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I, Miguel Andres Caicedo Caceres , hereby submit this original work as part of the requirements for the degree of Doctor of Philosophy in Mathematical Sciences.

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Well-posedness and Control of the Korteweg-de Vries Equation on a Finite Domain.

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Well-posedness and Control of the Korteweg-de Vries Equation on a Finite Domain

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Abstract

The Korteweg-de Vries equation is one of the most studied partial differential equations in past decades. This equation models unidirectional propagation of small finite amplitude long waves in a non-dispersive medium and has become the source of important breakthroughs in mechanics and nonlinear analysis and of many developments in algebra, analysis, geometry and physics.

This research focuses on an initial boundary value problem for the Korteweg-de Vries equation posed on a bounded interval with a nonlinear boundary condition. This nonlinearity is due to the presence of a moving wall at the left end point of the interval. By using the Kato smoothing property, sharp Kato smoothing property and the contraction mapping principle this initial boundary value problem is shown to be locally well posed in the L^2-based Sobolev space $H^s(0, L)$ for any $s \geq 0$; moreover it will be proved that an initial boundary value problem associated with the Korteweg-de Vries equation is local exactly controllable via the contraction mapping principle.

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My family played a fundamental role during this time. They were always present, not physically but in my heart and my mind, thanks to all of them.

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CHAPTER 1

Introduction

The Korteweg-de Vries equation (KdV-equation henceforth)

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1.1}$$

was derived by Boussinesq [9] and Korteweg and de Vries [40] as a model for long-crested small amplitude long waves propagating on the surface water. As is usual $u = u(x,t)$ is a real-valued function of the variables x and t which often correspond in applications to space and time, respectively and subscripts denote partial differentiation. In this document we will study the KdV-equation from two different approaches. One of these is well-posedness, which is existence, uniqueness and continuous dependence of the solutions of the KdV equation for $(x,t) \in (0,L) \times \mathbb{R}^+$, subject to the initial condition

$$u(x,0) = \phi(x)$$

and the non-homogeneous boundary conditions

$$u_{xx}(0,t) + u(0,t) - \frac{1}{6}u^2(0,t) = 0,$$

$$u(L,t) = 0,$$

$$u_x(L,t) = 0,$$

in the L^2-based Sobolev space $H^s(0, L)$. In order to prove well-posedness for this initial boundary value problem, we are going to use the approach developed by Bona, Sun and Zhang in [4]. This approach is strongly based on the smoothing properties of some linear problems associated with our initial boundary value problem; more precisely Kato and Sharp Kato smoothing properties, with these properties, some results for the KdV equation posed on the whole line and the contraction mapping principle, we will be able to prove well-posedness for this initial boundary value problem.

The second initial boundary problem considered in this research is the following:

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0, L), \quad t \geq 0, \\ u(x, 0) = u_0(x), \\ u_{xx}(0, t) = h_1(t), \quad u_x(L, t) = h_2(t), \quad u_{xx}(L, t) = h_3(t), \end{cases}$$

this problem will be studied from the control point of view. Basically, we are going to study whether the solution of the problem can be driven from a given initial state ($u(x, 0) = u_0(x)$) to a given final state ($u(x, T) = u_T(x)$) by using appropriated control inputs. For this problem the control inputs are the functions h_1, h_2 and h_3. Once we have proven well-posedness in the space $L^2(0, L)$, we consider a linearized system around the origin and the linear system will be proven to be exactly boundary controllable using one, two or three boundary inputs; moreover, the nonlinear system is shown to be locally exactly boundary controllable via the contraction mapping principle if the associated linearized system is exactly controllable. In addition to these mathematical results and according to the relevance of the KdV equation, we are going to provide a briefly, but not less important, historical review about this equation in the following section.

1.1 The Korteweg-de Vries Equation "A Historical review"

The history of the KdV equation begins with the Scottish engineer and naval architect John Scott Russell (1808 - 1882). During the 19th century, the study of water waves was of great interest due to the applications for naval architecture and engineering. In 1834 while observing a canal boat at the Edinburgh-Glasgow canal, Russell made a remarkable discovery that gave birth to the modern study of solitons, a wave moving in front of the canal boat, a particular one that kept his attention, in his own words:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of a vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

After this observation, Russell built water tanks in his home to study this phenomenon, trying to simulate this "Wave of Translation" in two different ways, either by using a weight in the left side of the tank, or by releasing some cumulation of water with the help of a floodgate located in the left side of the tank (see Figure 1.1).

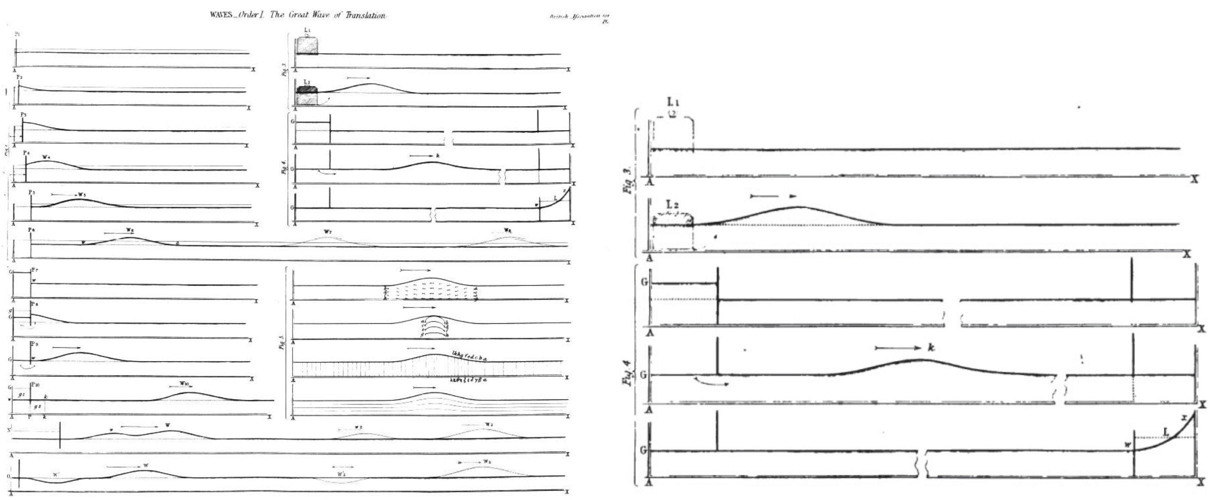

Figure 1.1: Russell's experimentation.

After his experimentation, Russell concluded:

- The waves are stable and can travel over very large distances.

- The speed depends on the size of the wave and its width on the depth of water.

- Unlike normal waves they will never merge.

- If a wave is too big for the depth of water, it splits into two, one big and one small.

- When two moved in the same direction, and the large one overtook the slower, smaller wave ahead of it, a nonlinear iteration occurred, after which both waves returned to their original shape.

The conclusions obtained by Russell were basically in conflict with the wave theory established at the time, and it challenged the theories of Newton and Bernoulli in hydrodynamics. Mathematician and astronomer Sir George Biddell Airy argued that long waves in a canal with rectangular cross section must necessarily change their form as time passes. George Gabriel Stokes believed that the only permanent wave should be basically sinusoidal and in 1849 published a "proof" that such a wave could not exist (he later retracted). Even with all these objections some researchers kept working with this challenge, Joseph Boussinesq was the first one that developed a mathematical theory to support Russell's observation, in 1871 he found a partial differential equation admitting a solitary wave solution. In 1876, Lord Rayleigh an English physicist obtained a different equation allowing the existence of solitary waves and finally in 1895 the Dutch mathematicians D. J. Korteweg and G. de Vries derived and published [40] a model equation for the motion of waves on the surface of a layer of fluid above a flat bottom, the Korteweg-de Vries equation (KdV equation):

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right), \tag{1.2}$$

here η is the surface elevation above the equilibrium level h, α a constant related to the uniform motion of the liquid, g is the gravitational acceleration, and $\sigma = h^3/3 - Th/\rho g$, with surface capillary tension T and density ρ. By using the transformations

$$t' = \frac{1}{2}\sqrt{\frac{g}{h\sigma}}t, \quad x' = -\frac{x}{\sqrt{\sigma}}, \quad u = -\frac{1}{2}\eta - \frac{1}{3}\alpha$$

we obtain the standard KdV equation

$$u_t + 6uu_x + u_{xxx} = 0.$$

The importance of the KdV equation was relegated for several decades and was just an insignificant part of the nonlinear wave theory during this time. In 1965, N. Zabuski and M. Kruskal at the Plasma Physics Laboratory in Princeton University demonstrated that certain solutions of the Fermi-Pasta-Ulam Lattice equations could be described in terms of the solutions of the KdV equation and they used the term soliton for the first time to describe these particle-like solitary waves. After this discovery, the nonlinear wave equations that had soliton solutions became an important field of research in both pure and applied mathematics. Several methods have been discovered while investigating the KdV equation.

In 1967, Gardner, Greene, Kruskal and Miura [49] developed a method for exactly solving the initial value problem of the KdV equation, the *Inverse Scattering Transform*, which is known as the nonlinear Fourier transform. Peter Lax [47] in 1968, developed a mathematical framework to apply the inverse-scattering transform to solve initial-value problems for partial differential equations. The KdV-equation has become the source of important breakthroughs in mechanics and nonlinear analysis and of many developments in algebra, analysis, geometry and physics. Among other applications this equation can be used as a model to study surface water waves, acoustic-gravity waves in a compressible heavy fluid, axisymmetric waves in rubber cords, hydromagnetic waves in cold plasma and recently as a model to study blood pressure waves in large arteries [24]-[25], [43], [62].

1.2 The KdV equation

1.2.1 The KdV equation in Eulerian Coordinates.

The KdV equation is a hyperbolic partial differential equation that can be used to model long water waves in a shallow canal with a rectangular cross section and air above it. In order to derive this equation we will use the basic equations of fluid mechanics (these equations are derived from the conservation of laws of mass, momentum and energy, see [18]) given by

$$\partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \tag{1.3}$$

$$\rho(\partial_t + \vec{u} \cdot \nabla)\vec{u} = -\nabla p + \vec{f} \tag{1.4}$$

and the following assumptions:

- The length of the canal is far greater than the width.
- The friction for the fluid along the boundaries of the canal is neglected.
- The flow has no viscosity (inviscid flow).

- The fluid is incompressible, homogeneous and irrotational.

For the equations of fluid mechanics, ρ is density, $\vec{u}$ the velocity of the fluid, p is the internal pressure and $\vec{f}$ is the external force density. The last three assumptions in mathematical terms are equivalent to

$$\partial_t \rho = 0, \qquad \nabla \rho = 0 \quad \text{and} \quad \nabla \times \vec{u} = 0.$$

The last equation allows us to consider the velocity in terms of some potential ϕ, that is, $\vec{u} = \nabla\phi$, by using the conditions given by incompressibility, irrotationality and (1.3), we have

$$\nabla \cdot \vec{u} = \nabla^2 \phi = \Delta\phi = 0.$$

Since we are considering an inviscid incompressible fluid (water) in a constant gravitational field, (x, y, z) as the space coordinates and the components of the velocity vector $\vec{u}$ by (u, v, w), we have $\vec{f} = -\rho g \vec{k}$, where g is the gravitational acceleration and $\vec{k}$ is the unit vector in the z direction, then we have the set of equations

$$\nabla \cdot \vec{u} = 0 \tag{1.5}$$

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} = \frac{1}{\rho}\nabla p - g\vec{k}, \tag{1.6}$$

using the identity

$$\vec{u} \times (\nabla \times \vec{u}) = \frac{1}{2}\nabla\big(\|\vec{u}\|^2\big) - (\vec{u} \cdot \nabla)\vec{u}$$

we have

$$\partial_t \vec{u} + \frac{1}{2}\nabla\big(\|\vec{u}\|^2\big) = \frac{1}{\rho}\nabla p - g\vec{k}.$$

Since $\vec{u} = \nabla\phi$, we can take the integral to the previous equation and obtain

$$\frac{p - p_0}{\rho_0} = b(t) - \phi_t - \frac{1}{2}\|\nabla\phi\|^2 - gz.$$

Here $b(t)$ is the constant of integration and p_0 is an arbitrary constant taken from $b(t)$. In order to apply some surface conditions, notice that we can eliminate $b(t)$ by choosing a new potential $\phi' = \phi - \int b(t)dt$, therefore

$$\frac{p - p_0}{\rho} = -\phi_t - \frac{1}{2}\|\nabla\phi\|^2 - gz.$$

Now, if the interface between air and water is described by $f(x, y, z, t) = 0$ and since the interface is defined by the property that fluid does not cross it, then we have that the normal velocity of the the surface must be equal to the normal velocity of the fluid, thus

$$\frac{-f_t}{\sqrt{f_x^2 + f_y^2 + f_z^2}} = \frac{uf_x + vf_y + wf_z}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

which implies

$$f_t + uf_x + vf_y + wf_z = 0.$$

If the surface is given by $z = \eta(x, y, t)$, then we can choose f as, $f(x, y, z, t) = \eta(x, y, t) - z$ and so we obtain the boundary condition

$$\eta_t + u\eta_x + v\eta_y = w. \tag{1.7}$$

Now, neglecting the motion of the air, we can assume, $p = p_0$, where p is the pressure in the water and p_0 corresponds to the pressure in the undisturbed air. Under this frame, we have the following two boundary conditions at the free surface

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z,$$

$$\phi_t + \frac{1}{2}\left(\|\nabla\phi\|^2\right) + g\eta = 0,$$

on $z = \eta(x, y, t)$. On a solid fixed boundary, the normal velocity of the fluid must vanish, therefore, $\vec{n} \cdot \nabla\phi = 0$, in particular, if the bottom is given by $z = -h_0(x, y)$, we have

$$\phi_z + \phi_x h_{0x} + \phi_y h_{0y} = 0. \tag{1.8}$$

Linearizing the free boundary conditions, we have

$$\eta_t = \phi_z, \qquad \phi_t + g\eta = 0.$$

Taking this linearization on $z = 0$, we have the following linear problem for ϕ

$$\begin{cases} \Delta\phi = 0, \qquad -h_0 < z < 0, \\ \phi_{tt} + g\phi_z = 0, \qquad y = 0, \\ \phi_z + h_{0x}\phi_x + h_{0y}\phi_y = 0, \qquad y = -h_0, \end{cases}$$

and once we have the solution ϕ of the previous system, the surface will be given by

$$\eta(x, y, t) = -\frac{1}{g}\phi_t(x, y, 0, t). \tag{1.9}$$

For water waves propagating horizontally, the elementary sinusoidal solutions are given by

$$\eta = Ae^{\mathbf{k}\cdot\mathbf{x}-i\omega t}, \qquad \phi = Z(z)e^{\mathbf{k}\cdot\mathbf{x}-i\omega t}.$$

From Laplace's equation, we have

$$Z'' - k^2 Z = 0, \qquad \text{where} \qquad k^2 = |\mathbf{k}|^2 = k_1^2 + k_2^2.$$

Now if the depth is constant ($z = -h_0$), then $Z'(z) = 0$, thus, Z is proportional to $\cosh k(h_0 + z)$. From (1.9), we obtain

$$A = \frac{i\omega}{g}Z(0)$$

and we can take

$$Z(z) = -\frac{ig}{\omega}A\frac{\cosh k(h_0 + z)}{\cosh kh_0}.$$

Then

$$\eta = Ae^{\mathbf{k}\cdot\mathbf{x}-i\omega t},$$

$$\phi = -\frac{ig}{\omega}A\frac{\cosh k(h_0 + z)}{\cosh kh_0}e^{\mathbf{k}\cdot\mathbf{x}-i\omega t},$$

the condition $\phi_{tt} + g\phi_z = 0$ on $z = 0$ gives the dispersion relation

$$\omega^2 = gk \tanh kh_0. \tag{1.10}$$

Now, approximating the vertical component of the momentum equation (1.6) by $\dfrac{\partial p}{\partial z} + \rho g = 0$ and integrating, we have

$$p - p_0 = \rho g(\eta - z),$$

with the above, the first two components of (1.6) become

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = -g\dfrac{\partial \eta}{\partial x}, \\ \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = -g\dfrac{\partial \eta}{\partial y}. \end{cases}$$

Notice that the right hand side of these equations is independent of z, so the rate of change of u and v are independent of z, which implies

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + g\dfrac{\partial \eta}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + g\dfrac{\partial \eta}{\partial y} = 0. \end{cases} \tag{1.11}$$

Integrating the conservation of mass equation (1.5), we have

$$\int_{-h_0}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0.$$

Since

$$\int_{-h_0}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = \frac{\partial}{\partial x} \int_{-h_0}^{\eta} u dz - u\Big|_{z=\eta} \frac{\partial \eta}{\partial x} - u\Big|_{z=h_0} \frac{\partial h_0}{\partial x} + \\ \frac{\partial}{\partial y} \int_{-h_0}^{\eta} v dz - v\Big|_{z=\eta} \frac{\partial \eta}{\partial y} - v\Big|_{z=h_0} \frac{\partial h_0}{\partial y} + w\Big|_{z=-h_0}^{z=\eta}$$

and using (1.7) and (1.8), we have

$$\frac{\partial}{\partial x} \int_{-h_0}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-h_0}^{\eta} v dz + \frac{\partial \eta}{\partial t} = 0$$

taking $h = h_0 + \eta$ and using the fact that u and v are independent of z, we have the conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{1.12}$$

Equations (1.11) and (1.12) are the shallow water equations for $\eta(\mathbf{x}, t)$ and $\vec{u}(\mathbf{x}, t)$, more precisely

$$\begin{cases} \vec{u}_t + \vec{u}\vec{u}_{\mathbf{x}} + gh_{\mathbf{x}} = 0, \\ h_t + (\vec{u}h)_{\mathbf{x}} = 0. \end{cases} \tag{1.13}$$

Let us consider the case of one dimensional waves with h_0 constant. Taking the derivative with respect to t of the second equation in (1.13) and using the first equation of this system, we can obtain the following equation

$$h_{tt} - (uu_x h - ghh_x - uh_t)_x = 0,$$

which is equivalent to

$$h_{tt} - (uu_x)_x h - uu_x h - g(hh_x)_x + (uh_t)_x = 0.$$

Linearizing, we have

$$h_{tt} - c_0^2 h_{xx} = 0,$$

and since $h = h_0 + \eta$,

$$\eta_{tt} - c_0^2 \eta_{xx} = 0.$$

In this approximation the dispersive effects do not appear, then we need to add a dispersive term $a\eta_{xxx}$ to the second equation in (1.13), to get the linearized system

$$\begin{cases} \eta_t + h_0 u_x = 0, \\ u_t + g\eta_x + a\eta_{xxx} = 0. \end{cases}$$

Eliminating u, we have

$$\eta_{tt} - c_0^2\eta_{xx} - ah_0\eta xxxx = 0.$$

Now, in order to obtain the dispersion relation (1.10), we can use the approximation

$$\omega^2 = c_0^2k^2 - \frac{1}{3}c_0^2h_0^2k^4.$$

Taking the corresponding derivatives and using the last equation, we can prove that $a = \dfrac{1}{3}c_0^2h_0$, therefore a more general system can be given by

$$\begin{cases} h_t + (uh)_x = 0, \\ u_t + uu_x + gh_x + \dfrac{1}{3}c_0^2h_0h_{xxx} = 0. \end{cases}$$

Taking the derivative with respect to t in the first equation and with respect to x in the second equation and combining the resulting equation, we obtain

$$h_{tt} - (u_x)^2h - uu_{xx}h - ghh_{xx} - \frac{1}{3}c_0^2h_0hh_{xxxx} + u_th_x + u_xh_t + uh_{tx} = 0.$$

Since $h_{tt} = c_0^2h_{xx}$, then

$$h_{tt} - (u_x)^2h - uu_{xx}h - ghh_{xx} - \frac{1}{3}h_0hh_{xxtt} + u_th_x + u_xh_t + uh_{tx} = 0,$$

and approximating

$$h_{tt} - ghh_{xx} - \frac{1}{3}h_0hh_{xxtt} = 0,$$

assuming $h \approx h_0$, we have

$$h_{tt} - c_0^2h_{xx} - \frac{1}{3}h_0^2h_{xxtt} = 0.$$

Finally using $\eta_{tt} - c_0^2\eta_{xx} = 0$ and $h = h_0 + \eta$, we obtain the well-known Boussinesq equation

$$\eta_{tt} - c_0^2\eta_{xx} - \frac{1}{3}c_0^2h_0^2\eta_{xxxx} = 0,$$

the dispersion relation for this equation is given by

$$\omega^2 = \frac{c_0^2k^2}{1 + (1/3)k^2h_0^2}.$$

This equation in particular includes waves moving to both left and right, for waves moving to the right the first two terms in the dispersion relation are

$$\omega = c_o k - \frac{1}{6} c_0 h_0^2 k^3.$$

Following a similar argument to the one used before, we can prove that this dispersion relation corresponds to the equation

$$\eta_t + c_0 \eta_x + \gamma \eta_{xxx} = 0, \tag{1.14}$$

where $\gamma = \frac{1}{6} c_0 h_0^2$. Now, waves moving to the right into undisturbed water of depth h_0 satisfy the Riemann invariant

$$u = 2\sqrt{g(h_0 + \eta)} - 2\sqrt{g h_0},$$

and using (1.13), we obtain

$$\eta_t + \left(3\sqrt{g(h_0 + \eta)} - 2\sqrt{g h_0}\right)\eta_x = 0.$$

Combining (1.14) and the last equation, we have

$$\eta_t + \left(3\sqrt{g(h_0 + \eta)} - 2\sqrt{g h_0}\right)\eta_x + \gamma \eta_{xxx} = 0.$$

Approximating the nonlinear terms to the first order, we will have one version of the KdV equation

$$\eta_t + c_0 \eta_x + \frac{3c_0}{2h_0} \eta \eta_x + \frac{1}{6} c_0 h_0^2 \eta_{xxx} = 0.$$

Setting

$$\bar{t} = \frac{\sqrt{6} c_0}{h_0} t, \qquad \bar{x} = \frac{\sqrt{6}}{h_0} x, \qquad \bar{\eta} = \frac{3}{2h_0} \eta,$$

we can get the KdV equation

$$\eta_t + \eta_x + \eta \eta_x + \eta_{xxx} = 0.$$

1.2.2 The KdV equation in Lagrangian coordinates.

In this subsection we are going to derive the KdV equation in Lagrangian coordinates. To achieve this goal we have to start by considering a Boussinesq system, which can be formulated as follows

$$\eta_t + [(1+\alpha\eta)u]_x = 0, \tag{1.15}$$

$$u_t + \alpha u u_x + \eta_x - \frac{1}{3}\beta u_{xxt} = 0. \tag{1.16}$$

In this set up:

- x is the Eulerian coordinate,
- t is the elapsed time,
- η is the deflection from rest position,
- u is the value averaged over the depth or the horizontal velocity,
- $\alpha = a/h_0$, $\beta = h_0^2/l^2$, where h_0 is the height of the surface fluid at rest, a is amplitude and l is wavelength.

