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Exact analytic solutions for nonlinear waves in cold plasmas

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Abstract. Large amplitude plasma oscillations are studied in a cold electron plasma. Using Lagrangian variables, a new class of exact analytical solutions is found. It turns out that the electric field amplitude is limited either by wave breaking or by the condition that the electron density always has to stay positive. The range of possible amplitudes is determined analytically.

In view of the many technological applications of low-temperature bounded plasmas, there has recently been an increased interest in this subfield of plasma physics. It is in this connection necessary to study large-amplitude disturbances of the oscillating quantities. The theory of nonlinear plasma waves, which was reviewed by Infeld and Rowlands a few years ago [1], concerns to a large extent studies of wave disturbances by means of expansions in the wave amplitudes. This leads to truncated approximate solutions. However, there are a few cases that can be solved exactly [1, 2]. In the present paper we point out another new interesting exact solution of the electron fluid equations. Such solutions can be used as a means to test numerical codes in regimes where it is usually difficult to determine their accuracy.

The method is described in [1, Sec. 6.4]. The basic equations describing one-dimensional large-amplitude plasma oscillations in a cold electron plasma with immobile ions are

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{e}{m}E \tag{2}$$

and

$$\frac{\partial E}{\partial x} = 4\pi e(n_0 - n), \tag{3}$$

where n is the electron density, u the electron velocity, E the electric field, n_0 the ion density, e the elementary charge and m the electron mass. The above equations can be combined to give

$$\frac{\partial E}{\partial t} + u\frac{\partial E}{\partial x} = 4\pi n_0 eu. \tag{4}$$

Next we change the Eulerian variables t, x to the Lagrangian variables τ, ξ where $t = \tau$ and $x = \xi + \int_0^\tau u(\xi, \tau')\, d\tau'$. Then we have

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau} - \frac{u}{R}\frac{\partial}{\partial \xi},$$

$$\frac{\partial}{\partial x} \to \frac{1}{R}\frac{\partial}{\partial \xi},$$

where $R = 1 + \int_0^\tau [\partial u(\xi, \tau')/\partial \xi]\, d\tau'$. From the above relations we find that without approximation (4) can be rewritten as

$$\frac{\partial^2 E}{\partial \tau^2} + \omega_{\rm p}^2 E = 0, \tag{5}$$

where $\omega_{\rm p}$ is the *unperturbed* plasma frequency $\omega_{\rm p} = (4\pi n_0 e^2/m)^{1/2}$. The most general solution to (5) is $E(\xi, \tau) = A(\xi)\cos(\omega_{\rm p}\tau) + B(\xi)\sin(\omega_{\rm p}\tau)$. If at time $t = 0$ ($\tau = 0, \xi = x$), $E(x, t) = E(\xi, 0)$ and $u(x, t) = u(\xi, 0)$, then for all τ and ξ

$$E(\xi, \tau) = E(\xi, 0)\cos(\omega_{\rm p}\tau) + \frac{m\omega_{\rm p}}{e}u(\xi, 0)\sin(\omega_{\rm p}\tau). \tag{6}$$

Furthermore, we have

$$x = \xi + \frac{u(\xi, 0)}{\omega_{\rm p}}\sin(\omega_{\rm p}\tau) - \frac{eE(\xi, 0)}{m\omega_{\rm p}^2}[1 - \cos(\omega_{\rm p}\tau)] \tag{7}$$

and the relation

$$n(x, 0) = n_0 - \left(\frac{1}{4\pi e}\right)\frac{\partial E}{\partial x}(x, 0) \tag{8}$$

such that the initial conditions are described by only two arbitrary functions $E(x, 0)$ and $u(x, 0)$.

In [1] two distinct solutions were examined in detail with $u(x, 0) = 0$:

1. $E(x, 0) = a\sin(kx)$ where a and k are constants;
2. $n(x, 0) = n_0[1 + rf(x)]$ where r is a dimensionless constant and the function $f(x)$ is $f(x) = 1$ for $0 \leqslant x < \pi/2k$, $f(x) = -1$ for $\pi/2k \leqslant x < 3\pi/2k$ and $f(x) = 1$ for $3\pi/2k \leqslant x < 2\pi/k$.

