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DANIELA GIACHETTI, PEDRO J. MARTÍNEZ-APARICIO, AND FRANÇOIS MURAT

Abstract. In this paper we consider semilinear elliptic equations with singularities, whose prototype is the following

$$\begin{cases} -div\, A(x)Du = f(x)g(u) + l(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$, $A \in L^\infty(\Omega)^{N\times N}$ is a coercive matrix, $g : [0, +\infty) \to [0, +\infty]$ is continuous, and $0 \leq g(s) \leq \frac{1}{s^\gamma} + 1$ $\forall s > 0$, with $0 < \gamma \leq 1$ and $f, l \in L^r(\Omega)$, $r = \frac{2N}{N+2}$ if $N \geq 3$, $r > 1$ if $N = 2$, $r = 1$ if $N = 1$, $f(x), l(x) \geq 0$ a.e. $x \in \Omega$.

We prove the existence of at least one nonnegative solution and a stability result; moreover uniqueness is also proved if $g(s)$ is nonincreasing or "almost nonincreasing".

Finally, we study the homogenization of these equations posed in a sequence of domains Ω^ε obtained by removing many small holes from a fixed domain Ω.

1 Introduction

We deal in this paper with nonnegative solutions to the following singular semilinear problem

$$\begin{cases} -div\, A(x)Du = F(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the model for the function $F(x, u)$ is

$$F(x, u) = f(x)g(u) + l(x),$$

for some continuous function $g(s)$ with $0 \leq g(s) \leq \frac{1}{s^\gamma} + 1$ for every $s > 0$, with $0 < \gamma \leq 1$, and some nonnegative functions $f(x)$ and $l(x)$ which belong to suitable Lebesgue spaces.

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Note that (except as far as uniqueness is concerned) we do not require g to be nonincreasing, so that functions g like $g(s) = \frac{1}{s^\gamma}(2 + \sin\frac{1}{s})$ can be considered.

In the present paper we are first interested in existence, uniqueness and stability results for this kind of problems. After this, we will study the asymptotic behaviour, as ε goes to zero, of a sequence of problems posed in domains Ω^ε obtained by removing many small holes from a fixed domain Ω, in the framework of [4].

As far as existence and regularity results for this kind of problems are concerned, we refer to the well known paper by M.G. Crandall, P.H. Rabinowitz and L. Tartar [5], and to the paper by L. Boccardo and L. Orsina [2] which inspired our work. We also refer to the references quoted in [5] and in [2], as well as in [1] which deals with the homogenization of this problem for a sequence of matrices $A^\varepsilon(x)$.

In [5] the authors show the existence of a classical positive solution if the matrix $A(x)$, the boundary $\partial\Omega$ and the function $F(x,s)$ are smooth enough; the function $F(x,s)$, which is not necessarily nonincreasing in s, is bounded from above uniformly for $x \in \bar{\Omega}$ and $s \geq 1$. Boundary behaviour of $u(x)$ and $|Du(x)|$ when x tends to $\partial\Omega$ is also studied.

In [2] the authors study the problem (1.1) with $F(x,u) = \frac{f(x)}{u^\gamma}$, $\gamma > 0$ and f in Lebesgue spaces. They prove existence, uniqueness and regularity results depending on the values of γ and on the sumability of f. Specifically, they prove the existence of strictly positive distributional solutions. In order to prove their results, they work by approximation and construct an increasing sequence $(u_n)_{n\in\mathbb{N}}$ of solutions to nonsingular problems

$$\begin{cases} -div\, A(x)Du_n = \dfrac{f_n(x)}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n(x) = \min\{f(x), n\}$.

This sequence satisfies, for any $\omega \subset\subset \Omega$, the following property

$$u_n(x) \geq u_{n-1}(x) \geq ... \geq u_1(x) \geq c_\omega > 0, \ \forall x \in \omega. \tag{1.2}$$

In order to prove that it is essential to assume that the nonlinearity $F(x,s)$ is nonincreasing in the s variable and to use, as a main tool, the strong maximum principle. Note that (1.2) provides the existence of a limit function $u = sup_n\, u_n$ which is strictly away from zero on any compact set ω of Ω; in addition, (1.2) implies that, on every such set ω, the functions $\frac{f_n(x)}{(u_n+\frac{1}{n})^\gamma}$ are uniformly bounded by an L^1- function h_ω.

This allow the authors to prove that the function u is a solution in the sense of distributions.

In the present paper, we are interested in giving existence and stability results without assuming that $F(x,s)$ is nonincreasing in the s variable and without using the strong maximum principle in the proofs of these results. The main interest of this lies in the fact that this kind of proofs provides the tools to deal with the homogenization of the problem

$$\begin{cases} -div\, A(x)Du^\varepsilon = F(x,u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

where Ω^ε is obtained by removing many small holes from Ω (see Theorem 5.1). Of course, the existence and stability results (see Theorem 4.1 and Theorem 4.2) have also an autonomous interest, due to the more general assumptions and to a different method in the proof.

Moreover, we point out that this method, which avoids using of strong maximum principle, also has a strong interest in other problems frameworks where one cannot expect the strict positivity of the solution on every compact set of Ω. Let us briefly describe some of these situations.

A first situation is the case of singular parabolic problems with p-laplacian type principal part, $p > 1$, and nonnegative data u_0 and f, whose model is the following

$$\begin{cases} u_t - div(|\nabla u|^{p-2}\nabla u) = f(x,t)\left(\dfrac{1}{u^\gamma} + 1\right) & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\gamma > 0$. In this case, due to the assumption $p > 1$ and the fact that the initial datum u_0 is not assumed to be strictly positive, no method of expansion of positivity can be applied and one cannot guarantee that the solution is strictly positive inside $\Omega \times (0,T)$ (see [3]).

A second situation deals with existence and homogenization for elliptic singular problems in an open domain Q of $\mathbb{R}^N$, which is made of an upper part Q_1^ε and a lower part Q_2^ε separated by an oscillating interface Γ^ε, when the boundary conditions at the interface Γ^ε are the continuity of the flux and the fact that this flux is proportional to the jump of the solution through the interface. Our method also applies in this case.

A third situation where our method applies is the case of a singular problem with a zeroth-order term whose coefficient is a nonnegative measure of $\mu \in H^{-1}(\Omega)$, namely

$$\begin{cases} u \geq 0 & \text{in } \Omega, \\ -div\, A(x)Du + \mu u = F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

Problem (1.3) naturally arises when performing the homogenization of (1.1) (where there is no zeroth-order term) posed on a domain Ω^ε obtained from Ω by perforating Ω by many small holes. Our method allows us to obtain results of existence, stability, uniqueness and homogenization, even if the strong maximum principle does not hold true in general in such a context (see [9] and [10]).

In the present paper we consider the case $0 < \gamma \leq 1$. We will deal with the case $\gamma > 1$ in the papers [9] and [10]. We point out that in the case $\gamma > 1$, where the singularity has a stronger behaviour, no global energy estimates are available for the solutions. This makes the problem more difficult, in particular from the point of view of the homogenization. For this reason, we will have to introduce a convenient (complicated) framework, in which we will prove existence, stability, uniqueness and homogenization results. Let us emphasize that despite the changes which are made necessary by this framework, the method of the present paper provides the guide to follow also in the case $\gamma > 1$.

The precise meaning of solutions is given in Definition 3.1. Note that the solutions are nonnegative.

The keystone in our proofs is the analysis of the behaviour of the singular terms near the singularity, which is done in Proposition 6.2 of Section 6.

On the other hand, if we suppose that $F(x,s)$ is "almost nonincreasing" in s) (see (2.4)), we prove the uniqueness of the solution (see Theorem 4.4).

Let us now come to the homogenization problem

$$\begin{cases} -div\, A(x)Du^\varepsilon = F(x,u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon. \end{cases}$$

The general questions we are concerned with are the following. Do the solutions u^ε converge to a limit u when the parameter ε tends to zero? If this limit exists, can it be characterized? Will the result be the same result as in the non singular case? In principle the answer is not obvious at all since, as ε tends to zero, the number of holes becomes greater and greater and the singular set for the right-hand side (which includes at least the holes' boundary) tends to "invade" the entire Ω.

Actually we will prove that a strange term appears in the limit of the singular problem in the same way as in the non singular case studied in [4]. This result is a priori not obvious at all, and a very different behaviour could have been expected.

We now describe the plan of the paper. Section 2 deals with the precise assumptions on problem (1.1). In Section 3 we give the precise definition of a solution to problem (1.1) which we will use in the whole of this paper. Section 4 is devoted to the statements of the existence, stability and uniqueness results; in addition a regularity result dealing with boundedness of solutions is stated in this Section. In Section 5 we give the statement of the homogenization result in a domain with many small holes and Dirichlet boundary condition as well as a corrector result. In Section 6 we prove a priori estimates. Section 7 is devoted to the proofs of the stability, existence and regularity Theorems 4.1, 4.2 and 4.3. In Section 8 we state and prove a comparison principle and we prove the uniqueness Theorem 4.4. Finally we prove in Section 9 the homogenization Theorem 5.1 and the corrector Theorem 5.4.

2 Assumptions

In this section, we give the assumptions on problem (1.1).

We assume that Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$ (no regularity is assumed on the boundary $\partial\Omega$ of Ω), that the matrix A satisfies

$$\begin{cases} A(x) \in L^\infty(\Omega)^{N\times N}, \\ \exists \alpha > 0,\ A(x) \geq \alpha I \quad \text{a.e. } x \in \Omega, \end{cases} \tag{2.1}$$

and the function F satisfies

(2.2)

$$\begin{cases} F : \Omega \times [0, +\infty[\to [0, +\infty] \text{ is a Carathéodory function,} \\ \text{i.e. } F \text{ satisfies} \\ i) \text{ for a.e. } x \in \Omega,\ s \in [0, +\infty[\to F(x, s) \in [0, +\infty] \text{ is continuous,} \\ ii) \,\forall s \in [0, +\infty[, x \in \Omega \to F(x, s) \in [0, +\infty] \text{ is measurable,} \end{cases}$$

(2.3)
$$\begin{cases} \exists\gamma, \exists h \text{ with} \\ i)\, 0 < \gamma \le 1, \\ ii)\, h \in L^r(\Omega),\ r = \frac{2N}{N+2} \text{ if } N \ge 3,\ r > 1 \text{ if } N = 2,\ r = 1 \text{ if } N = 1, \\ iii)\, h(x) \ge 0 \text{ a.e. } x \in \Omega, \\ \text{such that} \\ iv)\, 0 \le F(x,s) \le h(x)\left(\dfrac{1}{s^\gamma} + 1\right) \text{ a.e. } x \in \Omega, \forall s > 0. \end{cases}$$

Remark 2.1. The function $F = F(x,s)$ is a nonnegative Carathéodory function with values in $[0,+\infty]$. But, in view of (2.3 *iv*), the function $F(x,s)$ can take the value $+\infty$ only when $s = 0$ (or, in other terms, $F(x,s)$ is always finite when $s > 0$). □

On the other hand, for proving comparison and uniqueness results, we will assume that $F(x,s)$ is "almost nonincreasing" in s: denoting by λ_1 the first eigenvalue of the operator $-div\, {}^sA(x)D$ in $H^1_0(\Omega)$, where ${}^sA(x) = (A(x) + {}^tA(x))/2$ is the symmetrized part of the matrix $A(x)$, we will assume that

$$\text{(2.4)} \quad \begin{cases} \text{there exists } \lambda \text{ with } 0 \le \lambda < \lambda_1 \text{ such that} \\ F(x,s) - \lambda s \le F(x,t) - \lambda t \text{ a.e. } x \in \Omega,\ \forall s, \forall t,\ 0 \le t \le s. \end{cases}$$

Remark 2.2. Note that (2.4) holds with $\lambda = 0$ when F is assumed to be nonincreasing. But if in place of (2.4) one only assumes that the function

$$\text{(2.5)} \qquad s \in [0,+\infty] \to F(x,s) - \lambda_1 s \text{ is nonincreasing,}$$

uniqueness of the solution to problem (1.1) in general does not hold true, see Remark 8.2 below. □

Notation

We denote by $\mathcal{D}(\Omega)$ the space of the functions $C^\infty(\Omega)$ whose support is compact and included on Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

Since Ω is bounded, $\|Dw\|_{L^2(\Omega)^N}$ is a norm equivalent to $\|w\|_{H^1(\Omega)}$ on $H^1_0(\Omega)$. We set

$$\|w\|_{H^1_0(\Omega)} = \|Dw\|_{L^2(\Omega)^N}, \quad \forall w \in H^1_0(\Omega).$$

For every $s \in \mathbb{R}$ and every $k > 0$ we define

$$s^+ = \max\{s, 0\},\ s^- = \max\{0, -s\},$$
$$T_k(s) = \max\{-k, \min\{s, k\}\}, \quad G_k(s) = s - T_k(s).$$

For $l : \Omega \longrightarrow [0, +\infty]$ a measurable function we denote

$$\{l = 0\} = \{x \in \Omega : l(x) = 0\},\ \{l > 0\} = \{x \in \Omega : l(x) > 0\}.$$

3 Definition of a solution to problem (1.1)

We now give a precise definition of a solution to problem (1.1).