Introducing the normalized height $h = 1 + \alpha\eta$ and using (1.15), we have

$$h_t + \alpha(hu)_x = 0. \tag{1.17}$$

Taking the integral over the interval (a, b), we obtain the relation

$$\frac{d}{dt}\int_a^b h(t,x)dx = \alpha u(a)h(a) - \alpha u(b)h(b).$$

Notice that $\alpha u(a)$ is the fluid velocity at $x = a$ and $\alpha u(b)$ the fluid velocity at $x = b$. Following the work of L. Rosier in [52], we are going to express the Boussinesq system in mass Lagrangian coordinates. Let us denote by ξ the Lagrangian coordinate, therefore

$\xi \in [0, L]$ and $x = x(\tau, \xi)$, is the position at time $t = \tau$ of the fluid particle taken from ξ at $t = 0$. This coordinate is obtained by solving the system

$$\begin{cases} \dfrac{dx}{dt} = \alpha u(t, x), \\ x|_{t=0} = \xi. \end{cases}$$

Now, let us write the Boussinesq system in terms of the variables τ and ξ. In order to do this, let us consider the transformation, $\psi : (\tau, \xi) \to (t, x) = (\tau, x(\tau, \xi))$. The Jacobian matrix of ψ is given by

$$J = J(\tau, \xi) = \begin{pmatrix} 1 & 0 \\ \alpha u(t, x) & \dfrac{\partial x}{\partial \xi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha u(t, x) & j \end{pmatrix}. \tag{1.18}$$

Assuming enough regularity, we have

$$\frac{\partial j}{\partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial x}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \tau}\right) = \frac{\partial}{\partial \xi}[\alpha u(\tau, x(\tau, \xi))] = \alpha u_x(\tau, x(\tau, \xi)) j \tag{1.19}$$

and

$$\frac{\partial}{\partial \tau}[h(\tau, x(\tau, \xi))] = h_t(\tau, x) + h_x(\tau, x)\alpha u(\tau, \xi).$$

Using the two previous relations and (1.17), we can obtain

$$\frac{\partial}{\partial \tau}(\ln |h|) = -\alpha u_x(\tau, x) = \frac{\partial}{\partial \tau}(\ln |j|).$$

If h_0 is given by $h_0 = h|_{t=0}$ and since $j(0, \xi) = 1$, we obtain

$$j(\tau, \xi) = j(0, \xi)\frac{h(0, \xi)}{h(\tau, x(\tau, \xi))} = \frac{h_0(\xi)}{h(t, x)}.$$

With this, we can write the Jacobian J and its inverse in the following way

$$J = \begin{pmatrix} 1 & 0 \\ \alpha u(t, x) & \dfrac{h_0(\xi)}{h(t, x)} \end{pmatrix} \quad \text{and} \quad J^{-1} = \begin{pmatrix} 1 & 0 \\ -\dfrac{h(t, x)}{h_0(\xi)}\alpha u(t, x) & \dfrac{h(t, x)}{h_0(\xi)} \end{pmatrix}.$$

The mass Lagrangian coordinate ζ is defined by

$$\zeta = \int_0^{\xi} h_0(\eta) d\eta.$$

Taking the partial derivative with respect to τ and with respect to x, we get respectively

$$\frac{\partial \zeta}{\partial t} = h_0(\xi)\frac{\partial \xi}{\partial t} = -\alpha u(t,x)h(t,x)$$

and

$$\frac{\partial \zeta}{\partial x} = h_0(\xi)\frac{\partial \xi}{\partial x} = h(t,x),$$

which implies

$$\partial_t = \partial_\tau + \frac{\partial \zeta}{\partial t}\partial_\zeta = \partial_\tau - \alpha u h \partial_\zeta,$$
$$\partial_x = \frac{\partial \zeta}{\partial x}\partial_\zeta = h\partial_\zeta.$$

By using the above operators (1.17) becomes

$$h_\tau - \alpha u h h_\zeta + \alpha h h_\zeta u + \alpha h^2 u_\zeta = 0.$$

Since $h = 1 + \alpha\eta$, after some calculations we have

$$\eta_\tau + (1 + 2\alpha\eta)u_\zeta + O(\alpha^2) = 0. \tag{1.20}$$

For the second equation in Boussinesq system, we have the following

$$\begin{aligned} u_{xxt} &= (h_x u_\zeta)_t + (h^2 u_{\zeta\zeta})_t = h_{xt}u_\zeta + h_x(u_\zeta)_t + 2hh_t u_{\zeta\zeta} + h^2(u_{\zeta\zeta})_t \\ &= h_{xt}u_\zeta + h_x(u_{\zeta\tau} - \alpha u h u_{\zeta\zeta}) + 2h(h_\tau - \alpha u h h_\zeta)u_{\zeta\zeta} + h^2(u_{\zeta\zeta\tau} - \alpha u h u_{\zeta\zeta\zeta}). \end{aligned}$$

Since

$$h_{xt} = (hh_\zeta)_t = [h_\tau - \alpha u h h_\zeta]h_\zeta + h[h_{\zeta\tau} - \alpha u h h_{\zeta\zeta}],$$

we have

$$\begin{aligned} u_{xxt} = {} & h^2(u_{\zeta\zeta\tau} - \alpha u h u_{\zeta\zeta\zeta}) + 2h(h_\tau - \alpha u h h_\zeta)u_{\zeta\zeta} + \\ & hh_\zeta(u_{\zeta\tau} - \alpha u h u_{\zeta\zeta}) + [(h_\tau - \alpha u h h_\zeta)h_\zeta + h(h_{\zeta\tau} - \alpha u h h_{\zeta\zeta})]u_\zeta. \end{aligned}$$

Now, by using (1.16) and the previous calculations, we obtain

$$
\begin{aligned}
u_\tau - \alpha uhu_\zeta + \alpha uhu_\zeta + h\eta_\zeta - \frac{1}{3}\beta\{h^2(u_{\zeta\zeta\tau} - \alpha uhu_{\zeta\zeta\zeta}) + 2h(\alpha\eta_\tau - \alpha^2 uh\eta_\zeta)u_{\zeta\zeta} \\
+ h\alpha\eta_\zeta(u_{\zeta\tau} - \alpha uhu_{\zeta\zeta}) + [\alpha\eta_\zeta(\alpha\eta_\tau - \alpha^2 uh\eta_\zeta) + (\alpha\eta_{\zeta\tau} - \alpha^2 uh\eta_{\zeta\zeta})h]u_\zeta\} = 0.
\end{aligned}
$$

After some straightforward calculations and some simplifications, we have

$$
u_\tau + (1+\alpha\eta)\eta_\zeta - \frac{1}{3}\beta u_{\zeta\zeta\tau} + O(\alpha\beta, \alpha^2) = 0. \tag{1.21}
$$

According to (1.20) and (1.21), the Boussinesq system in mass Lagrangian coordinates to the first order is given by

$$
\begin{cases}
\eta_\tau + (1+2\alpha\eta)u_\zeta = 0, \\
u_\tau + (1+\alpha\eta)\eta_\zeta - \frac{1}{3}\beta u_{\zeta\zeta\tau} = 0.
\end{cases} \tag{1.22}
$$

Boussinesq's system includes waves moving to both left and right. However, to derive the KdV equation in mass Lagrangian coordinates we have to restrict to a wave moving to the right. Following the approach given in [61], we have to consider a solution u in the form

$$
u = \eta + \alpha A + \beta B + O(\alpha^2 + \beta^2), \tag{1.23}
$$

where A and B are functions of η and its ζ derivatives. With the previous assumption, the first and second equation in the Boussinesq system in Lagrangian coordinates become respectively

$$
\eta_\tau + (1+2\alpha\eta)(\eta_\zeta + \alpha A_\zeta + \beta B_\zeta + O(\alpha^2+\beta^2)) = 0
$$

and

$$
\eta_\tau + \alpha A_\tau + \beta B_\tau + (1+\alpha\eta)\eta_\zeta - \frac{1}{3}\beta(\eta_{\zeta\zeta\tau} + \alpha A_{\zeta\zeta\tau} + \beta B_{\zeta\zeta\tau} + O(\alpha^2+\beta^2)) = 0,
$$

therefore the system (1.22) becomes

$$
\begin{cases}
\eta_\tau + \eta_\zeta + \alpha(2\eta\eta_\zeta + A_\zeta) + \beta B_\zeta + O(\alpha^2+\beta^2) = 0, \\
\eta_\tau + \eta_\zeta + \alpha(\eta\eta_\zeta + A_\tau) + \beta\Big(B_\tau - \frac{1}{3}\eta_{\zeta\zeta\tau}\Big) + O(\alpha^2+\beta^2) = 0.
\end{cases}
$$

The equations in this system are consistent if

$$2\eta\eta_\zeta + A_\zeta = \eta\eta_\zeta + A_\tau + O(\alpha,\beta), \qquad B_\zeta = B_\tau - \frac{1}{3}\eta_{\zeta\zeta\tau} + O(\alpha,\beta),$$

hence, $A_\tau = -A_\zeta + O(\alpha,\beta)$ and $B_\tau = -B_\zeta + O(\alpha,\beta)$. Since $\eta_\tau = -\eta_\zeta + O(\alpha,\beta)$, we can write

$$A_\zeta = -\frac{1}{2}\eta\eta_\zeta + O(\alpha,\beta) \quad \text{and} \quad B_\zeta = \frac{1}{6}\eta\zeta\zeta\zeta + O(\alpha,\beta),$$

which implies

$$A = -\frac{1}{4}\eta^2 \quad \text{and} \quad B = \frac{1}{6}\eta_{\zeta\zeta}.$$

With the previous analysis, we have that the KdV system in mass Lagrangian coordinates is given by

$$\begin{cases} \eta_\tau + \eta_\zeta + \dfrac{3}{2}\alpha\eta\eta_\zeta + \dfrac{1}{6}\beta\eta_{\zeta\zeta\zeta} = 0, \\ u = \eta - \dfrac{1}{4}\alpha\eta^2 + \dfrac{1}{6}\beta\eta_{\zeta\zeta}. \end{cases}$$

By considering the variables

$$\bar{\zeta} = l\zeta, \qquad \bar{\eta} = a\eta, \qquad \bar{\tau} = \frac{l}{c_0}\tau, \qquad \text{and} \qquad \bar{u} = \frac{ga}{c_0}u,$$

where $c_0 = \sqrt{gh_0}$ is the sound speed in the fluid, we have

$$\eta_\tau = \frac{l}{ac_0}\bar{\eta}_{\bar{\tau}}, \quad \eta_\zeta = \frac{l}{a}\bar{\eta}_{\bar{\zeta}}, \quad \eta\eta_\zeta = \frac{l}{ah_0}\bar{\eta}\bar{\eta}_{\bar{\zeta}} \quad \text{and} \quad \frac{1}{6}\beta\eta_{\zeta\zeta\zeta} = \frac{h_0^2 l}{6a}\bar{\eta}_{\bar{\zeta}\bar{\zeta}\bar{\zeta}}$$

and the KdV system in mass Lagrangian coordinates becomes

$$\begin{cases} \bar{\eta}_{\bar{\tau}} + c_0\bar{\eta}_{\bar{\zeta}} + \dfrac{3c_0}{2h_0}\bar{\eta}\bar{\eta}_{\bar{\zeta}} + \dfrac{1}{6}c_0 h_0^2\bar{\eta}_{\bar{\zeta}\bar{\zeta}\bar{\zeta}} = 0, \\ \\ \bar{u} = \dfrac{g}{c_0}\left(\bar{\eta} - \dfrac{\bar{\eta}^2}{4h_0} + \dfrac{1}{6}h_0^2\bar{\eta}_{\bar{\zeta}\bar{\zeta}}\right). \end{cases}$$

Finally, setting

$$t = \frac{\sqrt{6}c_0}{h_0}\bar{\tau}, \quad x = \frac{\sqrt{6}}{h_0}\bar{\zeta}, \quad y = \frac{3}{2h_0}\bar{\eta} \quad \text{and} \quad v = \frac{3c_0}{2gh_0}\bar{u},$$

we obtain the system

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \\ v = y - \dfrac{1}{6}y^2 + y_{xx}. \end{cases}$$

1.3 Statement of Results

This research focuses on two initial boundary value problems on a bounded domain related with the Korteweg-de Vries (KdV) equation. The approach here is from two different perspectives: well-posedness and controllability. We are going to study a class of initial-boundary-value problem (IBVP) of the KdV equation posed on a finite domain $(0, L)$

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad \text{for } x \in (0, L) \text{ and } t > 0, \tag{1.24}$$

subject to the initial condition

$$u(x, 0) = \phi(x), \quad \text{for} \quad x \in (0, L) \tag{1.25}$$

and the nonhomogeneous nonlinear boundary conditions

$$u_{xx}(0, t) + u(0, t) - \frac{1}{6}u^2(0, t) = h(t), \quad u(L, t) = 0, \quad u_x(L, t) = 0, \quad \text{for} \quad t \geq 0, \tag{1.26}$$

where the initial value ϕ and the boundary data h are given functions. The IBVP (1.24)–(1.26) was derived by Rosier [52] as a model to investigate the motion of water waves in a long canal with a moving boundary at the left of the canal (wavemaker) and a fixed boundary at the right by using Lagrangian coordinates. It has been studied by Rosier [52] exclusively from the control point of view. In particular, viewing (1.24)-(1.26) as a distributed parameter control system with boundary value function $h(t)$ as a control input, Rosier investigated its controllability:

What waves can be generated by the wavemaker?

He provided the following answer:

Theorem 1.3.1. *(Rosier [52]) Let $T > 0$ be given and let*

$$\bar{y} \in C^0([0,T], H^3(0,L)) \cap C^1([0,T], L^2(0,L)) \cap H^1(0,T,H^1(0,L))$$

be a function satisfying

$$\begin{cases} \bar{y}_t + \bar{y}_x + \bar{y}\bar{y}_x + \bar{y}_{xxx} = 0, \quad \text{for} \quad 0 < x < L, \quad 0 < t < T, \\ \bar{y}|_{x=L} = \bar{y}_x|_{x=L} = 0. \end{cases}$$

Then there exists a number $r_0 > 0$ such that for any initial state $y_0 \in H^3(0,L)$ satisfying

$$y_0(L) = y_0'(L) = 0, \quad \|y_0 - \bar{y}(0)\|_{H^3(0,L)} < r_0,$$

there exists a control input $h \in C^0([0,T])$ such that the following IBVP

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \quad 0 < x < L, \quad 0 < t < T, \\ (y - \frac{1}{6}y^2 + y_{xx})|_{x=0} = h, \\ y|_{x=L} = 0, \qquad y_x|_{x=L} = 0, \\ y(0) = y_0, \end{cases} \tag{1.27}$$

possesses a solution $y \in L^2(0,T,H^3(0,L)) \cap H^1(0,T,H^1(0,L))$ satisfying

$$y(\cdot, T) = \bar{y}(\cdot, T).$$

In other words, any smooth trajectory of the KdV equation is locally reachable in finite time by choosing an appropriate boundary control input $h(t)$. To prove his result, Rosier considered the system

$$\begin{cases} w_t + w_x + ww_x + w_{xxx} = 0, \quad 0 < x < L, \quad 0 < t < T, \\ w|_{x=L} = 0, \qquad w_x|_{x=L} = 0, \\ w|_{t=0} = y_0. \end{cases} \tag{1.28}$$

He showed that under the assumptions of his theorem, the system (1.28) admits a solution $w \in L^2(0,T,H^3(0,L)) \cap H^1(0,T,H^1(0,L))$, satisfying

$$w(\cdot,T) = \bar{y}(\cdot,T).$$

Choosing

$$h(t) = w_{xx}(0,t) - \frac{1}{6}w^2(0,t) + w(0,t),$$

then $y(x,t) \equiv w(x,t)$ is a desired solution of (1.27). However, while the system (1.27) has been shown to be locally controllable in certain sense, there is still an elementary but important issue yet to be addressed:

Is the IBVP (1.27) well-posed in the sense of Hadamard?

More precisely, given an initial data ϕ in the space $H^s(0,L)$ and a boundary data h in certain appropriate space $H^{s'}(0,T)$, does the IBVP (1.27) admit a unique solution in the space $C([0,T];H^s(0,L))$? How does the solution depend on its initial value ϕ and boundary data h if it exists? In this paper, we will fill this gap by showing that the IBVP (1.27) is locally well-posed in the space $H^s(0,L)$ for any $s \geq 0$. The following theorem is our main result:

Theorem 1.3.2. *Let $T > 0$, $0 \leq s \leq 3$ and $\gamma > 0$ be given. There exists a $T^* \in (0,T]$ such that for any $\phi \in H^s(0,L)$ and $h \in H^{\frac{s-1}{3}}(0,T)$ satisfying*

$$\text{(1.29)} \qquad \begin{cases} \phi(L) = 0, & \text{if } \frac{1}{2} < s \leq 3, \\ \phi'(L) = 0, & \text{if } \frac{3}{2} < s \leq 3, \\ \phi''(0) - \frac{1}{6}\phi^2(0) + \phi(0) = h(0), & \text{if } \frac{5}{2} < s \leq 3, \end{cases}$$

and

$$\text{(1.30)} \qquad \|\phi\|_{H^s(0,L)} + \|h\|_{H^{\frac{s-1}{3}}(0,T)} \leq \gamma,$$

the IBVP (1.24)-(1.26) admits a unique solution

$$u \in C([0,T^*];H^s(0,L)) \cap L^2(0,T^*;H^{s+1}(0,L)).$$

Moreover, the corresponding solution map is Lipschitz continuous and the solution possesses the hidden regularities (the sharp Kato smoothing properties)

$$\partial_x^k u \in L_x^\infty(0,L;H^{\frac{s+1-k}{3}}(0,T^*)) \text{ for } k = 0,1,2.$$

The results in our theorem can be easily extended to any $s > 3$.

Beginning with the work of Bubnov [12, 13] in the late 1970s, the two-point boundary value problem of the KdV equation has been intensively studied (cf. [4, 6, 19, 20, 21, 27, 26, 28, 44, 45, 46] and the references therein) following the rapid advances of the study of the pure initial value problems of the equation posed either on $\mathbb{R}$ or on a bounded domain with periodic boundary conditions [8, 7, 35, 36, 37, 38, 39]. In particular, the following IBVP of the KdV equation posed on a bounded interval $(0,L)$,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0,L), \quad t \geq 0, \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{cases} \tag{1.31}$$

has been shown by Bona, Sun and Zhang [4] to be locally well-posed in the space $H^s(0,L)$ for any $s \geq 0$.

Theorem 1.3.3. *(Bona, Sun and Zhang) Let $T > 0$ and $s \geq 0$ be given. There exists a $r > 0$ such that for any given $s-$compatible*

$$\phi \in H^s(0,L), \quad h_1, h_2 \in H^{(s+1)/3}(0,T), \quad h_3 \in H^{s/3}(0,T)$$

satisfying

$$\|\phi\|_{H^s(0,L)} + \|h_1\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|_{H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (1.31) admits a unique solution $u \in C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$ *and, moreover, the corresponding solution map is analytically continuous.*

Later on this well-posedness result in the space $H^s(0,L)$ was extended to the case of $s > -3/4$ by Holmer [33], and then by Bona, Sun and Zhang [6], for $s > -1$. The proof of Theorem 1.3.3 was based on:

i. The Kato smoothing property of the associated linear IBVP

$$(1.32) \qquad \begin{cases} u_t + u_x + u_{xxx} = f, \quad u(x,0) = \phi(x), \qquad x \in (0,L), \qquad t \in (0,T), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0. \end{cases}$$

For any $\phi \in L^2(0,L)$ and $f \in L^2(0,T; L^2(0,L))$, the corresponding solution u of (1.32) not only belongs to the space $C([0,T]; L^2(0,L))$, but also belongs to the space $L^2(0,T; H^1(0,L))$. It is this Kato smoothing property that make it possible to establish the well-posedness of the nonlinear IBVP (1.31) in the space $H^s(0,L)$ via the contraction mapping principle.

ii. The explicit integral representation of the solution of the IBVP

$$(1.33) \qquad \begin{cases} u_t + u_x + u_{xxx} = 0, \quad u(x,0) = 0, \qquad x \in (0,L), \qquad t \in (0,T), \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{cases}$$

in terms of the boundary data $\vec{h} := (h_1, h_2, h_3)$:

$$u(x,t) = W_{bdr}(t)\vec{h},$$

where $W_{bdr}(t)$ is now called the boundary integral operator associated to the IBVP (1.33). It is this explicit integral representation of the boundary integral operator responsible for obtaining the well-posedness of the IBVP (1.31) in the space $H^s(0,L)$ with the boundary data $\vec{h}$ assuming the optimal regularities

$$h_1, \; h_2 \in H^{\frac{s+1}{3}}(0,T), \quad h_3 \in H^{\frac{s}{3}}(0,T).$$

We will prove Theorem 1.3.2 using the same approach as that developed by Bona, Sun and Zhang in [4] in proving Theorem 1.3.3. However, due to the presence of the nonlinear boundary condition, the Kato smoothing property is not strong enough to enable us to establish the well-posedness of the IBVP (1.24)-(1.26) via the contraction mapping principle. Instead, a so-called hidden regularity (also known as the sharp Kato smoothing property) of the following linear IBVP associated to the nonlinear IBVP (1.24)-(1.26)

$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = \phi(x), \quad x \in (0,L), \quad t \in (0,T), \\ u_{xx}(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \end{cases} \tag{1.34}$$

is needed. More precisely, we need to show that for any $\phi \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$, the solution $u \in C([0,T];L^2(0,L))$ of (1.34) possesses the hidden regularities,

$$\partial_x^k u \in L_x^\infty(0,L;H^{\frac{1-k}{3}}(0,T)) \text{ for } k = 0,1,2. \tag{1.35}$$

But it seems very difficult, if it not impossible, to show that (1.35) holds for solutions of (1.34) directly using energy estimates method. Instead of using this, we will invoke some harmonic analysis tools developed in the study of the pure initial value problems of the KdV equation. Consideration will be first given to the pure initial value problem of the linear KdV equation on the whole line $\mathbb{R}$,

$$u_t + u_{xxx} = g, \quad u(x,0) = \psi(x), \quad x,\, t \in \mathbb{R}. \tag{1.36}$$

Its solution u can be written as

$$u = W_{\mathbb{R}}(t)\psi + \int_0^t W_{\mathbb{R}}(t-\tau)g(\tau)d\tau,$$

where $W_{\mathbb{R}}(t)$ is the C_0-semigroup in the space $L^2(\mathbb{R})$ associated to (1.36), and is well-known to possess the sharp Kato smoothing property,

$$\partial_x^k u \in L_x^\infty(\mathbb{R};H^{\frac{1-k}{3}}(\mathbb{R})) \text{ for } k = 0,1,2,$$

whenever $\psi \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}, L^2(\mathbb{R}))$. Then we will turn to consider the IBVP

$$\left\{ \begin{array}{l} u_t + u_{xxx} = 0, \quad u(x,0) = 0, \quad x \in (0,L), \quad t \geq 0, \\ u_{xx}(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{array} \right. \tag{1.37}$$

An explicit integral representation of its solution in terms of boundary data $h_1,\ h_2,\ h_3$ will be derived and be denoted by

$$u(x,t) = W_{bdr}(t)\vec{h}, \qquad \vec{h} = (h_1, h_2, h_3).$$

In terms of the operators $W_{bdr}(t)$ and $W_{\mathbb{R}}(t)$, the solution u of (1.34) can be written as

$$u(x,t) = W_{\mathbb{R}}(t)\tilde{\phi} + \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau - W_{bdr}(t)(\vec{q} + \vec{p})$$

where $\tilde{\phi}$ and $\tilde{f}$ are extension of ϕ and f from $(0,L)$ to $\mathbb{R}$, respectively, and $\vec{q} = (q_1, q_2, q_3)$, $\vec{p} = (p_1, p_2, p_3)$ with

$$q_1(t) = \partial_x^2 W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=0}, \quad q_2(t) = W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=L}, \quad q_3(t) = \partial_x W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=L},$$

$$p_1(t) = \partial_x^2 \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=0}, \qquad p_2(t) = \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=L}$$

and

$$p_3 = \partial_x \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=L}.$$

Thus as long as we can show that the solution $u = W_{bdr}(t)\vec{h}$ of the IBVP (1.34) possesses the hidden regularity (1.35) whenever

$$h_1 \in H^{-\frac{1}{3}}(0,T), \quad h_2 \in H^{\frac{1}{3}}(0,T), \quad h_3 \in L^2(0,T),$$

then the solution of the IBVP

$$\left\{ \begin{array}{l} u_t + u_{xxx} = f, \quad u(x,0) = \phi, \quad x \in (0,L), \quad t \geq 0, \\ u_{xx}(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{array} \right. \tag{1.38}$$

will possess the hidden regularities (1.35) as long as $\phi \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and

$$h_1 \in H^{-\frac{1}{3}}(0,T), \quad h_2 \in H^{\frac{1}{3}}(0,T), \quad h_3 \in L^2(0,T).$$

With the help of the hidden regularities of the associated linear problem, the well-posedness of the nonlinear IBVP will be established via the contraction mapping principle.

The second part of this research is dedicated to the class of distributed parameter control systems described by the KdV equation posed on a bounded domain with nonhomogeneous Neumann boundary conditions

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L). \end{cases} \tag{1.39}$$

We will prove well-posedness for the IBVP (1.39) in the space $L^2(0,L)$ by using the aforementioned procedure. The main theorem related to well-posedness for system (1.39) is the following:

Theorem 1.3.4. *Let $T>0$ be given. For any $u_0 \in L^2(0,L)$ and*

$$\vec{h} := (h_1,h_2,h_3) \in \mathcal{H}_T := H^{-\frac{1}{3}}(0,T)\times L^2(0,T)\times H^{-\frac{1}{3}}(0,T),$$

the IBVP (1.39) admits a unique solution

$$u \in \mathcal{Z}_T := C([0,T];L^2(0,L)) \cap L^2(0,T,H^1(0,L)).$$

Moreover, there exists a positive constant C, such that

$$||u||_{\mathcal{Z}_T} \le C\left(||u_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T}\right).$$

In addition, the solution u possesses the following sharp trace estimates

$$\sup_{x\in(0,L)} ||\partial_x^r u(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \le C_r\left(||u_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T}\right), \tag{1.40}$$

for $r = 0,1,2$.

In addition to well-posedness, we are going to study system (1.39) from the control point of view.

Exact control problem: Given $T > 0$ and $u_0, u_T \in L^2(0,L)$. Is it possible to find appropriate control inputs h_j, $j = 1,2,3$, such that the corresponding solution u of (1.39) satisfies

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x)?$$

The study of the controllability and stabilization of the KdV equation started with the work of Russell and Zhang in [57], for a system with periodic boundary conditions and an internal control. Since then, both controllability and stabilization have been intensively studied (we refer the reader to [54] for a survey of results and [15] to a detailed presentation about control). In particular, the exact boundary controllability of the KdV equation on a finite domain was investigated in [14, 16, 23, 30, 31, 51, 52, 67]. The majority of these articles are concerned with the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u(0,t) = g_1(t),\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{1.41}$$

here, the boundary data g_1, g_2, g_3 can be chosen as control inputs. System (1.41) was first studied by Rosier [51] considering only the control input g_3 (i.e. $g_1 = g_2 = 0$). It was shown in [51] that the exact controllability of the linearized system holds in $L^2(0,L)$ if and only if L does not belong to the following countable set of critical lengths

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2 + kl + l^2} \ : k,\ l\ \in \mathbb{N}^* \right\}. \tag{1.42}$$

The analysis developed in [51] shows that if the linearized system is controllable, then the nonlinearized system is controllable as well. Notice that the converse is false, as it was proven in [14, 16, 23], that is, the (nonlinear) KdV equation is controllable even when L is a critical length, but the linearized system is not controllable.

Recently, Cerpa *et al.* in [17] proved similar results to those obtained by Rosier [51] for the system

$$
(1.43) \qquad \begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ y(0,t) = k_1(t),\ y_x(L,t) = k_2(t),\ y_{xx}(L,t) = k_3(t), & \text{in } (0,T), \\ y(x,0) = y_0(x), & \text{in } (0,L). \end{cases}
$$

The authors considered the system with one, two or three control inputs, and using the well-posedness results provided by Kramer *et al.* in [41] (see also [50]), they proved that the linear system associated with (1.43) is locally exactly controllable if and only if L does not belong to the following countable set of critical lengths

(1.44)

$$
\mathcal{F} := \left\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a,b\in\mathbb{C} \text{ satisfying } \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \right\}.
$$

Moreover, by using the contraction mapping principle, they showed that the nonlinear system (1.43) is locally exactly controllable.