In the present paper we consider again the case of $u(x, 0) = 0$ but with

$$E(x, 0) = \begin{cases} a\exp(-\kappa_1 x) & \text{for } 0 < x, \\ b\exp(\kappa_2 x) & \text{for } x < 0, \end{cases}$$

where a, b, κ_1 and κ_2 are constants. The major problem is obviously to invert the (x, ξ) transformation. With the above initial conditions we now have to solve

$$x = \begin{cases} \xi - a\beta(\tau)\exp(-\kappa_1\xi) & \text{for } \xi > 0, \\ \xi - b\beta(\tau)\exp(\kappa_2\xi) & \text{for } \xi < 0, \end{cases}$$

where $\beta(\tau) = [1 - \cos(\omega_{\rm p}\tau)]e/m\omega_{\rm p}^2$. To solve these equations we introduce the Lambert function. It has been adopted recently to investigate a number of other problems in physics [3]. In plasma physics it has been used to solve dispersion relations [4], but no dynamic equations. The Lambert function $W(z)$ is defined as the solution of the equation $y\exp y = z$ such that $y = W(z)$. As we are dealing with real quantities we can take z as real. A plot of $W(z)$ is shown in Fig. 1.

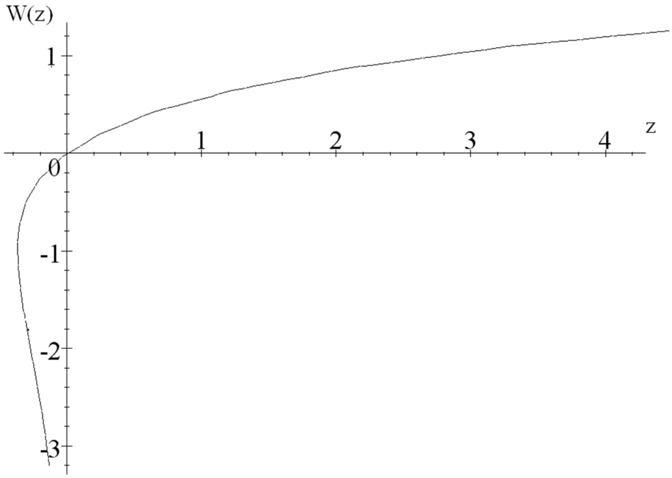

Figure 1. A plot of the real branch of the Lambert function.

We now find that the relation between x and ξ for $\xi > 0$ can be written in terms of the Lambert function as

$$\xi = x + \frac{1}{\kappa_1} W(a\beta\kappa_1 \exp(-\kappa_1 x)). \tag{9}$$

Then, from (7) together with the definition of the Lambert function we obtain

$$E(x,t) = \frac{1}{\beta\kappa_1} W[a\beta\kappa_1 \exp(-\kappa_1 x)] \cos(\omega_{\rm p} t), \tag{10}$$

which is valid for $\xi > 0$, that is for $W[a\beta\kappa_1 \exp(-\kappa_1 x)] < a\beta\kappa_1$. A similar treatment for $\xi < 0$ gives

$$\xi = x - \frac{1}{\kappa_2} W[-b\beta\kappa_2 \exp(\kappa_2 x)] \tag{11}$$

such that

$$E(x,t) = -\frac{1}{\beta\kappa_2} W[-b\beta\kappa_2 \exp(\kappa_2 x)] \cos(\omega_{\rm p} t) \tag{12}$$