Definition 3.1. Assume that the matrix A and the function F satisfy (2.1), (2.2) and (2.3). We will say that u is a solution to problem (1.1) if u satisfies

$$u \in H_0^1(\Omega), \tag{3.1}$$

$$u \geq 0 \quad \text{a.e. in } \Omega, \tag{3.2}$$

and

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\ \varphi \geq 0, \\ \displaystyle\int_\Omega F(x,u)\varphi < +\infty, \\ \displaystyle\int_\Omega A(x)DuD\varphi = \int_\Omega F(x,u)\varphi. \end{cases} \tag{3.3}$$

Remark 3.2. The nonnegative measurable function $F(x, u(x))$ can take infinite values when $u(x) = 0$. The integral $\displaystyle\int_\Omega F(x,u)\varphi$ is therefore correctly defined as a number in $[0, +\infty]$ for every measurable function $\varphi \geq 0$.

In (3.3) we require that this number is finite for every $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, when u is a solution to problem (1.1) in the sense of Definition 3.1. This in particular implies that

$$F(x, u(x)) \text{ is finite almost everywhere on } \Omega, \tag{3.4}$$

or in other terms that

$$\text{meas}\{x \in \Omega : u(x) = 0 \text{ and } F(x, 0) = +\infty\} = 0. \tag{3.5}$$

A result more precise than (3.5) will be given in Proposition 3.4, and an even much stronger result will be given in Proposition 3.5 and Remark 3.7 (note however that the strong maximum principle is used to obtain the results of Proposition 3.5 and Remark 3.7). □

Remark 3.3. Taking $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, in (3.3), any solution u to problem (1.1) in the sense of Definition 3.1 satisfies

$$F(x,u) \in L^1_{\text{loc}}(\Omega), \quad -div\, A(x)Du = F(x,u) \text{ in } \mathcal{D}'(\Omega).$$

□

Proposition 3.4. *Assume that the matrix A and the function F satisfy* (2.1)*,* (2.2) *and* (2.3)*. Then every solution u to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$meas\{x \in \Omega : u(x) = 0 \text{ and } 0 < F(x,0) \le +\infty\} = 0, \tag{3.6}$$

and

$$\int_\Omega F(x,u)\varphi = \int_{\{u>0\}} F(x,u)\varphi \quad \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0. \tag{3.7}$$

Proof. In Proposition 6.3 below we prove that for every u solution to problem (1.1) in the sense of Definition 3.1 one has

$$\int_{\{u=0\}} F(x,u)\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega), \varphi \ge 0, \tag{3.8}$$

which of course implies (3.7).

Writing now

$$\begin{cases} \{u=0\} = \\ = \Big\{\{u=0\} \cap \{F(x,0)=0\}\Big\} \cup \\ \cup \Big\{\{u=0\} \cap \{0 < F(x,0) \le +\infty\}\Big\} \end{cases}$$

implies that (3.8) is equivalent to

$$\int_{\{u=0\}\cap\{0<F(x,0)\le+\infty\}} F(x,u)\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega),\ \varphi \ge 0.$$

The latest assertion is equivalent to (3.6). Proposition 3.4 is therefore proved.

Note that (3.6) is also equivalent to

$$\begin{cases} \{x \in \Omega : u(x) = 0\} \subset \{x \in \Omega : F(x,0) = 0\} \\ \text{except for a set of zero measure,} \end{cases} \tag{3.9}$$

and also equivalent to

$$\begin{cases} \{x \in \Omega : 0 < F(x,0) \le +\infty\} \subset \{x \in \Omega : u(x) > 0\} \\ \text{except for a set of zero measure.} \end{cases}$$

□

The following Proposition 3.5 and Remark 3.7 assert that for every solution u to problem (1.1) in the sense of Definition 3.1, we can have two possibilities: either $u(x) > 0$ a.e. in Ω or $u \equiv 0$ in Ω.

This assertion is stronger than (3.9), but its proof uses the strong maximum principle.

As pointed out in the Introduction, the strong maximum principle is one of the main tools of the proof of the result obtained by by L. Boccardo and L. Orsina in [2], result which inspired the present paper.

Note that, in contrast with the proofs of the results in [2], the proofs of all the results in the present paper do not make use neither of the strong maximum principle nor of the results of Proposition 3.5 and Remark 3.7 below.

Proposition 3.5. *Assume that the matrix A and the function F satisfy* (2.1)*,* (2.2) *and* (2.3)*. Then every solution u to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$\text{either } u \equiv 0 \text{ or } meas\{x \in \Omega : u(x) = 0\} = 0. \tag{3.10}$$

Proof. We first recall the statement of the strong maximum principle Theorem 8.19 of [11], or more exactly of its variant where u is replaced by $-u$. In this variant, Theorem 8.19 of [11] becomes

$$\begin{cases} \text{Let } u \in W^{1,2}(\Omega) \text{ which satisfies } Lu \leq 0. \\ \text{If for some ball } B \subset\subset \Omega \text{ we have } \inf_B u = \inf_\Omega u \leq 0, \\ \text{then } u \text{ is constant in } \Omega. \end{cases} \tag{3.11}$$

In our situation one has $Lu = div\, A(x)Du$, and $Lu \leq 0$ is nothing but $-div\, A(x)Du \geq 0$. Therefore (3.11) implies

(3.12)

$$\begin{cases} \text{if } u \in H^1(\Omega) \text{ satisfies } -div\, A(x)Du \geq 0 \text{ in } \mathcal{D}'(\Omega), \\ \text{if } u \geq 0 \text{ a.e. in } \Omega \text{ and if } \inf_B u = 0 \text{ for some ball } B \subset\subset \Omega, \\ \text{then } u = 0 \text{ in } \Omega, \end{cases}$$

since when u is a constant in Ω with $\inf_B u = 0$, then $u = 0$ in Ω.

Therefore one has the alternative:

$$\begin{cases} \text{either } \inf_B u > 0 \text{ for every ball } B \subset\subset \Omega, \\ \text{or there exists a ball } B \subset\subset \Omega \text{ such that } \inf_B u = 0. \end{cases}$$

In the first case meas$\{x \in \Omega : u(x) = 0\} = 0$; in the second case (3.12) implies that $u \equiv 0$.

This proves (3.10). □

Remark 3.6. Actually the proof of Proposition 3.5 (which uses the strong maximum principle) provides a result which is much stronger than (3.10), namely

$$\begin{cases} \text{either } u \equiv 0, \\ \text{or for every ball } B \subset\subset \Omega \text{ one has } \inf_B u \geq c(u, B), \\ \quad \text{for some } c(u, B) \in \mathbb{R}, c(u, B) > 0. \end{cases} \tag{3.13}$$

Since the strong maximum principle continues to hold if the operator $-div\, A(x)Du$ is replaced by the operator $-div\, A(x)Du + a_0 u$, with $a_0 \in L^\infty(\Omega)$, $a_0 \geq 0$, both (3.10) and (3.13) continue to hold for such an operator.

But when $a_0 \geq 0$ does not belong to $L^\infty(\Omega)$ and is only a nonnegative element of $H^{-1}(\Omega)$ (this can be the case in the result of the homogenization with many small holes that we will perform in Section 5), the strong maximum principle does not hold anymore for the operator $-div\, A(x)Du + a_0 u$ (see [9] for a counter-example due to G. Dal Maso), and therefore (3.13) does not hold anymore for such an operator. □

Remark 3.7. If $u \equiv 0$ is a solution to problem (1.1) in the sense of Definition 3.1, then Proposition 3.4 implies that $F(x, 0) \equiv 0$.

Conversely, if $F(x, 0) \not\equiv 0$, $u \equiv 0$ is not a solution to problem (1.1) in the sense of Definition 3.1 and Proposition 3.5 (or more exactly (3.13)) then implies that

$$u(x) > 0 \text{ a.e. } x \in \Omega.$$

□

4 Statements of the existence, stability, uniqueness and regularity results

In this Section we state results of existence, stability and uniqueness of the solution to problem (1.1) in the sense of Definition 3.1. We also state a result (Proposition 4.3) concerning the boundedness of the solutions under a regularity assumption on the datum h.

Theorem 4.1. (Existence). *Assume that the matrix A and the function F satisfy* (2.1)*,* (2.2) *and* (2.3)*. Then there exists at least one solution u to problem* (1.1) *in the sense of Definition 3.1.*

The proof of Theorem 4.1 is based on a stability result (see Theorem 4.2 below), and on a priori estimates of $\|u\|_{H_0^1(\Omega)}$ and $\displaystyle\int_{\{u \leq \delta\}} F(x, u)v$ for every $v \in H_0^1(\Omega)$, $v \geq 0$, which are satisfied by every solution u to problem (1.1) in the sense of Definition 3.1 (see Propositions 6.1 and 6.2 in Section 6 below). This proof will be done in Section 7.

Theorem 4.2. (Stability). *Assume that the matrix A satisfies assumption* (2.1)*. Let F_n be a sequence of functions and F_∞ be a function which satisfy assumptions* (2.2) *and* (2.3) *for the same γ and h. Assume moreover that*

$$\text{(4.1)} \quad \text{a.e. } x \in \Omega,\ F_n(x, s_n) \to F_\infty(x, s_\infty) \text{ if } s_n \to s_\infty, s_n \geq 0, s_\infty \geq 0.$$

Let u_n *be any solution to problem* $(1.1)_n$ *in the sense of Definition 3.1, where* $(1.1)_n$ *is the problem* (1.1) *with* $F(x,u)$ *replaced by* $F_n(x,u_n)$.

Then there exists a subsequence, still labelled by n, *and a function* u_∞, *which is a solution to problem* $(1.1)_\infty$ *in the sense of Definition 3.1, such that*

$$u_n \to u_\infty \ \text{ in } H_0^1(\Omega) \text{ strongly.} \tag{4.2}$$

In the following Theorem we state regularity result for the solutions found in Theorem 4.1.

Proposition 4.3. (Boundedness). *Assume that the matrix* A *and the function* F *satisfy* (2.1), (2.2) *and* (2.3). *Assume moreover that the function* h *which appears in* (2.3) *satisfies*

$$h \in L^t(\Omega),\ t > \frac{N}{2} \ \text{ if } N \geq 2,\ \ t = 1 \text{ if } N = 1. \tag{4.3}$$

Then every u *solution to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$\|u\|_{L^\infty(\Omega)} \leq 1 + \frac{2}{\alpha} C(|\Omega|, t) \|h\|_{L^t(\Omega)}, \tag{4.4}$$

for a constant $C(|\Omega, t|)$ *which depends only on* $|\Omega|$ *and* t *and which is nondecreasing in* $|\Omega|$.

Finally, our uniqueness result is a consequence of the Comparison principle stated in Theorem 8.1 below. Note that these two results are the only results where the "almost nonincreasing" character in s of the function $F(x,s)$ is used in the present paper.

Theorem 4.4. (Uniqueness). *Assume that the matrix* A *and the function* F *satisfy* (2.1), (2.2) *and* (2.3). *Assume moreover that the function* F *also satisfies assumption* (2.4). *Then the solution to problem* (1.1) *in the sense of Definition 3.1 is unique.*

Remark 4.5. When assumptions (2.1), (2.2), (2.3) as well as (2.4) hold true, Theorems 4.1, 4.2 and 4.4 together assert that problem (1.1) is well posed in the sense of Hadamard in the framework of Definition 3.1. □

5 Statement of the homogenization result in a domain with many small holes and Dirichlet boundary condition

In this Section we deal with the asymptotic behaviour, as ε tends to zero, of nonnegative solutions to the singular semilinear problem

$$(5.0^\varepsilon) \qquad \begin{cases} -div\, A(x)Du^\varepsilon = F(x, u^\varepsilon) & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases}$$

where u^ε satisfies the homogenous Dirichlet boundary condition on the whole of the boundary of Ω^ε, when Ω^ε is a perforated domain obtained by removing many small holes from a given open bounded set Ω in $\mathbb{R}^N$, with a repartition of those many small holes producing a "strange term" when ε tends to 0.

We begin by describing in Subsection 5.1 the geometry of the perforated domains and the framework introduced in [4] (see also [6] and [12]) for solving this problem when the right-hand side is in $L^2(\Omega)$. We then state in Subsection 5.2 the homogenization result for the singular semilinear problem (5.0^ε).

5.1 The perforated domains

As before we consider a given matrix A which satisfies (2.1) and a given function F which satisfies (2.2) and (2.3).