The second aim here is to determine if system (1.39) possesses controllability results similar to those established for systems (1.41) and (1.43). It is natural to think of using the same approaches that have been effective for systems (1.41) and (1.43). However, these approaches will be difficult in our case and other tools will be required, specifically, we will apply the tools used in [17]. When we use only h_2 as a control input, the linear system associated to (1.39) is given by

$$
(1.45) \qquad \begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases}
$$

and its adjoint system by

(1.46)
$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ \psi(0,t) + \psi_{xx}(0,t) = 0,\ \psi_x(0,t) = 0,\ \psi(L,t) + \psi_{xx}(L,t) = 0, & t \in (0,T), \\ \psi(x,T) = \psi_T(x), & x \in (0,L). \end{cases}$$

It is well known that the exact controllability of system (1.45) is equivalent to the following observability inequality for the adjoint system (1.46):

$$||\psi_T||_{L^2(0,L)} \leq C||\psi_x(L,\cdot)||_{L^2(0,T)}. \tag{1.47}$$

However, the usual multiplier method and compactness argument, as used to deal with the control of system (1.45) only lead to the inequality

$$||\psi_T||^2_{L^2(0,L)} \leq C_1||\psi_x(L,\cdot)||^2_{L^2(0,T)} + C_2||\psi(0,\cdot)||^2_{L^2(0,T)}. \tag{1.48}$$

The issue now is how to remove the extra term in (1.48). To address this, the new approach used in [17] will play a crucial role in proving the observability inequality (1.47). This new approach turns out to be the hidden regularity (or the sharp Kato smoothing property) for solutions of the KdV equation. Specifically, we will prove the following result:

Theorem 1.3.5. *[Hidden regularities] For any $\psi_T \in L^2(0,L)$, the solution ψ of the IBVP* (1.46) *belongs to the space $\mathcal{Z}_T$ and possess the following sharp trace properties*

$$\sup_{x\in(0,L)} ||\partial_x^r \psi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r||\psi_0||_{L^2(0,L)}, \tag{1.49}$$

for $r = 0, 1, 2$.

Using h_2 as a control input, we will prove that system (1.39) is locally exactly controllable as long as $L \notin \mathcal{M}$, where $\mathcal{M}$ is defined as

$$\mathcal{M} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}\ :\ k,\ l\ \in \mathbb{N}^* \right\} \cup \{k\pi\ :\ k \in \mathbb{N}^*\} = \mathcal{N} \cup \{k\pi\ :\ k \in \mathbb{N}^*\}.$$

Theorem 1.3.6. *Let $T > 0$ and $L \notin \mathcal{M}$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0, L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_2 \in L^2(0, T)$ such that the system (1.39) *admits a unique solution*

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \qquad u(x, T) = u_T(x).$$

If we consider h_3 as a control input, system (1.39) is locally exactly controllable as well, but with a new set of critical lengths

(1.50)

$$\mathcal{R} = \Big\{L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} : X = e^a, Y = e^b \text{are solutions of}$$
$$AX^2 + BX + C = 0 \text{ and } Y = -\tfrac{b_3 + b_1 X}{b_2}\Big\}.$$

Here $A = a_1 b_1$, $B = a_1 b_3 - a_2 b_2 + a_3 b_1$ and $C = a_3 b_3$, where

$$a_1 := (b^2 - a^2)(a + b), \quad a_2 := b^2 e^c(2a + b), \quad a_3 := -a^2 e^c(a + 2b),$$

$$b_1 := -a^3(a + 2b), \quad b_2 := ab^2(2a + b) \text{ and } b_3 := -ae^c(b^2 - a^2)(a + b).$$

The theorem in this case is the following:

Theorem 1.3.7. *Let $T > 0$ and $L \notin \mathcal{R}$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0, L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_3 \in H^{-\frac{1}{3}}(0, T)$ such that the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0, T) \times (0, L), \\ u_{xx}(0, t) = 0,\ u_x(L, t) = 0,\ u_{xx}(L, t) = h_3(t), & \text{in } (0, T), \\ u(x, 0) = u_0(x), & \text{in } (0, L), \end{cases} \tag{1.51}$$

admits a unique solution $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$, *satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

Now, if $h_1 = 0$ and h_2, h_3 are the control inputs, system (1.39) is locally exactly controllable

Theorem 1.3.8. *Let* $T > 0$ *and* $L > 0$ *be given. There exists* $\delta > 0$ *such that for any* $u_0, u_T \in L^2(0,L)$ *with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_2 \in L^2(0,T)$ *and* $h_3 \in H^{-\frac{1}{3}}(0,T)$ *such that the system* (1.39) *admits a unique solution*

$$u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$$

satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

A similar result will be established if we consider h_1 and h_3 as control inputs. Finally, if we consider the three control inputs (h_1, h_2 and h_3), then system (1.39) is locally exactly controllable around any smooth solution of the KdV equation. More precisely

Theorem 1.3.9. *Let* $T > 0$ *and* $L > 0$ *be given. Assume that* $y \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ *satisfies*

$$y_t + y_x + yy_x + y_{xxx} = 0 \quad (x,t) \in \mathbb{R} \times \mathbb{R}.$$

Then, there exists $\delta > 0$ *such that for any* $y_0, y_T \in L^2(0,L)$ *with*

$$||u_0 - y(\cdot,0)||_{L^2(0,L)} + ||u_T - y(\cdot,T)||_{L^2(0,L)} \leq \delta,$$

one can find

$$h_1 \in H^{-\frac{1}{3}}(0,T), \;\; h_2 \in L^2(0,T), \;\; h_3 \in H^{-\frac{1}{3}}(0,T)$$

such that the system (1.39) *admits a unique solution*

$$u \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^1(0,L))$$

satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

The previous results are going to be established initially for the systems linearized around the origin by using the classical duality approach, more precisely, the Hilbert Uniqueness Method (H.U.M) introduced by J. L. Lions in [48]. This method reduces the proof of the exact controllability for (1.39) to proving an observability inequality for the solution of the adjoint system. To prove the observability inequality, we will use the compactness uniqueness argument developed by E. Zuazua in [48]. The exact controllability is extended to the nonlinear system by using the contraction mapping principle.

CHAPTER 2

Well-posedness of a non-linear boundary value problem for the Korteweg-de Vries equation posed on a finite domain

Considered in this article is an initial-boundary-value problem (IBVP) for the Korteweg-de Vries equation

$$u_t + u_x + uu_x + u_{xxx} = 0,$$

posed on a finite interval $I = (0, L)$ subject to the initial condition

$$u(x, 0) = \phi(x), \quad \text{for} \quad x \in (0, L),$$

and the nonhomogeneous nonlinear boundary conditions

$$u_{xx}(0, t) + u(0, t) - \frac{1}{6}u^2(0, t) = h(t), \quad u(L, t) = 0, \quad u_x(L, t) = 0, \quad \text{for} \quad t \geq 0,$$

which was derived by Rosier [52] as a model to investigate the motion of water waves in a long canal with a moving boundary at the left of the canal (wavemaker) and a fixed boundary at the right by using Lagrangian coordinates. It is shown here, using the hidden regularities (or sharp Kato smoothing properties) of the associated linear problem, that the IBVP is well-posed in the space $H^s(0, L)$ for any $s \geq 0$ via the contraction mapping principle and thus addresses a question left open by Rosier in [52].

2.1 Introduction

In this paper, we study a class of initial-boundary-value problem (IBVP) of the Korteweg-de Vries (KdV) equation posed on a finite domain $(0, L)$

$$u_t + u_x + uu_x + u_{xxx} = 0, \quad \text{for } x \in (0, L) \text{ and } t > 0, \tag{2.1}$$

subject to the initial condition

$$u(x, 0) = \phi(x), \quad \text{for} \quad x \in (0, L), \tag{2.2}$$

and the nonhomogeneous nonlinear boundary conditions

$$u_{xx}(0, t) + u(0, t) - \frac{1}{6}u^2(0, t) = h(t), \quad u(L, t) = 0, \quad u_x(L, t) = 0, \quad \text{for} \quad t \geq 0, \tag{2.3}$$

where the initial value ϕ and the boundary data h are given functions.

The IBVP (2.1)–(2.3) was derived by Rosier [52] as a model to investigate the motion of water waves in a long canal with a moving boundary at the left of the canal (wavemaker) and a fixed boundary at the right by using Lagrangian coordinates. It has been studied by Rosier [52] exclusively from the control point of view, the readers are referred to [66, 55, 56, 57, 67, 51, 23, 14, 30, 16, 31, 50, 53, 54, 68, 15] and the references therein for more studies of the KdV equation from the control point of view. In particular, viewed (2.1)-(2.3) as a distributed parameter control system with boundary value function $h(t)$ as a control input, Rosier investigated its controllability:

What waves can be generated by the wavemaker?

He provided the following answer [52]:

Theorem 2.1.1. *Let $T > 0$ be given and let*

$$\bar{y} \in C^0([0, T], H^3(0, L)) \cap C^1([0, T], L^2(0, L)) \cap H^1(0, T, H^1(0, L))$$

be a function satisfying

$$\begin{cases} \bar{y}_t + \bar{y}_x + \bar{y}\bar{y}_x + \bar{y}_{xxx} = 0, \quad \textit{for} \quad 0 < x < L, \quad 0 < t < T, \\ \bar{y}|_{x=L} = \bar{y}_x|_{x=L} = 0. \end{cases}$$

Then there exists a number $r_0 > 0$ such that for any initial state $y_0 \in H^3(0, L)$ satisfying

$$y_0(L) = y_0'(L) = 0, \quad \|y_0 - \bar{y}(0)\|_{H^3(0,L)} < r_0,$$

there exists a control input $h \in C^0([0, T])$ such that the following IBVP

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \quad 0 < x < L, \quad 0 < t < T, \\ (y - \frac{1}{6}y^2 + y_{xx})|_{x=0} = h, \\ y|_{x=L} = 0, \qquad y_x|_{x=L} = 0, \\ y(0) = y_0, \end{cases} \tag{2.4}$$

possesses a solution $y \in L^2(0, T, H^3(0, L)) \cap H^1(0, T, H^1(0, L))$ satisfying

$$y(\cdot, T) = \bar{y}(\cdot, T).$$

In other words, any smooth trajectory of the KdV equation is locally reachable in finite time by choosing an appropriate boundary control input $h(t)$.

To prove his result, Rosier considered the system

$$\begin{cases} w_t + w_x + ww_x + w_{xxx} = 0, \quad 0 < x < L, \quad 0 < t < T, \\ w|_{x=L} = 0, \qquad w_x|_{x=L} = 0, \\ w|_{t=0} = y_0, \end{cases} \tag{2.5}$$

instead of studying the system (2.4) directly. He showed that under the assumptions of Theorem 2.1.1, the system (2.5) admits a solution

$$w \in L^2(0, T, H^3(0, L)) \cap H^1(0, T, H^1(0, L))$$

satisfying

$$w(\cdot, T) = \bar{y}(\cdot, T).$$

Choosing

$$h(t) = w_{xx}(0,t) - \frac{1}{6}w^2(0,t) + w(0,t),$$

then $y(x,t) \equiv w(x,t)$ is a desired solution of (2.4) in Theorem 2.1.1. What Rosier has established in [52] implies, in fact, the null controllability of the boundary control system

$$\begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, \quad y(x,0) = y_0(x), \qquad 0 < x < L, \qquad 0 < t < T, \\ y|_{x=0} = h, \qquad y|_{x=L} = 0, \qquad y_x|_{x=L} = 0, \end{cases}$$

which has been studied further by Glass and Guerrero in [31]. However, while the system (2.4) has been shown to be locally controllable in a certain sense, there is still an elementary but important issue yet to be addressed:

Is the IBVP (2.4) well-posed in the sense of Hadamard?

More precisely, given an initial data ϕ in the space $H^s(0,L)$ and a boundary data h in a certain appropriate space, $H^{s'}(0,T)$, does the IBVP (2.4) admit a unique solution in the space $C([0,T]; H^s(0,L))$? How does the solution depend on its initial value ϕ and boundary data h if it exists?

In this paper, we will fill this gap to show the IBVP (2.4) is locally well-posed in the space $H^s(0,L)$ for any $0 \le s \le 3$.[1] The following theorem is our main result.

Theorem 2.1.2. *Let $T > 0$, $0 \le s \le 3$ and $\gamma > 0$ be given. There exists a $T^* \in (0,T]$ such that for any $\phi \in H^s(0,L)$ and $h \in H^{\frac{s-1}{3}}(0,T)$ satisfying*

$$\begin{cases} \phi(L) = 0, & \text{if} \quad \frac{1}{2} < s \le 3, \\ \phi'(L) = 0, & \text{if} \quad \frac{3}{2} < s \le 3, \\ \phi''(0) - \frac{1}{6}\phi^2(0) + \phi(0) = h(0), & \text{if} \quad \frac{5}{2} < s \le 3, \end{cases} \tag{2.6}$$

and

$$\|\phi\|_{H^s(0,L)} + \|h\|_{H^{\frac{s-1}{3}}(0,T)} \le \gamma, \tag{2.7}$$

[1] Our results can be easily extended to the case for any $s > 3$.

the IBVP (2.1) - (2.3) admits a unique solution

$$u \in C([0,T^*]; H^s(0,L)) \cap L^2(0,T^*; H^{s+1}(0,L)).$$

Moreover, the corresponding solution map is Lipschitz continuous and the solution possesses the hidden regularities (the sharp Kato smoothing properties)

$$\partial_x^k u \in L_x^\infty(0,L; H^{\frac{s+1-k}{3}}(0,T^*)) \text{ for } k = 0,1,2.$$

Beginning with the work of Bubnov [12, 13] in the late 1970s, the two-point boundary value problems of the KdV equation has been intensively studied (cf. [4, 6, 19, 20, 21, 27, 26, 28, 44, 45, 46] and the references therein) following the rapid advances of the study of the pure initial value problems for the Kdv equation posed either on $\mathbb{R}$ or on a bounded domain with periodic boundary conditions [8, 7, 35, 36, 37, 38, 39]. In particular, the following IBVP of the KdV equation posed on a bounded interval $(0,L)$,

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, \quad x \in (0,L), \quad t \geq 0, \\ u(x,0) = \phi(x), \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{cases} \tag{2.8}$$

has been shown by Bona, Sun and Zhang [4] to be locally well-posed in the space $H^s(0,L)$ for any $s \geq 0$.

Theorem 2.1.3. *(Bona, Sun and Zhang) Let $T > 0$ and $s \geq 0$ be given. There exists a $r > 0$ such that for any given $s-$compatible*

$$\phi \in H^s(0,L), \quad h_1, h_2 \in H^{(s+1)/3}(0,T), \quad h_3 \in H^{s/3}(0,T)$$

satisfying

$$\|\phi\|_{H^s(0,L)} + \|h_1\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_2\|_{H^{\frac{s+1}{3}}(0,T)} + \|h_3\|_{H^{\frac{s}{3}}(0,T)} \leq r,$$

the IBVP (2.8) admits a unique solution $u \in C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$ and, moreover, the corresponding solution map is analytically continuous.

Later on, this well-posedness result in the space $H^s(0,L)$ was extended to the case of $s > -3/4$ by Holmer [33], and then by Bona, Sun and Zhang [6], for $s > -1$. The proof of Theorem 2.1.3 was based on:

i. The Kato smoothing property of the associated linear IBVP

$$\begin{cases} u_t + u_x + u_{xxx} = f, \quad u(x,0) = \phi(x), \qquad x \in (0,L), \qquad t \in (0,T), \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0. \end{cases} \tag{2.9}$$

For any $\phi \in L^2(0,L)$ and $f \in L^2(0,T;L^2(0,L))$, the corresponding solution u of (2.9) not only belongs to the space $C([0,T];L^2(0,L))$, but also belongs to the space $L^2(0,T;H^1(0,L))$. It is this Kato smoothing property that makes it possible to establish the well-posedness of the nonlinear IBVP (2.8) in the space $H^s(0,L)$ via the contraction mapping principle.

ii. The explicit integral representation of the solution of the IBVP

$$\begin{cases} u_t + u_x + u_{xxx} = 0, \quad u(x,0) = 0, \qquad x \in (0,L), \qquad t \in (0,T), \\ u(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t) \end{cases} \tag{2.10}$$

in terms of the boundary data $\vec{h} := (h_1, h_2, h_3)$:

$$u(x,t) = W_{bdr}(t)\vec{h},$$

where $W_{bdr}(t)$ is now called the boundary integral operator associated to the IBVP (2.10). It is this explicit integral representation of the boundary integral operator responsible for obtaining the well-posedness of the IBVP (2.8) in the space $H^s(0,L)$ with the boundary data $\vec{h}$ assuming the optimal regularities

$$h_1, \ h_2 \in H^{\frac{s+1}{3}}(0,T), \quad h_3 \in H^{\frac{s}{3}}(0,T).$$

We will prove Theorem 2.1.2 using the same approach as that developed by Bona, Sun and Zhang in [4] in proving Theorem 2.1.3. However, due to the presence of the nonlinear boundary condition, the Kato smoothing property is not strong enough to enable us to establish the well-posedness of the IBVP (2.1)-(2.3) via the contraction mapping principle. Instead, a so-called hidden regularity (also known as the sharp Kato smoothing property) of the following linear IBVP associated to the nonlinear IBVP (2.1)-(2.3)

$$\begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = \phi(x), \quad x \in (0,L), \quad t \in (0,T), \\ u_{xx}(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \end{cases} \tag{2.11}$$

is needed. More precisely, we need to show that for any $\phi \in L^2(0,L)$, $f \in L^1(0,T; L^2(0,L))$, the solution $u \in C([0,T]; L^2(0,L))$ of (2.11) possesses the hidden regularities,

$$\partial_x^k u \in L_x^\infty(0,L; H^{\frac{1-k}{3}}(0,T)) \text{ for } k = 0,1,2. \tag{2.12}$$

But it seems very difficult, if it not impossible, to show that (2.12) holds for solutions of (2.11) directly using energy estimates methods. To get around this, we will invoke some harmonic analysis tools developed in the study of the pure initial value problems of the KdV equation. Consideration will be first given to the pure initial value problem of the linear KdV equation on the whole line $\mathbb{R}$,

$$u_t + u_{xxx} = g, \quad u(x,0) = \psi(x), \quad x,\, t \in \mathbb{R}. \tag{2.13}$$

Its solution u can be written as

$$u = W_{\mathbb{R}}(t)\psi + \int_0^t W_{\mathbb{R}}(t-\tau)g(\tau)d\tau,$$

where $W_{\mathbb{R}}(t)$ is the C_0-semigroup in the space $L^2(\mathbb{R})$ associated to (2.13), and is well-known to possess the sharp Kato smoothing property

$$\partial_x^k u \in L_x^\infty(\mathbb{R}; H^{\frac{1-k}{3}}(\mathbb{R})) \text{ for } k = 0,1,2,$$

whenever $\psi \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R}, L^2(\mathbb{R}))$. Then we will turn to consider the IBVP

$$\begin{cases} u_t + u_{xxx} = 0, \quad u(x,0) = 0, \quad x \in (0,L), \quad t \geq 0, \\ u_{xx}(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t). \end{cases} \tag{2.14}$$

An explicit integral representation of its solution in terms of boundary data $h_1,\ h_2,\ h_3$ will be derived and be denoted by

$$u(x,t) = W_{bdr}(t)\vec{h}, \qquad \vec{h} = (h_1, h_2, h_3).$$

In terms of the operators $W_{bdr}(t)$ and $W_{\mathbb{R}}(t)$, the solution u of (2.11) can be written as

$$u(x,t) = W_{\mathbb{R}}(t)\tilde{\phi} + \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau - W_{bdr}(t)(\vec{q}+\vec{p})$$

where $\tilde{\phi}$ and $\tilde{f}$ are extension of ϕ and f from $(0,L)$ to $\mathbb{R}$, respectively, and $\vec{q} = (q_1, q_2, q_3)$, $\vec{p} = (p_1, p_2, p_3)$ with

$$q_1(t) = \partial_x^2 W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=0}, \quad q_2(t) = W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=L}, \quad q_3(t) = \partial_x W_{\mathbb{R}}(t)\tilde{\phi}\Big|_{x=L},$$

$$p_1(t) = \partial_x^2 \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=0}, \qquad p_2(t) = \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=L}$$

and

$$p_3 = \partial_x \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau\Big|_{x=L}.$$

Thus as long as we can show that the solution $u = W_{bdr}(t)\vec{h}$ of the IBVP (2.11) possesses the hidden regularity (2.12) whenever

$$h_1 \in H^{-\frac{1}{3}}(0,T), \quad h_2 \in H^{\frac{1}{3}}(0,T), \quad h_3 \in L^2(0,T),$$

then the solution of the IBVP

$$(2.15) \qquad \begin{cases} u_t + u_{xxx} = f, \quad u(x,0) = \phi, \quad x \in (0,L), \quad t \geq 0, \\ u_{xx}(0,t) = h_1(t), \quad u(L,t) = h_2(t), \quad u_x(L,t) = h_3(t), \end{cases}$$

will possess the hidden regularities (2.12) as long as $\phi \in L^2(0,L)$, $f \in L^1(0,T; L^2(0,L))$ and

$$h_1 \in H^{-\frac{1}{3}}(0,T), \quad h_2 \in H^{\frac{1}{3}}(0,T), \quad h_3 \in L^2(0,T).$$

With the help of the hidden regularities of the associated linear problem, the well-posedness of the nonlinear IBVP will be established via the contraction mapping principle.

The paper is organized as follows: In Section 2, we will first derive an explicit integral representation of the boundary integral operator $W_{bdr}(t)$ associated to the IBVP (2.14). Then various estimates will be established for solutions of the IBVP (2.15) including the hidden regularities (2.12). The proof of our main result in this paper, Theorem 2.1.2, will be presented in section 3.

2.2 Linear problems

Consideration is first given to the following IBVP of the linear KdV equation with homogenous initial value and nonhomogenoeus boundary data

$$\begin{cases} w_t(x,t) + w_{xxx}(x,t) = 0, \quad x \in (0,L), \quad t \geq 0, \\ w(x,0) = 0, \\ w_{xx}(0,t) = h_1(t), \quad w(L,t) = h_2(t), \quad w_x(L,t) = h_3(t). \end{cases} \tag{2.16}$$

Following the approach developed in [3, 4], we derive an explicit formula for its solution in terms of the boundary values h_1, h_2 and h_3. Without loss of generality, we assume here that $L = 1$.

Applying the Laplace transform with respect to t, the IBVP (2.16) becomes

$$\begin{cases} s\hat{w}(x,s) + \hat{w}_{xxx}(x,s) = 0, \\ \hat{w}_{xx}(0,s) = \hat{h}_1(s), \quad \hat{w}(1,s) = \hat{h}_2(s), \quad \hat{w}_x(1,s) = \hat{h}_3(s), \end{cases}$$

where

$$\hat{w}(x,s) = \int_0^\infty e^{-st} w(x,t)dt$$

and

$$\hat{h}_j(s) = \int_0^\infty e^{-st} h_j(t)dt, \quad j = 1, 2, 3.$$

The solution $\hat{w}(x,s)$ can be written in the form

$$\hat{w}(x,s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s),\ j = 1, 2, 3$, are the three solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and the constants $c_j = c_j(s),\ j = 1, 2, 3$, solve the linear system

$$\begin{cases} c_1\lambda_1^2(s) + c_2\lambda_2^2(s) + c_3\lambda_3^2(s) = \hat{h}_1(s), \\ c_1 e^{\lambda_1(s)} + c_2 e^{\lambda_2(s)} + c_3 e^{\lambda_3(s)} = \hat{h}_2(s), \\ c_1\lambda_1(s)e^{\lambda_1(s)} + c_2\lambda_2(s)e^{\lambda_2(s)} + c_3\lambda_3(s)e^{\lambda_3(s)} = \hat{h}_3(s). \end{cases}$$

Let $\Delta(s)$ be the determinant of the coefficient matrix and $\Delta_j(s)$ be the determinants of the matrices that are obtained by replacing the jth-column of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s))^T$, $j = 1, 2, 3$. By Cramer's rule

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \qquad j = 1, 2, 3,$$

if $\Delta(s) \neq 0$. Taking the inverse Laplace transform of $\hat{w}$ we have

$$w(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{w}(x,s)ds = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)}e^{\lambda_j(s)x}ds$$

for any $r > 0$. Using the same arguments as those in [4] the solution $w(x,t)$ can be written as

$$w(x,t) = \sum_{m=1}^{3} w_m(x,t) \tag{2.17}$$

with

$$w_m(x,t) = \sum_{j=1}^{3} w_{j,m}(x,t) \quad \text{and} \quad w_{j,m}(x,t) = w_{j,m}^+(x,t) + w_{j,m}^-(x,t) \tag{2.18}$$

where for $m,\ j = 1, 2, 3$,

$$w_{j,m}^+(x,t) = \frac{1}{2\pi}\int_0^\infty e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta_{j,m}^+(\rho)}{\Delta^+(\rho)}\hat{h}_m^+(\rho)3\rho^2 d\rho, \quad w_{j,m}^-(x,t) = \overline{w_{j,m}^+(x,t)} \tag{2.19}$$

and

$$\hat{h}_m^+(\rho) = \hat{h}_m(i\rho^3),\ \Delta^+(\rho) = \Delta(i\rho^3),\ \Delta_{j,m}^+(\rho) = \Delta_{j,m}(i\rho^3),\ \lambda_j^+(\rho) = \lambda_j(i\rho^3). \tag{2.20}$$

Note that $w_m(x,t)$ solves the IBVP (2.16) with $h_j \equiv 0$ when $j \neq m$, $j, m = 1, 2, 3$.

Next we turn to estimate the solution $w(x,t)$ of the IBVP (2.16). The following technical lemma due to Bona, Sun and Zhang [3, 4] is needed, this lemma plays a similar role as the Plancherel theorem in estimating $w(x,t)$.

