Subtriviality of continuous fields of nuclear C*-algebras, with an appendix by E. Kirchberg

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Abstract

We extend in this paper the characterisation of a separable nuclear C*-algebra given by Kirchberg proving that given a unital separable continuous field of nuclear C*-algebras A over a compact metrizable space X, the $C(X)$-algebra A is isomorphic to a unital $C(X)$-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$, image of $\mathcal{O}_2 \otimes C(X)$ by a norm one projection.

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0 Introduction

In order to study deformations in the C*-algebraic framework, Dixmier introduced the notion of continuous field of C*-algebras over a locally compact space ([7]). In the same way as there is a faithful representation in a Hilbert space for any C*-algebra thanks to the Gelfand–Naimark–Segal construction, a separable continuous field of C*-algebras A over a compact metrizable space X always admits a continuous field of faithful representations π in a Hilbert $C(X)$-module, i.e. there exists a family of representations $\{\pi_x,\ x \in X\}$, in a separable Hilbert space H which factorize through a faithful representation of the fibre A_x such that for each $a \in A$, the map $x \mapsto \pi_x(a)$ is strongly continuous ([4, théorème 3.3]).

In a work on tensor products over $C(X)$ of continuous fields of C*-algebras over X ([16]), Kirchberg and Wassermann raised the question of whether the continuous field of C*-algebras A could be subtrivialized, i.e. whether one could find a continuous field of faithful representations π such that the map $x \mapsto \pi_x(a) \in L(H)$ is actually norm continuous for all a in A. In this case, given any C*-algebra B, the minimal tensor product $A \otimes B$ is a $C(X)$-subalgebra of the trivial continuous field $[L(H) \otimes B] \otimes C(X)$ and is therefore a continuous field with fibres $(A \otimes B)_x = A_x \otimes B$. They proved that a non-exact continuous field with exact fibres cannot be subtrivialized and they constructed such examples.

The non-trivial example of the continuous field of rotation algebras over the unit circle $\mathbb{T}$ had already been studied by Haagerup and Rørdam in [10]. More precisely, they constructed continuous functions u, v from $\mathbb{T}$ to the unitary group $U(H)$ of the infinite-dimensional separable Hilbert space H satisfying the commutation relation $u_t v_t = t v_t u_t$

for all $t \in \mathbb{T}$ and the uniform continuity condition $\max\{\|u_t - u_{t'}\|, \|v_t - v_{t'}\|\} < C'|t - t'|^{1/2}$ where C' is a computable constant.

Our purpose in the present paper is to show that the subtrivialization is always possible in the nuclear separable case through a generalisation of the following theorem of Kirchberg using $\mathcal{R}KK$-theory arguments:

Theorem 0.1 *([15]) A unital separable* C*-*algebra* A *is exact if and only if it is isomorphic to a* C*-*subalgebra of* $\mathcal{O}_2$. *Moreover the* C*-*algebra* A *is nuclear if and only if* A *is isomorphic to a* C*-*subalgebra of* $\mathcal{O}_2$ *containing the unit* $1_{\mathcal{O}_2}$ *of* $\mathcal{O}_2$*, image of* $\mathcal{O}_2$ *by a unital completely positive projection.*

As a matter of fact, we get an equivalent characterisation of nuclear separable continuous fields of C*-algebras (theorem 3.2) which is made possible thanks to $C(X)$-linear homotopy invariance of the bifunctor $\mathcal{R}KK(X; -, -)$ (theorem 1.6) and $C(X)$-linear Weyl-von Neumann absorption results (proposition 2.5). This also enables us to have a better understanding of the characterisation of separable continuous fields of nuclear C*-algebras given by Bauval in [2].

In an added appendix, the corresponding characterisation of exact separable continuous fields of C*-algebras as $C(X)$-subalgebras of $\mathcal{O}_2 \otimes C(X)$ given by Eberhard Kirchberg is described (theorem A.1).

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1 Preliminaries

1.1 $C(X)$-algebras

Let X be a compact Hausdorff space and $C(X)$ be the C*-algebra of continuous functions on X with complex values. We start by recalling the following definition.

Definition 1.1 *([13]) A* $C(X)$*-algebra is a* C*-*algebra* A *endowed with a unital morphism from* $C(X)$ *in the centre of the multiplier algebra* $M(A)$ *of* A.

Remark: We do not assume that $C(X)$ embeds into $M(A)$. For instance, there is a natural structure of $C([0, 2])$-algebra on the C*-algebra $C([0, 1])$.

For $x \in X$, define the kernel $C_x(X)$ of the evaluation map $ev_x : C(X) \to \mathbb{C}$ at x; denote by A_x the quotient of a $C(X)$-algebra A by the closed ideal $C_x(X)A$ and by a_x the image of an element $a \in A$ in the fibre A_x. Then the function

$$x \mapsto \|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\}$$

is upper semi-continuous for any $a \in A$ and the $C(X)$-algebra A is said to be a continuous field of C*-algebras over X if the function $x \mapsto \|a_x\|$ is actually continuous for every $a \in A$ ([7]).

Examples *1.* If A is a $C(X)$-algebra and D is a C*-algebra, the spatial tensor product $B = A \otimes D$ is naturally endowed with a structure of $C(X)$-algebra through the map $f \in C(X) \mapsto f \otimes 1_{M(D)} \in M(A \otimes D)$. In particular, if $A = C(X)$, the tensor product B is a trivial continuous field over X with constant fibre $B_x \simeq D$
2. Given a $C(X)$-algebra A, define the unital $C(X)$-algebra $\mathcal{A}$ generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$. It defines a continuous field of C*-algebras over X if and only if the $C(X)$-algebra A is continuous ([4, proposition 3.2]).

Remark: If A is a separable continuous field of non-zero C*-algebras (not necessarily unital) over the compact Hausdorff space X, the positive cone $C(X)_+$ and so the C*-algebra $C(X)$ are separable. Hence, the topological space X is metrizable.

Definition 1.2 *([4, 5]) Given a continuous field of* C*-*algebras A over the compact Hausdorff space X, a continuous field of representations of a $C(X)$-algebra D in the multiplier C*-algebra $M(A)$ of A is a $C(X)$-linear morphism $\pi : D \to M(A)$, i.e. for each $x \in X$, the induced representation π_x of D in $M(A_x)$ factorizes through the fibre D_x.*

If the $C(X)$-algebra D admits a continuous field of faithful representations π in the $C(X)$-algebra $M(A)$ where A is a continuous field over X, i.e. the induced representation of the fibre D_x in $M(A_x)$ is faithful for every point $x \in X$, the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|(\pi(d)a)_x\|, a \in A \,\text{such that}\, \|a\| \leq 1\}$$

is lower semi-continuous for all $d \in D$ and the $C(X)$-algebra D is therefore continuous.

In particular a separable $C(X)$-algebra D is continuous if and only if there exists a Hilbert $C(X)$-module $\mathcal{E}$ such that D admits a continuous field of faithful representations in the multiplier algebra $M(\mathcal{K}(\mathcal{E})) = \mathcal{L}(\mathcal{E})$ of the continuous field over X of compact operators $\mathcal{K}(\mathcal{E})$ acting on $\mathcal{E}$ ([4, théorème 3.3]).