Let Ω be an open and bounded set of $\mathbb{R}^N$ $(N \geq 2)$ and let us perforate it by holes: we obtain an open set Ω^ε. More precisely, consider for every ε, where ε takes its values in a sequence of positive numbers which tends to zero, some closed sets T_i^ε of $\mathbb{R}^N$, $1 \leq i \leq n(\varepsilon)$, which are the holes. The domain Ω^ε is defined by removing the holes T_i^ε from Ω, that is

$$\Omega^\varepsilon = \Omega - \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon.$$

We suppose that the sequence of domains Ω^ε is such that there exist a sequence of functions w^ε, a distribution $\mu \in \mathcal{D}'(\Omega)$ and two sequences of distributions $\mu^\varepsilon \in \mathcal{D}'(\Omega)$ and $\lambda^\varepsilon \in \mathcal{D}'(\Omega)$ such that

$$(5.1) \qquad w^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega),$$

$$(5.2) \qquad 0 \leq w^\varepsilon \leq 1 \text{ a.e. } x \in \Omega,$$

$$(5.3) \qquad \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ w^\varepsilon \psi \in H_0^1(\Omega^\varepsilon),$$

$$(5.4) \quad w^\varepsilon \rightharpoonup 1 \text{ in } H^1(\Omega) \text{ weakly, in } L^\infty(\Omega) \text{ weakly-star and a.e. in } \Omega,$$

$$(5.5) \qquad \mu \in H^{-1}(\Omega),$$

$$(5.6)\qquad \begin{cases} -div\,{}^tA(x)Dw^\varepsilon = \mu^\varepsilon - \lambda^\varepsilon \text{ in } \mathcal{D}'(\Omega), \\ \text{with } \mu^\varepsilon \in H^{-1}(\Omega),\ \lambda^\varepsilon \in H^{-1}(\Omega), \\ \mu^\varepsilon \geq 0 \text{ in } \mathcal{D}'(\Omega), \\ \mu^\varepsilon \to \mu \text{ in } H^{-1}(\Omega) \text{ strongly}, \\ \langle \lambda^\varepsilon, \tilde{z^\varepsilon} \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = 0\ \ \forall z^\varepsilon \in H_0^1(\Omega^\varepsilon); \end{cases}$$

here, as well as everywhere in the present paper, for every function z^ε in $L^2(\Omega)$, we define $\tilde{z^\varepsilon}$ as the extension by 0 of z^ε to Ω, namely by

$$(5.7)\qquad \tilde{z^\varepsilon}(x) = \begin{cases} z^\varepsilon(x) & \text{in } \Omega^\varepsilon, \\ 0 & \text{in } \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon. \end{cases}$$

Then $\tilde{z^\varepsilon} \in L^2(\Omega)$ and $\|\tilde{z^\varepsilon}\|_{L^2(\Omega)} = \|z^\varepsilon\|_{L^2(\Omega^\varepsilon)}$.

Moreover

$$(5.8)\qquad \begin{cases} \text{if } z^\varepsilon \in H_0^1(\Omega^\varepsilon), \text{ then } \tilde{z^\varepsilon} \in H_0^1(\Omega) \\ \text{with } \widetilde{Dz^\varepsilon} = D\tilde{z^\varepsilon} \text{ and } \|\tilde{z^\varepsilon}\|_{H_0^1(\Omega)} = \|z^\varepsilon\|_{H_0^1(\Omega^\varepsilon)}. \end{cases}$$

The meaning of assumption (5.3) is that

$$(5.9)\qquad w^\varepsilon = 0 \text{ on } \bigcup_{i=1}^{n(\varepsilon)} T_i^\varepsilon,$$

while the meaning of the last statement of (5.6) is that the distribution λ^ε only acts on the holes T_i^ε, $i = 1, \cdots, n(\varepsilon)$, since taking $z^\varepsilon \in \mathcal{D}'(\Omega^\varepsilon)$ in the first statement of (5.6) implies that

$$-div\,{}^tA(x)Dw^\varepsilon = \mu^\varepsilon \text{ in } \mathcal{D}'(\Omega^\varepsilon).$$

Taking $z^\varepsilon = w^\varepsilon\phi$, with $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$, as test function in (5.6) we have

$$\int_\Omega \phi A(x)Dw^\varepsilon Dw^\varepsilon + \int_\Omega w^\varepsilon A(x)Dw^\varepsilon D\phi = \langle \mu^\varepsilon, w^\varepsilon\phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

from which we easily deduce that

$$\int_\Omega \phi A(x)Dw^\varepsilon Dw^\varepsilon \to \langle \mu, \phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\ \forall \phi \in \mathcal{D}(\Omega),\ \phi \geq 0,$$

and therefore that

$$\mu \geq 0.$$

The distribution $\mu \in H^{-1}(\Omega)$ is therefore also a nonnegative measure. Moreover, since

$$\begin{cases} \forall \phi \in \mathcal{D}(\Omega), \phi \geq 0, \\ \displaystyle\int_\Omega \phi d\mu = \langle \mu, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leq \limsup_{\varepsilon} \int_\Omega \phi A(x) Dw^\varepsilon Dw^\varepsilon \leq \\ \leq \|\phi\|_{L^\infty(\Omega)} \limsup_{\varepsilon} \displaystyle\int_\Omega A(x) Dw^\varepsilon Dw^\varepsilon \leq C \|\phi\|_{L^\infty(\Omega)}, \end{cases}$$

the measure μ is a finite Radon measure, or in other terms $\mu \in \mathcal{M}_b(\Omega)$.

It is then (well) known[(1)] (see e.g. [7] Section 1 and [8] Section 2.2 for more details) that if $z \in H_0^1(\Omega)$, then z (or more exactly its quasi-continuous representative for the $H_0^1(\Omega)$ capacity) satisfies

$$z \in L^1(\Omega; d\mu) \text{ with } \langle \mu, z \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_\Omega z \, d\mu \,; \tag{5.10}$$

moreover if $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then z satisfies

$$z \in L^\infty(\Omega; d\mu) \text{ with } \|z\|_{L^\infty(\Omega; d\mu)} = \|z\|_{L^\infty(\Omega)}; \tag{5.11}$$

therefore when $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then z belongs to $L^1(\Omega; d\mu) \cap L^\infty(\Omega; d\mu)$ and therefore to $L^p(\Omega; d\mu)$ for every p, $1 \leq p \leq +\infty$.

When one assumes that the holes T_i^ε, $i = 1, \cdots, n(\varepsilon)$, are such that the assumptions (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) hold true, then (see [4] or [12], or [6] for a more general framework) for every $f \in L^2(\Omega)$, the (unique) solution of

$$\begin{cases} y^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ -div\, A(x) Dy^\varepsilon = f \text{ in } \mathcal{D}'(\Omega^\varepsilon), \end{cases} \tag{5.12}$$

satisfies

$$\widetilde{y^\varepsilon} \rightharpoonup y^0 \text{ in } H_0^1(\Omega),$$

where y^0 is the (unique) solution of

$$\begin{cases} y^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ -div\, A(x) Dy^0 + \mu y^0 = f \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

or equivalently of

$$\begin{cases} y^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ \displaystyle\int_\Omega A(x) Dy^0 Dz + \int_\Omega y^0 z \, d\mu = \int_\Omega f z \;\; \forall z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu). \end{cases} \tag{5.13}$$

(1) the reader who would not enter in this theory could continue reading the present paper assuming in (5.5) that μ is a function of $L^r(\Omega)$ (with $r = (2^*)'$ if $N \geq 3$, $r > 1$ if $N = 2$, and $r = 1$ if $N = 1$) and not only an element of $H^{-1}(\Omega)$.

Note that the "strange term" μy_0 (which is the asymptotic memory of the fact that $\tilde{y^\varepsilon}$ was zero on the holes) appears in the limit equation (5.13).

The model case where assumptions (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) are satisfied in the case where the matrix $A(x)$ is the identity (and where therefore the operator is $-div\, A(x)D = -\Delta$), where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where the holes T_i^ε are balls of radius r^ε (or more generally sets obtained by an homothety of ratio r^ε from a given bounded closed set $T \subset \mathbb{R}^N$) with r^ε given by

$$\begin{cases} r^\varepsilon = C_0 \varepsilon^{N/(N-2)} & \text{if } N \geq 3, \\ \varepsilon^2 \log r^\varepsilon \to -C_0 & \text{if } N = 2, \end{cases}$$

for some $C_0 > 0$ (taking $r^\varepsilon = \exp -\dfrac{C_0}{\varepsilon^2}$ is the model case for $N = 2$) which are periodically distributed on a N-dimensional cubic lattice of cubes of size 2ε, and where the measure μ is given by

$$\begin{cases} \mu = \dfrac{S_{N-1}(N-2)}{2^N} C_0^{N-2} & \text{if } N \geq 3, \\ \mu = \dfrac{2\pi}{4} \dfrac{1}{C_0} & \text{if } N = 2, \end{cases}$$

(see e.g. [4] and [12] for more details, and for other examples, in particular for the case where the holes are distributed on a manifold).

5.2 The homogenization result for the singular semilinear problem (5.0^ε)

The existence Theorem 4.1 asserts that when the matrix A and the function F satisfy assumptions (2.1), (2.2) and (2.3), then for given $\varepsilon > 0$, the singular semilinear problem (5.0^ε) posed on Ω^ε has at least one solution u^ε in the sense of Definition 3.1 (this solution is moreover unique if the function $F(x, s)$ also satisfies assumption (2.4)).

The following result asserts that the result of the homogenization process for the singular problem (5.0^ε) is very similar to the homogenization process for the "classical" problem (5.12).

Theorem 5.1. *Assume that the matrix A and the function F satisfy* (2.1), (2.2) *and* (2.3). *Assume also that the sequence of perforated domains Ω^ε satisfies* (5.1), (5.2), (5.3), (5.4), (5.5) *and* (5.6). *Then there exists a subsequence, still labelled by ε, such that for this subsequence the solution u^ε to problem* (5.0^ε) *in the sense of Definition 3.1, namely* *the* u^ε *such that*

$$\begin{cases} i)\, u^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ ii)\, u^\varepsilon(x) \geq 0 \;\; a.e. \;\; x \in \Omega^\varepsilon, \end{cases} \tag{5.14}$$

$$\begin{cases} \forall \varphi^\varepsilon \in H_0^1(\Omega^\varepsilon),\ \varphi^\varepsilon \geq 0, \\ \displaystyle\int_{\Omega^\varepsilon} F(x,u^\varepsilon)\varphi^\varepsilon < +\infty, \\ \displaystyle\int_{\Omega^\varepsilon} A(x)Du^\varepsilon D\varphi^\varepsilon = \int_{\Omega^\varepsilon} F(x,u^\varepsilon)\varphi^\varepsilon, \end{cases} \tag{5.15}$$

satisfies, for $\tilde{u^\varepsilon}$ *defined by* (5.7),

$$\tilde{u}^\varepsilon \rightharpoonup u^0 \;\; in\ H_0^1(\Omega)\ weakly, \tag{5.16}$$

for some u^0 *which is a solution of*

$$\begin{cases} i)\, u^0 \in H_0^1(\Omega) \cap L^2(\Omega; d\mu), \\ ii)\, u^0(x) \geq 0 \;\; a.e. \;\; x \in \Omega, \end{cases} \tag{5.17}$$

$$\begin{cases} \forall z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu),\ z \geq 0, \\ \displaystyle\int_{\Omega} F(x,u^0)z < +\infty, \\ \displaystyle\int_{\Omega} A(x)Du^0 Dz + \int_{\Omega} u^0 z d\mu = \int_{\Omega} F(x,u^0)\psi. \end{cases} \tag{5.18}$$

Remark 5.2. Requirements (5.17) and (5.18) are the adaptation of the Definition 3.1 of a solution to problem (1.1) to the case of problem

$$\begin{cases} -div\, A(x)Du^0 + \mu u^0 = F(x,u^0) & \text{in } \Omega^\varepsilon, \\ u^0 = 0 & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{5.19}$$

in which there is now a zeroth order term μu^0, where μ can be a non-negative measure of $H^{-1}(\Omega)$. Theorem 5.1 therefore expresses the fact that, when assumptions (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) hold true, the result of the homogenization process of the singular semilinear problem (5.0$^\varepsilon$) in Ω^ε with Dirichlet boundary condition on the whole of the boundary $\partial\Omega^\varepsilon$ is the singular semilinear problem (5.19), where the "strange term" μu^0 appears exactly as in the case of the "classical" problem (5.12) where the right-hand side belongs to $L^2(\Omega)$.

Note nevertheless that the result was not a priori obvious due to the presence of the term $F(x, u^\varepsilon)$, which is singular on the boundary $\partial\Omega^\varepsilon$ and, in particular, on the boundary of the holes, whose number increases more and more when ε goes to zero, "invading" the entire open set Ω. □

Remark 5.3. If $F(x,s)$ satisfies, in addition to (2.2) and (2.3), the further assumption (2.4), the solution u^ε to (5.14) and (5.15) is unique (see Theorem 4.4 above), and the solution u^0 to (5.17) and (5.18) is also unique, as it is easily seen from a proof very similar to the one made in Section 8 below.