Lemma 2.2.1. *For any $f \in L^2(\mathbb{R}^+)$, let Kf be the function defined by*

$$Kf(x) = \int_0^\infty e^{\gamma(\mu)x} f(\mu) d\mu$$

where $\gamma(\mu)$ is a continuous complex-valued function defined on $(0,\infty)$ satisfying the following two conditions:

1. *There exists $\delta > 0$ and $b > 0$ such that*

$$\sup_{0<\mu<\delta} \frac{|Re\gamma(\mu)|}{\mu} \geq b;$$

2. *There exists a complex number $\alpha + i\beta$ such that*

$$\lim_{\mu\to\infty} \frac{\gamma(\mu)}{\mu} = \alpha + i\beta.$$

Then there exists a constant C such that for all $f \in L^2(0,\infty)$,

$$\|Kf\|_{L^2(0,1)} \leq C(\|e^{Re\gamma(\cdot)} f(\cdot)\|_{L^2(\mathbb{R}^+)} + \|f(\cdot)\|_{L^2(\mathbb{R}^+)}).$$

The following proposition presents some estimates for w_1, w_2 and w_3.

Proposition 2.2.2. *Let $T > 0$ and $0 \le s \le 3$ be given. For any given*

$$h_1 \in H_0^{\frac{s-1}{3}}(\mathbb{R}^+), \quad h_2 \in H_0^{\frac{s+1}{3}}(\mathbb{R}^+), \quad h_3 \in H_0^{\frac{s}{3}}(\mathbb{R}^+),$$

we have

$$w_j \in C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$$

and

$$\partial_x^k w_j \in C_x([0,L]; H^{\frac{s+1-k}{3}}(0,T))$$

for $j = 1,2,3,$. Moreover, there exists a constant C such that, for $k = 0,1,2,$

$$\|w_1\|_{C([0,T];H^s(0,L))} + \|w_1\|_{L^2(0,T;H^{s+1}(0,L))} + \|\partial_x^k w_1\|_{L_x^\infty(0,L;H^{\frac{s+1-k}{3}}(0,T))} \le C\|h_1\|_{H^{\frac{s-1}{3}}(\mathbb{R}^+)}, \tag{2.21}$$

$$\|w_2\|_{C([0,T];H^s(0,L))} + \|w_2\|_{L^2(0,T;H^{s+1}(0,L))} + \|\partial_x^k w_2\|_{L_x^\infty(0,L;H^{\frac{s+1-k}{3}}(0,T))} \le C\|h_2\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \tag{2.22}$$

and

$$\|w_3\|_{C([0,T];H^s(0,L))} + \|w_3\|_{L^2(0,T;H^{s+1}(0,L))} + \|\partial_x^k w_3\|_{L_x^\infty([0,L];H^{\frac{s+1-k}{3}}(0,T))} \le C\|h_3\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}. \tag{2.23}$$

Proof. We only prove the Proposition for w_1; the proofs for w_2 and w_3 are similar. Note that

$$\lambda_1^+(\rho) = i\rho, \quad \lambda_2^+(\rho) = \frac{1}{2}\rho(\sqrt{3} - i), \quad \lambda_3^+(\rho) = \frac{1}{2}\rho(-\sqrt{3} - i),$$

$$\Delta^+(\rho) = \sqrt{3}\rho^3 e^{-i\rho} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(\sqrt{3}-i)} + \sqrt{3}\rho^3 e^{-\frac{1}{2}\rho(-\sqrt{3}-i)},$$

$$\Delta_{1,1}^+(\rho) = -\sqrt{3}\rho e^{-i\rho}, \quad \Delta_{2,1}^+(\rho) = \frac{1}{2}\rho(\sqrt{3} + 3i)e^{-\frac{1}{2}\rho(\sqrt{3}-i)}$$

and

$$\Delta_{3,1}^+(\rho) = \frac{1}{2}\rho(\sqrt{3} - 3i)e^{\frac{1}{2}\rho(\sqrt{3}+i)}.$$

Thus, as $\rho \to \infty$,

$$\frac{\Delta_{1,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}}, \quad \frac{\Delta_{2,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{1}{2}\rho\sqrt{3}}, \quad \frac{\Delta_{3,1}^+(\rho)}{\Delta^+(\rho)} \sim \rho^{-2},$$

By (3.27) and (2.18),

$$w_1(x,t) = w_1^+(x,t) + \overline{w_1^+(x,t)}$$

with

$$w_1^+(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho) d\rho.$$

It suffices to only estimate $w_1^+(x,t)$. Note that

$$\partial_x^3 w_1^+(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi} \int_0^\infty e^{i\rho^3 t} e^{\lambda_j^+(\rho)x} (\lambda_j^+(\rho))^3 \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho) d\rho.$$

Applying Lemma 2.2.1 yields that there exists a constant $C > 0$ such that for any $t > 0$,

$$\begin{aligned}
\|w_1^+(\cdot,t)\|_{L^2(0,L)}^2 &\le C \sum_{j=1}^{3} \int_0^\infty \left| \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{\mathrm{Re}\lambda_j^+(\rho)} + 1\right)^2 |3\rho^2 \hat{h}_1^+(\rho)|^2 d\rho \\
&\le C \int_0^\infty |\hat{h}_1^+(\rho)|^2 d\rho \le C \int_0^\infty \left| \int_0^\infty e^{-i\rho^3\tau} h_1(\tau) d\tau \right|^2 d\rho
\end{aligned}$$

(by letting $\mu = \rho^3$)

$$\le C \int_0^\infty \mu^{-2/3} \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\mu \le C \|h_1\|_{H^{-1/3}(\mathbb{R}^+)}^2$$

and

$$\begin{aligned}
\|\partial_x^3 w_1^+(\cdot,t)\|_{L^2(0,L)}^2 &\le C \sum_{j=1}^{3} \int_0^\infty \left| \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} \right|^2 \left(e^{\mathrm{Re}\lambda_j^+(\rho)} + 1\right)^2 |\lambda_j^+(\rho)|^6 |\rho^2 \hat{h}_1^+(\rho)|^2 d\rho \\
&\le C \int_0^\infty |\lambda_j^+(\rho)|^6 |\hat{h}_1^+(\rho)|^2 d\rho \\
&\le C \int_0^\infty \mu^{4/3} \left| \int_0^\infty e^{-i\mu\tau} h_1(\tau) d\tau \right|^2 d\mu \le C \|h_1\|_{H^{2/3}(\mathbb{R}^+)}^2.
\end{aligned}$$

We thus have

$$\sup_{0\le t\le T} \|w_1^+(\cdot,t)\|_{L^2(0,L)} \le C \|h_1\|_{H^{-1/3}(\mathbb{R}^+)}$$

and

$$\sup_{0\le t\le T} \|w_1^+(\cdot,t)\|_{H^3(0,L)} \le C \|h_1\|_{H^{2/3}(\mathbb{R}^+)}.$$

By interpolation, for $0 \le s \le 3$,

$$\sup_{0\le t\le T} \|w_1^+(\cdot,t)\|_{H^s(0,L)} \le C\|h_1\|_{H^{(s-1)/3}(\mathbb{R}^+)}.$$

In addition, for $k = 0, 1, 2$,

$$\begin{aligned}
\partial_x^k w_1^+(x,t) &= \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^\infty e^{i\rho^3 t}(\lambda_j^+(\rho))^k e^{\lambda_j^+(\rho)x}\frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)} 3\rho^2 \hat{h}_1^+(\rho)d\rho \\
&= \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^\infty e^{i\mu t}(\lambda_j^+(\theta(\mu)))^k e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1(i\mu)d\mu,
\end{aligned}$$

where $\theta(\mu)$ is the real solution of $\mu = \rho^3$ for $\rho \ge 0$. Applying Plancherel's Theorem (with respect to t), yields that for any $x \in (0, L)$, $0 \le s \le 3$, and $k = 0, 1, 2$,

$$\begin{aligned}
\|\partial_x^k w_1^+(x,\cdot)\|^2_{H^{\frac{s+1-k}{3}}(0,T)} &\le \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} \left|(\lambda_j^+(\theta(\mu)))^k e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \left|\hat{h}_1(i\mu)\right|^2 d\mu \\
&\le C\sum_{j=1}^{3} \int_0^\infty \left|(\lambda_j^+(\rho))^k \frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}\right|^2 \left|\hat{h}_1^+(\rho)\right|^2 \rho^{2s+4-2k} d\rho \\
&\le C\int_0^\infty \rho^{2s}|\hat{h}_1^+(\rho)|^2 d\rho \\
&\le C\|h_1\|^2_{H^{(s-1)/3}(\mathbb{R}^+)}.
\end{aligned}$$

Consequently, for $0 \le s \le 3$ and $k = 0, 1, 2$,

$$\sup_{0<x<L} \|\partial_x^k w_1^+(x,\cdot)\|_{H^{\frac{s+1-k}{3}}(0,T)} \le C\|h_1\|_{H^{(s-1)/3}(\mathbb{R}^+)}.$$

To prove the continuity of $\partial_x^k w_1^+$, from $(0, L)$ to the space $H^{\frac{s+1-k}{3}}(0,T)$, choose any $x_0 \in (0, L)$, $x \in (0, L)$ and note that

$$\partial_x^k w_1^+(x,t) - \partial_x^k w_1(x_0,t) = \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^\infty e^{i\mu t}(\lambda_j^+(\theta(\mu)))^k \left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1(i\mu)d\mu.$$

Applying Plancherel's Theorem with respect to t

$$\begin{aligned}
&\|\partial_x^k w_1^+(x,t) - \partial_x^k w_1^+(x_0,t)\|^2_{H^{\frac{s+1-k}{3}}(0,T)} \le \\
&\qquad \sum_{j=1}^{3} \frac{1}{2\pi}\int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} \left|\lambda_j^+(\theta(\mu))\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \left|\hat{h}_1(i\mu)\right|^2 d\mu,
\end{aligned}$$

Arguing as before,

$$\|\partial_x^k w_1^+(x,t) - \partial_x^k w_1^+(x_0,t)\|^2_{H^{\frac{s+1-k}{3}}(0,T)} \leq C\int_0^\infty \mu^{(2s-2)/3}\big|\hat{h}_1(i\mu)\big|^2 d\mu,$$

and then by Fatou's lemma, we can conclude

$$\lim_{x\to x_0} \|\partial_x^k w_1^+(x,t) - \partial_x^k w_1^+(x_0,t)\|^2_{H^{\frac{s+1-k}{3}}(0,T)} \leq$$
$$\sum_{j=1}^3 \frac{1}{2\pi}\int_0^\infty |\mu|^{\frac{2(s+1-k)}{3}} \left|\lambda_j^+(\theta(\mu)) \lim_{x\to x_0}\left(e^{\lambda_j^+(\theta(\mu))x} - e^{\lambda_j^+(\theta(\mu))x_0}\right)\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\right|^2 \big|\hat{h}_1(i\mu)\big|^2 d\mu = 0.$$

□

Let $T>0$ and $0\leq s\leq 3$ be given. Let

$$S(0,T) = \{g\in C^\infty[0,T]: \quad g'(0)=0\}, \quad \mathbb{S}(0,T) := S(0,T)\times S(0,T)\times S(0,T).$$

Note that $\mathbb{S}(0,T)$ is a subspace of the space $H^{(s-1)/3}(0,T)\times H^{(s+1)/3}(0,T)\times H^{s/3}(0,T)$. Let $\mathbb{H}_0^s(0,T)$ be the closure of $\mathbb{S}(0,T)$ under the norm of the space

$$H^{(s-1)/3}(0,T)\times H^{(s+1)/3}(0,T)\times H^{s/3}(0,T).$$

In addition, let

$$Z_{s,T} := C([0,T];H^s(0,L))\cap L^2(0,T;H^{s+1}(0,L)).$$

Then Proposition 2.2.2 can be rewritten as the following proposition

Proposition 2.2.3. *Let $T>0$ and $0\leq s\leq 3$ be given. For any $\vec{h}\in\mathbb{H}_0^s(0,T)$, the IBVP (2.16) admits a unique solution*

$$w(x,t) := [W_{bdr}\vec{h}](x,t)$$

belonging to the space $Z_{s,T}$ with

$$\partial_x^k w \in C_b([0,L];H^{\frac{s+1-k}{3}}(0,T)) \ for\ k=0,1,2.$$

Moreover there exists a constant C such that

$$\|w\|_{Z_{s,T}} + \sum_{k=0}^{2} \|\partial_x^k w\|_{C_b([0,L];H^{\frac{s+1-k}{3}}(0,T))} \leq C\|\vec{h}\|_{\mathbb{H}_0^s(0,T)},$$

for all $\vec{h} \in \mathbb{H}_0^s(0,T)$.

Next we consider the linear IBVP of the linear KdV equation with nonhomogeneous initial value

$$\begin{cases} v_t(x,t) + v_{xxx}(x,t) = f, \quad x \in (0,L), \quad t \geq 0, \\ v(x,0) = \phi(x), \\ v_{xx}(0,t) = h_1(t), \quad v(L,t) = h_2(t), \quad v_x(L,t) = h_3(t). \end{cases} \tag{2.24}$$

By the standard semigroup theory, its solution when $h_1 = h_2 = h_3 \equiv 0$ can be written as

$$v(x,t) = W_0(t)\phi(x) + \int_0^t W_0(t-\tau)f(\tau)d\tau,$$

where $\{W_0(t)\}_{t\geq 0}$ is the C_0-semigroup in the space $L^2(0,L)$ generated by the operator A defined by

$$Aq = -q'''$$

with the domain

$$D(A) = \{\nu \in H^3(0,L) : \ \nu''(0) = \nu(L) = \nu'(L) = 0\}.$$

For any $\phi \in L^2(0,L)$ and $f \in L^1(0,T;L^2(0,L))$, the solution v of (2.24), which is given by

$$v(x,t) = W_0(t)\phi(x) + \int_0^t W_0(t-\tau)f(\tau)d\tau,$$

belongs to the space $C([0,T];L^2(0,L))$. We need to show that the solution v of (2.24) also possess the sharp Kato smoothing properties, which however, seems hard to establish by using the classical energy estimate method directly due to the presence of the boundary conditions even in the homogenous case ($h_1 = h_2 = h_3 \equiv 0$). Following [4], we first consider the pure initial value problem (IVP) of the KdV equation posed on the whole line $\mathbb{R}$:

$$\begin{cases} p_t + p_{xxx} = g, \quad x \in \mathbb{R}, \quad t \geq 0, \\ p(x,0) = \psi(x). \end{cases} \tag{2.25}$$

Its solution can be written as

$$p(x,t) = W_{\mathbb{R}}(t)\psi(x) + \int_0^t W_{\mathbb{R}}(t-\tau)g(\tau)d\tau$$

where $W_{\mathbb{R}}(t)$ is the C_0-semigroup associated with the IVP (2.25). The solution of (2.25) is well-known to possess the following properties

Proposition 2.2.4. *Let $s \geq 0$, $T > 0$ and $L > 0$ be given. For any*

$$\psi \in H^s(\mathbb{R}) \text{ and } g \in L^1(0,T; H^s(\mathbb{R})),$$

the IVP (2.25) admits a unique solution

$$p \in C([0,T]; H^s(\mathbb{R})) \cap L^2(0,T; H^{s+1}(0,L))$$

with

$$\partial_x^j p \in L_x^\infty(\mathbb{R}; H^{(s+1-j)/3}(0,T)).$$

Moreover, there exists a constant $C > 0$ depending only on s, T and L such that

$$\|p\|_{C([0,T];H^s(\mathbb{R}))\cap L^2(0,T;H^{s+1}(0,L))} + \sum_{j=0}^{2} \|\partial_x^j p\|_{L_x^\infty(\mathbb{R};H^{(s+1-j)/3}(0,T))} \leq C\|\psi\|_{H^s(\mathbb{R})}.$$

For any $\phi \in H^s(0,L)$, let $\widetilde{\phi} = E\phi \in H^s(\mathbb{R})$ and $\tilde{f} = Ef \in L^1(0,T; H^s(\mathbb{R}))$ be their standard extensions from $H^s(0,L)$ to $H^s(\mathbb{R})$ and from $L^1(0,T; H^s(0,L))$ to $L^1(0,T; H^s(\mathbb{R}))$, respectively. Let

$$p = p(x,t) := [W_{\mathbb{R}}(t)\widetilde{\phi}](x) + \int_0^t W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau.$$

Set

$$p_{xx}(0,t) = g_1(t), \quad p(L,t) = g_2(t), \quad p_x(L,t) = g_3(t),$$

and

$$\vec{g} = (g_1, g_2, g_3).$$

Note that for $0 \le s \le 3$, if $\phi \in H^s(0,L)$ and

$$\vec{h} = (h_1, h_2, h_3) \in H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

satisfy

$$\left\{ \begin{array}{lll} \phi(L) = h_2(0), & \text{if} & \frac{1}{2} < s \le 3, \\ \phi'(L) = h_3(0), & \text{if} & \frac{3}{2} < s \le 3, \\ \phi''(0) = h_1(0), & \text{if} & \frac{5}{2} < s \le 3, \end{array} \right. \tag{2.26}$$

then $\vec{h} - \vec{g} \in \mathbb{H}_0^s(0,T)$, $w = W_{bdr}(t)(\vec{h} - \vec{g})$ is thus well defined and solves the IBVP(2.16) with boundary data $\vec{h}$ replaced by $\vec{h} - \vec{g}$. As a result, the solution v of the IBVP (2.24) can be expressed

$$v(x,t) = W_{\mathbb{R}}(t)\widetilde{\phi}(x) + \int_0^T W_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau + [W_{bdr}(\vec{h} - \vec{g})](x,t).$$

The following proposition follows from Propositions 2.2.3 and 2.2.4 directly.

Proposition 2.2.5. *Let $T > 0$ and $0 \le s \le 3$ be given. For any $\phi \in H^s(0,L)$, $f \in L^1(0,T; H^s(0,L))$ and*

$$\vec{h} = (h_1, h_2, h_3) \in H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

satisfying (2.26), the IBVP (2.24) admits a unique solution $v \in Z_{s,T}$ with

$$\partial_x^k v \in C_b([0,L]; H^{\frac{s+1-k}{3}}(0,T)) \ for \ k = 0, 1, 2.$$

Moreover there exists a constant C such that

$$\|v\|_{Z_{s,T}} + \sum_{k=0}^{2} \|\partial_x^k v\|_{C_b([0,L]; H^{\frac{s+1-k}{3}}(0,T))} \le C \|\phi\|_{H^s(0,L)}.$$

2.3 Non-linear Problems.

In this section, we consider the nonlinear IBVP

$$
\begin{cases}
u_t + u_x + uu_x + u_{xxx} = 0, \quad u(x,0) = \phi(x), \quad x \in (0,L), \quad t \in \mathbb{R}^+, \\
u_{xx}(0,t) + u(0,t) - \dfrac{1}{6}u^2(0,t) = h(t), \\
u(L,t) = 0, \quad u_x(L,t) = 0
\end{cases}
\tag{2.27}
$$

and present the proof of Theorem 2.1.2.

For any $T > 0$ and $s \geq 0$, let

$$X_{s,T} = H^s(0,L) \times H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s+1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T)$$

and $Y_{s,T}$ be the space consisting of all functions v in the space

$$C(0,T; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L))$$

with $\partial_x^j v \in L_x^\infty(0,L; H^{(s+1-j)/3}(0,T)),\ j = 0,1,2$. It is easy to verify that $Y_{s,T}$ is a Banach space with its norm defined as

$$\|v\|_{Y_{s,T}} := \|v\|_{C(0,T;H^s(0,L)) \cap L^2(0,T;H^{s+1}(0,L))} + \sum_{j=0}^{2} \|\partial_x^j v\|_{L_x^\infty(0,L;H^{s+1-j/3}(0,T))}.$$

In order to established the well-posedness of the IBVP (2.27), the following lemmas will be helpful, their proofs can be found in [4] and [34] respectively.

Lemma 2.3.1. *Let $0 \leq s \leq 3$ and $T > 0$ be given. There exists a constant C such that for any $T > 0$ and $u, v \in Y_{s,T}$,*

$$\int_0^T \|(u(\cdot,t)v(\cdot,t))_x\|_{H^s(0,L)} dt \leq C(T^{1/2} + T^{1/3})\|u\|_{Y_{s,T}}\|v\|_{Y_{s,T}}.$$

Lemma 2.3.2. *Let $0 \leq s \leq 3$ and $T > 0$ be given. There exist constants $C, \alpha > 0$ such that if $g,\ h \in H^{\frac{s+1}{3}}(0,T)$, then $gh \in H^{(s-1)/3}(0,T)$ and*

$$\|gh\|_{H^{(s-1)/3}(0,T)} \leq CT^\alpha \|g\|_{H^{(s+1)/3}(0,T)} \|h\|_{H^{(s+1)/3}(0,T)}. \tag{2.28}$$

Proof. For given $(\phi, h, 0, 0) \in X_{s,T}$, let $r > 0$ and $\theta > 0$ be two constants to be determined. Define the set

$$S^s_{\theta,r} := \{v \in Y_{s,\theta}, \|v\|_{Y_{s,\theta}} \le r\}.$$

Note that for any r and θ, the set $S^s_{\theta,r}$ is a closed, convex and bounded subset of the space $Y_{s,\theta}$ and therefore is a complete metric space in the topology induced from $Y_{s,\theta}$. Define a map Γ on $S^s_{\theta,r}$ by

$$\Gamma(v) = u(x,t)$$

for $v \in S_{\theta,r}$ where $u(x,t)$ is the unique solution of

$$\begin{cases} u_t + u_x + u_{xxx} = -vv_x, \quad u(x,0) = \phi(x), \quad x \in (0,L), \quad t \in \mathbb{R}^+, \\ u_{xx}(0,t) = h(t) - (v(0,t) + \dfrac{1}{6}v^2(0,t)), \\ u(L,t) = 0, \quad u_x(L,t) = 0. \end{cases}$$

Applying Proposition 2.2.5, Lemmas 2.3.1 and 2.3.2, for any $v \in S^s_{\theta,r}$, we have

$$\begin{aligned} \|\Gamma(v)\|_{Y_{s,\theta}} \le\ & C_0\|(\phi,h,0,0)\|_{X_{s,T}} + C_1\|v(0,t) - \frac{1}{6}v^2(0,t)\|_{H^{(s-1)/3}(0,T)} \\ & + C_2\int_0^\theta \|(vv_x + v_x)(\cdot,\tau)\|_{H^s(0,L)}d\tau \\ \le\ & C_0\|(\phi,h,0,0)\|_{X_{s,T}} + C_1\|v(0,t) - \frac{1}{6}v^2(0,t)\|_{H^{(s-1)/3}(0,T)} + C_2\int_0^\theta \|vv_x(\cdot,\tau)\|_{H^s(0,L)}d\tau \\ & + C_2\int_0^\theta \|v_x(\cdot,\tau)\|_{H^s(0,L)}d\tau \\ \le\ & C_0\|(\phi,h,0,0)\|_{X_{s,T}} + C_1\theta^{2/3}(\|v\|_{Y_{s,\theta}} + \|v\|^2_{Y_{s,\theta}}) + C_2(\theta^{1/2} + \theta^{1/3})\|v\|^2_{Y_{s,\theta}} \\ & + C_2\theta^{1/2}\|v\|_{Y_{s,\theta}}, \end{aligned}$$

where C_0, C_1 and C_2 are constants. Choosing $r > 0$ and $\theta > 0$ such that

$$\begin{cases} r = 4C_0\|(\phi,h,0,0)\|_{X_{s,T}}, \\ C_1\theta^{2/3} + C_2\theta^{1/2} \le 1/4, \\ C_2(\theta^{1/2} + \theta^{1/3})r \le 1/4, \\ C_1\theta^{2/3}r \le 1/4, \end{cases}$$

then for any $v \in S^s_{\theta,r}$

$$\|\Gamma(v)\|_{Y_{s,\theta}} \leq r.$$

With this choice of θ and r, we have Γ maps $S^s_{\theta,r}$ into $S^s_{\theta,r}$. Moreover, for any $v_1, v_2 \in S^s_{\theta,r}$

$$w(x,t) = \Gamma(v_1) - \Gamma(v_2)$$

solves

$$\begin{cases} w_t + w_x + w_{xxx} = -\frac{1}{2}((v_1+v_2)(v_1-v_2))_x, \quad u(x,0) = 0, \quad x \in (0,L), \quad t \in (0,T), \\ w_{xx}(0,t) = -(v_1(0,t) - v_2(0,t) + \dfrac{1}{6}(v_1(0,t) + v_2(0,t))(v_1(0,t) - v_2(,t)), \\ w(L,t) = 0, \quad w_x(L,t) = 0. \end{cases}$$

Applying Proposition 2.2.5 again leads to

$$\begin{aligned} \|\Gamma(v_1) - \Gamma(v_2)\|_{Y_{s,\theta}} &\leq C_1\theta^{2/3}\|v_2 - v_1\|_{Y_{s,\theta}} + \frac{1}{6}C_1\theta^{2/3}\|v_2 - v_1\|_{Y_{s,\theta}}\|v_2 + v_1\|_{Y_{s,\theta}} \\ &+ \frac{1}{2}C_2(\theta^{1/2} + \theta^{1/3})\|v_2 - v_1\|_{Y_{0,\theta}}\|v_2 + v_1\|_{Y_{s,\theta}} + C_2\theta^{1/2}\|v_2 - v_1\|_{Y_{s,\theta}} \\ &\leq C_1\theta^{2/3}\|v_2 - v_1\|_{Y_{s,\theta}} + \frac{1}{6}C_1\theta^{2/3}\|v_2 - v_1\|_{Y_{s,\theta}}2r \\ &+ \frac{1}{2}C_2(\theta^{1/2} + \theta^{1/3})\|v_2 - v_1\|_{Y_{s,\theta}}2r + C_2\theta^{1/2}\|v_2 - v_1\|_{Y_{s,\theta}} \\ &\leq \frac{1}{4}\|v_2 - v_1\|_{Y_{s,\theta}} + \frac{1}{12}\|v_2 - v_1\|_{Y_{s,\theta}} + \frac{1}{4}\|v_2 - v_1\|_{Y_{s,\theta}} + \frac{1}{4}\|v_2 - v_1\|_{Y_{s,\theta}} \\ &\leq \frac{5}{6}\|v_2 - v_1\|_{Y_{s,\theta}} \end{aligned}$$

for any $v_1, v_2 \in S^s_{\theta,r}$. This shows, that the map Γ is a contraction mapping of $S^s_{\theta,r}$, and its fixed point $u = \Gamma(u)$ is the unique solution of the IBVP 2.27 in $S^s_{\theta,r}$. The proof is complete. □

CHAPTER 3

Neumann boundary control of the Korteweg-de Vries equation on a bounded domain

This paper focused on the well-posedness and boundary controllability of the Korteweg-de Vries equation posed on a bounded domain with Neumann boundary conditions. We will consider the cases where one, two, or three of these boundary data are used as boundary control inputs. The approach used to prove well-posedness in the space $L^2(0, L)$, is the same approach developed in [4] for the Korteweg-de Vries system with Neumann boundary conditions. Once we have proven well-posedness, we consider the system linearized around the origin and the corresponding linear system will be proven to be exactly boundary controllable using one, two, or three boundary control inputs. Moreover, the nonlinear system is shown to be locally exactly boundary controllable via the contraction mapping principle if the associated linearized system is exactly controllable.