Let us also mention the characterisation of separable continuous fields of nuclear C*-algebras over a compact metrizable space X given by Bauval in [2] using a natural $C(X)$-linear version of nuclearity introduced by Kasparov and Skandalis in [14]§6.2 : a $C(X)$-linear completely positive σ from a $C(X)$-algebra A into a $C(X)$-algebra B is said to be $C(X)$-nuclear if and only if given any compact set F in A and any strictly positive real number ε, there exist an integer k and $C(X)$-linear completely positive contractions $T : A \to M_k(\mathbb{C}) \otimes C(X)$ and $S : M_k(\mathbb{C}) \otimes C(X) \to B$ such that for all $a \in F$, one has the inequality

$$\|\sigma(a) - (S \circ T)(a)\| < \varepsilon.$$

One can then state the following results. The first assertion is a simple $C(X)$-linear reformulation of the Choi-Effros theorem and the second one is due to Bauval.

Proposition 1.3 *Let X be a compact metrizable space and A be a separable $C(X)$-algebra.*

1. *([14]§6.2) Given a $C(X)$-algebra B and a closed ideal $J \subset B$, any contractive $C(X)$-nuclear map $A \to B/J$ admits a contractive $C(X)$-linear completely positive lift $A \to B$.*

2. *([2, théorème 7.2]) The $C(X)$-algebra A is a continuous fields of nuclear C*-algebras over X if and only if the identity map $id_A : A \to A$ is $C(X)$-nuclear.*

Remark: In assertion *1.*, the ideal $J = (C(X)B)J = C(X)J$ is a $C(X)$-algebra.

1.2 $C(X)$-extensions

Given a compact Hausdorff space X, we introduce a natural $C(X)$-linear version of the semi-group $Ext(-,-)$ defined by Kasparov ([12, 13]).

Call a morphism of $C(X)$-algebras a $*$-homomorphism between $C(X)$-algebras which is $C(X)$-linear.

Definition 1.4 *A $C(X)$-extension of a $C(X)$-algebra A by a $C(X)$-algebra B is a short exact sequence*

$$0 \to B \to D \xrightarrow{\pi} A \to 0$$

in the category of $C(X)$-algebras. The $C(X)$-extension is said to be trivial if the map π admits a cross section $s : A \to D$ which is a morphism of $C(X)$-algebras.

As in the C*-algebraic case a $C(X)$-extension $0 \to B \to D \to A \to 0$ of A by B defines unambiguously an homomorphism from D to the multiplier algebra $M(B)$ of B, which gives a morphism of $C(X)$-algebras $\sigma : A \to M(B)/B$ (called the Busby invariant of the extension) and the $C(X)$-extension is trivial if and only if the map σ lifts to a morphism of $C(X)$-algebras $A \to M(B)$. Conversely, given a morphism of $C(X)$-algebras $\sigma : A \to M(B)/B$, setting $D = \{(a, m) \in A \times M(B), \sigma(a) = q(m)\}$ where q is the quotient map $M(B) \to M(B)/B$, one has a $C(X)$-extension $0 \to B \to D \to A \to 0$ (see [12]§7).

Remark: A $C(X)$-extension $0 \to B \to D \to A \to 0$ induces for every $x \in X$ a C*-extension $0 \to B_x \to D_x \to A_x \to 0$.

In order to define the sum of two $C(X)$-extensions, recall that the Cuntz algebra $\mathcal{O}_2$ is the unital C*-algebra generated by two orthogonal isometries s_1 and s_2 subject to the relation $1 = s_1 s_1^* + s_2 s_2^*$ ([6]). Then if $\mathcal{K}$ is the C*-algebra of compact operators on the infinite-dimensional separable Hilbert space, one defines the sum of two $C(X)$-extensions σ_1 and σ_2 of the $C(X)$-algebra A by the stable $C(X)$-algebra $\mathcal{K} \otimes B$ through the choice of a unital copy of $\mathcal{O}_2$ in the multiplier algebra $M(\mathcal{K})$ of $\mathcal{K}$ to be the $C(X)$-extension

$$\sigma_1 \oplus \sigma_2 : a \mapsto q(s_1 \otimes 1)\sigma_1(a)q(s_1^* \otimes 1) + q(s_2 \otimes 1)\sigma_2(a)q(s_2^* \otimes 1) \in M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B),$$

where q is the quotient map $M(\mathcal{K} \otimes B) \to M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$.

Definition 1.5 *Given a compact Hausdorff space X and two $C(X)$-algebras A and B, $Ext(X;A,B)$ is the semi-group of $C(X)$-extensions of A by $\mathcal{K}\otimes B$ divided by the equivalence relation $\sim$ where $\sigma_1\sim\sigma_2$ if there exist a unitary $U\in M(\mathcal{K}\otimes B)$ of image $q(U)$ in the quotient $M(\mathcal{K}\otimes B)/(\mathcal{K}\otimes B)$ and two trivial $C(X)$-extensions π_1 and π_2 such that for all $a\in A$,*

$$(\sigma_2\oplus\pi_2)(a)=q(U)^*(\sigma_1\oplus\pi_1)(a)q(U)\ \text{(in } M(\mathcal{K}\otimes B)/(\mathcal{K}\otimes B)\text{)}.$$

Let $Ext(X;A,B)^{-1}$ be the group of invertible elements of $Ext(X;A,B)$, i.e. the group of classes of $C(X)$-extension σ such that there exists a $C(X)$-extension θ with $\sigma\oplus\theta$ trivial. One can generalise Kasparov's theorem of homotopy invariance of the group $Ext(A,B)^{-1}$ to the framework of $C(X)$-algebras as follows.

Theorem 1.6 *([12]) Assume that A is a separable $C(X)$-algebra and that B is a σ-unital $C(X)$-algebra. Then the group $Ext(X;A,B)^{-1}$ is isomorphic to the group $\mathcal{R}KK^1(X;A,B)$ and is therefore $C(X)$-linear homotopy invariant in both entries A and B.*

Proof : Let us first make the following observation. Given a $C(X)$-algebra B and a Hilbert B-module $\mathcal{E}$, denote by $\mathcal{L}(\mathcal{E})$ the set of bounded B-linear operators on $\mathcal{E}$ which admit an adjoint ([11]). Then any operator $T\in\mathcal{L}(\mathcal{E})$ is B-linear and so $C(X)$-linear. This argument provides a natural extension of the Stinespring-Kasparov theorem ([12]) to the framework of $C(X)$-algebras. Consequently, if A is a separable $C(X)$-algebra and B is a σ-unital $C(X)$-algebra, the class of a $C(X)$-extension $\sigma: A\to M(\mathcal{K}\otimes B)/(\mathcal{K}\otimes B)$ is invertible in $Ext(X;A,B)$ if and only if there is a $C(X)$-linear completely positive contractive lift $A\to M(\mathcal{K}\otimes B)$.

Let $\mathcal{R}\mathbb{E}(X;A,B)$ be the set of Kasparov $C(X)$–A,B-bimodules ([13, definition 2.19]), i.e. the set of Kasparov A,B bimodules $(\mathcal{E},F)$ such that the representation $A\to\mathcal{L}(\mathcal{E})$ is a $C(X)$-representation. Call a $C(X)$-linear operator homotopy an element $\{(\mathcal{E},F_t),0\le t\le 1\}\in\mathcal{R}\mathbb{E}(X;A,B\otimes C([0,1]))$ such that $t\mapsto F_t$ is norm continuous and define on $\mathcal{R}\mathbb{E}(X;A,B)$ the equivalence relation corresponding to the one defined by Skandalis in [18, definition 2]. The constructions given by Kasparov in [12, section 7] imply that, if the $C(X)$-algebra B is σ-unital, the group of equivalence classes $\mathcal{R}\widetilde{KK}(X;A,B\otimes\mathcal{C}_1)$ is isomorphic to $Ext(X;A,B)^{-1}$, where $\mathcal{C}_1$ is the first (graded) Clifford algebra.