Under this further assumption there is therefore no need to extract a subsequence in Theorem 5.1, and the convergence takes place for the whole sequence ε. □

Further to the homogenization result of Theorem 5.1, we will also prove the following corrector result, which, under the assumptions that $u^0 \in L^\infty(\Omega)$ and that the matrix A is symmetric, states that $w^\varepsilon u^0$ is a strong approximation in $H_0^1(\Omega)$ of $\tilde{u^\varepsilon}$.

Theorem 5.4. *Assume that the matrix A and the function F satisfy* (2.1)*,* (2.2) *and* (2.3)*. Assume also that the sequence of perforated domains Ω^ε satisfies* (5.1)*,* (5.2)*,* (5.3)*,* (5.4)*,* (5.5) *and* (5.6)*.*

Consider the subsequence u^ε of solutions to problem (5.0^ε)*, i.e.* (5.14) *and* (5.15)*, and its limit u^0 which is defined by* (5.16)*,* (5.17) *and* (5.18) *in Theorem 5.1.*

Assume moreover that

$$A(x) = {}^tA(x), \tag{5.20}$$

$$u^0 \in L^\infty(\Omega). \tag{5.21}$$

Then further to (5.16) *one has*

$$\tilde{u^\varepsilon} = w^\varepsilon u^0 + r^\varepsilon, \ \ \textit{where } r^\varepsilon \to 0 \textit{ in } H_0^1(\Omega) \textit{ strongly.} \tag{5.22}$$

Remark 5.5. If further to assumption (2.2) and (2.3), the function F is assumed to satisfy the regularity assumption (4.3), then in view of Proposition 4.3, the solutions u^ε of (5.0^ε) satisfy

$$\begin{cases} \|\tilde{u^\varepsilon}\|_{L^\infty(\Omega)} = \|u^\varepsilon\|_{L^\infty(\Omega^\varepsilon)} \leq \\ \leq 1 + \dfrac{2}{\alpha} C(|\Omega^\varepsilon|, t)\|h\|_{L^t(\Omega^\varepsilon)} \leq 1 + \dfrac{2}{\alpha} C(|\Omega|, t)\|h\|_{L^t(\Omega)}, \end{cases}$$

and therefore their limit u^0 satisfies assumption (5.21) with

$$\|u^0\|_{L^\infty(\Omega)} \leq 1 + \frac{2}{\alpha} C(|\Omega|, t)\|h\|_{L^t(\Omega)}.$$

□

6 A priori estimates

Proposition 6.1. **($H_0^1(\Omega)$ a priori estimate)**. *Assume that the matrix A and the function F satisfy* (2.1), (2.2) *and* (2.3). *Then every u solution to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$\|u\|_{H_0^1(\Omega)} \leq C(|\Omega|, N, \alpha, \gamma) \left(\|h\|_{L^{(2^*)'}(\Omega)}^{\frac{1}{1+\gamma}} + \|h\|_{L^{(2^*)'}(\Omega)} \right), \tag{6.1}$$

where $C(|\Omega|, N, \alpha, \gamma)$ is a increasing function of $|\Omega|$.

Proof. We take $\varphi = u$ as test function in (3.3), obtaining

$$\int_\Omega A(x)DuDu = \int_\Omega F(x,u)u \leq \int_\Omega h(x) \left(\frac{1}{u^\gamma} + 1 \right) u. \tag{6.2}$$

When $N \geq 3$, we use Sobolev's embedding Theorem $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, with 2^* defined by $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$, and the Sobolev's inequality

$$\|v\|_{L^{2^*}(\Omega)} \leq C_N \|Dv\|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

Note that $(2^*)' = 2N/(N+2)$.

Using (2.1) and (2.3 *iv*) and Hölder's inequality in (6.2) we get

$$\begin{cases} \alpha \displaystyle\int_\Omega |Du|^2 \leq \|h\|_{L^{(2^*)'}(\Omega)} \left(|\Omega|^{\frac{\gamma}{2^*}} \|u\|_{L^{2^*}(\Omega)}^{1-\gamma} + \|u\|_{L^{2^*}(\Omega)} \right) \leq \\ \leq \|h\|_{L^{(2^*)'}(\Omega)} \left(|\Omega|^{\frac{\gamma}{2^*}} C_N^{1-\gamma} \|Du\|_{L^2(\Omega)^N}^{1-\gamma} + C_N \|Du\|_{L^2(\Omega)^N} \right), \end{cases}$$

which implies estimate (6.1) using Young's inequality with $p = 1/(1-\gamma)$ when $0 < \gamma < 1$, namely

$$\begin{cases} X^{1-\gamma} \leq \dfrac{1}{p} (\lambda X)^{(1-\gamma)p} + \dfrac{1}{p'} \left(\dfrac{1}{\lambda^{1-\gamma}} \right)^{p'} = (1-\gamma)\lambda X + \dfrac{\gamma}{\lambda^{\frac{1-\gamma}{\gamma}}}, \\ \forall X > 0, \forall \lambda > 0. \end{cases}$$

The proof is similar when $N = 2$ and $N = 1$. □

In the following Proposition we give an estimate of $F(x,u)\varphi$ near the singular set $\{u = 0\}$.

To this aim we introduce for $\delta > 0$ the following function $Z_\delta : [0, +\infty[\to [0, +\infty[$ defined by

$$Z_\delta(s) = \begin{cases} 1, & \text{if } 0 \leq s \leq \delta, \\ -\frac{s}{\delta} + 2, & \text{if } \delta \leq s \leq 2\delta, \\ 0, & \text{if } 2\delta \leq s. \end{cases} \tag{6.3}$$

Proposition 6.2. (Control of $\displaystyle\int_{\{u\le\delta\}} F(x,u)v$ when δ is small). *Assume that the matrix A and the function F satisfy* (2.1), (2.2) *and* (2.3). *Then every u solution to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$\begin{cases}\forall\varphi\in H_0^1(\Omega),\ \ \varphi\ge 0,\\ \forall\delta>0,\ \displaystyle\int_{\{u\le\delta\}} F(x,u)\varphi\le\int_\Omega A(x)DuD\varphi Z_\delta(u).\end{cases}\tag{6.4}$$

Proof. The proof consists in taking $T_k(\varphi)Z_\delta(u)$, $\varphi\in H_0^1(\Omega),\varphi\ge 0$ as test function in (3.3). This function belongs to $H_0^1(\Omega)$ and we get

$$\begin{cases}\displaystyle\int_\Omega A(x)DuDT_k(\varphi)Z_\delta(u)=\\ =\displaystyle\frac{1}{\delta}\int_{\delta<u<2\delta} A(x)DuDuT_k(\varphi)+\int_\Omega F(x,u)T_k(\varphi)Z_\delta(u).\end{cases}$$

This implies that

$$\begin{cases}\forall\varphi\in H_0^1(\Omega),\ \ \varphi\ge 0,\ \forall k>0,\\ \forall\delta>0,\ \displaystyle\int_{\{u\le\delta\}} F(x,u)T_k(\varphi)\le\int_\Omega A(x)DuDT_k(\varphi)Z_\delta(u).\end{cases}\tag{6.5}$$

We now pass to the limit in (6.5) as k tends to infinity, using the strong convergence of $DT_k(u)$ to Du in $L^2(\Omega)^N$ in the right-hand side and Fatou's Lemma for on the left-hand side. This gives (6.4). □

As a consequence of Proposition 6.2 we have:

Proposition 6.3. *Assume that the matrix A and the function F satisfy* (2.1), (2.2) *and* (2.3). *Then every u which is solution to problem* (1.1) *in the sense of Definition 3.1 satisfies*

$$\int_{\{u=0\}} F(x,u)\varphi=0\quad\forall\varphi\in H_0^1(\Omega),\ \varphi\ge 0.\tag{6.6}$$

Proof. Since $\{u=0\}\subset\{u\le\delta\}$ for every $\delta>0$, inequality (6.4) implies that

$$\begin{cases}\forall\varphi\in H_0^1(\Omega),\ \ \varphi\ge 0,\ \forall k>0,\\ \forall\delta>0,\ 0\le\displaystyle\int_{\{u=0\}} F(x,u)\varphi\le\int_\Omega A(x)DuD\varphi Z_\delta(u).\end{cases}$$

When δ tends to zero,

$$Z_\delta(u)\to\chi_{\{u=0\}}\text{ a.e. in }\Omega,$$

but since $u \in H_0^1(\Omega)$, one has

$$Du = 0 \text{ a.e. on } \{u = 0\},$$

and therefore, since $A(x)DuD\varphi \in L^1(\Omega)$,

$$\int_\Omega A(x)DuD\varphi Z_\delta(u) \to 0 \text{ as } \delta \to 0.$$

This proves (6.6). □

7 Proofs of the stability, existence and regularity Theorems 4.1, 4.2 and 4.3

Proof of the stability Theorem 4.2.

Since all the functions $F_n(x,s)$ satisfy assumptions (2.2) and (2.3) for the same γ and h, every solution u_n to problem $(1.1)_n$ in the sense of Definition 3.1 satisfies the a priori estimates (6.1) and (6.4) of Propositions 6.1 and 6.2.

Therefore there exist a subsequence, still labelled by n, and a function u_∞ such that

$$u_n \rightharpoonup u_\infty \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega. \tag{7.1}$$

Since $u_n \geq 0$, we have also $u_\infty \geq 0$.

Since u_n satisfies $(3.3)_n$, we have

$$\int_\Omega A(x)Du_nD\varphi = \int_\Omega F_n(x,u_n)\varphi \;\; \forall \varphi \in H_0^1(\Omega),\; \varphi \geq 0. \tag{7.2}$$

Using the almost everywhere convergence (7.1) of u_n to u_∞, assumption (4.1) on the functions F_n and Fatou's Lemma gives

$$\int_\Omega F_\infty(x,u_\infty)\varphi \leq C < +\infty \;\; \forall \varphi \in H_0^1(\Omega),\; \varphi \geq 0, \tag{7.3}$$

which implies the first assertion of $(3.3)_\infty$.

It remains to prove the second assertion of $(3.3)_\infty$.

We write (7.2) as

$$\begin{cases} \forall \varphi \in H_0^1(\Omega),\; \varphi \geq 0, \\ \displaystyle\int_\Omega A(x)Du_nD\varphi = \int_{\{u_n \leq \delta\}} F_n(x,u_n)\varphi + \int_{\{u_n > \delta\}} F_n(x,u_n)\varphi. \end{cases} \tag{7.4}$$

We pass now to the limit as n tends to infinity for $\delta > 0$ fixed in (7.4). In the left-hand side we get

$$\int_\Omega A(x)Du_nD\varphi \to \int_\Omega A(x)Du_\infty D\varphi. \tag{7.5}$$

For what concerns the first term of the right-hand side of (7.4) we use the a priori estimate (6.4). Since $D\varphi Z_\delta(u_n)$ tends to $D\varphi Z_\delta(u_\infty)$ strongly in $L^2(\Omega)^N$ while $A(x)Du_n$ tends to $A(x)Du_\infty$ weakly in $L^2(\Omega)^N$, we obtain

$$\forall\delta > 0,\ \limsup_n \int_{\{u_n\le\delta\}} F_n(x,u_n)\varphi \le \int_\Omega A(x)Du_\infty D\varphi Z_\delta(u_\infty). \tag{7.6}$$

Since

$$Z_\delta(u_\infty) \to \chi_{\{u_\infty=0\}} \text{ a.e. in } \Omega, \text{ as } \delta\to 0,$$

and since $u \in H_0^1(\Omega)$ implies that $Du_\infty = 0$ almost everywhere on $\{x\in\Omega : u_\infty(x) = 0\}$, the right-hand side of (7.6) tends to 0 when δ tends to 0.