3.1 Introduction

In this paper we study a class of distributed parameter control system described by the KdV equation posed on a bounded domain with nonhomogeneous Neumann boundary conditions:

$$
(3.1) \qquad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L). \end{cases}
$$

One of the objectives of this article is to address the following problem:

Well-posedness:

Is it possible to prove local existence of solutions for (3.1) in the space $H^s(0,L)$ for $s \geq 0$? The study of well-posedness for the system (3.1) was motivated by the well-posedness of the KdV equation posed on a bounded domain with Dirichlet–Neumann boundary conditions, namely

$$
(3.2) \qquad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u(0,t) = g_1(t),\ u(L,t) = g_2(t),\ u_x(L,t) = g_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L). \end{cases}
$$

The case of the whole line was initiated by Gardner *et al.* [29] and Lax [47] in the mid-1960's via the inverse scattering theory, and by Sjöberg [59] and Temam [60] in the late 1960's using the then new methods for the analysis of nonlinear partial differential equations, and continued by many others since. The KdV equation posed on a finite interval, was initially studied by Bubnov in [10, 11], the authors studied a general two-point boundary-value problem posed on the interval $(0,1)$. More recently, Bona *et al.* [4], Rivas *et al.* [50] and Kramer *et al.* [41] worked on the KdV equation with different Dirichlet–Neumann boundary conditions. (See the cited references for a more extensive review of the literature.)

In this article, the nonhomogeneous boundary-value problem (3.1) is considered. Initially, the aim is to establish the well-posedness of (3.1) in the space $H^s(0,L)$ when the initial data is drawn from $H^s(0,L)$, for $s = 0$ and the boundary data (h_1, h_2, h_3) in the space $H^{s_1}(0,T) \times H^{s_2}(0,T) \times H^{s_3}(0,T)$ for some appropriate indices s_1, s_2 and s_3. As we will see later, the natural choices of s_1, s_2 and s_3 are

$$s_1 = s_3 = -\frac{1}{3} \quad \text{and} \quad s_3 = 0.$$

The main theorem related with the well-posedness is the following:

Theorem 3.1.1. *Let $T > 0$ be given. For any $u_0 \in L^2(0, L)$ and*

$$\vec{h} := (h_1, h_2, h_3) \in \mathcal{H}_T := H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T),$$

the IBVP (3.1) *admits a unique solution*

$$u \in \mathcal{Z}_T := C([0, T]; L^2(0, L)) \cap L^2(0, T, H^1(0, L)).$$

Moreover, there exists a positive constant $C > 0$, such that

$$||u||_{\mathcal{Z}_T} \leq C \left(||u_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T} \right)$$

and the solution u possesses the following sharp trace estimates

$$\sup_{x \in (0,L)} ||\partial_x^r u(x, \cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r \left(||u_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T} \right), \tag{3.3}$$

for $r = 0, 1, 2$.

The second result addresses the control theory:

Exact control problem:

Given $T > 0$ and $u_0, u_T \in L^2(0, L)$, is it possible to find appropriate control inputs h_j, $j = 1, 2, 3$ such that the corresponding solution u of (3.1) satisfies

$$u(x, 0) = u_0(x), \quad u(x, T) = u_T(x)?$$

The study of the controllability and stabilization of the KdV equation started with the work of Russell and Zhang in [57] who considered a system with periodic boundary conditions and an internal control. Since then, both the controllability and stabilization have been intensively studied (we refer the reader to [54] for a survey of results and [15] for a complete review of control). In particular, the exact boundary controllability of the KdV equation on a finite domain was investigated in [14, 16, 23, 30, 31, 51, 53, 67]. The majority of these articles are concerned with the system (3.2) in which the boundary data g_1, g_2, g_3 can be chosen as control inputs. System (3.2) was first studied by Rosier [51] considering only the control input g_3 (i.e. $g_1 = g_2 = 0$). It was shown in [51] that the exact controllability of the linearized system holds in $L^2(0, L)$ if and only if L does not belong to the following countable set of critical lengths:

$$\mathcal{N} := \left\{ \frac{2\pi}{\sqrt{3}} \sqrt{k^2 + kl + l^2} \; : k,\, l \in \mathbb{N}^* \right\}. \tag{3.4}$$

The analysis developed in [51] shows that if the linearized system is controllable, then the nonlinearized system is controllable as well. Notice that the converse is false, as proven in [14, 16, 23], that is, the (nonlinear) KdV equation is controllable even when L is a critical length, but the linearized system is not controllable.

The existence of a discrete set of critical lengths for which the exact controllability of the linearized equation fails was also noticed by Glass and Guerrero in [31] when g_2 is taken as a control input (i.e. $g_1 = g_3 = 0$). Finally, it is worth mentioning the result by Rosier [53] and Glass and Guerrero [30] for which g_1 is taken as a control input (i.e. $g_2 = g_3 = 0$). They proved that system (3.2) is then null controllable, but not exactly controllable, because of the strong smoothing effect.

Recently, Cerpa *et al.* in [17] proved similar results to those obtained by Rosier [51] for the system

$$
(3.5)\qquad \begin{cases} y_t + y_x + yy_x + y_{xxx} = 0, & \text{in } (0,T)\times(0,L),\\ y(0,t) = k_1(t),\ y_x(L,t) = k_2(t),\ y_{xx}(L,t) = k_3(t), & \text{in } (0,T),\\ y(x,0) = y_0(x), & \text{in } (0,L). \end{cases}
$$

More precisely, the authors consider the above system with one, two, or three controls. In addition, using the well-posedness properties proved by Kramer *et al.* in [41] (see also [50]), they also proved that the controls k_i, $i = 1, 2, 3$ belong to the space $H^s(0, T)$, for $s \in \mathbb{R}$ and the locally exactly controllability of the linear system associated to (3.5) holds if and only if, L does not belong to the following countable set of critical lengths

(3.6)

$$
\mathcal{F} := \left\{ L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} \text{ satisfying } \frac{e^a}{a^2} = \frac{e^b}{b^2} = \frac{e^{-(a+b)}}{(a+b)^2} \right\}.
$$

Moreover, they showed that the nonlinear system (3.5) is locally exactly controllable via the contraction mapping principle. In addition, Guilleron in [32], using Carleman estimates, showed that the linear system associated to (3.5) is null controllable only if $k_1(t)$ is used as a control input, that is, $k_2(t) = k_3(t) = 0$.

The second goal of this paper is to determine if the system (3.1) possesses similar controllability results to those established for systems (3.2) and (3.5). It is natural to think of using the same approaches that have been effective for systems (3.2) and (3.5). However, these approaches will be difficult in our case and other tools will be required, specifically, we will apply the tools used in [17]. When we use only h_2 as a control input, the linear system associated to (3.1) is given by

$$
(3.7)\qquad \begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L),\\ u_{xx}(0,t) = 0,\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = 0, & \text{in } (0,T),\\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases}
$$

with adjoint system

(3.8)
$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ \psi(0,t) + \psi_{xx}(0,t) = 0,\ \psi_x(0,t) = 0,\ \psi(L,t) + \psi_{xx}(L,t) = 0, & t \in (0,T), \\ \psi(x,T) = \psi_T(x), & x \in (0,L). \end{cases}$$

It is well known that the exact controllability of system (3.7) is equivalent to the following observability inequality for the adjoint system (3.8):

$$||\psi_T||_{L^2(0,L)} \leq C||\psi_x(L,\cdot)||_{L^2(0,T)}. \tag{3.9}$$

However, the usual multiplier method and compactness arguments, as those used in dealing with the control of system, (3.7) only lead to

$$||\psi_T||^2_{L^2(0,L)} \leq C_1||\psi_x(L,\cdot)||^2_{L^2(0,T)} + C_2||\psi(0,\cdot)||^2_{L^2(0,T)} \tag{3.10}$$

The issue now is how to remove the extra term in (3.10). To address this, the new approach used in [17] will play a crucial role in proving the observability inequality (3.9). This new approach turns out to be the hidden regularity (or the sharp Kato smoothing property) for solutions of the KdV equation. Specifically, we will prove the following result:

Theorem 3.1.2. *[Hidden regularities] For any $\psi_T \in L^2(0,L)$, the solution $\psi \in \mathcal{Z}_T$ of IBVP* (3.8) *possesses the following sharp trace properties:*

$$\sup_{x\in(0,L)} ||\partial_x^r \psi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r||\psi_0||_{L^2(0,L)}, \tag{3.11}$$

for $r = 0, 1, 2$.

Initially, we consider the case when only the control input h_2 is used and we will prove that system (3.1) is locally exactly controllable as long as $L \notin \mathcal{M}$, where $\mathcal{M}$ is defined as

$$\mathcal{M} := \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2}\ :\ k,\ l\ \in \mathbb{N}^* \right\} \cup \{k\pi\ :\ k \in \mathbb{N}^*\} = \mathcal{N} \cup \{k\pi\ :\ k \in \mathbb{N}^*\}. \tag{3.12}$$

Theorem 3.1.3. *Let $T > 0$ and $L \notin \mathcal{M}$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0, L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_2 \in L^2(0, T)$ such that the system (3.1) admits a unique solution

$$u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$$

satisfying

$$u(x, 0) = u_0(x), \qquad u(x, T) = u_T(x).$$

Remark 1. *The following systems*

$$\begin{cases} u_t + u_x + u_{xxx} = f, & \text{in } (0, T) \times (0, L), \\ u_{xx}(0, t) = h_1(t), \ u_x(L, t) = 0, \ u_{xx}(L, t) = 0, & \text{in } (0, T), \\ u(x, 0) = u_0(x), & \text{in } (0, L), \end{cases} \tag{3.13}$$

and

$$\begin{cases} y_t + y_x + y_{xxx} = f, & \text{in } (0, T) \times (0, L), \\ y(0, t) = k_1(t), \ y_x(L, t) = 0, \ y_{xx}(L, t) = 0, & \text{in } (0, T), \\ y(x, 0) = y_0(x), & \text{in } (0, L). \end{cases} \tag{3.14}$$

are equivalent in the following sense: For given $\{u_0, f, h_1\}$ one can find $\{y_0, f, k_1\}$ such that the corresponding solution u of (3.13) is exactly the same as the corresponding solution y for the system (3.14) and vice versa. Indeed, for given $u_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$ and $h_1(t) \in H^{-\frac{1}{3}}(0, T)$, system (3.13) admits a unique solution $u \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$. Let $y_0 = u_0$ and set $k_1(t) = h_1(t)$. Then, according to Proposition 3.2.8, we have $k_1(t) \in H^{\frac{1}{3}}(0, T)$. Due to the uniqueness of IBVP (3.14), with the selection $\{y_0, f, k_1\}$, the corresponding solution $y \in C([0, T]; L^2(0, L)) \cap$

$L^2(0,T;H^1(0,L))$ *of* (3.14) *must be equal to* u*, since* u *also solves* (3.14) *with the given auxiliary data* $\{y_0, f, k_1\}$*. On the other hand, for any given* $y_0 \in L^2(0,L)$*,* $f \in L^1(0,T;L^2(0,L))$ *and* $k_1(t) \in H^{\frac{1}{3}}(0,T)$*, let* $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *be the corresponding solution of the system* (3.14)*. From Proposition 3.2.9, we have* $y_{xx}(0,\cdot) \in H^{-\frac{1}{3}}(0,T)$*. Thus, if we set* $u_0 = y_0$ *and* $h_1(t) = k_1(t)$*, then* $h_1(t) \in H^{-\frac{1}{3}}(0,T)$ *and the corresponding solution* $u \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L))$ *of* (3.13) *must be equal to* y*, which also solves* (3.13) *with the auxiliary data* $\{u_0, f, h_1\}$*.*

With the techniques developed in [30, 32] and the previous remark, we are able to prove the null controllability for system (3.1), when $h_2(t) = h_3(t) = 0$.

Theorem 3.1.4 (Null Controllability)**.** *Let* $T > 0$ *be fixed. For* $\overline{u}_0 \in L^2(0,L)$*, we consider*

$$\overline{u} \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^1(0,L)),$$

the solution of

$$\begin{cases} \overline{u}_t + \overline{u}_x + \overline{u}_{xxx} + \overline{u}\overline{u}_x = 0, & \text{in } (0,T)\times(0,L), \\ \overline{u}_{xx}(0,t) = 0, \ \overline{u}_x(L,t) = 0, \ \overline{u}_{xx}(L,t) = 0, & \text{in } (0,T), \\ \overline{u}(x,0) = \overline{u}_0(x), & \text{in } (0,L). \end{cases} \tag{3.15}$$

Then, there exists $\delta > 0$ *such that for any* $u_0 \in L^2(0,L)$ *satisfying*

$$||u_0 - \overline{u}_0||_{L^2(0,L)} < \delta,$$

there exists $h_1(t) \in H^{-\frac{1}{3}}(0,T)$ *such that the solution* $u(x,t)$ *of the system*

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t), \ u_x(L,t) = 0, \ u_{xx}(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{3.16}$$

belongs to the space $\mathcal{Z}_T$ *and satisfies*

$$u(x,T) = \overline{u}(x,T) \quad \text{in } (0,L).$$

Using h_3 as a control input, the system

$$(3.17)\qquad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = 0,\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases}$$

is locally exactly controllable if L does not belong to the following countable set of critical lengths:

(3.18)

$$\mathcal{R} = \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2 + ab + b^2) \text{ with } a, b \in \mathbb{C} : X = e^a, Y = e^b \text{ are solutions of } AX^2 + BX + C = 0 \text{ and } Y = -\tfrac{b_3 + b_1 X}{b_2} \Big\}.$$

Here $A = a_1 b_1$, $B = a_1 b_3 - a_2 b_2 + a_3 b_1$ and $C = a_3 b_3$, where

$$(3.19)\qquad a_1 := (b^2 - a^2)(a+b), \quad a_2 := b^2 e^c (2a+b), \quad a_3 := -a^2 e^c (a+2b),$$

$$(3.20)\qquad b_1 := -a^3(a+2b), \quad b_2 := ab^2(2a+b) \text{ and } b_3 := -ae^c(b^2 - a^2)(a+b).$$

Theorem 3.1.5. *Let $T > 0$ and $L \notin \mathcal{R}$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0,L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_3 \in H^{-\frac{1}{3}}(0,T)$ such that the system (3.17) *admits a unique solution u in the space $\mathcal{Z}_T$, that satisfies*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

Something interesting to mention is the following: If h_3 is not considered, then the critical length phenomenon will not occur for the system

$$(3.21)\qquad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L). \end{cases}$$

The result is given in the following theorem:

Theorem 3.1.6. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0,L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_1 \in H^{-\frac{1}{3}}(0,T)$ and $h_2 \in L^2(0,T)$ such that the system (3.21) admits a unique solution $u \in \mathcal{Z}_T$, satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

Using h_2 and h_3 as control inputs,

$$(3.22) \qquad \begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T) \times (0,L), \\ u_{xx}(0,t) = 0,\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases}$$

we have the following local exact controllability result:

Theorem 3.1.7. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0,L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \leq \delta,$$

one can find $h_2 \in L^2(0,T)$ and $h_3 \in H^{-\frac{1}{3}}(0,T)$ such that the system (3.22) admits a unique solution $u \in \mathcal{Z}_T$, satisfying

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

In addition, considering h_1 and h_3 as control inputs, we have the system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = 0,\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{3.23}$$

and the following result:

Theorem 3.1.8. *Let $T > 0$ and $L > 0$ be given. There exists $\delta > 0$ such that for any $u_0, u_T \in L^2(0,L)$ with*

$$||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} \le \delta,$$

one can find $h_1, h_3 \in H^{-\frac{1}{3}}(0,T)$ such that the system (3.23) *admits a unique solution $u \in \mathcal{Z}_T$, satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

Remark 2. *If all three boundary control inputs are used, we can show that system* (3.1) *is locally exactly controllable around any smooth solution of the KdV equation. The following theorem holds using the same ideas of [17] and [64]:*

Theorem 3.1.9. *Let $T > 0$ and $L > 0$ be given. Assume that $y \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}))$ satisfies*

$$y_t + y_x + yy_x + y_{xxx} = 0 \quad (x,t) \in \mathbb{R}\times\mathbb{R}.$$

Then, there exists $\delta > 0$ such that for any $y_0, y_T \in L^2(0,L)$ with

$$||u_0 - y(\cdot,0)||_{L^2(0,L)} + ||u_T - y(\cdot,T)||_{L^2(0,L)} \le \delta,$$

one can find

$$h_1 \in H^{-\frac{1}{3}}(0,T),\ h_2 \in L^2(0,T),\ h_3 \in H^{-\frac{1}{3}}(0,T)$$

such that system (3.1) *admits a unique solution* $u \in \mathcal{Z}_T$*, satisfying*

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x).$$

Theorems 3.1.3-3.1.9 are going to be established initially for linearized systems around the origin by using the classical duality approach, that is, the Hilbert Uniqueness Method (H.U.M) introduced by J. L. Lions in [48]. This method reduces the proof of exact controllability for (3.1) to prove an observability inequality for the solution of the adjoint system. To prove the observability inequality, we will use the compactness uniqueness argument developed by E. Zuazua in [48]. The exact controllability is extended to the nonlinear system by using the contraction mapping principle.

This paper is organized as follows: In section 3.2, we present various linear estimates including hidden regularities for solutions of the linear systems associated to (3.1) and (3.2). The well-posedness of the nonlinear system (3.1), that is, Theorem 3.1.1 also will be presented in this section. The control theory is studied in section 3.3 and in this section some hidden regularities for solutions of the system (3.8) will be included. These regularities will play an important role in establishing our exact controllability results. Furthermore, we will prove that the associated linear systems are exactly controllable and the nonlinear systems are shown to be locally exactly controllable via the contraction mapping principle. Finally, in section 3.4 we will provide some remarks together with some open problems for further studies.

3.2 Well-posedness: Linear and Nonlinear problems

In this section, we study the well-posedness in $L^2(0,L)$ for the following IBVP:

$$
\begin{cases}
u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\
u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\
u(x,0) = u_0(x), & \text{in } (0,L).
\end{cases} \tag{3.24}
$$

In order to prove well-posedness for the IBVP (3.24), we need to established some smoothing properties for linear problems related with this IBVP.

3.2.1 The boundary integral operators

Consideration is first given to the following IBVP of the linear KdV equation with homogenous initial value and nonhomogenoeus boundary data:

$$
\begin{cases}
w_t + w_{xxx} = 0, & x\in(0,L), t\geq 0, \\
w_{xx}(0,t) = h_1(t),\ w_x(L,t) = h_2(t),\ w_{xx}(L,t) = h_3(t), & t>0, \\
w(x,0) = 0, & x\in(0,L).
\end{cases} \tag{3.25}
$$

Following the approach developed in [3, 4], we derive an explicit formula for its solution in terms of the boundary values h_1, h_2 and h_3.

Applying the Laplace transform with respect to t, the IBVP (3.25) becomes

$$
\begin{cases}
s\hat{w} + \hat{w}_{xxx} = 0, \\
\hat{w}_{xx}(0,s) = \hat{h}_1(s),\ \hat{w}_x(L,s) = \hat{h}_2(s),\ \hat{w}_{xx}(L,s) = \hat{h}_3(s),
\end{cases}
$$

where

$$\hat{w}(x,s) = \int_0^{+\infty} e^{-st} w(x,t)dt$$

and

$$\hat{h}_j(s) = \int_0^{+\infty} e^{-st} h(t)dt, \quad j=1,2,3.$$

The solution $\hat{w}(x,s)$ can be written in the form

$$\hat{w}(x,s) = \sum_{j=1}^{3} c_j(s) e^{\lambda_j(s)x},$$

where $\lambda_j(s)$, $j = 1,2,3$, are the three solutions of the characteristic equation

$$s + \lambda^3 = 0$$

and the constants $c_j = c_j(s)$, $j = 1,2,3$, solve the linear system

$$\begin{pmatrix} \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \lambda_3 e^{\lambda_3 L} \\ \lambda_1^2 e^{\lambda_1 L} & \lambda_2^2 e^{\lambda_2 L} & \lambda_3^2 e^{\lambda_3 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \hat{h}_3 \end{pmatrix}. \tag{3.26}$$

Let $\Delta(s)$ be the determinant of the coefficient matrix and $\Delta_j(s)$ be the determinants of the matrices that are obtained by replacing the jth-column of $\Delta(s)$ by the column vector $(\hat{h}_1(s), \hat{h}_2(s), \hat{h}_3(s))^T$, $j = 1,2,3$. By Cramer's rule

$$c_j = \frac{\Delta_j(s)}{\Delta(s)}, \qquad j = 1,2,3,$$

if $\Delta(s) \neq 0$. Taking the inverse Laplace transform of $\hat{w}$ we have

$$w(x,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\hat{w}(x,s)ds = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\frac{\Delta_j(s)}{\Delta(s)} e^{\lambda_j(s)x} ds$$

for any $r > 0$. Using the same arguments as those in [4] the solution $w(x,t)$ can be written as

$$w(x,t) = \sum_{m=1}^{3} w_m(x,t) \tag{3.27}$$

where $w_m(x,t)$ solves (3.25) with $h_j \equiv 0$ when $j \neq m$, $j,m = 1,2,3$. With the above, we can write w_m, for $m = 1,2,3$, in the following way:

$$w_m(x,t) = \sum_{j=1}^{3} \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{\Delta_{j,m}(s)}{\Delta(s)} e^{\lambda_j(s)x}\hat{h}_m(s)ds \equiv [W_{m,j}(t)h_m](x).$$

Notice that in the last two formulas, the right-hand sides are continuous with respect to r for $r \geq 0$ and they do not depend on r, thus we can take $r = 0$ in these formulas. Moreover,

$$w_{j,m}(x,t) = w_{j,m}^{+}(x,t) + w_{j.m}^{-}(x,t)$$

where

$$w_{j,m}^{+}(x,t) = \frac{1}{2\pi i}\int_{0}^{+i\infty} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds$$

and

$$w_{j,m}^{-}(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{0} e^{st}\frac{\Delta_{j,m}(s)}{\Delta(s)}\hat{h}_m(s)e^{\lambda_j(s)x}ds,$$

for $j, m = 1, 2, 3$. Here $\Delta_{j,m}(s)$ is obtained from $\Delta_j(s)$ by letting $\hat{h}_m(s) = 1$ and $\hat{h}_k(s) = 0$ for $k \neq m$, $k, m = 1, 2, 3$.

Using the substitution $s = i\rho^3$, $0 < \rho < +\infty$, the three roots of the characteristic equation are given by

$$\lambda_1(\rho) = i\rho, \quad \lambda_2(\rho) = -i\rho\left(\frac{1+i\sqrt{3}}{2}\right), \quad \lambda_3(\rho) = -i\rho\left(\frac{1-i\sqrt{3}}{2}\right). \tag{3.28}$$

Therefore $w_{j,m}^{+}$ has the following form

$$w_{j,m}^{+}(x,t) = \frac{1}{2\pi i}\int_{0}^{+\infty} e^{i\rho^3 t}\frac{\Delta_{j,m}^{+}(\rho)}{\Delta^{+}(\rho)}\hat{h}_m^{+}(\rho)e^{\lambda_j^{+}(\rho)x}3i\rho^2 d\rho$$

and

$$w_{j,m}^{-}(x,t) = \overline{w_{j,m}^{+}(x,t)},$$

where $\hat{h}_m^{+}(\rho) = \hat{h}_m(i\rho^3)$, $\Delta^{+}(\rho) = \Delta(i\rho^3)$, $\Delta_{j,m}^{+}(\rho) = \Delta_{j,m}(i\rho^3)$ and $\lambda_j^{+}(\rho) = \lambda_j(i\rho^3)$.

The following lemma give us a representation formula for the solution of the IBVP (3.25).

Lemma 3.2.1. *Given $\vec{h} = (h_1, h_2, h_3)$, the solution w of the IBVP* (3.25) *can be written in the form*

$$w(x,t) = [W_{bdr}\vec{h}](x,t) := \sum_{j,m=1}^{3}[W_{j,m}h_m](x,t).$$

Let $\vec{h} := (h_1, h_2, h_3) \in \mathcal{H}_T$ with

$$\mathcal{H}_T = H^{-\frac{1}{3}}(0,T) \times L^2(0,T) \times H^{-\frac{1}{3}}(0,T),$$

and $\mathcal{Z}_T$ as before. The following lemma holds for the solution of the system (3.25).

