to prove Theorem 4.12 we do not need to use any large cardinal axiom or even the consistency of such an axiom, like the axiom of analytic determinacy.*

§5. Concluding remarks.

We have proved that the determinacy of Gale-Stewart games whose winning sets are accepted by (real-time) 1-counter Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption.

On the other hand we have proved a similar result about the determinacy of Wadge games. We have also obtained an amazing result, proving that one can effectively construct a real-time 1-counter Büchi automaton $\mathcal{A}$ and a Büchi automaton $\mathcal{B}$ such that the sentence "the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is determined" is actually independent from ZFC.

Notice that it is still unknown whether the determinacy of Wadge games $W(L(\mathcal{A}), L(\mathcal{B}))$, where $\mathcal{A}$ and $\mathcal{B}$ are Muller tree automata (reading infinite labelled trees), is provable within ZFC or needs some large cardinal assumptions to be proved.

REFERENCES

[1] T. CACHAT, *Higher order pushdown automata, the Caucal hierarchy of graphs and parity games*, ***Proceedings of the 30th International Colloquium on Automata, Languages and Programming, ICALP 2003, Eindhoven, The Netherlands, june 30 - july 4, 2003.***, Lecture Notes in Computer Science, vol. 2719, Springer, 2003, pp. 556–569.

[2] A. CARAYOL, M. HAGUE, A. MEYER, C.-H. L. ONG, and O. SERRE, *Winning regions of higher-order pushdown games*, ***Proceedings of the twenty-third Annual IEEE Symposium on Logic in Computer Science, LICS 2008, 24-27 june 2008, Pittsburgh, PA, USA***, IEEE Computer Society, 2008, pp. 193–204.

[3] R.S. COHEN and A.Y. GOLD, *Theory of ω-languages, parts one and two*, ***Journal of Computer and System Science***, vol. 15 (1977), pp. 169–208.

[4] ———, *ω-computations on Turing machines*, ***Theoretical Computer Science***, vol. 6 (1978), pp. 1–23.

[5] J ENGELFRIET and H. J. HOOGEBOOM, *X-automata on ω-words*, ***Theoretical Computer Science***, vol. 110 (1993), no. 1, pp. 1–51.

[6] O. FINKEL, *Borel hierarchy and omega context free languages*, ***Theoretical Computer Science***, vol. 290 (2003), no. 3, pp. 1385–1405.

[7] ———, *Borel ranks and Wadge degrees of omega context free languages*, ***Mathematical Structures in Computer Science***, vol. 16 (2006), no. 5, pp. 813–840.

[8] ———, *The complexity of infinite computations in models of set theory*, ***Logical Methods in Computer Science***, vol. 5 (2009), no. 4:4, pp. 1–19.

[9] ———, *Highly undecidable problems for infinite computations*, ***Theoretical Informatics and Applications***, vol. 43 (2009), no. 2, pp. 339–364.

[10] ———, *The determinacy of context-free games*, ***29th International Symposium on Theoretical Aspects of Computer Science STACS 2012*** (Christoph Dürr and Thomas Wilke, editors), Leibniz International Proceedings in Informatics (LIPIcs), vol. 14, 2012, pp. 555–566.

[11] L. HARRINGTON, *Analytic determinacy and* $0^\sharp$, this JOURNAL, vol. 43 (1978), no. 4, pp. 685–693.

[12] J. E. HOPCROFT, R. MOTWANI, and J. D. ULLMAN, ***Introduction to automata theory, languages, and computation***, Addison-Wesley Publishing Co., Reading, Mass., 2001, Addison-Wesley Series in Computer Science.

[13] T. JECH, ***Set theory, third edition***, Springer, 2002.

[14] A. S. KECHRIS, ***Classical descriptive set theory***, Springer-Verlag, New York, 1995.

[15] H. LESCOW and W. THOMAS, *Logical specifications of infinite computations*, ***A Decade of Concurrency*** (J. W. de Bakker, Willem P. de Roever, and Grzegorz Rozenberg, editors), Lecture Notes in Computer Science, vol. 803, Springer, 1994, pp. 583–621.

[16] A. LOUVEAU and J. SAINT-RAYMOND, *The strength of Borel Wadge determinacy*, ***Cabal Seminar 81–85***, Lecture Notes in Mathematics, vol. 1333, Springer, 1988, pp. 1–30.

[17] D. A. MARTIN, *Measurable cardinals and analytic games*, ***Fundamenta Mathematicae***, vol. 66 (1969/1970), pp. 287–291.

[18] Y. N. MOSCHOVAKIS, ***Descriptive set theory***, North-Holland Publishing Co., Amsterdam, 1980.

[19] P.G. ODIFREDDI, ***Classical recursion theory, vol I***, Studies in Logic and the Foundations of Mathematics, vol. 125, North-Holland Publishing Co., Amsterdam, 1989.

[20] D. PERRIN and J.-E. PIN, ***Infinite words, automata, semigroups, logic and games***, Pure and Applied Mathematics, vol. 141, Elsevier, 2004.

[21] R. L. SAMI, *Analytic determinacy and* $0^\sharp$*. A forcing-free proof of Harrington's theorem*, ***Fundamenta Mathematicae***, vol. 160 (1999), no. 2, pp. 153–159.

[22] L. STAIGER, ω*-languages*, ***Handbook of formal languages, vol. 3***, Springer, Berlin, 1997, pp. 339–387.

[23] W. THOMAS, *On the synthesis of strategies in infinite games*, ***Proceedings of the International Conference STACS 1995***, Lecture Notes in Computer Science, vol. 900, Springer, 1995, pp. 1–13.

[24] ———, *Church's problem and a tour through automata theory*, ***Pillars of Computer Science, Essays Dedicated to Boris (Boaz) Trakhtenbrot on the Occasion of His 85th Birthday*** (Arnon Avron, Nachum Dershowitz, and Alexander Rabinovich, editors), Lecture Notes in Computer Science, vol. 4800, Springer, 2008, pp. 635–655.

[25] W. WADGE, *Reducibility and determinateness in the Baire space*, ***Ph.D. thesis***, University of California, Berkeley, 1983.

[26] I. WALUKIEWICZ, *Pushdown processes: games and model checking*, ***Information and Computation***, vol. 157 (2000), pp. 234–263.

EQUIPE DE LOGIQUE MATHÉMATIQUE
INSTITUT DE MATHÉMATIQUES DE JUSSIEU - PARIS RIVE GAUCHE UMR7586
CNRS ET UNIVERSITÉ PARIS DIDEROT PARIS 7
BÂTIMENT SOPHIE GERMAIN CASE 7012
75205 PARIS CEDEX 13, FRANCE..
E-mail: finkel@math.univ-paris-diderot.fr