We have proved that

$$\limsup_\delta \limsup_n \int_{\{u_n\le\delta\}} F_n(x,u_n)\varphi = 0. \tag{7.7}$$

Let us now observe that for every $\delta > 0$

$$\int_{\{u_\infty=0\}} F_n(x,u_n)\chi_{\{u_n\le\delta\}}\varphi \le \int_{\{u_n\le\delta\}} F_n(x,u_n)\varphi. \tag{7.8}$$

Since u_n converges almost everywhere to u_∞, one has, for every $\delta > 0$,

$$\chi_{\{u_n\le\delta\}} \to \chi_{\{u_\infty\le\delta\}} \text{ a.e. on } \{x\in\Omega : u_\infty(x) \ne \delta\},$$

and therefore

$$\chi_{\{u_n\le\delta\}} \to 1 \text{ a.e. on } \{x\in\Omega : u_\infty(x) = 0\},$$

while in view of assumption (4.1), one has

$$F_n(x,u_n(x)) \to F_\infty(x,u_\infty(x)) \text{ a.e. } x\in\Omega.$$

Applying Fatou's Lemma to the left-hand side of (7.8), we obtain

$$\int_{\{u_\infty=0\}} F_\infty(x,u_\infty)\varphi \le \limsup_n \int_{\{u_n\le\delta\}} F_n(x,u_n)\varphi,$$

which in view of (7.7) implies that

$$\int_{\{u_\infty=0\}} F_\infty(x,u_\infty)\varphi = 0. \tag{7.9}$$

Let us finally pass to the limit in n for $\delta > 0$ fixed in the second term of the right-hand side of (7.4), namely in

$$\int_{\{u_n>\delta\}} F_n(x,u_n)\varphi = \int_\Omega F_n(x,u_n)\chi_{\{u_n>\delta\}}\varphi.$$

Since in view of (2.3 *iv*)

$$0 \le F_n(x,u_n)\chi_{\{u_n>\delta\}}\varphi \le \ h(x)\left(\frac{1}{\delta^\gamma}+1\right)\varphi \ \text{ a.e. } x\in\Omega,$$

since in view of assumption (4.1) and of the almost everywhere convergence (7.1) of u_n to u_∞,

$$F_n(x,u_n)\varphi \to F_\infty(x,u_\infty)\varphi \text{ a.e. on } \Omega,$$

and finally since

$$\chi_{\{u_n>\delta\}} \to \chi_{\{u_\infty>\delta\}} \text{ a.e. on } \{x\in\Omega : u_\infty(x)\neq\delta\},$$

defining the set $\mathcal{C}$ by

$$\mathcal{C}=\{\delta>0,\ \text{meas}\{x\in\Omega : u_\infty(x)=\delta\}>0\,\},$$

and choosing $\delta\notin\mathcal{C}$, Lebesgue's dominated convergence Theorem implies that

(7.10)

$$\int_{\{u_n>\delta\}} F_n(x,u_n)\varphi \to \int_{\{u_\infty>\delta\}} F_\infty(x,u_\infty)\varphi \ \text{ as } n\to+\infty,\ \forall\lambda\notin\mathcal{C}.$$

Since the set $\mathcal{C}$ is at most a countable, choosing δ outside of the set $\mathcal{C}$ and using the fact that the set $\{x\in\Omega : u_\infty(x)>\delta\}$ monotonically shrinks to the set $\{x\in\Omega : u_\infty(x)>0\}$ as δ tends to 0, the fact that $F_\infty(x,u_\infty)\varphi$ belongs to $L^1(\Omega)$ (see (7.3)), and finally (7.9), we have proved that

(7.11)

$$\begin{cases}\displaystyle\int_{\{u_\infty>\delta\}} F_\infty(x,u_\infty)\varphi \to \int_{\{u_\infty>0\}} F_\infty(x,u_\infty)\varphi = \int_\Omega F_\infty(x,u_\infty)\varphi,\\ \text{as } \delta\to 0, \delta\notin\mathcal{C}.\end{cases}$$

Passing to the limit in each term of (7.4), first in n for $\delta>0$ fixed with $\delta\notin\mathcal{C}$, and then for $\delta\notin\mathcal{C}$ which tends to 0, and collecting the results obtained in (7.5), (7.7), (7.10) and (7.11), we have proved that

$$\int_\Omega A(x)Du_\infty D\varphi = \int_\Omega F_\infty(x,u_\infty)\varphi \ \ \forall\varphi\in H_0^1(\Omega),\ \varphi\ge 0,$$

which is nothing but the second assertion in $(3.3)_\infty$.

We have proved a weaker version of Theorem 4.2, where the strong $H_0^1(\Omega)$ convergence (4.2) is replaced by the weak $H_0^1(\Omega)$ convergence (7.1).

Let us now prove that (4.2) holds true. Indeed, further to (7.1) we have

$$\int_\Omega A(x)Du_nDu_n = \int_\Omega F_n(x,u_n)u_n dx.$$

Observe that in view of hypothesis (4.1) and of convergence (7.1) we have

$$F_n(x, u_n)u_n \to F_\infty(x, u_\infty)u_\infty \text{ a.e. } x \in \Omega,$$

and that the functions $F_n(x, u_n)$ are equi-integrable. Indeed for any measurable set $E \subset \Omega$, we have using (2.3 *iv*) and Hölder's inequality (see the proof of Theorem 4.1 above)

(7.12)

$$\begin{cases} 0 \le \int_E F_n(x, u_n)u_n \le \int_E h(x)\left(\frac{1}{u_n^\gamma} + 1\right)u_n \le \\ \le \|h\|_{L^{(2^*)'}(E)}(|\Omega|^{\frac{\gamma}{2^*}} C_N^{1-\gamma}\|Du_n\|_{L^2(\Omega)^N}^{1-\gamma} + C_N\|Du_n\|_{L^2(\Omega)^N}) \le \\ \le C\|h\|_{L^{(2^*)'}(E)}. \end{cases}$$

Therefore by Vitali's Theorem

$$F_n(x, u_n)u_n \to F_\infty(x, u_\infty)u_\infty \text{ in } L^1(\Omega) \text{ strongly.}$$

Taking u_∞ as test function in $(3.3)_\infty$ we have

$$\int_\Omega A(x)Du_\infty Du_\infty = \int_\Omega F_\infty(x, u_\infty)u_\infty dx.$$

Therefore

$$\int_\Omega A(x)Du_nDu_n \to \int_\Omega A(x)Du_\infty Du_\infty.$$

Together with (7.1), this implies the strong convergence (4.2).

This completes the proof of the stability Theorem 4.2. □

Proof of the existence Theorem 4.1.

Let u_n be a solution of

$$(7.13) \qquad \begin{cases} u_n \in H_0^1(\Omega), \\ -div\, A(x)Du_n = T_n(F(x, u_n^+)) \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where T_n is the truncation at height n.

Since $T_n(F(x, s^+))$ is a bounded Carathéodory function defined on $\Omega \times \mathbb{R}$, Schauder's fixed point theorem implies that problem (7.13) has at least a solution. Since $F(x, s^+) \ge 0$, this solution is nonnegative by the weak maximum principle, and therefore $u_n^+ = u_n$.

It is easy to see that u_n is a solution to problem $(1.1)_n$ in the sense of Definition 3.1, where $(1.1)_n$ is the problem (1.1) with $F(x, u)$ replaced by $F_n(x, u_n) = T_n(F(x, u_n))$.

Moreover it is easy to see, considering the cases where $s_\infty > 0$ and where $s_\infty = 0$, that the functions $F_n(x, s)$ satisfy assumption (4.1) with

$$F_\infty(x, s) = F(x, s).$$

The stability Theorem 4.2 then implies that there exists a subsequence of u_n whose limit u_∞ is a solution to problem (1.1) in the sense of Definition 3.1.

This proves the existence Theorem 4.1. □

Proof of the regularity Proposition 4.3.
We use $G_k(u)$, $k > 0$ as test function in (3.3), getting

$$\int_\Omega A(x)DG_k(u)DG_k(u) = \int_\Omega F(x,u)G_k(u) \quad \forall k > 0.$$

Setting $k = j+1$ with $j \geq 0$, this implies, using the coercivity (2.1) and the growth condition (2.3 *iv*), that

$$\begin{cases} \alpha \displaystyle\int_\Omega |DG_{j+1}(u)|^2 & \leq \displaystyle\int_\Omega h(x)\left(\frac{1}{u^\gamma}+1\right)G_{j+1}(u) \leq \\ & \leq \displaystyle\int_{\{u>1\}} h(x)\left(\frac{1}{u^\gamma}+1\right)G_{j+1}(u) \leq \\ & \leq 2\displaystyle\int_\Omega h(x)\,G_{j+1}(u),\ \forall j \geq 0. \end{cases} \tag{7.14}$$

Since

$$G_{j+1}(s) = G_j(G_1(s)), \quad \forall s \in \mathbb{R}, \forall j \geq 0,$$

and $G_1(u) \in H_0^1(\Omega)$, setting

$$\overline{u} = G_1(u),$$

we deduce from (7.14) that

$$\begin{cases} \overline{u} \in H_0^1(\Omega), \\ \alpha \displaystyle\int_\Omega |DG_j(\overline{u})|^2 \leq 2\int_\Omega h(x)G_j(\overline{u}) \quad \forall j \geq 0. \end{cases}$$

A result of G. Stampacchia (see e.g. the proof of Lemma 5.1 in [13]) then implies that when $h \in L^t(\Omega)$ (hypothesis (4.3)), there exists a function $C(|\Omega|, t)$ which is nondecreasing in $|\Omega|$ such that

$$\|\overline{u}\|_{L^\infty(\Omega)} \leq \frac{2}{\alpha}C(|\Omega|,t)\|h\|_{L^t(\Omega)}.$$

Combined with

$$u = T_1(u) + G_1(u) = T_1(u) + \overline{u},$$

this result implies that

$$\|u\|_{L^\infty(\Omega)} \leq 1 + \frac{2}{\alpha}C(|\Omega|,t)\|h\|_{L^t(\Omega)}.$$

This proves Proposition 4.3. □

8 Comparison principle and proof of the uniqueness Theorem 4.4

In this Section we prove a comparison result, assuming the "almost nonincreasing monotonicity" condition (2.4) of $F(x,s)$ with respect to s.

Theorem 8.1 (Comparison principle). *Assume that the matrix A satisfies* (2.1). *Let $F_1(x,s)$ and $F_2(x,s)$ be two functions satisfying* (2.2) *and* (2.3) *for the same γ and h. Assume moreover that*

$$\text{either } F_1(x,s) \text{ or } F_2(x,s) \text{ satisfies (2.4)}, \tag{8.1}$$

and that

$$F_1(x,s) \leq F_2(x,s) \ \text{ a.e. } x \in \Omega, \quad \forall s \geq 0. \tag{8.2}$$

Let u_1 and u_2 be solutions in the sense of Definition 3.1 to problem $(1.1)_1$ *and* $(1.1)_2$*, where* $(1.1)_1$ *and* $(1.1)_2$ *stand for* (1.1) *with $F(x,u)$ replaced by $F_1(x,u_1)$ and $F_2(x,u_2)$. Then*

$$u_1(x) \leq u_2(x) \ \text{ a.e. } x \in \Omega. \tag{8.3}$$

Proof of the uniqueness Theorem 4.4.
Applying this comparison principle to the case where $F_1(x,s) = F_2(x,s) = F(x,s)$, with $F(x,s)$ satisfying (2.4) immediately proves the uniqueness Theorem 4.4. □

Proof of Theorem 8.1.
Since $(u_1 - u_2)^+ \in H_0^1(\Omega)$, we can take it as test function in $(3.3)_1$ and add to both sides of $(3.3)_1$ the finite term $-\lambda \int_\Omega u_1(u_1-u_2)^+$. The same holds for $(3.3)_2$. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x)Du_i D(u_1-u_2)^+ - \lambda \int_\Omega u_i(u_1-u_2)^+ = \\ \displaystyle = \int_\Omega (F_i(x,u_i) - \lambda u_i)(u_1-u_2)^+, \ \ i = 1,2. \end{cases}$$

Taking the difference between these two equations it follows that

$$\begin{cases} \displaystyle\int_\Omega A(x)D(u_1-u_2)^+ D(u_1-u_2)^+ - \lambda \int_\Omega |(u_1-u_2)^+|^2 = \\ \displaystyle = \int_\Omega (F_1(x,u_1) - \lambda u_1) - (F_2(x,u_2) - \lambda u_2)(u_1-u_2)^+. \end{cases}$$

Using the coercivity (2.1) and the characterization of the first eigenvalue λ_1 of the operator $-div\, {}^sA(x)D$ in $H_0^1(\Omega)$, we get

$$(8.4)\qquad \begin{cases} (\lambda_1 - \lambda)\displaystyle\int_\Omega |(u_1-u_2)^+|^2 \leq \\ \leq \displaystyle\int_\Omega (F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2))(u_1-u_2)^+. \end{cases}$$

Let us prove that

$$(8.5)\quad ((F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2))(u_1-u_2)^+ \leq 0 \text{ a.e. } x\in\Omega,$$

or equivalently that

$$(8.6)\quad (F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2) \leq 0 \text{ a.e. } x\in\{u_1>u_2\}.$$

We first observe that since u_1 and u_2 are solutions to $(1.1)_1$ and $(1.1)_2$ in the sense of Definition 3.1, one has (see Remark 3.2)

$$(8.7)\quad F_1(x,u_1) \text{ and } F_2(x,u_2) \text{ are nonnegative and finite a.e. } x\in\Omega.$$

In order to prove (8.6), let us first consider the case where F_1 satisfies (2.4). In this case we have

$$(8.8)\qquad F_1(x,u_1)-\lambda u_1 \leq F_1(x,u_2)-\lambda u_2 \text{ a.e. } x\in\{u_1>u_2\}.$$

We observe that hypothesis (8.2) implies that

$$F_1(x,u_2)\leq F_2(x,u_2) \text{ a.e. } x\in\Omega,$$

and therefore that

$$F_1(x,u_2) \text{ is nonnegative and finite a.e. } x\in\Omega.$$

It is therefore licit to write that

$$(8.9)\qquad \begin{cases} (F_1(x,u_1)-\lambda u_1)-(F_2(x,u_2)-\lambda u_2) = \\ = (F_1(x,u_1)-\lambda u_1)-(F_1(x,u_2)-\lambda u_2)+ \\ +(F_1(x,u_2)-\lambda u_2)-(F_2(x,u_2)-\lambda u_2) \text{ a.e. } x\in\Omega. \end{cases}$$

Since the first line of the right-hand side of (8.9) is nonpositive on $\{u_1>u_2\}$ by (8.8), and since the second line of this right-hand side, namely $F_1(x,u_2)-F_2(x,u_2)$, is nonpositive by (8.2), we have proved (8.6).