Lemma 3.2.2. *Let $T > 0$ be given. There exists a constant $C > 0$ such that for any $\vec{h} \in \mathcal{H}_T$ the system* (3.25) *admits a unique solution $w \in \mathcal{Z}_T$. Moreover*

$$||w||_{\mathcal{Z}_T} + \sum_{j=0}^{2} ||\partial_x^j w||_{L^\infty(0,L;H^{\frac{1-j}{3}})} \leq C||\vec{h}||_{\mathcal{H}_T}.$$

Proof. As we stated above, the solution w can be written as

$$w(x,t) = w_1(x,t) + w_2(x,t) + w_3(x,t).$$

Let us prove Lemma 3.2.2 for w_1. Some straightforward calculations show that the asymptotic behavior of the ratios $\frac{\Delta^+_{j,m}(\rho)}{\Delta^+(\rho)}$ as $\rho \to +\infty$ are:

$\frac{\Delta^+_{1,1}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta^+_{2,1}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta^+_{3,1}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho L}$
$\frac{\Delta^+_{1,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$	$\frac{\Delta^+_{2,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$	$\frac{\Delta^+_{3,2}(\rho)}{\Delta^+(\rho)} \sim \rho^{-1}$
$\frac{\Delta^+_{1,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}$	$\frac{\Delta^+_{2,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}e^{-\frac{\sqrt{3}}{2}\rho L}$	$\frac{\Delta^+_{3,3}(\rho)}{\Delta^+(\rho)} \sim \rho^{-2}$

Since

$$w_1(x,t) = \frac{3}{\pi}\sum_{j=1}^{3}\int_0^{+\infty} e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta^+_{j,1}(\rho)}{\Delta^+(\rho)}\hat{h}_1^+(\rho)\rho^2 d\rho,$$

we have

$$\begin{aligned}\sup_{t\in(0,T)} ||w_1(\cdot,t)||^2_{L^2(0,L)} &\leq C\int_0^\infty \mu^{-2/3}|\hat{h}_1^+(i\mu)|^2 d\mu \\ &\leq C||h_1||^2_{H^{-\frac{1}{3}}(\mathbb{R}^+)} \\ &\leq C||\vec{h}||_{\mathcal{H}_T}.\end{aligned}$$

Furthermore, for $l=-1,0,1$ let us consider $\theta(\mu)$ the real solution of $\mu=\rho^3$, $\rho>0$, thus

$$\begin{aligned}\partial_x^l w_1(x,t) &= \frac{3}{\pi}\sum_{j=1}^{3}\int_0^{+\infty}\left(\lambda_j^+(\rho)^{\rho+1}\right)e^{i\rho^3 t}e^{\lambda_j^+(\rho)x}\frac{\Delta_{j,1}^+(\rho)}{\Delta^+(\rho)}\hat{h}_1^+(\rho)\rho^2 d\rho\\ &= \frac{3}{\pi}\sum_{j=1}^{3}\int_0^{+\infty}\left(\lambda_j^+(\theta(\mu))^{\rho+1}\right)e^{i\rho^3 t}e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1^+(i\mu)d\mu.\end{aligned}$$

Applying Plancherel's Theorem (with respect to t), yields that, for all $x\in(0,L)$

$$\begin{aligned}||\partial_x^{l+1}w_1(x,\cdot)||^2_{H^{-\frac{1}{3}}(0,T)} &\leq C\sum_{j=1}^{3}\int_0^{+\infty}\mu^{-\frac{2l}{3}}\left|(\lambda_j^+(\theta(\mu))^{\rho+1})e^{\lambda_j^+(\theta(\mu))x}\frac{\Delta_{j,1}^+(\theta(\mu))}{\Delta^+(\theta(\mu))}\hat{h}_1^+(i\mu)\right|^2 d\mu\\ &\leq C\int_0^{+\infty}\mu^{-\frac{2l}{3}}\left|h_1(i\mu)\right|^2 d\mu\\ &\leq C||h_1||^2_{H^{-\frac{l}{3}}(0,T)}\\ &\leq C||\vec{h}||^2_{\mathcal{H}_T},\end{aligned}$$

for $l=-1,0,1$. Therefore

$$\sup_{x\in(0,L)}||\partial_x^{l+1}w_1(x,\cdot)||_{H^{-\frac{l}{3}}(0,T)}\leq C||\vec{h}||^2_{\mathcal{H}_T},\ l=-1,0,1,$$

which ends the proof of Lemma 3.2.2 for w_1. The proofs for w_i, $i=2,3$ are similar. □

3.2.2 Linear estimates

In this subsection we consider the following initial boundary-value problem:

$$\begin{cases} v_t+v_{xxx}=f, & x\in(0,L), t>0,\\ v_{xx}(0,t)=0,\ v_x(L,t)=0,\ v_{xx}(L,t)=0, & t>0,\\ v(x,0)=\phi(x), & x\in(0,L).\end{cases}\tag{3.29}$$

The solution of the IBVP (3.29) can be expressed in terms of the solution of the following IVP:

$$\begin{cases} v_t+v_{xxx}=0, \quad x\in\mathbb{R},\ t\in\mathbb{R}^+,\\ v(x,0)=\phi.\end{cases}\tag{3.30}$$

The solution of this IVP is given by

$$v(x,t) = [W_{\mathbb{R}}(t)]\phi(x) = c\int_{\mathbb{R}} e^{i\xi^3 t} e^{ix\xi}\hat{\phi}(\xi)d\xi, \tag{3.31}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ.

Using this representation formula, we can write $W_0(t)$ in terms of $W_{\mathbb{R}}(t)$ and $W_{bdr}(t)$, where W_0 is the C_0-semigroup in the space $L^2(0,L)$ generated by the operator

$$Au = -u'''$$

with domain

$$\mathcal{D}(A) = \big\{u \in H^3(0,L) : u''(0) = u'(L) = u''(L) = 0\big\}.$$

For any $\phi \in H^s(0,L)$, let $\phi^* \in H^s(\mathbb{R})$ be its standard extension from $(0,L)$ to $\mathbb{R}$. Let $v = v(x,t)$ be the solution of

$$\begin{cases} v_t + v_{xxx} = 0, \quad x \in \mathbb{R},\ t \geq 0, \\ v(x,0) = \phi^*, \end{cases}$$

and set $g_1(t) = v_{xx}(0,t)$, $g_2(t) = v_x(L,t)$ and $g_3(t) = v_{xx}(L,t)$, $\vec{g} = (g_1, g_2, g_3)$ and

$$v_{\vec{g}} = v_{\vec{g}}(x,t) = [W_{bdr}(t)\vec{g}](x),$$

which is the corresponding solution of the non homogeneous boundary-value problem (3.25) with boundary data $h_j(t) = g_j(t)$ for $j = 1,2,3$ and $t \geq 0$. Then $v(x,t) - v_{\vec{g}}$ solves the IBVP (3.29). Thus this leads us to a particular representation of $W_0(t)$ in terms of $W_{bdr}(t)$ and $W_{\mathbb{R}}(t)$.

If $B : H^s(0,L) \to H^s(\mathbb{R})$ is the standard extension operator from, $H^s(0,L)$ to $H^s(\mathbb{R})$, then we have the following lemma:

Lemma 3.2.3. *Given $s \geq 0$ and $\phi \in H^s(0,L)$, let $\phi^* = B\phi$. Then*

$$W_0(t)\phi = W_{\mathbb{R}}(t)\phi^* - W_{bdr}(t)\vec{g}, \tag{3.32}$$

for any $t > 0$ and $x \in (0, L)$, where $\vec{g}$ is obtained from the trace of $W_{\mathbb{R}}(t)\phi^*$ at $x = 0, L$.

We can use $W_{\mathbb{R}}(t)$ and $W_{bdr}(t)$ to express the solution $v(x,t)$ of the non-homogeneous initial boundary-value problem

$$\begin{cases} v_t + v_{xxx} = f(x,t), \qquad x \in (0,L),\ t \geq 0, \\ v(x,0) = 0, \\ v_{xx}(0,t) = 0, \qquad v_x(L,t) = 0, \qquad v_{xx}(L,t) = 0. \end{cases} \tag{3.33}$$

More precisely,

Lemma 3.2.4. *If $f^*(\cdot,t) = Bf(\cdot,t)$, with B as was defined before, then the solution v of the problem (3.33) is given by*

$$v(x,t) = \int_0^t W_0(t-\tau)f(\tau)d\tau = \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\cdot,\tau)d\tau - W_{bdr}(t)\vec{v},$$

for any $x \in (0,L)$ and $t \geq 0$. Here $\vec{v} \equiv \vec{v}(t) = (v_1(t), v_2(t), v_3(t))$, with

$$v_1(t) = \partial_x^2 \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau\Big|_{x=0}, \qquad v_2(t) = \partial_x \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau\Big|_{x=L}$$

and

$$v_3(t) = \partial_x^2 \int_0^t W_{\mathbb{R}}(t-\tau)f^*(\tau)d\tau\Big|_{x=L}.$$

Lemma 3.2.3 and Lemma 3.2.4 are valid for $x \in (0,L)$ and $t \geq 0$ since some of the operators that we have constructed are defined only in this interval, moreover the only operator that is defined in the whole line is $W_{\mathbb{R}}(t)$ for any values of x and t.

Recall that

$$W_{bdr}(t)\vec{h} = \sum_{j,m=1}^{3} W_{j,m}h_j$$

and each $W_{j,m}h_j$ is of the form (see Lemma 3.2.1). Therefore by the extension method introduced in [5], the operator $W_{bdr}(t)$ can be extended as $\mathcal{W}_{bdr}(t)$ with

$$[\mathcal{W}_{bdr}(t)\vec{h}](x,t)$$

defined for any $t,\ x \in \mathbb{R}$ and

$$[W_{bdr}\vec{h}](x,t) = [\mathcal{W}_{bdr}(t)\vec{h}](x,t) \ \textit{for any} \ (x,t) \in (0,L) \times (0,T).$$

Next we present the spatial trace estimates for $W_{\mathbb{R}}(t)\phi$ and $\displaystyle\int_0^t W_{\mathbb{R}}(t-t')f(\cdot,t')dt'$

Proposition 3.2.5. *Let $s = 0$, there exists a constant C depending only on s such that*

$$\sup_{x\in\mathbb{R}} \|W_{\mathbb{R}}(t)\phi\|_{H_t^{\frac{1}{3}}(\mathbb{R})} \leq \|\phi\|_{L^2(\mathbb{R})}, \tag{3.34}$$

$$\sup_{x\in\mathbb{R}} \|\partial_x W_{\mathbb{R}}(t)\phi\|_{L_t^2(\mathbb{R})} \leq \|\phi\|_{L^2(\mathbb{R})} \tag{3.35}$$

and

$$\sup_{x\in\mathbb{R}} \|\partial_{xx} W_{\mathbb{R}}(t)\phi\|_{H_t^{-\frac{1}{3}}(\mathbb{R})} \leq \|\phi\|_{L^2(\mathbb{R})}. \tag{3.36}$$

Proposition 3.2.6. *Letting $s = 0$, $\psi \in C_0^\infty(\mathbb{R})$ and*

$$w(x,t) = \int_0^t W_{\mathbb{R}}(t-t')f(\cdot,t')dt',$$

there exists C depending only on s and ψ such that

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w(x,\cdot)\|_{H_t^{\frac{1}{3}}(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})},$$

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w_x(x,\cdot)\|_{L_t^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}$$

and

$$\sup_{x\in\mathbb{R}} \|\psi(\cdot)w_{xx}(x,\cdot)\|_{H_t^{-\frac{1}{3}}(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}.$$

The proofs of these propositions can be found in [3, 4].

3.2.3 Well-Posedness: Linear problems

With the results provided in the previous subsections, we are ready to prove some of the main results related to well-posedness. The first IBVP considered is

$$
\begin{cases}
v_t + v_{xxx} = f, & \text{in } (0,T)\times(0,L),\\
v_{xx}(0,t) = h_1(t),\ v_x(L,t) = h_2(t),\ v_{xx}(L,t) = h_3(t), & \text{in } (0,T),\\
v(x,0) = v_0(x), & \text{in } (0,L).
\end{cases}
\tag{3.37}
$$

Proposition 3.2.7. *Let $T>0$ be given. For any $v_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and $\vec{h} := (h_1,h_2,h_3) \in \mathcal{H}_T$, the IBVP (3.37) admits a unique solution $v \in \mathcal{Z}_T$. Moreover, there exists $C>0$ such that*

$$||v||_{\mathcal{Z}_T} \leq C\left(||v_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T} + ||f||_{L^1(0,T;L^2(0,L))}\right).$$

Proof. The proof of this proposition is a direct application of Lemma 3.2.2, Propositions 3.2.5 and 3.2.6. □

In addition, the solution v of (3.37) possesses the following hidden (or sharp trace) regularities:

Proposition 3.2.8. *Let $T>0$ be given. For any $v_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and $\vec{h} \in \mathcal{H}_T$, the solution v of the system (3.37) satisfies*

$$\sup_{x\in(0,L)} ||\partial_x^r v(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r\left(||v_0||_{L^2(0,L)} + ||\vec{h}||_{\mathcal{H}_T} + ||f||_{L^1(0,T;L^2(0,L))}\right), \tag{3.38}$$

for $r = 0,1,2$.

Proof. Note that the system

$$
\begin{cases}
v_t + v_{xxx} = f, & x \in (0,L), t>0,\\
v_{xx}(0,t) = h_1(t),\ v_x(L,t) = h_2(t),\ v_{xx}(L,t) = h_3(t), & t>0,\\
v(x,0) = v_0(x), & x\in(0,L),
\end{cases}
\tag{3.39}
$$

has solution $v(x,t)$ given by

$$v(t) = V_0(t)v_0 + W_{bdr}(t)\vec{h}(t) + \int_0^t V_0(t-\tau)f(\tau)d\tau,$$

where $\vec{h} = (h_1, h_2, h_3)$ and $V_0(t)$ is the C_0-semigroup in $L^2(0,L)$ generated by the operator

$$Bf = -f'''$$

with domain

$$\mathcal{D}(B) = \big\{f \in H^3(0,L) : f''(0) = f'(L) = f(L)'' = 0\big\}.$$

Therefore, $u(t) = V_0(t)v_0$ solves

$$\text{(3.40)} \qquad \begin{cases} u_t + u_{xxx} = 0, & x \in (0,L), t > 0, \\ u_{xx}(0,t) = 0,\ u_x(L,t) = 0,\ u_{xx}(L,t) = 0, & t > 0, \\ u(x,0) = v_0(x), & x \in (0,L), \end{cases}$$

$w(t) = W_{bdr}(t)\vec{h}$ solves

$$\text{(3.41)} \qquad \begin{cases} w_t + w_{xxx} = 0, & x \in (0,L), t > 0, \\ w_{xx}(0,t) = h_1(t),\ w_x(L,t) = h_2(t),\ w_{xx}(L,t) = h_3(t), & t > 0, \\ w(x,0) = 0, & x \in (0,L), \end{cases}$$

and $z(t) = \displaystyle\int_0^t V_0(t-\tau)f(\tau)d\tau$ solves

$$\text{(3.42)} \qquad \begin{cases} z_t + z_{xxx} = f, & x \in (0,L), t > 0, \\ z_{xx}(0,t) = 0,\ z_x(L,t) = 0,\ z_{xx}(L,t) = 0, & t > 0, \\ z(x,0) = 0, & x \in (0,L). \end{cases}$$

In order to complete the proof, we have to prove

$$\text{(3.43)} \quad ||\partial_x^r u||_{L^\infty_x(0,L;H^{\frac{1-r}{3}}(0,T))} + ||\partial_x^r z||_{L^\infty_x(0,L;H^{\frac{1-r}{3}}(0,T))} \leq C\left(||v_0||_{L^2(0,L)} + ||f||_{L^1(0,T;L^2(0,L))}\right),$$

for $r = 0, 1, 2$. To prove this, note that the solutions u and z of (3.40) and (3.42) respectively can be written as

$$u(t) = V_{\mathbb{R}}(t)\tilde{v}_0 - V_{bdr}(t)\vec{p}$$

and

$$z(t) = \int_0^t V_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau - V_{bdr}(t)\vec{q}.$$

Here

(i) $\tilde{v}_0$ and $\tilde{f}$ are the standard extensions of v_0 and f respectively:

$$\tilde{v}_0(x) = \begin{cases} v_0(x), & \text{if } x \in (0, L), \\ 0, & \text{if } x \notin (0, L), \end{cases} \qquad \tilde{f}(x,t) = \begin{cases} f(x,t), & \text{if } (x,t) \in (0, L) \times (0, T), \\ 0, & \text{if } x \notin (0, L). \end{cases}$$

(ii) $V_{\mathbb{R}}(t)$ is the C_0-semigroup associated to the initial value problem

$$\mu_t + \mu_{xxx} = 0, \qquad \mu(x, 0) = \tilde{v}_0(x), \quad x \in \mathbb{R}, t \in (0, T).$$

(iii) $\vec{p} = (p_1, p_2, p_3)$ with

$$p_1(t) = \mu_{xx}(0,t), \quad p_2(t) = \mu_x(L,t) \ \text{ and } \ p_3(t) = \mu_{xx}(L,t),$$

where $\mu(t) = V_{\mathbb{R}}(t)\tilde{v}_0$.

(iv) $\vec{q} = (q_1, q_2, q_3)$ with

$$q_1(t) = \tilde{z}_{xx}(0,t), \quad q_2(t) = \tilde{z}_x(L,t) \ \text{ and } \ q_3(t) = \tilde{z}_{xx}(L,t),$$

where $\tilde{z} = \displaystyle\int_0^t V_{\mathbb{R}}(t-\tau)\tilde{f}(\tau)d\tau$. Using the same argument as in [41], we have

$$||\partial_x^r \mu||_{L_x^\infty(\mathbb{R}; H^{\frac{1-r}{3}}(0,T))} \leq C||\tilde{v}_0||_{L^2(\mathbb{R})} \leq C||v_0||_{L^2(0,L)}$$

and

$$||\partial_x^r \tilde{z}||_{L_x^\infty(\mathbb{R}; H^{\frac{1-r}{3}}(0,T))} \leq C||\tilde{f}||_{L^1(0,T;L^2(\mathbb{R}))} \leq C||f||_{L^1(0,T;L^2(0,L))},$$

for $r = 0, 1, 2$. Furthermore, by Lemma 3.2.2,

$$||\partial_x^r V_{bdr}(t)\vec{p}||_{L_x^\infty(\mathbb{R};H^{\frac{1-r}{3}}(0,T))} \le C||\vec{p}||_{\mathcal{H}_T} \le C||v_0||_{L^2(0,L)}$$

and

$$||\partial_x^r V_{bdr}(t)\vec{q}||_{L_x^\infty(\mathbb{R};H^{\frac{1-r}{3}}(0,T))} \le C||\vec{q}||_{\mathcal{H}_T} \le C||f||_{L^1(0,T;L^2(0,L))},$$

thus, the proof of the Proposition 3.2.8 is complete. □

The next proposition states similar hidden (or sharp trace) regularities for the linear system

$$\begin{cases} y_t + y_x + y_{xxx} = f, & x \in (0,L), t > 0, \\ y(0,t) = g_1(t),\ y(L,t) = g_2(t),\ y_x(L,t) = g_3(t), & t > 0, \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{3.44}$$

associated to (3.2).

Proposition 3.2.9. *Let $T > 0$ be given. For any $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ and*

$$\vec{g} := (g_1, g_2, g_3) \in \mathcal{G}_T := H^{\frac{1}{3}}(0,T) \times H^{\frac{1}{3}}(0,T) \times L^2(0,T),$$

the IBVP (3.44) admits a unique solution $y \in \mathcal{Z}_T$. Moreover, there exists $C > 0$ such that

$$||y||_{\mathcal{Z}_T} \le C\left(||y_0||_{L^2(0,L)} + ||\vec{g}||_{\mathcal{G}_T} + ||f||_{L^1(0,T;L^2(0,L))}\right).$$

In addition, the solution y possesses the sharp trace estimates

$$\sup_{x\in(0,L)} ||\partial_x^r y(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \le C_r\left(||y_0||_{L^2(0,L)} + ||\vec{g}||_{\mathcal{G}_T} + ||f||_{L^1(0,T;L^2(0,L))}\right), \tag{3.45}$$

for $r = 0, 1, 2$.

The proof of Proposition 3.2.9 can be found in [64] (cf. also [4, 41]).

Remark 3. *Systems* (3.37) *and* (3.44) *are equivalent in the following sense: For given* $\{u_0, f, h_1, h_2, h_3\}$ *one can find* $\{y_0, f, g_1, g_2, g_3\}$ *such that the corresponding solution u of*

(3.37) *is exactly the same as the corresponding solution* y *of the system* (3.44) *and vice versa. In fact, for given* $u_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ *and* $\vec{h} \in \mathcal{H}_T$, *system* (3.37) *admits a unique solution* $u \in \mathcal{Z}_T$. *Let* $y_0 = u_0$ *and set*

$$g_1(t) = h_1(t), \quad g_3(t) = h_2(t), \quad g_2(t) = h_3(t).$$

Then, according to (3.38), *we have* $\vec{g} \in \mathcal{G}_t$. *Due to the uniqueness of IBVP* (3.44), *with the selection* $\{y_0, f, g_1, g_2, g_3\}$, *the corresponding solution* $y \in \mathcal{Z}_T$ *of* (3.44) *must be equal to* u *since* u *also solves* (3.44) *with the given auxiliary data* $\{y_0, f, g_1, g_2, g_3\}$. *On the other hand, for any given* $y_0 \in L^2(0,L)$, $f \in L^1(0,T;L^2(0,L))$ *and* $\vec{g} \in \mathcal{G}_T$, *let* $y \in \mathcal{Z}_T$ *be the corresponding solution of the system* (3.44). *From* (3.45), *we have* $y_{xx}(0,\cdot)$ *and* $y_{xx}(L,\cdot) \in H^{-\frac{1}{3}}(0,T)$. *Thus, if we set* $u_0 = y_0$ *and*

$$h_1(t) = u_{xx}(0,t), \quad h_2(t) = g_3(t), \quad h_3(t) = u_{xx}(L,T),$$

then $\vec{h} \in \mathcal{H}_T$ *and the corresponding solution* $u \in \mathcal{Z}_T$ *of* (3.37) *must be equal to* y *which also solves* (3.37) *with the auxiliary data* $(u_0, f, \vec{h})$.

3.2.4 Well-posedness: Nonlinear problem

Finally, we consider the well-posedness of the following nonlinear system:

$$\begin{cases} v_t + v_x + vv_x + v_{xxx} = 0, \quad x \in (0,L), \quad t > 0, \\ v(x,0) = \phi(x), \\ v_{xx}(0,t) = h_1(t), \; v_x(L,t) = h_2(t), \; v_{xx}(L,t) = h_3(t), \quad t \geq 0. \end{cases} \tag{3.46}$$

For given $T > 0$ and $s \geq 0$, let us define

$$X_{s,T} := H^s(0,L) \times H^{\frac{s-1}{3}}(0,T) \times H^{\frac{s}{3}}(0,T) \times H^{\frac{s-1}{3}}(0,T),$$

$$Z_{s,T} := C([0,T]; H^s(0,L)) \cap L^2(0,T; H^{s+1}(0,L)) \cap L^\infty(0,L; H^{\frac{s+1}{3}}(0,T))$$

and

$$\mathcal{Z}_{s,T} := Z_{s,T} \cap H^{\frac{s}{3}}(0,T; H^1(0,L)).$$

The following lemma is necessary to prove the main theorem of this section

Lemma 3.2.10. *(i) For $s \geq 0$ there exists a $C \geq 0$ such that for any $T > 0$ and $u, v \in Z_{s,T}$,*

$$\int_0^T \|uv_x\|_{H^s(0,L)}\, d\tau \leq C\big(T^{\frac{1}{2}} + T^{\frac{1}{3}}\big)\|u\|_{Z_{s,T}}\|v\|_{Z_{s,T}} \tag{3.47}$$

(ii) For $0 \leq s \leq 3$ there exists a $C \geq 0$ such that for any $T > 0$ and $u, v \in \mathcal{Z}_{s,T}$,

$$\|uv_x\|_{W^{\frac{s}{3},1}(0,T;L^2(0,1))} \leq C\big(T^{\frac{1}{2}} + T^{\frac{1}{3}}\big)\|u\|_{\mathcal{Z}_{s,T}}\|v\|_{\mathcal{Z}_{s,T}}. \tag{3.48}$$

The proof of this lemma can be found in [4, 42]. The following result guarantees the local well-posedness of the system (3.46):

Theorem 3.2.11. *Let $T > 0$ and $r > 0$. For $s = 0$, there exists a $T^* \in (0,T]$ such that for any $(\phi, \vec{h}) \in X_{s,T}$, the IBVP (3.46) admits a unique solution $v \in Z_{s,T^*}$. Moreover, the corresponding solution map is Lipschitz continuous.*

Proof. The proof of this lemma is based on the proof presented in [4, 42]. □

3.3 Control theory

As we shall see, whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system. The main difficulty when minimizing these functionals is to show that they are coercive. This turns out to be equivalent to the so-called observability property of the adjoint system, a property which is equivalent to the original control property of the state system. The next section is concerned with the adjoint system associated to (3.7).