On the other hand, given two graded $C(X)$-algebras A and B with A separable, the different steps of the demonstration of [18, theorem 19] provide us with an isomorphism between the two groups $\mathcal{R}\widetilde{KK}(X;A,B)$ and $\mathcal{R}KK(X;A,B)$ since proposition 2.21 of [13] defines an intersection product in $\mathcal{R}\widetilde{KK}$-theory and lemma 18 of [18] gives us the equality

$$(ev_0\otimes id_{C(X)})^*(1_{C(X)})=(ev_1\otimes id_{C(X)})^*(1_{C(X)})\text{ in }\mathcal{R}\widetilde{KK}(X;C([0,1])\otimes C(X),C(X)),$$

where $1_{C(X)}$ is the Kasparov $C(X),C(X)$-bimodule $(C(X),0)$ and $ev_t: C([0,1])\to\mathbb{C}$ is the evaluation map at $t\in[0,1]$. □

Remarks: *1.* Kuiper's theorem implies that the law of addition on the abelian group $Ext(X;A,B)^{-1}$ is independent of the choice of the unital copy of $\mathcal{O}_2$ in $M(\mathcal{K})$.

2. If A is a separable nuclear continuous field of C*-algebras over X and B is a $C(X)$-algebra, every $C(X)$-linear morphism from A to the quotient $M(\mathcal{K}\otimes B)/(\mathcal{K}\otimes B)$ is $C(X)$-nuclear and therefore admits a $C(X)$-linear completely positive lifting $A \to M(\mathcal{K}\otimes B)$ thanks to proposition 1.3. Accordingly one has the equality

$$Ext(X; A, B)^{-1} = Ext(X; A, B).$$

2 An absorption result

In this section we prove a continuous generalisation of a statement contained in [15] which will enable us to get a $C(X)$-linear Weyl-von Neumann type result (proposition 2.5). Let us start with the following definition of Cuntz.

Definition 2.1 *([6]) A simple* C*-*algebra* A *distinct from* $\mathbb{C}$ *is said to be purely infinite if and only if for any non-zero* $a, b \in A$*, there exist elements* $x, y \in A$ *such that* $a = xby$.

Then, we can state a proposition from Kirchberg's classification work, based on Glimm's lemma ([7], § 11.2). A sketch of proof can also be found in [1, proposition 5.1].

Proposition 2.2 *([15]) Let* A *be a purely infinite simple* C*-*algebra and* D *be a separable* C*-*subalgebra of* $M(A)$*. Assume that* $V : D \to A$ *is a nuclear contraction.*

Then there exists a sequence (a_n) *of elements in* A *of norm less than* 1 *such that* $V(d) = \lim_{n\to\infty} a_n^* d a_n$ *for all* $d \in D$*.*

Remark: A simple ring has by definition exactly two distinct two sided ideals and is therefore non-zero.

Corollary 2.3 *Let* A *be a continuous field of purely infinite simple* C*-*algebras over a compact Hausdorff space* X *and assume that* D *is a separable* $C(X)$*-subalgebra of the multiplier algebra* $M(A)$ *such that there is a unital* $C(X)$*-embedding of the* $C(X)$*-algebra* $\mathcal{O}_\infty \otimes C(X)$ *in the commutant* D' *of* D *in* $M(A)$ *and the identity map* $id_D : D \to M(A)$ *is a continuous field of faithful representations.*

If $V : D \to A$ *is a* $C(X)$*-nuclear contraction, there exists a sequence* (a_n) *in the unit ball of* A *with the property that for all* $d \in D$*,*

$$V(d) = \lim_{n\to\infty} a_n^* d a_n.$$

Proof : If F is a compact generating set for D, it is enough to prove that given a strictly positive real number $\varepsilon > 0$, there exists an element a in the unit ball of A such that $\|V(d) - a^* d a\| < \varepsilon$ for all $d \in F$.

For $x \in X$, the fibre A_x is a purely infinite simple C*-algebra and the map $d \mapsto V(d)_x \in A_x$ factorizes through $D_x \simeq (id_D)_x(D) \subset M(A_x)$ since id_D is a continuous field of faithful representations. As a consequence, the previous proposition implies that we can find an element $g \in A$ with $\|g\| \leq 1$ satisfying for all $d \in F$ the inequality

$$\|\big[V(d) - g^* d g\big]_x\| < \varepsilon.$$

Thus, by upper semi-continuity and compactness, there exist a finite open covering $\{U_1, \ldots, U_n\}$ of the space X and elements $g_1, \ldots, g_n$ in the unit ball of A such that for all $d \in F$ and $x \in U_i$, $1 \le i \le n$,

$$\| \Big[V(d) - g_i^* d g_i \Big]_x \| < \varepsilon.$$

Choose n orthogonal isometries $w_1, \ldots, w_n$ in the C*-algebra $\mathcal{O}_\infty \otimes 1_{C(X)} \subset D'$ and let $\{\phi_i\}$ be a partition of the unit $1_{C(X)}$ subordinate to the covering $\{U_i\}$ of X. The element $a = \sum_i \phi_i^{1/2} w_i g_i \in A$ verifies:

1. $a^* a = \sum_{i,j} \sqrt{\phi_i \phi_j}\, g_i^* w_i^* w_j g_j = \sum_i \phi_i g_i^* g_i \le 1_{M(A)}$,
2. for $d \in F$ and $x \in X$, $\| \Big[V(d) - a^* d a \Big]_x \| \le \sum_i \phi_i(x) \| \Big[V(d) - g_i^* d g_i \Big]_x \| < \varepsilon$. □

Let us mention the following technical corollary which will be needed in theorem 3.2.

Corollary 2.4 *If $p \in \mathcal{O}_2 \otimes C(X)$ is a projection such that for all points $x \in X$, p_x is non-zero, then there exists an isometry $u \in \mathcal{O}_2 \otimes C(X)$ such that $p = uu^*$.*

Proof : Let $\mathcal{D}_2 = \lim_{n \to \infty} {\mathcal{O}_2}^{\otimes n}$ be the infinite tensor product of $\mathcal{O}_2$.