Let us now consider the case where F_2 satisfies (2.4). In this case we have

$$(8.10)\qquad F_2(x,u_1)-\lambda u_1 \leq F_2(x,u_2)-\lambda u_2 \text{ a.e. } x\in\{u_1>u_2\}.$$

We observe that, together with the fact that $F_2(x, u_2)$ is finite almost everywhere on Ω (see (8.7)), this result implies that

$$F_2(x, u_1) \text{ is nonnegative and finite a.e. } x \in \{u_1 > u_2\}.$$

It is therefore licit to write that

(8.11)
$$\begin{cases} (F_1(x, u_1) - \lambda u_1) - (F_2(x, u_2) - \lambda u_2) = \\ = (F_1(x, u_1) - \lambda u_1) - (F_2(x, u_1) - \lambda u_1) + \\ +(F_2(x, u_1) - \lambda u_1) - (F_2(x, u_2) - \lambda u_2) \text{ a.e. } x \in \{u_1 > u_2\}. \end{cases}$$

Since the second line of the right-hand side of (8.11) is nonpositive on $\{u_1 > u_2\}$ by (8.10), and since the first line of this right-hand side, namely $F_1(x, u_1) - F_2(x, u_1)$, is nonpositive by (8.2), we have again proved (8.6).

In both cases we have proved that the right-hand side of (8.4) is nonpositive when assumptions (8.1) and (8.2) are assumed to hold true. Since $\lambda_1 - \lambda > 0$ by hypothesis (2.4), this implies that $(u_1 - u_2)^+ = 0$.

This proves (8.3). □

Remark 8.2. Consider the case where the matrix A satisfies (2.1) and is symmetric and where the function F is defined by

$$F(x, s) = \lambda_1 T_k(s) \quad \forall s \geq 0, \tag{8.12}$$

where T_k is the truncation at height $k > 0$, for some k fixed, and where λ_1 and ϕ_1 are the first eigenvalue and eigenvector of the operator $-div\, A(x)D$ in $H_0^1(\Omega)$, namely

$$\begin{cases} \phi_1 \in H_0^1(\Omega),\ \phi_1 \geq 0,\ \displaystyle\int_\Omega |\phi_1|^2 = 1, \\ -div\, A(x)D\phi_1 = \lambda_1 \phi_1. \end{cases} \tag{8.13}$$

The function F defined by (8.12) satisfies assumptions (2.2), (2.3) and (2.5).

Recall that ϕ_1, the solution to (8.13), belongs to $L^\infty(\Omega)$. Then for every t with $0 \leq t \leq k/\|\phi_1\|_{L^\infty(\Omega)}$, the function

$$u = t\phi_1$$

is a solution to (1.1) in the classical sense, and therefore in the sense of Definition 3.1.

This proves that uniqueness does not hold if assumption (2.4) is replaced by the weaker assumption (2.5). □

9 Proofs of the homogenization Theorem 5.1 and of the corrector Theorem 5.4

Proof of the homogenization Theorem 5.1.
First step

Theorem 4.1 asserts that for every $\varepsilon > 0$ there exists at least one solution to problem (5.0^ε) in the sense of Definition 3.1, namely a least one u^ε which satisfies (5.14) and (5.15).

Proposition 6.1 implies that

$$\begin{cases} \|\tilde{u}^\varepsilon\|_{H_0^1(\Omega)} = \|u^\varepsilon\|_{H_0^1(\Omega^\varepsilon)} \leq \\ \leq C(|\Omega^\varepsilon|, N, \alpha, \gamma) \left(\|h\|_{L^{(2^*)'}(\Omega^\varepsilon)}^{\frac{1}{1+\gamma}} + \|h\|_{L^{(2^*)'}(\Omega^\varepsilon)} \right) \leq \\ \leq C(|\Omega|, N, \alpha, \gamma) \left(\|h\|_{L^{(2^*)'}(\Omega)}^{\frac{1}{1+\gamma}} + \|h\|_{L^{(2^*)'}(\Omega)} \right). \end{cases} \tag{9.1}$$

Estimate (9.1) implies that there exists a function u^0, and a subsequence $\tilde{u^\varepsilon}$, still labelled by ε, which satisfies

$$\tilde{u^\varepsilon} \rightharpoonup u^0 \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega. \tag{9.2}$$

Observe that $u^0(x) \geq 0$ a.e. $x \in \Omega$.

Second step

In view of assumptions (5.1), (5.2) and (5.3), one has

$$w^\varepsilon\psi \in H_0^1(\Omega^\varepsilon)\cap L^\infty(\Omega^\varepsilon),\ \forall\psi\in H_0^1(\Omega)\cap L^\infty(\Omega),$$

and

$$\begin{cases}\|w^\varepsilon\psi\|_{H_0^1(\Omega^\varepsilon)} = \|w^\varepsilon\psi\|_{H_0^1(\Omega)} \le \\ \le \|w^\varepsilon\|_{L^\infty(\Omega)}\|D\psi\|_{L^2(\Omega)^N} + \|\psi\|_{L^\infty(\Omega)}\|Dw^\varepsilon\|_{L^2(\Omega)^N} \le \\ \le C^*(\|D\psi\|_{L^2(\Omega)^N} + \|\psi\|_{L^\infty(\Omega)}),\end{cases}$$

where

$$C^* = \max_\varepsilon\{1, \|Dw^\varepsilon\|_{L^2(\Omega)^N}\}.$$

We now fix $\psi\in H_0^1(\Omega)\cap L^\infty(\Omega)$, $\psi\ge 0$, and we use $\varphi^\varepsilon = w^\varepsilon\psi\in H_0^1(\Omega^\varepsilon)$, $w^\varepsilon\psi\ge 0$, as test function in (5.15). We obtain

$$\int_{\Omega^\varepsilon} A(x)Du^\varepsilon D\psi\, w^\varepsilon + \int_{\Omega^\varepsilon} A(x)Du^\varepsilon Dw^\varepsilon\,\psi = \int_{\Omega^\varepsilon} F(x,u^\varepsilon)w^\varepsilon\psi,$$

which using (5.8) implies that

$$\int_\Omega A(x)D\tilde{u^\varepsilon} D\psi\, w^\varepsilon + \int_\Omega A(x)D\tilde{u^\varepsilon} Dw^\varepsilon\,\psi = \int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\psi. \tag{9.3}$$

Equation (9.3) in particular implies by (9.1) and (5.4) that

$$\int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\psi \le C \tag{9.4}$$

where C is independent of ε.

We now claim that for a subsequence, still labelled by ε,

$$\chi_{\Omega_\varepsilon}\to 1 \text{ a.e. in } \Omega; \tag{9.5}$$

indeed, from $w^\varepsilon\chi_{\Omega_\varepsilon} = w^\varepsilon$ a.e. in Ω, which results from (5.3) (see (5.9)), and from (5.4) we get

$$\begin{cases}\chi_{\Omega_\varepsilon} = \chi_{\Omega_\varepsilon}w^\varepsilon + \chi_{\Omega_\varepsilon}(1-w^\varepsilon) = w^\varepsilon + \chi_{\Omega_\varepsilon}(1-w^\varepsilon) \rightharpoonup 1 \\ \text{in } L^\infty(\Omega) \text{ weakly-star,}\end{cases}$$

which implies that

$$\int_\Omega |\chi_{\Omega_\varepsilon} - 1| = \int_\Omega (1-\chi_{\Omega_\varepsilon}) \to 0,$$

which implies (9.5) (for a subsequence).

We deduce from (9.5) that for almost every x_0 fixed in Ω there exists $\varepsilon_0(x_0)$ such that $\chi_{\Omega_\varepsilon}(x_0) = 1$ for every $\varepsilon\le\varepsilon_0(x_0)$, which means that $x_0\in\Omega^\varepsilon$ for every $\varepsilon\le\varepsilon_0(x_0)$. This implies that

$$\widetilde{F(x,u^\varepsilon)}(x_0) = F(x,u^\varepsilon)(x_0) = F(x,\tilde{u^\varepsilon})(x_0) \quad \forall\varepsilon\le\varepsilon_0(x_0).$$

Therefore, using (9.2) we get for $\varepsilon < \varepsilon(x_0)$

$$\widetilde{F(x,u^\varepsilon)}(x_0) = F(x,\tilde{u^\varepsilon}(x_0)) \to F(x,u^0),$$

or in other terms

$$\widetilde{F(x,u^\varepsilon)} \to F(x,u^0) \text{ a.e. } x \in \Omega. \tag{9.6}$$

Using (9.4), (5.4) and (9.6) and applying Fatou's Lemma implies that

$$\int_\Omega F(x,u^0)\psi < +\infty \ \ \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega), \ \psi \geq 0. \tag{9.7}$$

Third step

Let us now fix $\phi \in \mathcal{D}(\Omega), \phi \geq 0$, and take $\psi = \phi$ in (9.3). Since in view of (5.6) one has

$$\begin{cases} \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}Dw^\varepsilon\,\phi = \int_\Omega {}^tA(x)Dw^\varepsilon D(\phi\tilde{u^\varepsilon}) - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\,\tilde{u^\varepsilon} = \\ \displaystyle = \langle \mu^\varepsilon, \phi\tilde{u^\varepsilon}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\,\tilde{u^\varepsilon}, \end{cases}$$

equation (9.3) can be rewritten, for any $\phi \in \mathcal{D}(\Omega), \phi \geq 0$, as

(9.8)

$$\begin{cases} \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}D\phi\,w^\varepsilon + \langle \mu^\varepsilon, \phi\tilde{u^\varepsilon}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\,\tilde{u^\varepsilon} = \\ \displaystyle = \int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\phi. \end{cases}$$

Using (9.2), (5.3), (5.4) and (5.6), we can easily pass to the limit in the left-hand side of (9.8), and we obtain in view of (5.10) (see footnote[(1)])

(9.9)

$$\begin{cases} \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}D\phi\,w^\varepsilon + \langle \mu^\varepsilon, \phi\tilde{u^\varepsilon}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\,\tilde{u^\varepsilon} \to \\ \displaystyle \to \int_\Omega A(x)Du^0D\phi + \langle \mu, \phi u^0\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \\ \displaystyle = \int_\Omega A(x)Du^0D\phi + \int_\Omega u^0\phi d\mu. \end{cases}$$

As far as the right-hand side of (9.8) is concerned we split it for any $\delta > 0$ as

(9.10)

$$\int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\phi = \int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\phi\chi_{\{0\leq\tilde{u^\varepsilon}\leq\delta\}} + \int_\Omega \widetilde{F(x,u^\varepsilon)}w^\varepsilon\phi\chi_{\{\tilde{u^\varepsilon}>\delta\}}.$$

Fourth step

We now use $\phi^\varepsilon = w^\varepsilon \phi Z_\delta(u^\varepsilon)$ as test function in (5.15), where $Z_\delta(s)$ is defined by (6.3) and where $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$. Note that $\phi^\varepsilon \in H_0^1(\Omega^\varepsilon)$, $\phi^\varepsilon \geq 0$ in view of (5.3). We get

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} F(x,u^\varepsilon) w^\varepsilon \phi Z_\delta(u^\varepsilon) = \\ \displaystyle= \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Dw^\varepsilon \, \phi Z_\delta(u^\varepsilon) + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\phi \, w^\varepsilon Z_\delta(u^\varepsilon) + \\ \displaystyle+ \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Du^\varepsilon \, Z'_\delta(u^\varepsilon) w^\varepsilon \phi, \end{cases}$$

which implies, since $Z_\delta(s) = 1$ for $0 \leq s \leq \delta$ and since Z_δ is nonincreasing, that

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} F(x,u^\varepsilon) w^\varepsilon \phi \chi_{\{0 \leq u^\varepsilon \leq \delta\}} \leq \\ \displaystyle\leq \int_{\Omega^\varepsilon} A(x) Du^\varepsilon Dw^\varepsilon \, \phi Z_\delta(u^\varepsilon) + \int_{\Omega^\varepsilon} A(x) Du^\varepsilon D\phi \, w^\varepsilon Z_\delta(u^\varepsilon). \end{cases}$$

In view of the definition (5.7) of the extension by zero and of (5.8), we get

$$\text{(9.11)} \quad \begin{cases} \displaystyle\int_{\Omega} \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \leq \tilde{u^\varepsilon} \leq \delta\}} \leq \\ \displaystyle\leq \int_{\Omega} A(x) D\tilde{u^\varepsilon} Dw^\varepsilon \, \phi Z_\delta(\tilde{u^\varepsilon}) + \int_{\Omega} A(x) D\tilde{u^\varepsilon} D\phi \, w^\varepsilon Z_\delta(\tilde{u^\varepsilon}). \end{cases}$$