3.3.1 The adjoint linear system

This subsection is devoted to study the properties of the backward adjoint system

(3.49)
$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ \psi(0,t) + \psi_{xx}(0,t) = 0, \ \psi_x(0,t) = 0, \ \psi(L,t) + \psi_{xx}(L,t) = 0, & t \in (0,T), \\ \psi(x,T) = \psi_T(x), & x \in (0,L). \end{cases}$$

Using the transformations $x' = L - x$ and $t' = T - t$, system (3.49) is equivalent to the following forward system:

(3.50)
$$\begin{cases} \varphi_t + \varphi_x + \varphi_{xxx} = 0, & (x,t) \in (0,L) \times (0,T), \\ \varphi(0,t) + \varphi_{xx}(0,t) = 0, \ \varphi_x(L,t) = 0, \ \varphi(L,t) + \varphi_{xx}(L,t) = 0, & t \in (0,T), \\ \varphi(x,0) = \varphi_0(x), & x \in (0,L). \end{cases}$$

We will prove that system (3.50) is well-posed in $\mathcal{Z}_T$ for $\varphi(x,0) = \varphi_0(x) \in L^2(0,L)$. However, before that we prove the next theorem which reveals that φ has a stronger trace regularity, more precisely,

$$\varphi(0,\cdot) \in H^{\frac{1}{3}}(0,T) \text{ (or } \varphi(L,\cdot) \in H^{\frac{1}{3}}(0,T)).$$

It will play an important role in establishing exact controllability of the system (3.1) as shown in the next section.

Theorem 3.3.1. *[Hidden regularities] For any $\varphi_0 \in L^2(0,L)$, the solution $\varphi \in \mathcal{Z}_T$ of IBVP* (3.50) *possesses the sharp trace properties*

(3.51)
$$\sup_{x \in (0,L)} ||\partial_x^r \varphi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \le C_r ||\varphi_0||_{L^2(0,L)},$$

for $r = 0, 1, 2$.

Remark 4. *Equivalently, the solution of the system* (3.49) *has the sharp trace estimates*

(3.52)
$$\sup_{x \in (0,L)} ||\partial_x^r \psi(x,\cdot)||_{H^{\frac{1-r}{3}}(0,T)} \le C_r ||\psi_T||_{L^2(0,L)},$$

for $r = 0, 1, 2$.

In order to prove Theorem 3.3.1, let us to consider the system

$$
\begin{cases}
w_t + w_{xxx} = f, & x \in (0, L) \times (0, T), \\
w_{xx}(0, t) = k_1(t),\ w_x(L, t) = k_2(t),\ w_{xx}(L, t) = k_3(t), & t \in (0, T), \\
w(x, 0) = w_0(x), & x \in (0, L).
\end{cases} \tag{3.53}
$$

Proposition 3.3.2. *If* $w_0 \in L^2(0, L)$, $f \in L^1(0, T; L^2(0, L))$ *and* $\vec{k} := (k_1, k_2, k_3) \in \mathcal{K}_T$ *with*

$$\mathcal{K}_T := H^{-\frac{1}{3}}(0, T) \times L^2(0, T) \times H^{-\frac{1}{3}}(0, T),$$

then system (3.53) *admits a unique solution* $w \in \mathcal{Z}_T$ *which, in addition, has the hidden (or sharp trace) regularities*

$$\partial_x^r w \in L^\infty(0, L; H^{\frac{1-r}{3}}(0, T)), \quad \textit{for} \quad r = 0, 1, 2.$$

Moreover, there exist constants $C, C_r > 0$, *such that*

$$||w||_{\mathcal{Z}_T} \leq C \left(||w_0||_{L^2(0,L)} + ||\vec{k}||_{\mathcal{K}_T} + ||f||_{L^1(0,T;L^2(0,L))} \right)$$

and

$$\sup_{x \in (0,L)} ||\partial_x^r w(x, \cdot)||_{H^{\frac{1-r}{3}}(0,T)} \leq C_r \left(||w_0||_{L^2(0,L)} + ||\vec{k}||_{\mathcal{K}_T} + ||f||_{L^1(0,T;L^2(0,L))} \right)$$

where,

$$||\vec{k}||^2_{\mathcal{K}_T} := \left(||k_1||^2_{H^{-\frac{1}{3}}(0,T)} + ||k_2||^2_{L^2(0,T)} + ||k_3||^2_{H^{-\frac{1}{3}}(0,T)} \right),$$

for $r = 0, 1, 2$.

Proof. The proof follows the same ideas developed in Section 3.2, more precisely, Propositions 3.2.7 and 3.2.8, therefore, it will be omitted. □

Now we turn to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let us consider the set

$$\mathcal{X}_T := \left\{u \in \mathcal{Z}_T : \partial_x^r u \in L_x^\infty(0, L; H^{\frac{1-r}{3}}(0, T)), r = 0, 1, 2\right\}$$

which is a Banach space equipped with the norm

$$||u||_{\mathcal{X}_T} := ||u||_{\mathcal{Z}_T} + \sum_{r=0}^{2} ||\partial_x^r u||_{L_x^\infty(0,L;H^{\frac{1-r}{3}}(0,T))}.$$

According to Proposition 3.3.2, for any $v \in \mathcal{X}_\beta$ where $0 < \beta \leq T$ and any $\varphi_0 \in L^2(0, L)$, the system

$$(3.54) \quad \begin{cases} w_t + w_{xxx} = -v_x, & x \in (0, L) \times (0, T), \\ w_{xx}(0, t) = -v(0, t),\ w_x(L, t) = 0,\ w_{xx}(L, t) = -v(L, t), & t \in (0, T), \\ w(x, 0) = \psi_0(x), & x \in (0, L), \end{cases}$$

admits a unique solution $w \in \mathcal{X}_\beta$ and, moreover,

$$||w||_{\mathcal{X}_\beta} \leq C \left(||\psi_0||_{L^2(0,L)} + ||v(0, \cdot)||_{H^{-\frac{1}{3}}(0,T)} + ||v(L, \cdot)||_{H^{-\frac{1}{3}}(0,T)} + ||v_x||_{L^1(0,\beta;L^2(0,L))} \right),$$

where the constant $C > 0$ depends only on T. Since

$$||v_x||_{L^1(0,\beta;L^2(0,L))} \leq C\beta^{1/2}||v||_{\mathcal{X}_\beta},$$

$$||v(0, \cdot)||_{H^{-\frac{1}{3}}(0,\beta)} \leq ||v(0, \cdot)||_{L^2(0,\beta)} \leq \beta^{2/3}||v(0, \cdot)||_{L^6(0,\beta)} \leq C\beta^{2/3}||v(0, \cdot)||_{H^{\frac{1}{3}}(0,\beta)} \leq C\beta^{2/3}||v||_{\mathcal{X}_\beta}$$

and

$$||v(L, \cdot)||_{H^{-\frac{1}{3}}(0,\beta)} \leq ||v(L, \cdot)||_{L^2(0,\beta)} \leq \beta^{2/3}||v(L, \cdot)||_{L^6(0,\beta)} \leq C\beta^{2/3}||v(L, \cdot)||_{H^{\frac{1}{3}}(0,\beta)} \leq C\beta^{2/3}||v||_{\mathcal{X}_\beta},$$

then, we can define the map

$$\begin{aligned} \Gamma : \quad \mathcal{X}_\beta &\longrightarrow \quad \mathcal{X}_\beta \\ v &\mapsto \quad \Gamma(v) = w, \end{aligned}$$

for any $v \in \mathcal{X}_T$ and $\beta \in (0, \max\{1, T\}]$. Here $w \in \mathcal{X}_\beta$ is the corresponding solution of (3.54) and

$$||\Gamma(v)||_{\mathcal{X}_\beta} \leq C_1 ||\psi_0||_{L^2(0,L)} + C_2 \beta^{1/2} ||v||_{\mathcal{X}_\beta},$$

where C_1 and C_2 are positive constants depending only on T. Choosing $r > 0$ and $\beta \in (0, \max\{1, T\}]$ such that

$$r = 2C_1 ||\psi_0||_{L^2(0,L)} \quad \text{and} \quad 2C_2 \beta^{1/2} \leq \frac{1}{2},$$

then, for any

$$v \in \mathcal{B}_{\beta,r} = \{v \in \mathcal{X}_\beta : ||v||_{\mathcal{X}_\beta} \leq r\},$$

we have

$$||\Gamma(v)||_{\mathcal{X}_\beta} \leq r.$$

Moreover, for any $v_1, v_2 \in \mathcal{B}_{\beta,r}$, we get

$$||\Gamma(v_1) - \Gamma(v_2)||_{\mathcal{X}_\beta} \leq 2C_2 \beta^{1/2} ||v_1 - v_2||_{\mathcal{X}_\beta} \leq \frac{1}{2} ||v_1 - v_2||_{\mathcal{X}_\beta}.$$

Therefore, the map Γ is a contraction mapping on $\mathcal{B}_{\beta,r}$. Its fixed point $w = \Gamma(v) \in \mathcal{X}_\beta$ is the desired solution for $t \in (0, \beta)$. As the chosen β is independent of ψ_0, the standard continuation extension argument yields that the solution w belongs to $\mathcal{X}_\beta$. The proof is complete. $\square$

Finally, we conclude this section with an elementary estimate for the solution of (3.50).

Proposition 3.3.3. *Any solution φ of the adjoint system* (3.50) *with initial data $\varphi_0 \in L^2(0, L)$ satisfies*

$$||\varphi_0||^2_{L^2(0,L)} \leq \frac{1}{T} ||\varphi||^2_{L^2((0,L)\times(0,T))} + ||\varphi_x(0, \cdot)||^2_{L^2(0,T)} + ||\varphi(0, \cdot)||^2_{L^2(0,T)}. \tag{3.55}$$

Proof. Multiplying the equation (3.50) by $(T-t)\varphi$ and integrating by parts over $(0,L)\times(0,T)$, we get

$$\frac{T}{2}\int_0^L \varphi_0^2 dx = \frac{1}{2}\int_0^T\int_0^L \varphi^2 dxdt + \int_0^T \left(\frac{T-t}{2}\right)\left(-\varphi^2(L)+\varphi^2(0)+\varphi_x^2(0)\right)dt,$$

thus (3.55) holds. □

Equivalently, the following estimate holds for solutions ψ of the system (3.49):

$$||\psi_T||^2_{L^2(0,L)} \leq \frac{1}{T}||\psi||^2_{L^2((0,L)\times(0,T))} + ||\psi_x(L,\cdot)||^2_{L^2(0,T)} + ||\psi(0,\cdot)||^2_{L^2(0,T)}. \tag{3.56}$$

Remark 5. *As a comparison, it is worth pointing out that for the adjoint system of* (3.44), *which is given by*

$$\begin{cases} \xi_t+\xi_x+\xi_{xxx}=0, & (x,t)\in(0,L)\times(0,T), \\ \xi(0,t)=0,\ \xi(L,t)=0,\ \xi_x(0,t)=0, & t\in(0,T), \\ \xi(x,T)=\xi_T(x), & x\in(0,L), \end{cases} \tag{3.57}$$

the following inequality holds:

$$||\xi_T||_{L^2(0,L)} \leq \frac{1}{T}||\xi||_{L^2((0,L)\times(0,T))} + ||\xi_x(L,\cdot)||^2_{L^2(0,T)}. \tag{3.58}$$

The extra term $||\psi(0,\cdot)||^2_{L^2(0,T)}$ *in* (3.56) *brings new challenges in establishing the observability inequality of the adjoint system* (3.49).

3.3.2 Exact boundary controllability results: The linear system

This part of the paper focuses on the analysis of the exact controllability property for the linear system corresponding to (3.1). More precisely, given $T>0$ and $u_0\in L^2(0,L)$, we study the existence of controls $(h_1,h_2,h_3)\in\mathcal{H}_T$ such that the solution u of the system

$$\begin{cases} u_t+u_x+u_{xxx}=0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t)=h_1(t),\ u_x(L,t)=h_2(t),\ u_{xx}(L,t)=h_3(t), & \text{in } (0,T), \\ u(x,0)=u_0(x), & \text{in } (0,L), \end{cases} \tag{3.59}$$

satisfies

$$u(T,\cdot) = u_T \ \text{ in } \ L^2(0,L). \tag{3.60}$$

Definition 1. *Let $T > 0$. System* (3.59) *is exactly controllable in time T if for any initial and final data $u_0 \in L^2(0,L)$, $u_T \in L^2(0,L)$, respectively, there exist control functions $(h_1,h_2,h_3) \in \mathcal{H}_T$ such that the solution of* (3.59) *satisfies* (3.60).

Remark 6. *Without loss of generality, we may study only the exact controllability property for the case $u_0 = 0$. Indeed, let u_0, u_T be arbitrarily in $L^2(0,L)$ and let $(h_1,h_2,h_3) \in \mathcal{H}_T$ be controls which lead the solution u of* (3.59) *from the zero initial data to the final state $u_T - W(T)u_0$ (recall that $W(t)$ is the mild solution corresponding to* (3.59)*). It follows immediately that these controls also lead to the solution $u + W(\cdot)u_0$ of* (3.59) *from u_0 to the final state u_T.*

Now, we analyze the following cases for the system (3.59):

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } \ (0,T)\times(0,L), \\ u_{xx}(0,t) = 0, \ u_x(L,t) = h_2(t), \ u_{xx}(L,t) = 0, & \text{in } \ (0,T), \\ u(x,0) = u_0(x), & \text{in } \ (0,L), \end{cases} \tag{3.61}$$

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } \ (0,T)\times(0,L), \\ u_{xx}(0,t) = 0, \ u_x(L,t) = 0, \ u_{xx}(L,t) = h_3(t), & \text{in } \ (0,T), \\ u(x,0) = u_0(x), & \text{in } \ (0,L), \end{cases} \tag{3.62}$$

$$\begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } \ (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t), \ u_x(L,t) = h_2(t), \ u_{xx}(L,t) = 0, & \text{in } \ (0,T), \\ u(x,0) = u_0(x), & \text{in } \ (0,L), \end{cases} \tag{3.63}$$

$$
(3.64) \qquad \begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } \ (0,T)\times(0,L), \\ u_{xx}(0,t) = 0, \ u_x(L,t) = h_2(t), \ u_{xx}(L,t) = h_3(t), & \text{in } \ (0,T), \\ u(x,0) = u_0(x), & \text{in } \ (0,L), \end{cases}
$$

and

$$
(3.65) \qquad \begin{cases} u_t + u_x + u_{xxx} = 0, & \text{in } \ (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t), \ u_x(L,t) = 0, \ u_{xx}(L,t) = h_3(t), & \text{in } \ (0,T), \\ u(x,0) = u_0(x), & \text{in } \ (0,L). \end{cases}
$$

As we mentioned in the introduction the proofs for the other cases can be obtained by following the ideas we will apply in the next section.

3.3.2.1 Exact boundary controllability for system (3.61)

In this section we study the exact controllability in time T, for the system (3.61). We first give an equivalent condition for the exact controllability property:

Lemma 3.3.4. *Let $u_T \in L^2(0,L)$. Then, there exists a control $h_2(t) \in L^2(0,T)$, such that the solution u of* (3.61) *satisfies* (3.60) *if and only if*

$$
(3.66) \qquad \int_0^L u(x,T)\psi_T dx = \int_0^T h_2(t)\psi_x(t,L)dt,
$$

for any $\psi_T \in L^2(0,L)$ and ψ being the solution of the backward system (3.49).

Proof. (3.66) is obtained multiplying the partial differential equation in (3.61) by the solution ψ of (3.49) and integrating by parts. □

Proposition 3.3.5. *Set*

$$
\mathcal{M} = \left\{ \frac{2\pi}{\sqrt{3}}\sqrt{k^2+kl+l^2} \ : k, \ l \ \in \mathbb{N}^* \right\} \cup \{k\pi \ : k \in \mathbb{N}^*\} = \mathcal{N} \cup \{k\pi \ : k \in \mathbb{N}^*\}.
$$

Let $T > 0$ and $L \notin \mathcal{M}$ be given. There exists a bounded linear operator

$$
\Psi : \ \ L^2(0,L) \times L^2(0,L) \longrightarrow \ \ L^2(0,T)
$$

such that for any $u_0, u_T \in L^2(0,L)$*, if one chooses* $h_2 = \Psi(u_0, u_T)$*, then system* (3.61) *admits a solution* $u \in \mathcal{Z}_T$ *satisfying* (3.60)*.*

To study the controllability property, as it is well known, the following observability inequality will play a fundamental role.

Lemma 3.3.6. *Let* $L \in (0,+\infty)\backslash\mathcal{M}$ *and* $T > 0$ *be given. There exists* $C(T,L) > 0$ *such that*

$$||\psi_T||_{L^2(0,L)} \leq C||\psi_x(L,t)||_{L^2(0,T)} \tag{3.67}$$

holds for any $\psi_T \in L^2(0,L)$*, where* ψ *is the solution of* (3.49) *with initial data* ψ_T*.*

Proof. We proceed by contradiction as in [51, Proposition 3.3]. If (3.67) does not hold, then there exists a sequence $\{\psi_T^n\}_{n\in\mathbb{N}} \in L^2(0,L)$ with

$$||\psi_T^n||_{L^2(0,L)} = 1, \forall n \in \mathbb{N} \tag{3.68}$$

such that the corresponding solutions of (3.49) satisfy

$$1 = ||\psi_T^n||_{L^2(0,L)} > n||\psi_x^n(L,t)||_{L^2(0,T)}, \tag{3.69}$$

which implies $||\psi_x^n(L,t)||_{L^2(0,T)} \to 0$, as $n \to \infty$. Theorem 3.3.1 and Proposition 3.3.2 imply that the sequences $\{\psi^n\}_{n\in\mathbb{N}}$ and $\{\psi^n(0,t)\}_{n\in\mathbb{N}}$ are bounded in the spaces $L^2(0,T;H^1(0,L))$ and $H^{\frac{1}{3}}(0,T)$ respectively. According to Proposition 3.3.3, we have

$$||\psi_T^n||_{L^2(0,L)} \leq \frac{1}{T}||\psi^n||^2_{L^2((0,L)\times(0,T))} + ||\psi_x^n(L,\cdot)||^2_{L^2(0,T)} + ||\psi^n(0,\cdot)||^2_{L^2(0,T)}. \tag{3.70}$$

Since $\psi_t^n = -\psi_x^n - \psi_{xxx}^n$ is bounded in $L^2(0,T;H^{-2}(0,L))$ and the embedding

$$H^1(0,L) \hookrightarrow L^2(0,L) \hookrightarrow H^{-2}(0,L),$$

then we can prove that the sequence $\{\psi^n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(0,T;L^2(0,L))$ (see [58]). Furthermore, the second term on the right in (3.70) converges to zero in $L^2(0,T)$, and by the compact embedding

$$H^{\frac{1}{3}}(0,T) \hookrightarrow L^2(0,T)$$

the sequence $\{\psi^n(0,t)\}_{n\in\mathbb{N}}$ has a convergent subsequence in $L^2(0,T)$. Therefore by (3.70), $\{\psi_T^n\}_{n\in\mathbb{N}}$ is an $L^2(0,L)$–Cauchy sequence, thus, at least for a subsequence, we have

$$\psi_T^n \longrightarrow \psi_T \text{ in } L^2(0,L). \tag{3.71}$$

By Theorem 3.3.1 it holds that

$$\psi_x^n(L,t) \longrightarrow \psi_x(L,t) \text{ in } L^2(0,T). \tag{3.72}$$

From (3.68), (3.71) and (3.72), we have ψ is a solution of

(3.73)
$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ \psi(0,t)+\psi_{xx}(0,t)=0, \quad \psi_x(0,t)=0, \quad \psi(L,t)+\psi_{xx}(L,t)=0, & \text{in } (0,T), \end{cases}$$

satisfying the additional boundary condition

$$\psi_x(L,t) = 0, \tag{3.74}$$

and

$$||\psi_T||_{L^2(0,L)} = 1. \tag{3.75}$$

Notice that (3.75) implies that the solutions of (3.73)-(3.74) cannot be identically zero. Therefore, by the following Lemma 3.3.7, one can conclude that $\psi \equiv 0$, therefore, $\psi_T(x) \equiv 0$, which contradicts (3.75). □

Lemma 3.3.7. *For any $T > 0$, let N_T denote the space of the initial states $\psi_T \in L^2(0,L)$ such that the mild solution ψ of* (3.73) *satisfies* (3.74). *Then, for $L \in (0,+\infty)\backslash\mathcal{M}$, $N_T = \{0\}, \forall T > 0$.*

Proof. The proof uses the same arguments as those given in [51]. Therefore, if $N_T \neq \{0\}$, the map $\psi_T \in \mathbb{C}N_T \longrightarrow A(\psi_T) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denote the complexification of N_T) has (at least) one eigenvalue, hence, there exists $\lambda \in \mathbb{C}$ and $\psi_0 \in H^3(0,L)\backslash\{0\}$ such that

$$\begin{cases} \lambda\psi_0 = -\psi_0' - \psi_0''', \\ \psi_0(0) + \psi_0''(0) = 0, \quad \psi_0(L) + \psi_0''(L) = 0, \quad \psi_0'(0) = 0, \quad \psi_0'(L) = 0. \end{cases} \tag{3.76}$$

To conclude the proof of Lemma 3.3.7, we prove that this does not hold if $L \notin \mathcal{M}$. To simplify the notation, henceforth we denote $\psi := \psi_0$. □

Lemma 3.3.8. *Let $L > 0$. Consider the assertion*

$$(\mathcal{F}) \quad \exists\lambda \in \mathbb{C},\ \exists\psi \in H^3(0,L)\backslash\{0\} \text{ such that } \begin{cases} \lambda\psi = -\psi' - \psi''', \\ \psi(0) + \psi''(0) = 0, \quad \psi(L) + \psi''(L) = 0, \\ \psi'(0) = 0, \quad \psi'(L) = 0. \end{cases}$$

Then, $(\mathcal{F})$ holds if and only if $L \in \mathcal{M}$.

Proof. We will use the argument developed in [51, Lemma 3.5]. Assume that ψ satisfies $\mathcal{F}$. Let us introduce the notation $\hat{\psi}(\xi) = \int_0^L \psi(\xi)e^{-ix\xi}dx$. Then, multiplying the equation (3.76) by $e^{-ix\xi}$, integrating by parts in $(0,L)$ and using the boundary condition we obtain

$$(\lambda + (i\xi) + (i\xi)^3)\hat{\psi}(\xi) = (i\xi)^2\psi(0) - (i\xi)^2\psi(L)e^{-iL\xi}. \tag{3.77}$$

Setting $\lambda = -ip$, $\alpha = \psi(0)$ and $\psi(L)$, we have

$$\hat{\psi}(\xi) = -i\xi^2\frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}. \tag{3.78}$$

Using the Paley-Wiener theorem (see [63, Section 4, page 161]) and the usual characterization of $H^2(\mathbb{R})$ by means of the Fourier transform we see that $\mathcal{F}$ is equivalent to the existence of $p \in \mathbb{C}$ and

$$(\alpha, \beta) \in \mathbb{C}^2\backslash\{(0,0)\},$$

such that

$$f(\xi) := \xi^2 \frac{\alpha - \beta e^{-iL\xi}}{\xi^3 - \xi + p}$$

satisfies

a) f is an entire function in $\mathbb{C}$;

b) $\displaystyle\int_{\mathbb{R}} |f(\xi)|^2 (1 + |\xi|^2)^2 d\xi < \infty$;

c) $\forall \xi \in \mathbb{C}$, we have that $|f(\xi)| \leq c(1 + |\xi|)^k e^{L|\operatorname{Im} \xi|}$ for some positive constants c and k.

Recall that f is an entire function if only if the roots ξ_0, ξ_1, ξ_2 of $Q(\xi) := \xi^3 - \xi + p$ are roots of

$$s(\xi) := \xi^2 (\alpha - \beta e^{-iL\xi}). \tag{3.79}$$

Notice that all the roots of $\alpha - \beta e^{-iL\xi}$ are simple roots, otherwise $\alpha = \beta = 0$, which implies $\psi(0) = \psi(L) = 0$. Using system (3.76) we conclude by the unique continuation property ($\psi(0) = \psi_x(0) = \psi_{xx}(0) = 0$ for any $t \in (0, T)$, see e.g. [65]) that $\psi \equiv 0$.

If we assume that $Q(\xi)$ and $\alpha - \beta e^{-iL\xi}$ share the same roots, then we can write the roots of $Q(\xi)$ in the following way

$$\xi_1 := \xi_0 + k\frac{2\pi}{L} \quad \text{and} \quad \xi_2 := \xi_1 + l\frac{2\pi}{L} \tag{3.80}$$

here k and l are positive integers, thus

$$Q(\xi) = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2), \tag{3.81}$$

which implies

$$\left\{ \begin{array}{rcl} \xi_0 + \xi_1 + \xi_2 & = & 0, \\ \xi_0\xi_1 + \xi_0\xi_2 + \xi_1\xi_2 & = & -1, \\ \xi_0\xi_1\xi_2 & = & -p. \end{array} \right. \tag{3.82}$$

Thus, as in [51], we have

$$
(3.83) \qquad \begin{cases} L &= 2\pi\sqrt{\dfrac{k^2+kl+l^2}{3}}, \\[2ex] \xi_0 &= -\dfrac{1}{3}(2k+l)\dfrac{2\pi}{L}, \\[2ex] p &= -\xi_0\left(\xi_0+k\dfrac{2\pi}{L}\right)\left(\xi_0+(k+l)\dfrac{2\pi}{L}\right). \end{cases}
$$

Now assume that $\xi = 0$ is a root of $Q(\xi)$, but not a root of $\alpha - \beta e^{-iL\xi}$. Then the roots of $Q(\xi)$ can be written as 0, ξ_1, $\xi_1 + k\frac{2\pi}{L}$ with k being a positive integer. We have

$$
(3.84) \qquad \begin{cases} \xi_1+\xi_2 &= 0, \\ \xi_1\xi_2 &= -1, \\ 0 &= -p, \end{cases}
$$

and, consequently, follows that

$$
(3.85) \qquad \begin{cases} L &= k\pi, \\ \xi_1 &= -k\dfrac{\pi}{L}, \\ p &= 0. \end{cases}
$$

Hence, $\mathcal{F}$ holds if and only if $L \in \mathcal{M}$. This completes the proof of Lemma 3.3.8 and, consequently, the proof of Lemma 3.3.7. □

Remark 7. *When $k = l$,* (3.83) *is reduced to $L = 2k\pi$, $\xi_0 = -1$ ($\xi_1 = 0$, $\xi_2 = 1$) and $p = 0$, hence, $\lambda = 0$. This yields to the unobservable steady solution of* (3.49)*: $\psi(x) = acos(x)$, for $a \in \mathbb{R}$. Note that the solution $\psi(x) = acos(x)$* ***is not the solution of*** (3.61)*. However, when we multiply the system* (3.61) *by the solution ψ of* (3.49) *and integrating by parts, we have*

$$
\int_0^L u_T(x)acos(x)dx = \int_0^L acos(x)u_0(x)dx.
$$

Now, if we consider $u_0 = 0$, we can conclude that $u_T(x) \neq a\cos(x)$, which means that the unobservable solutions of (3.61) *are the projections of the solutions of the adjoint system* (3.49).