Given a projection $q \in \mathcal{D}_2 \otimes C(X)$ such that $\|q_x\| = 1$ for all $x \in X$, we first show that there exists an element $v \in \mathcal{D}_2 \otimes C(X)$ satisfying $1_{\mathcal{D}_2 \otimes C(X)} = v^* q v$. Namely, by density of the algebraic tensor product

$$\Big[\bigcup_n {\mathcal{O}_2}^{\otimes n} \Big] \odot C(X) = \bigcup_n \Big[{\mathcal{O}_2}^{\otimes n} \odot C(X) \Big]$$

in the C*-algebra $\mathcal{D}_2 \otimes C(X)$ and functional calculus one can find an integer $n > 0$ and a projection $r \in {\mathcal{O}_2}^{\otimes n} \otimes C(X) \subset \mathcal{D}_2 \otimes C(X)$ such that $\|q - r\| < 1$, which implies that $r = s^* q s$ for some element $s \in \mathcal{D}_2 \otimes C(X)$. Take then a faithful state φ on ${\mathcal{O}_2}^{\otimes n}$ and consider the $C(X)$-linear completely positive map

$$V : [{\mathcal{O}_2}^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \to {\mathcal{O}_2}^{\otimes n+1} \otimes C(X)$$

defined by the formula $V(d) = (\varphi \otimes id_{C(X)})(d) 1_{{\mathcal{O}_2}^{\otimes n+1} \otimes C(X)}$ for $d \in [{\mathcal{O}_2}^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \simeq {\mathcal{O}_2}^{\otimes n} \otimes C(X)$. According to corollary 2.3, there exists an element $t \in {\mathcal{O}_2}^{\otimes n+1} \otimes C(X)$ such that

$$1_{\mathcal{D}_2 \otimes C(X)} = 1_{{\mathcal{O}_2}^{\otimes n+1} \otimes C(X)} = t^* r t = (st)^* q (st).$$

Consider now the set $\mathcal{P}$ of projections p in $\mathcal{O}_2 \otimes C(X)$ such that $p_x \neq 0$ for all points $x \in X$. If p belongs to $\mathcal{P}$, there exists an isometry $v \in \mathcal{O}_2 \otimes C(X)$ such that $p \ge vv^*$ since the K-trivial purely infinite separable unital nuclear C*-algebra $\mathcal{D}_2$ satisfying the U.C.T. is isomorphic to $\mathcal{O}_2$ ([15]). As a consequence, if t is the isometry $t = v(s_1 \otimes 1) v^*$, the projection $r = tt^*$ (Murray-von Neumann equivalent to $1_{\mathcal{O}_2 \otimes C(X)}$) verifies

$$p - r \ge r' = v(s_2 s_2^* \otimes 1) v^* \in \mathcal{P}.$$

The non-empty set $\mathcal{P}$ therefore satisfies the conditions (π_1)–(π_4) defined by Cuntz in [6]. But the C*-algebra $\mathcal{O}_2 \otimes C(X)$ is K_0-triviality thanks to [6, theorem 2.3] and the theorem 1.4 of [6] enables us to conclude. □

One now deduces from corollary 2.3 the following absorption results ([21, 12, 15]):

Proposition 2.5 *Let A be a σ-unital continuous field of purely infinite simple nuclear* C*-*algebras over a compact Hausdorff space X and let $\mathcal{K}$ be the* C*-*algebra of compact operators on the separable Hilbert space H. Denote by q the quotient map $M(\mathcal{K}\otimes A) \to M(\mathcal{K}\otimes A)/(\mathcal{K}\otimes A)$.*

1. *Assume that D is a unital separable $C(X)$-subalgebra of the multiplier algebra $M(\mathcal{K}\otimes A)$ with same unit such that there is a unital $C(X)$-embedding of the $C(X)$-algebra $\mathcal{O}_\infty \otimes C(X)$ in the commutant of D in $M(\mathcal{K}\otimes A)$ and the identity map id_D is a continuous field of faithful representations of D in $M(\mathcal{K}\otimes A)$.*

   (a) *If V is a unital $C(X)$-linear completely positive map from D in $M(\mathcal{K}\otimes A)$ which is zero on the intersection $D\cap(\mathcal{K}\otimes A)$, there exists a sequence of isometries s_n in $M(\mathcal{K}\otimes A)$ such that for every $d\in D$,*

   $$V(d) - s_n^* d s_n \in \mathcal{K}\otimes A \text{ and } V(d) = \lim_n s_n^* d s_n.$$

   (b) *If π is a unital morphism of $C(X)$-algebras from D into $M(\mathcal{K}\otimes A)$ which is zero on the intersection $D\cap(\mathcal{K}\otimes A)$, there exists a sequence of unitaries u_n in $M(\mathcal{K}\otimes A)$ such that for every $d\in D$,*

   $$(d\oplus\pi(d)) - u_n^* d u_n \in \mathcal{K}\otimes A \text{ and } (d\oplus\pi(d)) = \lim_n u_n^* d u_n.$$

   (c) *Let B be a $C(X)$-algebra and assume that the quotient $D/(D\cap(\mathcal{K}\otimes A))$ is isomorphic to the $C(X)$-algebra $\mathcal{B}$, where $\mathcal{B}$ is the unital $C(X)$-algebra generated by $C(X)$ and B in $M[B\oplus C(X)]$ ([4, définition 2.7]).*

   Then, if $\pi : B\to M(\mathcal{K}\otimes A)$ is a $C(X)$-linear homomorphism, there exists a unitary $U\in M(\mathcal{K}\otimes A)$ such that for all $b\in B\subset M(\mathcal{K}\otimes A)/(\mathcal{K}\otimes A)$,

   $$b\oplus(q\circ\pi)(b) = q(U)^*\, b\, q(U).$$

2. *Assume that the continuous field A is separable and let D be a separable $C(X)$-subalgebra of $M(A)$ containing A such that the identity representation $D\to M(A)$ is a continuous field of faithful representations and there is a unital $C(X)$-embedding of the $C(X)$-algebra $\mathcal{O}_\infty\otimes C(X)$ in the commutant of D in $M(A)$. Define the quotient $C(X)$-algebra $B = D/A$.*

   If $\pi : \mathcal{K}\otimes B\to M(\mathcal{K}\otimes A)$ is a morphism of $C(X)$-algebras, there exists a unitary $U\in M(\mathcal{K}\otimes A)$ such that for all $b\in\mathcal{K}\otimes B\subset M(\mathcal{K}\otimes A)/(\mathcal{K}\otimes A)$,

   $$b\oplus(q\circ\pi)(b) = q(U)^*\, b\, q(U).$$

Proof : *1.* It derives from corollary 2.3 by the same method as the one developed by Kasparov in [11, theorem 5 and 6]. Nevertheless, for the convenience of the reader we describe the different steps of the demonstration.
1.a) Let F be a compact generating set for D containing the unit $1_{M(\mathcal{K}\otimes A)}$. Then given a real number $\varepsilon > 0$, it is enough to find an element $a \in M(\mathcal{K}\otimes A)$ such that $V(d) - a^*da \in \mathcal{K}\otimes A$ and $\|V(d) - a^*da\| < 3\varepsilon$ for all $d \in F$.

Let $\{e_n\}$ be an increasing, positive, quasicentral, countable approximate unit in the ideal $\mathcal{K}\otimes A$ of the C*-algebra generated by $\mathcal{K}\otimes A + V(D)$. If we set $f_0 = (e_0)^{1/2}$ and $f_k = (e_k - e_{k-1})^{1/2}$ for $k \geq 1$, we can then assume, passing to a subsequence of (e_n) if necessary, that $\|V(d)f_k - f_kV(d)\| < 2^{-k}\varepsilon$ for all $k \in \mathbb{N}$ and $d \in F$. This implies that the series $\sum_k [V(d)f_k - f_kV(d)]f_k$ is convergent in $\mathcal{K}\otimes A$ and its norm is less than ε. Furthermore, the series $\sum_k [f_kV(d)f_k]$ is strictly convergent in $M(\mathcal{K}\otimes A)$ for all $d \in F$ since $\sum_k f_k^2$ is strictly convergent to 1.