valid for $W[-b\beta\kappa_2 \exp(\kappa_2 x)] > b\beta\kappa_2$. To be able to join the two solutions smoothly, we demand that the points $\xi = 0$ derived from (9) and (11) coincide. This happens if $a = b$, in which case the surface $x = x_0(t)$ separating the solutions (10) and (12) is given by $x_0(t) = a\beta(t)$. As a further consequence of the relation $a = b$, the electric field becomes continues at this interface. While we may have solutions with κ_1 different from κ_2, we focus here on the case with $\kappa_1 = \kappa_2 \equiv \kappa$ from now on. With these choices we thus we have an exact large-amplitude solution for all x and t with a continues electric field, written as

$$E(x,t) = \frac{\mathrm{sgn}[x - x_0(t)]}{\beta\kappa} W[a\beta\kappa\ \mathrm{sgn}[x - x_0(t)] \exp[-\mathrm{sgn}[x - x_0(t)]\kappa x]] \cos(\omega_{\rm p} t), \tag{13}$$

where $\mathrm{sgn}[x - x_0(t)] \equiv [x - x_0(t)]/|x - x_0(t)|$. For large $|x|$ or small β, that is $\cos(\omega_{\rm p}\tau) \to 1$, we can expand the Lambert function. Since to lowest order $W(x) = x$

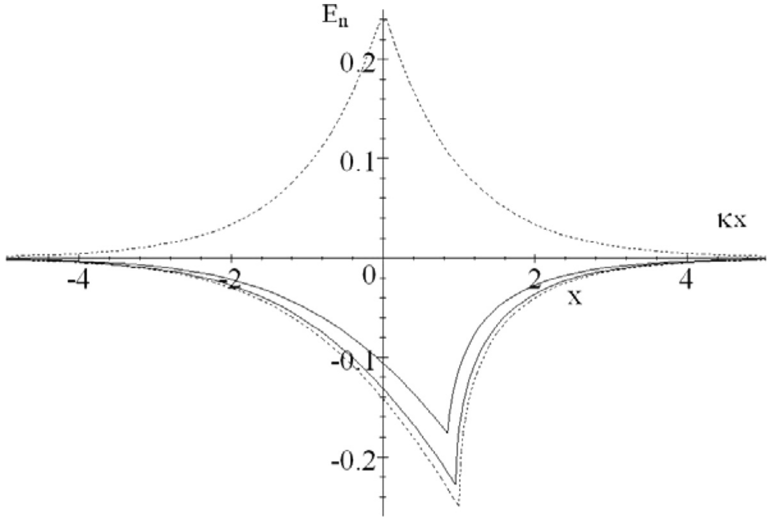

Figure 2. A plot of the normalized electric field $E_{\mathrm{n}} = 2e\kappa E/m\omega_{\mathrm{p}}^2$ as a function of κx for different times. The upper dotted curve corresponds to $\omega_{\mathrm{p}}t = 0$, the upper solid curve corresponds to $\omega_{\mathrm{p}}t = \pi/4$, the lower solid curve corresponds to $\omega_{\mathrm{p}}t = 3\pi/4$ and the lower dotted curve corresponds to $\omega_{\mathrm{p}}t = \pi$. To complete the picture we note that the electric field is zero everywhere for $\omega_{\mathrm{p}}t = \pi/2$ as well as for $\omega_{\mathrm{p}}t = 3\pi/2$.

(see [3, (3.1)]), we then have

$$E(x,t) = a\exp(-\kappa|x|)\cos(\omega_{\mathrm{p}}t).$$

We see from Fig. 1 that $dW/dz \to \infty$ as $z \to -1/\mathbf{e}$. (Note that we use the new symbol $\mathbf{e} = 2.718\ldots$ here, whereas our previous symbol e denotes the elementary charge.) Near that critical point we can write (see [3, (4.22)]) $W(z) = -1+p-p^2/3$, where $p^2 = 2(\mathbf{e}z+1)$, and thus $dW/dz \approx \mathbf{e}/[2(\mathbf{e}z+1)]^{1/2}$. This implies that $n(x,t)$ becomes infinite at this point, and this further implies from (12) that $a\beta\kappa_2 < 1/\mathbf{e}$. As this must hold for all times, we let $\beta(\tau) \to 2e/m\omega_{\mathrm{p}}^2$ to obtain a condition on the amplitude of the initial electric field, namely

$$a < 0.18\left(\frac{m\omega_{\mathrm{p}}^2}{e\kappa_2}\right) \equiv a_{\mathrm{c}}. \tag{14}$$

For $a > a_{\mathrm{c}}$, the solution becomes multivalued for certain values of t and x and, hence, unphysical.