Let us define the function $Y_\delta(s)$ by

$$Y_\delta(s) = \int_0^s Z_\delta(\sigma) d\sigma, \quad \forall s \geq 0,$$

and observe that $Y_\delta(u^\varepsilon) \in H_0^1(\Omega^\varepsilon)$ and $\widetilde{Y_\delta(u^\varepsilon)} = Y_\delta(\tilde{u^\varepsilon})$. Using (5.6), we have

$$\text{(9.12)} \quad \begin{cases} \displaystyle\int_{\Omega} A(x) D\tilde{u^\varepsilon} Dw^\varepsilon \phi Z_\delta(\tilde{u^\varepsilon}) = \int_{\Omega} {}^t\!A(x) Dw^\varepsilon DY_\delta(\tilde{u^\varepsilon}) \phi = \\ \displaystyle= \int_{\Omega} {}^t\!A(x) Dw^\varepsilon D(\phi Y_\delta(\tilde{u^\varepsilon})) - \int_{\Omega} {}^t\!A(x) Dw^\varepsilon D\phi \, Y_\delta(\tilde{u^\varepsilon}) = \\ \displaystyle= \langle \mu^\varepsilon, \phi Y_\delta(\tilde{u^\varepsilon}) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \int_{\Omega} {}^t\!A(x) Dw^\varepsilon D\phi \, Y_\delta(\tilde{u^\varepsilon}). \end{cases}$$

Using now (5.6), (9.2), the fact that

$$Y_\delta(\tilde{u^\varepsilon}) \rightharpoonup Y_\delta(u^0) \text{ in } H_0^1(\Omega) \text{ weakly},$$

and (5.4) proves that the right-hand side of (9.12) tends to

$$\langle \mu, \phi Y_\delta(u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

as ε tends to zero for $\delta > 0$ fixed.

Turning back to (9.11), passing to the limit in the last term of (9.11), and using (9.12) and the latest result, we have proved that for every $\delta > 0$ fixed

$$\begin{cases} \limsup_{\varepsilon} \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0\le \tilde{u^\varepsilon}\le\delta\}} \le \\ \le \langle \mu, \phi Y_\delta(u^0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega A(x)Du^0 D\phi\, Z_\delta(u). \end{cases} \tag{9.13}$$

We now pass to the limit in (9.13) as δ tends to zero.

For the first term of the right-hand side of (9.13), we use (5.10) (see footnote[(1)]) and the fact that $0 \le Y_\delta(s) \le \frac{3}{2}\delta$ for every $s \ge 0$; we get

$$0 \le \langle \mu, \phi Y_\delta(u^0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega \phi Y_\delta(u^0) d\mu \le \frac{3}{2}\delta \int_\Omega \phi d\mu \to 0 \text{ as } \delta \to 0.$$

For the second term of the right-hand side of (9.13) we have

$$\int_\Omega A(x)Du^0 D\phi Z_\delta(u^0) \to \int_\Omega A(x)Du^0 D\phi \chi_{\{u^0=0\}} = 0 \text{ as } \delta \to 0,$$

which results from to the fact that

$$0 \le Z_\delta(u^0) \le 1, \quad Z_\delta(u^0) \to \chi_{\{u^0=0\}} \text{ a.e. in } \Omega, \text{ as } \delta \to 0,$$

and then from the fact that $Du^0 = 0$ on the set $\{u^0 = 0\}$ since $u^0 \in H_0^1(\Omega)$.

As far as the first term of the right-hand side of (9.10) is concerned, we have we proved that

$$\lim_\delta \limsup_\varepsilon \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0\le \tilde{u^\varepsilon}\le\delta\}} = 0. \tag{9.14}$$

Fifth step

Let us now pass to the limit in the second term of the right-hand side of (9.10).

Observe that there is at most a countable set $\mathcal{C}^0$ of values of $\delta > 0$ such that

$$\text{meas}\{x \in \Omega : u^0(x) = \delta\} > 0 \text{ if } \delta \in \mathcal{C}^0.$$

From now on we will often choose $\delta > 0$ outside of this set $\mathcal{C}^0$.

Using (9.6), (5.4), (9.2), the fact that

$$\forall \delta > 0,\ \chi_{\{\tilde{u^\varepsilon}>\delta\}} \to \chi_{\{u^0>\delta\}} \text{ a.e. } x \notin \{u^0 = \delta\},$$

and therefore that

$$\forall \delta \notin \mathcal{C}^0,\ \chi_{\{\tilde{u^\varepsilon}>\delta\}} \to \chi_{\{u^0>\delta\}} \text{ a.e. } x \in \Omega,$$

and the estimate (see (2.3 *iv*))

$$\begin{cases} 0 \le \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi(x) \chi_{\{\tilde{u^\varepsilon} > \delta\}} \le \\ \le h(x) \left(\dfrac{1}{(\tilde{u^\varepsilon})^\gamma} + 1 \right) \phi(x) \chi_{\{\tilde{u^\varepsilon} > \delta\}} \le h(x) \left(\dfrac{1}{\delta^\gamma} + 1 \right) \phi(x) \text{ a.e. } x \in \Omega, \end{cases}$$

Lebesgue's dominated convergence Theorem implies that

$$\lim_{\varepsilon} \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u^\varepsilon} > \delta\}} = \int_\Omega F(x,u^0) \phi \chi_{\{u^0 > \delta\}} \; \forall \delta \notin \mathcal{C}^0.$$

Using (9.7) we now pass to the limit when $\delta \notin \mathcal{C}^0$ tends to zero. We obtain

$$\lim_{\delta \notin \mathcal{C}^0} \lim_{\varepsilon} \int_\Omega \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{\tilde{u^\varepsilon} > \delta\}} = \int_\Omega F(x,u^0) \phi \chi_{\{u^0 > 0\}}. \tag{9.15}$$

We now want to prove that

$$\int_{\{u^0 = 0\}} F(x,u^0) \phi = 0. \tag{9.16}$$

For almost everywhere $x_0 \in \{x \in \Omega : u^0(x) = 0\}$, one has

$$\tilde{u^\varepsilon}(x_0) \to 0 \text{ as } \varepsilon \to 0,$$

and therefore $\tilde{u^\varepsilon}(x_0) < \delta$ for every $\varepsilon < \varepsilon_0(x_0)$. This implies that

$$\chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} \to 1 \text{ a.e. } x \in \{x \in \Omega : u^0(x) = 0\}.$$

Using this fact, (9.6), (5.4) and Fatou's Lemma for $\delta > 0$ fixed we get

$$\int_{\{u^0 = 0\}} F(x,u^0) \phi \le \liminf_{\varepsilon} \int_{\{u^0 = 0\}} \widetilde{F(x,u^\varepsilon)} w^\varepsilon \phi \chi_{\{0 \le \tilde{u^\varepsilon} \le \delta\}} \; \forall \delta > 0,$$

which, passing to the limit with δ which tends to zero and using (9.14) gives (9.16), which implies that

$$\int_\Omega F(x,u^0) \phi \chi_{\{u^0 > 0\}} = \int_\Omega F(x,u^0) \phi. \tag{9.17}$$

Sixth step

We come back to (9.8). Collecting together (9.9), (9.10), (9.14), (9.15) and (9.17) we have proved that

$$\begin{cases} \forall \phi \in \mathcal{D}(\Omega),\ \phi \ge 0, \\ \displaystyle\int_\Omega A(x) D u^0 D\phi + \int_\Omega u^0 \phi d\mu = \int_\Omega F(x,u^0) \phi. \end{cases} \tag{9.18}$$

Seventh step

Let us now take $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\psi \geq 0$.

Consider a sequence ψ_n such that

$$\begin{cases} \psi_n \in \mathcal{D}(\Omega),\ \psi_n \geq 0,\ \|\psi_n\|_{L^\infty(\Omega)} \leq C, \\ \psi_n \to \psi\ H_0^1(\Omega) \text{ strongly, a.e. } x \in \Omega, \\ \text{and quasi-everywhere in } \Omega \text{ for the } H_0^1(\Omega) \text{ capacity.} \end{cases}$$

Define

$$\hat{\psi}_n = \inf\{\psi_n, \psi\};$$

then

$$\begin{cases} \hat{\psi}_n \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \hat{\psi}_n \geq 0,\ \|\hat{\psi}_n\|_{L^\infty(\Omega)} \leq C, \\ \operatorname{supp} \hat{\psi}_n \subset \operatorname{supp} \psi_n \subset\subset \Omega, \\ \hat{\psi}_n \to \psi\ H_0^1(\Omega) \text{ strongly, a.e. } x \in \Omega, \\ \text{and quasi-everywhere in } \Omega \text{ for the } H_0^1(\Omega) \text{ capacity.} \end{cases}$$

For the moment let n be fixed and let ρ_η be a sequence of mollifiers. For η sufficiently small the support of $\hat{\psi}_n \star \rho_\eta$ is included in a fixed compact K_n of Ω, and $\hat{\psi}_n \star \rho_\eta \in \mathcal{D}(\Omega)$, $\hat{\psi}_n \star \rho_\eta \geq 0$. We can therefore use $\phi = \hat{\psi}_n \star \rho_\eta$ as test function in (9.18). We get

$$\int_\Omega A(x) D u^0 D(\hat{\psi}_n \star \rho_\eta) + \int_\Omega u^0 (\hat{\psi}_n \star \rho_\eta) d\mu = \int_\Omega F(x, u^0)(\hat{\psi}_n \star \rho_\eta).$$

Let us pass to the limit in each term of this equation for n fixed as η tends to zero. In the right-hand side we use the facts that $F(x, u^0) \in L^1_{\text{loc}}(\Omega)$ (see (9.7)), that $\operatorname{supp}(\hat{\psi}_n \star \rho_\eta) \subset K_n$ and that $\|\hat{\psi}_n \star \rho_\eta\|_{L^\infty}(\Omega) \leq \|\hat{\psi}_n\|_{L^\infty}(\Omega)$, and the almost convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$ together with Lebesgue's dominated convergence Theorem. In the first term of the left-hand side we use the strong convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$ in $H_0^1(\Omega)$. This strong convergence implies (for a subsequence) the quasi-everywhere convergence for the $H_0^1(\Omega)$ capacity and therefore the μ-almost everywhere convergence of $\hat{\psi}_n \star \rho_\eta$ to $\hat{\psi}_n$; we use again Lebesgue's dominated convergence Theorem, this time in $L^1(\Omega; d\mu)$, and the fact that (see (5.11))

$$0 \leq u^0 (\hat{\psi}_n \star \rho_\eta) \leq u^0 \|\hat{\psi}_n\|_{L^\infty(\Omega; d\mu)} = u^0 \|\hat{\psi}_n\|_{L^\infty(\Omega)} \ \ \mu\text{-a.e. } x \in \Omega$$

to pass to the limit in the second term of the left-hand side. We have proved that

$$\int_\Omega A(x) D u^0 D\hat{\psi}_n + \int_\Omega u^0 \hat{\psi}_n d\mu = \int_\Omega F(x, u^0) \hat{\psi}_n. \tag{9.19}$$

We now pass to the limit in each term of (9.19) as n tends to infinity. This is easy in the right-hand side by Lebesgue's dominated convergence Theorem since $\hat{\psi}_n$ tends almost everywhere to ψ, since

$$0 \leq F(x,u^0)\hat{\psi}_n \leq F(x,u^0)\psi \text{ a.e. } x \in \Omega,$$

and since the latest function belongs to $L^1(\Omega)$ (see (9.7)). This is also easy in the first term of the left-hand side of (9.19) since $\hat{\psi}_n$ tends to ψ strongly in $H_0^1(\Omega)$. Also $\hat{\psi}_n$ converges to ψ quasi-everywhere in the sense of the $H_0^1(\Omega)$ capacity, therefore μ-almost everywhere and we easily pass to the limit in the second term of the left-hand side of (9.19) by Lebesgue's dominated convergence Theorem since (see (5.11))

$$0 \leq u^0\hat{\psi}_n \leq u^0\psi \leq u^0\|\psi\|_{L^\infty(\Omega;d\mu)} = u^0\|\psi\|_{L^\infty(\Omega)} \ \ \mu\text{-a.e. } x \in \Omega$$

and since $u^0 \in H_0^1(\Omega) \subset L^1(\Omega; d\mu)$ (see (5.10)).

We have proved that

$$\text{(9.20)} \qquad \begin{cases} \forall \psi \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \psi \geq 0, \\ \displaystyle\int_\Omega A(x)Du^0 D\psi + \int_\Omega u^0\psi d\mu = \int_\Omega F(x,u^0)\psi. \end{cases}$$

Eighth step

Let us finally prove that $u^0 \in L^2(\Omega; d\mu)$ and that (5.18) holds true.