Notice now that the maps $V_k(d) = f_kV(d)f_k$ are all $C(X)$-nuclear since the separable continuous field $\mathcal{K}\otimes A$ is nuclear. The corollary 2.3 therefore enables us to choose by induction $a_k \in \mathcal{K}\otimes A$ satisfying the following conditions:

1. $\forall d \in F \quad \|V_k(d) - a_k^*da_k\| < 2^{-k}\varepsilon$,
2. $\forall d \in F, \forall l < k \quad \|a_l^*da_k\| < 2^{-l-k}\varepsilon$,
3. $\sum_k a_k$ is strictly convergent toward an element $a \in M(A)$.

One then checks as in [11, theorem 5] that the limit a satisfies the desired properties.

1.b) Take a compact generating F for D containing $1_{M(\mathcal{K}\otimes A)}$ and consider the homomorphism $\pi' = 1\otimes\pi : D \to M(\mathcal{K}\otimes(\mathcal{K}\otimes A)) \simeq M(\mathcal{K}\otimes A)$. Given $\delta > 0$, one can find, thank to the previous assertion, an isometry $s \in M(\mathcal{K}\otimes A)$ such that

$$s^*ds - \pi'(d) \in \mathcal{K}\otimes A \text{ and } \|s^*ds - \pi'(d)\| < \delta \text{ for all } d \in K^*K.$$

As a consequence, if we fix $\varepsilon > 0$, the choice of δ small enough gives us the inequality $\|pd - dp\| < \varepsilon$, and so $\|d - [pdp + p^\perp dp^\perp]\| < 2\varepsilon$ for all $d \in F$, where $p = ss^*$ and $p^\perp = 1 - p$.

Define the unital map $\Theta : D \to M(p^\perp(\mathcal{K}\otimes A)p^\perp)$ by the formula $\Theta(d) = p^\perp dp^\perp$. According to the stabilisation theorem of Kasparov ([11, theorem 2]), one can construct a unitary $w \in M(\mathcal{K}\otimes A)$ verifying for all $d \in F$ the inequality

$$\|d - w^*[\pi'(d) \oplus \Theta(d)]w\| < 3\varepsilon.$$

To finish the demonstration, notice that the two homomorphisms π' and $\pi' \oplus \pi$ are unitarily equivalent.

1.c) Consider the unital extension $\widetilde{\pi}$ of π to $\mathcal{B}$. Then, the morphism $\widetilde{\pi}\circ q : D \to M(\mathcal{K}\otimes A)$ reduces the demonstration to the previous assertion.

2. The identity representation of the unital $C(X)$-algebra $\mathcal{D} = (\mathcal{K}\otimes D) + C(X) \subset M(\mathcal{K}\otimes A)$ is clearly a continuous field of faithful representations since the unital $C(X)$-representation $C(X) \to M(A)$ is a continuous field of faithful representations. Extend the map $\pi : \mathcal{K} \otimes B = (\mathcal{K} \otimes D)/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$ to a unital morphism of $C(X)$-algebras $\tilde{\pi} : \mathcal{D}/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$. Applying assertion *1.b)* to the unital homomorphism $d \mapsto (\tilde{\pi} \circ q)(d)$ from the $C(X)$-subalgebra $\mathcal{D} \subset M(\mathcal{K} \otimes A)$ to the multiplier algebra $M(\mathcal{K} \otimes A)$ now leads to the desired conclusion. □

3 The subtriviality

Given a separable continuous field of nuclear C*-algebras A over X, the strategy to prove the subtriviality of the $C(X)$-algebra A will be the same as the one developed by Kirchberg in [15] to prove theorem 0.1 whose main ideas of demonstration are also explained in [1, Théorème 6.1]. We associate to A a $C(X)$-extension by an hereditary C*-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$ (proposition 3.1) and then prove that after stabilisation, this $C(X)$-extension splits by $\mathcal{R}KK$-theory arguments (theorem 3.2).

3.1 Let us construct the fundamental $C(X)$-extension associated to an exact separable continuous field of C*-algebras.

Proposition 3.1 *Given a compact Hausdorff space X and a non-zero separable unital exact $C(X)$-algebra A, there exist a unital $C(X)$-subalgebra F of $\mathcal{O}_2 \otimes C(X)$ with same unit and an hereditary subalgebra I of $\mathcal{O}_2 \otimes C(X)$ such that I is an ideal in F and the $C(X)$-algebra A is isomorphic to the quotient $C(X)$-algebra F/I.*

Furthermore, if the topological space X is perfect (i.e. without any isolated point) and the $C(X)$-algebra A is continuous, the canonical map $F \to M(I)$ is a continuous field of faithful representations.

Proof : Thanks to the characterisation of separable exact C*-algebras obtained by Kirchberg (theorem 0.1), one may assume that the C*-algebra A is a C*-subalgebra of $\mathcal{O}_2$ containing the unit of $\mathcal{O}_2$.

Let $G \subset \mathcal{O}_2 \otimes C(X)$ be the trivial continuous field $A \otimes C(X)$ over X. Then the kernel of the $C(X)$-linear morphism $\pi : G \to A$ defined by $\pi(a \otimes f) = fa$ is the ideal $J = C_\Delta(X \times X)G$ where $C_\Delta(X \times X)$ is the ideal in $C(X \times X)$ of functions which are zero on the diagonal. Indeed suppose that $T \in G$ verifies $\pi(T) = 0$. Then given $\varepsilon > 0$, take a finite number of elements $a_i \in A$, $f_i \in C(X)$ such that $\|T - \sum_i a_i \otimes f_i\| < \varepsilon$; one has $\|T - \sum_i (1 \otimes f_i - f_i \otimes 1)(a_i \otimes 1)\| < \varepsilon + \|\pi(\sum_i a_i \otimes f_i)\| < 2\varepsilon$.

Define then the hereditary subalgebra $I = J[\mathcal{O}_2 \otimes C(X)]J$ in $\mathcal{O}_2 \otimes C(X)$ generated by J. It is a $C(X)$-algebra since it is closed by Cohen theorem (see e.g. [4, proposition 1.8]) and the product $(1_{\mathcal{O}_2} \otimes f)(bc) = b(1_{\mathcal{O}_2} \otimes f)c$ belongs to I for all $f \in C(X)$ and $b, c \in I$. If we set $F = I + G$, the intersection $G \cap I$ is reduced by construction to the subalgebra J, and so we have a $C(X)$-extension

$$0 \to I \to F \to A \to 0.$$

Assume now that the space X is perfect and that the $C(X)$-algebra A is continuous. We need to prove that the map $F_x \to M(I_x)$ is injective for each $x \in X$. Let $a \in G$ and $b \in I$ be two elements such that the sum $d = a + b \in F$ verifies for a given point $x \in X$ the equality

$$d_xI_x + I_xd_x = 0 \text{ (in } [\mathcal{O}_2 \otimes C(X)]_x \simeq \mathcal{O}_2 \otimes \mathbb{C}).$$

To end the proof, we have to show that d_x is zero. For every $f \in C_\Delta(X \times X)$, one has $(bf)_x = -(af)_x \in J_x$, whence $b_x \in J_x$ and so $d_x \in G_x$. But the representation of $G_x \simeq A$ in $M(J_x) \simeq M(C_x(X)A)$ is injective since X is perfect and A is continuous, from which we deduce that $d_x = 0$. □

Remark: With the previous notations, if the $C(X)$-algebra A is nuclear and ψ is a unital completely positive projection from $\mathcal{O}_2$ onto A, the map $\pi \circ (\psi \otimes id_{C(X)})$ is a unital $C(X)$-linear completely positive map from $\mathcal{O}_2 \otimes C(X)$ onto the $C(X)$-subalgebra A which is zero on the nuclear hereditary $C(X)$-subalgebra I.