Now, we turn to proving Proposition 3.3.5.

Proof of Proposition 3.3.5. Without loss of generality, we assume that $u_0 = 0$ (see Remark 6). Let us define the bounded linear map

$$\begin{aligned} \Xi : \quad L^2(0,L) &\longrightarrow \quad L^2(0,L) \\ \psi_T(\cdot) &\mapsto \quad \Xi(\psi_T(\cdot)) = u(\cdot, T), \end{aligned}$$

where u is the solution of (3.61) with $h_2(t) = \psi_x(L,t)$ and ψ the solution of the system (3.49). According to Lemmas 3.3.4 and 3.3.6, we obtain

$$(\Xi(\psi_T), \psi_T)_{L^2(0,L)} = ||\psi_x(L,\cdot)||^2_{L^2(0,T)} \geq C^{-1}||\psi_T||^2_{L^2(0,L)}. \tag{3.86}$$

and by the Lax-Milgran Theorem, we can conclude that Ξ is invertible. Now, for a given $u_T \in L^2(0,L)$, let us define $\psi_T := \Xi^{-1}u_T$ thus system (3.49) is solved with $\psi \in \mathcal{Z}_T$. If we set $h_2(t) = \psi_x(L,t)$ in system (3.49) the corresponding solution $u \in \mathcal{Z}_T$ satisfies (3.60) and this complete the proof of Proposition 3.3.5. $\square$

3.3.2.2 Exact boundary controllability for system (3.62)

In this subsection we study the exact controllability, in time T, for the system (3.62). We first give an equivalent condition for the exact controllability property:

Lemma 3.3.9. *Let $u_T \in L^2(0,L)$. Then, there exist a control $h_3(t) \in H^{-\frac{1}{3}}(0,T)$, such that the solution u of* (3.62) *satisfies* (3.60) *if and only if*

$$\int_0^L u(x,T)\psi_T dx = -\int_0^T h_3(t)\psi(t,L)dt, \tag{3.87}$$

for any $\psi_T \in L^2(0,L)$ and ψ being the solution of the backward system (3.49).

Proof. The relation (3.87) is obtained multiplying the equation in (3.62) by the solution ψ of (3.49) and integrating by parts. $\square$

Before presenting the main result of this section, we define the set

$$\mathcal{R} := \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a, b \in \mathbb{C} : X = e^a, Y = e^b \text{ are solutions of } AX^2 + BX + C = 0 \text{ and } Y = -\tfrac{b_3+b_1X}{b_2} \Big\}.$$

Here $A = a_1b_1$, $B = a_1b_3 - a_2b_2 + a_3b_1$ and $C = a_3b_3$, where

$$a_1 := (b^2-a^2)(a+b), \quad a_2 := b^2e^c(2a+b), \quad a_3 := -a^2e^c(a+2b) \tag{3.88}$$

$$b_1 := -a^3(a+2b), \quad b_2 := ab^2(2a+b) \text{ and } b_3 := ae^c(b^2-a^2)(a+b). \tag{3.89}$$

Thus, the following result holds:

Proposition 3.3.10. *Let $T > 0$ and $L \notin \mathcal{R}$ be given. There exists a bounded linear operator*

$$\Psi : \; L^2(0,L) \times L^2(0,L) \longrightarrow \; H^{-\frac{1}{3}}(0,T)$$

such that for any $u_0, u_T \in L^2(0,L)$, if one chooses $h_3 = \Psi(u_0,u_T)$, then system (3.62) *admits a solution $u \in \mathcal{Z}_T$ satisfying* (3.60).

As before, let us consider the following observability inequality.

Lemma 3.3.11. *Let $L \in (0,+\infty)\backslash\mathcal{R}$ and $T > 0$ be given. There exists $C(T,L) > 0$ such that*

$$||\psi_T||_{L^2(0,L)} \le C||\Delta_t^{\frac{1}{3}}\psi(L,t)||_{L^2(0,T)} \tag{3.90}$$

holds for any $\psi_T \in L^2(0,L)$, where ψ is the solution of (3.49) *with initial data ψ_T.*

Proof. We proceed by contradiction. If (3.90) does not holds, then there exists a sequence $\{\psi_T^n\}_{n\in\mathbb{N}} \in L^2(0,L)$ such that

$$||\psi_T^n||_{L^2(0,L)} = 1, \forall n \in \mathbb{N} \tag{3.91}$$

and

$$\int_0^T |\Delta_t^{\frac{1}{3}} \psi(L,t)|^2 dt \to 0 \text{ in } L^2(0,T), \tag{3.92}$$

where ψ^n is the solution of (3.49) with initial data ψ_T. Arguing as in the proof of Lemma 3.3.6 we can conclude that $\{\psi_T^n\}_{n\in\mathbb{N}}$ is an $L^2(0,L)$–Cauchy sequence. Then, at least for a subsequence, we have

$$\psi_T^n \longrightarrow \psi_T \text{ in } L^2(0,L) \tag{3.93}$$

and

$$\psi^n(L,t) \longrightarrow \psi(L,t) \text{ in } L^2(0,T), \tag{3.94}$$

thus ψ satisfies

(3.95)
$$\begin{cases} \psi_t + \psi_x + \psi_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ \psi(0,t) + \psi_{xx}(0,t) = 0, \quad \psi_x(0,t) = 0, \quad \psi(L,t) + \psi_{xx}(L,t) = 0, & \text{in } (0,T), \end{cases}$$

the additional boundary condition

$$\psi(L,t) = 0, \tag{3.96}$$

and

$$||\psi_T||_{L^2(0,L)} = 1. \tag{3.97}$$

Observe that (3.97) implies that the solution of (3.95)-(3.96) cannot be identically zero, therefore, by the following Lemma 3.3.12, one can conclude that $\psi \equiv 0$, thus, $\psi_T(x) \equiv 0$, this contradicts (3.97) which achieves the desired result. □

Lemma 3.3.12. *For any $T > 0$, let N_T denote the space of initial states $\psi_T \in L^2(0,L)$ such that the mild solution ψ of* (3.95) *satisfies* (3.96). *Then, for $L \in (0,+\infty)\backslash\mathcal{R}$, $N_T = \{0\}, \forall T > 0$.*

Proof. The proof uses the arguments given in [51]. Therefore, if $N_T \neq \{0\}$, the map $\psi_T \in \mathbb{C}N_T \longrightarrow A(\psi_T) \in \mathbb{C}N_T$ (where $\mathbb{C}N_T$ denotes the complexification of N_T) has (at least) one eigenvalue. Hence, there exists $\lambda \in \mathbb{C}$ and $\psi_0 \in H^3(0,L)\backslash\{0\}$ such that

$$\begin{cases} \lambda\psi_0 = -\psi_0' - \psi_0''', \\ \psi_0(0) + \psi_0''(0) = 0, \quad \psi_0(L) = 0, \quad \psi_0''(L) = 0, \quad \psi_0'(0) = 0. \end{cases} \tag{3.98}$$

For simplicity we will consider $\psi = \psi_0$.

The solution of (3.98) can be written as $\psi(x) = \sum_{j=1}^{3} C_j e^{\mu_j x}$ where the μ_j are the roots of the polynomial

$$P(\mu) := \lambda + \mu + \mu^3.$$

More explicitly, they satisfy

$$\begin{cases} \mu_0 + \mu_1 + \mu_2 &= 0, \\ \mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2 &= -1, \\ \mu_0\mu_1\mu_2 &= -p, \end{cases} \tag{3.99}$$

and the constants C_j, $j = 1,2,3$, solve the system

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ e^{\mu_1 L} & e^{\mu_2 L} & e^{\mu_3 L} \\ \mu_1^2 e^{\mu_1 L} & \mu_2^2 e^{\mu_2 L} & \mu_3^2 e^{\mu_3 L} \\ \mu_1^2 + 1 & \mu_2^2 + 1 & \mu_3^2 + 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.100}$$

Let us denote $a = L\mu_1$ and $b = L\mu_2$, then by (3.99), $c = L\mu_3 = -(a+b)$ and

$$L^2 = -(a^2 + ab + b^2).$$

Reducing the rows of the matrix, we have

$$\begin{pmatrix} 1 & b/a & -(a+b)/a \\ 0 & 1 & \frac{a}{ae^b - be^a}\left(e^c + \frac{a+b}{a}e^a\right) \\ 0 & 0 & A_1 \\ 0 & 0 & A_2 \end{pmatrix} \tag{3.101}$$

with

$$A_1 = c^2 e^c + a(a+b)e^a - (b^2 e^b - abe^a)\left(e^c + \frac{a+b}{a}e^a\right)\frac{a}{ae^b - be^a}$$

and

$$A_2 = c^2 + L^2 + \frac{a+b}{a}(a^2+L^2) - \left[(b^2+L^2) - \frac{b}{a}(a^2+L^2)\right]\left[e^c + \frac{a+b}{a}e^a\right]\frac{a}{ae^b - be^a},$$

therefore the system has nonzero solutions if and only if

$$A_1 = 0, \quad A_2 = 0,$$

or equivalently

$$\text{(3.102)} \qquad \begin{cases} (b^2-a^2)(a+b)e^a e^b + be^c(2a+b)e^a + ae^c(-a^2-2ab)e^b &= 0, \\ -a^3(a+2b)e^a + ab^2(2a+b)e^b + (b^2-a^2)(a+b)ae^c &= 0. \end{cases}$$

Setting $X := e^a$ and $Y := e^b$, we have the system

$$\text{(3.103)} \qquad \begin{cases} a_1 XY + a_2 X + a_3 Y &= 0, \\ b_1 X + b_2 Y + b_3 &= 0, \end{cases}$$

where a_i and b_i, for $i = 1, 2, 3$ were defined in (3.88) and (3.89), respectively. Thus, the set of nonzero solutions is empty if and only if L does not belong to

$$\mathcal{R} = \Big\{ L \in \mathbb{R}^+ : L^2 = -(a^2+ab+b^2) \text{ with } a, b \in \mathbb{C} : X = e^a, Y = e^b \text{ are solutions of } AX^2 + BX + C = 0 \text{ and } Y = -\tfrac{b_3 + b_1 X}{b_2} \Big\},$$

thus the proof is complete. □

Now, we prove Proposition 3.3.10.

Proof of Proposition 3.3.10. Without loss of generality, we assume that $u_0 = 0$ (see Remark 6). Let us define the bounded linear map

$$\begin{aligned} \Xi : \quad L^2(0,L) &\longrightarrow \quad L^2(0,L) \\ \psi_T(\cdot) &\mapsto \quad \Xi(\psi_T(\cdot)) = u(\cdot, T), \end{aligned}$$

where u is the solution of (3.62) with

$$h_3(t) = \Delta_t^{\frac{2}{3}} \psi(L,t)$$

and ψ the solution of the system (3.49). According to Lemmas 3.3.9 and 3.3.11, we have

$$(\Xi(\psi_T), \psi_T)_{L^2(0,L)} = ||\Delta_t^{\frac{2}{3}} \psi(L,\cdot)||^2_{L^2(0,T)} \geq C^{-1} ||\psi_T||^2_{L^2(0,L)}, \tag{3.104}$$

thus, the proof follows by using the Lax-Milgran Theorem. □

Remark 8. *When we consider two control inputs the critical length phenomenon will not occur. More precisely, the following result holds:*

Proposition 3.3.13. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Theta: \quad L^2(0,L) \times L^2(0,L) \longrightarrow \quad H^{-\frac{1}{3}}(0,T) \times L^2(0,T)$$

such that for any $u_0, u_T \in L^2(0,L)$, if one chooses

$$(h_1, h_2) = \Psi(u_0, u_T),$$

then the system (3.63) *admits a solution $u \in \mathcal{Z}_T$ satisfying* (3.60).

Proposition 3.3.14. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Pi: \quad L^2(0,L) \times L^2(0,L) \longrightarrow \quad L^2(0,T) \times H^{-\frac{1}{3}}(0,T)$$

such that for any $u_0, u_T \in L^2(0,L)$, if one chooses

$$(h_2, h_3) = \Psi(u_0, u_T),$$

then the system (3.64) *admits a solution $u \in \mathcal{Z}_T$ satisfying* (3.60).

Proposition 3.3.15. *Let $T > 0$ and $L > 0$ be given. There exists a bounded linear operator*

$$\Lambda: \quad L^2(0,L) \times L^2(0,L) \longrightarrow \quad H^{-\frac{1}{3}}(0,T) \times H^{-\frac{1}{3}}(0,T)$$

such that for any $u_0, u_T \in L^2(0,L)$, *if one chooses*

$$(h_1, h_3) = \Psi(u_0, u_T),$$

then the system (3.65) *admits a solution* $u \in \mathcal{Z}_T$ *satisfying* (3.60).

Note that Propositions 3.3.13-3.3.15 follow as a consequence of the following observability inequalities for the solution of the backward system (3.49)*:*

$$||\psi_T||_{L^2(0,L)} \leq C\left(||\Delta_t^{\frac{1}{3}}\psi(0,t)||_{L^2(0,T)} + ||\psi_x(L,t)||_{L^2(0,T)}\right), \tag{3.105}$$

$$||\psi_T||_{L^2(0,L)} \leq C\left(||\psi_x(L,t)||_{L^2(0,T)} + ||\Delta_t^{\frac{1}{3}}\psi(L,t)||_{L^2(0,T)}\right), \tag{3.106}$$

and

$$||\psi_T||_{L^2(0,L)} \leq C\left(||\Delta_t^{\frac{1}{3}}\psi(0,t)||_{L^2(0,T)} + ||\Delta_t^{\frac{1}{3}}\psi(L,t)||_{L^2(0,T)}\right). \tag{3.107}$$

The proofs of (3.105)-(3.107) *are similar to the proof of Lemma 3.3.6 (see also Lemma 3.3.11).*

3.3.3 Exact boundary controllability results: The nonlinear system

In this section we consider the nonlinear system

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L), \end{cases} \tag{3.108}$$

with one control input h_2, that is, $h_1 = h_3 = 0$. The proof of Theorem 3.1.3 will be presented.

Proof of Theorem 3.1.3. Rewrite the system (3.108) in its integral form

$$u(t) = W_0(t)u_0 + W_{bdr}(t)h_2 - \int_0^t W_0(t-\tau)(uu_x)(\tau,x)d\tau. \tag{3.109}$$

For any $v \in \mathcal{Z}_T$, let us define

$$\nu(T,v) := \int_0^T W_0(T-\tau)(vv_x)d\tau.$$

By Proposition 3.3.5, we can define, for any $u_0, u_T \in L^2(0,L)$,

$$h_2 = \Psi(u_0, u_T + \nu(T,v)),$$

thus,

$$v(t) = W_0(t)u_0 + W_{bdr}\Psi(u_0, u_t + \nu(T,v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau,x)d\tau$$

satisfies

$$v(x,0) = u_0(x), \qquad v(x,T) = u_T(x) + \nu(T,v) - \nu(T,v) = u_T.$$

This leads us to consider the map

$$\Gamma(v) = W_0(t)u_0 + W_{bdr}\Psi(u_0, u_t + \nu(T,v)) - \int_0^t W_0(t-\tau)(vv_x)(\tau,x)d\tau.$$

If we can show that the map Γ is a contraction in an appropriate metric space, then its fixed point v is a solution of (3.108) with $h_2 = \Psi(u_0, u_T + \nu(T,v))$ that satisfies

$$v(x,0) = u_0(x), \qquad v(x,T) = u_T.$$

Next, we show that this is indeed the case. Let

$$B_r = \{z \in \mathcal{Z}_T : ||z||_{\mathcal{Z}_T} \le r\}.$$

(i) Γ *maps* B_r *into itself.* From Proposition 3.2.7, we infer the existence of a constant $C_1 > 0$ such that for any $v \in \mathcal{Z}_T$, we have

$$||\Gamma(v)||_{\mathcal{Z}_T} \le C_1 \left(||u_0||_{L^2(0,L)} + ||\Psi(u_0, u_t + \nu(T,v))||_{L^2(0,L)} - \int_0^T ||vv_x||_{L^2(0,L)}(t)dt \right).$$

Since

$$||\Psi(u_0, u_t + \nu(T,v))||_{L^2(0,L)} \le C_2 \left(||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)} + ||\nu(T,v)||_{L^2(0,L)} \right)$$

and

$$||\nu(T,v)||_{L^2(0,L)} \leq \int_0^T ||vv_x||_{L^2(0,L)}(t)dt \leq C_3||v||^2_{\mathcal{Z}_T}$$

we infer that

$$||\Gamma(v)||_{\mathcal{Z}_T} \leq C_3\left(||u_0||_{L^2(0,L)} + ||u_T||_{L^2(0,L)}\right) + C_4||v||^2_{\mathcal{Z}_T},$$

for any $v \in \mathcal{Z}_T$ where C_3 and C_4 are constants depending only on T. By choosing r and δ such that

$$r = 2C_3\delta \quad \text{and} \quad 4C_3C_4\delta < \frac{1}{2},$$

we obtain that the operator Γ maps B_r into itself.

(ii) Γ *is a contraction.* Pick any $\tilde{v}, v \in B_r$. Thus we deduce that for some constant C, independent of v, $\tilde{v}$, and r, we have

$$||\Gamma(v) - \Gamma(\tilde{v})||_{\mathcal{Z}_T} \leq \gamma||v - \tilde{v}||_{\mathcal{Z}_T},$$

for $\gamma = 8C_3C_4\delta < 1$. Therefore the map Γ has a fixed point in B_r by the Banach fixed-point theorem. The proof of Theorem 3.1.3 is complete. □

Theorems 3.1.5, 3.1.6, 3.1.7 and 3.1.8 can be proved using the same arguments as in the proof of Theorem 3.1.3, therefore their proofs will be omitted.

3.4 Final comments and remarks

Our discussion has been focused on the boundary controllability of a class of boundary control system described by the KdV equation on a bounded domain $(0, L)$

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & \text{in } (0,T)\times(0,L), \\ u_{xx}(0,t) = h_1(t),\ u_x(L,t) = h_2(t),\ u_{xx}(L,t) = h_3(t), & \text{in } (0,T), \\ u(x,0) = u_0(x), & \text{in } (0,L). \end{cases} \tag{3.110}$$

The first study of controllability of a class of KdV equation on finite domain was made by Lionel Rosier in 1997 ([51]). In this article, the author studied the controllability of the system

$$
\begin{cases}
y_t + y_x + yy_x + y_{xxx} = 0, & \text{in } (0,T)\times(0,L),\\
y(0,t) = f_1(t),\ y(L,t) = f_2(t),\ y_x(L,t) = f_3(t), & \text{in } (0,T),\\
y(x,0) = y_0(x), & \text{in } (0,L),
\end{cases}
\tag{3.111}
$$

Rosier proved that if f_3 is used as a control, then the linear system associated to (3.111) is locally exactly controllable for $L \notin \mathcal{N}$, where $\mathcal{N}$ is defined as in (3.4).

In 2013, Cerpa *et al*, [17], considered the KdV equation with a different kind of boundary conditions

$$
\begin{cases}
v_t + v_x + vv_x + v_{xxx} = 0, & \text{in } (0,T)\times(0,L),\\
v(0,t) = g_1(t),\ v_x(L,t) = g_2(t),\ v_{xx}(L,t) = g_3(t), & \text{in } (0,T),\\
v(x,0) = v_0(x), & \text{in } (0,L).
\end{cases}
\tag{3.112}
$$

Using the techniques developed by Rosier and a new tool, the *sharp Kato smoothing property* for solutions of the KdV system (3.112), the authors proved that, if just one control, $g_2(t)$, acts on the boundary condition, the linear system associated to (3.112) is locally exactly controllable for $L \notin \mathcal{F}$, where $\mathcal{F}$ is defined as in (3.6). Note that in [17] the authors did not characterize the critical set $\mathcal{F}$. However, when we consider the KdV equation with new boundary conditions, we prove that system (3.110) with only one control input, $h_2(t)$, is locally exactly controllable if and only if $L \notin \mathcal{M}$. In this case, $\mathcal{M}$ is defined as in (3.12), that is, we can characterize the critical set. Actually, a more detailed picture of the control results obtained in these papers are presented in the following tables:

Controls				Properties
$h_1(t)$	$h_2(t)$	$h_3(t)$	Space of Control	Critical Length
0	$\star$	0	$h_2 \in L^2(0,T)$	$\mathcal{M}$
0	0	$\star$	$h_3 \in H^{-1/3}(0,T)$	$\mathcal{R}$
$\star$	0	$\star$	$h_1, h_3 \in H^{-1/3}(0,T)$	$\emptyset$
0	$\star$	$\star$	$h_2 \in L^2(0,T), h_3 \in H^{-1/3}(0,T)$	$\emptyset$
$\star$	$\star$	0	$h_1 \in H^{-1/3}(0,T), h_2 \in L^2(0,T)$	$\emptyset$
$\star$	$\star$	$\star$	$h_1 \in H^{-1/3}(0,T), h_2 \in L^2(0,T), h_3 \in H^{-1/3}(0,T)$	$\emptyset$

Table 1. Exact controllability results for the linear system associated to (3.110).

Controls				Properties
$f_1(t)$	$f_2(t)$	$f_3(t)$	Space of Control	Critical Length
0	0	$\star$	$f_3 \in L^2(0,T)$	$\mathcal{N}$
0	$\star$	0	$f_2 \in H^{1/3}(0,T)$	$\mathcal{N}^*$
$\star$	0	$\star$	$f_1 \in H^{1/3}(0,T), f_3 \in L^2(0,T)$	$\emptyset$
0	$\star$	$\star$	$f_2 \in H^{1/3}(0,T), f_3 \in L^2(0,T)$	$\emptyset$
$\star$	$\star$	0	$f_1, f_2 \in H^{1/3}(0,T)$	$\emptyset$
$\star$	$\star$	$\star$	$f_1, f_2 \in H^{1/3}(0,T), f_3 \in L^2(0,T)$	$\emptyset$

Table 2. Exact controllability results for the linear system associated to (3.111).

Controls				Properties
$g_1(t)$	$g_2(t)$	$g_3(t)$	Space of Control	Critical Length
0	$\star$	0	$g_2 \in L^2(0,T)$	$\mathcal{F}$
0	0	$\star$	$g_3 \in H^{-1/3}(0,T)$	$\mathcal{N}^*$
$\star$	0	$\star$	$g_1 \in H^{1/3}(0,T), g_3 \in H^{-1/3}(0,T)$	$\emptyset$
0	$\star$	$\star$	$g_2 \in L^2(0,T), g_3 \in H^{-1/3}(0,T)$	$\emptyset$
$\star$	$\star$	0	$g_1 \in H^{1/3}(0,T), g_2 \in L^2(0,T)$	$\emptyset$
$\star$	$\star$	$\star$	$g_1 \in H^{1/3}(0,T), g_2 \in L^2(0,T), g_3 \in H^{-1/3}(0,T)$	$\emptyset$

Table 3. Exact controllability results for the linear system associated to (3.112).

Moreover, systems (3.110), (3.111) and (3.112) possess another property of controllability: All theses system are null controllable when only one control input is considered, more precisely, $h_1(t)$, $f_1(t)$ and $g_1(t)$, respectively, as we mentioned in this article (for more references see [17, 30, 32]).

Observe that most of the results for the systems (3.110)-(3.112) have been established locally: *one can only guide a small amplitude initial state to a small amplitude terminal state by choosing appropriate boundary control inputs.* So, the following question arises naturally:

Question A:

Are the nonlinear systems (3.110)-(3.112) *globally exactly boundary controllable?*

In order to complete the study of the exact controllability of system (3.110) is necessary to investigate the so-called *critical length problems*. For system (3.111), Coron and Crépeau in [23], proved that this system is locally controllable around the origin for $L = 2k\pi$, if $f_3(t)$ is considered as a control input. The authors applied the *return method* which was introduced in [22] (see also [1, 2]). However, the minimal time required with this approach is far from being optimal. In addition, Cerpa in [14] considered the same system with only one control input ($f_3(t)$) and studied this problem with a critical length for which the linearized control system is not controllable, moreover, he proved that the time for local controllability of the system (3.111) is large enough. Due to Remark 7, we believe that with the same approach used in [14, 22] the controllability of the nonlinear system (3.110), with a control input $h_2(t)$, can be proved when $L \in \mathcal{M}$. However, this problem is still open.

Question B:

Is the nonlinear system (3.110)*, with only one control input* $h_2(t)$ *in action, exactly controllable on the critical set* $\mathcal{M}$*?*

Finally, if we consider the control acting in the boundary condition $u_{xx}(L,t)$, that is, $h_1(t) = h_2(t) = 0$, we have a new critical set for which we do not have a characterization, therefore, the following question is a still open problem:

Question C:

Is the nonlinear system (3.110) *with only one control input* $h_3(t)$ *in action exactly controllable when the length* L *of the spatial domain belongs to the critical set* $\mathcal{R}$*?*

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