Taking $\psi = T_n(u^0) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ in (9.20) we obtain

$$\int_\Omega A(x)Du^0 DT_n(u^0) + \int_\Omega u^0 T_n(u^0) d\mu = \int_\Omega F(x,u^0)T_n(u^0),$$

in which using the coercitivity (2.1) of A and the growth condition (2.3 *iv*) of F, we obtain

$$\int_\Omega |T_n(u^0)|^2 d\mu \leq \int_\Omega F(x,u^0)T_n(u^0) \leq \int_\Omega h(x)\left(\frac{1}{(u^0)^\gamma}+1\right)u^0 < +\infty.$$

Using Fatou's Lemma implies that

$$u^0 \in L^2(\Omega; d\mu).$$

Fix now some test function $z \in H_0^1(\Omega) \cap L^2(\Omega; d\mu)$, $z \geq 0$. Taking $T_n(z) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in (9.20) we have

$$\text{(9.21)} \qquad \int_\Omega A(x)Du^0 DT_n(z) + \int_\Omega u^0 T_n(z) d\mu = \int_\Omega F(x,u^0)T_n(z).$$

It is easy to pass to the limit in each term of the left-hand side of (9.21), since $T_n(z)$ tends to z in $H_0^1(\Omega) \cap L^2(\Omega; d\mu)$ and since $u^0 \in L^2(\Omega; d\mu)$.

Applying Fatou's Lemma to the right-hand side of (9.21), we obtain

$$\int_\Omega F(x,u^0)z \leq \int_\Omega A(x)Du^0 Dz + \int_\Omega u^0 z d\mu < +\infty,$$

which is the first statement of (5.18).

But since

$$0 \leq F(x,u^0)T_n(z) \leq F(x,u^0)z,$$

and since the latest function belongs to $L^1(\Omega)$, Lebesgue's dominated convergence Theorem implies that

$$\int_\Omega F(x,u^0)T_n(z) \to \int_\Omega F(x,u^0)z,$$

which completes the proof the second statement of (5.18).

The proof of Theorem 5.1 is now complete. □

Proof of the corrector Theorem 5.4.
First step

In view of hypothesis (5.21), the function $w^\varepsilon u^0$ belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$, and therefore the function r^ε defined by (5.22) belongs to $H_0^1(\Omega)$. By the coercivity assumption (2.1) and by the symmetry assumption (5.20) on the matrix A, we have

$$(9.22)\qquad \begin{cases} \alpha \displaystyle\int_\Omega |Dr^\varepsilon|^2 \leq \int_\Omega A(x)Dr^\varepsilon Dr^\varepsilon = \\ = \displaystyle\int_\Omega A(x)(D\tilde{u^\varepsilon} - D(w^\varepsilon u^0))(D\tilde{u^\varepsilon} - D(w^\varepsilon u^0)) = \\ = \displaystyle\int_\Omega A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} - 2\int_\Omega A(x)D\tilde{u^\varepsilon}D(w^\varepsilon u^0) + \\ + \displaystyle\int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0). \end{cases}$$

We will pass to the limit in each term of the right-hand side of (9.22).

Second step

As far as the first term of the right-hand side of (9.22) is concerned, taking $u^\varepsilon \in H_0^1(\Omega^\varepsilon)$ as test function in (5.15) and extending u^ε and $F(x,u^\varepsilon)$ by zero into $\tilde{u^\varepsilon}$ and $\widetilde{F(x,u^\varepsilon)}$ (see (5.8) and (5.7)), we get

$$(9.23)\qquad \int_\Omega A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} = \int_\Omega \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon}.$$

Since by (2.3 *iv*) we have

$$0 \leq F(x,u^\varepsilon)u^\varepsilon \leq h(x)\left(\frac{1}{(u^\varepsilon)^\gamma}+1\right)u^\varepsilon = h(x)((u^\varepsilon)^{1-\gamma}+u^\varepsilon),$$

using on the first hand (9.2) and (9.6), on the other hand the equi-integrability

$$0 \leq \int_E \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon} \leq C\|h\|_{L^{(2^*)'}(E)} \text{ for any measurable set } E\subset\Omega,\ \forall\varepsilon,$$

which results from the bound (9.1) and from inequality similar to (7.12), and finally Vitali's Theorem, we have

$$\int_\Omega \widetilde{F(x,u^\varepsilon)}\tilde{u^\varepsilon} \to \int_\Omega F(x,u^0)u^0. \tag{9.24}$$

On the other hand, taking $z=u^0$ as test function in (5.18) it follows that

$$\int_\Omega A(x)Du^0Du^0 + \int_\Omega (u^0)^2 d\mu = \int_\Omega F(x,u^0)u^0.$$

Thus, by (9.23), (9.24) and the previous equality we have, using (5.10) which holds true since $(u^0)^2\in H_0^1(\Omega)$ when $u^0\in L^\infty(\Omega)$,

$$\begin{cases} \displaystyle\int_{\Omega^\varepsilon} A(x)D\tilde{u^\varepsilon}D\tilde{u^\varepsilon} \to \int_\Omega A(x)Du^0Du^0 + \int_\Omega (u^0)^2 d\mu = \\ \displaystyle= \int_\Omega A(x)Du^0Du^0 + \langle\mu,(u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{cases} \tag{9.25}$$

Third step

Let us now pass to the limit in the third term of the right-hand side of (9.22). Using (5.6) we obtain

$$\begin{cases} \displaystyle\int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0) = \\ \displaystyle= \int_\Omega A(x)D(w^\varepsilon u^0)Dw^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon = \\ \displaystyle= \int_\Omega {}^tA(x)Dw^\varepsilon D(w^\varepsilon(u^0)^2) + \\ \displaystyle- \int_\Omega {}^tA(x)Dw^\varepsilon Du^0 w^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon = \\ = \langle\mu^\varepsilon, w^\varepsilon(u^0)^2\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \\ \displaystyle- \int_\Omega {}^tA(x)Dw^\varepsilon Du^0 w^\varepsilon u^0 + \int_\Omega A(x)D(w^\varepsilon u^0)Du^0 w^\varepsilon, \end{cases}$$

in which it is easy to pass to the limit in each term, obtaining

$$\begin{cases} \int_\Omega A(x)D(w^\varepsilon u^0)D(w^\varepsilon u^0) \to \\ \to \langle \mu, (u^0)^2 \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \int_\Omega A(x)Du^0 Du^0. \end{cases} \tag{9.26}$$

Fourth step

Passing to the limit in the second term of the right-hand side of (9.22) is a little bit more delicate. Let ϕ be any function such that $\phi \in \mathcal{D}(\Omega)$. We have

$$\begin{cases} \int_\Omega A(x)D\tilde{u^\varepsilon} D(w^\varepsilon u^0) = \int_\Omega A(x)D\tilde{u^\varepsilon} Du^0 w^\varepsilon + \\ + \int_\Omega A(x)D\tilde{u^\varepsilon} Dw^\varepsilon \phi + \int_\Omega A(x)D\tilde{u^\varepsilon} Dw^\varepsilon (u^0 - \phi). \end{cases} \tag{9.27}$$

It is easy to pass to the limit in the first term of the right-hand side of (9.27), obtaining

$$\int_\Omega A(x)D\tilde{u^\varepsilon} Du^0 w^\varepsilon \to \int_\Omega A(x)Du^0 Du^0. \tag{9.28}$$

For what concerned the second term of the right-hand side of (9.27), we have in view of (5.6)

$$\begin{cases} \int_\Omega A(x)D\tilde{u^\varepsilon} Dw^\varepsilon \phi = \int_\Omega {}^tA(x)Dw^\varepsilon D\tilde{u^\varepsilon} \phi = \\ = \int_\Omega {}^tA(x)Dw^\varepsilon D(\tilde{u^\varepsilon}\phi) - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\, \tilde{u^\varepsilon} = \\ = \langle \mu^\varepsilon, \tilde{u^\varepsilon}\phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega {}^tA(x)Dw^\varepsilon D\phi\, \tilde{u^\varepsilon}, \end{cases} \tag{9.29}$$

and therefore

$$\begin{cases} \int_\Omega A(x)D\tilde{u^\varepsilon} Dw^\varepsilon \phi \to \langle \mu, u^0\phi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \\ = \langle \mu, (u^0)^2 \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle \mu, u^0(\phi - u^0) \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \end{cases} \tag{9.30}$$

Taking now $w^\varepsilon(u^0 - \phi)^2$ as test function in (5.6), we have

$$\begin{cases} \int_\Omega {}^tA(x)Dw^\varepsilon Dw^\varepsilon (u^0 - \phi)^2 + 2\int_\Omega {}^tA(x)Dw^\varepsilon D(u^0 - \phi)(u^0 - \phi)w^\varepsilon = \\ = \langle \mu^\varepsilon, w^\varepsilon (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \end{cases}$$

which implies that

$$\int_\Omega {}^tA(x)Dw^\varepsilon Dw^\varepsilon (u^0 - \phi)^2 \to \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

By the coercivity (2.1), this implies that, for every $\phi \in \mathcal{D}(\Omega)$,

$$\limsup_{\varepsilon} \alpha \int_\Omega |Dw^\varepsilon|^2 |u^0 - \phi|^2 \leq \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

This result together with the bound (9.1) on $\|\tilde{u^\varepsilon}\|_{H_0^1(\Omega)}$ implies that for every $\phi \in \mathcal{D}(\Omega)$

$$(9.31) \quad \begin{cases} \limsup_{\varepsilon} \left| \int_\Omega A(x) D\tilde{u^\varepsilon} Dw^\varepsilon (u^0 - \phi) \right| \leq \\ \leq \|A\|_{L^\infty(\Omega)^{N\times N}} \|\tilde{u^\varepsilon}\|_{H_0^1(\Omega)} \dfrac{1}{\alpha} \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \leq \\ \leq c \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{cases}$$

where c depends on α, $\|A\|_{L^\infty(\Omega)^{N\times N}}$, etc.

Collecting the results obtained in (9.27), (9.28), (9.30) and (9.31), we have proved that for every $\phi \in \mathcal{D}(\Omega)$ one has

$$(9.32) \quad \begin{cases} \limsup_{\varepsilon} \left[\int_\Omega A(x) D\tilde{u^\varepsilon} D(w^\varepsilon u^0) + \right. \\ \qquad \left. - \int_\Omega A(x) Du^0 Du^0 - \langle \mu, (u^0)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right] \leq \\ \leq \left| \langle \mu, u^0 (\phi - u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| + c \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \end{cases}$$

where c does not depend on ϕ.

Fifth step

Using in (9.22) the results obtained in (9.25), (9.26) and (9.32), we have proved that for every $\phi \in \mathcal{D}(\Omega)$ one has

$$(9.33) \quad \begin{cases} \limsup_{\varepsilon} \alpha \|r^\varepsilon\|^2_{H_0^1(\Omega)} \leq \\ \leq 2 \left| \langle \mu, u^0 (\phi - u^0) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right| + 2c \langle \mu, (u^0 - \phi)^2 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \end{cases}$$

Approximating $u^0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ by a sequence of functions $\phi \in \mathcal{D}(\Omega)$ which converges to u^0 strongly on $H_0^1(\Omega)$ and weakly-star in $L^\infty(\Omega)$ proves that

$$r^\varepsilon \to 0 \text{ on } H_0^1(\Omega) \text{ strongly},$$

i.e. (5.22).

Theorem 5.4 is proved. □

Remark 9.1. The above proof of the corrector Theorem 5.4 works under the only assumption (5.21) on u^0, namely $u^0 \in L^\infty(\Omega)$. If we assume that the function F, in addition to assumption (2.2) and (2.3), verifies the regularity condition (4.3), then in view of Proposition 4.3 the solutions $\tilde{u}^\varepsilon$ are bounded in $L^\infty(\Omega)$, which implies (5.21) (see Remark 5.5).

Under such an assumption the above proof becomes also easier, since one can take $\phi = u^0$ in (9.27); indeed, even if u^0 does not belong to $\mathcal{D}(\Omega)$, the functions $\tilde{u}^\varepsilon$, u^0 and $\tilde{u}^\varepsilon u^0$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, which gives a meaning to each term in (9.29) with $\phi = u^0$. This allows one to continue the above proof with $\phi = u^0$, which makes it simpler. □

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Daniela Giachetti
Dipartimento di Scienze di Base e Applicate per l'Ingegneria
Facoltà di ingegneria, Sapienza Università di Roma
Via Scarpa 16, 00161 Roma, Italy.
E-mail address: daniela.giachetti@sbai.uniroma1.it

Pedro J. Martínez-Aparicio
Departamento de Matemática Aplicada y Estadística
Universidad Politécnica de Cartagena
Paseo Alfonso XIII 52, 30202 Cartagena (Murcia), Spain.
E-mail address: pedroj.martinez@upct.es

François Murat
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
Boîte Courrier 187, 75252 Paris Cedex 05, France.
E-mail address: murat@ann.jussieu.fr