3.2 We can now state the main theorem:

Theorem 3.2 *Let X be a compact metrizable space and A be a unital separable $C(X)$-algebra with a unital embedding of the $C(X)$-algebra $C(X)$ in A.*

The following assertions are equivalent:

1. *A is a continuous field of nuclear* C*-*algebras over X;*
2. *there exist a unital monomorphism of $C(X)$-algebras $\alpha : A \hookrightarrow \mathcal{O}_2 \otimes C(X)$ and a unital $C(X)$-linear completely positive map $E : \mathcal{O}_2 \otimes C(X) \to A$ such that $E \circ \alpha = id_A$.*

Proof : *2⇒1* By assumption the identity map $id_A = E \circ id_{\mathcal{O}_2 \otimes C(X)} \circ \alpha : A \to A$ is nuclear since the C*-algebra $\mathcal{O}_2 \otimes C(X)$ is nuclear and so the C*-algebra A is nuclear. Besides the $C(X)$-algebra A is isomorphic to the $C(X)$-subalgebra $\alpha(A)$ of the continuous field $\mathcal{O}_2 \otimes C(X)$ and is therefore continuous.

1⇒2 • Let us first deal with the case where the space X is perfect.

Given a unital nuclear separable continuous fields A over X which is unitaly embedded in the C*-algebra $\mathcal{O}_2$, consider the $C(X)$-extension

$$0 \to I \to F \xrightarrow{\pi} A \to 0$$

constructed in proposition 3.1 and take the associated $C(X)$-extension

$$0 \to \mathcal{K} \otimes I \otimes \mathcal{O}_2 \to D = (\mathcal{K} \otimes F \otimes 1_{\mathcal{O}_2}) + (\mathcal{K} \otimes I \otimes \mathcal{O}_2) \to \mathcal{K} \otimes A \to 0.$$

The $C(X)$-nuclear quotient map $\sigma = \sigma \circ id_{\mathcal{K}\otimes A}$ from the separable nuclear continuous field $\mathcal{K} \otimes A$ to the quotient $D/(\mathcal{K} \otimes I \otimes \mathcal{O}_2) \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)/(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$ admits a $C(X)$-linear completely positive lifting $\mathcal{K} \otimes A \to D$ $(\subset \mathcal{K} \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2)$ thanks to proposition 1.3. This means that the class of σ is invertible in $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)$ (see the second remark following theorem 1.6).

But the group $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$ is $C(X)$-linear homotopy invariant (theorem 1.6), hence zero since the endomorphism $\phi_2(a) = s_1as_1^* + s_2as_2^*$ of $\mathcal{O}_2$ is homotopic to the identity map $id_{\mathcal{O}_2}$ ([6, proposition 2.2]) and so $[\theta] = 2[\theta]$ in $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$ for any invertible extension θ of $\mathcal{K} \otimes A$ by $I \otimes \mathcal{O}_2$. As a consequence, the $C(X)$-extension defined by σ is stably trivial. Furthermore, the identity representation of

$D \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$ is a continuous field of faithful representations (proposition 3.1) and the assertion *2.* of proposition 2.5 implies that the quotient morphism $(id_{\mathcal{K}} \otimes \pi \otimes id_{\mathcal{O}_2})$ from D to $\mathcal{K} \otimes A$ admits a cross section α which is a morphism of $C(X)$-algebras.

This monomorphism α is going to enable us to conclude by standard arguments, using theorem 0.1 and the result of Elliott ([9]) that the C*-algebra $\mathcal{O}_2$ is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2$.

Choose a non-zero minimal projection e_{11} in the C*-algebra $\mathcal{K}$ of compact operators that we embed in $\mathcal{O}_2$ and let φ be a state on $\mathcal{O}_2$ such that $\varphi(e_{11}) = 1$. If we take a unital completely positive projection ψ of $\mathcal{O}_2$ onto the nuclear C*-subalgebra $A \subset \mathcal{O}_2$ (theorem 0.1), the composed map

$$E = (\varphi \otimes id_A) \circ \Big(id_{\mathcal{O}_2} \otimes [\pi \circ (\psi \otimes id_{C(X)})] \otimes \varphi \Big)$$

is a unital $C(X)$-linear completely positive map from $\mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2$ onto A. Take also an isometry $u \in \mathcal{O}_2 \otimes C(X)$ such that $\alpha(e_{11} \otimes 1_A) = uu^*$ (corollary 2.4) and define the unital $C(X)$-algebra morphism

$$\beta : A \longrightarrow \mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2 \simeq \mathcal{O}_2 \otimes C(X)$$

by the formula $\beta(a) = u^* \alpha(e_{11} \otimes a) u$. If $\widetilde{E} : \mathcal{O}_2 \otimes C(X) \to A$ is the completely positive unital map $d \mapsto E(udu^*)$, one gets for all $a \in A$ the equality

$$(\widetilde{E} \circ \beta)(a) = (E \circ \alpha)(e_{11} \otimes a) = a$$

• Let us now come back to the general case of a compact space X.

Define the continuous field $B = A \otimes C([0,1])$ over the perfect compact space $Y = X \times [0,1]$. According to the previous discussion, there exist a unital completely positive map $\widetilde{E} : \mathcal{O}_2 \otimes C(Y) \to B$ and a $C(X) \otimes C([0,1])$-linear monomorphism $\widetilde{\alpha} : B \to \mathcal{O}_2 \otimes C(Y)$ such that $\widetilde{E} \circ \widetilde{\alpha} = id_B$. If $ev_1 : C([0,1]) \to \mathbb{C}$ is the evaluation map at $x = 1 \in [0,1]$, define the two maps $E : \mathcal{O}_2 \otimes C(X) \to A$ and $\alpha : A \to \mathcal{O}_2 \otimes C(X)$ by

$$E(d) = (id_A \otimes ev_1) \circ \widetilde{E}(d \otimes 1_{C([0,1])}) \text{ and } \alpha(a) = (id_{\mathcal{O}_2 \otimes C(X)} \otimes ev_1) \circ \widetilde{\alpha}(a \otimes 1_{C([0,1])}).$$

Then E is a unital $C(X)$-linear completely positive map, α is a unital $C(X)$-linear monomorphism and one has the identity $E \circ \alpha = id_A$. □

Remark: Assume that X is a locally compact metrizable space and that the $C_0(X)$-algebra A is a nuclear continuous field of C*-algebras over X, where $C_0(X)$ denotes the algebra of continuous functions on X vanishing at infinity. If $\widetilde{X}$ is the Alexandroff compactification of X, the unital $C(\widetilde{X})$-algebra $\mathcal{A}$ generated by A and $C(\widetilde{X})$ in the multiplier algebra $M[A \oplus C(\widetilde{X})]$ is a separable unital continuous field of C*-algebras over $\widetilde{X}$ ([4, proposition 3.2]). By theorem 3.2, there exists therefore a $C(\widetilde{X})$-linear monomorphism $\alpha : \mathcal{A} \hookrightarrow \mathcal{O}_2 \otimes C(\widetilde{X})$ and the $C_0(X)$-algebra A is isomorphic to the $C_0(X)$-subalgebra $\alpha(A)$ of $\mathcal{O}_2 \otimes C_0(X)$.