Finally, from Poisson's equation (3) and (10), and using the relation $dW/dz = W/[z(1+W)]$, one obtains

$$n(x,t) = n_0\left[1 + \frac{W[a\beta\kappa\exp(-\kappa|x-x_0(t)|)]\cos(\omega_{\mathrm{p}}t)}{\{1+W[a\beta\kappa\exp(-\kappa|x-x_0(t)|)]\}[1-\cos(\omega_{\mathrm{p}}t)]}\right] \tag{15}$$

for $x > x_0(t)$. An analogous expression holds for $x < x_0(t)$. In Fig. 2 the solution for the electric field is shown for different times. In Fig. 3 the maximal and minimal density during a period is presented as a function of the amplitude. In addition to the condition of no wave breaking (14), we must ensure that the electron density is always positive for our solution to be applicable. Depending on the initial direction of the electric field, i.e. the sign of a, the initial electric field amplitude is limited

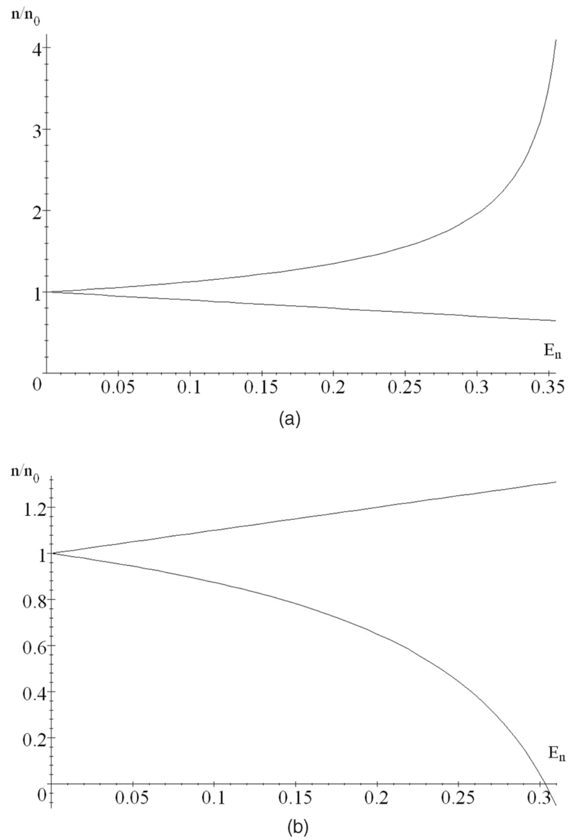

Figure 3. A plot of the maximum and minimum normalized electron density n/n_0 during a period as a function of the normalized initial electric field amplitude $E_\mathrm{n} = 2e\kappa|a|/m\omega_\mathrm{p}^2$. (a) Here $a < 0$, corresponding to an initial density depletion. In this case the maximum electric field amplitude is determined by the condition (14), which corresponds to the wave breaking amplitude $E_\mathrm{n} \approx 0.368$. (b) Here $a > 0$, corresponding to an initially positive density perturbation. In this case the maximum allowed electric field is determined by the condition that the electron density should stay positive. The limiting case $n = 0$ occurs for a slightly lower amplitude than in (a), with $E_\mathrm{n} \approx 0.304$.

by the condition of no wave breaking (for a negative a) or by the condition of a positive density (for a positive a), as illustrated in Fig. 3(a) and (b).

The present results significantly contribute to the existing zoo of previous solutions [1, 2, 5].

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