4 Concluding remarks

4.1 A $C(X)$-subalgebra of $\mathcal{O}_2 \otimes C(X)$ is by construction exact and continuous. Conversely, if A is a non-zero exact separable unital continuous field of C*-algebras over a perfect metrizable compact space X, one has by proposition 3.1 a $C(X)$-extension

$$0 \to I \to F \to A \to 0$$

where F is a $C(X)$-subalgebra of $\mathcal{O}_2 \otimes C(X)$. If the identity map $A \to A = F/I$ admits a $C(X)$-linear completely positive lifting $A \to F$, the same method as the one used in theorem 3.2 will imply that the exact continuous field A is isomorphic to a $C(X)$-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$.

It is therefore interesting to know whether this map admits a $C(X)$-linear completely positive lifting in the not discrete case.

4.2 Let us have a look at one of the technical problems involved, the Hahn-Banach type extension property in the continuous field framework for finite type $C(X)$-submodules.

Let A be a separable unital continuous field of C*-algebras over a compact metrizable space X and let D be a finitely generated $C(X)$-submodule which is an operator system. Assume that $\phi : D \to C(X)$ is a $C(X)$-linear unital completely positive map. Then for $x \in X$, there exists, thanks to [4, proposition 3.13], a continuous field of states Φ^x on A, i.e. a $C(X)$-linear unital positive map from A to $C(X)$, such that for all $d \in D$,

$$\Phi^x(d)(x) = \phi(d)(x).$$

As a consequence, given $\varepsilon > 0$ and a finite subset $\mathcal{F}$ of D, one can build by continuity and compactness a continuous field of states Φ on A such that

$$\max\{\|\Phi(d) - \phi(d)\|, d \in \mathcal{F}\} < \varepsilon.$$

But one cannot find in general any continuous field of states on A whose restriction to D is ϕ. Indeed, consider the $C(\widetilde{\mathbb{N}})$-algebra $A = \mathbb{C}^2 \otimes C(\widetilde{\mathbb{N}})$ where $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of the space $\mathbb{N}$ of positive integers. Define the positive element $a \in C_\infty(\widetilde{\mathbb{N}})A \subset A$ by the formulas

$$a_n = \begin{cases} (\frac{1}{n+1}, 0) & \text{if } n \text{ even} \\ (0, \frac{1}{n+1}) & \text{if } n \text{ odd} \end{cases}$$

and let ϕ be the $C(\widetilde{\mathbb{N}})$-linear unital completely positive map with values in $C(\widetilde{\mathbb{N}})$ defined on the $C(\widetilde{\mathbb{N}})$-submodule generated by the two $C(\widetilde{\mathbb{N}})$-linearly independent elements 1_A and a through the formula

$$\phi(a)(n) = \tfrac{1}{n+1} \text{ if } n < \infty \text{ and } \phi(a)(\infty) = 0.$$

Suppose that the continuous field of states Φ is a $C(\widetilde{\mathbb{N}})$-linear extension of ϕ to A. Then as A. Bauval already noticed it, one has the contradiction

$$\begin{aligned} 1 = \Phi(1_A)(\infty) &= \Phi\left((1,0) \otimes 1\right)(\infty) + \Phi\left((0,1) \otimes 1\right)(\infty) \\ &= \lim_{n\to\infty} \Phi\left((1,0) \otimes 1\right)(2n+1) + \lim_{n\to\infty} \Phi\left((0,1) \otimes 1\right)(2n) \\ &= 0 + 0 = 0. \end{aligned}$$

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In this appendix, we solve in proposition A.3 the lifting question raised in paragraph 4.1 through a continuous generalisation of joint work of E.G. Effros and U. Haagerup on lifting problems for C*-algebras ([8], see also [22]). This result enables us to state the following characterisation of separable exact continuous fields of C*-algebras:

Theorem A.1 *Let X be a compact metrizable space and A be a (unital) separable continuous field of* C*-*algebras over* X.

Then the C*-*algebra A is exact if and only if there exists a (unital) monomorphism of $C(X)$-algebras $A \hookrightarrow \mathcal{O}_2 \otimes C(X)$.*

Let us start with a technical $C(X)$-linear version of Auerbach's theorem ([17, proposition 1.c.3]) for a continuous field of C*-algebras A over X which gives us local bases over $C(X)$ with continuous coordinate maps for particular free $C(X)$-submodules of finite type in A.

Define a $C(X)$–operator system in A to be a $C(X)$-submodule which is an operator system.

Lemma A.2 *([8, lemma 2.4]) Let A be a separable unital continuous field of* C*-*algebras over a compact metrizable space X, $E \subset A$ be a $C(X)$–operator system and assume that there exists an integer $n \in \mathbb{N}^*$ such that for all $x \in X$, the dimension* $\dim E_x$ *of the operator system $E_x \subset A_x$ equals n. Then the following holds.*

Given any point $x \in X$, there exist an open neighbourhood $\mathcal{U}$ of x in X, self-adjoint $C(X)$-linear contractions $\varphi_i : A \to C(X)$ and self-adjoint elements $f_i \in E$ with $\|f_i\| \leq 2$ for $1 \leq i \leq n$ such that

$$\forall a \in C_0(\mathcal{U})E,\ a = \textstyle\sum_i \varphi_i(a) f_i.$$

Furthermore, there exists a continuous field of states $\Psi : A \to C(X)$ such that the restriction of the map $2n\Psi - id_A$ to the operator system E is completely positive.

Proof : Let us fix a point $x \in X$. Then there exist, thanks to Auerbach's theorem, a normal basis $\{r_1, \ldots, r_n\}$ of the fibre E_x where each r_i is self adjoint and norm one hermitian functionals $\phi_j : A_x \to \mathbb{C}$, $1 \leq j \leq n$, such that $\phi_j(r_i) = \delta_{i,j}$.

Consider the polar decomposition $\phi_j = \phi_j^+ - \phi_j^-$ where ϕ_j^+ and ϕ_j^- are positive functionals such that $1 = \|\phi_j\| = \|\phi_j^+\| + \|\phi_j^-\|$. By [4] lemme 3.12, there exist $C(X)$-linear positive maps φ_j^+ and $\varphi_j^- : A \to C(X)$ which extend the functionals ϕ_j^+ and ϕ_j^- on the fibre A_x to the $C(X)$-algebra A with the property that $\varphi_j^+(1) = \|\phi_j^+\|$ and $\varphi_j^-(1) = \|\phi_j^-\|$. Take also n norm 1 self-adjoint elements $e_i \in E$ satisfying the equality $(e_i)_x = r_i$ and define the matrix $T = [\varphi_j(e_i)]_{i,j} \in M_n(\mathbb{R}) \otimes C(X)$.

One has by construction $T_x = 1_{M_n(\mathbb{R})}$; the set $\mathcal{U}_1 \subset X$ of points $y \in X$ for which the spectrum of $T_y \in M_n(\mathbb{R})$ is included in the open set $\{z \in \mathbb{C}, |z| > 1/2\}$ is therefore an open neighbourhood of x in X ([4, proposition 2.4 *b)*]). If η is a continuous function on X with values in $[0, 1]$ which is 0 outside $\mathcal{U}_1$ and 1 on an open neighbourhood $\mathcal{U}$ of the point $x \in X$, the self-adjoint elements $f_1, \ldots, f_n$ of norm less than 2 are then well defined in $C_0(\mathcal{U}_1)E$ by the formula

$$T\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \eta \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

and satisfy the relation $\varphi_j(f_i)(y) = \delta_{i,j}$ for each $y \in \mathcal{U}$ since the matrix T_y is invertible, whence the desired equality for every $a \in C_0(\mathcal{U})E$.

Keeping the same fixed point x, define now the continuous field of states $\Phi = \frac{1}{n}\sum_i(\varphi_i^+ + \varphi_i^-)$. Then one gets for all $a \in C_0(\mathcal{U})E$ the equality:

$$(2n\Phi - id_A)(a) = \sum_{1\le i\le n} \Big[\varphi_i^+(a)(2-f_i) + \varphi_i^-(a)(2+f_i)\Big].$$

The restriction of the map $(2n\Phi - id_A)$ to $C_0(\mathcal{U})E$ is therefore completely positive and an appropriate partition of the unit $1_{C(X)}$ enables us to conclude. □

Noticing that a $C(X)$-linear map $\sigma : A \to B$ between $C(X)$-algebras is completely positive if and only if each induced map $\sigma_x : A_x \to B_x$ is completely positive (see for instance [4, proposition 2.9]), the lemma A.2 allows us to state a continuous version of theorem 2.5 of [8]. Replacing then the continuous field A by $A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ (where $M_{2^\infty}(\mathbb{C}) = \varinjlim_{n\to\infty} M_{2^n}(\mathbb{C})$) and working with finitely generated $C(X)$–operator systems $E_k \subset A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ for which the function $x \mapsto \dim(E_k)_x$ is continuous, one derives the following desired $C(X)$-linear completely positive lifting result.

Proposition A.3 *([8, theorem 3.4]) Suppose that A and B are two unital separable exact continuous fields of* C*-algebras over a compact space X with $A = B/J$ for some nuclear ideal J in B.*

Then there exists a $C(X)$-linear unital completely positive lifting $A \to B$ of id_A.

Proof : Let us define the two continuous fields $\mathcal{A} = A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ and $\mathcal{B} = B \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$. It is clearly enough to find a $C(X)$-linear unital completely positive cross section θ of the quotient morphism $\mathcal{B} \to \mathcal{A}$ (by [4, theorem 3.3]).

Consider a dense sequence $\{a_k\}$ in the self-adjoint part of $\mathcal{A}$ where each a_k belongs to the dense subalgebra $A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ of $\mathcal{A}$ and $a_1 = 1$. Let us show that we may assume inductively that $C(X)$–operator system E_n generated by the a_k, $1 \le k \le n$, satisfies the equality $\dim(E_n)_x = n$ for every $n \in \mathbb{N}^*$ and every $x \in X$. The inductive step is the following. Given $n \ge 2$, there exists by construction an integer l such that $E_n \subset A \oplus M_{2^l}(\mathbb{C}) \otimes C(X)$. Set $a_n' = a_n + 2^{-n-1} d_l \otimes 1_{C(X)}$ where

$$d_l = 1_{M_{2^l}(\mathbb{C})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2^{l+1}}(\mathbb{C}) \subset M_{2^\infty}(\mathbb{C}).$$

Then the $C(X)$-module $E_n' = E_{n-1} + C(X)a_n'$ verifies for each $x \in X$ the equality $\dim(E_n')_x = \dim(E_{n-1})_x + 1$.

Using proposition 1.3, one can now finish the proof by the same method as the one developed by E.G. Effros and U. Haagerup in [8].3 (see also [22, theorem 6.10]). □

References

[1] *C. Anantharaman-Delaroche*, Classification des C*-algèbres purement infinies [d'après E. Kirchberg], Séminaire Bourbaki **805** (1995).

[2] *A. Bauval*, $\mathcal{R}KK(X)$-nucléarité (d'après G. Skandalis), *to appear in* K-theory.

[3] *B. Blackadar*, K-theory for Operator Algebras, M.S.R.I. Publications **5**, Springer Verlag, New York (1986).

[4] *E. Blanchard*, Déformations de C*-algèbres de Hopf, Bull. Soc. Math. France **124** (1996), 141–215.

[5] *E. Blanchard*, Tensor products of $C(X)$-algebras over $C(X)$, Astérisque **232** (1995), 81–92.

[6] *J. Cuntz*, K-theory for certain C*-algebras, Ann. of Math. **113** (1981), 181–197.

[7] *J. Dixmier*, Les C*-algèbres et leurs représentations, Gauthiers-Villars Paris (1969).

[8] *E.G. Effros and U. Haagerup*, Lifting problems and local reflexivity for C*-algebras, Duke Math. J. **52** (1985), 103–128.

[9] *G. Elliott*, On the classification of C*-algebras of real rank zero, J. Reine angew. Math. **443** (1993), 179–219.

[10] *U. Haagerup and M. Rørdam*, Perturbations of the rotation C*-algebras and of the Heisenberg commutation relation, Duke Math. J. **77** (1995), 627–656.

[11] *G.G. Kasparov*, Hilbert C^*-modules: theorems of Stinespring and Voiculescu, J. Operator Theory **4** (1980), 133–150.

[12] *G.G. Kasparov*, The operator K-functor and extensions of C*-algebras, Math. U.S.S.R. Izv. **16** (1981), 513–572. Translated from Izv. Acad. Nauk S.S.S.R., Ser. Math. **44** (1980), 571–636.

[13] *G.G. Kasparov*, Equivariant KK-theory and the Novikov conjecture, Invent. Math. **91** (1988), 147–201.

[14] *G.G. Kasparov and G. Skandalis*, Groups acting on buildings, Operator K-theory and Novikov conjecture, K-theory **4** (1991), 303–337.

[15] *E. Kirchberg*, The classification of purely infinite C*-algebras using Kasparov's theory, preliminary version (3rd draft), Humboldt Universität zu Berlin (1994).

[16] *E. Kirchberg and S. Wassermann*, Operations on continuous bundles of C*-algebras, Math. Annalen **303** (1995), 677–697.

[17] *J. Lindenstrauss and L. Tzafriri*, Classical Banach Spaces I, Ergebnisse Math., Springer Verlag, 1977.

[18] *G. Skandalis*, Some remarks on Kasparov theory, J. Funct. Anal. **56** (1984), 337–347.

[19] *G. Skandalis*, Kasparov's bivariant K-theory and applications, Expo. Math. **9** (1991), 193–250.

[20] *G. Skandalis*, Une notion de nucléarité en K-théorie (d'après J. Cuntz), K-theory **1** (1988), 549–573.

[21] *D. Voiculescu*, A non-commutative Weyl-von Neumann theorem, Rev. Roum. Math. Pures et Appl. **21** (1976), 97–113.

[22] *S. Wassermann*, Exact C*-algebras and Related Topics, Lecture Notes Series 19, GARC, Seoul National University, 1994.

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